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## Framed sheaves on a quadric surface

Nguyen Thuc Huy Le  
*University of Massachusetts Amherst*

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# FRAMED SHEAVES ON A QUADRIC SURFACE

A Dissertation Presented

by

NGUYEN-THUC-HUY LE

Submitted to the Graduate School of the  
University of Massachusetts Amherst in partial fulfillment  
of the requirements for the degree of

DOCTOR OF PHILOSOPHY

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Department of Mathematics and Statistics

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A Dissertation Presented

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NGUYEN-THUC-HUY LE

Approved as to style and content by:

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Paul Hacking, Chair

---

Tom Braden, Member

---

Eyal Markman, Member

---

David Kastor, Outside Member

---

Farshid Hajir, Department Head  
Department of Mathematics and Statistics

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# ABSTRACT

## FRAMED SHEAVES ON A QUADRIC SURFACE

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NGUYEN-THUC-HUY LE

B.Sc., UNIVERSITY OF CAMBRIDGE

M.Sc., KING'S COLLEGE LONDON

M.S., UNIVERSITY OF MASSACHUSETTS AMHERST

Ph.D., UNIVERSITY OF MASSACHUSETTS AMHERST

Directed by: Professor Paul Hacking

Nakajima [42, chapter 2] obtains a quiver variety description for the moduli of torsion free sheaves on the projective plane with fixed rank and Chern classes and a framing along a fixed line (such a sheaf is called a *framed sheaf*.) In this thesis, we study the moduli of framed sheaves on the projective plane with a framing along a fixed smooth *conic*. This is embedded as the fixed locus of a certain involution on the moduli space  $\mathcal{M}$  of framed sheaves on the smooth quadric surface with a framing along a fixed hyperplane section. We obtain a description for  $\mathcal{M}$  in terms of a hyperkähler quotient (see Hitchin et al. [20, section 3.E] ) of the coadjoint orbit

$$L = \{(\widetilde{W}_I, \widetilde{W}_{II}) \mid \widetilde{W}_I \cap \widetilde{W}_{II} = 0\} \subset G(n, \widetilde{W}) \times G(n, \widetilde{W}),$$

where  $\widetilde{W}$  is a fixed  $2n$ -dimensional vector space. Then the involution above is given by simply switching  $\widetilde{W}_I$  and  $\widetilde{W}_{II}$ . We then study in details an example with low rank

and second Chern class. We also compute the Poincaré polynomial of  $\mathcal{M}$ . Finally, we conjecture that the hyperkähler quotient of the twistor deformation of  $L$  gives a twistor deformation of  $\mathcal{M}$  to  $\mathcal{M}_{\mathbb{F}_2}$ . Because  $H^*(\mathcal{M}_{\mathbb{F}_2})$  can be explicitly described using a torus action, we expect that this twistor family can be used to describe the action of the involution on  $H^*(\mathcal{M})$ .

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# CHAPTER 1

## INTRODUCTION

Nakajima [42, chapter 2] considers the moduli of rank  $r$  torsion free sheaves  $E$  on  $\mathbb{P}^2$  with a fixed second Chern class together with an isomorphism

$$E|_l \simeq \mathcal{O}_{\mathbb{P}^1}^{\oplus r}$$

along a fixed line  $l \subset \mathbb{P}^2$ , which he calls a *framing at infinity*. He obtains an elegant ADHM description (see Atiyah, Drinfeld, Hitchin and Manin [4]) for this space and computes its Poincaré polynomial via an Atiyah-Bott type calculation (cf. Atiyah and Bott [3, page 23].) This is the first instance of his *quiver varieties* (see Nakajima [41, section 2].)

One can ask for a description in terms of a quiver for the next simplest case, the moduli space  $\mathcal{M}_{\mathbb{P}^2}(r, m)$  of torsion free sheaves on  $\mathbb{P}^2$  with a framing along a fixed *conic*  $C$ , with fixed rank and Chern classes, indicated by  $r$  and  $m$  (see section 2.1 for the precise definition.) In this thesis, a *framing* of a sheaf  $E$  along a smooth rational curve  $T$  is an isomorphism

$$\phi : E|_T \xrightarrow{\sim} \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus r}.$$

Such a pair  $(E, \phi)$  is called a *framed pair* (the reason we consider this framing rather than Nakajima's is outlined in appendix A.)

Let  $\mathcal{M}_{BM}(r, n)$  be the fine moduli space of torsion free sheaves on  $\mathbb{P}^1 \times \mathbb{P}^1$  with a framing along the *diagonal*, where  $r$  and  $n$  indicate the rank and Chern classes

(see section 2.1.) The existence, smoothness and quasi-projectivity of  $\mathcal{M}_{BM}(r, n)$  and  $\mathcal{M}_{\mathbb{P}^2}(r, m)$  follow from Bruzzo and Markushevich [10, theorem 3.1], whose work builds on the construction of fine moduli spaces of framed sheaves by Huybrechts and Lehn [25, 24]. Our key observation is the following, which is inspired by Kim [28, page 88],

$$\mathcal{M}_{\mathbb{P}^2}(r, m) \text{ is the fixed locus of a certain involution on } \mathcal{M}_{BM}(r, 2m).$$

This involution is induced by the following involution on  $\mathbb{P}^1 \times \mathbb{P}^1$

$$(z_1, z_2) \mapsto (z_2, z_1).$$

To understand this better, we construct in section 2.2 a moduli space  $\mathcal{M}(r, n)$ , which is isomorphic to  $\mathcal{M}_{BM}(r, n)$ , as follows.

Let  $\widetilde{W}$ ,  $V$  and  $W$  be fixed complex vector spaces of dimensions  $2n$ ,  $n - \frac{r}{2}$  and  $r$  respectively ( $r$  is even.) We fix an identification

$$\widetilde{W} = V \oplus V \oplus W.$$

There is a coadjoint orbit  $L_n$  of  $GL(\widetilde{W})$  which sits inside the product of Grassmannians

$$G(n, \widetilde{W}) \times G(n, \widetilde{W})$$

as a Zariski open subset ( $L_n$  is symplectic by McDuff and Salamon [37, page 168].)

The diagonal action of  $GL(V)$  on  $\widetilde{W}$  induces one on  $L_n \subset G(n, \widetilde{W}) \times G(n, \widetilde{W})$ . We consider a moment map

$$\mu_{r,n} : L_n \longrightarrow \text{End}(V)^*$$

for this action and define

$$\mathcal{M}(r, n) \stackrel{def}{=} U(r, n)/GL(V)$$

where  $U(r, n)$  is a certain Zariski open subset of  $\mu_{r,n}^{-1}(0)$  and show that this parametrizes torsion free sheaves on  $\mathbb{P}^1 \times \mathbb{P}^1$  which are framed along the diagonal. That is,

$$\mathcal{M}(r, n) \simeq \mathcal{M}_{BM}(r, n).$$

In this way, the induced involution on  $\mathcal{M}(r, n)$  can be understood simply as

$$(\widetilde{W}_I, \widetilde{W}_{II}) \mapsto (\widetilde{W}_{II}, \widetilde{W}_I).$$

Such an explicit description is potentially fruitful for the study of the fixed locus  $\mathcal{M}_{\mathbb{P}^2}(r, m)$ .

On  $\mathcal{M}(r, n)$  ( $\simeq \mathcal{M}_{BM}(r, n)$ ) there is a holomorphic symplectic structure such that  $\mathcal{M}_{\mathbb{P}^2}(r, m)$  is a holomorphic Lagrangian subvariety in the case  $n = 2m$  (see section 2.1.) In chapter 5 we conjecture that  $\mathcal{M}(r, n)$  is in fact a hyperkähler quotient in the sense of Hitchin et al. [20, section 3.D] (this would agree with the fact that  $\mathcal{M}(r, n)$  is isomorphic to a certain quiver variety  $\mathcal{M}_\xi(\underline{v}, \underline{w})$ , which is a hyperkähler quotient in a different way.) We conjecture that a similar moduli space  $\mathcal{M}_{\mathbb{F}_2}(r, n)$  of framed sheaves on the Hirzebruch surface  $\mathbb{F}_2$  is a hyperkähler quotient of the cotangent bundle  $T^*G(n, \widetilde{W})$  and the twistor deformation of this hyperkähler structure has general fibers isomorphic to  $\mathcal{M}(r, n)$ . Because  $H^*(\mathcal{M}_{\mathbb{F}_2}(r, n))$  can be explicitly described using a torus action, we expect that this twistor family can be used to describe the action of the involution on  $H^*(\mathcal{M}(r, n))$ .

The case  $n = 2m = 2$  is worked out in details in chapter 3. The Poincaré polynomial for  $\mathcal{M}(r, n)$  is obtained in chapter 4 by referring to the torus fixed-point method of Nakajima and Yoshioka [44, section 3] and in particular the explicit computation of Bruzzo et al. [11, theorem 4.4].

**Conventions.** Throughout, an *algebraic variety* means a noetherian reduced scheme of finite type over  $\mathbb{C}$ .

The following computation is used frequently in this thesis in computations involving Chern classes and Euler characteristics, and hence will be referred to simply as the *Hirzebruch-Riemann-Roch formula*.

$$\begin{aligned}
\chi(E) &= (ch(E).td(X))_2 \\
&= \left( \left( r + c_1(E) + \frac{1}{2}(c_1(E)^2 - 2c_2(E)) \right) \cdot \left( 1 - \frac{1}{2}K_X + \chi(\mathcal{O}_X) \right) \right)_2 \\
&= r\chi(\mathcal{O}_X) + \frac{1}{2}c_1(E).(c_1(E) - K_X) - c_2(E),
\end{aligned}$$

where  $X$  is a smooth projective algebraic surface and  $E$  a rank  $r$  coherent sheaf on  $X$ .

## CHAPTER 2

### MODULI OF FRAMED SHEAVES ON A SMOOTH QUADRIC SURFACE

#### 2.1. Moduli of framed sheaves on $\mathbb{P}^2$ as the fixed locus of an involution

Let  $r$ ,  $m$  and  $n$  be positive integers with  $r$  even. Let  $\mathcal{M}_{\mathbb{P}^2}(r, m)$  be the moduli space parametrizing pairs  $(G, \phi)$  which consist of a rank  $r$  torsion free sheaf  $G$  on  $\mathbb{P}^2$  with second Chern class  $m + \frac{r}{4} - \frac{r^2}{8}$  and an isomorphism

$$\phi : G|_C \xrightarrow{\sim} \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus r},$$

where  $C$  is a fixed smooth conic in  $\mathbb{P}^2$ . Two pairs  $(G_1, \phi_1)$  and  $(G_2, \phi_2)$  in  $\mathcal{M}_{\mathbb{P}^2}(r, m)$  are said to be *isomorphic* if there exists an isomorphism  $f : G_1 \xrightarrow{\sim} G_2$  which restricts on  $C$  to the following commutative diagram

$$\begin{array}{ccc} G_1|_C & \xrightarrow{f|_D} & G_2|_C \\ & \searrow \phi_1 & \swarrow \phi_2 \\ & \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus r} & \end{array} .$$

In this case, we denote

$$(G_1, \phi_1) \xrightarrow{\sim} (G_2, \phi_2).$$

Let  $\mathcal{M}_{BM}(r, n)$  be the moduli space parametrizing pairs  $(E, \phi)$  which consist of a rank  $r$  torsion free sheaf  $E$  on  $\mathbb{P}^1 \times \mathbb{P}^1$  with first Chern class  $-\frac{r}{2}(1, 1)$ , second Chern class  $n + \frac{r}{2} - \frac{r^2}{4}$  and an isomorphism

$$\phi : E|_D \xrightarrow{\sim} \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus r},$$

where  $D$  is the diagonal in  $\mathbb{P}^1 \times \mathbb{P}^1$ . The notion of *isomorphic* pairs in  $\mathcal{M}_{BM}(r, n)$  is similar to that in  $\mathcal{M}_{\mathbb{P}^2}(r, m)$ . Elements of  $\mathcal{M}_{\mathbb{P}^2}(r, m)$  and  $\mathcal{M}_{BM}(r, n)$  are often called *framed pairs*. These moduli spaces are instances of a construction by Bruzzo and Markushevich [10, theorem 3.1].

Let  $i : \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$  be the involution  $(z_1, z_2) \mapsto (z_2, z_1)$ . Let  $p : \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^2$  be the quotient by  $i$ . Then  $p$  is a double covering which ramifies along the diagonal  $D$ , a smooth hyperplane section, and the branch locus is a smooth conic, which after a change of coordinates, can be taken to be  $C$ .

We study  $\mathcal{M}_{\mathbb{P}^2}(r, m)$  by pulling back via  $p^*$  to  $\mathbb{P}^1 \times \mathbb{P}^1$ . More precisely, we have

**Proposition 2.1.1.**  *$\mathcal{M}_{BM}(r, n)$  is a smooth holomorphic symplectic variety and there is a closed embedding of  $\mathcal{M}_{\mathbb{P}^2}(r, m)$  in  $\mathcal{M}_{BM}(r, 2m)$  as a holomorphic Lagrangian subvariety given by*

$$\begin{aligned} \mathcal{M}_{\mathbb{P}^2}(r, m) &\xrightarrow{p^*} \mathcal{M}_{BM}(r, 2m) \\ (G, \phi) &\mapsto (p^*G, p^*\phi) \end{aligned}$$

Moreover, there is an induced involution

$$\begin{aligned} \mathcal{M}_{BM}(r, n) &\xrightarrow{i_{\mathcal{M}}} \mathcal{M}_{BM}(r, n) \\ (E, \phi) &\mapsto (i^*E, i^*\phi) \end{aligned}$$

and the image of  $\mathcal{M}_{\mathbb{P}^2}(r, m)$  under  $p^*$  is equal to the fixed locus of  $i_{\mathcal{M}}$ .

*Proof.* By Huybrechts and Lehn [24, theorem 4.1], the obstruction to smoothness for  $\mathcal{M}_{BM}(r, n)$  at  $(E, \phi)$  is

$$\text{Ext}^2(E, E(-D)) \simeq \text{Hom}(E, E(-D))^* = 0, \quad (2.1.2)$$

where the last equality follows from equation (2.2.32). Hence  $\mathcal{M}_{BM}(r, n)$  is smooth. Sala [47, theorem 6.1] constructs a holomorphic symplectic form  $\Omega_{\mathcal{M}}(\tau)$  on  $\mathcal{M}_{BM}(r, n)$ ,

which depends on the choice of a nonzero rational 2-form  $\tau \in H^0(\omega_{\mathbb{P}^1 \times \mathbb{P}^1}(2D)) \simeq \mathbb{C}$  (see also Mukai [38, theorem 0.1] for the unframed case and Bottacin [8, theorem 4.3] for the framed locally free case.) We describe  $\Omega_{\mathcal{M}}(\tau)$  on a fiber. By Huybrechts and Lehn [24, theorem 4.1],

$$T_{(E, \phi)} \mathcal{M}(r, n) \simeq \text{Ext}^1(E, E(-D)).$$

Then  $\Omega_{\mathcal{M}}(\tau)$  is given at  $(E, \phi)$  by

$$\begin{aligned} \text{Ext}^1(E, E(-D)) \times \text{Ext}^1(E, E(-D)) &\xrightarrow{\circ} \text{Ext}^2(E, E(-2D)) \\ &\xrightarrow{\text{tr}} H^2(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-2D)) \simeq H^2(\mathbb{P}^1 \times \mathbb{P}^1, \omega_{\mathbb{P}^1 \times \mathbb{P}^1}) \simeq \mathbb{C}, \end{aligned}$$

where the first isomorphism comes from the isomorphism

$$\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-2D) \xrightarrow{\tilde{\tau}} \omega_{\mathbb{P}^1 \times \mathbb{P}^1}.$$

Next, we claim that

$$\text{The morphism } p^* \text{ maps } \mathcal{M}_{\mathbb{P}^2}(r, m) \text{ bijectively onto the fixed locus of } i_{\mathcal{M}}. \quad (2.1.3)$$

That  $p^*$  is a regular map is due to an argument similar to that in the proof of lemma 4.2.1 below. Since  $p \circ i = p$ , we have

$$i^*(p^*G) = p^*G$$

for any  $(G, \phi) \in \mathcal{M}_{\mathbb{P}^2}(r, m)$ . Hence  $p^*$  maps  $\mathcal{M}_{\mathbb{P}^2}(r, m)$  into the fixed locus of  $i_{\mathcal{M}}$ . To prove that the image of  $p^*$  is equal to this fixed locus, we claim that for any fixed



point  $(E, \phi)$  of  $i_{\mathcal{M}}$  (i.e.  $i^*(E, \phi) \simeq (E, \phi)$ ) there exists a map  $g : E \rightarrow E$  which makes the diagram

$$\begin{array}{ccc} E & \xrightarrow{g} & E \\ \downarrow & & \downarrow \\ \mathbb{P}^1 \times \mathbb{P}^1 & \xrightarrow{i} & \mathbb{P}^1 \times \mathbb{P}^1 \end{array} \quad (2.1.4)$$

commute and such that  $g^2 = id_E$  and  $g|_D = id_{E|_D}$ . In fact, let us define (2.1.4) to be the composition of the two commutative diagrams

$$\begin{array}{ccc} E & \xrightarrow{\sim} & i^*E \\ & \searrow & \downarrow \\ & & \mathbb{P}^1 \times \mathbb{P}^1 \end{array} \quad \text{and} \quad \begin{array}{ccc} i^*E & \xrightarrow{i^*} & E \\ \downarrow & & \downarrow \\ \mathbb{P}^1 \times \mathbb{P}^1 & \xrightarrow{i} & \mathbb{P}^1 \times \mathbb{P}^1 \end{array}$$

and show that it has the desired properties. Restricting it to  $D$  we obtain the commutative diagram

$$\begin{array}{ccc} E|_D & \xrightarrow{g|_D} & E|_D \\ \searrow \phi & & \swarrow \phi \\ & \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus r} & \end{array} \quad (2.1.5)$$

which implies that  $g|_D = id_{E|_D}$ . Composing (2.1.4) with itself we obtain

$$\begin{array}{ccc} E & \xrightarrow{g^2} & E \\ \downarrow & & \downarrow \\ \mathbb{P}^1 \times \mathbb{P}^1 & \xlongequal{\quad} & \mathbb{P}^1 \times \mathbb{P}^1 \end{array} \quad .$$

$g^2$  fixes the framing  $\phi$ , since  $g|_D = id_{E|_D}$ . By the proof of lemma 2.2.29, the only automorphism of  $(E, \phi)$  is  $id_E$ . Thus  $g^2 = id_E$ . Hence  $p^*$  is surjective onto the fixed locus of  $i_{\mathcal{M}}$ . The injectivity of  $p^*$  follows from the fact that

$$(p_*p^*G)^{\mathbb{Z}/2\mathbb{Z}} \simeq G.$$

The fixed locus of  $i_{\mathcal{M}}$  is smooth by Donovan [14, lemma 4.1] since  $\mathcal{M}_{BM}(r, 2m)$  is smooth. Similarly to (2.1.2),  $\mathcal{M}_{\mathbb{P}^2}(r, m)$  is smooth. By Zariski's main theorem (see for example Mumford [39, page 209]),  $p^*$  is an isomorphism onto its image, which is

closed. Hence  $p^*$  is a closed embedding. We observe that  $\Omega_{\mathcal{M}}(\tau)$  is anti-invariant. In fact, in local coordinates,  $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-2D) \xrightarrow{\tau} \omega_{\mathbb{P}^1 \times \mathbb{P}^1}$  is given (up to a scalar factor) by

$$\tau = \frac{dz \wedge dw}{(z - w)^2}.$$

The involution  $i$  interchanges  $z$  and  $w$ . Hence  $i^*\tau = -\tau$ , which implies that  $i_{\mathcal{M}}^*\Omega_{\mathcal{M}}(\tau) = -\Omega_{\mathcal{M}}(\tau)$ . Hence the fixed locus of  $i_{\mathcal{M}}$  is a holomorphic Lagrangian subvariety of  $\mathcal{M}_{BM}(r, 2m)$ .  $\square$

This motivates us to seek a description of  $\mathcal{M}_{BM}(r, n)$  in which the involution  $i_{\mathcal{M}}$  can be made explicit. This is the content of our main result, theorem 2.2.6.

## 2.2. Moduli of framed sheaves on $\mathbb{P}^1 \times \mathbb{P}^1$

Let  $\widetilde{W}$ ,  $V$  and  $W$  be fixed complex vector spaces of dimensions  $2n$ ,  $n - \frac{r}{2}$  and  $r$  respectively. We fix an identification

$$\widetilde{W} = V \oplus V \oplus W. \tag{2.2.1}$$

Let  $G(n, \widetilde{W})$  be the Grassmannian of  $n$ -dimensional subspaces in  $\widetilde{W}$  and  $L_n$  the Zariski open subset of  $G(n, \widetilde{W}) \times G(n, \widetilde{W})$  which consists of complementary pairs of subspaces. For each  $(\widetilde{W}_I, \widetilde{W}_{II}) \in L_n$  there is a natural identification

$$\widetilde{W} = \widetilde{W}_I \oplus \widetilde{W}_{II}. \tag{2.2.2}$$

We define  $\mu_{r,n} : L_n \rightarrow \text{End}(V)$  by

$$\mu_{r,n}(\widetilde{W}_I, \widetilde{W}_{II}) \stackrel{def}{=} p_0 i_I p_I i_0 + p_1 i_{II} p_{II} i_1 - id_V, \tag{2.2.3}$$

where  $i_0$  (resp.  $i_1$ ) is the inclusion of the first (resp. second) term of (2.2.1),  $p_0$  (resp.  $p_1$ ) is the projection onto the first (resp. second) term of (2.2.1),  $i_I$  (resp.  $i_{II}$ ) is the

inclusion of the first (resp. second) term of (2.2.2) and  $p_I$  (resp.  $p_{II}$ ) is the projection onto the first (resp. second) term of (2.2.2).

Hereafter, we denote by  $X$  the quadric surface  $\mathbb{P}^1 \times \mathbb{P}^1$ . Let  $(X_0 : X_1)$  (resp.  $(Y_0 : Y_1)$ ) be coordinates on the first (resp. second) factor of  $X$ . For each  $(\widetilde{W}_I, \widetilde{W}_{II}) \in L_n$  we have a sequence

$$\begin{array}{ccc}
& \mathcal{O}_X(-1, 0) \otimes \widetilde{W}_I & \\
0 \rightarrow \mathcal{O}_X(-1, -1) \otimes V \xrightarrow{a} & \oplus & \xrightarrow{b} \mathcal{O}_X \otimes V \rightarrow 0, \quad (2.2.4) \\
& \mathcal{O}_X(0, -1) \otimes \widetilde{W}_{II} &
\end{array}$$

where

$$a = \begin{pmatrix} Y_0 p_I i_0 + Y_1 p_I i_1 \\ X_0 p_{II} i_0 + X_1 p_{II} i_1 \end{pmatrix},$$

$$b = (-X_0 p_1 i_I + X_1 p_0 i_I \mid -Y_0 p_1 i_{II} + Y_1 p_0 i_{II}).$$

Let  $U(r, n)$  be the Zariski open subset of  $\mu_{r,n}^{-1}(0)$  which corresponds to the morphism  $b$  being surjective.

**Lemma 2.2.5.**  $(\widetilde{W}_I, \widetilde{W}_{II}) \in U(r, n)$  iff (2.2.4) is a monad.

(We say

$$0 \rightarrow A \xrightarrow{c} B \xrightarrow{d} C \rightarrow 0$$

is a *monad* if it is exact at  $A$  and  $C$  and  $dc = 0$  and, in that case, call  $\ker(d)/\text{Im}(c)$  the *cohomology* of this monad.)

*Proof.* Let  $Z$  be the locus of points  $p \in X$  such that  $a \otimes \mathbb{C}(p)$  is not injective. Then  $Z$  is closed in  $X$  and disjoint from  $D$ , since  $a \otimes \mathbb{C}(p)$  is injective for all  $p \in D$ . But  $D$  is ample so  $D.C > 0$  for any curve  $C \subset Z$ . Hence  $Z$  contains no such curve. It follows that  $Z$  is of codimension at least 2 in  $X$ . Hence  $a$  is injective as a sheaf morphism.

We have

$$\begin{aligned}
ba &= (-X_0p_1i_I + X_1p_0i_I)(Y_0p_Ii_0 + Y_1p_Ii_1) + (-Y_0p_1i_{II} + Y_1p_0i_{II})(X_0p_{II}i_0 + X_1p_{II}i_1) \\
&= -X_0Y_0p_1(i_Ip_I + i_{II}p_{II})i_0 + X_1Y_1p_0(i_Ip_I + i_{II}p_{II})i_1 + X_1Y_0(p_0i_Ip_Ii_0 - p_1i_{II}p_{II}i_1) + \\
&X_0Y_1(-p_1i_Ip_Ii_1 + p_0i_{II}p_{II}i_0) \\
&= X_0Y_0p_1i_0 + X_1Y_1p_0i_1 + X_1Y_0(p_0i_Ip_Ii_0 + p_1i_Ip_Ii_1 - p_1i_1) + X_0Y_1(-p_1i_Ip_Ii_1 + p_0i_0 - \\
&p_0i_Ip_Ii_0) \\
&= (X_1Y_0 - X_0Y_1)\mu_{r,n},
\end{aligned}$$

where we have used the fact that  $i_Ip_I + i_{II}p_{II} = id_{\widetilde{W}}$  in the third and fourth equalities and the facts that  $p_1i_0 = p_0i_1 = 0$  and  $p_1i_1 = p_0i_0 = id_V$  in the last equality. Hence

$$ba = 0 \text{ iff } \mu_{r,n} = 0.$$

Finally, the surjectivity of  $b$  follows from the definition of  $U(r, n)$ .  $\square$

$GL(V)$  acts on  $L_n$  by simultaneously acting on the two copies of  $V$  in (2.2.1). This action restricts to one on  $U(r, n) \subset \mu_{r,n}^{-1}(0)$ , which is set-theoretically free by lemma 2.2.29. Let  $\mathcal{M}(r, n)$  be the set-theoretical quotient  $U(r, n)/GL(V)$ . By proposition 2.2.48 below,  $\mathcal{M}(r, n)$  is an algebraic space. Our main result is

**Theorem 2.2.6.**  *$\mathcal{M}(r, n)$  is isomorphic to  $\mathcal{M}_{BM}(r, n)$ . The involution  $i_{\mathcal{M}}$  in section 2.1 becomes*

$$(\widetilde{W}_I, \widetilde{W}_{II}) \mapsto (\widetilde{W}_{II}, \widetilde{W}_I).$$

**Remark 2.2.7.** In fact, there is a description for  $\mathcal{M}(r, n) \simeq \mathcal{M}_{BM}(r, n)$  as a quiver variety by Nakajima [43, page 709] and lemma 4.2.1 below. However, our result makes it clear how the involution  $i_{\mathcal{M}}$  acts.

We first construct a bijective map  $\mathcal{M}(r, n) \rightarrow \mathcal{M}_{BM}(r, n)$ . For  $(\widetilde{W}_I, \widetilde{W}_{II}) \in U(r, n)$  we denote by  $E_{\widetilde{W}_I, \widetilde{W}_{II}}$  the cohomology of monad (2.2.4). The restriction of this monad to the diagonal  $D \simeq \mathbb{P}_{Z_0, Z_1}^1$  is of the form

$$0 \rightarrow \mathcal{O}_D(-2) \otimes V \xrightarrow{\begin{pmatrix} Z_0 \\ Z_1 \\ 0 \end{pmatrix}} \mathcal{O}_D(-1) \otimes (V \oplus V \oplus W) \xrightarrow{(Z_1, -Z_0, 0)} \mathcal{O}_D \otimes V \rightarrow 0, \quad (2.2.8)$$

which is also a monad, where  $Z_i$  is the restriction of  $X_i$  to  $D$  for  $i = 0, 1$ . Since the Koszul complex

$$0 \rightarrow \mathcal{O}_D(-2) \otimes V \xrightarrow{\begin{pmatrix} Z_0 \\ Z_1 \end{pmatrix}} \mathcal{O}_D(-1) \otimes (V \oplus V) \xrightarrow{(Z_1, -Z_0)} \mathcal{O}_D \otimes V \rightarrow 0$$

is exact, the cohomology of (2.2.8) is  $\mathcal{O}_{\mathbb{P}^1}(-1) \otimes W$ . Hence we obtain a natural isomorphism

$$\phi_{\widetilde{W}_I, \widetilde{W}_{II}} : E_{\widetilde{W}_I, \widetilde{W}_{II}}|_D \xrightarrow{\sim} \mathcal{O}_{\mathbb{P}^1}(-1) \otimes W,$$

which is a framing of  $E_{\widetilde{W}_I, \widetilde{W}_{II}}$  along the diagonal  $D$ . We define

$$\begin{aligned} F_{r,n} : \mathcal{M}(r, n) &\longrightarrow \mathcal{M}_{BM}(r, n) \\ (\widetilde{W}_I, \widetilde{W}_{II}) &\mapsto (E_{\widetilde{W}_I, \widetilde{W}_{II}}, \phi_{\widetilde{W}_I, \widetilde{W}_{II}}). \end{aligned}$$

**Lemma 2.2.9.**  *$F_{r,n}$  is well-defined.*

*Proof.* For each  $g \in GL(V)$  let

$$g' := g \oplus g \oplus id_W \in GL(V \oplus V \oplus W) = GL(\widetilde{W}).$$

We have a commutative diagram, which is an isomorphism of monads,

$$\begin{array}{ccccccc} & & \mathcal{O}_X(-1, 0) & \otimes & \widetilde{W}_I & & \\ 0 & \longrightarrow & \mathcal{O}_X(-1, -1) \otimes V & \longrightarrow & \oplus & \longrightarrow & \mathcal{O}_X \otimes V \longrightarrow 0 \\ & & \downarrow g & & \downarrow g' & & \downarrow g \\ & & \mathcal{O}_X(0, -1) & \otimes & \widetilde{W}_{II} & & \\ 0 & \longrightarrow & \mathcal{O}_X(-1, -1) \otimes V & \longrightarrow & \oplus & \longrightarrow & \mathcal{O}_X \otimes V \longrightarrow 0, \\ & & \downarrow g & & \downarrow g' & & \downarrow g \\ & & \mathcal{O}_X(-1, 0) & \otimes & g' \widetilde{W}_I & & \\ & & \mathcal{O}_X(0, -1) & \otimes & g' \widetilde{W}_{II} & & \end{array} \tag{2.2.10}$$

This gives an isomorphism

$$E_{\widetilde{W}_I, \widetilde{W}_{II}} \xrightarrow{\sim} E_{g' \widetilde{W}_I, g' \widetilde{W}_{II}},$$

which preserves the framing because  $g'$  acts trivially on  $W$ . □

The next task is to construct a set-theoretical inverse for  $F_{r,n}$ .

**Proposition 2.2.11.** *For a pair  $(E, \phi) \in \mathcal{M}_{BM}(r, n)$  there is a canonical monad*

$$\begin{array}{ccc} & \mathcal{O}_X(-1, 0) \otimes \widetilde{W}_{I,E} & \\ & \otimes & \\ 0 \rightarrow \mathcal{O}_X(-1, -1) \otimes V_E \xrightarrow{a_E} & \oplus & \xrightarrow{b_E} \mathcal{O}_X \otimes V'_E \rightarrow 0, \quad (2.2.12) \\ & \mathcal{O}_X(0, -1) \otimes \widetilde{W}_{II,E} & \end{array}$$

of which the cohomology is  $E$ , where

$$\begin{aligned} V_E &= H^1(E(-1, -1)), \quad V'_E = H^1(E), \\ \widetilde{W}_{I,E} &= H^1(E(-1, 0)), \quad \widetilde{W}_{II,E} = H^1(E(0, -1)). \end{aligned}$$

The proof uses

**Lemma 2.2.13.**  $H^q(E(-1, -1)) = H^q(E(-1, 0)) = H^q(E(0, -1)) = H^q(E) = 0$  for  $q = 0, 2$ . Moreover,

$$\begin{aligned} \dim_{\mathbb{C}} \widetilde{W}_{I,E} &= \dim_{\mathbb{C}} \widetilde{W}_{II,E} = n, \\ \dim_{\mathbb{C}} V_E &= \dim_{\mathbb{C}} V'_E = n - \frac{r}{2}. \end{aligned}$$

*Proof.* Since  $E$  is torsion free and  $X$  is a smooth surface, we have the exact sequence

$$0 \longrightarrow E \longrightarrow E^{\vee\vee} \longrightarrow E^{\vee\vee}/E \longrightarrow 0, \quad (2.2.14)$$

where  $E^{\vee\vee}$  is locally free and  $E^{\vee\vee}/E$  is supported at a finite set of points. The short exact sequence

$$0 \longrightarrow E^{\vee\vee}(-(k+1)D) \longrightarrow E^{\vee\vee}(-kD) \longrightarrow E^{\vee\vee}(-kD)|_D \longrightarrow 0$$

yields the long exact sequence

$$\begin{aligned} 0 &\longrightarrow H^0(E^{\vee\vee}(-(k+1)D)) \longrightarrow H^0(E^{\vee\vee}(-kD)) \longrightarrow H^0(E^{\vee\vee}(-kD)|_D) \\ &\longrightarrow H^1(E^{\vee\vee}(-(k+1)D)) \longrightarrow H^1(E^{\vee\vee}(-kD)) \longrightarrow H^1(E^{\vee\vee}(-kD)|_D) \\ &\longrightarrow H^2(E^{\vee\vee}(-(k+1)D)) \longrightarrow H^2(E^{\vee\vee}(-kD)) \longrightarrow 0. \end{aligned}$$

We have  $H^0(E^{\vee\vee}(-kD)|_D) = H^0(\mathcal{O}_{\mathbb{P}^1}(-2k-1)^{\oplus r}) = 0$  if  $k \geq 0$ . Hence

$$H^0(E^{\vee\vee}(-kD)) = H^0(E^{\vee\vee}(-(k+1)D))$$

for  $k \geq 0$ . By Serre vanishing theorem, Serre duality and the fact that  $D$  is ample, we have  $H^0(E^{\vee\vee}(-kD)) = H^2(E^{\vee\vee\vee}(kD))^* = 0$  for  $k$  sufficiently large. Hence

$$H^0(E^{\vee\vee}) = H^0(E^{\vee\vee}(-D)) = H^0(E^{\vee\vee}(-2D)) = \dots = 0.$$

We have  $H^1(E^{\vee\vee}(-kD)|_D) \simeq H^0(E^{\vee\vee\vee}|_D \otimes \mathcal{O}_{\mathbb{P}^1}(2k-2))^* = H^0(\mathcal{O}_{\mathbb{P}^1}(2k-1)^{\oplus r})^* = 0$  for  $k \leq 0$ , where the first isomorphism is by Serre duality on  $D$ . Hence

$$H^2(E^{\vee\vee}(-kD)) = H^2(E^{\vee\vee}(-(k+1)D))$$

for  $k \leq 0$ . By Serre vanishing theorem we have  $H^2(E^{\vee\vee}(lD)) = 0$  for  $l$  sufficiently large. Hence

$$H^2(E^{\vee\vee}(-D)) = H^2(E^{\vee\vee}) = H^2(E^{\vee\vee}(D)) = \cdots = 0.$$

(2.2.14) yields the long exact sequence

$$\begin{aligned} 0 &\longrightarrow H^0(E) \longrightarrow H^0(E^{\vee\vee}) \longrightarrow H^0(E^{\vee\vee}/E) \\ &\longrightarrow H^1(E) \longrightarrow H^1(E^{\vee\vee}) \longrightarrow 0 \\ &\longrightarrow H^2(E) \longrightarrow H^2(E^{\vee\vee}) \longrightarrow 0, \end{aligned}$$

which gives that

$$H^0(E) = H^2(E) = 0.$$

Similarly,  $H^0(E(-D)) = H^2(E(-D)) = 0$  and  $H^q(E(-1, 0)) = H^q(E(0, -1)) = 0$  for  $q = 0, 2$ .

The dimensions of  $\widetilde{W}_{I,E}$ ,  $\widetilde{W}_{II,E}$ ,  $V_E$  and  $V'_E$  are obtained from a Chern class computation using the Hirzebruch-Riemann-Roch formula.  $\square$

*Proof of proposition 2.2.11.* We use a Beilinson-type spectral sequence associated to the resolution of the diagonal for  $X = \mathbb{P}^1 \times \mathbb{P}^1$ . Let  $\Delta_{X \times X}$  be the diagonal in  $X \times X$  and  $p_i : X \times X \rightarrow X$  the projection onto the  $i$ th factor for  $i = 1, 2$ . We have  $X \times X = \mathbb{P}^1_{X_0:X_1} \times \mathbb{P}^1_{Y_0:Y_1} \times \mathbb{P}^1_{Z_0:Z_1} \times \mathbb{P}^1_{W_0:W_1}$  and  $\Delta_{X \times X}$  is given in  $X \times X$  by the equations

$$X_0Z_1 - Z_0X_1 = Y_0W_1 - W_0Y_1 = 0.$$

Using these equations, we obtain a Koszul complex

$$\begin{aligned} 0 \rightarrow \mathcal{O}_{X \times X}(-1, -1, -1, -1) \rightarrow & \begin{array}{c} \mathcal{O}_{X \times X}(-1, 0, -1, 0) \\ \oplus \\ \mathcal{O}_{X \times X}(0, -1, 0, -1) \end{array} \rightarrow \mathcal{O}_{X \times X} \rightarrow \mathcal{O}_{\Delta_{X \times X}} \rightarrow 0, \end{aligned}$$



which is a resolution of  $\mathcal{O}_{\Delta_{X \times X}}$ . Note that

$$\mathcal{O}(-1, -1, -1, -1) = \wedge^2(\mathcal{O}(-1, 0, -1, 0) \oplus \mathcal{O}(0, -1, 0, -1)).$$

We put

$$C^p := \wedge^{-p}(\mathcal{O}(-1, 0, -1, 0) \oplus \mathcal{O}(0, -1, 0, -1)).$$

Let  $L^\bullet \bullet$  be a double complex each column of which is a resolution of the corresponding term in the complex  $p_1^*E \otimes C^\bullet$  (this is called a *Cartan-Eilenberg resolution* of  $p_1^*E \otimes C^\bullet$ , see Weibel [51, pages 145-146].) There are two associated spectral sequences

$$'E_1^{pq} = R^q p_{2*}(p_1^*E \otimes C^p),$$

$$''E_2^{pq} = R^q p_{2*}(\mathcal{H}^p(p_1^*E \otimes C^\bullet)),$$

which both converge to  $\mathbb{R}^{p+q} p_{2*}(p_1^*E \otimes C^\bullet)$  (the notation  $\mathbb{R}^i p_{2*}(D^\bullet)$  is the *ith hyperdirect image* which is defined as follows: one chooses a Cartan-Eilenberg resolution  $L^\bullet \bullet$  of  $D^\bullet$ , applies  $p_{2*}$  to  $L^\bullet \bullet$  and defines  $\mathbb{R}^i p_{2*}(D^\bullet)$  as the *ith* cohomology sheaf of the total chain complex associated to  $p_{2*}(L^\bullet \bullet)$ , see Okonek et al. [45, page 243].) We have

$$p_{2*}(p_1^*E \otimes \mathcal{O}_{\Delta_{X \times X}}) = p_{2*}(p_1^*E|_{\Delta_{X \times X}}) = E.$$

Hence

$$''E_2^{pq} = \begin{cases} E & \text{if } (p, q) = (0, 0) \\ 0 & \text{otherwise} \end{cases}.$$

Since

$$R^q p_{2*}(p_1^*E \otimes \mathcal{O}(a, b, c, d)) = R^q p_{2*}(p_1^*(E(a, b)) \otimes p_2^* \mathcal{O}_X(c, d)) = H^q(E(a, b)) \otimes \mathcal{O}_X(c, d),$$

' $E_1$  is given by

$$\begin{array}{ccccccc}
& & & H^q(E(-1, 0)) & \otimes & \mathcal{O}_X(-1, 0) & \\
0 \rightarrow & H^q(E(-1, -1)) \otimes \mathcal{O}_X(-1, -1) & \rightarrow & & \otimes & & \rightarrow H^q(E) \otimes \mathcal{O}_X \rightarrow 0. \\
& & & H^q(E(0, -1)) & \otimes & \mathcal{O}_X(0, -1) & \\
& & & & & & (2.2.15)
\end{array}$$

This spectral sequence degenerates at  $E_2$  by lemma 2.2.13. Hence  $E$  is the cohomology of the monad (2.2.12).  $\square$

Since  $E|_D \simeq \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus r}$  we have  $H^0(E|_D) = H^1(E|_D) = 0$ , and the long exact sequence of cohomology for the exact sequence

$$0 \longrightarrow E(-1, -1) \xrightarrow{X_0 Y_1 - X_1 Y_0} E \longrightarrow E|_D \longrightarrow 0$$

yields a natural isomorphism

$$\alpha_E : V_E = H^1(E(-1, -1)) \xrightarrow{\sim} H^1(E) = V'_E.$$

Restricting monad (2.2.12) to  $D$ , we obtain the following monad

$$0 \longrightarrow \mathcal{O}_D(-2) \otimes V_E \xrightarrow{a_E|_D} \mathcal{O}_D(-1) \otimes (\widetilde{W}_{I,E} \oplus \widetilde{W}_{II,E}) \xrightarrow{b_E|_D} \mathcal{O}_D \otimes V'_E \longrightarrow 0, \quad (2.2.16)$$

of which the cohomology is  $E|_D$ .

Let  $\widetilde{W}_E := (\widetilde{W}_{I,E} \oplus \widetilde{W}_{II,E})$ . Let  $(Z_0 : Z_1)$  be coordinates on  $D \simeq \mathbb{P}^1$ . Let

$$\begin{aligned}
a_E|_D &= a_{E,1}Z_0 + a_{E,2}Z_1 \\
b_E|_D &= b_{E,1}Z_0 + b_{E,2}Z_1
\end{aligned}$$

where  $a_{E,i} \in \text{Hom}(V_E, \widetilde{W}_E)$  and  $b_{E,i} \in \text{Hom}(\widetilde{W}_E, V'_E)$ .

**Lemma 2.2.17.** (Nakajima [42, pages 21-22])

(i) There is a natural inclusion

$$W = H^0(E(1)|_D) \xrightarrow{c_{E,\phi}} \widetilde{W}_E$$

such that  $\text{Image}(c_{E,\phi}) = \ker(b_{E,1}) \cap \ker(b_{E,2})$ .

(ii) The map

$$\gamma_{E,\phi} := a_{E,1} \oplus a_{E,2} \oplus c_{E,\phi} : V_E \oplus V_E \oplus W \longrightarrow \widetilde{W}_E$$

is an isomorphism.

(iii) Fixing an isomorphism  $\theta_E : V \rightarrow V_E$ , then the isomorphism  $\psi_{E,\phi} := \gamma_{E,\phi} \circ (\theta_E \oplus \theta_E \oplus \text{id}_W) : (V \oplus V \oplus W) \xrightarrow{\sim} \widetilde{W}_E$  makes the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}_D(-2) \otimes V & \xrightarrow{\begin{pmatrix} Z_0 \\ Z_1 \\ 0 \end{pmatrix}} & \mathcal{O}_D(-1) \otimes \begin{pmatrix} V \\ \oplus \\ V \\ \oplus \\ W \end{pmatrix} & \xrightarrow{(Z_1, -Z_0, 0)} & \mathcal{O}_D \otimes V \longrightarrow 0 \\ & & \downarrow \theta_E & & \downarrow \psi_{E,\phi} & & \downarrow \alpha_E \circ \theta_E \\ 0 & \longrightarrow & \mathcal{O}_D(-2) \otimes V_E & \longrightarrow & \mathcal{O}_D(-1) \otimes \widetilde{W}_E & \longrightarrow & \mathcal{O}_D \otimes V'_E \longrightarrow 0 \end{array} \quad (2.2.18)$$

commute. This isomorphism of monads is unique up to a change of basis in  $V$ .

The inverse of (2.2.18) induces the framing

$$E|_D \xrightarrow{\phi} \mathcal{O}_{\mathbb{P}^1}(-1) \otimes W.$$

Using lemma 2.2.17, we define

$$\begin{aligned} G_{r,n} : \mathcal{M}_{BM}(r,n) &\longrightarrow \mathcal{M}(r,n) \\ (E, \phi) &\mapsto (\widetilde{W}_I, \widetilde{W}_{II}) := (\psi_{E,\phi}^{-1}(\widetilde{W}_{I,E}), \psi_{E,\phi}^{-1}(\widetilde{W}_{II,E})). \end{aligned}$$

**Lemma 2.2.19.**  $G_{r,n}$  is well-defined.

The proof uses the following lemma, which is a clear consequence of lemma 2.2.17.

**Lemma 2.2.20.** *For  $(\widetilde{W}_I, \widetilde{W}_{II}) = G_{r,n}(E, \phi)$ , the following diagram*

$$\begin{array}{ccccccc}
& & \mathcal{O}_X(-1, 0) & \otimes & \widetilde{W}_I & & \\
0 & \longrightarrow & \mathcal{O}_X(-1, -1) \otimes V & \longrightarrow & \oplus & \longrightarrow & \mathcal{O}_X \otimes V \longrightarrow 0 \\
& & \downarrow \theta_E & & \downarrow \left( \psi_{E,\phi}|_{\widetilde{W}_I} \oplus \psi_{E,\phi}|_{\widetilde{W}_{II}} \right) & & \downarrow (\alpha_E \circ \theta_E) \\
& & \mathcal{O}_X(0, -1) & \otimes & \widetilde{W}_{II} & & \\
0 & \longrightarrow & \mathcal{O}_X(-1, -1) \otimes V_E & \longrightarrow & \oplus & \longrightarrow & \mathcal{O}_X \otimes V'_E \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & \mathcal{O}_X(-1, 0) & \otimes & \widetilde{W}_{I,E} & & \\
& & \mathcal{O}_X(0, -1) & \otimes & \widetilde{W}_{II,E} & & 
\end{array} \tag{2.2.21}$$

(the first row is monad (2.2.4) and the second row is monad (2.2.12)) is commutative and restricts to the diagram (2.2.18) on  $D$ . The inverse of (2.2.21) induces the framing

$$E|_D \xrightarrow{\phi} \mathcal{O}_{\mathbb{P}^1}(-1) \otimes W.$$

*Proof of lemma 2.2.19.* Suppose we have an isomorphism

$$(E, \phi) \xrightarrow[g]{} (E', \phi')$$

between framed pairs. Let

$$(\widetilde{W}_I, \widetilde{W}_{II}) := (\psi_{E,\phi}^{-1}(\widetilde{W}_{I,E}), \psi_{E,\phi}^{-1}(\widetilde{W}_{II,E})),$$

$$(\widetilde{W}'_I, \widetilde{W}'_{II}) := (\psi_{E',\phi'}^{-1}(\widetilde{W}_{I,E'}), \psi_{E',\phi'}^{-1}(\widetilde{W}_{II,E'})).$$

By the following commutative diagram

$$\begin{array}{ccccccc}
& & \mathcal{O}_X(-1, 0) & \otimes & \widetilde{W}_{I,E} & & \\
0 & \longrightarrow & \mathcal{O}_X(-1, -1) \otimes V_E & \longrightarrow & \oplus & \longrightarrow & \mathcal{O}_X \otimes V'_E \longrightarrow 0 \\
& & \downarrow g_{E,E'} & & \downarrow h_{E,E'} & & \downarrow g'_{E,E'} \\
& & \mathcal{O}_X(0, -1) & \otimes & \widetilde{W}_{II,E} & & \\
0 & \longrightarrow & \mathcal{O}_X(-1, -1) \otimes V_{E'} & \longrightarrow & \oplus & \longrightarrow & \mathcal{O}_X \otimes V'_{E'} \longrightarrow 0 \\
& & & & \mathcal{O}_X(-1, 0) & \otimes & \widetilde{W}_{I,E'} \\
& & & & \mathcal{O}_X(0, -1) & \otimes & \widetilde{W}_{II,E'}
\end{array} \tag{2.2.22}$$

of which the vertical arrows are isomorphisms induced by  $g$ , we obtain the following isomorphism of monads

$$\begin{array}{ccccccc}
& & \mathcal{O}_X(-1, 0) & \otimes & \widetilde{W}_I & & \\
0 & \longrightarrow & \mathcal{O}_X(-1, -1) \otimes V & \longrightarrow & \oplus & \longrightarrow & \mathcal{O}_X \otimes V \longrightarrow 0 \\
& & \downarrow G & & \downarrow H & & \downarrow G' \\
& & \mathcal{O}_X(0, -1) & \otimes & \widetilde{W}_{II} & & \\
0 & \longrightarrow & \mathcal{O}_X(-1, -1) \otimes V & \longrightarrow & \oplus & \longrightarrow & \mathcal{O}_X \otimes V \longrightarrow 0 \\
& & & & \mathcal{O}_X(-1, 0) & \otimes & \widetilde{W}'_I \\
& & & & \mathcal{O}_X(0, -1) & \otimes & \widetilde{W}'_{II}
\end{array} \tag{2.2.23}$$

where the rows are of the form of monad (2.2.4) and

$$\begin{aligned}
G &:= \theta_{E'}^{-1} \circ g_{E,E'} \circ \theta_E, \\
H &:= (\psi_{E',\phi'}^{-1} \circ h_{E,E'} \circ \psi_{E,\phi}) \Big|_{\widetilde{W}_I} \oplus (\psi_{E',\phi'}^{-1} \circ h_{E,E'} \circ \psi_{E,\phi}) \Big|_{\widetilde{W}_{II}}, \\
G' &:= (\alpha_{E'} \circ \theta_{E'})^{-1} \circ g'_{E,E'} \circ (\alpha_E \circ \theta_E) = \theta_{E'}^{-1} \circ (\alpha_{E'}^{-1} \circ g'_{E,E'} \circ \alpha_E) \circ \theta_E = G.
\end{aligned}$$

Viewing  $H = \psi_{E',\phi'}^{-1} \circ h_{E,E'} \circ \psi_{E,\phi} : \widetilde{W} \rightarrow \widetilde{W}$ , we extract from (2.2.23) the following commutative diagrams

$$\begin{array}{ccc}
V & \xrightarrow{p_I i_0} & \widetilde{W}_I \\
G \downarrow & & \downarrow H|_{\widetilde{W}_I} \\
V & \xrightarrow{i'_I p'_I i_0} & \widetilde{W}
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
V & \xrightarrow{p_{II} i_0} & \widetilde{W}_{II} \\
G \downarrow & & \downarrow H|_{\widetilde{W}_{II}} \\
V & \xrightarrow{i'_{II} p'_{II} i_0} & \widetilde{W}
\end{array},$$

where we use the notations of (2.2.3) and  $i'_I, p'_I, i'_{II}$  and  $p'_{II}$  are defined similarly for  $\widetilde{W}'_I, \widetilde{W}'_{II} \subset \widetilde{W}$ . Hence

$$Hi_0 = Hi_I p_I i_0 + Hi_{II} p_{II} i_0 = i'_I p'_I i_0 G + i'_{II} p'_{II} i_0 g = i_0 G.$$

It follows that

$$p_0 Hi_0 = p_0 i_0 G = G.$$

Similarly,

$$p_1 Hi_1 = G.$$

On the other hand, diagram (2.2.22), lemma 2.2.17 and the fact that  $g$  maps  $\phi$  to  $\phi'$  give the commutative diagram

$$\begin{array}{ccc}
W & \xrightarrow{c_{E,\phi}} & \widetilde{W}_E \\
\parallel & & \downarrow h_{E,E'} \\
W & \xrightarrow{c_{E',\phi'}} & \widetilde{W}_{E'}
\end{array}$$

It follows that (2.2.23) preserves the framing. That is,

$$p_W Hi_W = id_W,$$

where  $p_W$  is the projection of  $\widetilde{W}$  onto the  $W$ -factor and  $i_W$  is the inclusion of  $W$  into  $\widetilde{W}$ . Hence

$$H = G \oplus G \oplus id_W,$$

with respect to the decomposition  $\widetilde{W} = V \oplus V \oplus W$ . Since  $H \cdot (\widetilde{W}_I, \widetilde{W}_{II}) = (\widetilde{W}'_I, \widetilde{W}'_{II})$  by (2.2.23), it follows that  $(\widetilde{W}_I, \widetilde{W}_{II}) = (\widetilde{W}'_I, \widetilde{W}'_{II})$  in  $U(r, n)/GL(V)$ . Hence  $G_{r,n}$  is well-defined.  $\square$

**Proposition 2.2.24.**  $F_{r,n}$  is a bijection.

*Proof.* First we show that

$$F_{r,n} \circ G_{r,n} = id_{\mathcal{M}_{BM}(r,n)}. \quad (2.2.25)$$

In fact, for  $(\widetilde{W}_I, \widetilde{W}_{II}) = G_{r,n}(E, \phi)$ , lemma 2.2.20 gives that

$$(E_{\widetilde{W}_I, \widetilde{W}_{II}}, \phi_{\widetilde{W}_I, \widetilde{W}_{II}}) \simeq (E, \phi).$$

This proves (2.2.25). Next we show that

$$G_{r,n} \circ F_{r,n} = id_{\mathcal{M}(r,n)}. \quad (2.2.26)$$

For  $(\widetilde{W}_I, \widetilde{W}_{II}) \in \mathcal{M}(r, n)$  let

$$(E, \phi) := F_{r,n}(\widetilde{W}_I, \widetilde{W}_{II}) = (E_{\widetilde{W}_I, \widetilde{W}_{II}}, \phi_{\widetilde{W}_I, \widetilde{W}_{II}}).$$

By Okonek et al. [45, lemma II.4.1.3], there is an isomorphism of monads

$$\begin{array}{ccccccc}
& & \mathcal{O}_X(-1, 0) & \otimes & \widetilde{W}_I & & \\
0 & \longrightarrow & \mathcal{O}_X(-1, -1) \otimes V & \longrightarrow & \oplus & \longrightarrow & \mathcal{O}_X \otimes V \longrightarrow 0 \\
& & \downarrow \wr & & \downarrow \wr & & \downarrow \wr \\
0 & \longrightarrow & \mathcal{O}_X(-1, -1) \otimes V_E & \longrightarrow & \oplus & \longrightarrow & \mathcal{O}_X \otimes V'_E \longrightarrow 0, \\
& & \mathcal{O}_X(0, -1) & \otimes & \widetilde{W}_{II} & & \\
& & \mathcal{O}_X(-1, 0) & \otimes & \widetilde{W}_{I,E} & & \\
& & \mathcal{O}_X(0, -1) & \otimes & \widetilde{W}_{II,E} & & 
\end{array} \quad (2.2.27)$$

which induces the identity morphism  $E = E$ , since the cohomologies of both rows are  $E$ . Let

$$(\widetilde{W}'_I, \widetilde{W}'_{II}) := G_{r,n}(E, \phi).$$

By lemma (2.2.20), we have the following isomorphism of monads

$$\begin{array}{ccccccc}
& & \mathcal{O}_X(-1, 0) & \otimes & \widetilde{W}'_I & & \\
0 \longrightarrow & \mathcal{O}_X(-1, -1) \otimes V & \longrightarrow & \oplus & \longrightarrow & \mathcal{O}_X \otimes V & \longrightarrow 0 \\
& \downarrow \theta_E & & & & \downarrow (\alpha_E \circ \theta_E) & \\
& \mathcal{O}_X(0, -1) & \otimes & \widetilde{W}'_{II} & & & \\
& & \downarrow (\psi_{E,\phi}|_{\widetilde{W}'_I} \oplus \psi_{E,\phi}|_{\widetilde{W}'_{II}}) & & & & \\
0 \longrightarrow & \mathcal{O}_X(-1, -1) \otimes V_E & \longrightarrow & \oplus & \longrightarrow & \mathcal{O}_X \otimes V'_E & \longrightarrow 0, \\
& & & & & \downarrow & \\
& & \mathcal{O}_X(0, -1) & \otimes & \widetilde{W}_{II,E} & & 
\end{array} \tag{2.2.28}$$

the inverse of which induces the framing  $E|_D \xrightarrow{\phi} \mathcal{O}_{\mathbb{P}^1}(-1) \otimes W$  by lemma 2.2.20. Composing (2.2.27) with the inverse of (2.2.28) we obtain an isomorphism of monads, which is of the form of (2.2.23) and clearly preserves the framing. By the same method as in the proof of lemma 2.2.19, we obtain that  $(\widetilde{W}'_I, \widetilde{W}'_{II}) = (\widetilde{W}_I, \widetilde{W}_{II})$  in  $U(r, n)/GL(V)$ . Hence  $G_{r,n} \circ F_{r,n} = id_{\mathcal{M}(r,n)}$ . It follows that  $F_{r,n}$  is bijective.  $\square$

The next step is to give  $\mathcal{M}(r, n)$  the structure of a smooth algebraic variety.

**Lemma 2.2.29.** *The action of  $GL(V)$  on  $U(r, n)$  is set-theoretically free.*

The proof uses the following lemma, which is proved by the same method as lemma 2.2.13.

**Lemma 2.2.30.** *For a rank  $l$  torsion free sheaf  $G$  on  $X = \mathbb{P}^1 \times \mathbb{P}^1$  such that  $G|_D \simeq \mathcal{O}_{\mathbb{P}^1}^{\oplus l}$ ,*

$$H^0(G(-D)) = 0.$$



*Proof of lemma 2.2.29.* Suppose  $g \in GL(V)$  fixes  $u = (\widetilde{W}_I, \widetilde{W}_{II}) \in U(r, n)$ . We have the following isomorphism of monads

$$\begin{array}{ccccccc}
& & \mathcal{O}_X(-1, 0) & \otimes & \widetilde{W}_I & & \\
0 & \longrightarrow & \mathcal{O}_X(-1, -1) \otimes V & \longrightarrow & \oplus & \longrightarrow & \mathcal{O}_X \otimes V \longrightarrow 0 \\
& & \downarrow g & & \downarrow g \oplus g \oplus id_W & & \downarrow g \\
0 & \longrightarrow & \mathcal{O}_X(-1, -1) \otimes V & \longrightarrow & \oplus & \longrightarrow & \mathcal{O}_X \otimes V \longrightarrow 0, \\
& & & & \mathcal{O}_X(0, -1) & \otimes & \widetilde{W}_{II} \\
& & & & \mathcal{O}_X(-1, 0) & \otimes & \widetilde{W}_I \\
& & & & \mathcal{O}_X(0, -1) & \otimes & \widetilde{W}_{II}
\end{array} \tag{2.2.31}$$

which induces an automorphism of  $E_u$ , which we also call  $g$ , such that the diagram

$$\begin{array}{ccc}
E_u|_D & \xrightarrow{g|_D} & E_u|_D \\
& \searrow \phi_u & \swarrow \phi_u \\
& \mathcal{O}_{\mathbb{P}^1}(-1) \otimes W & 
\end{array}$$

commutes.

It follows that

$$g|_D = id_{E_u|_D}.$$

Note that the sheaf  $\mathcal{E}nd(E_u)$  is trivial along  $D$ . By lemma 2.2.30 we have

$$H^0(X, \mathcal{E}nd(E_u)(-D)) = 0. \tag{2.2.32}$$

The exactness of

$$0 \rightarrow \mathcal{E}nd(E_u)(-D) \rightarrow \mathcal{E}nd(E_u) \rightarrow \mathcal{E}nd(E_u|_D) \rightarrow 0$$

implies that the restriction map

$$H^0(\mathcal{E}nd(E_u)) \rightarrow H^0(\mathcal{E}nd(E_u|_D))$$

is injective. Hence  $g|_D$  lifts uniquely to the automorphism  $g = id_{E_u}$ . By the isomorphism (2.2.31) and Okonek et al. [45, lemma II.4.1.3], it follows that  $g = id_{E_u}$  is induced by identity of  $GL(V)$ .  $\square$

**Proposition 2.2.33.**  *$U(r, n)$  can be given the structure of a smooth algebraic variety of dimension  $n^2 + nr - \frac{r^2}{4}$ .*

The proof uses lemmas 2.2.34 and 2.2.47 below.

**Lemma 2.2.34.** *There is a holomorphic symplectic form  $\omega_{L_n}$  on  $L_n$  such that  $\mu_{r,n}$  is a moment map for the action of  $GL(V)$  on  $(L_n, \omega_{L_n})$ .*

(For a complex manifold  $M$  with a holomorphic symplectic form  $\omega$  and the holomorphic action of a Lie group  $G$ , a holomorphic mapping  $\mu : M \rightarrow Lie(G)^*$  is called a *moment map* for the action of  $G$  if

$$\langle d\mu, \xi \rangle = i_{\vec{\xi}}\omega$$

where  $\vec{\xi}$  is the vector field on  $M$  which corresponds to the infinitesimal action of  $\xi$  and  $\langle \cdot, \cdot \rangle$  is the natural pairing  $TM \times T^*M \rightarrow \mathbb{C}$ .)

**Remark 2.2.35.** In lemma 2.2.34, we have identified

$$End(V) \simeq End(V)^* \tag{2.2.36}$$

via the trace pairing  $tr : End(V) \times End(V) \rightarrow \mathbb{C}$ .

*Proof of lemma 2.2.34.* For each  $(\widetilde{W}_I, \widetilde{W}_{II}) \in L_n$ , there is a natural identification for the tangent space

$$T_{(\widetilde{W}_I, \widetilde{W}_{II})} L_n \simeq \text{Hom}(\widetilde{W}_I, \widetilde{W}_{II}) \oplus \text{Hom}(\widetilde{W}_{II}, \widetilde{W}_I).$$

We can define a holomorphic 2-form  $\omega_{L_n}$  on  $L_n$  by

$$(\omega_{L_n})_{(\widetilde{W}_I, \widetilde{W}_{II})}((\alpha, \beta), (\alpha', \beta')) \stackrel{\text{def}}{=} \text{tr}(\alpha' \beta) - \text{tr}(\alpha \beta'),$$

which is nowhere degenerate. We first show that  $\omega$  is closed. Fix an element  $(\widetilde{W}_I, \widetilde{W}_{II}) \in L_n$ . We define

$$\begin{aligned} q : GL(\widetilde{W}) &\longrightarrow L_n \\ C &\mapsto (C\widetilde{W}_I, C\widetilde{W}_{II}). \end{aligned}$$

Let

$$C = \left( \begin{array}{c|c} C_I & * \\ \hline * & C_{II} \end{array} \right)$$

be an element of  $GL(\widetilde{W})$  and

$$H := \left( \begin{array}{c|c} * & \beta \\ \hline \alpha & * \end{array} \right), \quad H' := \left( \begin{array}{c|c} * & \beta' \\ \hline \alpha' & * \end{array} \right)$$

elements of  $\text{End}(\widetilde{W})$ , where the display of these matrices is with respect to the decomposition  $\widetilde{W} = \widetilde{W}_I \oplus \widetilde{W}_{II}$ . We define the left multiplication by  $C' \in GL(\widetilde{W})$

$$\begin{aligned} L_{C'} : GL(\widetilde{W}) &\longrightarrow GL(\widetilde{W}) \\ A &\mapsto C'A. \end{aligned}$$

We have a trivialization

$$\begin{aligned} TGL(\widetilde{W}) &\xrightarrow{\sim} GL(\widetilde{W}) \times \text{End}(\widetilde{W}) \\ (C', d(L_{C'})_{id_{\widetilde{W}}}(K)) &\leftrightarrow (C', K) \end{aligned} \tag{2.2.37}$$

of the tangent bundle  $TGL(\widetilde{W})$ . Then

$$\begin{aligned} d(q \circ L_C)_{id_{\widetilde{W}}}(H) &= \left. \frac{d}{dt} \right|_{t=0} (C(id_{\widetilde{W}} + tH)\widetilde{W}_I, C(id_{\widetilde{W}} + tH)\widetilde{W}_{II}) \\ &= \left. \frac{d}{dt} \right|_{t=0} ((id_{\widetilde{W}} + tCHC^{-1})C\widetilde{W}_I, (id_{\widetilde{W}} + tCHC^{-1})C\widetilde{W}_{II}) \\ &= (C_{II}\alpha C_I^{-1}, C_I\beta C_{II}^{-1}) \in \text{Hom}(C\widetilde{W}_I, C\widetilde{W}_{II}) \oplus \text{Hom}(C\widetilde{W}_{II}, C\widetilde{W}_I), \end{aligned}$$

where in the last equality we have used the fact that

$$\left. \frac{d}{dt} \right|_{t=0} ((id_{\widetilde{W}} + tK)U_I, (id_{\widetilde{W}} + tK)U_{II}) = (\gamma, \delta),$$

for any  $(U_I, U_{II}) \in L_n$  and  $K = \left( \begin{array}{c|c} * & \delta \\ \hline \gamma & * \end{array} \right) \in \text{End}(\widetilde{W})$  is displayed with respect to the decomposition  $\widetilde{W} = U_I \oplus U_{II}$ . Hence

$$\begin{aligned} &(q^*\omega_{L_n})_C(d(L_C)_{id_{\widetilde{W}}}(H), d(L_C)_{id_{\widetilde{W}}}(H')) \\ &= (\omega_{L_n})_{(C\widetilde{W}_I, C\widetilde{W}_{II})}(d(q \circ L_C)_{id_{\widetilde{W}}}(H), d(q \circ L_C)_{id_{\widetilde{W}}}(H')) \\ &= (\omega_{L_n})_{(C\widetilde{W}_I, C\widetilde{W}_{II})}((C_{II}\alpha C_I^{-1}, C_I\beta C_{II}^{-1}), (C_{II}\alpha' C_I^{-1}, C_I\beta' C_{II}^{-1})) \\ &= \text{tr}(C_{II}\alpha' C_I^{-1} C_I\beta C_{II}) - \text{tr}(C_{II}\alpha C_I^{-1} C_I\beta' C_{II}) \\ &= \text{tr}(\alpha'\beta) - \text{tr}(\alpha\beta'). \end{aligned}$$

It follows that  $q^*\omega_{L_n}$  is constant with respect to the trivialization (2.2.37) and hence closed. Hence  $\omega_{L_n}$  is also closed.

Next we prove that  $\mu_{r,n}$  is a moment map for the action of  $GL(V)$  on  $(L_n, \omega_{L_n})$ . Let  $a$  be an element in  $\text{Lie}(GL(V)) = \text{End}(V)$ . Then  $a$  can be considered as an element

of  $End(\widetilde{W})$  via the diagonal action of  $GL(V)$  on  $\widetilde{W} = V \oplus V \oplus W$ . We write  $A$  for the image of  $a$  in  $End(\widetilde{W})$ . That is,

$$A = \left( \begin{array}{c|c|c} a & 0 & 0 \\ \hline 0 & a & 0 \\ \hline 0 & 0 & 0 \end{array} \right),$$

with respect to the decomposition  $\widetilde{W} = V \oplus V \oplus W$ . In other words,

$$A = i_0 a p_0 + i_1 a p_1.$$

As a tangent vector field on  $L_n$  we have  $A = (\alpha, \beta) \in Hom(\widetilde{W}_I, \widetilde{W}_{II}) \oplus Hom(\widetilde{W}_{II}, \widetilde{W}_I)$  where  $\alpha = p_{II} A i_I$  and  $\beta = p_I A i_{II}$ . In other words,

$$A = \left( \begin{array}{c|c} * & \beta \\ \hline \alpha & * \end{array} \right),$$

with respect to the decomposition  $\widetilde{W} = \widetilde{W}_I \oplus \widetilde{W}_{II}$ . Since  $End(V)$  is identified with  $End(V)^*$  via (2.2.36), we need to prove that

$$d(tr(\mu_{r,n} a))_{(\widetilde{W}_I, \widetilde{W}_{II})}(\alpha', \beta') = i_A \omega(\alpha', \beta') \quad (2.2.38)$$

for a fixed  $a \in End(V)$ . The right hand side of (2.2.38) is

$$\begin{aligned} i_{(\alpha, \beta)} \omega(\alpha', \beta') &= tr(\alpha' \beta) - tr(\alpha \beta') \\ &= tr(\alpha' p_I A i_{II}) - tr(p_{II} A i_I \beta') \\ &= tr(i_{II} \alpha' p_I A) - tr(i_I \beta' p_{II} A). \end{aligned}$$

For the left hand side of (2.2.38) we first compute

$$\begin{aligned}
d(\operatorname{tr}(\mu_{r,n}a))_{(\widetilde{W}_I, \widetilde{W}_{II})}(\alpha', 0) &= d(\operatorname{tr}(p_0 i_I p_I i_0 a) + \operatorname{tr}(p_1 i_I p_I i_1 a) - \operatorname{tr}(a))_{(\widetilde{W}_I, \widetilde{W}_{II})}(\alpha', 0) \\
&= d(\operatorname{tr}(i_I p_I i_0 a p_0) + \operatorname{tr}(i_I p_I i_1 a p_1))_{(\widetilde{W}_I, \widetilde{W}_{II})}(\alpha', 0) \\
&= d(\operatorname{tr}(i_I p_I A))_{(\widetilde{W}_I, \widetilde{W}_{II})}(\alpha', 0) \\
&= \operatorname{tr}(i_{II} \alpha' p_I A),
\end{aligned}$$

where the last equality holds because  $d(i_I)(\alpha', 0) = i_{II} \alpha'$  and  $d(p_I)(\alpha', 0) = 0$ . Since

$$\begin{aligned}
p_0 i_I p_I i_0 + p_0 i_{II} p_{II} i_0 &= id_V, \\
p_1 i_I p_I i_1 + p_1 i_{II} p_{II} i_1 &= id_V,
\end{aligned}$$

we can write  $\mu_{r,n}$  alternatively as  $-p_0 i_{II} p_{II} i_0 - p_1 i_{II} p_{II} i_1 + id_V$  and compute

$$d(\operatorname{tr}(\mu_{r,n}a))_{(\widetilde{W}_I, \widetilde{W}_{II})}(0, \beta') = -\operatorname{tr}(i_I \beta' p_{II} A)$$

similarly to above. Hence (2.2.38) holds.  $\square$

**Remark 2.2.39.** For  $(\widetilde{W}_I, \widetilde{W}_{II}) \in L_n$  we can define  $A(\widetilde{W}_I, \widetilde{W}_{II}) \in \operatorname{Lie}(GL(\widetilde{W})) = \operatorname{End}(\widetilde{W})$  by

$$\begin{aligned}
A(\widetilde{W}_I, \widetilde{W}_{II})|_{\widetilde{W}_I} &:= \frac{1}{2} id_{\widetilde{W}_I}, \\
A(\widetilde{W}_I, \widetilde{W}_{II})|_{\widetilde{W}_{II}} &:= -\frac{1}{2} id_{\widetilde{W}_{II}}
\end{aligned} \tag{2.2.40}$$

and extend linearly. We identify

$$\operatorname{End}(\widetilde{W}) \simeq \operatorname{End}(\widetilde{W})^* \tag{2.2.41}$$

(via the trace pairing  $\operatorname{tr} : \operatorname{End}(\widetilde{W}) \times \operatorname{End}(\widetilde{W}) \rightarrow \mathbb{C}$ .) Then (2.2.40) yields an injective map

$$A : L_n \rightarrow \operatorname{End}(\widetilde{W})$$

whose image is a coadjoint orbit for  $GL(\widetilde{W})$ . Hence there exists a holomorphic symplectic form on  $L_n$  by McDuff and Salamon [37, page 168], which we call  $\omega'_{L_n}$ .

**Lemma 2.2.42.**  $\omega'_{L_n} = \omega_{L_n}$ .

*Proof.* We use the following formula for  $\omega'_{L_n}$  (see McDuff and Salamon [37, page 168]),

$$\omega_\eta(ad(\xi)^*\eta, ad(\xi')^*\eta) = \langle \eta, [\xi, \xi'] \rangle, \quad (2.2.43)$$

where we identify  $T_\eta\mathcal{O} = \{ad(\xi)^*\eta \mid \xi \in Lie(G)\}$  for any coadjoint orbit  $\mathcal{O} \subset Lie(G)^*$  and any  $\eta \in \mathcal{O}$ .

We fix  $\eta = (\widetilde{W}_I, \widetilde{W}_{II}) \in L_n$ . In the identification (2.2.41),  $\eta$  corresponds to

$$tr(v \cdot \bullet) \in End(\widetilde{W})$$

for some  $v \in End(\widetilde{W})$ . In this proof, we identify  $L_n$  with the image of the map  $A$  (hence  $\eta = A(\widetilde{W}_I, \widetilde{W}_{II})$ .) Since

$$\begin{aligned} L_n &= \{Ad(C)^*\eta \mid C \in GL(\widetilde{W})\} \\ &= \{tr(v \cdot Ad(C)(\bullet)) \mid C \in GL(\widetilde{W})\} \\ &= \{tr(vC(\bullet)C^{-1}) \mid C \in GL(\widetilde{W})\} \subset End(\widetilde{W})^*, \end{aligned}$$

we can identify

$$\begin{aligned} T_\eta L_n &= \{tr(v[\xi, \bullet]) \mid \xi \in End(\widetilde{W})\} \subset End(\widetilde{W})^* \\ &= \{[v, \xi] \mid \xi \in End(\widetilde{W})\} \subset End(\widetilde{W}), \end{aligned}$$

where in the last equality we have used the identification (2.2.41) and the fact that

$$tr(v[\xi, \bullet]) = tr([v, \xi] \cdot \bullet).$$

Formula (2.2.43) becomes

$$(\omega'_{L_n})_\eta([v, \xi], [v, \xi']) = tr(v[\xi, \xi']), \quad (2.2.44)$$

where

$$\xi = \left( \begin{array}{c|c} \gamma & \beta \\ \alpha & \delta \end{array} \right) \text{ and } \xi' = \left( \begin{array}{c|c} \gamma' & \beta' \\ \alpha' & \delta' \end{array} \right)$$

are in  $\text{End}(\widetilde{W})$  (the display of the matrices in this proof is with respect to the decomposition  $\widetilde{W} = \widetilde{W}_I \oplus \widetilde{W}_{II}$ .) The right hand side of (2.2.44) is

$$\begin{aligned} &= \text{tr} \left( \left( \begin{array}{c|c} \frac{1}{2}id_{\widetilde{W}_I} & 0 \\ 0 & -\frac{1}{2}id_{\widetilde{W}_{II}} \end{array} \right) \left( \begin{array}{c|c} \beta\alpha' - \beta'\alpha + [\gamma, \gamma'] & * \\ * & \alpha\beta' - \alpha'\beta + [\delta, \delta'] \end{array} \right) \right) \\ &= \text{tr}(\alpha'\beta - \alpha\beta'). \end{aligned}$$

We have

$$[v, \xi] = \left[ \left( \begin{array}{c|c} \frac{1}{2}id_{\widetilde{W}_I} & 0 \\ 0 & -\frac{1}{2}id_{\widetilde{W}_{II}} \end{array} \right), \left( \begin{array}{c|c} \gamma & \beta \\ \alpha & \delta \end{array} \right) \right] = \left( \begin{array}{c|c} 0 & \beta \\ \alpha & 0 \end{array} \right),$$

and similarly,

$$[v, \xi'] = \left( \begin{array}{c|c} 0 & \beta' \\ \alpha' & 0 \end{array} \right).$$

Hence

$$(\omega'_{L_n})_{(\widetilde{W}_I, \widetilde{W}_{II})}((\alpha, \beta), (\alpha', \beta')) = \text{tr}(\alpha'\beta - \alpha\beta') = \omega_{L_n}((\alpha, \beta), (\alpha', \beta')). \quad \square$$

Let  $V(r, n) \subset L_n$  be a Zariski open subset such that  $V(r, n) \cap \mu_{r,n}^{-1}(0) = U(r, n)$ . By theorem 2.2.6, the symplectic form  $\omega_{L_n}|_{V(r,n)}$  on  $V(r, n)$  descends by Marsden and Weinstein [35, theorem 1] to one on the (symplectic) quotient  $\mathcal{M}(r, n) = U(r, n)/GL(V)$ , which we call  $\omega_{\mathcal{M}}$ . The following is immediate from the form of monad (2.2.4).

**Lemma 2.2.45.** (i) *The involution  $(z_1, z_2) \mapsto (z_2, z_1)$  on  $X = \mathbb{P}^1 \times \mathbb{P}^1$  induces the involution*

$$(\widetilde{W}_I, \widetilde{W}_{II}) \mapsto (\widetilde{W}_{II}, \widetilde{W}_I) \quad (2.2.46)$$

*on  $\mathcal{M}(r, n)$ , which coincides with the involution  $i_{\mathcal{M}}$  in section 2.1.*



(ii)  $\mathcal{M}_{\mathbb{P}^2}(r, m)$  is a holomorphic Lagrangian subvariety of  $\mathcal{M}(r, 2m)$  with respect to the symplectic form  $\omega_{\mathcal{M}}$ .

*Proof of part (ii).* This follows from proposition 2.1.1 and the fact that

$$i_{\mathcal{M}}^* \omega_{L_n} = -\omega_{L_n}.$$

□

We believe the next result is standard. Nevertheless, we include here a proof for the reader's convenience.

**Lemma 2.2.47.** *Let  $Y$  be a complex manifold with a holomorphic symplectic form  $\omega$  and  $G$  a complex Lie group acting holomorphically on  $X$  such that there exists a holomorphic moment map  $\mu : Y \rightarrow \mathfrak{g}^*$  for this action, one has*

$$\dim_{\mathbb{C}} T_y(G.y) = \dim_{\mathbb{C}} \text{Im}(d_y \mu)$$

for all  $y \in Y$  such that the orbit  $G.y \subset Y$  is an immersed submanifold.

*Proof.* We first note that  $T_y(G.y)$  is the image of the Lie algebra  $\mathfrak{g}$  under the infinitesimal action

$$\eta \mapsto \vec{\eta}_y.$$

Let

$$\begin{aligned} T_y(G.y)^{\perp} &:= \{v \in T_y Y \mid \omega_y(v, w) = 0 \ \forall w \in T_y(G.y)\} \\ &= \{v \in T_y Y \mid \omega_y(v, \vec{\eta}_y) = 0 \ \forall \eta \in \mathfrak{g}\}. \end{aligned}$$

By the definition of a moment map,

$$\omega_y(v, \vec{\eta}_y) = d_y \mu(v)(\eta)$$

for all  $v \in T_y Y$  and  $\eta \in \mathfrak{g}$ . Hence

$$\ker(d_y \mu) = T_y(G \cdot y)^\perp.$$

It follows that

$$\begin{aligned} \dim_{\mathbb{C}} \operatorname{Im}(d_y \mu) &= \dim_{\mathbb{C}} T_y Y - \dim_{\mathbb{C}} \ker(d_y \mu) \\ &= \dim_{\mathbb{C}} T_y Y - \dim_{\mathbb{C}} T_y(G \cdot y)^\perp \\ &= \dim_{\mathbb{C}} T_y(G \cdot y), \end{aligned}$$

since  $\omega_y$  is nondegenerate for all  $y \in Y$ . □

*Proof of proposition 2.2.33.* Since the  $GL(V)$ -action on  $U(r, n)$  is set-theoretically free by lemma 2.2.29, its orbits are immersed submanifolds, as in the proof of proposition III.1.5.10 in Bourbaki [9]. By lemmas 2.2.34 and 2.2.47,

$$\dim_{\mathbb{C}} \operatorname{Im}(d_u \mu_{r,n}) = \dim_{\mathbb{C}} T_u(GL(V) \cdot u) = \dim_{\mathbb{C}} GL(V)$$

for all  $u \in U(r, n) \subset \mu_{r,n}^{-1}(0)$ . It follows that  $d_u \mu_{r,n}$  is surjective for all  $u \in U(r, n)$ . Hence  $U(r, n)$  is a smooth algebraic variety. The dimension of  $U(r, n)$  is

$$\begin{aligned} 2 \dim_{\mathbb{C}} G(n, \widetilde{W}) - \dim_{\mathbb{C}} \operatorname{End}(V) &= 2n^2 - \left(n - \frac{r}{2}\right)^2 \\ &= n^2 + nr - \frac{r^2}{4}. \end{aligned}$$

□

**Proposition 2.2.48.**  $\mathcal{M}(r, n) = U(r, n)/GL(V)$  can be given the structure of a smooth algebraic space of dimension  $2nr - \frac{r^2}{2}$ . Moreover,  $U(r, n)$  is a principal  $GL(V)$ -bundle over  $\mathcal{M}(r, n)$ .

*Proof.* Following Okonek et al. [45, theorem II.4.1.9], we first prove that the image  $\Gamma$  of the following map

$$\begin{aligned} \gamma: U(r, n) \times GL(V) &\longrightarrow U(r, n) \times U(r, n) \\ (u, g) &\longmapsto (u, g.u) \end{aligned}$$

is closed in  $U(r, n) \times U(r, n)$ .

Let  $\zeta$  be the tautological bundle over the Grassmannian  $G(n, \widetilde{W})$ . Let  $pr_1$  and  $pr_2$  be the projections of  $G(n, \widetilde{W}) \times G(n, \widetilde{W})$  onto the first and second factors respectively. Denote also by  $pr_1$  and  $pr_2$  their restriction to the locally closed subset  $U(r, n) \subset G(n, \widetilde{W}) \times G(n, \widetilde{W})$ . Let  $pr_U$  and  $pr_X$  be the projections of  $U(r, n) \times X$  onto the first and second factors respectively. We consider the following complex

$$\begin{aligned} 0 \rightarrow pr_X^* \mathcal{O}_X(-1, -1) \otimes V \xrightarrow{A} & \oplus \begin{array}{c} pr_X^* \mathcal{O}_X(-1, 0) \otimes pr_U^* pr_1^* \zeta \\ pr_X^* \mathcal{O}_X(-1, 0) \otimes pr_U^* pr_2^* \zeta \end{array} \xrightarrow{B} \mathcal{O}_{U(r, n) \times X} \otimes V \rightarrow 0 \end{aligned} \quad (2.2.49)$$

where  $A$  (resp.  $B$ ) is defined at each point  $(\widetilde{W}_I, \widetilde{W}_{II}) \in U(r, n)$  as the morphism  $a$  (resp.  $b$ ) in monad (2.2.4). The complex (2.2.49) is in fact a monad by Nakayama's lemma. Let  $\widetilde{\mathcal{E}}$  be its cohomology. Then  $\widetilde{\mathcal{E}}$  is a family of framed sheaves on  $X$ , parametrized by  $U(r, n)$ , as follows. Since  $ker(B)$  is locally free, it is flat over  $U(r, n)$ . We have a natural isomorphism

$$ker B \otimes \mathbb{C}(u) \simeq ker \left( \mathcal{O}_X(-1, 0) \otimes \widetilde{W}_I \oplus \mathcal{O}_X(0, -1) \otimes \widetilde{W}_{II} \xrightarrow{b} \mathcal{O}_X \otimes V \right)$$

for any  $u = (\widetilde{W}_I, \widetilde{W}_{II}) \in U(r, n)$ . Hence the morphism

$$A \otimes \mathbb{C}(u) : V \otimes \mathcal{O}_X(-1, -1) \longrightarrow ker B \otimes \mathbb{C}(u)$$

is injective for all  $u \in U(r, n)$ . It follows that  $\tilde{\mathcal{E}}$  is flat over  $U(r, n)$  (see for example Matsumura [36, 20.E].) Hence there is a natural isomorphism

$$\tilde{\mathcal{E}} \otimes \mathbb{C}(u) \simeq E_u \quad (2.2.50)$$

at each point  $u \in U(r, n)$ .

By Bruzzo and Markushevich [10, theorem 3.1],  $\mathcal{M}_{BM}(r, n)$  is a fine moduli space of framed sheaves. Hence the universal family  $\mathcal{E}_{BM}$  on  $\mathcal{M}_{BM}(r, n)$  pulls back to  $\tilde{\mathcal{E}}$  via a unique morphism  $F'_{r,n} : U(r, n) \rightarrow \mathcal{M}_{BM}(r, n)$ . It follows by (2.2.50) that

$$F'_{r,n} = F_{r,n}.$$

(more precisely,  $F'_{r,n}$  descends to  $F_{r,n}$ .) Let  $V$  be the set-theoretical fiber product defined by the following commutative diagram

$$\begin{array}{ccc} V & \longrightarrow & \mathcal{M}_{BM}(r, n) \\ \downarrow & & \downarrow \Delta_{\mathcal{M}} \\ U(r, n) \times U(r, n) & \xrightarrow{F'_{r,n} \times F'_{r,n}} & \mathcal{M}_{BM}(r, n) \times \mathcal{M}_{BM}(r, n), \end{array}$$

where  $\Delta_{\mathcal{M}}$  is the diagonal map. It follows from the definition of  $F_{r,n}$  that

$$V = \{(u, u') \mid (E_u, \phi_u) \simeq (E_{u'}, \phi_{u'})\}.$$

By proposition 2.2.24, this is equal to

$$\Gamma = \{(u, u') \mid u' = g.u \text{ for some } g \in GL(V)\}. \quad (2.2.51)$$

Since  $\mathcal{M}_{BM}(r, n)$  is separated,  $\Delta_{\mathcal{M}}(\mathcal{M}_{BM}(r, n))$  is closed in  $\mathcal{M}_{BM}(r, n) \times \mathcal{M}_{BM}(r, n)$ . Therefore,  $\Gamma$  is closed in  $U(r, n) \times U(r, n)$ .

As in the proof of proposition 2.2.33,  $\gamma$  is injective on tangent spaces. It follows that  $\gamma$  is a closed immersion. By Laumon and Moret-Bailly [33, corollaire 8.1.1], the set-theoretical quotient  $\mathcal{M}(r, n) = U(r, n)/GL(V)$  is a smooth algebraic space and  $U(r, n)$  is a principal  $GL(V)$ -bundle in the category of algebraic spaces. The dimension of  $\mathcal{M}(r, n)$  is

$$\begin{aligned} \dim_{\mathbb{C}} U(r, n) - \dim_{\mathbb{C}} GL(V) &= n^2 + nr - \frac{r^2}{4} - \left(n - \frac{r}{2}\right)^2 \\ &= 2nr - \frac{r^2}{2}. \end{aligned}$$

□

*Proof of theorem 2.2.6.* It is clear that both  $\mathcal{E}$  and the universal monad (2.2.49) are  $GL(V)$ -equivariant. Let  $\pi : U(r, n) \rightarrow \mathcal{M}(r, n)$  be the quotient map by the action of  $GL(V)$ . Since  $U(r, n)$  is a principal  $GL(V)$ -bundle over  $\mathcal{M}(r, n)$ , by Huybrechts and Lehn [26, theorem 4.2.14] the monad (2.2.49) descends to a monad

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{C} \longrightarrow 0 \tag{2.2.52}$$

on  $\mathcal{M}(r, n) \times X$ , each term of which pulls back via  $\pi$  to the corresponding term of (2.2.49). The family  $\tilde{\mathcal{E}}$  also descends to one, which we call  $\mathcal{E}$ , on  $\mathcal{M}(r, n) \times X$  which is the cohomology of (2.2.52). It follows that  $\pi^* \mathcal{E} \simeq \tilde{\mathcal{E}}$ .

*Claim.* *If  $G$  is a locally free  $GL(V)$ -equivariant sheaf on  $U(r, n)$  then  $G$  descends to a locally free sheaf  $G_0$  on  $\mathcal{M}(r, n)$ .*

In fact, since  $\pi^* G_0 \simeq G$ , the fibers of  $G_0$  all have the same dimension. The claim follows. Therefore, all the terms of monad (2.2.52) are locally free. Hence  $\mathcal{E}$  is flat over  $\mathcal{M}(r, n)$  and

$$\mathcal{E} \otimes \mathbb{C}([u]) \simeq E_u$$

for any  $u \in U(r, n)$  and  $[u]$  its image in  $\mathcal{M}(r, n) = U(r, n)/GL(V)$ . As in the proof of proposition 2.2.48,  $\mathcal{E}$  induces the map  $F_{r,n} : \mathcal{M}(r, n) \rightarrow \mathcal{M}_{BM}(r, n)$ . Hence  $F_{r,n}$  is a morphism of algebraic spaces. In particular, it is a holomorphic mapping between smooth complex manifolds (their smoothness was proved in proposition 2.2.48 and proposition 2.1.1.) Since  $F_{r,n}$  is bijective by proposition 2.2.24, it is an isomorphism of complex manifolds by Griffiths and Harris [17, page 19]. Hence it is an isomorphism of algebraic spaces (cf. Serre [48, proposition 9].)  $\square$

**Remark 2.2.53.** In particular,  $\mathcal{M}(r, n)$  is a smooth quasi-projective algebraic variety.

## CHAPTER 3

### EXAMPLES

In this chapter we consider moduli of rank 2 sheaves. To simplify notations we define

$$\begin{aligned}\mathcal{M}(n) &\stackrel{def}{=} \mathcal{M}(2, n), \\ \mathcal{M}_{\mathbb{P}^2}(m) &\stackrel{def}{=} \mathcal{M}_{\mathbb{P}^2}(2, m).\end{aligned}$$

#### 3.1. The case $r = n = 2m = 2$

**Definition 3.1.1.** For a fixed ample class  $H$  on a surface  $Y$ , a rank 2 torsion free sheaf  $E$  is said to be *Gieseker stable* (resp. *Gieseker semistable*) if

$$\chi(F \otimes \mathcal{O}_Y(kH)) < (\text{resp. } \leq) \frac{\chi(E \otimes \mathcal{O}_Y(kH))}{2}, \quad k \gg 0$$

for all nonzero subsheaves  $F \subset E$  of rank 1.

$E$  is said to be  $\mu$ -stable (resp.  $\mu$ -semistable) if

$$c_1(F).H < (\text{resp. } \leq) \frac{c_1(E).H}{2}$$

for all nonzero subsheaves  $F \subset E$  of rank 1.

**Remark 3.1.2.** (Huybrechts and Lehn [26, lemma 1.2.13])

$$\mu\text{-stable} \Rightarrow \text{Gieseker stable} \Rightarrow \text{Gieseker semistable} \Rightarrow \mu\text{-semistable}.$$

On the quadric surface  $X = \mathbb{P}^1 \times \mathbb{P}^1$  we fix the ample class  $H = (1, 1)$  and define Gieseker (semi)stability according to definition 3.1.1. Let

$$\mathcal{M}^{ss}(n) \stackrel{def}{=} \{(E, \phi) \in \mathcal{M}(n) \mid E \text{ is Gieseker semistable}\}$$

On  $\mathbb{P}^2$  all choices of an ample class give the same stability condition. Due to the following lemma, hereafter the sheaves in  $\mathcal{M}_{\mathbb{P}^2}(m)$  are referred to simply as being *stable*.

**Lemma 3.1.3.** *The sheaves in  $\mathcal{M}_{\mathbb{P}^2}(m)$  are all  $\mu$ -stable*

*Proof.* Let  $(G, \phi) \in \mathcal{M}_{\mathbb{P}^2}(m)$  such that  $G$  is not  $\mu$ -stable. Then there is a rank 1 subsheaf  $L \subset G$  such that

$$c_1(L) \cdot H' \geq \frac{c_1(G) \cdot H'}{2},$$

for any hyperplane section  $H' \subset \mathbb{P}^2$ . Since  $L^{\vee\vee} = \mathcal{O}_{\mathbb{P}^2}(d)$  for some  $d \in \mathbb{Z}$ , we have

$$d \geq -\frac{1}{2}.$$

But  $L|_C$  is a subsheaf of  $G|_C \simeq \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$ . We may assume  $L$  is locally free near  $C$  and  $L|_C = \mathcal{O}_{\mathbb{P}^1}(2d)$ . Then

$$2d = c_1(L|_C) \leq c_1(G|_C) = -2.$$

This is a contradiction. Hence  $G$  is  $\mu$ -stable. □

Let  $\mathcal{M}'(n)$  be the (unframed) moduli space of rank 2 Gieseker semistable torsion free sheaves on  $X$  with first Chern class  $(-1, -1)$  and second Chern class  $n$ . Let  $\mathcal{M}'_{\mathbb{P}^2}(m)$  be the (unframed) moduli space of rank 2 stable torsion free sheaves on  $\mathbb{P}^2$  with first Chern class  $-1$  and second Chern class  $m$ . The existence of these moduli spaces are proved by Gieseker [16].



Let

$$D'(n) \stackrel{def}{=} \{E \in \mathcal{M}'(n) \mid E|_D \not\cong \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}\},$$

$$D'_{\mathbb{P}^2}(m) \stackrel{def}{=} \{G \in \mathcal{M}'_{\mathbb{P}^2}(m) \mid G|_C \not\cong \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}\}.$$

where  $C$  is the fixed conic from section 2.1. There are forgetful maps

$$F : \mathcal{M}^{ss}(n) \longrightarrow \mathcal{M}'(n) \setminus D'(n),$$

$$F_{\mathbb{P}^2} : \mathcal{M}_{\mathbb{P}^2}(m) \longrightarrow \mathcal{M}'_{\mathbb{P}^2}(m) \setminus D'_{\mathbb{P}^2}(m),$$

which remove the framing data. Huh [22, page 2102] proves that

$$\mathcal{M}'(2) \simeq \mathbb{P}^3.$$

By Huh [22, proposition 3.2],  $D'(2)$  is a hyperplane in  $\mathbb{P}^3$ , which we denote by  $K$ .

Then

$$\mathcal{M}'(2) \setminus D'(2) = \mathbb{P}^3 \setminus K \simeq \mathbb{A}^3.$$

**Proposition 3.1.4.** (i)  $\mathcal{M}^{ss}(2) \simeq Q \setminus Q'$ , where  $Q$  and  $Q'$  are quadric hypersurfaces in  $\mathbb{P}^7$  ( $Q$  is smooth.)

(ii) The forgetful map  $F : \mathcal{M}^{ss}(2) \rightarrow \mathcal{M}'(2) \setminus D'(2) \simeq \mathbb{A}^3$  is a trivial  $PGL(2)$  torsor.

That is,

$$\mathcal{M}^{ss}(2) \simeq \mathbb{A}^3 \times PGL(2).$$

(iii)  $\mathcal{M}_{\mathbb{P}^2}(1)$ , which is the fixed locus of the involution  $i_{\mathcal{M}}$  in  $\mathcal{M}(2)$ , is isomorphic to  $PGL(2)$ .

**Remark 3.1.5.** The fact that  $\mathcal{M}_{\mathbb{P}^2}(1) \simeq PGL(2)$  can be seen directly since the only rank 2 stable torsion free sheaf on  $\mathbb{P}^2$  with first Chern class  $-1$  and second Chern class 1 is  $T_{\mathbb{P}^2}(-2)$  (see Okonek et al. [45, example II.3.2.1].) Nevertheless, we explicitly compute below the involution  $i_{\mathcal{M}}$  and its fixed locus because we believe this demonstrates the effectiveness of the key idea of this thesis (section 2.1.)

**Lemma 3.1.6.** For  $(\widetilde{W}_I, \widetilde{W}_{II}) \in \mathcal{M}(2)$  one has that

$$E_{\widetilde{W}_I, \widetilde{W}_{II}} \text{ is Gieseker semistable iff } W \cap \widetilde{W}_I = W \cap \widetilde{W}_{II} = 0.$$

*Proof.* Let  $E = E_{\widetilde{W}_I, \widetilde{W}_{II}}$ .

*Claim.*  $W \cap \widetilde{W}_I \neq 0$  (resp.  $W \cap \widetilde{W}_{II} \neq 0$ ) iff there is a nonzero morphism from  $\mathcal{O}_X(-1, 0)$  (resp.  $\mathcal{O}_X(0, -1)$ ) to  $E$ .

*Proof:* Suppose  $U := W \cap \widetilde{W}_I \neq 0$ . The restriction of the morphism

$$\mathcal{O}_X(-1, 0) \otimes \widetilde{W}_I \xrightarrow{X_1 p_0 i_I - X_0 p_1 i_I} \mathcal{O} \otimes V$$

(in the monad (2.2.4)) to  $\mathcal{O}_X(-1, 0) \otimes U$  is 0, since  $p_0|_W$  and  $p_1|_W$  are both 0. Hence this morphism factors through  $\ker(b)$ . It follows that

$$H^0(\ker(b)(1, 0)) \neq 0.$$

From the short exact sequence

$$0 \rightarrow \mathcal{O}_X(-1, -1) \otimes V \rightarrow \ker(b) \rightarrow E \rightarrow 0,$$

we can identify

$$H^0(\ker(b)(1, 0)) \simeq H^0(E(1, 0)),$$

since  $H^0(\mathcal{O}_X(0, -1)) = H^1(\mathcal{O}_X(0, -1)) = 0$ . This gives a nonzero morphism  $\mathcal{O}_X(-1, 0) \rightarrow E$  as desired.

Conversely, suppose there is a nonzero morphism  $\mathcal{O}_X(-1, 0) \rightarrow E$ . The short exact sequence

$$0 \rightarrow \mathcal{O}_X(-1, -1) \otimes V \rightarrow \ker(b) \rightarrow E \rightarrow 0$$

yields, via the associated long exact sequence for cohomology, a surjection

$$H^0(\ker(b)(1, 0)) \rightarrow H^0(E(1, 0)).$$

We obtain a nonzero morphism  $\mathcal{O}_X(-1, 0) \rightarrow \ker(b)$ . Composing with  $\ker(b) \hookrightarrow \mathcal{O}_X(-1, 0) \otimes \widetilde{W}_I \oplus \mathcal{O}_X(0, -1) \otimes \widetilde{W}_{II}$  we get an injection  $\mathcal{O}_X(-1, 0) \hookrightarrow \mathcal{O}_X(-1, 0) \otimes \widetilde{W}_I \oplus \mathcal{O}_X(0, -1) \otimes \widetilde{W}_{II}$ , or simply  $\mathcal{O}_X(-1, 0) \hookrightarrow \mathcal{O}_X(-1, 0) \otimes \widetilde{W}_I$ . That is, we obtain a one dimensional subspace  $U' \subset \widetilde{W}_I$ . Restricting this to the diagonal  $D$  we get

$$\mathcal{O}_{\mathbb{P}^1}(-1) \hookrightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \otimes \widetilde{W}.$$

The image of this is mapped to 0 by  $b|_D$  and hence lies in  $\mathcal{O}_{\mathbb{P}^1}(-1) \otimes W$ . Equivalently,  $U' \subset W$ . Hence  $U' \subset W \cap \widetilde{W}_I$ . The claim is proved.

We specialize to the case  $(\widetilde{W}_I, \widetilde{W}_{II}) \in \mathcal{M}(2)$ . Suppose  $W \cap \widetilde{W}_I \neq 0$ . By the claim above, there is a nonzero morphism  $\mathcal{O}_X(-1, 0) \rightarrow E$ . This is saturated because  $H^0(E) = 0$  and  $H^0(E(1, -1)) = 0$  by a computation similar to lemma 2.2.13. Hence there is a short exact sequence

$$0 \rightarrow \mathcal{O}_X(-1, 0) \rightarrow E \rightarrow \mathcal{O}_X(0, -1) \otimes \mathcal{I}_Z \rightarrow 0$$

for some zero dimensional subscheme  $Z$ . We have  $\text{length}(Z) = c_2(E(1, 0)) = 1$  by Griffiths and Harris [17, page 727]. Using the Hirzebruch-Riemann-Roch formula, we find that the Hilbert polynomial of  $\mathcal{O}_X(0, -1) \otimes \mathcal{I}_Z$  is less than half that of  $E$ . Hence  $E$  is not Gieseker semistable.

Conversely, suppose  $E$  is not Gieseker semistable. There is a rank 1 destabilizing sheaf  $L = \mathcal{I}_Z \otimes \mathcal{O}_X(a, b)$ , with  $a, b \in \mathbb{Z}$  and  $Z \subset X$  is a zero-dimensional subscheme, and a nonzero morphism  $L \hookrightarrow E$  which is saturated. By the Hirzebruch-Riemann-Roch formula,

$$\begin{aligned} \chi(L \otimes \mathcal{O}_X(k, k)) &= k^2 + k(a + b + 2) + (a + 1)(b + 1) - l(Z), \\ \chi(E(k, k)) &= 2k^2 + 2k - 1. \end{aligned}$$

Since  $\chi(L \otimes \mathcal{O}_X(k, k)) \geq \chi(E(k, k))$  for  $k \gg 0$  and  $L \hookrightarrow E$  is saturated, we have

$$a + b = -1,$$

or  $b = -1 - a$  and hence

$$k^2 + k - (a + 1)a - l(Z) \geq \frac{1}{2}(2k^2 + 2k - 1).$$

It follows that

$$(a + 1)a = l(Z) = 0.$$

That is,  $L = \mathcal{O}_X(-1, 0)$  or  $\mathcal{O}_X(0, -1)$ . By the claim above,  $W \cap \widetilde{W}_I \neq 0$  or  $W \cap \widetilde{W}_{II} \neq 0$ .  $\square$

*Proof of proposition 3.1.4.* For  $(\widetilde{W}_I, \widetilde{W}_{II}) \in \mathcal{M}^{ss}(2)$  (i.e.  $W \cap \widetilde{W}_I = W \cap \widetilde{W}_{II} = 0$ ), let  $E = E_{\widetilde{W}_I, \widetilde{W}_{II}}$ . We represent  $\widetilde{W}_I$  and  $\widetilde{W}_{II}$  by matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ a & c \\ b & d \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ e & g \\ f & h \end{pmatrix} \quad (3.1.7)$$

where the first two rows correspond to the two copies of  $V$  in  $\widetilde{W}$  and the last two rows correspond to  $W \subset \widetilde{W}$ . Since  $\widetilde{W}_I \cap \widetilde{W}_{II} = 0$ , we have

$$\begin{vmatrix} a - e & c - g \\ b - f & d - h \end{vmatrix} \neq 0.$$

The morphism  $b$  in monad (2.2.4) is given by

$$b = \left( X_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - X_0 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mid Y_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - Y_0 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right),$$

which is clearly surjective. Let  $A := \begin{pmatrix} a & c & e & g \\ b & d & f & h \end{pmatrix}$  and define  $A_{ij}$  to be the determinant of the 2 by 2 matrix which consists of the  $i$ th and  $j$ th columns of  $A$ . Let

$$M := \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ a & c & e & g \\ b & d & f & h \end{pmatrix}.$$

Then

$$\text{Adj}(M) = \begin{pmatrix} A_{34} + A_{23} & A_{24} & * & * \\ -A_{13} & -A_{14} + A_{34} & * & * \\ A_{12} - A_{14} & -A_{24} & * & * \\ A_{13} & A_{12} + A_{23} & * & * \end{pmatrix}.$$

Up to a scalar factor, the morphism  $a$  in (2.2.4) is given by

$$a = \left( \begin{array}{l} Y_0 \begin{pmatrix} A_{34} + A_{23} \\ -A_{13} \end{pmatrix} + Y_1 \begin{pmatrix} A_{24} \\ -A_{14} + A_{34} \end{pmatrix} \\ X_0 \begin{pmatrix} A_{12} - A_{14} \\ A_{13} \end{pmatrix} + X_1 \begin{pmatrix} -A_{24} \\ A_{12} + A_{23} \end{pmatrix} \end{array} \right),$$

The condition  $ba = 0$  is equivalent to

$$A_{12} = A_{34}, \tag{3.1.8}$$

which is the equation of a smooth quadric hypersurface  $Q \subset \mathbb{P}_{a,b,c,d,e,f,g,h}^7$

(=  $(\mathbb{A}_{a,b,c,d,e,f,g,h}^8 \setminus 0) / \mathbb{C}^*$ -action.)

Let  $\alpha = A_{34} + A_{23}$  and  $\beta = -A_{13}$  and  $\gamma = A_{24}$  and  $\delta = -A_{14} + A_{34}$ . Then

$$a = \left( \begin{array}{l} Y_0 \begin{pmatrix} \alpha \\ \beta \end{pmatrix} + Y_1 \begin{pmatrix} \gamma \\ \delta \end{pmatrix} \\ X_0 \begin{pmatrix} \delta \\ -\beta \end{pmatrix} + X_1 \begin{pmatrix} -\gamma \\ \alpha \end{pmatrix} \end{array} \right). \tag{3.1.9}$$

We have

$$\begin{aligned}
0 \neq \begin{vmatrix} a-e & c-g \\ b-f & d-h \end{vmatrix} &= ad - bc + eh - fg - ah - de + bg + cf \\
&= A_{12} + A_{34} - A_{14} + A_{23} = \alpha + \delta.
\end{aligned} \tag{3.1.10}$$

It follows that

$$\mathcal{M}^{ss}(2) \simeq Q \setminus Q', \tag{3.1.11}$$

where  $Q' = \{\alpha + \delta = 0\}$  is a quadric hypersurface in  $\mathbb{P}_{a,b,c,d,e,f,g,h}^7$ .

For part (ii), we note that the sheaves in  $\mathcal{M}^{ss}(2)$  are Gieseker stable by Huh [22, page 2100] and hence simple (see Huybrechts and Lehn [26, corollary 1.2.8].) Hence the forgetful map  $F : \mathcal{M}^{ss}(2) \rightarrow \mathcal{M}'(2) \setminus D'(2)$  is a  $PGL(2)$  torsor, which is trivial by the following

*Claim.* For all  $d$  and  $n$ ,

$$H_{et}^1(\mathbb{C}^d, PGL(n, \mathcal{O}_{\mathbb{C}^d})) = 0,$$

where  $PGL(n, \mathcal{O}_{\mathbb{C}^d})$  is the sheaf of groups on  $\mathbb{C}^d$  defined by

$$\begin{aligned}
PGL(n, \mathcal{O}_{\mathbb{C}^d})(U) &:= \text{Aut}_U(U \times \mathbb{P}^n) \\
&= \{\text{morphisms of } U \text{ into the algebraic group } PGL(n, \mathbb{C})\}
\end{aligned} \tag{3.1.12}$$

for any Zariski open subset  $U \subset \mathbb{C}^d$  (c.f. Beauville [5, page 28].)

Proof: Similarly to (3.1.12), we can define the sheaves  $GL(n, \mathcal{O}_{\mathbb{C}^d})$ ,  $SL(n, \mathcal{O}_{\mathbb{C}^d})$  and  $PSL(n, \mathcal{O}_{\mathbb{C}^d})$ . We have

$$H_{et}^1(\mathbb{C}^d, GL(n, \mathcal{O}_{\mathbb{C}^d})) = 0,$$

since vector bundles on  $\mathbb{C}^d$  are trivial by Quillen-Suslin theorem (see for example Artin [2, theorem 8.1].) The short exact sequence

$$1 \rightarrow \mathcal{O}_{\mathbb{C}^d}^\times \rightarrow GL(n, \mathcal{O}_{\mathbb{C}^d}) \rightarrow PGL(n, \mathcal{O}_{\mathbb{C}^d}) \rightarrow 1$$

yields the following exact sequence for etale cohomology

$$0 \rightarrow H_{et}^1(PGL(n, \mathcal{O}_{\mathbb{C}^d})) \rightarrow H_{et}^2(\mathcal{O}_{\mathbb{C}^d}^\times). \quad (3.1.13)$$

Let  $\underline{\mu}_n$  be the constant sheaf on  $\mathbb{C}^d$  with values in the group  $\mu_n$  of  $n$ th roots of unity.

The commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \underline{\mu}_n & \longrightarrow & SL(n, \mathcal{O}_{\mathbb{C}^d}) & \longrightarrow & PSL(n, \mathcal{O}_{\mathbb{C}^d}) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \parallel & & \\ 1 & \longrightarrow & \mathcal{O}_{\mathbb{C}^d} & \longrightarrow & GL(n, \mathcal{O}_{\mathbb{C}^d}) & \longrightarrow & PGL(n, \mathcal{O}_{\mathbb{C}^d}) & \longrightarrow & 1 \end{array}$$

yields the commutative diagram

$$\begin{array}{ccc} H_{et}^1(PSL(n, \mathcal{O}_{\mathbb{C}^d})) & \longrightarrow & H_{et}^2(\underline{\mu}_n) \\ \parallel & & \downarrow \\ H_{et}^1(PGL(n, \mathcal{O}_{\mathbb{C}^d})) & \longrightarrow & H_{et}^2(\mathcal{O}_{\mathbb{C}^d}^\times) \end{array}, \quad (3.1.14)$$

where the bottom map is the map (3.1.13.) We have

$$\underline{\mu}_n \simeq \mathbb{Z}/n\mathbb{Z}$$

as constructible sheaves in the etale topology on  $\mathbb{C}^d$ . By Artin [1, theorem 5.2] we have

$$H_{et}^2(\underline{\mu}_n) \simeq H_{an}^2(\underline{\mu}_n) = 0.$$

By the commutativity of (3.1.14), the map (3.1.13) is 0. Hence  $H_{et}^1(PGL(n, \mathcal{O}_{\mathbb{C}^d})) = 0$ , as claimed.

For part (iii), let  $P_{ij}$  (resp.  $P'_{ij}$ ) be Plucker coordinates on the first (resp. second) factor of  $G(2, \widetilde{W}) \times G(2, \widetilde{W}) \supset \mu_{2,2}^{-1}(0)$ . Modulo the  $\mathbb{C}^*$ -action

$$\begin{aligned} & ((P_{12} : P_{13} : P_{14} : P_{23} : P_{24} : P_{34}), (P'_{12} : P'_{13} : P'_{14} : P'_{23} : P'_{24} : P'_{34})) \mapsto \\ & ((t^2 P_{12} : t P_{13} : t P_{14} : t P_{23} : t P_{24} : P_{34}), (t^2 P'_{12} : t P'_{13} : t P'_{14} : t P'_{23} : t P'_{24} : P_{34})), \end{aligned}$$

$\mathcal{M}(2)$  is given by the following conditions

$$\begin{aligned} P_{12}P'_{34} &= P'_{12}P_{34}, \\ P_{12}P_{34} + P_{14}P_{23} - P_{13}P_{24} &= P'_{12}P'_{34} + P'_{14}P'_{23} - P'_{13}P'_{24} = 0, \\ P_{12}P'_{34} + P_{34}P'_{12} + P_{14}P'_{23} - P_{13}P'_{24} + P_{23}P'_{14} - P_{24}P'_{13} &\neq 0, \end{aligned}$$

where the first equality is from equation (3.1.8), the second and third equalities are the equations of the Grassmannians and the inequality is extended from condition (3.1.10) above.

The involution  $i_{\mathcal{M}}$  interchanges  $\widetilde{W}_I$  and  $\widetilde{W}_{II}$ . Clearly,  $\mathcal{M}^{ss}(2)$  is invariant under  $i_{\mathcal{M}}$  and so is  $\mathcal{M}(2) \setminus \mathcal{M}^{ss}(2)$ . By lemma 3.1.6,

$$\mathcal{M}(2) \setminus \mathcal{M}^{ss}(2) = \{P_{12}P'_{12} = 0\} \subset \mathcal{M}(2). \quad (3.1.15)$$

In the identification (3.1.11), the action of  $i_{\mathcal{M}}$  on  $\mathcal{M}^{ss}(2)$  is given by

$$(a : b : c : d : e : f : g : h) \mapsto (e : f : g : h : a : b : c : d) \quad (3.1.16)$$

The fixed locus of (3.1.16) is  $M_1 \sqcup M_{-1}$  where

$$M_1 = \{(a : b : c : d : a : b : c : d)\},$$

$$M_{-1} = \{(a : b : c : d : -a : -b : -c : -d)\}.$$

But  $M_1$  is disjoint from  $\mathcal{M}^{ss}(2)$ . Hence the fixed locus of  $i_{\mathcal{M}}$  in  $\mathcal{M}^{ss}(2)$  is

$$\left\{ (a : b : c : d : -a : -b : -c : -d) \left| \left| \begin{array}{cc} a & b \\ c & d \end{array} \right| \neq 0 \right. \right\} \simeq PGL(2).$$

Similarly, the fixed locus of  $i_{\mathcal{M}}$  in  $\mathcal{M}(2) \setminus \mathcal{M}^{ss}(2)$  satisfies that  $P_{ij} = -P'_{ij}$  and hence that

$$P_{12} = P'_{12} = 0 \text{ (by (3.1.15))},$$



$$P_{14}P_{23} - P_{13}P_{24} = 0,$$

$$P_{14}P_{23} - P_{13}P_{24} \neq 0,$$

and thus is empty. Hence the fixed locus of  $i_{\mathcal{M}}$  in  $\mathcal{M}(2)$  is  $PGL(2)$ .  $\square$

### 3.2. $D'_{\mathbb{P}^2}(2)$

Stromme [49, page 406] proves that  $D'_{\mathbb{P}^2}(n)$  is a divisor in  $\mathcal{M}'_{\mathbb{P}^2}(n)$ . Let  ${}^{lf}\mathcal{M}'_{\mathbb{P}^2}(2)$  be the locally free locus in  $\mathcal{M}'_{\mathbb{P}^2}(2)$ . Hulek [23, proposition 8.2] proves that

$${}^{lf}\mathcal{M}'_{\mathbb{P}^2}(2) \simeq Sym^2\mathbb{P}^2 \setminus \Delta_{\text{sym}} \quad (3.2.1)$$

where  $\Delta_{\text{sym}}$  is the diagonal. His method uses the notion of a *jumping line of the second kind*. This is a line, over the first order neighborhood of which, the sheaf has a nontrivial global section. He finds that the locus of these lines is a union of two lines in  $(\mathbb{P}^2)^*$ , which determine the sheaf uniquely. Huh [21, proposition I.4] extends this to an isomorphism

$$\mathcal{M}'_{\mathbb{P}^2}(2) \simeq Hilb^2\mathbb{P}^2. \quad (3.2.2)$$

**Proposition 3.2.3.**  $D'_{\mathbb{P}^2}(2)$  is the pullback of a  $(1, 1)$ -divisor under the Hilbert-Chow morphism  $Hilb^2\mathbb{P}^2 \rightarrow Sym^2\mathbb{P}^2 = \mathbb{P}^2 \times \mathbb{P}^2/S_2$ .

(Here a  $(1, 1)$ -divisor on  $Sym^2\mathbb{P}^2$  means a divisor which pulls back to a  $(1, 1)$ -divisor on  $\mathbb{P}^2 \times \mathbb{P}^2$  under the quotient  $\mathbb{P}^2 \times \mathbb{P}^2 \rightarrow \mathbb{P}^2 \times \mathbb{P}^2/S_2 = Sym^2\mathbb{P}^2$ .)

The proof uses

**Lemma 3.2.4.** (i) (Vitter [50, page 382]) Any rank 2  $\mu$ -stable bundle  $G$  on  $\mathbb{P}^2$  with  $c_1 = -H$  and  $c_2 = 2$  can be obtained from a short exact sequence

$$0 \rightarrow G \rightarrow U \otimes \mathcal{O}_{\mathbb{P}^2} \xrightarrow{\psi} j_* \mathcal{O}_L(2) \rightarrow 0, \quad (3.2.5)$$

where  $L$  is a unique line in  $\mathbb{P}^2$ ,  $U$  is a vector space of dimension 2 and  $j : L \hookrightarrow \mathbb{P}^2$  is the inclusion map.

(ii) The morphism  $\psi$  yields a map

$$f : \mathbb{P}(U) \simeq \mathbb{P}^1 \rightarrow L$$

which is a double covering which ramifies at two points  $q_1$  and  $q_2$ . Then a line  $L' \subset \mathbb{P}^2$  is a jumping line of the second kind of  $G$  (i.e.  $H^0(G|_{L'^2}) \neq 0$ ) iff it passes through either  $q_1$  or  $q_2$ .

(iii) The isomorphism (3.2.1) becomes

$$\begin{aligned} {}^{\text{lf}}\mathcal{M}'_{\mathbb{P}^2}(2) &\longrightarrow \text{Sym}^2 \mathbb{P}^2 \setminus \Delta_{\text{sym}} \\ G &\mapsto \{q_1, q_2\}. \end{aligned}$$

*Proof.* We give a detailed proof of part (i), following what is outlined in Vitter [50, page 382]. First we claim

$$h^0(G(1)) = 2. \quad (3.2.6)$$

Since  $G^\vee$  is stable,  $H^0(G^\vee(-4)) = 0$ . By Serre duality,

$$H^2(G(1)) = 0.$$

Therefore,  $h^0(G(1)) - h^1(G(1)) = \chi(G(1)) = 2$  by Hirzebruch-Riemann-Roch formula and hence

$$h^0(G(1)) \geq 2.$$

Let  $\sigma \in H^0(G(1))$  be a nonzero section. There is an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{\sigma} G \xrightarrow{\sigma^\wedge} I_{Z_\sigma} \longrightarrow 0, \quad (3.2.7)$$

where  $Z_\sigma$  is the zero locus of  $\sigma$ . Since  $H^0(G) = 0$  by stability,  $\sigma$  cannot be factored through  $\mathcal{O}_{\mathbb{P}^2}$ . It follows that  $Z_\sigma$  contains no divisor, and hence is of dimension 0. Therefore,

$$\text{length}(Z_\sigma) = c_2(G(1)) = 2.$$

Twisting (3.2.7) by  $\mathcal{O}_{\mathbb{P}^2}(1)$  and taking the long exact sequence of cohomology, we obtain

$$H^1(G(1)) \simeq H^1(I_{Z_\sigma}(1)).$$

The short exact sequence

$$0 \rightarrow I_{Z_\sigma}(1) \rightarrow \mathcal{O}_{\mathbb{P}^2}(1) \rightarrow \mathcal{O}_{Z_\sigma} \rightarrow 0$$

yields the following long exact sequence of cohomology

$$0 \rightarrow H^0(I_{Z_\sigma}(1)) \rightarrow H^0(\mathcal{O}_{\mathbb{P}^2}(1)) \rightarrow H^0(\mathcal{O}_{Z_\sigma}) \rightarrow H^1(I_{Z_\sigma}(1)) \rightarrow 0,$$

which implies that  $H^1(I_{Z_\sigma}(1)) = 0$  since  $h^0(I_{Z_\sigma}(1)) = 1$  and  $h^0(\mathcal{O}_{Z_\sigma}) = 2$ . Hence

$$H^1(G(1)) = 0.$$

For each nonzero section  $\sigma \in H^0(G(1))$  let  $L_\sigma$  be the line passing through  $Z_\sigma$ . Tensoring the short exact sequence

$$0 \rightarrow I_{Z_\sigma} \rightarrow \mathcal{O}_{\mathbb{P}^2} \rightarrow \mathcal{O}_{Z_\sigma} \rightarrow 0$$

with  $\mathcal{O}_{L_\sigma}$  we obtain

$$0 \rightarrow \text{Tor}_1(\mathcal{O}_{L_\sigma}, \mathcal{O}_{Z_\sigma}) \rightarrow I_{Z_\sigma}|_{L_\sigma} \rightarrow \mathcal{O}_{L_\sigma} \rightarrow \mathcal{O}_{Z_\sigma} \rightarrow 0.$$

This yields a natural surjection

$$I_{Z_\sigma} \rightarrow \mathcal{O}_{L_\sigma}(-2) = \ker(\mathcal{O}_{L_\sigma} \rightarrow \mathcal{O}_{Z_\sigma}). \quad (3.2.8)$$

Restricting (3.2.7) to  $L_\sigma$  we obtain a surjection

$$G|_{L_\sigma} \twoheadrightarrow I_{Z_\sigma}|_{L_\sigma},$$

which composes with (3.2.8) and yields the surjection

$$G|_{L_\sigma} \twoheadrightarrow \mathcal{O}_{L_\sigma}(-2),$$

of which the kernel is  $\mathcal{O}_{L_\sigma}(1)$ . Since  $\text{Ext}^1(\mathcal{O}_{L_\sigma}(-2), \mathcal{O}_{L_\sigma}(1)) = 0$ , we have  $G|_{L_\sigma} \simeq \mathcal{O}_{L_\sigma}(1) \oplus \mathcal{O}_{L_\sigma}(-2)$ . By Hulek [23, proposition 8.2]  $G$  has a unique *jumping line of the first kind*, i.e. the *unique* line  $L \subset \mathbb{P}^2$  such that

$$G|_L \simeq \mathcal{O}_L(1) \oplus \mathcal{O}_L(-2).$$

Hence

$$L_\sigma = L$$

for any  $\sigma \in H^0(G(1))$ . Let  $\sigma$  and  $s$  be a basis of  $H^0(G(1))$ . Then  $\sigma \wedge s \in H^0(\wedge^2(G(1))) \simeq H^0(\mathcal{O}_{\mathbb{P}^2}(1))$ . It follows that  $Z_{\sigma \wedge s}$  is a line  $L'$ . We obtain the following exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \xrightarrow{(\sigma \ s)} G(1) \longrightarrow j'_* \mathcal{O}_{L'}(a) \longrightarrow 0, \quad (3.2.9)$$

where  $j' : L' \hookrightarrow \mathbb{P}^2$  is the inclusion map and  $a$  is an integer. By the formula in Friedman [15, page 30],

$$2 = c_2(G(1)) = c_2(j'_* \mathcal{O}_{L'}(a)) = L'^2 - j'_* c_1(\mathcal{O}_{L'}(a)) = 1 - a.$$

Hence  $a = -1$ . Restricting (3.2.9) to  $L'$  we obtain a surjection

$$G(1)|_{L'} \twoheadrightarrow \mathcal{O}_{L'}(-1)$$

with kernel  $\mathcal{O}_{L'}(2)$ . It follows that  $L'$  is a jumping line of the first kind of  $G$  and thus equals to  $L$ .

From the exact sequence (3.2.9),  $\mathcal{O}_{\mathbb{P}^2}^{\oplus 2}$  is an *elementary modification* of  $G(1)$  along  $L$  (see Friedman [15, page 41].) Performing another elementary modification as in loc. cit. (i.e. taking dual of the morphism  $(\sigma \ s)$  in (3.2.9)) we obtain an exact sequence

$$0 \longrightarrow G \longrightarrow \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \longrightarrow j_* \mathcal{O}_L(2) \longrightarrow 0, \quad (3.2.10)$$

since  $G(1)|_L \simeq \mathcal{O}_L(2) \oplus \mathcal{O}_L(-1)$ . The uniqueness of the line  $L$  in (3.2.10) follows from the fact that performing another elementary modification in the obvious way to a sequence of the form (3.2.10) gets us back to one of the form (3.2.9) (see the remark in Friedman [15, page 41]), where the line ought to be the unique jumping line of the first kind, as proved above.

For part (ii), suppose  $L'$  passes through  $q_1$  and  $L' \neq L$ . Let  $(X_0 : X_1 : X_2)$  be a homogeneous coordinate system on  $\mathbb{P}^2$ . We can assume  $L = \{X_0 = 0\}$ ,  $L' = \{X_1 = 0\}$ ,  $q_1 = (0 : 0 : 1)$  and  $q_2 = (0 : 1 : 0)$ . Then

$$\begin{aligned}\psi &= (X_1^2 \ X_2^2), \\ f(0 : x_1 : x_2) &= (x_1^2 : x_2^2).\end{aligned}$$

Tensoring (3.2.5) with  $\mathcal{O}_{L^2}$  we obtain

$$0 \longrightarrow \text{Tor}_1(\mathcal{O}_L(2), \mathcal{O}_{L^2}) \longrightarrow G|_{L^2} \longrightarrow \mathcal{O}_{L^2}^{\oplus 2} \xrightarrow{\begin{pmatrix} 0 & X_2^2 \end{pmatrix}} \mathcal{O}_L(2)|_{L^2} \longrightarrow 0.$$

Let

$$K := \ker \left( \mathcal{O}_{L^2}^{\oplus 2} \xrightarrow{\begin{pmatrix} 0 & X_2^2 \end{pmatrix}} \mathcal{O}_L(2)|_{L^2} \right).$$

Then  $H^0(K) \neq 0$ . Since  $\text{Tor}_1(\mathcal{O}_L(2), \mathcal{O}_{L^2})$  is supported on a zero-dimensional subscheme, it follows that  $H^0(G|_{L^2}) \neq 0$  because it surjects onto  $H^0(K)$ . So  $L'$  is a jumping line of the second kind. The proof of part (ii) is finished since the lines  $L'$  such that  $L' \cap \{q_1, q_2\} \neq \emptyset$  are the only jumping lines of the second kind by Hulek [23, proposition 8.2].

Finally, part (iii) follows directly from the first two parts.  $\square$

*Proof of proposition 3.2.3.* Let  $G \in \mathcal{M}'_{\mathbb{P}^2}(2)$  be locally free. First we give a characterization for the jumping conics of  $G$ , i.e. the conics  $C$  for which  $G|_C \simeq \mathcal{O}_C \oplus \mathcal{O}_C(-2)$  (here  $\mathcal{O}_C(a) := \mathcal{O}_C(1)^{\otimes a}$ .)

*Claim.* (Vitter [50, page 382]) *A smooth conic  $C$  which intersects  $L$  at two distinct points  $p_1$  and  $p_2$  is a jumping conic iff  $\ker \psi_{p_1} = \ker \psi_{p_2}$  as subspaces of  $U$ . If  $C$  and  $L$  have a double intersection at  $p$ ,  $C$  is a jumping conic iff  $\ker \psi_p = \ker \psi'_p$  ( $\psi_p$  (resp.  $\psi_q$ ) is the evaluation at  $p$  (resp. at  $q$ ) of  $\psi$ .)*

Suppose  $C \cap L$  consists of two distinct points  $p_1$  and  $p_2$ . Restricting (3.2.5) to  $C$  we obtain

$$0 \longrightarrow G|_C \longrightarrow U \otimes \mathcal{O}_C \xrightarrow{\psi_{p_1} \oplus \psi_{p_2}} \mathbb{C}_{p_1} \oplus \mathbb{C}_{p_2} \longrightarrow 0,$$

If  $\ker\psi_{p_1} = \ker\psi_{p_2}$  then  $G|_C$  contains a copy of  $\mathcal{O}_C$  and hence is isomorphic to  $\mathcal{O}_C \oplus \mathcal{O}_C(-2)$ . If  $\ker\psi_{p_1} \neq \ker\psi_{p_2}$  then  $\psi_{p_1} \oplus \psi_{p_2}$  can be split into a direct sum of

$$\psi_{p_1}|_{\ker\psi_{p_2}} : \mathcal{O}_C \longrightarrow \mathbb{C}_{p_1}$$

and

$$\psi_{p_2}|_{\ker\psi_{p_1}} : \mathcal{O}_C \longrightarrow \mathbb{C}_{p_2}.$$

Each of these has kernel  $\mathcal{O}_C(-1)$ . It follows that  $G|_C \simeq \mathcal{O}_C(-1)^{\oplus 2}$ .

Now suppose  $C$  is tangent to  $L$  at  $p$ . Let  $x$  be a local coordinate of  $C$  near  $p$ .

Restricting (3.2.5) to  $C$  we obtain

$$0 \longrightarrow G|_C \longrightarrow U \otimes \mathcal{O}_C \xrightarrow{\Psi} \mathbb{C}[x]/(x^2) \longrightarrow 0,$$

or equivalently,

$$0 \longrightarrow G(1)|_C \longrightarrow U \otimes \mathcal{O}_C(2) \xrightarrow{\Psi} \mathbb{C}[x]/(x^2) \longrightarrow 0.$$

$\Psi$  is determined by the one-jet  $j_1(\psi)$  of the morphism  $\psi$  in (3.2.5) as follows. Choose a basis for  $U$ . Then a local section of  $\mathcal{O}_C(2)^{\oplus 2}$  is given by

$$h(x) = \begin{pmatrix} f(x) \\ g(x) \end{pmatrix}$$

for some degree 2 polynomials  $f(x)$  and  $g(x)$ , and the one-jet of  $\psi$  is given by

$$j_1(\psi) = \begin{pmatrix} a + bx \\ c + dx \end{pmatrix}$$

for some  $a, b, c, d \in \mathbb{C}$ . Then  $\Psi$  is given by

$$\Psi(h) = j_1((a + bx)f(x) + (c + dx)g(x)).$$

We prove that  $C$  is a jumping conic iff  $\ker\psi(0) = \ker\psi'(0)$  (iff  $ad = bc$ .) Changing the local frames so that  $a = 0$  and  $c = 1$ , this condition becomes  $b = 0$ . We obtain

$$\Psi(h) = g(0) + (bf(0) + g'(0) + dg(0))x.$$

$h$  is in the kernel  $G(1)|_C$  of  $\Psi$  iff

$$g(0) = 0,$$

$$bf(0) + g'(0) = 0.$$

Suppose  $b = 0$ , then a local section of  $G(1)|_C$  is given by

$$\begin{pmatrix} f(x) \\ x^2 g_2(x) \end{pmatrix}.$$

Hence  $G(1)|_C \simeq \mathcal{O}_C(2) \oplus \mathcal{O}_C$ . Suppose  $b \neq 0$ , then a local section of  $G(1)|_C$  is given by

$$\begin{pmatrix} -\frac{g'(0)}{b} + xf_1(x) \\ xg_1(x) \end{pmatrix},$$

which after a change of local frames becomes

$$\begin{pmatrix} xf_1(x) \\ xg_1(x) \end{pmatrix}.$$

Hence  $G(1)|_C \simeq \mathcal{O}_C(1)^{\oplus 2}$ . The claim is proved.



We choose coordinates so that  $C = V(x_0x_2 - x_1^2)$  and write the ramification points of the map  $f$  in lemma 3.2.4(ii) as  $q_i = (a_{i0} : a_{i1} : a_{i2})$  for  $i = 1, 2$ . Suppose  $C$  intersects  $L$  at two distinct points,  $p_1$  and  $p_2$ . By the claim above,  $C$  is a jumping conic of  $G$  iff  $f(p_1) = f(p_2)$ . A point  $sq_1 + tq_2$  lies on  $C$  iff

$$(sa_{10} + ta_{20})(sa_{12} + ta_{22}) = (sa_{11} + ta_{21})^2. \quad (3.2.11)$$

In suitable coordinates,  $f$  maps  $q_1$  (resp.  $q_2$ ) to 0 (resp.  $\infty$ ), i.e.

$$f(sq_1 + tq_2) = (s^2 : t^2).$$

Let  $p_i = s_{i1}q_1 + s_{i2}q_2$ . Then  $f(p_1) = f(p_2)$  implies that  $\{(s_{11} : s_{12}), (s_{21} : s_{22})\} = \{(s : t), (s : -t)\}$  for some  $s, t$ . Hence the coefficient of  $st$  in (3.2.11) vanishes, i.e.  $a_{12}a_{20} + a_{10}a_{22} - a_{11}a_{21} = 0$ .

Next suppose that  $C$  intersects  $L$  doubly at  $p$ . By the claim above,  $C$  is a jumping conic iff  $p$  is a ramification point of  $f$ . So either  $a_{10}a_{12} = a_{11}^2$  or  $a_{20}a_{22} = a_{21}^2$ . By the double intersection of  $C$  and  $L$  and equation (3.2.11), we have

$$a_{12}a_{20} + a_{10}a_{22} - a_{11}a_{21} = 0. \quad (3.2.12)$$

$$\begin{aligned} \text{Let } Z &:= \{\xi \in \text{Hilb}^2\mathbb{P}^2 \mid \xi \text{ is supported at a single point}\} \\ &= \{G \in \mathcal{M}'_{\mathbb{P}^2}(2) \mid G \text{ is not locally free}\}, \end{aligned}$$

where the last equality is by (3.2.1) and (3.2.2). Then  $Z$  is the inverse image of the diagonal in  $\text{Sym}^2\mathbb{P}^2$  under the Hilbert-Chow morphism  $\text{Hilb}^2\mathbb{P}^2 \rightarrow \text{Sym}^2\mathbb{P}^2$ . Hence  $Z$  is the exceptional divisor of this morphism. We show that  $Z$  is not entirely contained

in  $D'_{\mathbb{P}^2}(2)$ . Let  $p \in \mathbb{P}^2 \setminus C$ . Consider a rank 2 torsion free sheaf  $G$  on  $\mathbb{P}^2$  which is defined by the following exact sequence

$$0 \longrightarrow G \longrightarrow T_{\mathbb{P}^2}(-2) \longrightarrow \mathbb{C}(p) \longrightarrow 0.$$

Restricting to  $C$  we obtain

$$G|_C \simeq T_{\mathbb{P}^2}(-2)|_C \simeq \mathcal{O}_C(-1)^{\oplus 2},$$

where the last isomorphism follows from remark 3.1.5. Moreover,

$$c_2(G) = c_2(T_{\mathbb{P}^2}(-2)) - c_2(\mathbb{C}(p)) = 2.$$

It follows that  $Z \setminus D'_{\mathbb{P}^2}(2)$  contains  $G$  and hence is nonempty. That means  $D'_{\mathbb{P}^2}(2) \not\subset Z$ . So  $D'_{\mathbb{P}^2}(2)$  is the strict transform of the (1,1)-divisor on  $Sym^2\mathbb{P}^2$  which is given by equation (3.2.12). On the other hand, this (1,1)-divisor clearly does not contain the diagonal in  $Sym^2\mathbb{P}^2$ . Hence its pull back to  $Hilb^2\mathbb{P}^2$  is equal to its strict transform, which is  $D'_{\mathbb{P}^2}(2)$ .  $\square$

# CHAPTER 4

## POINCARÉ POLYNOMIAL

### 4.1. Quiver varieties and ALE spaces

We give an overview of quiver varieties and ALE spaces of type  $\hat{A}_{k-1}$ , following faithfully Nakajima [41, 43]. For an integer  $k \geq 2$ , let  $Q$  be the quiver whose underlying graph is of affine type  $\hat{A}_{k-1}$ ; i.e.  $Q$  is

$$\begin{array}{ccc} 0 & \longrightarrow & 1 \\ \circ & \longleftarrow & \circ \end{array}$$

for  $k = 2$  and is

$$\begin{array}{ccccccc} & & & & 0 & & \\ & & & & \circ & & \\ & \swarrow & & & \nwarrow & & \\ 1 & & 2 & & \dots & & k-2 & & k-1 \\ \circ & \longrightarrow & \circ & \longrightarrow & \dots & \longrightarrow & \circ & \longrightarrow & \circ \end{array}$$

for  $k \geq 3$ . Let  $I = \{0, \dots, k-1\}$  be the set of vertices of  $Q$  and  $E$  the set of edges. For each  $e \in E$ , let  $s(e)$  be the source and  $t(e)$  the target of  $e$ . Let  $V_i$  and  $W_i$  be fixed complex vector spaces for each  $i \in I$ . Let  $\underline{v}$  (resp.  $\underline{w}$ )  $\in \mathbb{Z}_{\geq 0}^I$  be the dimension vectors of  $\{V_i\}$  (resp.  $\{W_i\}$ ), i.e.  $v_i = \dim_{\mathbb{C}} V_i$  and  $w_i = \dim_{\mathbb{C}} W_i$  for each  $i \in I$ . We denote by  $\overline{Q}$  the *double* of  $Q$ , which is the quiver with the same set of vertices as  $Q$  and the set of edges  $\overline{E} \stackrel{\text{def}}{=} E \cup E^{op}$  where  $E^{op}$  is the set of edges of  $Q$  with *reversed* directions. Let

$$\begin{aligned} \text{End}(\underline{v}) &\stackrel{\text{def}}{=} \bigoplus_{e \in \overline{E}} \text{Hom}(V_{s(e)}, V_{t(e)}), \\ L(\underline{w}, \underline{v}) &\stackrel{\text{def}}{=} \bigoplus_{i \in I} \text{Hom}(W_i, V_i), \quad L(\underline{v}, \underline{w}) \stackrel{\text{def}}{=} \bigoplus_{i \in I} \text{Hom}(V_i, W_i). \end{aligned}$$

Let

$$M(\underline{v}, \underline{w}) \stackrel{def}{=} End(\underline{v}) \oplus L(\underline{w}, \underline{v}) \oplus L(\underline{v}, \underline{w}).$$

be the *representation space* for  $\overline{Q}$ . The group  $GL(\underline{v}) \stackrel{def}{=} \prod_{i \in I} GL(V_i)$  acts on  $M(\underline{v}, \underline{w})$  by changing the basis in each  $V_i$ , i.e.

$$g \cdot (B, a, b) \stackrel{def}{=} \left( \left( g_{t(e)} B_e g_{s(e)}^{-1} \right)_{e \in \overline{E}}, (g_i a_i)_{i \in I}, (b_i g_i^{-1})_{i \in I} \right) \quad (4.1.1)$$

for each  $g_i \in GL(V_i)$ ,  $B_e \in Hom(V_{s(e)}, V_{t(e)})$ ,  $a_i \in Hom(W_i, V_i)$  and  $b_i \in Hom(V_i, W_i)$ .

There is a symplectic form on  $M(\underline{v}, \underline{w})$  given by

$$tr((B, a, b), (B', a', b')) \stackrel{def}{=} \sum_{e \in \overline{E}} tr(\epsilon(e) B_e B_{\overline{e}}) + \sum_{i \in I} tr(a_i b'_i - a'_i b_i), \quad (4.1.2)$$

where  $\overline{e}$  is the reversed edge of  $e$  and  $\epsilon(e)$  is defined as 1 if  $e \in E$  and as  $-1$  otherwise.

Let  $End(\underline{v}) \stackrel{def}{=} \bigoplus_{i \in I} End(V_i)$  be the Lie algebra of  $GL(\underline{v})$ . With respect to the symplectic form (4.1.2), there is a moment map  $\mu_{\mathbb{C}} : M(\underline{v}, \underline{w}) \rightarrow End(\underline{v})$  for the action (4.1.1) which is given by

$$\mu_{\mathbb{C}}(B, a, b) \stackrel{def}{=} \epsilon B B + ab,$$

where

$$\epsilon B B \stackrel{def}{=} \left( \sum_{e \in \overline{E}, t(e)=i} \epsilon(e) B_e B_{\overline{e}} \right)_{i \in I} \quad \text{and} \quad ab \stackrel{def}{=} (a_i b_i)_{i \in I}.$$

Let  $\xi_{\mathbb{C}} = (\xi_{\mathbb{C}, i}) \in \mathbb{C}^I$ . The corresponding element in the center of the Lie algebra  $End(\underline{v})$  is

$$\bigoplus_{i \in I} \xi_{\mathbb{C}, i} id_{V_i},$$

which we also denote by  $\xi_{\mathbb{C}}$ . The group  $GL(\underline{v})$  acts on  $\mu_{\mathbb{C}}^{-1}(\xi_{\mathbb{C}})$ .

Let  $\xi_{\mathbb{R}} = (\xi_{\mathbb{R},i}) \in \mathbb{R}^I$ . There is a notion of  $\xi_{\mathbb{R}}$ -(semi)stability on  $M(\underline{v}, \underline{w})$  in Nakajima [43, page 702], which by King [29, proposition 3.1] turns out to be the same as  $\xi_{\mathbb{R}}$ -(semi)stability in geometric invariant theory.

**Definition 4.1.3.** The quotient

$$\mathcal{M}_{\xi}(\underline{v}, \underline{w}) \stackrel{def}{=} \mu_{\mathbb{C}}^{-1}(\xi_{\mathbb{C}}) //_{\xi_{\mathbb{R}}} GL(\underline{v})$$

is called the *Nakajima quiver variety* associated with  $\underline{v}$ ,  $\underline{w}$  and  $\xi$ , where the right hand side is a geometric invariant theory quotient in the sense of King [29, proposition 3.1].

**Remark 4.1.4.** Fixing a Hermitian metric on each  $V_i$ , one can consider the corresponding Hermitian matrices  $U(V_i)$ . There is a real moment map

$$\mu_{\mathbb{R}} : M(\underline{v}, \underline{w}) \longrightarrow \bigoplus_{i \in I} U(V_i) =: U(\underline{v})$$

by Nakajima [41, formulas 2.5], where we have used the notations of remark B.1.6. By Nakajima [41, page 371],

$$\mathcal{M}(\underline{v}, \underline{w}) = (\mu_{\mathbb{R}}^{-1}(\xi_{\mathbb{R}}) \cap \mu_{\mathbb{C}}^{-1}(\xi_{\mathbb{C}})) / U(\underline{v})$$

is a hyperkähler quotient in the sense of Hitchin et al. [20, section 3.D].

For a fixed dimension vector  $\underline{v}$ , let

$$\begin{aligned} R_+ &\stackrel{def}{=} \{\underline{\theta} = (\theta_i) \in \mathbb{Z}_{\geq 0}^I \mid \theta^T C \theta \leq 2\} \setminus \{0\}, \\ R_+(\underline{v}) &\stackrel{def}{=} \{\underline{\theta} \in R_+ \mid \theta_i \leq \dim_{\mathbb{C}} V_i \text{ for all } i\}, \\ D_{\underline{\theta}} &\stackrel{def}{=} \{\underline{x} = (x_i) \in \mathbb{R}^I \mid \underline{x} \cdot \underline{\theta} = 0\} \text{ for } \underline{\theta} \in R_+, \end{aligned}$$

where  $C$  is the  $\widehat{A}_{k-1}$ -Cartan matrix associated to  $Q$ , i.e. for  $k = 2$

$$C = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

and for  $k \geq 3$

$$C = \begin{pmatrix} 2 & -1 & 0 & \dots & -1 \\ -1 & 2 & -1 & \dots & 0 \\ 0 & -1 & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \dots & 2 \end{pmatrix}.$$

Since  $Q$  is of affine type,  $R_+$  is the set of positive roots and  $D_{\underline{\theta}}$  is the wall defined by the root  $\underline{\theta}$ . Although  $R_+$  is infinite,  $R_+(\underline{v})$  is finite.

An element  $\xi = (\xi_{\mathbb{R}}, \xi_{\mathbb{C}}) \in (\mathbb{R} \oplus \mathbb{C})^I$  is called *generic* if

$$\xi \notin \bigcup_{\underline{\theta} \in R_+(\underline{v})} (\mathbb{R} \oplus \mathbb{C}) \otimes D_{\underline{\theta}}.$$

For a fixed  $\xi_{\mathbb{C}}$ , a connected component of the generic locus

$$(\mathbb{R} \oplus \xi_{\mathbb{C}})^I \setminus \bigcup_{\underline{\theta} \in R_+(\underline{v})} (\mathbb{R} \oplus \xi_{\mathbb{C}}) \otimes D_{\underline{\theta}}$$

is called a *chamber*.

**Proposition 4.1.5.** (i) (Nakajima [41, theorem 2.8]) If  $\xi$  is generic then  $\mathcal{M}_{\xi}(\underline{v}, \underline{w})$  is a smooth connected complex algebraic variety.

(ii) (Nakajima [43, lemma 1.4]) If  $\xi$  and  $\xi'$  lie in the same chamber then  $\mathcal{M}_{\xi}(\underline{v}, \underline{w})$  is isomorphic to  $\mathcal{M}_{\xi'}(\underline{v}, \underline{w})$ .

(iii) (Nakajima [41, corollary 4.2]) If  $\xi$  and  $\xi'$  are both generic then  $\mathcal{M}_{\xi}(\underline{v}, \underline{w})$  is diffeomorphic to  $\mathcal{M}_{\xi'}(\underline{v}, \underline{w})$ .

We consider the case when  $\underline{w} = 0$ ,  $\underline{v} = \underline{\delta} := \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$  is the dimension vector of the irreducible representations of the cyclic group  $\Gamma = \mathbb{Z}/k\mathbb{Z} \subset SU(2)$  (which corresponds to the quiver  $Q$  via the McKay correspondence, see Nakajima [42, page 50]) and

$$\xi^0 = (\xi_{\mathbb{R}}^0, \xi_{\mathbb{C}}^0) \in (\mathbb{R} \oplus \mathbb{C}) \otimes D_{\underline{\delta}} \setminus \bigcup_{\underline{\theta} \in R_+ \setminus \underline{\delta}} (\mathbb{R} \oplus \mathbb{C}) \otimes D_{\underline{\theta}}. \quad (4.1.6)$$

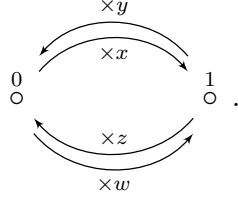
**Definition 4.1.7.** [43, page 704] The quotient

$$X_{\xi^0} \stackrel{def}{=} \mu_{\mathbb{C}}^{-1}(-\xi_{\mathbb{C}}) //_{(-\xi_{\mathbb{R}})} (GL(\underline{\delta})/\mathbb{C}^*)$$

is called the *ALE space* of type  $\widehat{A}_{k-1}$  with parameter  $\xi^0$ .

**Example 4.1.8.** When  $k = 2$ ,  $\underline{v} = \underline{\delta} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$  and  $\underline{w} = 0$  we have  $a = 0$ ,  $b = 0$  and  $B = (x, y, z, w) \in Hom(V_0, V_1) \oplus Hom(V_1, V_0) \oplus Hom(V_1, V_0) \oplus Hom(V_0, V_1) \simeq \mathbb{C}^4$ .

Here  $\overline{Q}$  is



The  $GL(\underline{\delta})/\mathbb{C}^*$  ( $\simeq \mathbb{C}^*$ )-action is given for  $\lambda \in \mathbb{C}^*$  by

$$\lambda.(x, y, z, w) = (\lambda x, \lambda^{-1}y, \lambda^{-1}z, \lambda w).$$

When  $\xi = (\xi_{\mathbb{R}}, \xi_{\mathbb{C}})$  satisfies (4.1.6) with  $\xi_{\mathbb{C}} \neq 0$  we have

$$X_{\xi} = \{xy - zw = -\xi_{\mathbb{C}}\}/\mathbb{C}_{\lambda}^*.$$

Letting  $a = xy$ ,  $b = zw$ ,  $c = xz$  and  $d = yw$  we have

$$X_{\xi} = \{ab = cd\} \setminus \{a - b = 0\} \subset \mathbb{P}_{a:b:c:d}^3,$$

which is a smooth quadric surface with a hyperplane section removed.

When  $\xi^0 = (\xi'_{\mathbb{R}}, 0)$  satisfies (4.1.6) (i.e.  $\xi'_{\mathbb{R},0} + \xi'_{\mathbb{R},1} = 0$  and  $\xi'_{\mathbb{R}} \neq 0$ ) we have

$$X_{\xi^0} = \begin{cases} (\{xy = zw\} \setminus \{x = w = 0\}) / \mathbb{C}_{\lambda}^* & \text{if } \xi'_{\mathbb{R},0} > 0 \\ (\{xy = zw\} \setminus \{y = z = 0\}) / \mathbb{C}_{\lambda}^* & \text{if } \xi'_{\mathbb{R},0} < 0 \end{cases},$$

according to the definition of  $(-\xi'_{\mathbb{R}})$ -stability in Nakajima [43, page 702]. Since the two cases are isomorphic, we can assume  $\xi'_{\mathbb{R},0} > 0$ . Then  $X_{\xi^0}$  is isomorphic to the Hirzebruch surface  $\mathbb{F}_2$  with the positive line  $C_{\infty}$  at infinity removed. In fact, since a point in the latter has the representation

$$((t_0 : t_1), x_0) \in \mathbb{P}^1 \times \mathbb{C}$$

where the action of  $\mathbb{C}_{\lambda}^* \times \mathbb{C}_{\mu}^*$  is given by

$$((t_0 : t_1), x_0) \mapsto ((\lambda t_0 : \lambda t_1), \lambda^{-2} x_0)$$

(see Reid [46, page 20]) we have the isomorphism

$$\begin{aligned} \mathbb{F}_2 \setminus C_{\infty} &\longrightarrow X_{\xi^0} \\ ((t_0 : t_1), x_0) &\mapsto (x, y, z, w) = (t_0, x_0 t_1, x_0 t_0, t_1). \end{aligned}$$

**Proposition 4.1.9.** *(Kronheimer [31, corollaries 2.10, 3.2 and 3.12]) For  $\xi^0$  which satisfies (4.1.6),  $X_{\xi^0}$  is a smooth complex surface (i.e. a 4-dimensional hyperkähler manifold); it is diffeomorphic to the minimal resolution of  $X_0 = \mathbb{C}^2 / \Gamma$  where  $\Gamma = \mathbb{Z} / k\mathbb{Z}$ , and the hyperkähler metric is ALE.*



Here the ALE condition means that there is a compact subset  $K \subset X_{\xi^0}$  and a diffeomorphism

$$X_{\xi^0} \setminus K \xrightarrow{\sim} \left( \mathbb{C}^2 \setminus \overline{B_r(0)} \right) / \Gamma, \quad (4.1.10)$$

under which the metric is approximated by the standard Euclidean metric on  $X_0 = \mathbb{C}^2 / \Gamma$ . As in Nakajima [43, page 709], we define

$$\overline{X}_{\xi^0} \stackrel{def}{=} X_{\xi^0} \cup l_{\infty},$$

where  $l_{\infty} = \mathbb{P}^1 / \Gamma$ . We endow the structure of a differential orbifold so that  $(X_{\xi^0} \setminus K) \cup l_{\infty}$  is identified with  $(\mathbb{P}^2 \setminus \overline{B_r(0)}) / \Gamma$  via the diffeomorphism (4.1.10). Here *orbifold* means that we remember the action of  $\Gamma$  on the tubular neighborhood  $\tilde{U} = \mathbb{P}^2 \setminus \overline{B_r(0)}$  of  $l_{\infty}$ . Nakajima [43, page 709] also gives the structure of a complex analytic orbifold on  $\overline{X}_{\xi^0}$ .

**Example 4.1.11.** We continue example 4.1.8 ( $k = 2$ .) Let  $\xi = (\xi_{\mathbb{R}}, \xi_{\mathbb{C}})$  and  $\xi^0 = (\xi'_{\mathbb{R}}, 0)$  satisfy (4.1.6) with  $\xi_{\mathbb{C}} \neq 0$ . Then  $\overline{X}_{\xi}$  (resp.  $\overline{X}_{\xi^0}$ ) is a smooth quadric surface  $\mathbb{P}^1 \times \mathbb{P}^1$  (resp. a Hirzebruch surface  $\mathbb{F}_2$ ), where we remember the action of  $\Gamma = \mathbb{Z}/2\mathbb{Z}$  on a tubular neighborhood of  $l_{\infty}$ , which in this case is a hyperplane section which can be taken to be the diagonal  $D$  (resp. the positive section at infinity.) We view the orbifold  $\overline{X}_{\xi}$  as a Deligne-Mumford stack

$$\mathcal{X}_{\xi} \xrightarrow{q} \mathbb{P}^1 \times \mathbb{P}^1,$$

which is an isomorphism outside a tubular neighborhood  $N$  of the diagonal  $D$  and over  $N$  is the stack  $[\tilde{N}/(\mathbb{Z}/2\mathbb{Z})]$  where  $\tilde{N} \xrightarrow{p} N$  is a double covering of  $N$  which is ramified along  $\tilde{D} := p^{-1}D$ . Then

$$q^*D = 2\tilde{D}.$$

Similarly, we view the orbifold  $\overline{X}_{\xi^0}$  as a Deligne-Mumford stack

$$\mathcal{X}_{\xi^0} \xrightarrow{q_0} \mathbb{F}_2$$

with coarse space  $\mathbb{F}_2$  as follows (cf. Bruzzo et al. [11, section 4.2].) Let  $C$ ,  $C_\infty$  and  $F$  be the exceptional divisor, the line at infinity and the fiber in  $\mathbb{F}_2$  respectively. The Picard group of  $\mathbb{F}_2$  is generated by  $C_\infty$  and  $F$  and

$$C = C_\infty - 2F.$$

We define  $q_0$  to be an isomorphism outside a tubular neighborhood  $N_0$  of  $C_\infty$  and over  $N_0$  the stack  $[\widetilde{N}_0/(\mathbb{Z}/2\mathbb{Z})]$  where  $\widetilde{N}_0 \xrightarrow{p_0} N_0$  is a double covering of  $N_0$  which is ramified along  $\widetilde{C}_\infty := p_0^{-1}C_\infty$ . Then

$$q_0^*C_\infty = 2\widetilde{C}_\infty.$$

**Definition 4.1.12.** (Nakajima [43, page 711]) For  $\xi^0$  which satisfies (4.1.6), a *framing* on a torsion free sheaf  $E$  on  $\overline{X}_{\xi^0}$  is an isomorphism

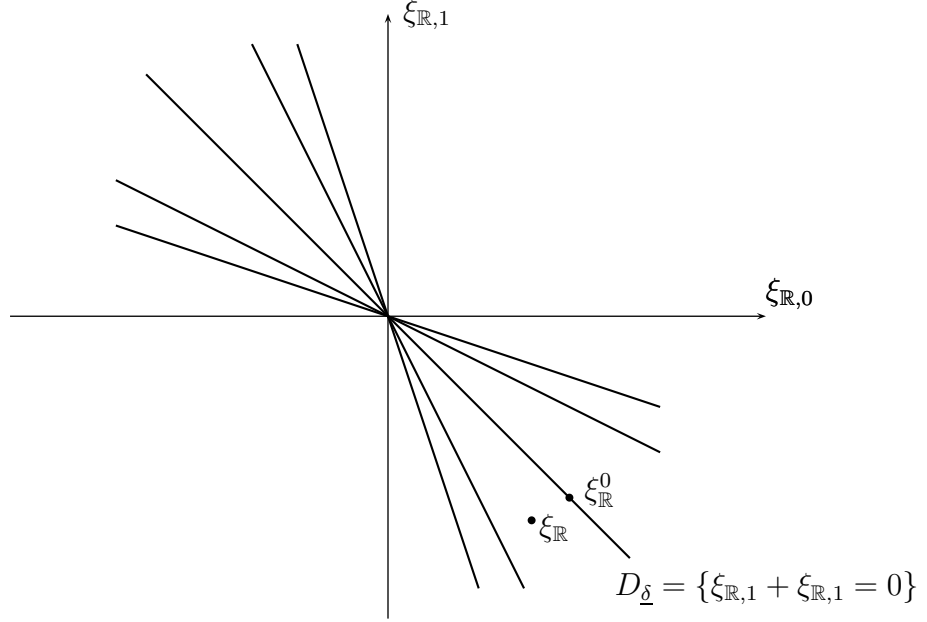
$$\phi : E|_{l_\infty} \xrightarrow{\sim} \mathcal{O}_{\mathbb{P}^1} \otimes \rho$$

where  $\rho$  is a representation of  $\Gamma = \mathbb{Z}/k\mathbb{Z}$ .

**Remark 4.1.13.** When  $E$  is a Hermitian vector bundle, its associated Hermitian connection  $A$  approximates a flat unitary connection  $A_0$  on a neighborhood of  $l_\infty$  ( $A_0$  is the canonical connection on  $X_0 = \mathbb{C}^2/\Gamma$  by Kronheimer and Nakajima [32, proposition 2.2(i)].) The connection  $A_0$  is determined by its holonomy, and hence corresponds to a unitary representation of  $\Gamma = \mathbb{Z}/k\mathbb{Z}$ .

Let  $R_i$  be the irreducible representations of  $\Gamma$  where  $R_0$  is trivial,  $\mathcal{R}_0$  the trivial line bundle on  $\overline{X}_{\xi^0}$  and  $\mathcal{R}_i$  the so-called *tautological* line bundle on  $\overline{X}_{\xi^0}$  (see Nakajima [43, page 705]) which restricts to  $\mathcal{O}_{\mathbb{P}^1} \otimes R_i$  over  $l_\infty$  for each  $i \neq 0$ .

We take a parameter  $\xi_{\mathbb{R}}$  from the chamber containing  $\xi_{\mathbb{R}}^0$  in its closure with  $\xi_{\mathbb{R}} \cdot \underline{\delta} < 0$ . This chamber is uniquely determined.



**Figure 4.1.** Chambers (when  $k = 2$  and  $\xi_{\mathbb{C}}^0 = 0$ )

**Proposition 4.1.14.** (Nakajima [43, page 709])  $\mathcal{M}_{(\xi_{\mathbb{R}}, \xi_{\mathbb{C}}^0)}(\underline{v}, \underline{w})$  is a fine moduli space parametrizing framed torsion free sheaves  $E$  on  $\overline{X}_{\xi^0}$ , where  $\underline{w}$  corresponds to a framing

$$E|_{l_\infty} \xrightarrow{\sim} \mathcal{O}_{\mathbb{P}^1} \otimes \bigoplus_i R_i^{\oplus w_i}$$

and  $\underline{v}$  is given by the Chern classes of  $E$  via the formulas

$$\begin{aligned} c_1(E) &= \sum_{i \neq 0} u_i c_1(\mathcal{R}_i) \quad \text{where } \underline{u} = \underline{w} - C\underline{v}, \\ ch_2(E) &= \sum_i u_i ch_2(\mathcal{R}_i) - 2\underline{v} \cdot \underline{\delta} ch_2(\mathcal{O}(l_\infty)). \end{aligned} \tag{4.1.15}$$

We include here a proof of formulas (4.1.15) since the corresponding formulas in Nakajima [43, formulas 1.9] have a typo.

*Proof of formulas (4.1.15).* Let  $i^*$  be the index such that  $R_{i^*}$  is the dual representation of  $R_i$  and

$$\mathcal{R} \stackrel{def}{=} \bigoplus_i \mathcal{R}_{i^*} \otimes R_i$$

which corresponds to the regular representation  $R \stackrel{def}{=} \bigoplus_i R_{i^*} \otimes R_i$  of  $\Gamma$ . The sheaf  $E$  is the cohomology of the monad

$$0 \longrightarrow (V \otimes \mathcal{R})^\Gamma(-l_\infty) \longrightarrow (V \otimes Q \otimes \mathcal{R})^\Gamma \oplus (W \otimes \mathcal{R})^\Gamma \longrightarrow (V \otimes \mathcal{R})^\Gamma(l_\infty) \longrightarrow 0 \quad (4.1.16)$$

on  $\overline{X}_{\xi^0}$ , according to Nakajima and Kronheimer [32, equation 4.3] and Nakajima [43, equation 3.1]. Let

$$V = \bigoplus_i V_i \otimes R_i$$

be the decomposition of  $V$  into  $\Gamma$ -equivariant modules. Then

$$\underline{\delta} = (\dim_{\mathbb{C}} R_0, \dots, \dim_{\mathbb{C}} R_{k-1}).$$

The monad (4.1.16) can be written as

$$\begin{aligned} 0 \longrightarrow \left( \bigoplus_i V_i \otimes \mathcal{R}_i \right) (-l_\infty) &\longrightarrow \left( \bigoplus_{i,j} a_{ij} V_i \otimes \mathcal{R}_j \right) \oplus \left( \bigoplus_i W_i \otimes \mathcal{R}_i \right) \\ &\longrightarrow \left( \bigoplus_i V_i \otimes \mathcal{R}_i \right) (l_\infty) \longrightarrow 0, \end{aligned}$$

where  $(Q \otimes R)_i = \bigoplus_j a_{ij} R_j$ . By the McKay correspondence,

$$a_{ij} = 2\delta_{ij} - C_{ij},$$

where  $C = (C_{ij})$  is the Cartan matrix of  $Q$ . It follows that

$$\begin{aligned} ch(E) &= \sum_{i,j} (2\delta_{ij} - C_{ij}) v_i ch(\mathcal{R}_j) + \sum_i w_i ch(\mathcal{R}_i) \\ &\quad - \sum_i v_i \left( ch(\mathcal{R}_i) + (dim_{\mathbb{C}} R_i) ch\left(\mathcal{O}_{\overline{X}_{\xi^0}}(-l_{\infty})\right) \right) - \sum_i v_i \left( ch(\mathcal{R}_i) + (dim_{\mathbb{C}} R_i) ch\left(\mathcal{O}_{\overline{X}_{\xi^0}}(l_{\infty})\right) \right) \\ &= - \sum_{i,j} C_{ij} v_i ch(\mathcal{R}_j) + \sum_i w_i ch(\mathcal{R}_i) - 2 \sum_i v_i \delta_i ch\left(\mathcal{O}_{\overline{X}_{\xi^0}}(l_{\infty})\right). \end{aligned}$$

This explains formula (4.1.15).  $\square$

**Remark 4.1.17.** The above proof is valid for any finite subgroup  $\Gamma \subset SU(2)$ .

## 4.2. The Poincaré polynomial of $\mathcal{M}(r, n)$

We consider the case  $k = 2$ . Then the group  $\Gamma = \mathbb{Z}/2\mathbb{Z}$  has two irreducible representations,  $R_0$  and  $R_1$  ( $R_0$  is trivial.) Let  $\xi = (\xi_{\mathbb{R}}, \xi_{\mathbb{C}})$  and  $\xi^0 = (\xi'_{\mathbb{R}}, 0)$  satisfy (4.1.6), as in examples 4.1.8 and 4.1.11. We compute the Poincaré polynomial of  $\mathcal{M}(r, n)$ , using section 4.1. Let

$$\mathcal{M}_{\mathcal{X}_{\xi}}(r, n) \stackrel{def}{=} \mathcal{M}_{\xi} \left( \binom{n-\frac{r}{2}}{n}, \binom{0}{r} \right).$$

**Lemma 4.2.1.**  $\mathcal{M}(r, n)$  is isomorphic to  $\mathcal{M}_{\mathcal{X}_{\xi}}(r, n)$ .

*Proof.* By proposition 4.1.14,  $\mathcal{M}_{\mathcal{X}_{\xi}}(r, n)$  parametrizes framed torsion free sheaves on  $\mathcal{X}_{\xi}$  (see example 4.1.11) with certain Chern classes and the framing along  $\tilde{D}$  being given by  $R_1^{\oplus r}$ . Since

$$\mathcal{O}_{\mathcal{X}_{\xi}}(\tilde{D})|_{\tilde{D}} = \mathcal{O}_{\tilde{D}}(1) \otimes R_1,$$

we define

$$\begin{aligned} f : \mathcal{M}(r, n) &\longrightarrow \mathcal{M}_{\mathcal{X}_{\xi}}(r, n) \\ (E, \phi) &\mapsto (q^*E \otimes \mathcal{O}_{\mathcal{X}_{\xi}}(\tilde{D}), q^*\phi). \end{aligned}$$

as a map of sets, where  $q$  is the projection  $\mathcal{X}_{\xi} \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$  defined in example 4.1.11.

*Claim.*  $f$  is well-defined.

In fact,

$$f(E)|_{\tilde{D}} \simeq \mathcal{O}_{\tilde{D}} \otimes R_1^{\oplus r}.$$

Let

$$\underline{u} = \begin{pmatrix} 0 \\ r \end{pmatrix} - \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \underline{v}.$$

Solving the following equations (formulas (4.1.15))

$$0 = c_1(f(E)) = u_1 c_1(\mathcal{R}_1),$$

$$-n + \frac{r}{4} = ch_2(f(E)) = u_1 ch_2(\mathcal{R}_1) - 2\underline{v} \cdot \underline{\delta} ch_2(\mathcal{O}(\tilde{D})),$$

we obtain

$$\underline{v} = \begin{pmatrix} n - \frac{r}{2} \\ n \end{pmatrix}.$$

The claim is proved. Similarly, we have the following map of sets

$$\begin{aligned} g : \mathcal{M}_{\mathcal{X}_\xi}(r, n) &\longrightarrow \mathcal{M}(r, n) \\ (F, \phi) &\longmapsto \left( q_* \left( F \otimes \mathcal{O}_{\mathcal{X}_\xi}(-\tilde{D}) \right), p_* \phi \right), \end{aligned}$$

which is well-defined ( $\tilde{N} \xrightarrow{p} N$  was defined in example 4.1.11.) Since  $q_* \mathcal{O}_{\mathcal{X}_\xi} = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}$ , we have  $q_* q^* E = E$  for all  $E \in \mathcal{M}(r, n)$ . Hence

$$g \circ f = id_{\mathcal{M}(r, n)}.$$

Since  $F|_{\tilde{D}} \simeq \mathcal{O}_{\tilde{D}} \otimes R_0$  for all  $F \in \mathcal{M}_{\mathcal{X}_\xi}(r, n)$ , we have  $q^* q_* \left( F \otimes \mathcal{O}_{\mathcal{X}_\xi}(-\tilde{D}) \right) = F \otimes \mathcal{O}_{\mathcal{X}_\xi}(-\tilde{D})$ . Hence

$$f \circ g = id_{\mathcal{M}_{\mathcal{X}_\xi}(r, n)}.$$

Thus  $f$  is bijective. By proposition 4.1.14, there exists a universal family  $\mathcal{F}$  over  $\mathcal{M}_{\mathcal{X}_\xi}(r, n) \times \mathcal{X}_\xi$ , which parametrizes framed torsion free sheaves on  $\mathcal{X}_\xi$ . We define over  $\mathcal{M}(r, n) \times \mathcal{X}_\xi$  the following sheaf

$$\mathcal{F}' := (id_{\mathcal{M}(r, n)} \times q)^* \mathcal{E} \otimes \left( \mathcal{O}_{\mathcal{M}(r, n)} \boxtimes \mathcal{O}_{\mathcal{X}_\xi}(\tilde{D}) \right),$$

where  $\mathcal{E}$  is the universal family of framed torsion free sheaves on  $\mathbb{P}^1 \times \mathbb{P}^1$  from the proof of theorem 2.2.6. Then  $\mathcal{F}'$  is flat over  $\mathcal{M}(r, n)$ , since  $\mathcal{E}$  is. We have

$$\mathcal{F}' \otimes \mathbb{C}(u) \simeq q^* (\mathcal{E} \otimes \mathbb{C}(u)) \otimes \mathcal{O}_{\mathcal{X}_\xi}(\tilde{D}) = \mathcal{F} \otimes \mathbb{C}(f(u)) \quad (4.2.2)$$

for all  $u \in \mathcal{M}(r, n)$ , where the isomorphism is due to the flatness of  $\mathcal{E}$  and the equality the definition of  $f$ . The universal family  $\mathcal{F}$  pulls back to the family  $\mathcal{F}'$  via a unique morphism  $f' : \mathcal{M}(r, n) \rightarrow \mathcal{M}_{\mathcal{X}_\xi}(r, n)$ , which is equal to  $f$  by (4.2.2) above. By proposition 4.1.5(i) and the choice of  $\xi$ ,  $\mathcal{M}_{\mathcal{X}_\xi}(r, n) = \mathcal{M}_\xi\left(\binom{n-\frac{r}{2}}{n}, \binom{0}{r}\right)$  is smooth. Hence  $f$  is a bijective morphism between smooth algebraic varieties. By Zariski's main theorem (see for example Mumford [39, page 209]),  $f$  is an isomorphism. □

Since the Picard group of  $\mathcal{X}_{\xi^0}$  is generated by  $\tilde{C}_\infty$  and  $F$  (see example 4.1.11), hereafter we make the formal identification

$$\frac{1}{2}C \stackrel{def}{=} \tilde{C}_\infty - F,$$

which then gives the identification

$$\mathbb{Z} \cdot \frac{1}{2}C \oplus \mathbb{Z} \cdot F = \mathbb{Z} \cdot \tilde{C}_\infty \oplus \mathbb{Z} \cdot F$$

for the Picard group of  $\mathcal{X}_{\xi^0}$ .

Let  $\widetilde{\mathfrak{M}}^2(r, k, m)$  be the fine moduli space of rank  $r$  framed torsion free sheaves on  $\mathcal{X}_{\xi^0}$  with  $c_1 = kC$  ( $k \in \mathbb{Z} \cdot \frac{1}{2}$ ), discriminant  $m$  and the framing being trivial along  $\widetilde{C}_\infty$ , considered by Bruzzo et al. [11, section 3.1] and Bruzzo et al. [12, theorem 1.1]. There they apply the torus fixed-point method of Nakajima and Yoshioka [44, section 3] to obtain

**Proposition 4.2.3.** *(Bruzzo et al. [11, theorem 4.4]) The Poincaré polynomial of  $\widetilde{\mathfrak{M}}^2(r, k, m)$  is*

$$P_t(\widetilde{\mathfrak{M}}^2(r, k, m)) = \sum \prod_{\alpha=1}^r t^{2(|Y_\alpha| - l(Y_\alpha))} \prod_{i=1}^{\infty} \frac{t^{2(m_i^{(\alpha)} + 1)} - 1}{t^2 - 1} \prod_{\alpha < \beta} t^{2(l'_{\alpha, \beta} + |Y_\alpha| + |Y_\beta| - n'_{\alpha, \beta})}$$

where the summation runs over  $r$ -tuple of pairs  $((k_1, Y_1), \dots, (k_r, Y_r))$  with  $m = \sum_{\alpha=1}^r |Y_\alpha| + \frac{1}{r} \sum_{\alpha < \beta} (k_\alpha - k_\beta)^2$  and  $\sum_{\alpha=1}^r k_\alpha = k$  where the  $Y_i$  are Young diagrams (for a Young diagram  $Y$ ,  $|Y|$  and  $l(Y)$  denote respectively the number of boxes and the number of columns),

$$l'_{\alpha, \beta} = \begin{cases} [n_{\alpha, \beta}]^2 + 2[n_{\alpha, \beta}]\{n_{\alpha, \beta}\} & \text{if } n_{\alpha, \beta} \geq 0 \\ [n_{\alpha, \beta}]^2 + 2[n_{\alpha, \beta}]\{n_{\alpha, \beta}\} - \delta_{2\{n_{\alpha, \beta}\}, 0} & \text{otherwise} \end{cases},$$

where  $n_{\alpha, \beta} = k_\alpha - k_\beta$ ,  $[ ]$  (resp.  $\{ \}$ ) denotes the integral part (resp. fractional part) of a number,  $\delta_{ij}$  is the Kronecker delta,

$$n'_{\alpha, \beta} = \begin{cases} \# \text{ of columns of } Y_\alpha \text{ that are longer than } k_\alpha - k_\beta & \text{if } k_\alpha - k_\beta \geq 0 \\ \# \text{ of columns of } Y_\beta \text{ that are longer than } k_\beta - k_\alpha - 1 & \text{otherwise} \end{cases},$$

and  $m_i^{(\alpha)}$  is the number of columns in  $Y_\alpha$  that have length  $i$ .

Let

$$\mathcal{M}_{\mathbb{F}_2}(r, n) \stackrel{\text{def}}{=} \widetilde{\mathfrak{M}}^2\left(r, \frac{r}{2}, n - \frac{r}{4}\right). \quad (4.2.4)$$

**Proposition 4.2.5.** *The Poincaré polynomial of  $\mathcal{M}(r, n)$  is  $P_t(\mathcal{M}_{\mathbb{F}_2}(r, n))$ .*



*Proof.* Since  $\mathcal{M}(r, n)$  is diffeomorphic to  $\mathcal{M}_{\xi^0} \left( \binom{n-\frac{r}{2}}{n}, \binom{0}{r} \right)$  by proposition 4.1.5(iii) and lemma 4.2.1, it is enough to prove that  $\mathcal{M}_{\xi^0} \left( \binom{n-\frac{r}{2}}{n}, \binom{0}{r} \right)$  is isomorphic to  $\widetilde{\mathfrak{M}}^2 \left( r, \frac{r}{2}, n - \frac{r}{4} \right)$ . A framed sheaf  $(F, \phi)$  in  $\mathcal{M}_{\xi^0} \left( \binom{n-\frac{r}{2}}{n}, \binom{0}{r} \right)$  is one on the stack  $\mathcal{X}_{\xi^0} \xrightarrow{q} \mathbb{F}_2$  with the framing

$$\phi : F|_{\tilde{C}_\infty} \xrightarrow{\sim} \mathcal{O}_{\mathbb{P}^1} \otimes R_1^{\oplus r}.$$

Equations (4.1.15) give  $c_1(F) = 0$  and  $ch_2(F) = -n + \frac{r}{4}$ . Hence the discriminant of  $F$  is

$$-ch_2(F) + \frac{1}{2r}c_1(F)^2 = n - \frac{r}{4}.$$

Since  $\mathcal{O}_{\mathcal{X}_{\xi^0}} \left( \frac{1}{2}C \right) |_{\tilde{C}_\infty} \simeq \mathcal{O}_{\tilde{C}_\infty} \otimes R_1$ , we have the following bijective map of sets

$$\mathcal{M}_{\xi^0} \left( \binom{n-\frac{r}{2}}{n}, \binom{0}{r} \right) \ni (F, \phi) \mapsto \left( F \otimes \mathcal{O}_{\mathcal{X}_{\xi^0}} \left( \frac{1}{2}C \right), \phi \right) \in \widetilde{\mathfrak{M}}^2 \left( r, \frac{r}{2}, n - \frac{r}{4} \right),$$

which is an isomorphism by an argument similar to the proof of lemma 4.2.1.  $\square$

**Example 4.2.6.**

$$P_t(\mathcal{M}(2, 2)) = P_t \left( \widetilde{\mathfrak{M}}^2 \left( 2, 1, \frac{3}{2} \right) \right) = 2t^6 + 3t^4 + 2t^2 + 1.$$

## CHAPTER 5

### A TWISTOR FAMILY

We describe a one-parameter deformation of  $T^*G(n, \widetilde{W})$ , following Markman [34, section 2.1]. There is a unique nontrivial extension

$$0 \longrightarrow T^*G(n, \widetilde{W}) \longrightarrow E \xrightarrow{j} \mathcal{O}_{G(n, \widetilde{W})} \longrightarrow 0 \quad (5.1)$$

over  $G(n, \widetilde{W})$ . The morphism  $j$  can be understood as a map from the total space of  $E$  to  $\mathbb{C}$ . Let  $E_t$  be the fiber over  $t$  of  $j$ . We have

$$E_0 = T^*G(n, \widetilde{W}).$$

Let  $\underline{\widetilde{W}}$  be the trivial bundle on  $G(n, \widetilde{W})$  with fiber  $\widetilde{W}$ . Recall our notation  $\zeta$  for the tautological bundle on  $G(n, \widetilde{W})$  in the proof of proposition 2.2.48. By Markman [34, section 2.1],

$$E = \left\{ (t, (U, \theta)) \in \mathbb{C}_t \times \mathcal{H}om(\underline{\widetilde{W}}, \zeta) \mid U \in GL(n, \widetilde{W}), \theta \in \mathcal{H}om(\widetilde{W}, U) \text{ and } \theta|_U = t \cdot id_U \right\}. \quad (5.2)$$

The fiber of  $E_t$  over  $U \in G(n, \widetilde{W})$  is

$$\left\{ \theta \in \mathcal{H}om(\widetilde{W}, U) \mid \theta|_U = t \cdot id_U \right\},$$

which for  $t \neq 0$  is isomorphic to

$$\left\{ U' \in G(n, \widetilde{W}) \mid U' \cap U = 0 \right\}$$

by taking  $U' = \ker(\theta)$ . Hence for  $t \neq 0$  we have

$$E_t \simeq L_n \tag{5.3}$$

( $L_n$  is defined in section 2.2.) In fact,

$$E \times_{\mathbb{C}^*} \mathbb{C}^* \simeq L_n \times \mathbb{C}^*. \tag{5.4}$$

The following is our key observation for this section (see appendix B.1 for the definition of a hyperkähler manifold.)

**Proposition 5.5.** *For each  $t \in \mathbb{C}$  the fiber  $E_t$  can be given the structure of a hyperkähler manifold.*

*Proof.* Each general fiber  $E_t \simeq L_n$  is a coadjoint orbit of  $GL(\widetilde{W})$  (see remark 2.2.39) and hence admits a hyperkähler structure by Biquard [6, théorème 1] and Kovalev [30, theorem 1.1]; while the special fiber  $E_0 \simeq T^*G(n, \widetilde{W})$  carries a hyperkähler structure by Burns [13, section 3]. □

Using description (5.2) we define

$$\begin{aligned} \mu : E &\longrightarrow \text{End}(V) \\ (t, U, \theta) &\mapsto p_0 i_U \theta i_0 + p_1 i_U \theta i_1 - t \cdot \text{id}_V, \end{aligned}$$

where  $i_U : U \hookrightarrow \widetilde{W}$  is the inclusion map and  $i_i$  and  $p_i$  are defined as in equation (2.2.3). Let

$$\mu_t := \mu|_{E_t}.$$

In the identification (5.3), for  $t \neq 0$  we have

$$\mu_t = t\mu_{r,n}$$

( $\mu_{r,n}$  is defined in section 2.2), which is a moment map for the action of  $GL(V)$  on  $E_t \simeq L_n$ .

In this chapter, we give a direction for our future study. By (5.11) below,

$$\mathcal{M}(r, n) = \mu_{r,n}^{-1}(0) //_{\chi} GL(V)$$

where  $\chi$  is the character of  $GL(V)$  given by (5.8) and the notation  $\mu_{r,n}^{-1}(0) //_{\chi} GL(V)$  is defined in definition 5.6. This suggests that a Kempf-Ness-theorem [27, theorems 0.1 and 0.2] (see also appendix B.2) argument may be used to give  $\mathcal{M}(r, n) = \mu_{r,n}^{-1}(0) //_{\chi} GL(V)$  the structure of a hyperkähler manifold as follows. We use the notations in remark B.1.3. Let  $(I, \omega_{\mathbb{C}})$  be the natural holomorphic symplectic structure on the coadjoint orbit  $L_n$  as in McDuff and Salamon [37, page 168] (recall that in section 2.2 we called this  $\omega'_{L_n}$ .) By Biquard [6, page 275], there is a real symplectic form  $\omega_{\mathbb{R}}$  on  $L_n$  which together with  $\omega_{\mathbb{C}}$  gives rise to a hyperkähler structure on  $L_n$  (Biquard and Gauduchon [7, théorème 3] gives a simple formula for  $\omega_{\mathbb{R}}$ .) This corresponds to a real moment map

$$\mu_{\mathbb{R}} : \mu_{r,n}^{-1}(0) \longrightarrow U(V)$$

( $U(V)$  is a unitary subgroup of  $GL(V)$  obtained by fixing a Hermitian metric on  $V$ , which is a maximal compact subgroup of  $GL(V)$ ) in the sense of remark B.1.6 (here  $\mu_{\mathbb{C}} := \mu_{r,n}$ .) Then similarly to Nakajima [42, corollary 3.22], it may be possible to use a Kempf-Ness-theorem argument to prove that  $\mathcal{M}(r, n)$  is a hyperkähler quotient in the sense of Hitchin et al. [20, section 3.D] (see appendix B.1.)

Since  $\mu_t = t\mu_{r,n}$ , we can consider for each  $t \neq 0$  a GIT quotient

$$\mu_t^{-1}(0)//_{\chi}GL(V) \simeq \mu_{r,n}^{-1}(0)//_{\chi}GL(V) = \mathcal{M}(r,n).$$

Since  $\mu_0$  is also a (holomorphic) moment map by lemma 5.12 below, we speculate that the GIT quotient

$$\mu_0^{-1}(0)//_{\chi}GL(V)$$

is also a hyperkähler quotient and that the family

$$\{\mu_t^{-1}(0)//_{\chi}GL(V)\}_{t \in \mathbb{C}}$$

is the twistor deformation (see Hitchin [19, section 2]) of the hyperkähler structure on  $\mu_0^{-1}(0)//_{\chi}GL(V)$ . This is the content of conjecture 5.14 below. The discussion after conjecture 5.14 will then make it clear why we are interested in such a twistor family.

**Definition 5.6.** For the action of a Lie group  $G$  on an algebraic variety  $M$  and a character  $\chi$  of  $G$  we define

$$M//_{\chi}G$$

to be the GIT quotient with respect to the linearization

$$g \cdot (x, z) = (g \cdot x, \chi(g) \cdot z) \text{ for } g \in G \text{ and } (x, z) \in M \times \mathbb{C} \quad (5.7)$$

on the trivial line bundle  $M \times \mathbb{C}$  over  $M$ .

For each  $t \neq 0$  the action of  $GL(V)$  on  $E_t \simeq L_n$  restricts to one on  $\mu_t^{-1}(0)$ . Let  $\chi : GL(V) \rightarrow \mathbb{C}^*$  be the character

$$\chi(g) = \det(g). \quad (5.8)$$

**Proposition 5.9.** *For each  $t \neq 0$ ,*

$$\mu_t^{-1}(0) //_{\chi} GL(V) \simeq \mathcal{M}(r, n).$$

The proof uses

**Lemma 5.10.** *For  $G$  a reductive group acting on an affine variety  $X$  and  $L_1$  and  $L_2$  two  $G$ -linearized line bundles on  $X$  such that*

$$\emptyset \neq X^{ss}(L_1) \subset X^s(L_2)$$

*one has*

$$X^s(L_1) = X^{ss}(L_1) = X^s(L_2) = X^{ss}(L_2).$$

*(here  $X^{ss}(L_i)$  is the geometric invariant theory (GIT) semistable locus with respect to  $L_i$  as in Mumford et al. [40, page 37] and  $X^s(L_i)$  is the properly stable locus, i.e. the stable points which have zero dimensional stabilizers.)*

*Proof.* For any  $G$ -linearized line bundle  $L$  we have a projective morphism

$$X //_L G = \text{Proj} \left( \bigoplus_{n \geq 0} H^0(X, L^n)^G \right) \longrightarrow \text{Spec} \left( H^0(X, \mathcal{O}_X)^G \right) =: X // G.$$

Since  $X^{ss}(L_1) \subset X^s(L_2)$ , every point of  $X^{ss}(L_1)$  has zero dimensional stabilizer.

Hence

$$X^{ss}(L_1) = X^s(L_1).$$

We have

$$X //_{L_1} G = X^s(L_1)/G \hookrightarrow X^s(L_2)/G \hookrightarrow X^{ss}(L_2)/G = X //_{L_2} G,$$

where both inclusions are open immersions. Since  $X//_{L_1}G \rightarrow X//G$  is proper, it follows by Hartshorne [18, corollary II.4.8(e)] that the inclusion  $X//_{L_1}G \hookrightarrow X//_{L_2}G$  above is also proper. Since  $X//_{L_1}G \neq \emptyset$ , we have  $X//_{L_1}G = X//_{L_2}G$  and thus

$$X^{ss}(L_1) = X^s(L_1) = X^s(L_2) = X^{ss}(L_2).$$

□

*Proof of proposition 5.9.* Via the identification (5.3) we only need to prove that

$$\mu_{r,n}^{-1}(0)//_{\chi}GL(V) = \mathcal{M}(r, n). \quad (5.11)$$

We denote by  $S_{\chi}$  the semistable locus in  $\mu_{r,n}^{-1}(0)$  with respect to the linearization (5.7) on the trivial line bundle  $\mu_{r,n}^{-1}(0) \times \mathbb{C}$  over  $\mu_{r,n}^{-1}(0)$ . Similarly to lemma 3.25 in Nakajima [42], the proof is finished by the following

*Claim.*  $S_{\chi} = U(r, n)$ .

Suppose  $x = (\widetilde{W}_I, \widetilde{W}_{II}) \in \mu_{r,n}^{-1}(0) \setminus U(r, n)$  then the morphism  $b$  in monad (2.2.4) is not surjective at an off-diagonal point  $p \in X \setminus D$ . We represent  $\widetilde{W}_I$  and  $\widetilde{W}_{II}$  by matrices

$$\begin{pmatrix} A \\ \hline B \\ \hline * \end{pmatrix} \text{ and } \begin{pmatrix} C \\ \hline D \\ \hline * \end{pmatrix}$$

where the rows correspond to the decomposition  $\widetilde{W} = V \oplus V \oplus W$  (here  $A, B \in \text{Hom}(\widetilde{W}_I, V)$  and  $C, D \in \text{Hom}(\widetilde{W}_{II}, V)$ .) After a change of coordinates,  $p$  can be taken to be  $((1 : 0), (0 : 1))$  and

$$b \otimes \mathbb{C}(p) = (-A \mid D).$$

Since  $b \otimes \mathbb{C}(p)$  is not surjective, there is a basis of  $V$  for which

$$(A \mid D) = \left( \begin{array}{ccc|cc|ccc} & & & * & * & & & \\ & & & * & * & & & \\ & & & * & * & & & \\ \hline & & & 0 & 0 & 0 & \dots & 0 \\ \hline & & & 0 & 0 & 0 & \dots & 0 \end{array} \right).$$

We change the bases in  $\widetilde{W}_I$  and  $\widetilde{W}_{II}$  so that

$$B = \left( \begin{array}{ccc|c} & & & * \\ & & & * \\ & & & * \\ \hline & & & 1 \\ \hline 0 & \dots & 0 & \end{array} \right), \quad C = \left( \begin{array}{c|ccc} * & & & \\ * & & & \\ * & & & \\ \hline 1 & 0 & \dots & 0 \end{array} \right).$$

With respect to the above basis for  $V$ , we define

$$g(t) := \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & 1 & 0 \\ 0 & \dots & 0 & t \end{pmatrix}.$$

Then  $g(t) \cdot (\widetilde{W}_I \mid \widetilde{W}_{II}) =$

$$= \left( \begin{array}{ccc|cc|ccc} & & & * & * & & & \\ & & & * & * & & & \\ & & & * & * & & & \\ \hline & & & 0 & t^{-1} & 0 & \dots & 0 \\ \hline & & & * & * & & & \\ & & & * & * & & & \\ & & & * & * & & & \\ \hline & & & 0 & t^{-1} & 0 & 0 & \dots & 0 \\ \hline & & & * & * & * & & * \end{array} \right) = \left( \begin{array}{ccc|cc|ccc} & & & t* & t* & & & \\ & & & t* & t* & & & \\ & & & t* & t* & & & \\ \hline & & & 0 & 1 & 0 & \dots & 0 \\ \hline & & & t* & t* & & & \\ & & & t* & t* & & & \\ & & & t* & t* & & & \\ \hline & & & 0 & 1 & 0 & 0 & \dots & 0 \\ \hline & & & * & t* & t* & & * \end{array} \right).$$



Hence the limit

$$\lim_{t \rightarrow 0} g(t).(\widetilde{W}_I, \widetilde{W}_{II})$$

lies inside  $L_n$ . On the other hand,  $\det(g(t)).z = tz \rightarrow 0$ . Hence the orbit  $GL(V).(x, z)$  is not closed. It follows that

$$S_\chi \subset U(r, n).$$

Since the action of  $GL(V)$  on  $U(r, n) \subset \mu_{r,n}^{-1}(0)$  is free, the quotient  $\pi : U(r, n) \rightarrow \mathcal{M}(r, n)$  is a principal  $GL(V)$ -bundle and  $\mathcal{M}(r, n)$  is quasi-projective (see sections 2.1 and 2.2), by Mumford et al. [40, converse 1.13] there exists a  $GL(V)$ -linearized line bundle

$$\mathcal{L} \in \text{Pic}^{GL(V)}(\mu_{r,n}^{-1}(0))$$

such that the corresponding GIT stable locus, which we call  $S_{\mathcal{L}}$ , contains  $U(r, n)$ . By lemma 5.10, we have

$$S_\chi = U(r, n) = S_{\mathcal{L}}.$$

□

On the special fiber  $E_0 = T^*G(n, \widetilde{W})$  there is an action of  $GL(\widetilde{W})$ , which is lifted from the one on  $G(n, \widetilde{W})$ . This yields an action of  $GL(V)$  on  $T^*G(n, \widetilde{W})$  via the decomposition  $\widetilde{W} = V \oplus V \oplus W$ .

**Lemma 5.12.**  $\mu_0$  is a moment map for the action of  $GL(V)$  on  $T^*G(n, \widetilde{W})$ .

The proof uses the following result which we believe is standard, nevertheless we include here a proof for the reader's convenience.

**Lemma 5.13.** *For a complex Lie group  $G$  acting holomorphically on a complex manifold  $M$ , the map*

$$\begin{aligned}\mu : T^*M &\longrightarrow \mathfrak{g}^* \\ (m, \alpha) &\mapsto \alpha \circ d(\sigma_m)_e,\end{aligned}$$

where  $m \in M$ ,  $\alpha \in T_m^*M$ ,  $\sigma_m : G \rightarrow M$  is defined by  $\sigma_m(g) = g.m$  and  $e$  is the identity of  $G$ , is a holomorphic moment map for the lifted action of  $G$  on  $T^*M$  with the canonical holomorphic symplectic structure.

*Proof.* The lifted action of  $G$  on  $T^*M$  is given by

$$g.(m, \alpha) = (g.m, (g^{-1})^*\alpha)$$

for any  $g \in G$ ,  $m \in M$  and  $\alpha \in T_m^*M$ . Let  $\pi : T^*M \rightarrow M$  be the natural projection. The *tautological* 1-form  $\tau$  on  $T^*M$  is given by

$$\tau_{(m,\alpha)}(v) = \alpha(d\pi(v))$$

for any  $m \in M$ ,  $\alpha \in T_m^*M$  and  $v \in T_{(m,\alpha)}T^*M$ . The canonical holomorphic symplectic form on  $T^*M$  is given by

$$\omega = -d\tau.$$

For any  $\lambda \in \mathfrak{g}$  let  $\vec{\lambda}_M$  (resp.  $\vec{\lambda}_{T^*M}$ ) be the vector field corresponding to the infinitesimal  $\lambda$ -action on  $M$  (resp.  $T^*M$ .) We have

$$i_{\vec{\lambda}_{T^*M}}\tau = \tau(\vec{\lambda}_{T^*M}) = \alpha(d\pi(\vec{\lambda}_{T^*M})) = \alpha(\vec{\lambda}_M).$$

We define a map  $H_\lambda : T^*M \rightarrow \mathbb{C}$  by

$$H_\lambda(m, \alpha) \stackrel{def}{=} \alpha(d(\sigma_m)_e(\lambda)) = \alpha(\vec{\lambda}_M) = i_{\vec{\lambda}_{T^*M}}\tau.$$

Then

$$\begin{aligned} dH_\lambda &= d(i_{\tilde{\lambda}_{T^*M}} \tau) = \mathcal{L}_{\tilde{\lambda}_{T^*M}} d\tau - i_{\tilde{\lambda}_{T^*M}} d\tau \\ &= i_{\tilde{\lambda}_{T^*M}} \omega, \end{aligned}$$

since the action of  $G$  preserves  $\omega = -d\tau$ . This means  $H_\lambda$  is a Hamiltonian for the action of  $G$  on  $T^*M$ .  $\square$

*Proof of lemma 5.12.* By (5.2), we have

$$E_0 \times_{G(n, \widetilde{W})} \text{Spec } \mathbb{C}(U) = \left\{ \theta \in \text{Hom}(\widetilde{W}, U) \mid \ker(\theta) = U \right\}$$

An element of  $E_0 = T^*G(n, \widetilde{W})$  can be viewed as one on the trivial bundle  $G(n, \widetilde{W}) \times \text{End}(\widetilde{W})$  by the identification

$$(U, \theta) \mapsto (U, i_U \theta),$$

Applying lemma 5.13 for the action of  $GL(V)$  on  $G(n, \widetilde{W})$  and identifying  $\text{Lie}(GL(V)) = \text{End}(V)$  with its dual by the trace pairing

$$(X, Y) \mapsto \text{tr}(XY),$$

we obtain the moment map

$$\begin{aligned} \mu' : T^*G(n, \widetilde{W}) &\longrightarrow \text{End}(V)^* \\ (U, i_U \theta) &\mapsto (a \mapsto \text{tr}((i_U \theta)A)), \end{aligned}$$

where  $A$  is the image in  $\text{End}(\widetilde{W})$  of  $a \in \text{End}(V)$ , as in the proof of lemma 2.2.34.

Recall that

$$A = i_0 a p_0 + i_1 a p_1.$$

Hence

$$\begin{aligned}\mu'(U, i_U\theta)(a) &= \text{tr}(i_U\theta(i_0ap_0 + i_1ap_1)) \\ &= \text{tr}((p_0i_U\theta i_0 + p_1i_U\theta i_1)a).\end{aligned}$$

Since  $\mu_0 = p_0i_U\theta i_0 + p_1i_U\theta i_1$ , via the identification  $\text{End}(V) \simeq \text{End}(V)^*$  by the trace pairing we have

$$\mu' = \mu_0.$$

□

**Conjecture 5.14.** (i) *The one-parameter family  $E \xrightarrow{j} \mathbb{C}$  from (5.1) is the twistor deformation (see Hitchin [19, section 2]) for the hyperkähler structure on  $E_0 = T^*G(n, \widetilde{W})$  from proposition 5.5.*

(ii) *For each  $t \in \mathbb{C}$ ,*

$$\mu_t^{-1}(0) //_{\chi} GL(V)$$

*is a hyperkähler quotient.*

(iii) *There is an isomorphism*

$$\mathcal{M}_{\mathbb{F}_2}(r, n) \simeq \mu_0^{-1}(0) //_{\chi} GL(V)$$

*( $\mathcal{M}_{\mathbb{F}_2}(r, n)$  was defined in (4.2.4) and  $\chi$  in (5.8).)*

(iv) *The one-parameter family  $\mu^{-1}(0) //_{\chi} GL(V) \xrightarrow{j} \mathbb{C}$  (which descends from the family  $E \xrightarrow{j} \mathbb{C}$  from part (i)) is the twistor deformation for the hyperkähler structure on  $\mu_0^{-1}(0) //_{\chi} GL(V) \simeq \mathcal{M}_{\mathbb{F}_2}(r, n)$  obtained in part (ii).*

The involution  $i_{\mathcal{M}}$  on  $\mathcal{M}(r, n)$  induces one on the cohomology ring  $H^*(\mathcal{M}_{\mathbb{F}_2}(r, n)) \simeq H^*(\mathcal{M}(r, n))$ , which we call  $\Phi$ . We believe conjecture 5.14(iii) implies that

$$\Phi \text{ is induced by a correspondence } \overline{Z}_0 \subset \mathcal{M}_{\mathbb{F}_2}(r, n) \times \mathcal{M}_{\mathbb{F}_2}(r, n). \quad (5.15)$$

**Remark 5.16.** (5.15) can be useful for studying  $H^*(\mathcal{M}_{\mathbb{P}^2}(r, m))$ , which is related to the induced involution on  $H^*(\mathcal{M}(r, 2m)) \simeq H^*(\mathcal{M}_{\mathbb{F}_2}(r, 2m))$ . Presumably, the latter involution can be studied by descending to  $\mathcal{M}_{\mathbb{F}_2}(r, 2m)$  the correspondence (5.17) below, which is studied in Markman [34, section 2.1].

We outline our idea for a proof. Here by *Markman* we mean section 2.1 in Markman [34]. For each  $t \neq 0$  there is an involution on  $E_t \simeq L_n$ , which we denote by  $i_t$ . By the trivialization (5.4), it follows that  $i_t$  lifts to an involution on  $E \times_{\mathbb{C}^*} \mathbb{C}^*$ . We take Zariski closure to obtain a correspondence

$$\widetilde{Z} \subset E \times E.$$

Let  $\widetilde{Z}_t := \widetilde{Z} \cap (E_t \times E_t)$ . Then  $\widetilde{Z}_t$  induces  $i_t$  for each  $t \neq 0$ ; while a certain irreducible component  $\overline{Z}_0$  of  $\widetilde{Z}_0$  induces the birational map

$$E_0 = T^*G(n, \widetilde{W}) \dashrightarrow T^*G(n, \widetilde{W}^*) \simeq T^*G(n, \widetilde{W}) = E_0 \quad (5.17)$$

in Markman. Resolving the indeterminacy loci we obtain a smooth variety  $\widetilde{E}_0$  and a commutative diagram

$$\begin{array}{ccc} & \widetilde{E}_0 & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ E_0 & \dashrightarrow & E_0. \end{array}$$

Let  $pr_i$  be the projection from  $E_0 \times E_0$  onto the  $i$ th factor. We have the commutative diagram

$$\begin{array}{ccccc} & & \widetilde{E}_0 & & \\ & & \downarrow (\pi_1, \pi_2) & & \\ \pi_1 \swarrow & & & & \searrow \pi_2 \\ E_0 & \xleftarrow{pr_1|_{\overline{Z}_0}} & \overline{Z}_0 & \xrightarrow{pr_2|_{\overline{Z}_0}} & E_0 \\ & \searrow & & \swarrow & \\ & & \mathcal{N} & & \end{array}$$

where  $\mathcal{N}$  is a certain variety defined in Markman and the bottom two arrows are resolutions of singularities. Since the middle arrow is surjective and the maps  $\pi_i$  are proper, it follows that the maps  $pr_i|_{\overline{Z}_0}$  are proper. Hence  $\overline{Z}_0$  induces an involution  $\tilde{\Phi}$  on the cohomology ring of  $E_0 = T^*G(n, \widetilde{W})$ . This method can be repeated with  $E$ ,  $E_t$  and  $E_0$  replaced by  $\mu^{-1}(0)//_{\chi}GL(V)$ ,  $\mu_t^{-1}(0)//_{\chi}GL(V)$  and  $\mu_0^{-1}(0)//_{\chi}GL(V)$  ( $\simeq \mathcal{M}_{\mathbb{F}_2}(r, n)$ ) to deduce (5.15).

# APPENDIX A

## A TECHNICAL OBSTRUCTION

We explain the reason for not considering in this thesis the trivial framing as in Nakajima [42, chapter 2]. Let  $E$  be a rank  $r$  torsion free sheaf on  $\mathbb{P}^1 \times \mathbb{P}^1$  with a framing

$$\phi : E|_D \simeq \mathcal{O}_{\mathbb{P}^1}^{\oplus r}$$

along the diagonal  $D$ . One would expect a monad description for  $E$  as in proposition 2.2.11. Unfortunately, the crucial cohomology vanishing result - lemma 2.2.13 - does not hold in this case because

$$H^0(E(-1, -1)) \neq 0.$$

One may remedy this by introducing a twist

$$E \mapsto E' := E(-1, 0)$$

so that the framing becomes

$$\phi : E'|_D \simeq \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus r}.$$

Then  $E'$  satisfies a cohomology vanishing result as in lemma 2.2.13 and is the cohomology of a monad of the form (2.2.12). However, the moduli space of such  $E'$  does not have an involution.

## APPENDIX B

### HYPERKÄHLER QUOTIENTS AND KEMPF-NESS THEOREM

#### B.1. Hyperkähler quotients

We define the notion of a *hyperkähler quotient*, following the exposition in Nakajima [42, pages 37-38].

**Definition B.1.1.** Let  $X$  be a  $2n$ -dimensional manifold. A *Kähler structure* of  $X$  is  $(g, I)$  where  $g$  is a Riemannian metric and  $I$  an almost complex structure on  $X$ , which satisfies that

- (i)  $g(Iv, Iw) = g(v, w)$  for all  $v, w \in TX$ ,
- (ii)  $I$  is integrable,
- (iii) The 2-form  $\omega$  defined by

$$\omega(v, w) \stackrel{\text{def}}{=} g(Iv, w) \text{ for all } v, w \in TX$$

is closed. (This  $\omega$  is called the *Kähler form* associated with  $(g, I)$ .)

**Definition B.1.2.** Let  $X$  be a  $4n$ -dimensional manifold. A *hyperkähler structure* on  $X$  is given by the data  $(g, I, J, K)$  where  $g$  is a Riemannian metric and  $I, J, K$  almost complex structures on  $X$  such that

- (i)  $g(Iv, Iw) = g(Jv, Jw) = g(Kv, Kw) = g(v, w)$  for all  $v, w \in TX$ ,
- (ii)  $I^2 = J^2 = K^2 = IJK = -1$ ,



(iii)  $\nabla_g I = \nabla_g J = \nabla_g K = 0$  ( $\nabla_g$  is the Levi-Civita connection of  $g$ .)

We call a manifold with a hyperkähler structure a *hyperkähler manifold*.

**Remark B.1.3.**  $(g, I), (g, J), (g, K)$  define Kähler structures  $\omega_1, \omega_2, \omega_3$  respectively. One can pick a particular complex structure, say  $I$ , let  $\omega_{\mathbb{R}} := \omega_1$  and combine the other Kähler forms as  $\omega_{\mathbb{C}} := \omega_2 + \sqrt{-1}\omega_3$ . As in Nakajima [42, page 38], the pair  $(I, \omega_{\mathbb{C}})$  is then a holomorphic symplectic structure on  $X$ .

We now give the definition of a hyperkähler quotient by Hitchin et al. [20]. Let  $(X, g, I, J, K)$  be a hyperkähler manifold, and  $\omega_1, \omega_2, \omega_3$  the kähler forms associated to  $I, J, K$  respectively. Let  $G$  be a compact Lie group which acts on  $X$  preserving  $g, I, J, K$ .

**Definition B.1.4.** A map

$$\mu = (\mu_1, \mu_2, \mu_3) : X \rightarrow \mathbb{R}^3 \otimes \text{Lie}(G)^*$$

is said to be a *hyperkähler moment map* if it satisfies

- (i)  $\mu$  is  $G$ -equivariant with respect to the coadjoint action of  $G$  on  $\text{Lie}(G)^*$ ,
- (ii)  $\langle d\mu_i(v), \xi \rangle = \omega_i(\vec{\xi}, v)$  for any  $v \in TX$ , any  $\xi \in \text{Lie}(G)$  and  $i = 1, 2, 3$ , where  $\vec{\xi}$  is the vector field on  $X$  which corresponds to the infinitesimal  $\xi$ -action.

Let  $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \otimes \text{Lie}(G)^*$  be a fixed point of the coadjoint action. Then  $\mu^{-1}(\xi)$  is invariant under the action of  $G$ .

**Theorem B.1.5.** (Hitchin et al. [20]) *Suppose the  $G$ -action on  $\mu^{-1}(\xi)$  is free. Then the quotient space  $\mu^{-1}(\xi)/G$  is a smooth manifold and has a Riemannian metric and a hyperkähler structure induced from those on  $X$ .*

$(\mu^{-1}(\xi)/G)$  is thus called a *hyperkähler quotient*.

**Remark B.1.6.** In light of remark B.1.3, we may denote  $\mu_{\mathbb{R}} := \mu_1$ ,  $\mu_{\mathbb{C}} := \mu_2 + \sqrt{-1}\mu_3$ ,  $\xi_{\mathbb{R}} := \xi_1$  and  $\xi_{\mathbb{C}} = \xi_2 + \sqrt{-1}\xi_3$ . Then  $\mu_{\mathbb{R}}$  is a real moment map with respect to  $\omega_{\mathbb{R}}$  and  $\mu_{\mathbb{C}}$  is a holomorphic moment map with respect to  $\omega_{\mathbb{C}}$ . The hyperkähler quotient  $\mu^{-1}(\xi)/G$  is sometimes written as

$$(\mu_{\mathbb{R}}^{-1}(\xi_{\mathbb{R}}) \cap \mu_{\mathbb{C}}^{-1}(\xi_{\mathbb{C}})) / G.$$

## B.2. Kempf-Ness theorem

The following is Kempf-Ness theorem [27, theorems 0.1 and 0.2] (the statement we give below is taken from the exposition in Mumford et al. [40, page 148].)

**Theorem B.2.1.** *Let  $X$  be a projective variety,  $G$  a reductive group acting on  $X$ ,  $K \subset G$  a maximal compact subgroup,  $\omega$  a  $K$ -invariant kähler form on  $X$  and  $\mu : X \rightarrow \text{Lie}(K)^*$  a moment map for the  $K$ -action on  $X$ , such that there is an embedding  $X \subset \mathbb{P}^N$  for which  $G$  acts via a group homomorphism*

$$\rho : G \longrightarrow GL(N + 1)$$

*such that  $\rho$  restricts to a unitary action of  $K$  and  $\omega$  is the restriction of the Fubini-Study form on  $\mathbb{P}^N$ . Then there is a homeomorphism*

$$\mu^{-1}(0)/K \xrightarrow{\sim} X//G,$$

*where the right hand side is the GIT quotient of  $X$  by  $G$  for the  $G$ -linearized line bundle  $\mathcal{O}_{\mathbb{P}^N}(1)|_X$ .*

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