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MULTIPLE REPRESENTATIONS OF THE FUNDAMENTAL THEOREM OF CALCULUS AS ENACTED IN THE CURRICULUM, SENSE-MAKING AND GENDER

Ileana Vasu
University of Massachusetts Amherst

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MULTIPLE REPRESENTATIONS OF THE FUNDAMENTAL THEOREM OF
CALCULUS AS ENACTED IN THE CURRICULUM, SENSE-MAKING AND
GENDER

A Dissertation Presented

by

ILEANA VASU

Submitted to the Graduate School of the
University of Massachusetts Amherst in partial fulfillment
of the requirements for the degree of

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DEDICATION

To my family:

George and Laura (mare),

Mary and Laura (mica),

Bunicu and bunica,

Mica and Dade
ACKNOWLEDGMENTS

I am grateful to my parents, George and Laura, who have provided me with moral and emotional support in my life, and who have made many sacrifices for their children. I am also grateful to my siblings, Mary and Laura, and to my other family and friends who have supported me graciously in my progress. I am especially indebted to my sister, Laura Vasu, for her invaluable help formatting and editing this document, and to my father George Vasu for his expert advice.

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ABSTRACT

MULTIPLE REPRESENTATIONS OF THE FUNDAMENTAL THEOREM OF CALCULUS AS ENACTED IN THE CURRICULUM, SENSE-MAKING AND GENDER

FEBRUARY 2018

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Directed by: Professor Sandra Madden

Multiple representations of the Fundamental Theorem of Calculus (FTC) are deemed essential to creating mathematical habits of mind, but not all classroom instruction includes them. This study articulates the relationship between college students’ experience with multiple representations of the FTC, gained through the enacted curriculum, and their use of multiple representations when problem solving or discussing the FTC. It suggests that students’ use of multiple representations directly relates to their curricular experience, which outweighs a student’s own inclination towards any particular representation. It further suggests that the relationship between classroom experience with a particular representation of the FTC, and its subsequent use in problem solving and discussion, is stronger for female students than for male students. Results in the literature indicate that female students tend to gravitate toward the representations they are exposed to through the enacted curriculum, while male students tend to be risk takers and may explore alternate representations. This study suggests that rich cognitive demand tasks that include multiple representations and are supported by an active learning environment help students develop a fuller understanding of the FTC. A
mixed methods design is used, which includes lesson observations at three colleges, classroom assessments, and semi-structured think-aloud interviews with nine students – three from each college – as they problem-solve around the FTC. The study contributes to the existing literature on Calculus education by providing a more complete picture of the ways in which an enacted college curriculum that includes multiple representations of the FTC supports deeper learning and understanding of Calculus for all students, particularly female students.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>ACKNOWLEDGEMENTS</td>
<td>v</td>
</tr>
<tr>
<td>ABSTRACT</td>
<td>vi</td>
</tr>
<tr>
<td>LIST OF TABLES</td>
<td>xvii</td>
</tr>
<tr>
<td>LIST OF FIGURES</td>
<td>xx</td>
</tr>
<tr>
<td>CHAPTER</td>
<td></td>
</tr>
<tr>
<td>1. INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>Overview</td>
<td>1</td>
</tr>
<tr>
<td>Introduction to the Landscape</td>
<td>3</td>
</tr>
<tr>
<td>Statement of the Problem</td>
<td>10</td>
</tr>
<tr>
<td>Motivation for Studying the Fundamental Theorem</td>
<td>11</td>
</tr>
<tr>
<td>Conclusion</td>
<td>12</td>
</tr>
<tr>
<td>2. REVIEW OF LITERATURE</td>
<td>15</td>
</tr>
<tr>
<td>Multiple Representations</td>
<td>15</td>
</tr>
<tr>
<td>Representations: Definition and Role of Representations</td>
<td>15</td>
</tr>
<tr>
<td>Choosing Representations</td>
<td>16</td>
</tr>
<tr>
<td>Transfer Among Mathematical Representations</td>
<td>18</td>
</tr>
<tr>
<td>Multiple Representations in Mathematics</td>
<td>21</td>
</tr>
<tr>
<td>Diagrams and Representations</td>
<td>23</td>
</tr>
<tr>
<td>Visualization and Imagery</td>
<td>24</td>
</tr>
<tr>
<td>Representational Framework for Learning Mathematics</td>
<td>26</td>
</tr>
</tbody>
</table>
3. METHODS .................................................................................................................. 74

Introduction ................................................................................................................. 74
Research Questions ...................................................................................................... 74
  Research Question 1 .................................................................................................. 74
  Research Question 2 .................................................................................................. 75
Conceptual Framework ............................................................................................... 77
Setting and Participants ............................................................................................. 79
  Setting ....................................................................................................................... 79
    Site 1: Riverside Community College ................................................................. 79
    Site 2: Hudson County Community College ...................................................... 80
    Site 3: College of Southern New England ......................................................... 81
  Participants ............................................................................................................... 81
Data Collection ........................................................................................................... 82
  Lesson Observation Protocol (LOP) and Classroom Portraits ......................... 83
  The Classroom Portrait .......................................................................................... 85
  Background Questionnaire ..................................................................................... 85
  Five Problems Involving the FTC ......................................................................... 86
  Think-Alouds ........................................................................................................... 86
Data Analysis ................................................................................................................ 87
  Operationalization of Concept Variables ............................................................ 87
    Enacted Curriculum Variables .......................................................................... 88
    Cognitive Representational Preference (CP) ..................................................... 89
    Perceived Representational Instruction (PR) ...................................................... 90
Accommodated Needs Variables .......................................................... 91
FTC Understanding Concept Variables ............................................. 91
Individual and Total Representation Variables ............................... 91
Students’ Prior Knowledge ................................................................. 92
Participants’ Baseline for Prior Knowledge ....................................... 93
Overview of Research Methodology .................................................. 94
Qualitative Analysis ........................................................................... 97
Coding for the Enacted Curriculum and Analytic Memos .................... 98
Stages for the coding process for the enacted curriculum .................. 98
  Stage 1: Developing the codes ......................................................... 99
  Stage 2: Coding ............................................................................. 100
  Stage 3: Connecting across classrooms .......................................... 101
Quantitative Data Analysis ................................................................ 102
Researcher Profile ............................................................................. 108

4. RESULTS ON THE ENACTED CURRICULUM .............................. 110
  Classroom Portraits ........................................................................ 120

Riverside Community College (RCC) .............................................. 121
Hudson County Community College (HCCC) ................................. 134
College of Southern New England (CSNE) ...................................... 150
Themes and Patterns in the Enacted FTC Curriculum ....................... 157
  Patterns Emerging from the Analysis of the Enacted Curriculum ...... 158

5. RESULTS ON STUDENT UNDERSTANDING .............................. 163
  Chapter Organization ....................................................................... 163
Student Understanding of the FTC .......................................................... 163

Descriptive Statistics ........................................................................... 164

RCC Assessments ................................................................................ 164

HCCC Assessments ............................................................................. 165

CSNE Assessments ............................................................................. 166

Boxplots for Student Understanding of the FTC ................................. 167

Inferential Statistics on Student Understanding ................................. 170

ANOVA Results on Student Understanding ..................................... 170

Regression Results on Student Understanding of the FTC, Students’

  Cognitive Preference and Perceived Representational Instruction..... 171

  Cognitive Preference and Student Understanding ....................... 172

  Perceived Representational Instruction and Student Understanding.. 174

  Needs Met: Accommodated Preference and Student Understanding . 174

  Effect of Prior Knowledge ............................................................... 178

Gender and Student Understanding of the FTC ................................. 179

Boxplots by Gender ............................................................................. 180

Boxplots by Gender and Site .............................................................. 182

Inferential Statistics on Student Understanding and Gender ............ 184

ANOVA Results on Student Understanding and Gender ............... 185

Total Score Results ............................................................................ 185

Individual Representational Results and Gender ............................ 186

Regression Results on Student Understanding of the FTC, Students’

  Cognitive Preference and Gender.................................................. 187
Dan’s Problem Solving (RCC) ................................................................. 234
Deliana’s Problem Solving (HCCC)......................................................... 235
Dave’s Problem Solving (HCCC)............................................................. 236
Dawn’s Problem Solving (HCCC)........................................................... 237
Problem E from FTC Assessment.......................................................... 238
Emily’s Problem Solving (RCC)............................................................. 239
Emanuel’s Problem Solving (CSNE)......................................................... 239

6. CONCLUSIONS.................................................................................... 242
Overview.................................................................................................. 242
Answering the Research Questions ....................................................... 245
Research Question 1 ............................................................................. 245
Enacted Multiple Representations and Classroom Discourse .............. 248
FTC Assessments and the Enacted Curriculum..................................... 250
Think-Alouds and the Enacted Curriculum ........................................... 253
Effect of Cognitive Preference............................................................... 257
Effect of Prior Knowledge .................................................................... 260
Conclusions about Research Question 1 .............................................. 261
Research Question 2 ............................................................................. 262
Comparisons of Female Groups at the Three Sites.............................. 264
Comparison of Female and Male Students within Site......................... 266
Comparison of Male and Female Differences Across Sites............... 269
Effect of Enacted Curriculum on Female Students’ Choices ............... 272
Effect of Female Cognitive Preference ................................................ 274
Conclusions about Research Question 2 ............................................. 276
Limitations of the Study ........................................................................ 277
Discussion .................................................................................................. 279
Implications and New Directions .............................................................. 280
Recommendations .................................................................................... 283

APPENDICES

A. MODIFIED SAMPI (LOP) .................................................................. 284
B. FIVE PROBLEMS INVOLVING THE FTC ........................................... 308
A. APPENDIX C THINK-ALOUD PROBLEMS AND PROTOCOL ............ 310
D. BACKGROUND QUESTIONNAIRE ...................................................... 314
E. MATH RUBRICK FOR THE FIVE FTC PROBLEMS ......................... 315
REFERENCES ......................................................................................... 323
LIST OF TABLES

<table>
<thead>
<tr>
<th>TABLE</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Numerical values for ( y = x^2 ).</td>
<td>17</td>
</tr>
<tr>
<td>2. Representational direct translations model by Janvier, 1987.</td>
<td>20</td>
</tr>
<tr>
<td>3. Mathematical Representations (Gray &amp; Tall, 1994).</td>
<td>27</td>
</tr>
<tr>
<td>4. Several frameworks for learning mathematics.</td>
<td>35</td>
</tr>
<tr>
<td>5. Cognitive demand of mathematical tasks (Stein, 2000).</td>
<td>43</td>
</tr>
<tr>
<td>6. Study timeline.</td>
<td>82</td>
</tr>
<tr>
<td>7. Sample multiple representations score.</td>
<td>89</td>
</tr>
<tr>
<td>8. Five cognitive representational preference types.</td>
<td>89</td>
</tr>
<tr>
<td>9. Types of perceived representational instruction.</td>
<td>90</td>
</tr>
<tr>
<td>10. Variable scores on the FTC assessment.</td>
<td>91</td>
</tr>
<tr>
<td>11. Categories for students’ prior knowledge score.</td>
<td>93</td>
</tr>
<tr>
<td>12. Summary of methods used to answer research questions.</td>
<td>96</td>
</tr>
<tr>
<td>13. Categories and codes in enacted curriculum.</td>
<td>100</td>
</tr>
<tr>
<td>14. An example of dialogue coding.</td>
<td>102</td>
</tr>
<tr>
<td>15. Variables involved in the study.</td>
<td>107</td>
</tr>
<tr>
<td>16. LOP summary.</td>
<td>110</td>
</tr>
<tr>
<td>17. RCC visit and curricular objectives summary.</td>
<td>122</td>
</tr>
<tr>
<td>18. Enacted curriculum representation scores at RCC.</td>
<td>134</td>
</tr>
<tr>
<td>19. Lesson objectives for HCCC.</td>
<td>134</td>
</tr>
<tr>
<td>20. MR scores for HCCC.</td>
<td>149</td>
</tr>
<tr>
<td>21. Lesson objectives for CSNE.</td>
<td>150</td>
</tr>
</tbody>
</table>
22. MR scores for CSNE. ................................................................. 157
23. Enacted curriculum LOP scores for all three schools. ................. 160
24. Patterns and themes in the enacted curriculum at all three schools. 161
25. RCC scores........................................................................... 164
26. HCCC scores........................................................................... 166
27. CSNE scores. .......................................................................... 167
28. ANOVA results of student understanding.................................... 171
29. Total Score (TS) as a function of Cognitive Preference (CP), [F (5, 49) = 7.23; R-square = 0.43]................................................................. 172
30. Regression of Total Score and Cognitive Preference. .................. 173
31. Total Score (TS) as a function of Perceived Instruction (PR), [F (5, 49) = 0.68, R-square = 0.07]................................................................. 174
32. Total Needs Met by Site. ............................................................ 175
33. Regression results for Scores, Needs, Site, and Gender. ............. 176
34. Regression for Total Score, Needs, Gender, and Site. ..................... 177
35. Regression for Graphical Score, Needs, Gender, and Site. ............. 178
36. Five-number summary for the Total Scores by Gender and overall 180
37. ANOVA for gender and site across all scores on FTC assessment. ............. 186
38. Regression results for Total Score as a function of Cognitive Preference, Site and Gender................................................................. 188
39. Regression results for Total Score of female students as a function of Cognitive Preference, Site and Gender (R-square = 0.61)......................... 188
40. Regression results for Total Score for male students as a function of Cognitive Preference, Site and Gender (R-square = 0.42) .......................... 189

41. Summary of Student Think-Alouds. ............................................................ 228

42. Researcher scores on the enacted curriculum. ........................................... 249

43. Mean Scores for Female Students at the Three Sites .................................. 265

44. Mean Scores for Students at the Three Sites. ............................................. 267

45. Female student connection with the enacted curriculum. .......................... 273
**LIST OF FIGURES**

<table>
<thead>
<tr>
<th>FIGURE</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Graph of the parabola ( y = x^2 ).</td>
<td>17</td>
</tr>
<tr>
<td>2. Representations (Janvier, 1987).</td>
<td>20</td>
</tr>
<tr>
<td>3. Types of representations systems (Lesh, Post &amp; Behr).</td>
<td>22</td>
</tr>
<tr>
<td>4. Diagram of five representations of functions (Van de Walle, 2004).</td>
<td>22</td>
</tr>
<tr>
<td>5. The square of a binomial (Gardner, 1973).</td>
<td>24</td>
</tr>
<tr>
<td>7. SAT score as a composition of group and gender (Inzlicht &amp; Ben-Zeev, 2000).</td>
<td>50</td>
</tr>
<tr>
<td>8. Gender identification and levels of implicit stereotyping may affect exam scores and career goals (Kiefer &amp; Sekaquaptewa, 2007).</td>
<td>51</td>
</tr>
<tr>
<td>9. Fundamental Theorem I in graphical form.</td>
<td>59</td>
</tr>
<tr>
<td>10. FTC, Part II ( \int_a^b f(x) , dx = F(b) - F(a) ) if ( F'(x) = f(x) ).</td>
<td>60</td>
</tr>
<tr>
<td>11. Area under the velocity graph is the distance traveled.</td>
<td>61</td>
</tr>
<tr>
<td>12. ( f(x) = 1/x ) (Initial problem graph) (Haciomeroglu, Aspinwall, &amp; Presmeg 2009).</td>
<td>68</td>
</tr>
<tr>
<td>13. Graphs from Amy, Bob, and Jack. (Figures from Aspinwall, 2007).</td>
<td>68</td>
</tr>
<tr>
<td>14. Conceptual framework for classroom discourse and culture.</td>
<td>78</td>
</tr>
<tr>
<td>15. Concept variable map.</td>
<td>87</td>
</tr>
<tr>
<td>16. Teacher’s graph of ( f' ).</td>
<td>125</td>
</tr>
<tr>
<td>17. Matt's graph.</td>
<td>126</td>
</tr>
<tr>
<td>18. Graph of ( f' ) used to generate graph of ( f ) at RCC.</td>
<td>129</td>
</tr>
<tr>
<td>19. Graph of the anti-derivative of ( f'(x) ) with ( f(0) = 100 ).</td>
<td>132</td>
</tr>
<tr>
<td>20. Problems projected on the board at HCCC on Day 1.</td>
<td>137</td>
</tr>
</tbody>
</table>
21. Investigative task A – small group................................................................. 142
22. Investigative task B – small group................................................................. 146
23. Professor Smith’s picture.................................................................................. 154
24. Boxplots of the total and individual representation scores by site. From left to right:
   Total, Graphical, Numerical, Verbal, Contextual, and Symbolic scores. .......... 169
25: Boxplots of the total score and gender. From left to right: Total, Graphical,
   Numerical, Verbal, Contextual, and Symbolic scores. ................................. 181
26: Boxplots for class, gender, and total score. From left to right: Total, Graphical,
   Numerical, Verbal, Contextual, and Symbolic scores. ................................ 183
27. Interaction, class and gender............................................................................ 185
28. Graph for problem A from the *Think-Aloud*. ............................................. 190
29. Anne’s computations for $F(x)$....................................................................... 192
30. Anne’s picture of $F(x)$................................................................................... 192
31. Allison’s result for part defining $F(x)$............................................................. 194
32. Allison’s computation....................................................................................... 196
33. Christine’s answer to the first question.......................................................... 198
34. Christine’s drawing of $g(x)$.......................................................................... 199
35. Christine’s work on the regions of increase or decrease................................. 200
36. Dan’s demonstration for the velocity graph..................................................... 203
37. Dan’s computation of the distance traveled..................................................... 203
38. Dan’s graph of the distance traveled............................................................... 205
39. Deliana’s graph of velocity............................................................................. 208
40. Deliana’s part 2............................................................................................... 208
41. Deliana’s graph of the minimum. ................................................................. 209
42. Deliana’s minimum and maximum distance computations. ...................... 210
43. Dave’s calculation of the distance traveled by the ladybug. ...................... 212
44. Dave’s work around the bug’s acceleration. ............................................. 213
45. Dave's graph of the bug’s velocity. ............................................................. 214
46. Dave’s conclusion about the bug’s distance from A. .............................. 215
47. Dawn's representation of the velocity function. ....................................... 217
48. Dawn’s solution to finding the distance. .................................................. 217
49. Dawn’s conclusions about the bug’s direction of travel. ......................... 218
50. Dawn’s representation for the distance away from A. ............................ 219
51. Emily’s picture of $G(x)$. .......................................................................... 222
52. Emily's calculations. .................................................................................. 222
53. Emanuel’s work on part 1. ....................................................................... 223
54. Emanuel’s computations for the antiderivative $G(x)$ and its extrema. ...... 224
55. Emanuel’s work on concavity. ................................................................... 225
56. Graph given in problem A from the FTC Assessment. ............................ 231
57. Christine’s work on problem C. ................................................................. 233
58. Deliana’s work on problem D. ................................................................. 236
59. Dave's partial work on Problem D. ............................................................ 237
60. Dawn’s work on problem D. ................................................................. 238
61. Emanuel’s work on problem E. ................................................................. 240
62. Average cognitive preference and total score by site. .............................. 259
63. Total Score and cognitive preference by site and gender. ....................... 275
CHAPTER 1

INTRODUCTION

Overview

This study explores the role of multiple representations of the Fundamental Theorem of Calculus (FTC) in mathematics teaching and learning, with a special interest toward gender. The literature review presented in this work identifies a gap in research as to the connection between multiple representations of the FTC as enacted in the curriculum and the meaning that students make of the FTC. A study attempting to fill this gap is outlined. The intent is to make more explicit the links between the use of an enacted curriculum that features multiple representations of the FTC and student understanding, particularly in female students.

Traditional mathematics curricula and teaching styles may be partly responsible for the conceptual difficulties students encounter with Calculus, which in turn may contribute to many students opting out of the so-called science, technology, engineering, and mathematics (STEM) disciplines (Boaler & Staples, 2008; Ellis & Rasmussen, 2014). Women opt out of STEM disciplines at a higher rate than male students, with many citing difficulties in understanding Calculus as a major contributing factor (MAA, 2012), and research into gender differences in classroom environments has shown that traditional teaching methods may disadvantage female students (Manicom, 1992; Ellis & Rasmussen, 2014).

The Fundamental Theorem of Calculus connects two major concepts in Calculus: the derivative and the definite integral. It is considered the main theorem of differential and integral Calculus (Thompson, 1994) and is the main theorem students learn in their
first Calculus course. Understanding the FTC involves the ability to make connections among concepts such as derivatives, integrals and their representations (Pantozzi, 2010). Students learn to use the FTC in computations (Kaput, 1994; Smith, 2008), but they often are confused about what the theorem actually means (Bressoud, 2005; Pantozzi, 2010).

Despite calls for fundamental changes in mathematics teaching overall, and for the inclusion of multiple representations of the FTC in Calculus instruction, such as graphical, verbal, numerical, contextual and symbolical (NCTM, 2000, 2014; NRC 1990), Calculus is often still taught in a traditional manner, with an emphasis on drill and symbolic manipulation and with little time devoted to rich mathematical understanding. Research suggests that students’ understanding of Calculus is often limited to algorithms and procedures with little to no understanding of the underlying concepts (Bloch, 2003).

The representational forms of the FTC that students use in the classroom, and students’ own preferences, each play a role in developing student understanding of the FTC (Bloch, 2003; Even, 1998), but research into student learning of the FTC has focused mostly on reporting the conceptual difficulties that students have. Relatively few studies explore the methods students use to negotiate and create meaning around the FTC and relate them to the curriculum presented (Carlson, Smith, & Presson, 2003). A review of the literature suggested it would be compelling to further explore the connections between mathematical representations of the Fundamental Theorem of Calculus as they appear in the enacted curriculum and student understanding of the FTC.

The study presented in this dissertation suggests that students need to see more than one representation of the FTC to develop a complete understanding of it, and that students’ understanding is directly connected to their classroom experience, regardless of
any representational preferences they may have. For female students, the use of multiple representations and the ability to exhibit what one researcher calls “versatile thinking” (Tall, 1997) is even more closely related to the enacted curriculum.

This study contributes to the existing literature and to a more complete understanding of how the use of multiple representations of the FTC in the classroom environment supports learning for all students, and how interaction and collaboration among students can increase the persistence and enhance the academic achievement of female students in mathematics.

**Introduction to the Landscape**

During the past 30 years, mathematics educators have accumulated overwhelming evidence that something is wrong in mathematics education. Many high school and college graduates are either unaffected or negatively affected by their mathematics experiences (Schoenfeld, 1995). Moreover, international assessments suggest that U.S. students are not well prepared compared to European, Asian or Australian students (Baldi, Jin, Skemer, Green & Herget, 2007; Fleischman, Hopstock, Pelczar & Shelley, 2010; Lemke, 2004; TIMSS, 2008). Results from the Program for International Student Assessment (PISA), an international test of students’ knowledge taken at around age fifteen, show U.S. students scoring lower in mathematics than students in other countries (Baldi et al., 2007; Fleischman et al, 2010; Lemke, 2004). Since 2003, when PISA started, U.S. performance in mathematics literacy and in problem solving has been lower than the average for the participating countries. In each mathematics literacy subscale (space and shape, change and relationship, quantity, and uncertainty), U.S. students performed lower than the mean score for other countries in all years the test was given.
The U. S. also had greater percentages of students scoring at the lowest of the test’s six proficiency levels than almost all other countries (Baldi et al., 2007; Fleischman et al, 2010).

In his autobiography, *My Early Life*, Winston Churchill describes studying for the college math entrance exams:

When I look back upon those care-laden months, their prominent features rise from the abyss of memory. Of course I had progressed far beyond Vulgar Fractions and the Decimal System. We were arrived in an Alice in Wonderland world, at the portals of which stood A Quadratic Equation. Further dim chambers lighted by sullen, sulphurous fires were reputed to contain a dragon called the "Differential Calculus.” I have never met any of these creatures since. With my third and successful examination they passed away like the phantasmagoria of a fevered dream. (1930)

Churchill never connected mathematics to anything in the rest of his life, so he forgot it. Our students may be in good company, but connections in mathematics are essential for the promotion of understanding and relevance. In the view of this researcher, understanding mathematics and doing mathematics are one and the same. We learn mathematics by doing it, by negotiating and by creating meaning, and not by simply memorizing and imitating what the teacher does:

Mathematics is a living subject which seeks to understand patterns that permeate both the world around us and the mind within us. Although the language of mathematics is based on rules that must be learned, it is important for motivation that students move beyond rules to be able to express things in the language of mathematics. (NCTM, 2000)

The current crisis in mathematics education is particularly important given the need to expand access to STEM careers to diverse groups, and mathematics and science educators are facing a related concern: traditionally mathematics has been a male dominated field (Kirkman, Maxwell, & Rose, 2004). The lack of interest and representation of women in STEM fields means these fields, and ultimately society,
cannot benefit from the contributions of a diversified group of people (Mura, 1995). The imbalance continues to propagate the image of STEM as a man’s world and denies a large group of the population access to higher paying jobs and better lives (NSF, 2006).

Although the gender gap in mathematics achievement has improved in the last 30 years, with women today earning the same or higher grades as men at all levels of the K-12 curriculum (Byrnes, 2005) and comparable numbers of female and male students taking mathematics through the level of Calculus (Hanna, 2003), data from the National Science Foundation’s Division of Resources and Statistics show that gender imbalance still exists in STEM fields and may even be worsening, especially at the post-baccalaureate level. At the highest levels of degree completion, women continue to be underrepresented in mathematics and science; the ratio of women to men among those receiving doctorates in STEM fields is much smaller than the corresponding ratio upon entering STEM fields (NCES, 2010). In recent years, the proportion of women in science has declined at every career level (NAS, 2006).

For example, in 2004, women were awarded 27% of the degrees in computer science, 45% of the degrees in mathematical sciences and 41% of the degrees in physical sciences at the undergraduate level (NSF, 2010). By 2010, the percentages had decreased, particularly in computer science, where women now earned only 18% of the degrees (NSF, 2010). At the doctoral level women are also losing ground, obtaining only 17% of the doctoral degrees awarded in computer science, 23% of the doctoral degrees in mathematical sciences, and 22.4% of the degrees in physical science in 2010 (NSF, 2010). The “Annual Survey of the Mathematical Sciences in the US,” as reported by the Notices of the American Mathematical Society, which includes fields like statistics and
biostatistics with a traditionally higher percentage of women, still showed the proportion of females among the Citizen Doctoral Recipients in Mathematics declining from 34% in 2003–2004 to 27% in 2012–2013 (Velez, Maxwell, & Rose, 2014). Some fields within engineering are especially troublesome: for example, only 5.6% of the doctoral degrees in aeronautical engineering are awarded to women (NSF, 2010).

Today we have come to an understanding that women’s abilities and skills are not the result of their biological makeup, but rather a reflection of a complex network of societal, cultural, ethnic and curricular factors (Boaler, 1997; Fennema & Hart, 1994; Mura, 1995), and interestingly, PISA results show that while the gender gap in mathematics remains in many countries, some countries consistently show different results. The gender gap on the PISA tests ranged from a difference of 9 scale score points in Colombia to 62 scale score points in Albania in 2006 (PISA, 2003, 2009). Fifteen-year old female students in the United States scored lower than males overall and smaller percentages of them performed at the highest level of proficiency (PISA, 2003, 2009). On the other hand, in Iceland, Norway, Finland, and Sweden, women performed significantly better than men on the mathematics literacy test; six years later the same was true in Malaysia, Thailand, Qatar and Jordan (PISA, 2012).

In parts of the Southern Americas, the gender gap in science and mathematics is much smaller or is not seen at all. For example, in Río Piedras and Mayagüez, Puerto Rico, approximately 36% of graduates of engineering programs are women and approximately 60% of graduates of other science and mathematics programs are women (Rosario, Scott & Vogeli, 2015). Unfortunately, the situation in the United States stands in stark contrast. Although more female students in the U.S. have been entering the
STEM pipeline by taking advanced math and science courses in high school, this trend does not continue into the collegiate years. In 2004, female high school graduates were more likely than male graduates to have completed some advanced math courses (e.g., trigonometry, pre-calculus, or Calculus) yet the number of female students who chose to major in STEM disciplines was still small (NSF, 2010).

While more students, both male and female, are taking more math early on – many college students have already taken introductory Calculus in high school (MAA, 2012) – interest in mathematical careers among American students is declining. This is a worrisome trend as society comes to depend more and more on these fields (National Academy of Science, 2006). At a time when mathematics is the “critical filter for employment and full participation in our society (Valero & Zelenbergen, 2004),” getting a sound mathematics education is essential and Calculus is a course vital to STEM fields of study.

Unfortunately, for many students, Calculus is a gatekeeper rather than a pump to STEM disciplines. A 2002 study by the Conference Board of Mathematical Sciences found that remedial mathematics enrollment at two-year and four-year institutions increased 76% from 1980 to 2000, while Calculus enrollment during the same period remained essentially flat (CBMS, 2002). This indicates that more students are spending time in courses designed to prepare them for Calculus, but they are either not continuing their math studies at the college level or they are dropping out of Calculus (McGowan & Tall, 2010).

The gender profile of the students who opt out of STEM majors after taking Calculus is disproportionately female (MAA, 2012; Rasmussen, 2012). Although
generally poor support of students interested in STEM and explicit gender bias have been cited as barriers preventing women from completing their education in a STEM field (Boaler, 1994; Bressoud 2011; Bryant 2011), many suggest that this trend may be a consequence of pedagogical methods that do not support learning (Boaler & Staples, 2008; Bressoud et al., 2012; Schoenfeld, 1995). In a large-scale study of students at US postsecondary institutions, the Higher Education Research Institute (2010) found that more than 50% of students who start in a STEM major do not complete their major in five years. Students who opted out of STEM majors cited learning difficulties in Calculus as the main reason (Bressoud, 2012). The same study found that females represent 41% of those who entered college intending to major in STEM but 57% of those who choose to opt out (Bressoud, 2011; Rasmussen, 2012).

Some researchers (Boaler 1994; Bressoud 2010, 2012) have expressed the notion that students’ difficulties are a consequence of the curriculum: “Our traditional courses have unfortunately graduated too many students who have been unable to communicate what they were doing, or to translate a problem communicated to them in words without variables” (Kennedy, 1997). In terms of classroom instruction, traditional lecture is not especially effective in teaching mathematics and is not recommended by organizations such as NCTM (Boaler 1994; Bressoud 2010, 2012). Lecture and rote learning has been found to be particularly ineffective for women (Bressoud, 2012).

According to Boaler (1994), mathematics is almost never taught in a context that would allow for skills to appear naturally. “Story” problems are seldom used in classrooms, and when they do appear, they are rarely realistic, so students tend to ignore them (Boaler, 1994; Boaler & Staples, 2008; Drake & Sherin, 2006). As a result, instead
of seeing mathematics as something useful, women see it as something meaningless (Boaler, 1994), where exercises are unrealistic and the emphasis is on memorization.

Unfortunately, most college Calculus courses continue to use a traditional lecture-based format. According to a large-scale survey of Calculus instruction in the U.S., conducted by the Mathematical Association of America (MAA) in 2012, more than 70% of Calculus classes are lecture based and emphasize symbolic and procedural work in exam questions used to assess students. Two-thirds of the instructors surveyed believe that “understanding ideas in calculus comes AFTER achieving procedural fluency” (Bressoud, 2010, 2012), and most college calculus professors believe that “calculus students learn best from lectures, provided they are clear and well-prepared” (Bressoud, 2011).

On the contrary, much research has shown that when students are taught a set of rules to follow without a meaningful way to support them, they place the symbolic representation first and allow for little or no contextual or conceptual meaning. This practice encourages students to memorize procedures and use symbols without understanding the mathematical principles (Schoenfeld, 1992, 1985). Skemp (1987) describes the type of understanding where concepts are taught in isolation as “rules without reasons”. Many students seem to forget mathematical concepts almost as soon as they learn them. Perhaps students lacked a true understanding of the fundamental concepts to begin with (Schoenfeld, 1992).

The reform movement in mathematics, which began in the late 1980s, attempted to overcome this problem by advocating for profound changes in teaching to emphasize relational understanding between different mathematical concepts and between different
representational forms of the same concept (NCTM, 2000). Its recommendations include a greater emphasis on collaborative learning and on the use of technology. Reform curricula stress sense-making and problem solving, and reform educational materials differ significantly from traditional ones by promoting multiple representations through the inclusion of graphs, tables, story problems, symbols, and verbal explanations in the presentation of mathematical ideas to help build connections and to convey mathematical understanding (NCTM 2000, 2009, 2014).

**Statement of the Problem**

Educators must successfully implement Calculus reform to better support learning for all, especially women, as Calculus remains a gatekeeper for students pursuing STEM degrees. The proportion of students who drop out of STEM degrees is disproportionately female. Reasons female students cite for their decision to leave STEM disciplines are poor instruction and lack of consideration of gender-specific issues in learning (Bressoud, 2012). Multiple representations of mathematical ideas can provide students with a better understanding of mathematical concepts (Janvier, 1987) and allow for a perception of mathematics as active and engaging, rather than rote and procedural (Hiebert & Carpenter, 1992).

Calls for reform in the way Calculus is presented articulate the need for instruction that includes graphical, numerical, analytical and contextual representations of functions, derivatives and integrals (Aspinwall, Shaw & Presmeg, 1997; Hughes-Hallett, 1994; Sevimli & Delice, 2011; Weber & Dorko, 2014). In theory, this is provided by many reform curricula, but in practice, despite an emphasis on varied pedagogical practices and multiple representations in the classroom, mathematical representations are
seldom meaningfully included in Calculus teaching (MAA, 2012). Because multiple representations are necessary in learning and understanding mathematics, students’ experience with multiple representations in the enacted curriculum may directly affect their ability to make sense of mathematics, and therefore may be a key to student retention in Calculus courses. The fact that female students are achieving at the same level as their male counterparts in the high school curriculum but continue to be underrepresented in mathematics and sciences (Besecke & Reilly, 2006; NSF, 2007) suggests, that to increase women’s participation in mathematics, we may need to change classroom practices and culture.

The review of the literature and subsequent study gives attention to the relationship between the enacted curriculum, multiple representations and student understanding. The study brings to light some of the reasons why Calculus remains a challenge for many students and offers some possible solutions. In particular, the study advocates for more explicit inclusion of multiple representations in the enacted curriculum, for the use of collaborative learning, and for deliberate inclusion of contextual and cognitively rich tasks to engage all learners and in particular female students.

**Motivation for Studying the Fundamental Theorem**

The mathematical form of the Fundamental Theorem of Calculus (FTC) connects the derivative and the definite integral. The theorem is called “fundamental” because it represents a central idea in Calculus. The FTC says (in one of its applications) that to find the distance traveled by an object traveling at a non-constant positive velocity between two instances of time, say \( t = 0 \) and \( t = 4 \), one can find the area bounded by the velocity
graph and the $t$ axis between those two times. In practice, this area under the curve is
given in terms of a definite integral of the velocity function on that interval (or
approximated by adding up the area under rectangles under the graph of the curve as the
width of the rectangles shrinks to zero.) In other words, $\int_{0}^{4} v(t) \, dt = d(4) - d(0)$.

The present study focuses on the Fundamental Theorem of Calculus for two
reasons. First, this theorem is a foundation for Calculus and a major component in
undergraduate Calculus study. Secondly, the current version of the theorem is a result of
centuries of refinement from an original understanding that was far from the current one
(Bressoud, 2012).

**Conclusion**

Despite three decades of recommendations for reform, Calculus instruction
remains largely unchanged (Bressoud, 2010, 2012). Traditional Calculus instruction has
several distinguishing characteristics: uniformity in curriculum, an emphasis on rote
practice and memorization, heavy symbolical usage, and teacher-directed lectures
(Bressoud, 2010). Students’ classroom activities are restricted to note-taking, and
problem solving is relegated to a minimal portion of the homework completed outside the
classroom. Studies indicate that students in traditional courses have a superficial
understanding of Calculus (Lithner, 2003; Cipra 1988). Traditional university calculus is
driven by content features, sequencing of topics, and specific lessons by topic (Ferrini-
Mundy, 1991). Students in these courses are given straight-forward tasks that test the
ability to sketch, graph, calculate, or solve basic problems, and are rarely engaged in
higher order thinking (Bressoud, 2012; Ferrini-Mundy & Graham, 1991).
Women may be opting out of STEM careers as a response to a male dominated, hostile environment (Bryant, 2011), but they may also be responding to instructional methods that are not sufficiently responsive to female students as learners. Study of the use of different mathematical representations of the Fundamental Theorem of Calculus in the classroom can provide important insight into how students understand Calculus in different contexts and to examine how this understanding may be a function of gender. If there are certain representations of mathematical ideas around the FTC that support women’s learning, and if there are certain ways these representations may be used to make this support possible, we need to know about them, and to actively incorporate them in teaching practices.

Tall and Dubinski contend that learning of the FTC should be grounded in Actions and Processes that then can be encapsulated into mathematical Objects and Schemes (Tall, 1994; Dubinsky, 1996). Their theory, which has come to be known as APOS theory, informed the current study. When integrated in the enacted curriculum, multiple mathematical representations of the FTC can form a mechanism to facilitate the transition from students’ initial image of a mathematical concept, which Tall calls the procept, to the concept itself, and that this transition is a function of the learning opportunities students have in the classroom.

By articulating the relationship between the use of multiple representations in the enacted Calculus curriculum and the learning preferences of female students, thought can be given to how to teach women in a more effective and productive manner. This study sheds new light on the way in which multiple representations are intended to support learning and how they appear in the enacted curriculum, and articulates the connection
between different representations of the FTC and the meaning students make of the theorem. To understand what support mechanisms in the enacted curriculum make for more effective learning and a richer Calculus experience, the structure of lessons should be examined and altered. Studying what actually happens in the classroom and the ways students use different representations provides insight into how students build ideas and contributes to our understanding of the difficulties students face when studying mathematics (Davis & Maher, 1997).
CHAPTER 2

REVIEW OF LITERATURE

Research about teaching and learning that has shaped the Calculus curriculum and the crucial role of multiple representations in learning, especially as they impact female students, was used to inform research questions in this study and provided a strong platform for the exploration of the nexus between the enacted curriculum and multiple representations. The literature review naturally led to the question of how students’ experience with multiple representations in today's classroom affected their understanding of the Fundamental Theorem of Calculus.

Multiple Representations

Representations: Definition and Role of Representations

A representation is a structure or a scheme that stands for something else (Goldin & Shteingold, 2001; Palmer, 1978). For example, a picture of this author or the author’s written or spoken name could all stand for the author. Road maps represent roads and towns, and diagrams of car engines, printers, and cameras also stand in for the actual objects themselves. Mathematics uses different representations for the same concept. Symbolic, graphical, and numerical representations are the most common representations used in mathematics. The symbolic ones, formulas, are also called analytical representations. Numerical representations can be tables of data.

Systems of representation can be classified as internal or external (Goldin & Janvier, 1998). External systems are representations that can be used to communicate to other people. The Cartesian coordinate system, algebraic formulas, area models for multiplication, or data tables are all part of external systems of representations.
Conversely, internal systems are related to the learner’s way of making sense of the particulars of a mathematical notation. Internal representations are what happens inside one’s head and can only be manipulated by the learner. External representations are situated in the classroom (Cobb, Yackel & Wood, 1992), and can be seen in physical situations that embody mathematical ideas (Goldin & Janvier, 1998); they may be manipulated by others (Goldin, 1998) and they are shared. Meaning in mathematics is communicated through representations. Different representations (graphs, symbols, and data) for a concept may communicate different ideas or meanings for the same concept. And of course, the representation (or notation) is devoid of meaning unless people assign one to it (Pantozzi, 2010).

Kaput (1994) discusses connections between mental structures and notation systems. Notation systems are what we mean by external representations and are used to organize mental structures. The mental structures are the internal representations of a learner by which that person manages his or her learning. For Kaput, learning occurs only if the notational system (or external representations) occurs in connection with the mental structures (organizing internal schemes) of an individual. In these terms, a notation needs interpretation for the learner. This means that the correspondence between the signified and signifier needs to be developed for learning to occur and this correspondence procedure is something that can be addressed in the curriculum.

Choosing Representations

When choosing representations, it is important to know what one wants to communicate or illustrate with the given representation. Various aspects of the world could be highlighted, and not all aspects of the represented world need to be modeled in a
representation. Palmer (1978) includes five aspects of a representational system: (1) the represented world, (2) the representing world, (3) the aspects of the represented world being modeled, (4) the aspects of the representing world doing the modeling, and (5) the correspondences between the two worlds.

In a mathematics course, students are exposed to different representations for the same concept. Although representations may be similar or equivalent, students may not immediately see that this is so. For example, the graph of the parabola $y = x^2$ or $f(x) = x^2$, its algebraic formula, or a numerical correspondence may all emphasize different aspects of this mathematical concept. The graph may emphasize the symmetry of the parabola, or the fact that the range of the function is all positive real numbers and zero, in a way that is not as obvious from the algebraic formula. A table of values is discrete so it does not show the continuity of the function but probably emphasizes the idea of “squaring” in a way that is harder for a graph to convey.

![Figure 1. Graph of the parabola $y = x^2$.](image)

<table>
<thead>
<tr>
<th>x:</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>y:</td>
<td>9</td>
<td>4</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>4</td>
<td>9</td>
</tr>
</tbody>
</table>
Although representations cannot be understood in isolation (Goldin & Shteingold, 2001), and the interaction between two systems of representations is fundamental to learning, various representations can pose difficulties for students especially when the representations convey apparent contradictory information (Sfard, 1998; Tall 1992, 2004; Thompson, 1994).

The FTC is expressed in multiple ways through various representations. Students are exposed to these representations in the classroom or in talking to others, or in learning from their textbooks (Thompson, 1994). However, significant issues arise when interpretations are not concordant with their representations or with other interpretations. For example, two people may have different interpretations or meanings for the same representation: novice learners may have already assigned meanings to the symbols that are not the intended meanings of the representations. How the learners communicate about a given representation may be fruitful in creating meaning. How students come to choose and understand representations is still a fertile subject of inquiry.

Transfer Among Mathematical Representations

It is possible to learn about an environment, by studying maps and descriptions about the place and the activities of its people. Such learning, however, is largely limited to interactions with symbolic expressions—speaking or writing or drawing maps—and is very different from the abilities needed to live and work in the environment successfully. “In learning a conceptual domain, it is possible to confuse representations of concepts with the concept themselves and learn how to manipulate symbolic expressions rather than how to find and use conceptual resources” (Greeno, 1991).
Translation among mathematical representations is such an “obvious component for mathematical understanding,” yet its exercise is often omitted from instruction (Lesh, Landau & Hamilton, 1983). Paradoxically, although students face difficulties using representations it is only through representations that they create meaning (Thompson, 1994). When thinking of linear functions, for example, one student may have in mind a graph that is linear, another student a formulaic representation of the form $f(x) = mx+b$, and the third a numerical format where the rate of change of the dependent variable (with respect to the independent one) is constant. In problem solving, the choice of representation can make a significant difference in the problem-solving trajectory (whether or not the problem is successfully solved). Translations (transforming one representation to another), and transformations within the same representation are important to the development of mathematical concepts and to the application of mathematical ideas (Lesh, Post, & Behr, 1987). Researchers (Janvier, 1987; Lesh, 1987) say translation is best developed in pairs. For example, a graphical representation can be translated to a verbal one or vice-versa.

To use an analogy of Janvier (1987), a representation from a set of possible representations is like a star shaped iceberg, with one point of the star showing, and all other modes being hidden, as seen in Figure 2. Janvier uses the word “schematization,” or illustration, to widen the concept of translation, and the word “contamination” to illustrate a common problem experienced when features of one representation are being transferred to another. A good representation system captures exactly the features of a problem that are important for that context rather than trying to represent everything (Goldin & Shteingold, 2001).
Translations can be direct or indirect. Direct translations go from one representation to another without the assistance of an intermediary representation. Indirect translations involve translations that use one (or more) intermediary modes of representation to go from what Janvier calls source to target. The source could be the graphical representation and the target the verbal representation. For example, Janvier suggests that students are exposed to all sort of direct translations as in the grid below.

Table 2. Representational direct translations model by Janvier, 1987.

<table>
<thead>
<tr>
<th>From-To</th>
<th>Situations, pictures and verbal descriptions</th>
<th>Tables</th>
<th>Graphs</th>
<th>Formulas</th>
</tr>
</thead>
<tbody>
<tr>
<td>Situations, pictures, verbal descriptions</td>
<td>–</td>
<td>Measuring</td>
<td>Modeling</td>
<td>Analytical Modeling</td>
</tr>
<tr>
<td>Tables</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Graphs</td>
<td>Reading</td>
<td></td>
<td></td>
<td>Fitting</td>
</tr>
<tr>
<td></td>
<td>Interpretation</td>
<td>Reading Off</td>
<td></td>
<td>Curve</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Fitting</td>
</tr>
<tr>
<td>Formulas</td>
<td>Parameter Recognition</td>
<td>Computing</td>
<td>Sketching</td>
<td></td>
</tr>
</tbody>
</table>

Sfard (1997) positions a developed mathematical concept, which is static, at the center of Janvier’s star, and the points of the star (or the representations) are what she calls “conceptions.” Other researchers (e.g. David Tall, Patrick Thompson and Ed Dubinski) have also formed related views about translating among representations. Despite the difficulties entailed in changing representations (Schoenfeld, 1989;
Sfard, Tall, 1997; Tall, 1992), Tall notes that “switching one representation to another is a hallmark of mathematical success” (Tall 1992). He terms this movement between representations “versatile thinking” (Tall 1992).

**Multiple Representations in Mathematics**

When students can move between different representations, they are sometimes able to develop new concepts (Lesh, Post, & Behr, 1987). In addition, multiple representations can also be used to diagnose students’ learning difficulties or to identify learning opportunities and to provide access to mathematics to people of various learning styles. The types of representations as presented by Lesh, Post and Behr are shown in Figure 3 and include: 1) Scripts, which refer to knowledge around real-world situations, context; 2) Manipulatives such as fraction bars that can serve to model situations; 3) Static Pictures or Diagrams that can be internalized as images; 4) Symbols that can be used to write the mathematical relationship and; 5) Language that can be used to verbalize the mathematical idea.

The diagram was revised for functions by Van de Walle in 2004 and Smith, Silver & Stein in 2005. Functions are fundamental objects in the study of mathematics and are building blocks for Calculus concepts. According to Smith, Silver & Stein (2005), functions can be examined by (1) a physical or pictorial representation of the pattern; (2) a chart or table of data; (3) an equation or formulaic representation; (4) a graphical representation and; (5) a language based representation as shown in the second diagram. Here, the context is the situation outside the world of mathematics that would give rise to that mathematical object within mathematics. As seen in Figure 4, the vertices are all
connected, suggesting the interdependence of these representations. These ideas are very close to Janvier’s star of 1987.

Figure 3. Types of representations systems (Lesh, Post & Behr).

Figure 4. Diagram of five representations of functions (Van de Walle, 2004).
Diagrams and Representations

Expertise in mathematics almost always involves navigating through different representations with ease (Sfard, 1997; Tall, 2004). Students who are not able to fruitfully work with these various representations may never reach expert status in this subject (NCTM, 2000; Schoenfeld, 1992). Thus, educators should use them regularly in the curriculum to enable students to become more proficient navigating among different representations. Of importance are mathematical diagrams. These are used by mathematicians to emphasize certain mathematical relations or properties of the represented concept, while omitting others as the situation dictates. Familiar and frequently used mathematical diagrams include knot diagrams, Venn diagrams, and circuits. Some diagrams aid learners with visualization of particular properties.

Martin Gardener (1993) offers a wonderful passage about the role of diagrams in learning mathematics:

There is no more effective aid in understanding certain algebraic identities than a good diagram. One could, of course, manipulate algebraic symbols to obtain proofs, but in many cases a dull proof can be supplemented by a geometric analogue so simple and beautiful that the truth of a theorem is almost seen at a glance. (Gardner, 1993)

The square of a binomial formula and its representation (proof) via geometry are illustrated below: \(a^2 + 2ab + b^2 = (a + b)^2\). The area of the large square is \((a + b)^2\) and it can be seen as the sum of the areas of two smaller squares and two rectangles, which is \(a^2 + 2ab + b^2\). As an aside, Greek mathematicians initially proved their algebraic identities such as the one above, by relating them to geometrical forms like this one, showing algebra and geometry complementing each other in a profound way.
Figure 5. The square of a binomial (Gardner, 1973).

**Visualization and Imagery**

Schoenfeld (1992) and other promoters of reform have shown the extent to which mathematics pedagogy does not tap into imagery or visual representations. Visualization deals with perception and manipulation of mental images (Aspinwall, 1994; Presmeg, 1986) and when students are not exposed to these means of thinking, they may be unable to solve a problem because they do not make a model. First, imagery does not arise spontaneously. Most students do not know how to draw diagrams, or are reluctant to do so (Cuoco, Goldenberg & Mark, 1996; Presmeg, 2001). Many students do not use analogies in solving transfer problems even if they are able to generate these analogies (Novick & Holyoak, 1991). However, there is a definite relationship between students’ success in mathematical problem solving and the type of drawing or diagram they provide as shown by Edens & Potter (2008) and van Garderen (2006). These studies indicate there is a positive correlation between students’ problem-solving ability and the type of visual representation they create. Students who create schematic representations, or abstract drawings that include only relevant and helpful information, score higher than students who create pictorial representations or images for the same problem. Students need to learn they should construct representations, but the task of generating the correct
kind of diagrams and of adapting them to appropriate problems seems then to be a very
difficult one.

Visualization may be at times unspoken and thus hard to transfer from participant
to researcher. It may be of a “personal nature” and again hard to articulate or to
understand (Presmeg & Balderas-Canas, 2001). Presmeg and Balderas-Canas studied
visualization and affect in non-routine problem solving by four mathematics education
graduate students. Imagery evidenced in the study came in three types: graphical, gestural
and verbal, and was exemplified in two types of processes, namely that of “making
sense” of a problem and that of “solving” a specific problem. The study included an
analysis of both the solution process for each participant and for each problem, and of the
cognitive and affective issues regarding each participant’s use of imagery. Visualization,
or the creation and interpretation of images to depict and communicate information and
advance understanding, was reported by all participants, even in places where no diagram
was drawn. This is an interesting study in that it not only reports that students do use
imagery, but it also shows how and why they used such visualizations.

Although the study does not look at the participants’ undergraduate mathematics
curriculum, it does report cognitive and affective responses in the use of diagrams and
images that could indicate inexperience with mathematical representations other than
symbolic ones. Almost exclusively, the participants only used imagery in the initial
preparatory stage of the problem or as hindsight to check that the answer they obtained
made sense. However, they were reluctant, afraid, or unsure to use it while solving the
problems, and relied on methods with which they were more familiar, such as algebra. If
participants had been encouraged to use visualization (like experts do), their problem-
solving trajectory may have been different. The results of the student Think-Alouds in this study show a similar pattern, when participants revert to symbolic use for most of their problem-solving.

**Representational Framework for Learning Mathematics**

Gray and Tall (1994) show how procepts (the combination of processes and concepts) are linked to multiple representations and in fact permeate ALL concepts in mathematics. In Table 3, functions, derivatives and integrals are all examples of procepts and all of them can be viewed through five representations: visuo-spatial, numeric, symbolic, graphic, and formal. The numerical, symbolical, and graphical representations ideas have been discussed. However, Gray and Tall illustrate how these representations permeate mathematics from algebra through Calculus. The visuo-spatial component, which the bottom of the diagram labels as “real world-calculus,” is the context.

Table 3 contains no “verbal” category, but it does indicate a possible kinesthetic domain that one may add to the existing representations, which appears in the first “enactive,” or “experiencing”.

This table provides a scheme to evaluate and compare current texts and curricular materials and many mathematical ideas fit nicely in this representation. Tall’s description of concepts like functions, derivatives, and integrals as procepts emphasizes enactment or activity needed to move from processes to concepts.
Table 3. Mathematical Representations (Gray & Tall, 1994).

<table>
<thead>
<tr>
<th>Procepts</th>
<th>Representations</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Visual-spatial</td>
</tr>
<tr>
<td>Change: FUNCTION</td>
<td>doing</td>
</tr>
<tr>
<td></td>
<td>undoing</td>
</tr>
<tr>
<td>Rate of change: DERIVATIVE</td>
<td>doing</td>
</tr>
<tr>
<td></td>
<td>undoing</td>
</tr>
<tr>
<td>Cumulative growth: INTEGRAL</td>
<td>doing</td>
</tr>
<tr>
<td></td>
<td>undoing</td>
</tr>
<tr>
<td>REAL-WORLD CALCULUS</td>
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</tbody>
</table>

Representational Preference and Problem-Solving Methods in Calculus

Few studies relate representational preferences to problem solving trajectories by students when participants are given different representational input for problems. In a study of 26 Calculus students by Sevimli and Delice (2011) students were permitted to
choose different representations to solve definite integral problems. The study investigated the relationship between preferred representation and differences in cognitive processes involved in the definite integral problem-solving process through multiple representations. Sevimli and Delice’s research dealt with the effect of the input representations in the problem statement on learner preferences. Results show that participants generally preferred algebraic representations and that the visual preference tendencies were influenced by input representations. More specifically, their findings showed harmonic participants, or those adept with both analytic and pictorial representation, preferred numerical representations when input representation in the problem was numeric, and that otherwise, harmonic and analytic participants had similar preference tendencies with both preferring solutions that use algebraic representations regardless of the representational input used. However, the visual participants’ preferences changed according to the input representations. These participants preferred to solve problems in the representation in which the problem was presented, and interestingly, the visual participants also believed that the input representation was the best way to solve the given problem. The study did not examine the relation to gender or curriculum.

Another study on mathematical Calculus performance, preference and gender by Haciomeroglu and Chicken (2012) examined 183 Advanced Placement calculus students in five high schools. The study showed that students’ visual preferences were not influenced by gender, but found statistically significant differences in visual preference scores among high- and low-performing students and suggested that stronger preference for visual thinking was associated with higher mathematical performances.
(Haciomeroglu & Chicken, 2012). The study did not examine the relationship to the enacted curriculum, or to multiple representations of problems.

Outside the calculus realm, Yerushalmy and Schwartz (1992) examined the relationship between curriculum and student preference using representations in problem solving with functions by looking at a reform and a traditional algebra curriculum and student preference for a particular representation when problem-solving. One of the most significant influences on students’ preference was the curricular emphasis on the manipulation of the algebraic representation. Students often viewed graphs as end-products or something extra that was unconnected to the algebraic representation (Yerushalmy & Schwarz, 1992). Traditional curricula emphasize an equation-to-graph direction in teaching functional representations and may have hindered the students’ ability to think of the graph as a correct way to solve a problem (Yerushalmy & Schwarz, 1992). Many Calculus curricula also favor functional representations (Bressoud, 2012), but studies that connect students’ reasoning in Calculus with the classroom experience still need to be developed.

**Aspects of Learning Mathematics**

Piaget, a pioneer of cognitive psychology, shifted our understanding of knowledge development from having a passive model in which the mind acts as a file cabinet to one in which the mind actively constructs and reconstructs knowledge. Learning, like all life processes, requires adaptation and assimilation and the role of the tasks being learned, along with the person’s prior knowledge need to be considered (Dossey, 1992). Because not all learning is cerebral, a theory that only focuses on the individual, in the researcher’s view misses the participatory aspects of learning
mathematics. Thus, a situated perspective was adopted in this study (Cobb & Bowers, 1999; Greeno, Collins & Resnick, 1996; Greeno, 1997; Sfard, 1998).

**Situated Learning**

Social environments that establish an *interactive social context* for discussing, reflecting upon, and collaborating in the mathematical thinking necessary to solve a problem also motivate mathematical thinking (Pea, 1987). The environment, the classroom and the classmates, along with the classroom discussion are all part of the learning process. Lave & Wegner (1991) argues that learning is situated—in other words, learning is connected to the context. Learners need to become involved or engaged in a “community of practice”. A community of practice is determined by: what it is about, how it functions, and what it has produced (Borko, 2004). The situated perspective considers both the individual learner and the learner’s interaction with the community in which structures are formed. Learning and cognition are thus situated processes. According to Greeno (1997) and Brown (1989), both the cognitive and the situated aspect of learning need to be addressed by mathematics education researchers.

**Learning Mathematics as a Social Activity**

Hatano (1991) offers a sound basis for both designing and evaluating curricular materials and classroom objectives in mathematics teaching. He considered knowledge as development by construction that involves restructuring, is constrained so that successive revisions of knowledge by different individuals produce similar but not identical knowledge based on internal (prior knowledge) or external (cultural views) constraints, is acquired domain by domain, and is situated in contexts instead of being a purely cognitive process. Teachers, resources, representations (texts, technology, symbols, and
notation) affect (Hatano, 1991). Hatano’s last point suggests an evaluative stance is based on a situative approach but that considers cognitive studies as well (Greeno, 1997, Hatano, 1991). According to the situative approach, students participate in the development of mathematical practices in their classroom community (Cobb & Yackel, 1996).

The organizing metaphor for situated learning is position with regard to social circumstances (Sfard, 1998). Learning occurs in terms of participation in a social community and mathematical reasoning is developed by communal practices. This metaphor perspective focuses on knowing as an activity situated with respect to an individual’s position in social affairs. In the researcher’s view, knowing and understanding mathematics are used interchangeably. Knowing and understanding mathematics need to be taken “as aspects of participation in social practices, particularly discourse practices in which people engage in sense-making and problem solving using mathematical representations, concepts, and methods as resources.” (Boaler & Greeno, 2000)

But part of knowing mathematics involves being able to move easily among multiple representations of mathematical ideas (Sfard 1998; Tall, 2010). If one adopts a situative framework, as was done by this study, discussions about these representations are a part of the classroom culture—the activities, the student dialogue, and the learning arena. On the other hand, some cognitive aspects of learning mathematics inform the work at hand. While adopting Hatano’s view on situated learning and the construction of mathematical ideas, the works of Ed Dubinski, Ana Sfard, and David Tall are especially useful in thinking about the development of mathematical ideas.
Cognitive Aspects of Learning Mathematics

In studies about learning, mathematics researchers explored the connection between the act of developing a mathematical concept and the mathematical concept itself. Anna Sfard sees that mathematics has both an operational and a structural aspect (Sfard, 1987).

According to Sfard (1997), we need to look at abstract notions in two ways—structurally (static) and operationally (process), and learning involves an interplay between the two. The operational way comes first, as it is more algorithmic, while the structural method is more abstract and comes later (Sfard, 1997). Advanced mathematical constructs, Sfard argued, are totally inaccessible to our senses and can only be seen with our mind’s eyes. This claim led to a pedagogical question as to what class experiences will elicit this process. Functions, numbers, and geometry diagrams are among many representations that need to be translated or given life by the viewer’s mind. In Kaput’s language, these are notations that need to be interpreted for learning to occur (Kaput, 1991). Being able to see these invisible objects (functions, graphs, logarithms, integrals, differential equations, and variables) is a necessary constituent of mathematical ability; lack of this capacity may be one of the major reasons mathematics appears practically impermeable to so many “well-formed minds” (Sfard 1997).

Sfard distinguishes between the ideas of concept (notion) and that of conceptions again with an eye to the fact that mathematical ideas are developed through activity. Concepts, she said, correspond to the official, absolute (and static) mathematical (or scientific) ideas or theoretical constructs. The whole “cluster of internal representations
and associations evoked by the concept” (Sfard, 1997) are the internal, and subjective ideas that exist in different subjects at various times called conceptions.

Seeing a mathematical entity as an object means being capable of referring to it as if it were a real thing—a static structure, existing somewhere in space and time. It also means being able to recognize the idea “at a glance” and to manipulate it as a whole. In contrast, interpreting a notion as a process implies regarding it as a potential. The structural conception is static and integrative, while the operational is dynamic, sequential, and detailed. According to Sfard, the two views are, in fact, complementary. The conceptions belong to the physical/operational domain while concepts live in the structural realm.

Gray and Tall (1994) link this combination of process and concept into a new thing—an initial idea about the concept, which they call procept. Because mathematics is both operational and structural, it may be understood not only as a set of tools whose procedural/operational mastery can lead to solutions to real problems, but also as a structural picture of mathematics—a picture of patterns and relationships (Hatano, 1991). The writer has found students espouse the former algorithmic view, while mathematicians or experts often embrace the structural and more abstract view.

David Tall (2004) explores the three worlds of mathematics which represent stages experienced in mathematical reasoning. These stages consist of: conceptual embodied, “which is world based on perception, action and thought experiment”; proceptual symbolic, “a world of algebraic manipulation compressing processes”; and finally, axiomatic formal, “a world of set-theoretic concept definitions and having its own forms of proof that may be blended together to give a rich variety of ways to think.
mathematically”. Tall's approach expands a prior theory of Dubinsky and colleagues (Asiala, Cottrill, Dubinsky & Schwingendorf, 1996) that suggested a learning development through Actions, Processes, Objects, and Schema (APOS), and the later BAPOS theory in which Chae (2003) added Base Objects (B) as an initial stage of learning. According to these researchers, Actions are routinized as Processes, encapsulated as Objects and embedded in a Schema of knowledge. BAPOS begins with Base Objects on which the individual performs Actions that are coordinated into processes and represented by symbols having meaning as mental Objects, within a wider Schema.

For Ana Sfard learning starts with initial ideas or conceptions, which are processed during an operational stage of learning and developed into structural mathematical objects (Sfard, 1991). She identifies three stages in this development, which are interiorization, condensation and reification. In the first stage, learners perform routine processes on familiar conceptions until the process becomes a mental entity (for example learning to convert from the Cartesian to the polar form of a complex number). In the second stage, a process can be viewed as an entity. So, a complex number could be viewed as an object together with different forms or processes that could be applied to it. The last stage, reification, is the stage when one can identify a mathematical concept as a structural whole. Whereas the first two stages happen gradually, reification is defined as an Eureka moment, “an ontological shift, a sudden ability to see something familiar in a new light” (Sfard, 1991).

Sfard repeatedly draws out the dual nature of mathematics as an object and process. This complementarity can only be inferred in the other researchers like
Dubinsky and Chae and Tall. A possible suggestion as to how Sfard’s notion of mathematics as process and structure falls in alignment with Tall’s Three Worlds of Mathematics and with the (B)APOS theory suggested by Dubinsky and Chae, is outlined in Table 4.

Table 4. Several frameworks for learning mathematics.

<table>
<thead>
<tr>
<th>David Tall Three Worlds of Mathematics</th>
<th>Ed Dubinsky &amp; David Chae (B)APOS Theory</th>
<th>Ana Sfard’s Dual Nature of Mathematics as Process and Structure*</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conceptual Embodied</td>
<td>Base Objects and Actions</td>
<td><strong>Operational State</strong> Conceptions (based on limited or personal knowledge about some but not all representations of a given concept) are interiorized.</td>
</tr>
<tr>
<td>Proceptual Symbolic</td>
<td>Objects and Processes</td>
<td><strong>Operational State</strong> Condensation and Reification of Mathematical Conceptions</td>
</tr>
<tr>
<td>Axiomatic Formal</td>
<td>Schemes</td>
<td><strong>Structural State</strong> (Mathematical Concept can be viewed abstractly to encompass all representations)</td>
</tr>
</tbody>
</table>

**Curriculum**

Conflict, crisis and compromise have shaped the mathematics curriculum of today. Much controversy revolved around who math should be for, what is mathematics and how it should be presented. Some see mathematics as a tool for dealing with the real world. Others consider mathematics a set of rules, axioms and deductive proofs that are divorced from intuition, and whose connections with the sciences or with the real world need to be developed after a grasp of the fundamentals (Jones, 1970). Depending on the views of mathematics, the principal issues in curriculum are approached or answered differently. Although in theory the intended curriculum emphasizes multiple
representations and mathematical habits of mind, in practice most enacted curricula at the level of College Calculus still employ a symbolic and drill approach (MAA, 2012).

**Traditional View**

The traditional approach follows an absolutist framework and focuses on procedural knowledge that is built on definitions, symbols, and isolated skills, along with routine algorithms that support solving different problems. Traditional teaching is primarily expository teaching with little attempt to first focusing on building deep, connected meaning to support mathematical concepts (Skemp, 1987). According to the traditional view, mathematics is procedural and students’ experiences with new mathematical ideas begin with definitions and theorems (Ross, 2001; Wu, 1999). Doing and knowing mathematics then, means memorizing formulas, remembering and following correct rules, and getting the right answer, rather than problem solving and making sense.

**Reform View**

Reform-based teaching focuses on a balanced approach that first emphasizes teaching for conceptual understanding and then on skills. Multiple representations are key. Students meaningfully learn and understand Calculus when they can “move comfortably between symbolic, verbal, numerical, and graphical representations of mathematical ideas” (Roberts, 1996).

The proposed learning framework is BAPOS, developed by Dubinski (1996) and modified by Chae (2003), in which the learner moves from Base Objects through Actions and Processes to Objects and Schemas. Students are required to reason flexibly and to make connections to what they already know. In a reform curriculum, students are
encouraged to extend their prior knowledge and transfer it to new situations (NCTM, 2000). It focuses on mathematical habits of mind (Cuoco, 1996) by seeking solutions, exploring patterns, and formulating conjectures, rather than just memorizing procedures.

Proponents of reform teaching methods want students to move beyond just looking for an answer into concept development, and thus make sense of mathematics (NCTM, 2000).

Hughes-Hallett et al.’s *Calculus: Single and Multivariable* (1991, 2008, 2012) is perhaps the most renowned reform text in Calculus. It is written under the sponsorship of NSF and collaboration of 15 authors from the Harvard NSF-sponsored Calculus Reform Consortium. This is important as it represents an area of agreement for a sizeable number of professional mathematicians. One of the principles is the "Rule of Four," which stipulates that topics should be taught graphically, numerically, verbally and analytically, so that ideas are balanced, and students see each major idea from several angles (Hughes-Hallett, 1991). The "Rule of Four" is about multiple representations.

Research supports the approach (Hiebert, 1999, 2003). Students who develop conceptual knowledge first, perform better on procedural tests later; they show greater retention and an increased likelihood to use ideas in other contexts (Grouws & Cebulla, 2000). By contrast, “if students over-practice procedures before they understand them, it is more difficult to make sense of them later” (Hiebert et al., 2003); and “if students are initially drilled too much on isolated skills, they have a harder time making sense of them later.” (Grouws & Cebulla, 2000). In a reform environment, math is an activity and knowing means actively doing mathematics. Students apply their prior knowledge to generate conjectures and they question these conjectures and self-correct, when necessary (Lakatos, 1963; Lampert, 1990).
Curriculum Definitions

Educators refer to different forms of curriculum, among which are the intended, the written, the enacted, and the achieved curriculum. For the purposes of this work the following definitions based on Valverde (2002) are used:

1. **Intended Curriculum:** The learning standards or expectations for a course or school program, as established by local, state, national or college agents. The intended curriculum may also include or guide the development of textbooks and assessments.

2. **Written Curriculum:** The materials developed by publishers to implement the intended curriculum.

3. **Enacted Curriculum:** The curriculum enacted by teachers in everyday teaching as they make decisions to implement the intended curriculum. The enacted (or implemented) curriculum is viewed here as situated and as referring to students’ opportunity to learn.

4. **Achieved Curriculum:** What students learned from a curricular unit.

Differences in philosophical orientations toward learning and teaching influence the intended, written and the enacted curricula. Mathematics curricula have experienced several shifts in content emphasis over the past century. Early in the twentieth century intended and implemented curricula focused on drill and practice, while midway through the century they began to focus on meaningful mathematics (Kloosterman & Walcott, 2010). After the back-to-basics short-lived movement in the 1970’s, the publication of *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989), moved the focus to conceptual understanding once again. Curricular materials have a powerful
influence on the intended curriculum. However, they are not the determining factor of what students learn. Rather, how the curriculum is enacted is a significant indicator of what the students have the opportunity to learn (Thompson & Senk, 2010; Valverde, 2002). Even at the K-12 level most teachers do not teach all the topics in their books, and often teach different lessons. Teachers use materials based on their various contextual and personal perceptions about the needs of their students and their own understanding of the mathematics (Stein, Remillard & Smith, 2007).

In the *intended curriculum* set forth by organizations such as NCTM (2000) students are asked to explain, justify or evaluate conjectures, to look for multiple solutions to problems and to use multiple representations of concepts. This pedagogy has been outlined in the *written curriculum* in several publications such as NCTM’s Principles and Standards for School Mathematics (2000) and NSF-funded curriculum materials like the Harvard reform-based Hughes-Hallett Calculus texts, the Core-Plus Mathematics (CPMP) and the Integrated Mathematics Program (IMP) curriculum materials (CPMP, 2015; Hughes-Hallett, Gleason, McCallum, et al., 2008; IMP, 2009; NCTM, 2000). Reform pedagogy does not stress algorithms, but rather asks students to make arguments to explain why answers make sense and to revise solutions (Lampert, 1990; NCTM, 2000; Schoenfeld, 1992; Sowder, 1998).

Professionally we realize that not all tasks provide the same opportunities for learning and those with highest cognitive demand are the hardest to implement and hardest to implement well (National Council of Teachers of Mathematics, 2014).
Multiple Representations in the Enacted Curriculum

Pestalozzi claimed that education is ineffective unless it comes from the learner and praised an inductive approach (Wiloughby, 2010), but instead of evidence that learning should start with the learner’s reality, classroom practice and the enacted curriculum have seldom approached that goal (Wiloughby, 2010).

Gantner (2001) and others (Leitzel & Tucker, 1994) report that colleges have changed their approaches to teaching Calculus over the past 20 years, placing a greater “emphasis on concepts” and “multiple representations of functions,” but other studies show the opposite (MAA, 2012), which may account for many conceptual difficulties students still face (Tall, 2004, 2009, 2010: Thompson & Silverman, 2007). Multiple representations are often missing or underutilized in teaching practice, as evidenced by the fact that most teachers still prefer to teach procedures (Arcavi, 2003; MAA, 2012). In a traditional curriculum, students learn the things that will get them through the exams (Tall, 1992) and when students meet difficulties, they concentrate on the procedural aspects that are usually set in examinations (Tall, 1991).

Mathematician P. Halmos has said that “the best way to learn is to do; and the worst way to teach is to talk” (Bressoud, 2011; Halmos, Moise & Piranian, 1975), but despite changes in teaching recommendations more than two–thirds of the Calculus professors still believe lecturing is the best way to teach (Bressoud, 2010). There is a uniformity to the college Calculus classroom. Most use similar texts. Stewart, a traditional text, was used in approximately 43% of the classes, and Hughes-Hallett, a reform text, in about 20%, with a variety of other texts used in the remaining other 37% of classes (Bressoud, 2011). Eighty percent of the classrooms had no more than 40
students and 70% of the instructors were male. In 90% of the exams, over 70% of the items were coded as either “remember” or “recall and apply procedure” and 89.4% of all exam items required students to perform symbolic computations with little emphasis on graphical or verbal representations (Rasmussen, 2005). Words such as “understand,” “analyze,” and “explain” were nearly absent.

In the enacted curriculum many instructors may choose algorithmic presentations to present to their students (Arcavi, 2003) because they seem efficient and because it is what they know. In addition, some mathematical communities hold the belief that “visual solutions are not mathematical” (Guzman, 2002).

The absence of visual representations in instruction can be problematic. Learners need to create what are called visual schemes to encapsulate information even if this is not directly addressed in the curriculum (Kaput, 1994). Images constructed in the learner’s mind for a specific mathematical object are recalled by the learner when the mathematical object appears. When the image the learner has is faulty, difficulties can arise (Aspinwall, Shaw, & Presmeg, 1997; Carlson, Jacobs, Coe, & Hsu, 2002; Thompson, 1994). Because a diagram or picture can enhance learning, giving students the chance to create and to discuss diagrams within the mathematics classroom is very important.

Calculus reform has demonstrated encouraging results. In recent years, women students are more likely to take advanced mathematics courses by the time they graduate (Bryant, 2011; NCES, 2005). Calculus concepts, such as rate of change and accumulated growth may be introduced as early as middle school. Early exposure to Calculus related concepts makes it more likely to reach more learners (NCTM, 2000). College-bound high school students are more likely to take Calculus as a high-school course prior to
encountering it for the first time in college. A 2011 NSF sponsored survey of Calculus I college courses that included more than 14,000 students, found that 61% had taken calculus in high school. Still, theory does not always agree with practice. In the classroom, most time remains dedicated to lecture and assessment methods continue to emphasize procedural work (MAA, 2012). If we are to improve access to STEM careers for all students, a first step is to revisit the way we teach Calculus.

Thus, there remain significant gaps between the intended curriculum (envisioned by curriculum makers), the enacted curriculum (what happens in the classroom), and the achieved curriculum (what students get out of the classroom experience) (Aspinwall, 2009; Tall, 2012; Thompson, 1994).

**Cognitive Demand of Mathematical Tasks**

The cognitive demand of a task is a concept that was developed in a study of classrooms participating in the QUASAR project, which was a national educational reform project which aimed to foster and study the development of enhanced mathematics instructional programs for students who were attending middle schools in economically disadvantaged areas (Silver & Stein, 1996). Mathematical tasks should be worthwhile. They should engage learners into discovering new mathematical grounds and be chosen at a level appropriate to the students’ preparation. One measure of how a task is maintained in the enacted curriculum in terms of its cognitive demand is the Task Analysis Guide and the Mathematics Task Framework which was used in large scale empirical studies (Stein & Smith, 1998; Stein, Smith, Henningsen, & Silver, 2000). The Task Analysis Guide for Mathematics developed by Stein et. al. was used (Stein, M.K. et al., 2000) has been useful in this work by providing an additional lens through which to
analyze the enacted curriculum. The tasks or activities are organized into lower level demand task and higher level demand tasks, as illustrated in Table 5.

Table 5. Cognitive demand of mathematical tasks (Stein, 2000).

<table>
<thead>
<tr>
<th>Lower Level Demands</th>
<th>Higher Level Demands</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Memorization</strong></td>
<td><strong>Procedures with Connections</strong></td>
</tr>
<tr>
<td>• Involves reproducing previously learned facts, rules, formulas or definitions or committing these to memory.</td>
<td>• Students are guided for particular understanding or content.</td>
</tr>
<tr>
<td>• Cannot be solved using procedures because a procedure does not exist or because the time frame for the task is too short.</td>
<td>• Focus students’ attention on the use of procedures for the purpose of developing deeper levels of understanding of mathematical concepts and ideas.</td>
</tr>
<tr>
<td>• Is not ambiguous. Such tasks involve the exact reproduction of previously seen material, and what is to be reproduced is clearly and directly stated.</td>
<td>• Suggest explicitly or implicitly pathways to follow that are broad general procedures that have close connections to underlying conceptual ideas as opposed to narrow algorithms that are opaque with respect to underlying concepts.</td>
</tr>
<tr>
<td>• Has no connection to the concepts or meaning that underlie the facts, rules, formulas or definitions being used.</td>
<td><strong>Procedures with Connections (contd.)</strong></td>
</tr>
<tr>
<td></td>
<td>• Usually are represented in multiple ways, such as visual diagrams, manipulatives, symbols, and problem situations. Making connections among multiple representations helps develop meaning.</td>
</tr>
<tr>
<td></td>
<td>• Require some degree of cognitive effort. Although general procedures may be followed, they cannot be followed mindlessly. Students need to engage with conceptual ideas that underlie the procedures to complete the task successfully and that develop understanding.</td>
</tr>
<tr>
<td><strong>Procedures without Connections</strong></td>
<td><strong>Doing Mathematics</strong></td>
</tr>
<tr>
<td>• Is algorithmic or scripted. The use of a procedure or script is specifically called for or is evident from prior instruction and/or experience.</td>
<td>• Require complex and non-algorithmic thinking—a predictable, well-rehearsed approach or pathway is not explicitly suggested by the task, task instructions, or a worked-out example.</td>
</tr>
<tr>
<td>• Requires limited cognitive demand for successful completion. Limited ambiguity exists about what needs to be done and how to do it.</td>
<td>• Require students to explore and understand the nature of mathematical concepts, processes, or relationships.</td>
</tr>
<tr>
<td>• Have no connection to the concepts or meaning that underlie the procedure being used.</td>
<td>• Demand self-monitoring or self-regulation of one’s own cognitive processes.</td>
</tr>
<tr>
<td>Lower Level Demands</td>
<td>Higher Level Demands</td>
</tr>
<tr>
<td>-----------------------------------------------------------------------------------</td>
<td>-------------------------------------------------------------------------------------</td>
</tr>
<tr>
<td>• Are focused on producing correct answers instead of on developing mathematical understanding.</td>
<td>• Require students to access relevant knowledge and experiences and make appropriate use of them in working through the task.</td>
</tr>
<tr>
<td>• Require no explanations, or explanations that focus solely on describing the procedure that was used</td>
<td>• Require students to analyze the task and actively examine task constraints that may limit possible solution strategies and solutions.</td>
</tr>
<tr>
<td></td>
<td>• Require considerable cognitive effort and may involve some level of anxiety for the student because of the unpredictable nature of the task required.</td>
</tr>
<tr>
<td></td>
<td>• Engaged in the practices of doing mathematics.</td>
</tr>
</tbody>
</table>

Even with worthwhile tasks, the way tasks are enacted in the curriculum may change the nature of the task, which in turn may affect student learning. The enacted curriculum should scaffold the development of new ideas, press students to provide explanations and make meaningful connections. Demand can be altered by a multitude of factors, including the level of the task, time allotted to the task, too much or too little time allotted for working on the task, a lack of accountability, and a shift in focus from ideas to correct answers (Henningsen & Stein, 2002).

**Conclusion**

Knowledge is situated (Hatano, 1991; Sfard, 1987) so that the right context (Boaler, 1997) is essential in creating meaningful learning. Contexts, activities and lessons ensure that the concept developed is used in an appropriate way. Situations co-produce knowledge through activity. Teaching must include students’ prior knowledge or have an experiential basis that allows for active participation and discourse to construct knowledge. Following Sfard, the view adopted in this paper is that mathematics is both operational and structural, and assumes that mathematical concepts have multiple
representations which need to be addressed in classroom instruction. Learning is a process that moves the learner through stages of learning from a more concrete and experiential stage to a more abstract one. Multiple representations are the medium for this transfer and are fundamental to understanding mathematics (Van de Walle, 2004; Sfard, 1997; Tall, 2010). Mathematical representations and technology are usually imbedded in the enacted curriculum. How this is done is in turn is influenced by the views of mathematics that curriculum makers and teachers have, and different views of mathematics (traditional or reform) have been shown to color how mathematics is taught. Even after Calculus Reform, most teachers and curricula emphasize traditional teaching, procedural work and have little use of multiple representations and other sense-making activities. The Fundamental Theorem of Calculus (FTC) is a difficult theorem for students to master in Calculus.

**Gender and Mathematics**

Women are less likely than men to major in mathematics or pursue careers that require mathematics (U.S. Department of Education, 2005). In addition, once in these fields, women are twice as likely as men to drop out of them. (Halpern, Wai & Saw, 2005; Leder, 1992; Meyer & Koehler, 1990). Many factors affect women’s participation in mathematics. Women have been traditionally excluded from the fields of mathematics and technology (Armstrong & Price, 1982), and despite years of intentional work to make STEM fields more interesting, women still largely avoid them (NSF, 2011). There are many factors that contribute to these statistics, but for many women Calculus is still a “filter,” rather than a “pump” (NCES, 2005).
Historical Role of Women in STEM

Historically, women have been given little access to opportunities to develop the skills, knowledge and the social connections to advance in STEM disciplines. In the nineteenth and early twentieth century, most education was reserved for male elite and women were restricted to domestic curricula (Delamont, 1989). In the early part of the twentieth century, many colleges and universities did not admit women (Armstrong & Kahl, 1979; Bosse & Hurd, 2002). The lack of access to advanced study in any discipline, and later in the sciences, was part of an indication of the cultural and commonly held belief that women were less intellectually capable than men (Clements, 1979). Male scientists and mathematicians had both access and control of most professional activities in this domain (Bosse & Hurd, 2002), and the few women who succeeded in carving out significant careers in mathematics or in science are precisely those women who did have access to some of the resources their male counterparts enjoyed. Typically, these women had male mentors and had connections with men in these fields. Emily Noether, Sophia Kowaleska and Julia Robinson are some examples in mathematics. They were in contact with gate openers, who afforded opportunities, support, and credibility (Bosse & Hurd, 2002).

Today’s Situation for Women in STEM

Many older studies about women and mathematics found that male students outperformed female students in mathematical tests, and that differences emerge between 13–16 years of age. The meta-analysis of Hyde, Fennema & Lamon (1990) shows that while females showed a slight superiority in performance in the elementary and middle school years, this was replaced by a moderate male superiority in the high school years.
that increased in the college years. The gender difference in understanding of concepts was essentially zero, but there was a moderate disparity in problem-solving in the high school and college years. The moderate difference persisted on tests with mixed cognitive levels (Hyde, Fennema & Lamon, 1990). As a note, the meta-analysis did not support prior claims that “males outperform females,” (Halpern, 2013), but revealed the complexity of the issue. While females were better overall in computation, there were no differences in conceptual understanding, and gender differences favoring males in problem-solving did not emerge until high school. Because of the complexity, the reviewer limits her work on gender to the pedagogical—using multiple representations, technology, small group interactions, and the classroom culture—while acknowledging that males have controlled and dominated STEM fields.

A five-year study, started in 2012, of Calculus I instruction at universities in the United States done by the Mathematical Association of America (MAA), shows college calculus instruction to be troublesome for all students and especially for women (Bressoud, 2011). Half of the 34,000 students surveyed in the first year of the study got D’s and F’s even though 61% of those students had Calculus in high school and 58% thought they would get A’s. Rasmussen, Ellis, and Duncan further analyzed the data to understand why STEM students who were in Calculus I initially had decided to switch out of STEM (Rasmussen, 2012). Findings about the gender makeup of switchers reveal that switchers are disproportionally female. Only 41.5% of the STEM intending students were female. However, 56.1% of the switchers were female (Rasmussen, 2012). This finding raises serious concerns as to why women, already under-represented in STEM, are disproportionally choosing to switch out of the STEM trajectory. Students who
switched from STEM after Calculus I reported being less engaged during the class than the students who persisted and the majority cite their experience as a main factor (Rasmussen & Ellis, 2013). Switchers reported their instructors were less likely to actively engage them, they were less likely to contribute to class discussion and they more frequently found themselves to be lost in class (Rasmussen & Ellis, 2013).

To reap the benefits of a fully participatory society, mathematics and mathematics education needs to be equitable and accessible to all learners. Women may have different learning styles than men and prefer collaboration (Rosser, 1993). Some of the factors pushing women away from STEM careers include cultural influences; lack of confidence, support and role models, educational treatment that favors male students, and teaching out of context (Fennema & Sherman, 1977, 1978).

The world of mathematics is a male dominated world and it is not collaborative. Professional mathematicians seldom work in groups. For the most part, they work in isolation. The majority of faculty at colleges and universities are men (Kirkman, Maxwell, & Rosse, 2004; MAA, 2012). Female students in STEM cite that having female faculty mentors and collaboration as a great benefit to them in the field (Bryant, 2011), so the lack of female mentors is a big problem for female students in mathematics. Mathematical discourse has been often characterized as “confrontational” and “competitive,” which is not compatible with the way that female students learn (Boaler, 1997, 2002; Bryant, 2011). In addition, the very idea that mathematics is a male domain is so prevalent in our society that women may be reluctant to participate in a field that would label them as socially deviant (Damarin, 2000). According to Damarin (2000), mathematicians are often portrayed as abnormal and socially incompetent, so women
who show an interest in mathematics, may be afraid to be viewed as socially deviant, removed from social or family relationships, and have their femininity called into question (Bryant, 2011; Damarin, 2008).

**Stereotype Threat**

Being outnumbered in a course or in a testing environment may cause females to suffer from stereotype threat. Stereotype threat occurs when targets of stereotypes alleging intellectual inferiority are consistently reminded of the possibility of confirming these stereotypes (Aronson et al., 1999; Spencer, Steele, & Quinn, 1999). A study by Inzlicht and Ben-Zeev (2000) argues that placing women in an environment where they have any contact with male students may create a threatening environment and impinge on their performance. In their study, participants completed a difficult math or verbal test in three-person groups, each of which included two additional people of the same sex as the participant (same-sex condition) or of the opposite sex (minority condition). Female participants in the minority condition experienced performance deficits in the math test only. Male students performed equally well on the math test in the two conditions. Women’s deficits were proportional to the number of males in their group.
Figure 6. Placing women in a minority affects math test aptitude (Inzlicht & Ben-Zeev, 2000).

Figure 7. SAT score as a composition of group and gender (Inzlicht & Ben-Zeev, 2000).
If we are to extrapolate, these results may be related to women studying different representations of the FTC in a typical classroom. Amy Kiefer and Denise Sekaquaptewa (2007) examined the effects of gender identification and implicit and explicit gender stereotyping among undergraduate women enrolled in college-level Calculus courses. Women’s gender identification and gender stereotyping regarding math aptitude were assessed after the course’s first midterm exam. Implicit, but not explicit, stereotyping interacted with gender identification to affect women’s performance on their final exams and their desire to pursue math-related careers. Women who showed low gender identification and low implicit gender stereotyping performed best on the final exam, while women with high scores on both factors were the least inclined to pursue math-based careers.

Figure 8. Gender identification and levels of implicit stereotyping may affect exam scores and career goals (Kiefer & Sekaquaptewa, 2007).
These findings suggest that implicit and explicit stereotypes affect the behavior and implicit beliefs about women’s mathematical competence and may thus contribute (along with other factors) to women’s underrepresentation in mathematics. As learning is connected to societal and classroom behavior, this problem can only in part be addressed by looking at curricular issues.

**Female Learning and Reform Curricula**

Boaler (1997) argues that in the past the non-participation of women in mathematics was dealt with by suggesting ways in which they could change to become more competitive, more confident and essentially more masculine. However, researchers are now explaining that the reason for low interest in STEM for many women is not because of ability or a deficit in knowledge, but more linked with pedagogy (Boaler, 1997; Bryant, 2011; Mura, 1995). They suggest teaching without context, as done in most mathematics courses, may affect girls more than boys and account for their disinterest in furthering math-related careers. According to Boaler (1997), girls fail to pursue math not because they are not good at it, but because “they won’t accept a system which merely encourages rote learning of symbols and equations that mean little or nothing to them” (Boaler, 1997; Bryant, 2011). Boaler’s interviews with underachieving girls illustrate how their underachievement may be linked to the way these women were taught mathematics. She conducted case studies of two schools and longitudinal studies of a year group in each of these schools as the students moved from the 9–13 age group to the 11–16 age group (Boaler, 1997). The aim of her research was to consider the relative effectiveness of two approaches—one traditional textbook and one reform problem based—on the students’ ability to transfer concepts and on their attitude towards
mathematics. Her qualitative and quantitative results indicated that attitudinal and achievement scores in the traditional curriculum predominantly affected girls, who cited disaffection with the lessons and continually scored lower than the boys in their classes. The difference in achievement was greater at the top of the age scale around the age of 16. The in-depth interviews suggested that girls and boys expressed a strong dislike for the textbook coursework. However, the girls expressed the difference as a “quest for understanding,” while for the boys it involved a lack of interest in the “school mathematics game” (Boaler, 1997). Understanding and sense-making were regarded as the most important aspects of learning by 91% of the girls, compared with 65% of the boys, and only 4% of the girls, compared with 24% of the boys, regarded understanding rules and memorizing as most important, with $p < 0.001$ (Boaler, 1997).

Overwhelmingly, both girls and boys wanted to work at their own pace and for girls this was linked with a desire for understanding. As far as achievement is concerned, Boaler’s study illustrated significant gender differences in the two types of classroom environments. In the traditional classroom environment, there were significant disparities in achievement for girls and boys with differences favoring males, while in the problem based approach, there were no significant differences. Boaler found a larger percent of the girls became disaffected by the traditional curriculum, and their disillusionment was related to the closed approach to teaching, which did not allow them to think, and to the competitive learning environment. He also found girls related discussion and collaboration to understanding; their achievement was related to a lack of interest in an approach they did not like; they attributed their lack of understanding to their inability to change the pedagogical traditions in their institution, not themselves; and the disparity
between preferred modes of instruction was greatest for the highest ability girls (Boaler, 1997).

Gender differences in classroom environments emerged in several studies that indicated women perceived more involvement and achieved significantly better in courses that used active learning strategies (Blackett & Tall, 1998; Joiner, Malone & Haimes, 2002). Taken together, the results of Rasmussen et al. and those of Joiner may have implications for the classroom environment: more research is needed to explore how classroom experiences are perceived by students, and how pedagogical activities, when implemented, are implemented in an equitable fashion.

Gallagher (1998) suggests that female students tend to be more conservative in strategies they apply to mathematical problem solving and are more likely than males to adhere to problem solving strategies learned in school. Thus, the lack of models in the classroom to demonstrate the exploratory nature of problem solving will differentially hurt female students more than male students. Problem-solving is critical to many mathematical-related tasks, and what we teach and how we teach becomes an issue of gender equity. Many of the problems on the SAT-M exam can be classified as either “conventional” or “unconventional” (Gallagher, 1998). Conventional problems are routine text problems and can be answered by algorithmic methods; unconventional items are presented infrequently in textbooks and require unusual use of a familiar algorithm or insight. A mixed gendered group of highly able students showed differences in their approach to different conventional and unconventional SAT-M items (Gallagher, 1998). Female students were more likely than male students to correctly solve conventional
problems using algorithmic strategies; male students were more likely than female students to correctly solve the unconventional ones.

Blackett and Tall (1991) showed that versatile learning (this is Tall’s term but it refers to non-routine thinking) in trigonometry (rather than Calculus) using interactive computer graphics would lead to a greater improvement in the performance of girls over boys. The experiment was carried out with 15–year–old students in two schools with matched entry standards, each subdivided by ability into mixed gender groups. They were given three tests—one pretest, and two post-tests—with the last post-tests given eight weeks after the course. The computer representation enabled the students to explore the relationship between numerical and geometric data in an interactive manner. Students were encouraged to make dynamic links between visual and numerical data which is less apparent in a traditional approach. Results showed that experimental boys and girls improved more than control boys or girls for all groups. Interestingly, the control girls’ performance deteriorated compared with that of the control boys’; however, the experimental girls’ performance improved in comparison with the experimental boys’. All save the least able girls in the experimental group eventually surpassed their male counterparts. On the second post-test, control students’ performance deteriorated much more than the experimental group’s performance for all students. The difference between control and experimental boys on the delayed post-tests was statistically non-significant in all groups tested, while the difference between girls was statistically significant. This suggests that multiple representations presented with technology may be beneficial to all learners, but may have very important positive effects on female performance.
Gender differences also emerged in a study by Joiner, Malone and Haimes (2002) which looked at two reform curricula (one computer based, and one without computer), and compared them to a traditional Calculus course. Females had a higher expectation of interaction than male students within the classroom. Relative to males, females also perceived significantly more involvement in the reform classes and significantly less innovation by the teachers. In addition, females were found to achieve significantly better than males in both types of Calculus reform classrooms (Joiner, Malone & Haimes, 2002). These interesting results could potentially be significant in the efforts to increase the numbers of female students in upper level mathematics. The correspondence between curriculum, persistence, and gender suggested by Joiner makes a compelling case for reform courses.

In recent years there has been a great deal of evidence to indicate that students exposed to active learning strategies and collaborative learning in their STEM classes learn better both in high-school (Boaler, 1998; Boaler & Staples 2008) and in college (Ruiz-Primo, Iverson, Talbot & Shephard, 2011; Joiner & Malone, 2001; Kwon, Rasmussen & Allen, 2005). Despite these results, mathematics classroom practice at the college level has been, for the most part, highly traditional (Bressoud, 2012). A study of Inquiry-Based Learning (IBL) as implemented in over 100 courses and at multiple institutions, showed that despite variations in the implementation of the IBL course, students in IBL courses had significantly improved course learning and attitudinal outcomes compared with students in control non-IBL courses (Laursen, Hassi, Kogan & Weston, 2014). According to this study, female students in non-IBL courses reported much lower cognitive and affective gains than male students in the same class. In
contrast, female students in the active curricula, had statistical cognitive and affective gains that were identical to those of male IBL students, and higher collaborative gains.

Pre- to post- changes in students’ interest and confidence differed by gender with more women reporting a substantial decrease in their confidence and intent to take more mathematics, while the opposite was true of IBL courses. IBL pedagogies, benefits all students, and make for a more equitable mathematics classroom.

The next of this review examines the Fundamental Theorem of Calculus.

**Fundamental Theorem of Calculus**

The development of the fundamental theorem started with the Italian mathematician Nicole Oresme around 1350. The statement of the theorem continued changing as various mathematicians such as Isaac Barrow (1670), James Gregory (1668), Isaac Newton (1666), and Augustin–Louis Cauchy (1823) thought about it (Bressoud, 2012). Each examined various aspects of the theorem. The Riemann form of the definite integral, which is the form that appears in current Calculus texts, was not adopted by mathematicians until 1870. Clearly, the FTC has posed significant challenges in the mathematical community, and it should not be surprising that it is also difficult for students (Orton, 1983; Thompson, 1994).

The main Calculus concepts of the rate of change (differentiation) and cumulative growth (integration) deal with functions and how things change. The FTC ties differentiation and integration together by showing they are essentially inverse processes. The statement of the FTC is usually done in two parts, and both the statement and its mathematical explanations are included in the appendix. This section discusses the two parts of the theorem in light of several representations to illustrate that multiple
representations can offer access to understanding this theorem. Without such access students may remain in an “operational” state in their understanding and work within one tip of Janvier’s star, leaving connections to the other tips unexplored, and thus have an incomplete understanding of this theorem.

Statement and Multiple Representations of the Fundamental Theorem

As suggested several times in this review, meaning is made and communicated using representations, and the choice of presentation and discussion may provide different understandings for the subject. This section focuses on the statement and representations of the FTC.

The symbolic representation is the most abstract because students must understand notation in addition to being able to understand concepts such as derivative, differentiability, continuity and so on. The verbal form of the theorem does not contain intimidating symbols, but students still need to understand the meaning of the derivative, of the integral, and the idea of an inverse.

Verbally, part I of the theorem says that: The derivative of the integral of a continuous function f from a to x, is the function f itself. In this sense the derivative and the integral are inverse processes, and the derivative undoes the integral. The verbal statement of the Fundamental Theorem also has a symbolic representation:

If f is a continuous function on [a, b], then the integral function defined by $F(x) = \int_a^x f(t)dt$ is a differentiable function on the interval (a, b) and has derivative given by $F'(x) = \frac{d}{dx}(\int_a^x f(t)dt) = f(x)$

When we look at this theorem graphically or pictorially, its statement appears to be more intuitive. The area under the graph of f(x) from a to x is given by an integral function, namely the function $F(x) = \int_a^x f(t)dt$. The graphical representation of $F(x) = \int_a^x f(t)dt$.
\[ \int_a^x f(t) \, dt \] is the signed area under \( f(x) \) from \( a \) to \( x \) (where the area is positive if it is above the horizontal axis and negative if below). In this context, the theorem says that the rate of change (or the derivative) at which the area changes as we move \( x \), is equal to the height of the function at that point. By analogy, if you unroll a nonrectangular carpet (shaped like the shaded area below) at a constant speed, the rate at which you unroll rug area equals the height of the carpet you are unrolling at that moment.

![Graphical representation of the Fundamental Theorem of Calculus](image)

Figure 9. Fundamental Theorem I in graphical form.

The second part of the FTC deals with the definite integral of a piece-wise continuous function and says symbolically that 
\[ \int_a^b f(x) \, dx = F(b) - F(a), \]
where \( F(x) \) is any particular anti-derivative of \( f(x) \). Here \( F(x) \) is an anti-derivative of \( f \) means that \( F(x) \) has derivative equal to \( f(x) \). Verbally represented, the statement says that the definite integral from \( a \) to \( b \) of a rate of change of a function on the interval \([a, b]\) gives the total change in the function. That is, if we multiply the rate at which \( F \) changes as \( x \) changes (this rate is \( f(x) \)) by the total change in \( x \), and we integrate (or add the changes) we will then find the total amount by which \( F \) changed.

Graphically \( \int_a^b f(x) \, dx \) is represented by the signed area between two instances, \( a \) and \( b \). The theorem says if \( f(x) \) represents the rate of change (derivative) of a quantity \( F(x) \), the total change in the quantity \( F(x) \) on \([a, b]\) (in other words \( F(b) - F(a) \)) is the
same as the signed area bounded by the graph of the derivative of $F(x)$ and the $x$ axis between $x=a$ and $x=b$. 

![Graph showing the area under the curve of $f(x)$ between $a$ and $b$.]

Figure 10. FTC, Part II $\int_a^b f(x)\,dx = F(b) - F(a)$ if $F'(x) = f(x)$

In context, the second part of the FTC appears when there is a need to calculate accumulated growth, such as the total amount of liquid accumulated at a time dependent rate (gallon/minute) between two moments of time, say from time $= 0$ minutes to time $= 25$ minutes, or the distance covered by a projectile moving between two moments in time. It might be that a weight is distributed along a bridge at a linear density (lbs./foot), and we want to calculate the total weight on the bridge between distances of 10 feet to 35 feet.

**Context and Numerical:** Context may be illustrated by using velocity as an example. Velocity is the rate of change of position over time; in other words, velocity is the derivative of position with respect to time. For example, if the speed of a car is constant at 65 miles per hour and if the car travels for an hour, it will then cover $1 \text{ hour} \cdot 65 \text{mi/h} = 65$ miles. Suppose the car odometer is broken but we want to keep track of the distance using the speedometer. This time, the velocity of the car varies, so we need to be cleverer than just multiplying the speed and the time to get the distance. This can be done.
If the car starts out going 10 miles per hour, one can assume it maintains that speed (with minimal change) for a brief period—say, 1 minute. By multiplying 10 miles per hour by one minute (1 min = 1/60 hours) we get the distance traveled in the first minute (in this case $10 \text{ mi/hr} \times \frac{1}{60} \text{ hrs.} = \frac{1}{6} \text{ miles}$). If at the end of a minute one looks at the speedometer to observe that the car now travels at 12 miles per hour, one can determine the distance traveled during the second minute by multiplying 12 miles per hour by $\frac{1}{60}$ of an hour giving $\frac{1}{5}$ miles. For the third minute one may have 15 miles per hour for a minute or $\frac{1}{60}$ hours, and thus travel $\frac{15}{60}$ miles = $\frac{1}{4}$ miles.

Proceeding like this for the entire duration of the trip, the distance could be found by adding the component distances $\frac{1}{6} \text{mi} + \frac{1}{5} \text{mi} + \frac{1}{4} \text{mi} + \ldots$

**Pictorially,** if the velocity is represented by the curve in Figure 11 and if the time intervals are on the x axis, then one could assume that on each subinterval, the velocity is constant and find the distance traveled in that interval by multiplying the velocity by the time interval (represented on the x axis). Graphically, for each time minute subinterval, the change in distance is represented by the area of the rectangle (because area equals length [time duration] times height [speed]). The total distance is approximately equal to the sum the areas of the rectangles. The actual (exact) total distance is the area under the velocity graph.

Figure 11. Area under the velocity graph is the distance traveled.
If the car’s velocity varies very rapidly, there is a need to “divide up the time” into smaller intervals to get an accurate estimate. The time intervals may be every 10 seconds, or every second, or twice a second. As we divide up the time into ever smaller intervals and sum up the products of speed and time in each of those (tiny) intervals, our total sum will certainly approach the correct answer, which is the total distance traveled, and which corresponds to the area under the curve.

Today's textbook authors present integration in numerous ways. Most textbooks begin with differentiation since differentiation is simpler than integration (Hughes-Hallett, 2002), but researchers have suggested alternative approaches to introduce differentiation, integration, and the fundamental theorem of Calculus without a clear picture as to whether one method is more beneficial than another (Tall, 1985, 1986, 1990, 1991; Bressoud, 1992).

More detailed proofs of the FTC are in the appendix. Although the proofs are mine, they are consistent with proofs given in current mathematical texts.

**Students’ Understanding and Difficulties with Calculus**

Students’ difficulties with Calculus are largely conceptual. Difficulties include the derivative and rates of change (Baker, Cooley & Ztrigueros, 2000; Tall, 1992), limits (Tall, 1991), and the integral (Aspinwall, Shaw & Presmeg, 1997; Thompson, 1994). Baker, Cooley and Ztrigueros (2000) show that students can be proficient at differentiating a function and at finding critical values, but they may not be able to conceptualize these ideas or to work with them if they are not presented in equation form. In their study, the researchers analyzed students’ understanding of calculus concepts used to solve non-routine problems. Students were given a problem that required them to graph a function given a set of conditions regarding the first and second derivatives of the function and horizontal or vertical asymptotes. Many of the students in their study were proficient at routine processes that involve calculations. However, the non-routine problems showed that when their understanding of the derivative was incomplete students reverted to procedural knowledge rather than conceptual understanding (Baker et. al., 2000). When students come to conceptual conflicts in problems, two consequences were possible: they could reconcile the old and the new by re-constructing a new coherent knowledge structure, or they could keep the conflicting elements in separate compartments and never let them be brought simultaneously to the conscious mind (Tall, 1992). Students usually choose the latter option (Tall, 1992), because the former is harder to do, and because even if they can see an area of conflict, they have difficulty letting go of what is already in their minds (Aspinwall, Shaw & Presmeg, 1997). A study by Aspinwall (1997) shows how a student’s incorrect image of what a graph looks like creates difficulty for graphing of the derivative. Her student rejected the Calculus
knowledge he had in favor of his incorrect image when this knowledge conflicted with his prior knowledge.

Bressoud (1992) suggested that for some students, the FTC looks more like a definition than a theorem. Students may thus interpret the definite integral as the difference between two anti-derivatives evaluated at specific endpoints, without thinking how such a calculation would involve the area under a specific curve (Bressoud, 1992, 2005). As early as 1917, Graham noted that a student can simply set an integral sign before an expression, evaluate a definite integral, and think no further about the meaning of what he or she has done. This is not so hard to imagine. In algebra courses, students learn that to solve an equation like \((x-3)(x+5) = 0\), one can set each factor equal to zero and then solve two linear equations. Students then sometimes attempt to do the same when solving \((x-3)(x+5) = 7\) because they are incorrectly generalizing that if \(ab=0\) then \(a=0\) or \(b=0\), to \(ab = 7\). Calculus students exhibit a similar behavior. A common error many students make is to think that the integral of a product is the product of the integrals because the integral of a sum is equal to the sum of the integral. Prenowitz (1953) noted that the mode and order of presentation of ideas in calculus would give students certain impressions. He noted, for example, that anti-differentiation can be presented as “the essential idea of integral calculus,” and that “the limit of a sum” would then be “just an interpretation or application of this idea,” leaving the learner without a sense of the importance of the FTC.

Tall (1994, 2004, 2009, and 2010) lists many other difficulties students encounter in the study of Calculus, each worthy of extended investigation:

- Restricted mental images of functions
• Understanding Leibniz notation – a “useful fiction”

• Difficulties in translating real-world problems into Calculus formulation

• Difficulties in selecting and using appropriate representations

• Algebraic manipulation – or lack thereof

• Difficulties in absorbing complex new ideas in a limited time

• Difficulties in handling quantifiers in multiply quantified definitions

• Even students who are able to perform well on routine Calculus may have difficulties on non-routine problems (Selden, Selden, & Mason, 1994; Tall, Smith & Piez, 2008)

Many questions also remain about the role of transferability and of visualization in Calculus (Aspinwall, 2009, Keller & Hirsch, 1998; Presmeg, 2006).

Some studies show that students in Calculus are proficient at differentiating, but have a poor understanding of other representations such as graphical ones. In graphing the derivative, students’ graphical understanding of the derivative soon leaves them, as they revert to procedural knowledge rather than conceptual understanding (Baker, Cooley & Triegueros, 2000). If students do use a graphical method, they tend to use only the first derivative to get all their information about the graph, ignoring, or worse yet, completely misunderstanding, the second derivative. Notably, they cannot understand the relation between the first and second derivative and how this relationship relates to concavity (Aspinwall 2004; Baker, Cooley & Triegueros, 2000; Thompson, 1994).

Presentation of the Fundamental Theorem

The FTC has taken a long time to develop mathematically, and different approaches may be selected to explain this theorem.
According to Thompson (1994), students’ poor concepts of rate of change and its connection to the derivative leads to their poor understanding of the integral and of the fundamental theorem because the FTC deals with rates of change.

In a series of studies of students’ understanding of Calculus, Thompson (1994) reported a teaching experiment with 19 college senior and graduate mathematics students. In his study students had inappropriate images of Riemann sums and were unable to understand Riemann sums in relation to rate of change. Students considered a Riemann sum static. Thompson concluded that students had weak concepts of rate of change and poorly developed and coordinated images. The study suggested that students construct “images” of accumulation, rate of change, and rate of accumulation prior to their coordination and synthesis into the FTC (Thompson, 1994). For Thompson, an “image” is constituted by coordinated fragments of experience from kinesthesia, prior concepts, smell, touch, taste vision and hearing. Images thus construed are also affected by students' past experiences (fear, joy, puzzlements) and are less defined than schemes of action or operation because fear, touch, and smell are more variable and situated (Thompson, 1994). Thompson’s paper did not address the role of curriculum in generating students’ incomplete or incorrect images of the FTC, but it does suggest that a carefully constructed curriculum may play a vital role in developing a complete understanding of the FTC.

Kruteskii (1976) observed three types of reasoning in Calculus: analytic reasoning, geometric reasoning and harmonic reasoning (Kruteskii, 1976). His research categorized students by looking at the development of their verbal–logical component, the development of their visual–pictorial component and how they solved various
categories of problems. Students who were classified as analytic reasoners showed weak development of the visual–pictorial component and primarily solved word problems through algebraic means and by writing equations. Students with geometric reasoning demonstrated a very strong development of the visual component and solved word problems through a graphical/pictorial approach. Students who were harmonic reasoners solved word problems through a combination of graphical and algebraic methods. They showed strong development in both the verbal–logical and the visual–pictorial realms. The verbal–logical realm corresponds to the verbal and analytic points of Janvier’s star, whereas the visual–pictorial is related to the graphical point.

Haciomeroglu, Aspinwall and Presmeg (2009) confirmed results from Thompson and Kruteskii—students show preferential treatments to one representation. They also demonstrate that establishing reversible relations can greatly enhance students’ understanding of the relationship between the derivative and the integral and perhaps clarify misconceptions. Reversibility refers to the ability of establishing two–way relations as opposed to one–way relations which function only in a single direction.

Haciomeroglu, Aspinwall, & Presmeg (2009) asked students to draw the picture of the antiderivative of the graph in Figure 12. Initial graphs by three students (Amy, Bob, and Jack) are below in Figure 13. Amy displays analytical reasoning: she incorrectly knows that “the antiderivative of $1/x$ is $\ln x$ “up to a constant,” but misses including the antiderivative for the left side of the graph. She corrects when asked to go backwards.
Bob and Jack employed imagery to transform the derivative graph into the antiderivative graph, but were unsure what to do around $x = 0$. This study identified only two types of reasoning—visual (Bob and Jack) and analytic (Amy). Participants of Aspinwall’s study attempted to solve the tasks using one representation instead of even thinking of translating among representations. Since the participants’ knowledge was strongly associated with one mathematical representation and weakly associated with other mathematical representations, their one-sided representation or over reliance on one representation impeded their understanding of derivative graphs.
A similar study, done by the author as a graduate student, investigated aspects of student thinking about the FTC in four students who completed Calculus II. The study was motivated by the studies of Hamericoglu (2007) and Hamericoglu, Aspinwall and Presmeg (2009) and investigated the role and presence of visual skills in translating from the graphs of derivatives of functions to the graphs of the corresponding functions. Findings of this study showed that two of the students reasoned entirely graphically, while the other two students showed entirely analytical reasoning. Visual thinking played a key role in the graphical reasoners’ thinking, but was almost absent in the analytical thinkers. The visual students derived all information about their graphs from graphical representation of the derivative (slope of the tangent graph) and relied on this representation only to draw the graph of the function. These students were not as fluent in the procedural portion. On the other hand, visual thinking was not apparent in the analytical reasoners except in their ability to transfer ideas from a graph to an equation. These reasoners did not use any of graphical knowledge to draw their new graphs. The analytical reasoners scored higher on the procedural portion. The four participants in this earlier study had been exposed to various curricula, which incorporated both traditional and reform ideas. However, in this task, the participants were not able to move between the two representations and stayed confined within one type of reasoning. Indeed, none of these four students could fully reason around the different representations of a function as described by Van de Walle (2007). Each student was fluent in some of the domains and not the others. Harmonic thinkers, as defined by Haciomeroglu (2009) and Kruteskii (1976), were not among the four students interviewed. A more detailed study, such as the one here, would explore the connection between students’ classroom experience in
Calculus, their performance on exams, and the how transfer tasks among representations (when present in the curriculum) affects their problem-solving trajectory.

Pantozzi (2009) investigated graphical and analytical problem-solving ability and how students communicate meaning to each other about the FTC. Students in his study (all of whom completed Calculus three years prior) had to communicate the FTC to a novice student. In representing integrals and derivatives as graphs, students communicated certain meanings of the FTC more prominently than others. The meaning of integrals as accumulation functions (of the area under a graph) was the primary meaning they used. The study showed that when allowed to choose the manner of the presentation students could coordinate results and multiple representations to make sense of the FTC together, forming connections in ways that acknowledge Calculus as a cohesive body of knowledge. This finding suggests that discussion and connected experiences for students can encourage them to engage in these kinds of active mathematical behaviors. Students’ conversations, and the connections they made, also point to the importance of collaborative work in mathematics, and remind researchers that mathematical learning is a situated activity and a social one.

**Conclusion**

More than 50% of students who start as STEM majors do not complete it in five years (NCES, 2011). More women than men opt out of STEM degrees and careers (Bressoud, 2012) and many students who left STEM cite Calculus instruction as a reason for their decision (Bressoud, 2012).

Multiple representations are key to understanding the FTC. Still, many students are not learning about multiple representations in the manner recommended by NCTM
Despite recommendations for teachers to include the “Rule of Four,” verbal, graphical, numerical and symbolic, in Calculus teaching. But despite the presence of the “Rule of Four,” or five if one were to add the contextual representation, in textbooks, evidence suggests that students still have a lot of difficulty dealing with multiple representations of the FTC. While they are able to do computations, many students do not know what the computations mean. In a traditional classroom, the learning is teacher-directed, with the teacher setting the agenda and the students following his or her directions. In contrast, NCTM's reform recommendations situate learning, and the roles of the teacher and learners are far different (De Kock, Sleegers & Voeten, 2004). More importantly, if learning is a social activity, then the actual text used in the classroom is less important than the interactions in the classroom community. To examine the enacted curriculum, one needs to look at what happens in individual classrooms.

Many studies are dedicated to the types of difficulties that students have regarding the FTC. However, they fail to convey the relationship between the enacted curriculum and the associated sense students make of various Calculus concepts. How this transition happens is a function of the situated learning environment students have in their classrooms. One area of further investigation identified by the review is to articulate the relationship between how mathematical representations appear in the enacted curriculum and the meaning students make of the FTC. Such an articulation would be beneficial in exploring how representations can be used in the classroom to support
learning, and the relationship between multiple representations of the FTC and female achievement and persistence.

**Important Findings from the Literature Review**

The literature review indicates that mathematics education researchers are in general agreement that learning progresses from the concrete to the abstract and that this process needs to be experientially grounded. Additionally, many researchers have developed ideas about how to promote relational and higher order thinking, but these ideas have not necessarily been translated into practice.

Multiple representations are important to mathematical thinking and current curriculum recommendations emphasize this. Despite recommendations, most Calculus courses appear to be taught in a traditional format and most Calculus exams appear to test for procedures.

A literature review also indicates that female students (in Calculus) may prefer reform classrooms. Female students’ problem-solving trajectory and use of representations may be more aligned with the course they take.

Studies also show students have difficulties transferring ideas among representations. However, most of the studies emphasize translation from a symbolic to graphical form or from a graphical to a symbolic form. Curriculum is also important, and curriculum includes both the text and the classroom environment. Additionally, technology is found to be important in enabling the use of many representations and in grounding ideas.

The review of the literature demonstrates a lack of studies that examine the relation between the enacted curriculum and students’ understanding of the FTC across
multiple representations, as well as a lack of literature examining the relation between gender and understanding of the FTC across representations.

The current study seeks to close this gap by exploring the relationship between the use of multiple representations in the FTC as present in the enacted curriculum and student understanding of the FTC and gender.

A close examination of the relation between how representations are used in “successful” Calculus courses and students’ understanding may be helpful in identifying classroom practices that appear to be more useful than others, either for the entire class or for female students in particular.
CHAPTER 3

METHODS

Introduction

This chapter is a discussion of the methods of this study designed to address gaps identified in the literature review and to operationalize the research questions for the study. The chapter is organized as follows: purpose of the study, research questions, conceptual framework, setting and participants, data collection and data analysis, and researcher’s background.

This study examined students’ experience with multiple representations of the Fundamental Theorem of Calculus (FTC) through the enacted calculus curriculum and student understanding of the FTC. The statement of the FTC is usually done in two parts, both of which have been discussed in the literature review section of the paper. The FTC is often presented to students using various representations, verbal, symbolic, contextual, graphical, and numerical. While all representations are meaningful to experts, some are abstract and require additional decoding for novices. The study addresses the following research questions.

Research Questions

Research Question 1

What is the nature of the relationship between students’ use of multiple representations (MR) in the enacted curriculum and student understanding of the FTC?

Sub-questions:

1.1 In what ways do MR appear in the enacted curriculum?
1.2 What is the nature of the relationship between the use of MR in the classroom and students’ overall understanding of the FTC?

1.3 What roles do other factors, such as representational cognitive preference and perceived representational instruction, play in the ways students communicate about the FTC?

Research Question 2

To what extent does students’ gender influence their use of MRs and their understanding of the FTC?

Sub-questions:

2.1 What is the relation between the use of multiple representations of the FTC in the classroom and female students understanding of the FTC?

2.2 What is the relation between the use of multiple representations in the classroom and female students’ use of multiple representations?

2.3 What is the role of other factors, cognitive preference, perceived representational instruction, and accommodated preference in female student understanding of the FTC?

The underlying hypothesis of the study is that when students are offered substantive experiences working with multiple representations (MR) on tasks related to the FTC, they are more likely to gain a deeper understanding of the theorem. Multiple representations support learning (Tall, 2012; Janvier 1987); students need to see more than one representation for complete understanding. This study hypothesizes that when teachers gravitate toward one or two representations of the FTC, students’ understanding and problem-solving trajectory tend to be more in line with that of their teachers’ despite
their own cognitive representational preference. It is further hypothesized that female students’ problem-solving trajectory aligns more closely with the enacted curriculum than does the problem-solving trajectory of male students.

To answer the research questions, the study examined the enacted FTC curriculum and student understanding of the FTC using multiple representations in three calculus courses during the fall of 2015. A convergent mixed methods approach was used to analyze the results. Quantitative methods were used to investigate the relationship of students’ experience with multiple representations of the FTC. Qualitative methods were used to create a classroom portrait for the enacted curriculum in each of the courses observed. An overview of the study is provided in what follows:

1. During the Fall of 2015, three Calculus sections from different institutions were observed during the duration of the enacted FTC curriculum. The classes had student enrollments ranging from 14 – 22 students per class.

2. Field notes and analytic memos were taken at each class meeting, and a lesson observation protocol (Lesson Observation Protocol) described in Appendix A was completed for each site.

3. At the end of the FTC unit, students in each class were administered a background and cognitive preference questionnaire (Background Questionnaire) presented in Appendix D, and an assessment of student understanding of each of the five (verbal, numerical, contextual, symbolic and graphical) representations of FTC (Five Problems on the FTC) presented in Appendix B.

4. One week after the end of the FTC unit, a subset of students (three from each course – one male and two female) participated in semi-structured interviews.
(Think-Alouds) that further probed students’ understanding of the FTC and problem-solving choices. The Think-Alouds are presented in Appendix C.

5. Based on the researcher’s field notes, the lesson observation protocol completed by the researcher, the Background Questionnaires and student semi-structured interviews, Classroom Portraits that captured the essence of enacted FTC curriculum and the use of multiple representations of the FTC were created for each of the three Calculus sites.

6. Student understanding of Multiple Representations of the FTC captured in the Five Problems Involving the FTC assessment and in the Think-Alouds was analyzed both qualitatively and quantitatively and compared to the Classroom Portraits to answer the research questions.

**Conceptual Framework**

The working hypothesis for this study was that students benefit from substantial experiences with multiple representations (MR) in their study of mathematics as enacted in their courses, in order to have a deeper or a complete understanding of the Fundamental Theorem of Calculus (FTC). The study also posited that students’ understanding of the FTC is connected to their experiences in the classroom. Based on the review of literature, the study hypothesized that female students’ use of mathematical representations would align itself more closely with their classroom experiences than male students’ use of similar representations.

The conceptual framework diagram, Figure 14, shows learning as situated in the classroom activities and discourse. The diagram illustrates the working hypothesis by showing how the Enacted Curriculum may not provide students (and in particular female
students) with a complete understanding of the Fundamental Theorem of Calculus. This incomplete knowledge is represented with a dashed arrow. The Fundamental Theorem of Calculus acts as a potential gate-keeper to mathematical discourse. However, when the Enacted Curriculum is supported by the holistic use of Multiple Representations and ambitious classroom discourse, female students are more likely to develop strategies and skills that give them access to a complete understanding. The strategies include modeling of ideas, risk taking, forming connections with the contextual meanings of the theorem and students’ preferential learning styles as well as encouraging mathematical habits of mind.

**Figure 14. Conceptual framework for classroom discourse and culture.**
Setting and Participants

Setting

The setting for this study was three Calculus courses at three colleges in the northeastern United States during the Fall of 2015: Riverside Community College (RCC), Hudson County Community College (HCCC), and College of Southern New England (CSNE). The decision to select different institutions and different instructors, was driven by the research question that focused on enacted curricula. The schools chosen draw students from different demographics, have different teachers, and offer a broader representation of institutional Calculus teaching practices. Snapshots of the institutions follow.

Site 1: Riverside Community College

Demographics. Riverside Community College (RCC) is located in a semi-urban setting. Calculus courses at RCC average 20-32 students per class. RCC had over 7,000 students at the time of the study, two-thirds of which were full time students. Forty-four percent of full-time students at RCC were 18–21 years, 38% were between 22–26 years, and the remaining 18% were older. Sixty-three percent of the student population identified as White and 23% identified as Hispanic/Latinos. The other 14% was made up of African Americans, Native Americans and Asians. The gender make-up was approximately 62% female and 38% male. Most students taking calculus were STEM majors. The STEM major composition was about 80% male and 20% female. The average class size for courses in STEM disciplines at RCC was 20–30 students per class. The number of student in the RCC Calculus course was 22, with 16 male and 6 female students. The instructor used a Hughes-Hallet calculus text for the class.
**Instructor 1: Professor Rohlin.** Professor Rohlin is a white middle-aged male holding a PhD. in mathematics from a foreign university. He immigrated to the United States approximately 10 years prior to this study and had been teaching calculus for the past seven years. Professor Rohlin has a theoretical mathematics background.

**Site 2: Hudson County Community College**

**Demographics.** At the time of this study Hudson County Community College (HCCC) had approximately 9,000 students, with an average student age of 26 years. Students at HCCC were mostly under the age of 30, with 22% of the student population between the ages of 18-19 years of age, 19% of the population between 20-21 years of age, 31% of the population between 22 and 29 years of age, 23% over the age of 29, and 5% under 18. Forty-eight percent of the student population identified as White and 29% identified as Hispanic/Latinos, 17% was made up of African Americans, and the rest of the students fell in the Other category, which included Native Americans and Asian students. The gender make-up of the student population was approximately 62% female and 38% male. Most students taking calculus were STEM majors. The STEM major composition was more than 70% male. Fifty-eight percent of the student population was female and 42% was male. Calculus courses were taught using one of the Stewart mathematics texts, and supplemented by one hour a week of lab. The average class size for courses in STEM disciplines at RCC was 20–30 students per class. The number of student in the HCCC Calculus course was 14, with 8 male students and 6 female students.

**Instructor 2: Professor Brown.** Professor Brown is a middle-aged white male professor in the math department at Hudson County Community College. He has more
than 12 years of college teaching experience and teaches Calculus I, II, and III on a regular basis. He has a theoretical mathematics background with an interest in topology.

**Site 3: College of Southern New England**

**Demographics.** College of Southern New England (CSNE) is a four-year private institution which offers a few doctorate degrees in select fields. Students enrolled the at the time of the study were typically traditional in age (18-21), included an even mix of male and female students who resided primarily on campus or on college-owned property. Sixty percent of the undergraduate students came from the Northeast corridor of the USA. The ethnic make-up for traditional programs was 78% white, 15% African American, and 7% other, as reported in the college website. The average class size for courses in STEM disciplines at CSNE was 20–30 students per class. Approximately 52% of the students at CSNE are female and 48% were male. The number of student in the CSNE Calculus course was 19, with 7 male students and 12 female students. The instructor used a Lial and Greenwell calculus text for the class.

**Instructor 3: Professor Smith.** Professor Smith is a young, white, male professor at CSNE. He has two years of college teaching experience and is teaching Calculus for the fourth time. He has an interest in making mathematics accessible to more students.

**Participants**

Participants in this study were students enrolled in Calculus courses at each of the three college sites above during the fall semester of 2015. At the end of the Fall 2015 semester when this study occurred, the three classes had a combined total of 55 students: 19 students (12 female and 7 male) from CSNE, 22 students (16 male and 6 female) from RCC, and 14 students (8 male and 6 female) from HCCC.
From these 55 students, a subset of 9 participants (three from each school) were also selected to participate in a Think-Aloud Interview involving multiple representations and the Fundamental Theorem of Calculus and Post Interviews. The students selected for the Think-Aloud were purposefully chosen to represent mid-level grades on this the FTC assessment. Students in three classes had equivalent levels of familiarity with Calculus as assessed through the baseline for prior knowledge discussed prior in this section.

**Data Collection**

The data collected included: 1) researcher’s field notes on the enacted curriculum at the three sites, collected over the period of instruction of the FTC, 2) a Lesson Observation Protocol completed at the end of the FTC instruction, which was used to construct classroom portraits 3) a student Background Questionnaire, 4) an FTC assessment, Five Problems Involving the FTC, used to assess student learning of the FTC and multiple representations, 5) a semi-structured Think Aloud interviews with a subset of the participants. A timeline for the data collection is provided in the Table 6 below.

Table 6. Study timeline.

<table>
<thead>
<tr>
<th>Site</th>
<th>Lesson Observations</th>
<th>Number of Classes</th>
<th>Background Questionnaire</th>
<th>FTC Assessment</th>
<th>Think – Alouds</th>
</tr>
</thead>
<tbody>
<tr>
<td>RCC</td>
<td>Dec. 1 –10</td>
<td>6 (50 min)</td>
<td>Dec. 8</td>
<td>Dec. 9</td>
<td>Dec. 17 – 23</td>
</tr>
<tr>
<td>HCCC</td>
<td>Dec. 3 –10</td>
<td>5 (50 min)</td>
<td>Dec. 9</td>
<td>Dec. 10</td>
<td>Dec. 18 – 24</td>
</tr>
<tr>
<td>CSNE</td>
<td>Dec. 7 –17</td>
<td>4 (75 min)</td>
<td>Dec. 16</td>
<td>Dec. 17</td>
<td>Dec. 28 – 31</td>
</tr>
</tbody>
</table>

Field notes were recorded during each class meeting and analytic memos were created immediately after each meeting. The Lesson Observation Protocol (Appendix A) was based on a tool developed by Western Michigan State University (Jenness & Barley, 1999) and was completed at the end of the instruction period, based on the classroom observations and field notes. At the end of the FTC unit, the Background Questionnaire
(Appendix D), and The Five Problems on the Fundamental Theorem of Calculus (Appendix B) were administered to each student.

The Background Questionnaire (Appendix D) was designed to assess students’ cognitive multiple representational preference (CP) and their perceived representational instruction (PR) after instruction. The Five Problems Involving the Fundamental Theorem of Calculus represent a similar problem given in five representations: graphical, numerical, contextual, verbal and symbolic. Students were unaware that the problems were related. Nine participants (three students, two female and one male, from each course) were also video recorded in a 30-minute Think-Aloud semi-structured interview about one week after the FTC unit.

The data collection tools and modifications are described in more detail below.

Lesson Observation Protocol (LOP) and Classroom Portraits

This research used a Lesson Observation Protocol (Appendix A) (LOP) based on a modified Science and Mathematics Program Improvement (SAMPI) Lesson Observation Tool for making lesson observations. The SAMPI was developed by Western Michigan University and was supported by grants from the Michigan Department of Education Michigan Goals 2000 program (Jenness & Barley, 2003). The SAMPI protocol consists of three main sections: Information about the lesson and classroom, key elements of the lesson, and an optional section that allows one to provide a summary of the lesson. The Information about the Lesson and Classroom section contains questions regarding the classroom arrangement, the purpose of the lesson, and the classroom resources. The second section, Key Elements of the Lesson, contains five sub-categories: planning and organization of the lesson, implementation of the lesson,
content of the lesson, classroom culture, and use of technology. In these five sections scores are based on a seven-point Likert scale. This section of the SAMPI was modified for this study. The modifications included questions pertaining to the use of multiple representations and student gender. A list of modified indicators appears at the end of Appendix A. The LOP also included an added a section, *Character of Multiple Representation in the Lesson*, which captured the presence and quality of multiple representations in each observed lesson. The *Lesson Observation Protocol* was completed at the end of the instruction on the FTC and was based on the complete set of lesson observations at each site, and represents a summary of the classroom observations. The researcher was familiar with the protocol, and had been trained on the SAMPI instrument on which the LOP was based, so special attention to the LOP elements was taken while capturing field notes.

Data captured by the LOP, along with the researcher’s field notes and the *Background Questionnaire*, were used to create *Classroom Portraits* describing the enacted curriculum at each site. Since lessons were observed for the entire duration of the FTC instruction, the *Classroom Portraits* represent a summary of the observations. Each portrait is intended to provide the researcher and the reader with a sense of the enactment of the FTC at that site. To construct the portraits, the researcher read and analyzed the data and used deductive and inductive coding to generate themes and patterns. Initial themes were suggested by the LOP, the research questions and by the researcher’s orientation. They included classroom culture, multiple representations, and teacher-student and student-student interactions. Additional themes, such as the perspective on mathematics, were generated during the coding process.
The Classroom Portrait

The classroom portrait includes data analyses from field notes and Lesson Observation Protocol (LOP) for each of the three courses along the dimensions suggested by the SAMPI (Planning/Organization, Implementation, Classroom Culture, Technology, and Multiple Representations). Since lessons were observed for the entire duration of the FTC instruction, the classroom portrait represents a summary of the observations. The classroom portrait is intended to provide the researcher and the reader with a sense of the enactment of the FTC at each site. Elements of the classroom portrait include:

- Multiple representations in the enacted curriculum as noted in the LOP. This includes an overall score (OER) and individual representation enactment scores (VER, SER, GER, NER, SER), and additional field notes regarding the quality and use of multiple representations.

- Other salient LOP dimensions regarding the classroom culture, lesson implementation, student participation, and sensitivity to issues of gender.

- The Think-Aloud protocol and the student written work produced during the Think-Aloud.

Background Questionnaire

At the end of FTC unit, all students in the three sections observed were presented a Background Questionnaire (Appendix D), which asked questions about their cognitive representational preference (CP) across representations, their perception of the representational instruction on the FTC that their curriculum included, and their prior knowledge with Calculus (PK).
Five Problems Involving the FTC

The day after the completion of the Background Questionnaire, students were given the Five Problems on the Fundamental Theorem of Calculus (Appendix D) and asked to work on them in class. This set of problems consisted of similar problems across the five representational formats: verbal, graphical, numerical, symbolic and contextual.

Students used codes on their answers to maintain participant confidentiality. Their responses were collected and scored. Twenty percent of the answers were scored by the researcher and a colleague with mathematics background to provide for inter-rater reliability. The scoring of the solutions followed a Scoring Rubric that anticipated answers and partial answers (Appendix E). Each student had an overall score and individual problem scores for each of the representations: a verbal, a graphical, a numerical, a symbolic and a contextual score. This data was used for quantitative analysis as described under the quantitative methods section.

Think-Alouds

Nine students (two females and one male from each class) participated in this semi-structured interview (Appendix C), which was video recorded and transcribed verbatim. During this interview students were asked to choose one problem out of the five presented in Appendix C. The five problems presented were similar but ranged across the five representations in this study: graphical, numerical, verbal, contextual and symbolic. During the Think-Aloud, students solved the problem of their choice while explaining their problem-solving strategies aloud. When each participant finished the problem, she or he was asked several questions to better understand their choice, problem solving trajectory and to further investigate their understanding of the FTC. The protocol
was adapted from Aspinwall, Shaw, & Presmeg (1997). A semi-structured interview was employed to ask the same set of questions of each participant, but allowing for follow-up questions (Ritchie, Lewis, Nicholls, & Ormston 2013). Follow-up questions further probed the students’ understanding. Student artifacts produced during the *Think-Aloud* were also collected for analysis.

**Data Analysis**

A convergent mixed methods design using both qualitative and quantitative data simultaneously to converge upon the results of the analysis was used (Cresswell, Plano-Clark & Garrett, 2011). A concept variable map, as shown in Figure 15 was created to show how the variables are related to the data collection tools. The study in question examined the relationship between the “*Enacted Curriculum*” and “*FTC Understanding*” as moderated by “*Gender*”. The variables used to measure each concept are indicated below the concept.

Figure 15. Concept variable map.

**Operationalization of Concept Variables**

The variables of enacted curriculum and student understanding were operationalized to allow them to be measured and explained by observation and tools.
This operationalization of these concept variables allowed for qualitative and quantitative analysis of the enacted curriculum and student understanding of the FTC.

**Enacted Curriculum Variables**

Under the Enacted Curriculum concept, the presence of each type of representation in the lesson, and the extent to which the teacher and students used each representation, was recorded in the LOP tool as an individual representation score. This score reflected the quality, time, and nature of the use of representations in the enacted curriculum on a 7-point scale as captured by the LOP. These representational scores are labeled in this paper as VER (verbal), GER (graphic), CER (context), NER (numerical), and SER (symbolic). The letter “E” is used to indicate the word “enacted”. The scoring was done over the period of instruction of the Fundamental Theorem of Calculus.

The Overall Multiple Representations score (OER), was a composite of the individual representation scores. To calculate the OER, a normed score for each representation was developed using the following process. A normed score of 1 was assigned for raw scores of 4 (mid-range) or higher on the LOP’s overall rating for that representation. A normed score of 0 was assigned for raw scores below 4. The score of 4 was chosen as the limiting value for counting the representation in the MR score since the LOP is rated on a 7-point scale, with a score of 4 representing a mid-level score. The Overall Multiple Representation Score is the sum of the normed scores across the five representations as illustrated in Table 7. Other independent variables included students’ cognitive preference (CP), perceived representational instruction (PR), student gender, and students’ class. These variables were self-reported in the *Background Questionnaire* and they are described in the next section.
Table 7. Sample multiple representations score.

<table>
<thead>
<tr>
<th>Representation score</th>
<th>GER</th>
<th>CER</th>
<th>SER</th>
<th>NER</th>
<th>VER</th>
</tr>
</thead>
<tbody>
<tr>
<td>LOP score</td>
<td>4</td>
<td>3</td>
<td>5</td>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>Normed score</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Overall (OER)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>3</td>
</tr>
</tbody>
</table>

Cognitive Representational Preference (CP)

In the Background Questionnaire, students recorded their preference for each representation on a Likert scale ranging from 1 to 4; 4 represented strong agreement, and 1 represented strong disagreement with that representation based on the statements in Table 8. There were five representational categories for each student: Graphical Cognitive Preference (GCP), Numerical Cognitive Preference (NCP), Verbal Cognitive Preference (VCP), Contextual Cognitive Preference (CCP), and Symbolical Cognitive Preference (SCP)

Table 8. Five cognitive representational preference types.

<table>
<thead>
<tr>
<th>Cognitive Preference</th>
<th>Likert Agreement Statements</th>
</tr>
</thead>
<tbody>
<tr>
<td>Graphical Cognitive Preference</td>
<td>I like problems or ideas presented in graphical ways.</td>
</tr>
<tr>
<td>Numerical Cognitive Preference</td>
<td>I like when problems and ideas in Calculus are presented using tables of values.</td>
</tr>
<tr>
<td>Contextual Cognitive Preference</td>
<td>I like when problems and ideas in Calculus are presented through stories and real-life contexts.</td>
</tr>
<tr>
<td>Symbolical Cognitive Preference</td>
<td>I like when problems and ideas in Calculus are presented using symbolical means such as formulas, integrals and derivative symbols.</td>
</tr>
<tr>
<td>Verbal Cognitive Preference</td>
<td>I like when problems and ideas in Calculus are presented through verbal mathematical explanations.</td>
</tr>
</tbody>
</table>
**Perceived Representational Instruction (PR)**

The Background Questionnaire recorded the students’ perception of the use of representation in the enacted curriculum. The students responded to the statements in Table 9 using a Likert scale with 4 representing strong agreement, and 1 representing strong disagreement with a statement regarding the perceived use of that representation in the enacted curriculum. There were five scores for each student: Graphical Perceived Representation (GPR), Numerical Perceived Representation (NPR), Verbal Perceived Representation (VPR), Contextual Perceived Representation (CPR), and Symbolical Perceived Representation (SPR).

Table 9. Types of perceived representational instruction.

<table>
<thead>
<tr>
<th>Instruction Type</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Graphical Perceived Representation</strong></td>
<td>The lessons included graphs, charts, pictures and drawings in presenting the Fundamental Theorem of Calculus.</td>
</tr>
<tr>
<td><strong>Numerical Perceived Representation</strong></td>
<td>The lessons included numerical (tables of data, sequences) in presenting the Fundamental Theorem of Calculus.</td>
</tr>
<tr>
<td><strong>Verbal Perceived Representation</strong></td>
<td>The lessons used verbal presentation (mathematical explanations of concepts) and words like “integrals”, “anti-derivatives” in teaching the Fundamental Theorem of Calculus.</td>
</tr>
<tr>
<td><strong>Contextual Perceived Representation</strong></td>
<td>The lessons used stories, word problems and real contexts in presenting the Fundamental Theorem of Calculus.</td>
</tr>
<tr>
<td><strong>Symbolical Perceived Representation</strong></td>
<td>The lesson used mathematical symbols (algebraic formulas, integrals and derivatives) in presenting the Fundamental Theorem of Calculus.</td>
</tr>
</tbody>
</table>
**Accommodated Needs Variables**

To measure accommodated needs, also referred to as needs met, this research used the difference between the standardized score for the instruction on any of the five representations (GR, NR, VR, CR, SR), and standardized scores for students’ cognitive preference for that representation (GPR, NPR, VPR, CPR, NPR).

**FTC Understanding Concept Variables**

The *Five Problems on the FTC* consisted of five problems on the FTC, one for each representation (graphical, numerical, verbal, contextual, and symbolic). Students were asked to solve all five problems. These were scored using the scoring rubric in Appendix E.

**Individual and Total Representation Variables**

The Individual Representation Variables used in this analysis are presented below.

Table 10. Variable scores on the FTC assessment.

<table>
<thead>
<tr>
<th>Variable Name</th>
<th>Assesses</th>
</tr>
</thead>
<tbody>
<tr>
<td>Graphical Representation Score (GS)</td>
<td>score on the graphical representation problem (Problem A)</td>
</tr>
<tr>
<td>Numerical Representation Score (NS)</td>
<td>score on the numerical representation problem (Problem B)</td>
</tr>
<tr>
<td>Verbal Representation Score (VS)</td>
<td>score on the verbal representation problem (Problem C)</td>
</tr>
<tr>
<td>Contextual Representation Score (CS)</td>
<td>score on the contextual representation problem (Problem D)</td>
</tr>
<tr>
<td>Symbolical Representation Score (SS)</td>
<td>score on the symbolical representation problem (Problem E)</td>
</tr>
<tr>
<td>Total Score (TS)</td>
<td>the mean of the five individual representation scores on each representation</td>
</tr>
</tbody>
</table>
Students’ Prior Knowledge

The student’s prior mathematical background captured in the *Background Questionnaire* was used to construct a baseline for students’ prior knowledge used in the analysis. A variable named *Prior Knowledge of the FTC* was created. This was based on students’ answers to the first two questions in the Background Questionnaire, namely:

1. Did you take Calculus in high school?
   If so, when, where and what course? (E.g. HS, AP, Honors)
2. Have you been exposed to the FTC before?
   What sort of representations do you recall using at that time?

Students’ answers were divided in two categories as shown in Table 11. Student Prior Knowledge was compared using Chi-Square tests to compare the proportion of students with no prior knowledge of the FTC and those with little prior knowledge of the FTC across sites and gender.

The researcher had also wanted to construct a variable for *prior knowledge of multiple representations* based on Question 6 of the *Background Questionnaire* (In your prior mathematics classes, to what extent did you find multiple representations of mathematical concepts utilized effectively?). Student responses on this category, however, were inconsistent and the data was inconclusive. One source of confusion was that students did not know what classes to talk about when they answered about the effective utilization of multiple representations. For example, one female student at CSNE answered “I have seen multiple representations in all my classes”, another student said, “I used tables and graphs,” and other students answered “no” or “yes”. The question may have been poorly worded. This question was too vague and could not be used in the
analysis. One could not know based on their answers if when students said that they used tables and graphs if that meant that they used only those representations, or that only those representations were used effectively. Also, when they answered “no,” it was not clear if they meant they did not use multiple representations in all their classes, in or in some classes, or if representations were not used effectively.

Table 11. Categories for students’ prior knowledge score.

<table>
<thead>
<tr>
<th>FTC Prior Knowledge Score</th>
<th>Score 0</th>
<th>Score 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Interpretation</td>
<td></td>
<td></td>
</tr>
<tr>
<td>No knowledge or prior experience with the FTC</td>
<td></td>
<td>Some experience and knowledge of the FTC</td>
</tr>
<tr>
<td>Examples</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Student had not taken Calculus before and gave answers such as:</td>
<td></td>
<td>Student has taken Calculus before or provided answers such as:</td>
</tr>
<tr>
<td>• No</td>
<td></td>
<td>• Yes,</td>
</tr>
<tr>
<td>• Unsure</td>
<td></td>
<td>• A little bit</td>
</tr>
<tr>
<td>• no answer</td>
<td></td>
<td>• Heard of it before</td>
</tr>
<tr>
<td>Student answered that they had taken Calculus but did not get to the FTC. (Only one student from the three classes was in this category.)</td>
<td></td>
<td>• I think so</td>
</tr>
<tr>
<td>• Yes, but did not get to the FTC</td>
<td></td>
<td>• I was exposed to it a little</td>
</tr>
</tbody>
</table>

**Participants’ Baseline for Prior Knowledge**

Student Prior Knowledge was compared using Chi-Square tests to compare the proportion of students with no prior knowledge of the FTC and those with little prior knowledge of the FTC across sites and gender. The three classes were homogenous with respect to the proportions of students who had prior knowledge of the FTC and those who did not, suggesting that the classes were similar as far as their prior knowledge was
concerned. It was concluded that there was not strong enough evidence to suggest that the classes differed with respect to prior knowledge of the FTC. These results will be presented in Chapter 5.

**Overview of Research Methodology**

Qualitative analysis was employed first to help set the stage for the quantitative analysis. The qualitative analysis was also helpful in corroborating results of the quantitative analysis, in helping understand these results, and in interpreting anomalous or unexpected results. An essential element in the qualitative analysis was the classroom portrait, introduced earlier in this work. Since the research questions sought to find a relationship between the classroom practices with multiple representations and students’ understanding, qualitative methods were used to create a rich description of each classroom, and to develop the classroom portrait. Qualitative methods were mostly used to describe concepts on the left side of the concept variable map presented in Figure 15 which made up the Enacted Curriculum. Students’ understanding of the FTC (at the right side of the concept variable map) was measured by both quantitative and qualitative means.

Quantitative methods were used to make comparisons of student understanding of the FTC across the three classes and across gender. The associated classroom portraits were used to connect classroom experience with multiple representations and students’ understanding. To characterize the distribution of scores on the Five Problems Involving the FTC at the three locations, descriptive statistics such as mean, median, mode, variance and range for all assessment scores were calculated. The students’ individual overall assessment score (TS) and five individual representation scores (verbal, graphical,
numerical, contextual, and symbolic) from the Five Problem Involving the FTC were 
used to make comparisons between the classes at the three locations, and between male 
and female students. To compare student understanding at the three sites, two-way 
ANOVA for the total score as all sub-scores obtained on the FTC Assessments at the 
three sites were performed, followed by multiple comparisons where ANOVA yielded 
significance. Fisher LSD was used to control for the family-wise error for the main effect 
of site. While ANOVA results do not establish causation, the results obtained were 
compared with the researcher scores on the enacted curriculum, and field notes to 
establish patterns and make connections. A summary of the study’s research questions 
and the qualitative or quantitative methods used to answer the research questions is in
Table 12 on the next page. A more detailed description of each method will be provided later in this chapter.
Table 12. Summary of methods used to answer research questions.

<table>
<thead>
<tr>
<th>Questions and Sub-Questions</th>
<th>Instruments Used</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. What is the nature of the relationship between students’ use of multiple representations in the enacted curriculum and student understanding of the FTC?</td>
<td></td>
</tr>
<tr>
<td>1a. In what ways do MR appear in the enacted curriculum?</td>
<td>• LOP lesson observations and field notes were used to measure the depth and quality of multiple representations in the enacted curriculum.</td>
</tr>
<tr>
<td></td>
<td>• Classroom Portrait for the enacted curriculum.</td>
</tr>
<tr>
<td>1b. What is the nature of the relationship between the use of MR in the classroom and students’ overall understanding of the FTC?</td>
<td>• Chi-Square for difference in proportions by site to compare the proportions of students with prior knowledge of Calculus and without to establish the three baseline Calculus knowledge at the three sites was not significantly different.</td>
</tr>
<tr>
<td></td>
<td>• Descriptive Statistics for the FTC Assessments (TS, GS, VS, NS, CS, SS)</td>
</tr>
<tr>
<td></td>
<td>• One Way ANOVA for all scores in the FTC Assessments at the three sites (TS, VS, GS, NS, CS, SS), followed by Fisher LSD where ANOVA yielded significance. One-way ANOVA results do not establish causation, ANOVA results were compared with the researcher scores on the enacted curriculum.</td>
</tr>
<tr>
<td></td>
<td>• <em>Think-Alouds</em> – qualitative are used to corroborate and extend results</td>
</tr>
<tr>
<td></td>
<td>• Field Notes used to extend results</td>
</tr>
<tr>
<td>1c. What role do other factors such as students’ representational cognitive preference and their perceived representational instruction play in the ways they communicate about the FTC?</td>
<td>• Regression analysis on Total Score on the FTC assessment (TS) as a function of individual representation scores (GS, VS, CS, NS, SS), site, cognitive preference, perceived representational instruction, and accommodated preference.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Questions and Sub-Questions</th>
<th>Instruments Used</th>
</tr>
</thead>
<tbody>
<tr>
<td>2. To what extent does students’ gender influence their use of MRs and their understanding of the FTC?</td>
<td></td>
</tr>
<tr>
<td>2a. What is the relation between the use of multiple representations of the FTC in the classroom and female students understanding of the FTC?</td>
<td>• Two-way ANOVA for difference in means by gender and site for each of the representational scores</td>
</tr>
<tr>
<td>2b. What is the relation between the use of multiple representations in the classroom and female students’ use of multiple representations?</td>
<td>• <em>Think-Alouds</em>, field notes</td>
</tr>
<tr>
<td>2c. What is the role of other factors, such as cognitive, preference, perceived representational instruction, and accommodated preference and female student understanding of the FTC?</td>
<td>• Regression analysis on Total Score on the FTC assessment (TS) as a function of individual representation scores (GS, VS, CS, NS, SS), site, cognitive preference, perceived representational instruction, and accommodated preference.</td>
</tr>
</tbody>
</table>
Qualitative Analysis

For the qualitative portion of the study, an epistemological framework was adopted based on interpretative or constructivist ideas (Merriam, 2009). Under this assumption, there is no single observable reality. What was of interest was the interpretation of events (in this case, the enacted curriculum and the fundamental theorem) as constructed by students. The motive of the study was to “describe, decode, translate and otherwise come to terms with the meaning, not the frequency of certain naturally occurring phenomena” (Van Maanen, 1979).

The qualitative methods included direct observation, and semi-structured interviews with a subset of the students. The approaches to collecting the qualitative data were richly descriptive. Direct observation helped the researcher collect data as it happened naturally in the enacted curriculum, the typical context for this study. The FTC assessment taken by all students was collected and analyzed as an additional artifact that helped clarify and enrich the scores that students received in the FTC assessment. The semi-structured interviews were effective collecting data about the participants’ backgrounds and perspectives, their problem-solving trajectory and their understanding of the FTC. Video recordings of students’ during the Think-Alouds, field notes and note contexts of quotes were used. The interviews were transcribed verbatim to capture the participants’ complete thinking about the FTC. At the end of the observations, a table was created with all indicators from the LOP to compare results at the sites.

The process of analyzing the qualitative data provided by the LOP, field notes, and the Think-Aloud Protocol was initially deductive with the data organized into themes and categories to form tentative hypotheses. The first stage was an analysis of the enacted
curriculum from the LOP and the field notes used to generate a classroom portrait for each class. The process adopted a flexible stance in altering the hypothesis as data is continuously deconstructed. The analysis was multi-phased, consisting of retrospective analysis during open and axial coding (Corbin & Strauss, 2008).

Coding for the Enacted Curriculum and Analytic Memos

Field notes were taken during each observation visit. Attention was paid to the classroom environment, resources available to students, lesson content and implementation, classroom culture and mathematical discourse, female participation, student sense-making, and the use of multiple representations. This aligned generally with the sections of the LOP. Mathematical processes written on the board were all captured by the researcher’s notes, and to the extent possible, student-teacher dialogues were written down in as much detail as possible. Analytic memos were also recorded to formulate and document thinking around the classroom events and to help tell the story. At the end of each session, notes and analytic memos were reviewed.

For example, while taking notes from a class on Day 2 at College of Southern New England (CSNE), the following analytic memo was recorded:

“There is little or no communication between students in this class. I am wondering how much of the teaching style has led to this or whether the student prior background factors into students not reaching out to each other as resources for learning.”

Stages for the coding process for the enacted curriculum

A deductive coding process was begun after reviewing of the field notes, LOP observations, analytic memos, and transcripts from the Think-Aloud. The decision to use deductive coding was based on the following hypotheses: 1) when students are exposed
to substantive experiences with multiple representations, they are more likely to gain a deeper understanding of the FTC; 2) students’ use of the FTC tends to be aligned with the enacted curriculum; 3) the tendency to align the use of the multiple representation is particularly true for female students.

The deductive coding process occurred in three stages: developing the codes, applying the codes to each setting, and connecting the codes across the three class locations. The following is an explanation of these three stages.

**Stage 1: Developing the codes**

Categories for the deductive coding process were based on the research questions and on the theoretical framework. Three major categories for data analysis were identified as: multiple representations, student learning process, and classroom culture. Several codes were developed within each category. Multiple representations refer to the type, frequency and depth of representations used in the enacted curriculum. In the context of this study, student learning processes include aspects of classroom discourse including student reasoning, negotiating, explaining, questioning, interpreting and evaluating ideas related to the FTC. Classroom culture involves the nature of the classroom environment as it relates to student participation, respect and equity, student collaboration and classroom management.

The analysis was not confined to the preliminary codes. During the coding of the transcripts, inductive codes were also generated. These were either separate from the initial codes or they expanded on codes previously established. For example, “student initiated discourse” and “connections among representation” were added as codes under “classroom culture.” Table 13 illustrates the codes used in the analysis.
Table 13. Categories and codes in enacted curriculum.

<table>
<thead>
<tr>
<th>Category</th>
<th>Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Classroom Culture</td>
<td>Instructor question, student responds, instructor evaluates (IRE)</td>
</tr>
<tr>
<td></td>
<td>Teacher Lecture (TL)</td>
</tr>
<tr>
<td></td>
<td>Access and Equity (AE)</td>
</tr>
<tr>
<td>Multiple Representation</td>
<td>Student or teacher use of verbal, contextual, graphical, numerical, or symbolical representations (VMR, CMR, GMR, CMR, SMR)</td>
</tr>
<tr>
<td></td>
<td>Representational Facility (RF)</td>
</tr>
<tr>
<td></td>
<td>Connections among Representations (CR)</td>
</tr>
<tr>
<td>Student Learning Process</td>
<td>Student Initiated Discourse (SID)</td>
</tr>
<tr>
<td></td>
<td>Interpretation (I)</td>
</tr>
<tr>
<td></td>
<td>Justification or Explanation (JE)</td>
</tr>
<tr>
<td></td>
<td>Female Student Learning (FSL)</td>
</tr>
</tbody>
</table>

**Stage 2: Coding**

Once satisfied with the codes, the field notes were reread and coded accordingly. For example, when a teacher asked a question, the students responded, and the teacher either elaborated or evaluated, it was labeled as Initiation, Response, Evaluation (IRE). When students asked questions of the instructor or of other students, the interaction was labeled as student initiated discourse (SID). Representation Facility (RF) was used to code instances where students showed an understanding of a specific representation, whereas Connections Among Representations (CR) was used to code instances where either the teacher or the students translated or transferred among two or more representations. The Access and Equity (AE) code was used to indicate teacher moves to steer the classroom dialogue in such a way as to facilitate learning for all students.
Table 14 provides an example of the work undertaken to connect codes from lesson two at Riverside Community College.

**Stage 3: Connecting across classrooms**

This final stage began with suggesting the emergent themes for each location and clustering them under headings that related to the research questions regarding the enacted curriculum of the FTC. Areas of agreement and potential conflict with the hypotheses were identified. Comparisons of the enacted curriculum were made across sites. Data was scrutinized to ensure that results were representative of what had been observed and without researcher bias. Overarching themes were identified.

Information from the *Background Questionnaire* regarding student preference and prior knowledge was used to triangulate or extend results and to identify potential points of bias.

Apart from teacher practices, student in-class activities, and assessments, there are other factors that influence students’ understanding of the FTC. Variables influencing the results of the study may include homework, outside help, student life, etc.; to the extent to possible, these variables were observed to corroborate, expand, or further explain results. Homework discussed during the lesson instruction, for example, was part of the researcher’s field notes and used to expand the classroom portrait of enacted curriculum at each site. Some questions around the students’ prior mathematical background and outside class help were also included in a *Background Questionnaire*.
Table 14. An example of dialogue coding.

<table>
<thead>
<tr>
<th>Example</th>
<th>Codes</th>
<th>Categories</th>
<th>Interpretation</th>
</tr>
</thead>
</table>
| **Dan:** Decreasing, but \( f' \) is increasing  
**Teacher:** But what does that mean? Yes, Dan? | IRE, AE | Class Culture | Dialogue shows Dan as role model in the class. |
| **Dan:** Decreasing \( f \), but concave up. | JE, FSL | Student Learning | Female students feel free to ask questions and volunteer answers. |
| **Sophie:** Why?  
**Jack:** Since \( f' \) is negative but increasing. | AE | Class Culture | Classroom discourse follows IRE pattern. |
| **Teacher:** Think about the slope of tangent. Try to imagine it. Can someone draw \( f' \)? | IRE, GMR | Class Culture, Multiple Representations | Students show their understanding with explanations and justifications. |
| **Matt's Graph of \( f \):** ![Graph](image) | GMR | Multiple Representations | Students make connections between graphical representations and the first and second derivative test. |
| **Teacher:** Thank you, Matt. So, \( f' \) is the slope of the tangent to the \( f \). So, \( f \) has to be decreasing since \( f' \) is negative, and concave up since \( f' \) is increasing, meaning \( f'' \), the second derivative is positive.  
**Teacher:** Is this the only graph? | IRE, JE | Student Learning, Class Culture | Classroom discourse follows IRE pattern. |
| **Matt:** It can start anywhere | GMR, SL | Multiple Representations  
Student learning | Students justify |
| **Teacher:** Why? | IRE | Class Culture | Classroom discourse follows IRE pattern. |
| **Teacher:** What if I asked to draw the me with \( f(a) = 0 \)? What are you supposed to do? | IRE | Class Learning | Classroom discourse follows IRE pattern. |
| **Sophie:** Nothing dictates where it starts  
**Dan:** Translate the graph down and start at 0. I will do it | RF, FSL, GMR, AE | Class Discourse, Student Learning | Students are engaged and student answers are encouraged and valued. |

**Quantitative Data Analysis**

Most of the quantitative methods were used to analyze *Students Understanding of the FTC*. The quantitative analysis portion of this research included summary statistics as
well as statistical analysis of the relationship of the variables. Boxplots for the assessment scores were created as a graphical representation of the data and compared to the enacted curriculum scores for each site. A summary of the basic descriptive statistical tools used in the study includes, but is not limited to, the following:

- Summary statistics of the total assessment score on the FTC assessment, and the summary statistics for the five representation assessment scores to compare the means and medians for the scores at each site.
- Boxplots for the total assessment scores (TS) at the three sites and by gender to compare the shapes of distributions. The visual representation was helpful in making comparisons between the shapes of the distributions.
- Boxplots for the individual representation assessment scores (VS, GS, CS, NS, and SS) to assess the level of proficiency across representations. This analysis was helpful in determining whether students within a class were more proficient on one or more of the representations.

To further analyze the relationships of the scores between the students in the three classes (n = 55) and gender, parametric testing was deemed most appropriate for the data. One- and two-way ANOVA were used to determine if any of the independent variables (gender, site, or gender and site interaction) had a significant impact on student understanding as measured by this assessment. A Fisher LSD test was performed on any statistically significant difference to determine where the difference occurred. Statistical differences in the data were examined using inferential statistics. It should be noted that certain requirements for an ANOVA-type test are not met by the study but that it was still possible to use this type of analysis. For instance, one-way ANOVA requires the students
to be randomly assigned to the three classes (Mertler & Vannatta, 2004). This study does not meet the random-assignment requirement. However, the researcher had no influence on how students were selected for the classes at the three locations and therefore, no bias in student assignments to classes was introduced. In addition, the decision to use ANOVA rather than non-parametric tests was because the data was more accurately represented by the mean and closer to a normal distribution, each group had a sufficiently close variance, and the groups were independent of each other.

ANOVA also assumes the three distributions are normal and have the same variance, which was determined after the data was collected. One of the course scores had a wider variance and the number of students in this course was only 14 rather than 15 suggested as a minimum number for ANOVA. This departure from the standard requirements was considered not great enough to disqualify the use of this test in the analysis. Non-parametric methods, such as the Kruskal-Wallis test may be more appropriate if the distributions appeared skewed (Mertler & Vannatta, 2004), but this was not the case.

The study employed ANOVA in the following ways:

- One-way ANOVA for differences in means in total assessment scores (TS) to see if there was a significant difference in medians in the three classes. When there was a significant difference in means, then the Fisher LSD procedure was used to determine which pair of scores were different (Levine, Berenson & Stephan, 1999). Summary statistics and tests of differences in means were not sufficient to answer the research question regarding the nature of the relationship between usage of multiple representations in the enacted
curriculum and student understanding of the FTC. However, the results of these tests, in addition to the classroom portrait, the overall enacted curriculum score (OER) for each site introduced in what follows, and the rich description from field notes helped to generate conjectures as to why these differences occurred.

- One-way ANOVA for differences in means of the individual representational scores (VS, GS, NS, SS, and CS) at the three sites helped identify which of the representations calculated in the total score (TS) contributed to the results of the previous one-way ANOVA, and to compare differences in assessment of various representations at the three sites.

- One-way ANOVA for differences in mean scores between the male and female scores at each site. This test helped answer research Question 2 regarding the extent to which students’ gender influences their use of MRs and their understanding of the FTC.

- Chi-square test for differences in proportions of prior knowledge (as defined earlier) at the three sites to ensure students had similar background knowledge prior to the course.

- Two-way ANOVA to compare differences in performance on the FTC assessment by gender and site and to examine the interaction of gender and site. This was done both for the total score (TS) and for the individual representation scores.

During the quantitative analysis, multiple regression models were pursued to investigate the relation between assessment scores, cognitive preference (CP),
representational preference (PR), accommodated representational preference (zdiff), and
gender. The regression analysis was useful in identifying other factors such as cognitive
preference, and perceived representational instruction that play a role in student
understanding. Models that were analyzed include regression models for the score on the
FTC assessment (TS) as a function of students’ cognitive preference for each
representation (CP), their perceived representational instruction on each representation
(PR), their accommodated cognitive (zdiff) preference, and gender.

\[ TS = F (GCP, VCP, NCP, CCP, SCP, gender, site) \]

\[ TS = F (GPR, VPR, NPR, CPR, SPR, gender, site) \]

There was a linear relationship between student score and the independent
variables. This was tested visually by scatterplots of the scores against each of the
independent variables. The independent variables were not highly correlated and the data
follows homoscedasticity assumptions in that the variance was similar across the
independent variables.

Multiple regressions were also used to address the connection with student-
accommodated representational preference and its relationship to student understanding
as a possible factor contributing to student understanding. If students prefer a specific
representation and if that representation is accommodated in the enacted curriculum, the
expectation is that students are more likely to be engaged and have a better understanding
of the topic under analysis. Thus, additional multiple regression analyses were also
conducted to gain an understanding of how the individual representational assessment
scores correlated with students’ accommodated preference for that representation.

Verbal: \[ TS = F (zdiff, site, gender) \]
Verbal: \( VS = F (\text{zdiff}, \text{site}, \text{gender}) \)

Graphical: \( GS = F (\text{zdiff}, \text{site}, \text{gender}) \)

Contextual: \( CS = F (\text{zdiff}, \text{site}, \text{gender}) \)

Symbolic: \( SS = F (\text{zdiff}, \text{site}, \text{gender}) \)

Table 15. Variables involved in the study.

<table>
<thead>
<tr>
<th>Variables</th>
<th>Scale</th>
<th>Type</th>
<th>Who Supplies it</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Overall Enacted Multiple Representation (OER) Score</strong></td>
<td>Ordinal scale 0 - 5</td>
<td>Group Independent</td>
<td>Researcher 1 score / instruction period</td>
</tr>
<tr>
<td><strong>Individual Enacted Representation (ER) Score</strong></td>
<td>Ordinal scale 1 - 7</td>
<td>Group Independent</td>
<td>Researcher 1 score / instruction period</td>
</tr>
<tr>
<td>Verbal (VER), Graphical (GER), Contextual (CER), Symbolic (SER) and Numerical (NER)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Cognitive Representational Preference (CP) Scores</strong></td>
<td>Ordinal Scale (Likert) 1-4</td>
<td>Individual Independent</td>
<td>Students 1 score / student</td>
</tr>
<tr>
<td>Verbal (VCP), Graphical (GCP), Contextual (CCP), Symbolic (SCP) and Numerical (NCP)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Perceived Representational Instruction (PR) Scores</strong></td>
<td>Ordinal 1 - 4</td>
<td>Individual Independent</td>
<td>Students 1 score / student</td>
</tr>
<tr>
<td>Verbal (VPR), Graphical (GPR), Contextual (CPR), Symbolic (SPR), and Numerical (NPR)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Gender</strong></td>
<td>Categorical</td>
<td>Individual Independent</td>
<td>Students 1 score / student</td>
</tr>
<tr>
<td><strong>Total Score (TS)</strong></td>
<td>Interval 0 – 75</td>
<td>Dependent</td>
<td>Researcher 1 score /student assessment</td>
</tr>
<tr>
<td><strong>Student Individual Representation Scores</strong></td>
<td>Interval 0 – 12 (VSR) 0 – 15 (GSR) 0 – 18 (CSR) 0 – 15 (SSR) 0 – 15 (NSR)</td>
<td>Dependent</td>
<td>Researcher 1 score / each assessment</td>
</tr>
<tr>
<td>Verbal (VS), Graphical (GS), Contextual (CS), Symbolic (SS) and Numerical (NS)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Student Prior Knowledge of Calculus</strong></td>
<td>Categorical Categorical NK or LK</td>
<td>Independent</td>
<td>Students 1 score/ student</td>
</tr>
<tr>
<td><strong>Needs (zdiff)</strong></td>
<td>Continuous</td>
<td>Independent</td>
<td>Students &amp; Researcher</td>
</tr>
<tr>
<td>Verbal (zdiffv), Graphical (zdiffg), Contextual (zdiffc), Symbolic (zdiffs), Numerical (zdiffn)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The results were then compared with the *classroom portrait* for each class to generate conjectures about why differences occurred, and to build a possible explanation.
for the connection between student understanding of the FTC and the use of multiple representations in the enacted curriculum.

The variables for the Enacted Curriculum and for FTC Understanding are recapped in Table 15, along with their type. They include variables for enacted multiple representation, cognitive preference, perceived representational instruction, FTC assessment, prior knowledge, and representational needs.

**Researcher Profile**

The researcher profile presented here seeks to inform readers of the background, beliefs and potential biases the researcher may have brought with her. The researcher’s attitude toward multiple representations, mathematics and problem solving was formed by her experiences as a student of mathematics, as a doctoral student of mathematics education, and as a long-time mathematics teacher.

The researcher is a middle-aged Caucasian female of Eastern European immigrant background. She came to this country around college age. She grew up in a family with an academic background, but her experiences in this country as an immigrant place her in a nontraditional category. She was initially interested in pursuing a doctorate in mathematics with the original goal of doing research in topology. Life moved her to the field of teaching. The investigator started teaching in 1995, initially as part-time faculty at different colleges, and then moved to a full-time job at a community college.

In these 20 years, she acquired a great deal of experience teaching students of various backgrounds and learning styles. She enjoys teaching and has a special interest in thinking about how people learn mathematics, and what teachers can do to empower students to be successful at mathematics. The researcher’s experiences as a woman in
mathematics have not always been positive. She witnessed and experienced bias, discrimination, lack of role models, and stereotyping, and that is why looking at female students’ understanding of Calculus is particularly interesting to her.

The investigator believes knowledge is acquired by construction, and, in mathematics, that construction happens as a result of engagement with the subject. Learning may be constrained by internal factors such as prior knowledge, or external ones such as cultural views, and is situated in contexts instead of being a purely cognitive process. Teachers, resources, and representations (texts, technology, symbols, and notation) all affect knowledge. Students should have “opportunities to study mathematics as an exploratory, dynamic, evolving discipline rather than as a rigid, absolute, closed body of laws to be memorized (…) and to recognize that mathematics is really about patterns.” (NCTM, 2000).
CHAPTER 4

RESULTS ON THE ENACTED CURRICULUM

This chapter discusses the Enacted Curriculum at the sites observed. The chapter begins with the summative results of the coding using the LOP, then the classroom portraits for each location are described. Finally, the comparison of emergent themes across sites follows.

Table 16 includes observations across sites. These ratings, along with field notes were used to build the classroom portraits.

Table 16. LOP summary.

<table>
<thead>
<tr>
<th></th>
<th>RCC</th>
<th>HCCC</th>
<th>CSNE</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>PLANNING &amp; ORGANIZATION OF THE LESSON</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1. Does the lesson come directly from a pre-packaged program?</td>
<td>Yes. Hughes Hallett (adapted by instructor)</td>
<td>Yes. Stewart</td>
<td>Yes. Lial Chapter 7</td>
</tr>
<tr>
<td>2. Rate the adequacy of classroom resources to support the lesson.</td>
<td>4 Desks and chairs, teacher computer ample space</td>
<td>4 Desks and chairs, teacher computer ample space</td>
<td>5 Adequate resources</td>
</tr>
<tr>
<td>3. Did organization provide substantive teacher-student interactions?</td>
<td>Yes. IRE Pattern</td>
<td>Yes. Teacher asked questions.</td>
<td>No. Teacher lectured entire time.</td>
</tr>
<tr>
<td>4. Did organization provide substantive student-student interactions?</td>
<td>No. No group work. No student interaction.</td>
<td>Yes. Students had time to work in groups.</td>
<td>No. Teacher lectured entire time.</td>
</tr>
<tr>
<td>Question</td>
<td>RCC</td>
<td>HCCC</td>
<td>CSNE</td>
</tr>
<tr>
<td>----------</td>
<td>--------------</td>
<td>-----------------------------------------</td>
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</tr>
<tr>
<td>6. Was the lesson organized to address student experiences, developmental levels, preparedness, and/or learning styles regardless of gender?</td>
<td>Yes. Teacher had plan that he seldom deviated from. However, he moved fairly slowly through material.</td>
<td>No. Teacher was well liked, knew students, but moved through material fast as some students appeared lost.</td>
<td>Yes. Teacher reminds students of various concepts they learned. Student-student interaction.</td>
</tr>
<tr>
<td>7. Was the lesson organized to appropriately address issues of access, equity, and/or diversity?</td>
<td>Don’t know.</td>
<td>Yes. Lectures self-contained. One student had a note taker, followed teacher to board to see.</td>
<td>Yes. No access or equity issues came up.</td>
</tr>
<tr>
<td>8. Did the lesson incorporate student and/or teacher use of technology (i.e., computers, video/digital cameras, monitoring equipment, calculators)?</td>
<td>Yes. Graphing calculator and computer</td>
<td>Yes. Graphing Calculator</td>
<td>Yes. Graphing calculator document camera</td>
</tr>
<tr>
<td>9. Other comments about lesson planning/organization or other indicators of importance.</td>
<td></td>
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<td></td>
</tr>
</tbody>
</table>

**IMPLEMENTATION OF THE LESSON**

<table>
<thead>
<tr>
<th>Implementation</th>
<th>RCC</th>
<th>HCCC</th>
<th>CSNE</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. The students appeared confident of their understanding of the lesson.</td>
<td>6 Students seemed engaged and interacted well with teacher.</td>
<td>4 Students volunteer to come to the board and put up answers, answer questions, discuss solutions in groups. Some students do not participate in group work fully.</td>
<td>1 Students were taking notes and seemed to follow but did not have opportunity to show understanding.</td>
</tr>
<tr>
<td>2. Periods of teacher-student interaction were probing and substantive (emphasized higher-order thinking and exposed students' prior knowledge).</td>
<td>4 Teacher constantly asked for explanations, but sometimes problems or questions were easy.</td>
<td>5 Teacher had students solve problems at the board.</td>
<td>1 Teacher does not probe students or challenge them.</td>
</tr>
<tr>
<td>3. Classroom management was effective in engaging all students in the lesson.</td>
<td>6 Teacher called on all students.</td>
<td>4 Some students were lost, could not take notes, some students silent.</td>
<td>2 Students took notes but did not engage. Student behavior is managed, but not their learning.</td>
</tr>
<tr>
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<td>RCC</td>
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</tr>
<tr>
<td>4</td>
<td>The pace of the lesson was appropriate for the developmental levels</td>
<td>5 Good pace, everyone engaged — for me a little slow.</td>
<td>4 Teacher engaged students and approached the lesson as a lecture, mostly. He was skilled but moved too fast.</td>
</tr>
<tr>
<td></td>
<td>of the students.</td>
<td></td>
<td>3 Students seemed to follow teacher.</td>
</tr>
<tr>
<td>5</td>
<td>Periods of student-student interaction were focused on pertinent</td>
<td>2 No interaction or little interaction among students.</td>
<td>1 No small group work or collaboration.</td>
</tr>
<tr>
<td></td>
<td>lesson content and enhanced individual understanding of it.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>The lesson was organized so there was adequate time for students</td>
<td>3 Students were not rushed.</td>
<td>4 Students worked well together when in groups. Some sideline conversations during lecture.</td>
</tr>
<tr>
<td></td>
<td>and/or the teacher to reflect on the lesson and its content.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>The lesson was organized so there was adequate time for wrap-up</td>
<td>4 Students got out on time on most occasions with one minute to wrap-up.</td>
<td>2 Lessons frequently ended with students still working on problems.</td>
</tr>
<tr>
<td></td>
<td>and closure of the lesson.</td>
<td></td>
<td>4 Teacher summarized lessons, told students about future lessons.</td>
</tr>
<tr>
<td>8</td>
<td>Teacher makes connections between the content and the students’</td>
<td>2 Connections to knowing about students not present.</td>
<td>2 No context.</td>
</tr>
<tr>
<td></td>
<td>culture, community and families.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>The teacher communicates high expectations for all students,</td>
<td>5 Questions all students.</td>
<td>2 Uses encouraging words and communicates high expectations but does not challenge students</td>
</tr>
<tr>
<td></td>
<td>challenging all students to engage in problem solving, question</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>and the generation of knowledge.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>Female students were engaged in sense-making of this lesson.</td>
<td>6 Calls on all students. Several (2) females act as role models.</td>
<td>5 Females attentive; some participate, others text.</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>2 Females copied notes, on occasion asked questions.</td>
</tr>
<tr>
<td>11</td>
<td>Teacher is sensitive to issues of gender when facilitating this</td>
<td>4 Lesson objectives sensitive to gender.</td>
<td>3 No issues of gender arose. Female students appeared comfortable.</td>
</tr>
<tr>
<td></td>
<td>lesson.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>12. Students regardless of gender were given equal attention.</td>
<td>RCC</td>
<td>HCCC</td>
<td>CSNE</td>
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<td>---------------------------------------------------------------</td>
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</tr>
<tr>
<td>6 All students expected to defend ideas without judgement. Teacher values answers.</td>
<td>4 Teacher responds to some more active members more often. Does not appear gender biased.</td>
<td>4 Whenever students have questions, the teacher answers them, but there is little opportunity to participate.</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>13. Other comments about lesson implementation.</th>
<th>RCC</th>
<th>HCCC</th>
<th>CSNE</th>
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</thead>
</table>

<table>
<thead>
<tr>
<th>CONTENT OF THE LESSON</th>
<th>RCC</th>
<th>HCCC</th>
<th>CSNE</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. The content of the lesson was important and worthwhile.</td>
<td>6 All content important and adequate. Skills include problem solving, graphing, applications of the FTC.</td>
<td>6 All content important and adequate. Skills include problem-solving, investigative tasks.</td>
<td>4 Lessons focused on important content, FTC but lacked higher order thinking demands for students.</td>
</tr>
<tr>
<td>2. Students were intellectually engaged with important ideas related to the focus of the lesson.</td>
<td>6 All focused or participating.</td>
<td>6 All focused or participating.</td>
<td>2 Students texting, copying notes. Not engaged in math discourse.</td>
</tr>
<tr>
<td>3. The subject matter was portrayed as a dynamic body of knowledge enriched by conjecture, investigation, analysis, and/or proof/justification.</td>
<td>4 Students had opportunity to create mathematics. Teacher carefully scaffolds lessons, at times not allowing students to grapple with more difficult ideas.</td>
<td>5 Lectures based on proofs, usually launched by investigation.</td>
<td>1 No student opportunity to address alternate solutions, justify or prove.</td>
</tr>
<tr>
<td>4. The students had understanding of the concepts and content of the lesson and the topical/conceptual area being addressed by the lesson.</td>
<td>5 Students answer teacher questions well as a group. Some students just focused on procedure.</td>
<td>4 Some students understand, others left behind.</td>
<td>3 Students seem to follow and on occasion ask appropriate questions.</td>
</tr>
<tr>
<td>Count</td>
<td>RCC</td>
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</tr>
<tr>
<td>5. The lesson had concept/content connections with this and/or previous or future lessons in the overall unit or topic being addressed.</td>
<td>4 Connection to prior content in Calculus I like 1st derivative test, concavity, etc.</td>
<td>6 Many connections to prior lessons, courses, and mathematics topics.</td>
<td>3 Connections to derivative, max, min.</td>
</tr>
<tr>
<td>6. The lesson included connections between this lesson and/or other areas of the same subject and/or other subjects.</td>
<td>3 Some connections to physics.</td>
<td>6 Connections to physics, real life.</td>
<td>3 Few connections to geometry and algebra.</td>
</tr>
<tr>
<td>7. The lesson incorporated applications of the lesson to real-world situations.</td>
<td>2 Only one physics example; even feelings “can be solved in terms of math.”</td>
<td>6 Many applications.</td>
<td>2 Only once did the teacher mention a velocity problem.</td>
</tr>
<tr>
<td>8. The lesson included abstractions (theories and models) as appropriate.</td>
<td>5 Abstractions to functions, family of functions.</td>
<td>6 Abstractions and generalizations left to the students as class work. Not much scaffolding.</td>
<td>4 Abstractions limited to symbolical notation. No modeling.</td>
</tr>
<tr>
<td>9. The lesson included the following representations (check all that apply):</td>
<td>Numerical, verbal, graphical, contextual, analytical/formula</td>
<td>Numerical, verbal, graphical, contextual, analytical/formulas</td>
<td>Numerical, verbal, graphical, analytical/formulas.</td>
</tr>
<tr>
<td>10. The students responded positively to learning the concepts and the content of the lesson.</td>
<td>6 All participate, eager to show what they know</td>
<td>4 Most students engaged, some fell behind</td>
<td>2 Students take notes. Some off task. No active engagement</td>
</tr>
<tr>
<td>11. Other comments about lesson content or other indicators of importance.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Overall Rating for Content of the Lesson</strong></td>
<td>5</td>
<td>5</td>
<td>2</td>
</tr>
</tbody>
</table>

CLASSROOM CULTURE IN WHICH THE LESSON WAS CONDUCTED
<table>
<thead>
<tr>
<th></th>
<th>RCC</th>
<th>HCCC</th>
<th>CSNE</th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>Active participation of all students was encouraged and valued.</td>
<td>5 Teacher constantly asks class: Can anyone explain that? Who can help Max? You may need to think about this.</td>
<td>2 No efforts made to get students engaged.</td>
</tr>
<tr>
<td>2</td>
<td>The teacher showed respect for and valued students’ ideas, questions, and/or contributions to the lesson regardless of gender.</td>
<td>6 Female students encouraged to participate.</td>
<td>3 Teacher accepts questions without making judgement. However, no ideas are solicited from students so little opportunity to participate.</td>
</tr>
<tr>
<td>3</td>
<td>Students showed respect for and valued each other’s ideas, questions, and/or contributions to the lesson.</td>
<td>2 IRE but students did not dismiss each other’s answers.</td>
<td>Don’t know. Students do not interact</td>
</tr>
<tr>
<td>4</td>
<td>The classroom climate for the lesson encouraged students to generate ideas, questions, conjectures, and/or propositions.</td>
<td>5 Students encouraged to go to board.</td>
<td>2 Students do not have opportunity to participate.</td>
</tr>
<tr>
<td>5</td>
<td>Student-student interactions reflected collaborative working relationships.</td>
<td>1 No group work.</td>
<td>1 Students do not work together.</td>
</tr>
<tr>
<td>6</td>
<td>Teacher-female student interactions reflected collaborative working relationships.</td>
<td>6 Teacher asks questions equally of male and female students.</td>
<td>2 Teachers and students do not work together.</td>
</tr>
<tr>
<td>7</td>
<td>The teacher's language and behavior showed sensitivity to issues of gender, race/ethnicity, special needs, and/or socio-economic status.</td>
<td>4 In language teacher does not stereotype students. No issues come up.</td>
<td>4 In language teacher does not stereotype students. No issues come up.</td>
</tr>
<tr>
<td><strong>RCC</strong></td>
<td><strong>HCCC</strong></td>
<td><strong>CSNE</strong></td>
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</tr>
<tr>
<td>8. Teacher-student interactions reflect teacher knowledge of and appreciation for students’ lives outside of the classroom including knowledge of family, culture and the life of the community.</td>
<td>5 Knows students by name. Knows and relates examples to their major.</td>
<td>5 Teacher knows names and some of the students’ majors and interests.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>3 Little knowledge of student family and culture.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9. Female students asserted themselves with confidence</td>
<td>6 Female students act as leaders in their class.</td>
<td>5 Most female students participated. One discouraged. One shy but very attentive.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2 There is no opportunity for female students to assert themselves.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10. All students have the opportunity to participate in the lesson regardless of gender.</td>
<td>4 All students participate. Large group discussion shows some female leaders and some who are silent.</td>
<td>6 Participation in group work by all.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1 No opportunity to participate.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>11. Other comments about classroom culture or other indicators of importance.</td>
<td>Overall Rating for Classroom Culture</td>
<td>6 5 2</td>
<td></td>
</tr>
<tr>
<td><strong>USE OF TECHNOLOGY TO SUPPORT THE LESSON</strong></td>
<td><strong>1.</strong> List the major types(s) of technology hardware used by the teacher and students to support the lesson.</td>
<td>Graphing calculator Overhead</td>
<td>Students: TI graphing calculator</td>
</tr>
<tr>
<td></td>
<td>Teacher: document camera TI Students: TI</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2. List the major type(s) of software or programs being used to support the lesson.</td>
<td>None</td>
<td>None</td>
<td>None</td>
</tr>
<tr>
<td>3. Student technology use arrangement:</td>
<td>At desks/TI only</td>
<td>At desks/TI only</td>
<td>At tables/TI only</td>
</tr>
<tr>
<td></td>
<td>RCC</td>
<td>HCCC</td>
<td>CSNE</td>
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</tr>
<tr>
<td>4. Indicate the primary intended purpose(s) for which technology was used.</td>
<td>Main use of technology is to compute answers, evaluate integrals etc.</td>
<td>Teacher does not use technology, but the students do on occasion to check answers, see graphs and evaluate integrals.</td>
<td>Presentation: Teacher and/or students present (PowerPoint, video, music, publication) Visualization: Use graphing calculators or visualizations software to see or manipulate relationships or objects.</td>
</tr>
<tr>
<td>5. If this lesson is part of a curriculum unit or series of lessons, is technology used to support other lessons in the unit or series?</td>
<td>N/A</td>
<td>N/A</td>
<td>3 Yes.</td>
</tr>
<tr>
<td>6. In using the technology and/or accessing information through technology, were students limited to specific procedures or sources devised by the teacher or directed by the instructional materials? (Note: This may vary by grade or student skill level.)</td>
<td>3 Students use technology only to check teacher led discussion</td>
<td>4 Students use the technology individually and in groups. No limitations in use</td>
<td>3 Yes. Teacher shows students steps</td>
</tr>
<tr>
<td>7. Technology resources were adequate to support the lesson.</td>
<td>4 All students have TI’s.</td>
<td>4 All students have TI’s.</td>
<td>4 All students have TI’s.</td>
</tr>
<tr>
<td>8. Technology use was effectively integrated into this lesson (not an “add-on” or novelty).</td>
<td>Not applicable</td>
<td>Not applicable</td>
<td>4 Students use technology to analyze info.</td>
</tr>
<tr>
<td>9. The use of technology enhanced student learning of the lesson’s core concepts/content.</td>
<td>2 Calculator used for calculations.</td>
<td>5 Students use GC to visualize and work on problems.</td>
<td>4 Technology used significantly to understand Riemann Sums and approximations.</td>
</tr>
<tr>
<td>10. The use of technology supported real-world application of the lesson concepts/content.</td>
<td>2 No real work applications.</td>
<td>4 Some applications that appear real life.</td>
<td>2 Few real-life applications.</td>
</tr>
<tr>
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<td>RCC</td>
<td>HCCC</td>
<td>CSNE</td>
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</tr>
<tr>
<td>11. Technology use enhanced the ability of students to collaborate with each other.</td>
<td>2 Students checked answers</td>
<td>4 Students discussed graphing calculator solutions</td>
<td>1 No collaboration</td>
</tr>
<tr>
<td>12. Classroom management was effective in engaging female students in the use of the technology.</td>
<td>2 Students use technology only to check but not to generate ideas.</td>
<td>4 Many students use technology in many ways: to graph and interpret results, check answers, to investigate behavior of functions.</td>
<td>1 Teacher did not engage students in use of technology.</td>
</tr>
<tr>
<td>13. The teacher shows skills and ability in using technology (consider both technical skills and lesson design).</td>
<td>1 Teacher does not use technology.</td>
<td>1 Teacher does not use technology.</td>
<td>4 Teacher operates technology effectively.</td>
</tr>
<tr>
<td>14. Other comments about use of technology or other indicators of importance.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Overall Rating for Use of Technology to Support the Lesson</td>
<td>NA</td>
<td>NA</td>
<td>4</td>
</tr>
</tbody>
</table>

**MULTIPLE REPRESENTATIONS IN SUPPORT THE LESSON**

<table>
<thead>
<tr>
<th></th>
<th>RCC</th>
<th>HCCC</th>
<th>CSNE</th>
</tr>
</thead>
<tbody>
<tr>
<td>N1. The lesson included Numerical Representations</td>
<td>3 Little numerical representations</td>
<td>3 Few numerical representations in class present: error estimate</td>
<td>5 Teacher used a great deal of numerical data</td>
</tr>
<tr>
<td>N2. Students were encouraged to use numerical representations.</td>
<td>3 Some encouragement—prompted to make calculation as part of large groups.</td>
<td>4 How often should we calculate delta t to estimate distance to within 0.1 feet? Try it.</td>
<td>4 Teacher encourages students to use representations.</td>
</tr>
<tr>
<td>N3. Students use numerical representations to reason make conjectures, analyze or justify their solution</td>
<td>4 Students reason with numerical representations throughout class period.</td>
<td>2 Unless prompted by teacher, none.</td>
<td>3 Students follow and work on the calculator but they do not generate solutions themselves</td>
</tr>
<tr>
<td>Overall Rating for Numerical Representations</td>
<td>3</td>
<td>3</td>
<td>4</td>
</tr>
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<td></td>
<td>RCC</td>
<td>HCCC</td>
<td>CSNE</td>
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</tr>
<tr>
<td><strong>Graphical Representations</strong></td>
<td></td>
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</tr>
<tr>
<td>G1. The lesson included Graphical Representations.</td>
<td>6 Teacher uses a lot of problems that include graphs.</td>
<td>6 Graphical connections between verbal and symbols for all concepts.</td>
<td>3 Some use of graphs in classroom presentation to make meaning.</td>
</tr>
<tr>
<td>G2. Students were encouraged to use Graphical representations.</td>
<td>6 Students are encouraged to draw and use graphs.</td>
<td>6 A great deal of encouragement. Can you draw a picture?</td>
<td>2 Students rarely encouraged to make graphical connections.</td>
</tr>
<tr>
<td>G3. Students use to reason make conjectures, analyze or justify their solutions.</td>
<td>6 Students draw and reason with graphs.</td>
<td>6 Students explain with pictures.</td>
<td>1 Students do not have opportunity to use graphs.</td>
</tr>
<tr>
<td><strong>Overall Rating for Graphical Representations in the Lesson</strong></td>
<td>6</td>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td><strong>Verbal Representations</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>V1. The lesson included Verbal Representations.</td>
<td>4 Teacher talked a lot and used a lot of math concepts</td>
<td>5 Teacher uses common words (like distance, volume) other than mathematical terms.</td>
<td>3 Teacher talks a lot</td>
</tr>
<tr>
<td>V2. Students were encouraged to use Verbal representations</td>
<td>4 Teacher encourages students to explain their reasons verbally</td>
<td>4 Students put up solutions but not always encouraged to explain.</td>
<td>1 Students are not asked to explain their reasons verbally.</td>
</tr>
<tr>
<td>V3. Students use Verbal representations to reason make conjectures, analyze or justify their solutions.</td>
<td>4 Students volunteer to come to board and explain solutions.</td>
<td>4 Students explore problems in group work and volunteer to come to the board and present their thinking. They respond to each other’s thinking.</td>
<td>1 Students seldom ask questions or respond verbally to concepts.</td>
</tr>
<tr>
<td><strong>Overall Rating for Verbal Representations in the Lesson</strong></td>
<td>4</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td><strong>Contextual Representations</strong></td>
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</table>
### Contextual Representations

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<thead>
<tr>
<th></th>
<th>RCC</th>
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<th>CSNE</th>
</tr>
</thead>
<tbody>
<tr>
<td>C1. The lesson included Contextual Representations</td>
<td>2</td>
<td>6</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>Two contextual problems over the course of instruction.</td>
<td>Sample stories and word problems used.</td>
<td>None.</td>
</tr>
<tr>
<td>C2. Students were encouraged to use.</td>
<td>1</td>
<td>6</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>Students have no opportunity.</td>
<td>Sample group work with context.</td>
<td>Students have no opportunity.</td>
</tr>
<tr>
<td>C3. Students use contextual representations to reason make conjectures, analyze or justify their solutions.</td>
<td>1</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>Students have no opportunity</td>
<td>Students apply contextual solutions</td>
<td>Students have no opportunity</td>
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</table>

### Overall Rating for Contextual Representations in the Lesson

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<tr>
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<tbody>
<tr>
<td></td>
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<td>6</td>
<td>1</td>
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</table>

### Symbolic Representations

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<tr>
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<th>RCC</th>
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<th>CSNE</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1. The lesson included Symbolic Representations.</td>
<td>6</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>Teacher uses many symbolic representations when presenting the FTC.</td>
<td>Teacher uses a moderate amount of symbolic representations when presenting the FTC.</td>
<td>Teacher uses few symbolic representations when presenting the FTC.</td>
</tr>
<tr>
<td>S2. Students were encouraged to use symbolic representations</td>
<td>6</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>Students volunteer to come to board and explain solutions symbolically.</td>
<td>Students volunteer to come to board and explain solutions symbolically.</td>
<td>Students do not explain solutions symbolically. Just the teacher.</td>
</tr>
<tr>
<td>S3. Students use symbolic representations to reason make conjectures, analyze or justify their solutions.</td>
<td>4</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>Students have some opportunity</td>
<td>Students have some opportunity</td>
<td>Students have no opportunity</td>
</tr>
</tbody>
</table>

### Overall Rating for Symbolic Representations in the Lesson

<table>
<thead>
<tr>
<th></th>
<th>RCC</th>
<th>HCCC</th>
<th>CSNE</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
</tbody>
</table>

### Classroom Portraits

Examination of the scoring on the LOP shows some differences and similarities across sites. For example, at RCC and at HCCC, field notes of the classroom observation reflected more classroom interaction and participation. At RCC and at CSNE students...
had little opportunity to interact with each other, but at RCC there was more interaction with the teacher.

Analysis of the LOP data reported in the previous section along with the field notes collected, was used to create the Classroom Portraits. As indicated in the Concept Variable Map in Figure 15, Chapter 3, each classroom portrait includes: 1) a section on the overall implementation, 2) one on the classroom culture and discourse, and 3) one on multiple representations. These elements, themes, and patterns were then compared across sites to develop the classroom portraits.

**Riverside Community College (RCC)**

Six classes at Riverside Community College were observed. Classes were taught by professor Rohlin, a white middle-aged professor, who had taught Calculus at the college five times before. The RCC Calculus class had 24 students, 18 male and 6 female. Most of the students enrolled in the course were taking Calculus as a required course for their STEM major, according to the data collected in the *Background Questionnaire*. Most students in the class were on the engineering track, but there were several math, biology and computer science majors. According to the field notes, 20%–25% of the class was non-white compared to the overall average at the college of 37% non-white. Most students were in their early twenties, a traditional age for students at a community college.

**Overall Impression at RCC.** All lessons observed at RCC were run as a whole class lecture, with desks arranged in columns and rows facing the instructor. The instructor used a Hughes-Hallett textbook (Hughes-Hallett & Gleason, 2012) on his iPad. Some students had their own tablets with the text on it, but most had paper copies of the
book. The teacher began the class with a five-minute homework review, but on most occasions students did not ask any homework questions. During the two weeks observed, students asked questions on homework on only one day. Investigative tasks were not a part of the lesson plan on any of the occasions observed, and most of the classroom interaction followed the traditional classroom discourse pattern: teacher initiation, student response, then teacher evaluation or feedback (IRE/F) pattern. The dates and the topics covered, along with the number of students present are included in Table 15 below. All names in the discussion of the observations are pseudonyms.

Table 17. RCC visit and curricular objectives summary.

<table>
<thead>
<tr>
<th>Date</th>
<th>Objective</th>
<th>Attendance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Day 1</td>
<td>Properties of definite integrals, even and odd functions, average value of function on interval</td>
<td>18 M, 6 F</td>
</tr>
<tr>
<td>12/1/2015</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Day 2</td>
<td>Review of first and second derivative tests, connections between the derivative and the function in preparation for graphing anti-derivatives</td>
<td>15 M, 6 F</td>
</tr>
<tr>
<td>12/2/2015</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Day 3</td>
<td>Graphing Anti-derivatives</td>
<td>15 M, 6 F</td>
</tr>
<tr>
<td>12/4/2015</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Day 4</td>
<td>Continuing graphing Anti-derivatives and FTC I</td>
<td>15 M, 6 F</td>
</tr>
<tr>
<td>12/7/2015</td>
<td>Applying the FTC to compute definite integrals</td>
<td></td>
</tr>
<tr>
<td>Day 5</td>
<td>Formulas for computing definite integrals using FTC I</td>
<td>14 M, 6 F</td>
</tr>
<tr>
<td>12/8/2015</td>
<td>Proof of the FTC</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Applications of the FTC</td>
<td></td>
</tr>
<tr>
<td>Day 6</td>
<td>More FTC II examples</td>
<td>17 M, 5 F</td>
</tr>
<tr>
<td>12/9/2015</td>
<td>Applications to differential equations</td>
<td></td>
</tr>
</tbody>
</table>

During all the observed classes, students neither worked in groups nor did they engage in dialogue about mathematics with their peers. Rather, they were engaged with and by the teacher. The professor included everyone in the conversation and referred to students by name. It was notable that the teacher called on every student in the class on a
regular basis. Two students, a white male student, Dan, and an Asian female, Christine, were quick to respond to the questions the instructor posed. On several occasions, the teacher asked the two students to delay answering to allow others to have the chance to contribute. No gender issues arose in class and the teacher gave equal attention to both male and female students.

The content of the lessons included many connections with prior algebra, pre-calculus, and beginning Calculus math concepts: even and odd functions, polynomials, derivatives and limits. However, those were the only connections with other disciplines or real-world examples given. A single application having to do with differential equations, where the teacher derived the equation for projectile motion was observed on the sixth day of observation. The few investigations observed in the lessons were always directed by the teacher.

The students responded positively to the teacher and to learning the concepts he introduced. They regularly volunteered answers and displayed and explained their solutions during the class lecture. Students dialogued with the teacher easily and did not appear afraid to ask questions or to offer their understanding of math concepts. For example, when the teacher showed that the definite integral from -a to a of an odd function is zero because the areas to the left and right of the y-axis cancel, one student raised his hand and said, “so if we knew the function was odd and we had to integrate from -3 to 5, then we would only need to integrate from 3 to 5, right?” The student not only showed that he understood the concepts, but that he was engaged in the lesson in a meaningful way.
RCC Class Interaction and Discourse. The class was characterized by whole group activities. No small group work amongst the students was observed. The whole class work was teacher led, wherein the teacher presented a mini lecture, then asked for volunteers, called on students, or asked questions regarding the lesson. The teacher repeatedly used an IRE pattern. He called on most students and did not usually let them self-select in offering explanations or in answering his questions. The class was led at a pace appropriate for learning, as the students seemed engaged, paid attention to the teacher, interacted with him, and were able to keep up. Every student was given an opportunity to speak either by volunteering or by being called on by the teacher.

In the RCC Calculus class, students worked significantly with graphical, verbal and symbolic representations. They were invited to the board to graph anti-derivatives or invited to explain solutions. In a 50-minute class, the teacher initiated an average of 22 IRE questions either directed to the whole class or to specific students and more than 70% of the class time was spent in activities involving IRE discussion. On average, there were only one or two student initiated discourse instances in each class visited. The following excerpt illustrates the dialogue (mostly IRE) that characterized the class and the level of engagement of the class. The example also shows the graphical representations that appeared in much of the teacher’s presentation. The teacher had drawn a derivative of a graph and then asked his students to think about the graph of the potential parent functions. The graph derivative drawn by him is below.
Then, the teacher asked, “How about this graph? How can we graph the parent function $f'$?” Four students, Christine, Sophie, Dan and Matt engaged in the conversation, with the rest of the students paying attention to their explanations:

*Christine:* Decreasing, but $f'$ is increasing

*Teacher:* But what does that mean?

*Christine:* Decreasing $f$, but concave up.

*Sophie:* Why?

*Dan:* Since $f'$ is negative but increasing.

*Teacher:* Think about the slope of tangent. Imagine it. Can someone draw $f$ not prime?

The dialogue shows Christine as a leader in her class, as she often had volunteered to solve mathematical problems. It is also indicative of the instructor initiated classroom discourse. Based on observation, students, and female students in particular, asked questions when they needed clarification. As indicated in the following dialogue, two out of the six female students in the class were involved in the discussion. The class proceeded with Matt volunteering to draw the graph below, showing an appropriate understanding of the mathematics.
The teacher offered no direct evaluation but addressed the class with a summary explanation, repeating what the students had observed.

*Teacher:* Thank you, Matt. So, \( f' \) is the slope of the tangent to \( f \). So, \( f \) has to be decreasing since \( f' \) is negative, and concave up since \( f' \) is increasing, meaning \( f'' \), the second derivative is positive.

Then the teacher asked follow-up questions about the placement of the graph drawn by Matt.

*Teacher:* Is this the only graph?

*Matt:* It can start anywhere.

*Teacher:* Why?

*Sophie:* Nothing dictates where it starts.

*Teacher:* What if I ask to draw the one with \( f(a) = 0 \). What are you supposed to do?

*Christine:* Translate the graph down and start at 0. I will do it.

Christine then drew the graph placing \( f(a) \) to start at 0, as the teacher paused to give the students the chance to think about the solution. The example shows that the teacher made sure to provide his student with time to think about and to dialogue about mathematics. Active participation was valued and encouraged through questions such as, “Can anyone explain?” or “Who can help?” and there was a high expectation that students engage in the class. As seen in the example above, some female students had
equal air time and they felt comfortable answering questions and volunteering to come to the board.

**Multiple representations at RCC.** In each class at RCC a significant use of graphical representations from simple to complex graphs was observed. Symbolic representations also appeared in every lecture extensively. Whenever a proof was given, there was much emphasis on the analytical representation as the most mathematical one. For example, in proving that the anti-derivative of $f(x) = 1/x$ is $F(x) = \ln|x| + C$, and not just $\ln x$, the teacher chose an analytical explanation rather than a graphical one.

*Teacher:* $f(x) = 1/x$ is defined for all $x$ unequal to zero, but $F(x) = \ln x$ is defined only as $x > 0$. What do we do with all the $x$’s? Do we just ignore the negative ones? We can’t.

Then the teacher explained and wrote on the board that $\ln(-x)$ is defined for negative $x$, so when $x$ is negative, one can write $(\ln(-x))' = \ln(-x)' (\ln(x))' = \frac{1}{x}(-1)$, from the chain rule, $= \frac{1}{x}$. The teacher then said:

*Teacher:* So, $(\ln(-x))' = \frac{1}{x}$ if $x < 0$ and also $(\ln(x))' = \frac{1}{x}$ if $x > 0$.

Turning to the students he then asked:

*Teacher:* What function has this property? The absolute value of $x$, or $|x|$. So, if $f'(x) = 1/x$, then $f(x) = \ln |x| + C$

Good to everyone? Starting today, any time we talk about the antiderivative of $1/x$, you should write what?

*Students:* $\ln|x|$

The field notes reflected that some students did not follow the explanation. A graphical explanation that an antiderivative for $1/x$ is $\ln|x|$ could have been offered, but it was not. Students had just learned to graph anti-derivatives, so they could have graphed
the antiderivative and justified, at least qualitatively and by symmetry, that the antiderivative was $\ln|x|$.

Verbal representations complemented the symbolical or graphical ones, again with the interaction always between teacher and students. The teacher talked through concepts, and in response to the teacher, students went to the board and verbally explained while writing solutions. The students did not discuss concepts with each other, and although professor Rohlin valued and asked for student explanations throughout the lessons; he skillfully scaffolded examples so students arrived easily at the correct conclusions.

One observation worth noting is that across all classroom visits at RCC, students were only asked to answer specific questions. They were never asked to work on a problem with many steps from start to finish. Questions addressed by the teacher were generally smaller in scope, usually asking for a specific reason, clarification, or procedure. There were no circumstances where students were asked to devise their own problem-solving strategies or to grapple with important mathematical ideas. An example illustrating the careful scaffolding is described below. It involves the graph from the board on Day 4, which is illustrated in Figure 18. For this reason, the cognitive demand of the tasks in the RCC class was later evaluated as procedures with connections (Stein & Smith, 1998).

In this problem the class was asked to draw the function $f$ corresponding to the derivative graph provided in the figure above given that $f(0) = 100$. Commenting that “Now we put all we learned together to try to get a more complicated graph”, the teacher then asked the class about the behavior of $f(x)$ on each subinterval, from 0 to 10, 10 to
20, 20 to 25, 25 to 30 and so on. He often asked students to explain, clarify, or justify answers. By questioning students about the sign and zeros of the derivative on each subinterval, the teacher led the class to produce a basic anti-derivative graph illustrating the concavity and regions of increase and decrease of the graph.

*Teacher:* What can I say about the graph from 0 to 10?

*Sam:* It’s increasing and concave up.

*Teacher:* What about 10?

*Dan:* It’s inflection point.

*Teacher:* Can anyone explain this? Except you (pointing to Dan).

*Dan:* At 10, slope of derivative changes from positive to negative.

*Teacher:* Jessica? Sally? Does the derivative change sign at 10?

*Sally:* No. Just direction, so it is concavity.

![Graph of f' used to generate graph of f at RCC.](image)

The professor continued leading students toward the solution in a similar manner, with students engaged in sense making. Graphical, numerical, verbal representations and symbolic representations were included in the problem selected, as the students discussed slopes, derivatives, and concavity. Students showed they understood the concepts they
were asked to discuss and gave correct answers and explanations, but the cognitive
demand for the students was not high, as illustrated below. When Fatina could not answer
a question, the teacher called on someone else to explain. Although the teacher did not go
back to the original student to clarify if she understood the explanation or not, the
additional explanations seemed to offer multiple entry points to mathematical discourse.

*Teacher:* Yes, the slope of the derivative is the second derivative. So, the function
is still increasing from 10 to 20, but it is concave down. What do we have there, at
20?

*Sam:* The same reason as the other one.

*Teacher:* Do we have a second derivative there?

*Max:* No. Sharp points.

*Teacher:* And another local min for the graph. Why is 30 a local min? Fatina?

*Fatina:* I am not sure.

*Teacher:* What do you think, Zack?

*Zack:* Because it changes from negative to positive.

*Teacher:* What changes?

*Zack:* The derivative.

*Teacher:* Does it make sense? Yes or no?

*Students:* Yes.

As the class continued, the teacher proceeded to add the *y*-values of inflection
points of the maxima and minima. He showed the students how the fundamental theorem
of calculus can be used to compute the *y*-values of some of the points in question. Then
he asked students to apply this process to figure out the other local extrema.

As the class developed, the teacher went on to discuss symbolical representations
and mentioned that the class would try to get a more accurate graph by using the FTC
exactly. In the conversation that follows, the teacher applied the FTC to find the y coordinates of critical points of $F(x)$.

Teacher: The FTC says, if $F'(x) = f(x)$ then $\int_a^b f(x)\,dx = F(b) - F(a)$. Assume we know the value of $\int_a^b f(x)\,dx$ and we know the $F(a)$. Then, 

$$F(b) = F(a) + \int_a^b f(x)\,dx.$$ 

Let’s use this idea to find this: $f(10) = f(0) + \int_0^{10} f'(x)\,dx$

He then showed that $f(10) = 100 + \frac{1}{2}(10)(20) = 200$ by using areas of the triangle to compute the integral and continued:

Teacher: $f(20)$ we can do 2 ways: 

$$f(20) = f(0) + \int_0^{20} f'(x)\,dx$$ 

or $f(20) = f(10) + \int_{10}^{20} f'(x)\,dx = 200 + 100 = 300$

Only after careful scaffolding, did the teacher turn the exercise over to the students. In doing so, he called on several of them to complete the task.

Teacher: I will ask you to find $f(25), f(30),$ and $f(35)$.

Teacher: 25 is, Sally?

Sally: 275

Teacher: 30, Jessica?

Jessica: 250

Teacher: 35?

Sam: 275

Teacher: Now we can draw the complete graph.

As the students answered, Professor Rohlin wrote: $f(25) = 300 - 25 = 275$; $f(30) = 250$; $f(35) = 275$, and then drew the graph in

Figure 19, which students copied down. While drawing the graph, the professor also summarized the lesson.
**Teacher:** So, we discussed two things:

1) How to use the FTC to find the values of the function.
2) Using those values and behavior of the derivative we can draw the graph of the antiderivative.

![Graph of the anti-derivative of \( f'(x) \) with \( f(0) = 100 \).](image)

**Figure 19.** Graph of the anti-derivative of \( f'(x) \) with \( f(0) = 100 \).

**Background Questionnaire at RCC.** The Background Questionnaire revealed that for most of the students (17/24), Calculus was a new subject they have not taken in high-school. Of the 24 students in the class, four male students and three female students had taken Calculus in high school, either AP Calculus or Honors Calculus. Their goals for taking the class were similar, but most students indicated they wanted to be successful in their future career, that they needed the course for their degree or their personal learning. More discussion of the background of the students will appear in the next chapter.

**Summary of the Enacted Curriculum at RCC.** The act of talking in class can help develop student understanding. Describing, justifying and explaining solutions can help students develop an improved understanding (Lampert, 1997; Cazden, 2001) and Riverside CC students fully participated in the development of mathematical ideas.
The teacher skillfully applied the FTC to support graph making and analytical reasoning. He did not make many contextual connections for the students as to why the FTC may be true, and few physical explanations, such as projectile motion, were used in instruction. The FTC was applied to get more accurate numerical or symbolic answers for graphical representations.

The tasks students were engaged in during the enactment of the FTC at Riverside Community College are characterized as procedures with connections as the tasks and the teacher’s enactment included ample pathways to follow broad general procedures with close connections to underlying conceptual ideas as opposed to narrow algorithms that are opaque with respect to underlying concepts (Stein, M.K., 2000). The teacher regularly engaged the class in meaning making and used connections to student prior knowledge of mathematical concepts to help foster understanding of the FTC and its relation to other mathematical ideas.

The teacher made use of graphs, numerical and verbal explanations, and formulas to develop mathematical ideas. Explanations and other teacher moves included graphs and their features, language, and symbolical representations in meaningful ways. The teacher encouraged all students to use many representations of the FTC as he called on them in class. Students were actively seen using graphical, numerical, and verbal explanations, and to a lesser extent symbolic representations to reason, analyze and justify their solutions. Contextual representations were seldom present, and there were no examples where students were asked to examine patterns, or to make conjectures. The LOP scores (Table 18) reflect these conclusions. The overall enacted curriculum score for the RCC curriculum is 3 out of 5. (Recall that this score counts the representations for
which the score given by the researcher is 4 or higher on the SAMPI instrument.) Even though the teacher used symbolic representations extensively, the students did not. For this reason, symbolic representation was rated lower.

Table 18. Enacted curriculum representation scores at RCC.

<table>
<thead>
<tr>
<th>Graphic</th>
<th>Numeric</th>
<th>Verbal</th>
<th>Context</th>
<th>Symbolic</th>
<th>Overall</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>4</td>
<td>4</td>
<td>2</td>
<td>4</td>
<td>4</td>
</tr>
</tbody>
</table>

**Hudson County Community College (HCCC)**

Five classes were observed at Hudson County Community College over a 10-day period in December 2015. Classes were one hour and 15 minutes in length and were taught by Professor Brown, a middle aged white male with more than 30 years of college teaching experience. There were 14 students in the class with 6 female and 8 male students. There were seven students of color in this class. Many of the students appeared to be in their mid-twenties. The dates attended, and the lesson objectives for each class observed are summarized in

Table 19.

Table 19. Lesson objectives for HCCC.

<table>
<thead>
<tr>
<th>Day and Date</th>
<th>Objective</th>
<th>Class</th>
</tr>
</thead>
<tbody>
<tr>
<td>Day 1</td>
<td>Definite integral: How we measure distance. Riemann sums. First Fundamental Theorem of Calculus: works with any rate of change, context applications.</td>
<td>6F, 12M</td>
</tr>
<tr>
<td>12/3/2015</td>
<td>Applications of the FTC I Properties of the definite integral and average value</td>
<td>6F, 12M</td>
</tr>
<tr>
<td>Day 2</td>
<td>Graphing Antiderivatives Graphing Anti-derivatives and antiderivative formulas Equations of motion</td>
<td>6F, 13M</td>
</tr>
<tr>
<td>12/4/2015</td>
<td>Context problems (three velocity acceleration, two population, two geometry Second Fundamental Theorem Proof of the FTC</td>
<td>5F, 11M</td>
</tr>
</tbody>
</table>
Overall Impression at HCCC. The HCCC lessons were run in various teaching formats: as a whole class lecture, as large group discussions, and as small group collaborative or interactive sessions. The interactive nature of this class immediately stood out. The class was arranged in rows facing the teacher, but when groups were in session, students moved the desks toward each other to better work together. The instructor used a Stewart textbook (Stewart, 2015) but made comments that he did not like the blue boxes in the book. He wrote on white board with colored markers from a very large multicolored box. Each class started with homework questions and about 15–20 minutes were spent on homework. On these occasions, the teacher either answered the questions himself, or asked the class as a whole if anyone had done it and if they would not mind sharing the solution. Another important feature of the HCCC curriculum was the use of investigative tasks. A major part of two of the five lessons observed was devoted to investigative tasks, and problem solving by students either individually or in groups occurred in every class observed.

The students responded to the teacher’s questions and supplied answers and explanations, questioned him when they were not following, and occasionally stayed after class to ask more questions. The teacher addressed students by name and in a respectful and caring manner. Because he computed fast, skipping some steps, and at times, it was difficult to read his writing, students had to stop him on occasion and ask for clarification. Professor Brown often used historical examples or references (Newton, Archimedes, or other scientists or mathematicians) when talking about mathematical concepts. He also made frequent references to stories (or problems situated in context) in
his presentation and his lessons included many connections to other subjects or real-world examples.

All but one of the female students sat in the middle of the class closer to the teacher. Several students were heard making comments such as “last night’s homework was hard”, “I got this”, or “this is cool” referring to the curricular content. One student commented on the second day of observation that he “did not like math in high-school, but now [he] did.” This was an indication that students were engaged in sense making, felt math was interesting, and were active in the learning process.

**Interaction and classroom discourse at HCCC.** The HCCC Calculus class as orchestrated by the teacher encouraged, expected and valued student participation as students often were engaged in small-group work on mathematical problems. Students had frequent opportunities to interact with the teacher, as they readily asked, or answered, the teacher’s questions. In addition, students often worked together in groups to make sense of what they were learning. Every student was given the opportunity to speak up when they wanted to contribute. A few students remained quiet and did not volunteer answers to the teacher’s questions; however, throughout the lessons, students were invited to go to the board to share answers and to explain their solutions. During the lessons, the teacher explicitly prompted an explanation by either requesting a student explanation at the onset, or by asking the students, who did not volunteer, to explain an answer. In Professor Brown’s class, students were expected to explain their thinking, so the teacher only had to prompt explanations from target students on a few occasions. As illustrated through student dialogue, students offered arguments on their own (and not just solutions) and reinforced each other’s explanation without interference from the
teacher. However, the teacher was attentive to the class discourse throughout, and intervened occasionally to ask students to give others the chance to explain a solution. He was observed encouraging students to understand what each other said, instead of looking to him as the ultimate source of knowledge.

For example, after talking about the first part of the FTC. \( \int_{a}^{b} F'(t) dt = F(b) - F(a) \), students were given five problems to work on. They were asked to work on problems individually and check with each other. Then they were asked to volunteer answers and solutions. The teacher projected the problems illustrated in Figure 20 and encouraged students” to take a few moments to think about them” and then discuss. The first three problems were computational in nature, and the last two were applications. This was the first time students had the occasion to use the FTC or needed to think about anti-derivatives:

<table>
<thead>
<tr>
<th>1) ( \int_{0}^{\pi/2} \cos(t) , dt )</th>
<th>2) ( \int_{2}^{3} t^2 , dt )</th>
<th>3) ( \int_{0}^{\pi/4} \sec^2(t) , dt )</th>
</tr>
</thead>
</table>

4) The population of Hicksville is 10,000 in 2013 and is growing at a rate of \( e^{-0.1} \) people per year. What will be the population in the year 2017?

5) Today there are 300 tons of water in a hemispherical reservoir which drains at a rate of \( r'(t) = -5t \) gallons per day (negative for draining). Write an expression for the water in the reservoir after \( t \) days.

As the students worked together, the teacher walked around and helped groups. At one point, one student (Ben) asked out about whether he was supposed to use the FTC.

It was unclear whether Ben’s question was directed to the class or to the teacher. However, when the teacher did not give a concrete answer, another student (Dawn) answered, indicating that class participation was the norm. Multiple approaches, entry
points to the problems, and discussion about the relationship between a calculator answer and the pencil and paper answer appeared naturally in the process of problem solving. The teacher’s response to Ben of “maybe” allowed the following dialogue between five students:

\[ \text{Dawn: } \text{Yes… I put negative sine (referring to question 1). I’ll put it the board?} \]
\[ \text{Teacher: } \text{Wait for others. Work on the rest.} \]
\[ \text{Dawn: } \text{What gives secant squared? (referring to question 3)} \]
\[ \text{Billy: } \text{It’s the tangent.} \]
\[ \text{Dawn: } \text{Oh yeah.} \]
\[ \text{Walt: } \text{I don’t get what I should do.} \]
\[ \text{Teacher: } \text{Can anyone help?} \]
\[ \text{Dawn: } \text{I will do it.} \]

Then Dawn ran to the board to write the solution as:

\[ \int_{0}^{\pi/2} \cos(t) \, dt = - \]
\[ \sin(t) \bigg|_{0}^{\pi/2} = -1 \]

As she went back to her seat, another student, Ben, questioned her answer:

\[ \text{Ben: } \text{I got 1 with the calculator.} \]
\[ \text{Cindy: } \text{The area under the cosine graph from 0 to pi/2 is positive.} \]
\[ \text{Teacher: } \text{Hmm…. What’s going on?} \]
\[ \text{Ben: } \text{I know. You (motioning toward Dawn) want to put sin(t) for the answer, and not negative sin(t).} \]
\[ \text{Dawn: } \text{Why? The derivative of sin is cosine. Oh, you are right. I got it the opposite way.} \]
\[ \text{Teacher: } \text{Can one of you share with the rest of the class?} \]
\[ \text{Dawn: } \text{Yes. You find a function with derivative cosine. That function is sine, because the derivative sine is cosine t.} \]

Then Dawn recorded the following on the board.

140
\[
\int_0^{\pi/2} \cos(t) \, dt = \sin(t) \bigg|_0^{\pi/2} = \sin(\pi/2) - \sin(0) = 1
\]

The teacher turned to the class and asked if everyone was good. Cindy nodded and said,

*Cindy:* And now the area is positive. And this matches Ben’s answer too.

The teacher went on to explain that technically, the answer could be \( \sin(t) + C \), where \( C \) is any constant, and that “the \( C \)’s cancel when evaluating the difference”, since the definite integral gives you the change in the function. The exchange showed students generating various multiple representations (verbal, numerical, graphical and symbolical), being able to share their solutions and to negotiate correct answers when their answers did not agree. The teacher took a secondary role, but he directed the students to listen to each other and to help each other understand. It also demonstrated students were used to constructing and deconstructing their own understanding. They were not given a list of anti-derivatives to apply to the FTC. Rather, students figured out on their own the meaning of the symbols and how to apply them with the teacher supporting their sense-making.

The class continued with two other students volunteering to explain the other two symbolic problems, then discussion turned to the context problems. As the class drew to close, the teacher asked the class directly about their thinking on contextual problem 4.

When one student (Ben) volunteered an answer, the teacher questioned him and asked the rest of the class for more clarification. He used encouraging words to talk about the students' contribution and he decontextualized the solution to the more abstract form of the theorem.

*Ben:* For this (problem 4), can’t you say that \( P(4) = 10,000 + \int_0^4 e^{-t} \, dt \)?
Teacher: Why do you say that?

Ben: You told us that the population at time zero is 10,000. I used that theorem.

At this point the teacher turned to the class and asked for someone else to explain and Cindy responded:

Teacher: Yes. Can anyone explain this?

Cindy: I think he just solved for $P(4)$ that’s the population on 2017. He rewrote the theorem as $P(4) = P(0) + \int_0^4 e^{-1t} dt$?

The teacher praised the students and used this occasion to generalize and contextualize their thoughts:

Teacher: Yes. I liked the way you approached that problems. Thank you. So, another way we can use the FTC is $F(b) = F(a) + \int_a^b F'(t) dt$. Can anyone put that in non-math words? (silence)

Teacher: Nobody? Ok. I shall try. This says that the final value $F(b)$ is the initial value $F(a)$ plus the change—this is the definite integral. So, to find the population in 2017, we need to have the population in 2013, then add to it the change in population from 2013 to 2017. You can think about the other one for tomorrow.

On several occasions, Professor Brown asked students to break into small groups and work on problems. In these cases, students were allowed to self-select into groups of three or four, and the female students grouped themselves into two groups of three. One of the two female student groups also had a male student in it. As the groups were working on problems, the teacher stopped by various groups to check their progress.

Approximately 50% of the class time on Days 3, 4, and 5 was spent with students working in groups, while the remainder was divided between whole group discussion and lecture. The class discourse in Professor Brown’s class was consistent with the position of the National Council of Teacher of Mathematics which maintains all students “should have equitable opportunities to learn mathematics” (NCTM, 2014). The teacher was
supportive of the shared discourse and willing to forgo his role of “master” of the content knowledge. Professor Brown’s class functioned as a community of practice (Lave & Wenger, 1991), which provided students with an environment that supported their identity as learners and doers of mathematics.

**Multiple Representations at HCCC.** Graphical representations were frequently used in every class observed at HCCC. Contextual problems and symbolical representations also appeared in every lecture. Verbal representations were used to a secondary extent, but their use by students was prominent. When used by the teacher, they complemented the other representations used. A major component of the class was making connections among various representations. The teacher did not “model” all types of problems for the students. Rather, he offered rich problems for students to work on in small groups, so that they solidified concepts for themselves, and the enactment of the mathematical tasks would be classified as doing mathematics, in accordance with Stein (2000).

The following two investigative tasks occurred on Days 3, 4 and 5 of observation of the HCCC classroom. On both these occasions, the teacher directed students to small group work that occurred toward the end of the class. The students were allowed to self-select to form groups. It can be inferred that when students generate multiple solutions and multiple representations when they are afforded rich tasks, and that they are able to clarify areas of confusion on their own. On Day 4, students were presented with the Investigative Task A illustrated in Figure 21 below. The students spent 20 minutes working in small groups on the problem, toward the end of class.
Alpha, Beta, and Gamma are contestants in a road race, each traveling in home-built automobiles. Alpha’s velocity is shown by \( \ldots \ldots \ldots \). Beta’s by \( \ldots \ldots \ldots \ldots \). Gamma’s is not shown because he is traveling steady at 35 miles per hour. Four hours later one crosses the finish line and the race ends.

a) Who is leading after 2 hours?
b) Who wins the race?
c) How long is the race course?
d) At the moment the race ends, how far is the runner up?
e) On the same graph, graph the distance traveled by each of the contestants as a function of time.

---

Figure 21. Investigative task A – small group.

Three female students (Cindy, Sara and Dawn), who worked cooperatively and collaboratively on the problem, were the focus of the observation. The problem related the graph of the velocity to the distance traveled by three people in the context of a story problem. Students were asked to arrive at certain conclusions about the distance traveled. During the initial period of problem solving, the three female students disagreed over what was being asked. The incident outlined below includes two such examples of disagreement.

Initially, Cindy confused the velocity with the distance traveled. Sara agreed with this incorrect answer. Dawn offered several ways, including dimensional analysis, to explain her thinking as she explained her solution to Sara and Cindy. Dawn argued that
they could not simply read the values on the graph, but had to look at the area under the velocity graph to the distance traveled:

*Cindy:* Who is ahead? Gamma, because Beta is at 30, and Alpha at 20.

*Sara:* I agree.

*Dawn:* No, it’s not the velocity. It’s the distance. That’s the area.

*Sara:* You are right. Let me see (she starts counting squares).

*Cindy:* Why is that?

*Dawn:* (Turning to her) Look at the units. Velocity is miles per hour and time is in hours. You multiply velocity and time to get the distance. That’s the area. So, we add up the squares. Do you see?

Cindy nodded in agreement and then Sara continued Dawn’s solution but did not know how to find the distance for Beta (dotted line).

*Sara:* After 2 hours, Alpha is at 40 miles, and Gamma is at 70 miles or 2 times 35. How do we find Beta?

As the discussion ensued, students use graphical and geometrical reasoning to find the distance traveled by Beta. They provided two explanations, one involving a conservation of area argument, and the other a similar triangles argument, both initially supplied by Dawn. They did not use symbolical computation.

*Dawn:* You can split it into a rectangle and triangle to find it, if we can find what it (the velocity) is at \( t = 2 \).

*Cindy:* I think it is 35 but it could be 30. Let’s say 35. So, 2 times 35 for rectangle and 2 times 25 divided by 2 for the triangle. That’s 70 plus 25…

*Sara:* It is 30 because the area of the triangle on top matches the triangle on the bottom right.

At this point, Sara caught on to Dawn’s argument, but Cindy did not, so Sara continued the explanation, while Dawn provided a second solution.

*Cindy:* What do you mean?
Sara: See these triangles? They are the same.

Dawn: Or you can do similar triangles. See the large triangle with the smaller inside?

Cindy acknowledged that she understood and completed the solution to the first part saying:

Cindy: That’s cool. So, $30 \times 2 + 30 = 90$ miles for Beta. He’s ahead.

Another area of disagreement occurred when the students discussed the length the race. As the students proceeded, they negotiated the relationships involved and could translate the graphical representation to the contextual setting and to resolve their differences. All three students participated and they took care to explain their reasoning to each other. Cindy took a little longer to understand, but she appeared to feel safe to ask Dawn and Sara for explanations. Dawn and Sara in turn, acted as mentors to Cindy. Sara suggested that they needed to find who finished first and that would provide them the length of the race. She was careful to ask Cindy if she was following. Dawn joined in to provide some potential solutions, but in doing so she realized she may have to readjust her thinking. This was a prime example of students working together to create meaning. Sara first suggested that to determine the length of the race they needed to first look at areas.

Sara: I think we can figure out the next part – how long is the race first.

We take the areas to find the distance. Do you get this, Cindy?

Cindy: Yes. I think so.

Dawn continued Sara’s thought but Sara intervened and suggested they figure out the distance by where Beta stops.
Dawn: It does not matter what graph you use. It’s the same distance. So, take Beta’s. Area is ½(60mi/h)(4h) = 120 mi. Or if you do Gamma’s you get (35mi/h)(4h) = 140 mi. Oops. Gamma is ahead. That does not work.

Sara: It does. I get it. How long is the race? It is still 120 miles. That’s because that is when Beta stops. So, it is 120 miles. Gamma wins. Does he?

Cindy: How about Alpha? Could he win?

Dawn: Alpha’s distance is 80 miles from the bottom rectangle plus 30 from the top triangle, so he is not even done!

Sara: But he is speeding up.

There were multiple ways to interpret this problem. One could argue that the course is 120 miles and that Gamma finishes first, but one could also suggest that Beta continues until the end, so that the course is 140 miles and that Beta stopped when she saw Gamma win. The class ended before the students finished, and they were instructed to finish the problem for homework, so the rest of the solution was not observed. The next day, homework was not collected or discussed by the class as a whole, but after class one student was heard asking the teacher for help with the last part of the solution.

During the last observation, students worked on another investigative task that involved several representations. This task dealt with the second part of the FTC. The students were once again asked to work in small groups, and Jane, Cindy, Sara and Ben worked together. The task asked students to reason abstractly and to make connections between the abstract form of the second part of the FTC, which they had just learned, and to apply it to a graphical representation, while making connections to other calculus concepts such as maxima and minima, inflection points and average value. The problem is in Figure 22 below.
While the students worked on the problem, the teacher once again took a back role and directed students to help each other out. The episode illustrated how students were able to clarify, solidify and strengthen their understanding. Initially, they confused the function $F(x)$ with its derivative, $f(x)$, a common mistake for Calculus novices (Baker, Cooley & Trigueros, 2000). During the group work, students use the definitions and the theorem to construct their arguments and to explore the truth of their conjectures. Students in this group participated equally and inclusively. The students initially read the problem by themselves, then Dawn initiated the conversation. Cindy did not at first understand Dawn’s answers. The teacher did not jump in to explain her solution, but rather allowed this to be resolved by the group.

*Dawn:* So, we need to find $F(0)$ and $F(-1)$. That’s tedious. $F(0) = 0$, $F(-1) = 1.5$

*Cindy:* No. I got $F(0) = 2$, $F(-1) = 1$. Isn’t that right Jane, or Mr. Brown?

*Teacher:* See if your group can figure it out, first.

*Cindy:* Can you give me a hint?

*Dawn:* Is there a difference between little $f$ and big $F$?

Figure 22. Investigative task B – small group.
**Cindy:** Yes. One is the integral of the other.

**Sara:** Yes. That is right. So, you just plug in the value and to find the integral. That is the area.

Then Sara said \( F(0) = 0 \) because “there is no area” and for \( F(-1) \) the area from 0 to -1 needed to be found. It was 1.5 and similar to something they had seen before.

**Sara:** That is one and a half little squares. It’s 1.5. Like the problem we did yesterday. Remember?

As the students continued to solve the problem, they had the opportunity to strengthen their understanding and to review properties of definite integrals such as switching the limits of integration and the meanings of the first and second derivatives. Sara corrected her explanation. Both Dawn and Sara encouraged Cindy to persevere and offered her alternate solutions.

**Dawn:** Actually it’s -1.5. I stand corrected. Good thing you explained it. You have to go from 0 to -1 and that is opposite direction. We need to switch the sign. Right? Now you do \( F(2) \), Cindy.

**Cindy:** Uh, I think is … It is hard.

**Sara:** You can do it. Just find the area. Do it in two strips.

**Cindy:** Okay. I guess I can. I never could in high school. So, it’s 2.5 plus the area of this other triangle. What’s that?

**Sara:** That’s 3 over 2. Take the whole rectangle and divide by 2.

**Cindy:** Ok. So, 2.5 plus 3/2 or 1.5. So, it’s 4. I got it.

When Ben wanted to move to another part of the problem, Cindy was not afraid to stop him, indicating that she was comfortable with the group.

**Ben:** (Who had been sitting quietly). What’s critical points?

**Cindy:** Wait for me. What does he mean?

**Dawn:** That’s where the derivative is zero. So that’s easy. -3.5, -2, and 2

**Cindy:** Not -2 and 3?
Dawn: No. See the theorem we learned? The derivative of $F$ is little $f$. You need to find the zeros of little $f$ to find where $F'=0$

Cindy: That is easy. I could not do it all by myself.

Once again Ben wanted to move ahead, but Dawn and Sara provided explanations for Cindy to catch up. They took mentoring roles and modeled their reasoning for the rest of the group. Students reviewed the first and second derivative tests and made connections among graphical, verbal and symbolical representations.

Ben: So, inflection is where the second derivative is zero.

Dawn: Yes, it’s $x = -2$.

Cindy: Wait a bit. I am still on question b.

Sara: What do you have Cindy?

Cindy: So you explained this. I need to find the zeros of little $f$. That’s -3.5, -2, and 2

Dawn: Yes. Because the critical points are where the first derivative of big $F$ is zero, and the first derivative of big $F$ is just little $f$.

Cindy: Ok. I am with you. What did you say, Ben?

Ben: For part c we need to find the inflections. It’s the second derivative. That means we look at the slope of little $f$ and see where that is zero, and that is at $x = -2$.

Cindy: I got that. You are right.

Dawn: And also, $x = 3$. The slope changes sign, so it’s an inflection.

Class ended before the students had the chance to fully finish the solution, but the time students had to work on the problem was sufficient for them to contextualize the problem and to recognize and apply both the FTC and other prior concepts to provide viable argument for their solutions. All five representations (graphical, numerical, symbolic, contextual and verbal) were involved in each task, and during the group work, students were observed using and connecting these representations in meaningful ways,
indicating a deepening understanding of the FTC. As the period ended before the problem was fully solved, Mr. Brown instructed students that he would later post solutions in Moodle.

**Background Questionnaire.** The background questionnaire at HCCC revealed that six of the 14 students had seen Calculus in high school, and that most students were taking the course as a major requirement. Of six female students, three had taken Calculus in high school. Most students indicated that they enjoyed math and that they were motivated by a desire to succeed in their careers.

**Summary of the Enacted FTC Curriculum at HCCC.** The HCCC curriculum used multiple representations to discuss the FTC. Students were heard and participation was considered an important vehicle for understanding. When students “make conjectures and reason with others about mathematics, ideas and knowledge are developed collaboratively, revealing that mathematics is constructed within an intellectual community” (NCTM, 2014). The nature of investigations was rich and complex, rather than low-level factual or recall based. For these reasons, the teacher was rated high on the multiple representation section of the LOP.

Table 20. MR scores for HCCC.

<table>
<thead>
<tr>
<th>Class</th>
<th>Graphical</th>
<th>Numerical</th>
<th>Verbal</th>
<th>Contextual</th>
<th>Symbolic</th>
<th>Overall</th>
</tr>
</thead>
<tbody>
<tr>
<td>HCCC</td>
<td>5</td>
<td>3</td>
<td>4</td>
<td>6</td>
<td>4</td>
<td>4</td>
</tr>
</tbody>
</table>

The group work and tasks chosen for this course, along with the enactment, required students to explore and understand the nature of mathematical concepts, processes, or relationships and engaged them in accessing relevant knowledge and experiences, limit or expand possible solution strategies (Stein, 2000).
Codes of RF, FSL, I, JE, CR, and MR in the coding of the discourse were used amply to indicate that the class discourse showed examples of representational facility, female student learning, interpretation, justification or explanation, connections among representations and many multiple representations.

**College of Southern New England (CSNE)**

Four classes were observed over a two-week period in December 2015. Classes were one hour and 15 minutes in length and taught by Professor Smith. Professor Smith is a young white male professor with two years college teaching experience and was teaching Calculus for the second time. The make-up of the CSNE class was two-thirds female and one-third male. Of the 22 students in the class, 13 were females and nine males. There were two students of color in the class, and all students were of traditional age, roughly 18–21. The dates attended, and the lesson objectives for each class observed are summarized in

Table 21.

**Table 21. Lesson objectives for CSNE.**

<table>
<thead>
<tr>
<th>Day and Date</th>
<th>Objective</th>
<th>Class</th>
</tr>
</thead>
<tbody>
<tr>
<td>Day 1 12/9/2015</td>
<td>Finish Antiderivatives. Riemann Sums and definition of the definite integral</td>
<td>14F, 5M</td>
</tr>
<tr>
<td>Day 2 12/11/2015</td>
<td>Area and the Definite Integral. The First Fundamental Theorem</td>
<td>14F, 5M</td>
</tr>
<tr>
<td>Day 4 12/7/2015</td>
<td>The Second FTC</td>
<td>12F, 6M</td>
</tr>
</tbody>
</table>
**Overall Impression at CSNE.** The CSNE lessons were always run as a whole class lecture, with long tables/desks arranged in rows facing the teacher. There were four rows of such tables and students sat four to six in a row. The instructor used the 7th edition of the Lial, Greenwall, and Ritchie text, *Calculus with Applications*. He used a document camera occasionally projecting either the textbook, or a graphing calculator on the screen. Like the RCC class, the CSNE class began with a five-minute homework review, but on most occasions students did not ask any homework questions. Investigative tasks were not a part of the lesson plan during any of the observations, and there was very little classroom interaction. Most of class was spent in lecture, with the teacher writing on the document camera and students writing notes. All names in the write-up of the observations are pseudonyms.

The students spent most their time writing down what was projected on the board via the document camera. They responded positively to the teacher, and for the most part took notes on his explanations. The teacher rarely asked students to provide answers or volunteer to put something up on the board. Students were never asked to solve a problem in class by themselves. Students were not questioned about ideas from other math classes, such as the equation of a circle, the shape of an exponential function, etc. Rather the teacher put equations on the board explaining that “all students need to know this,” and proceeded with the lecture.

There were few connections with other subjects or real-world examples. An application to velocity and distance was referred to as something that “we did not have time to get to.” No investigative tasks were observed. Math concepts were referred to as formulas, and math appeared as a set of procedures to be memorized. Mathematical
knowledge was seen as something that flows from the teacher as evidenced through phrases such as “The last trick I want to show you,” referring to the FTC, or “You need lots of practice writing these out,” referring to Riemann sums and summation notation.

**Interaction and classroom discourse at CSNE.** The CSNE Calculus class was driven by the teacher. Students did not interact with each other, and rarely asked questions of the teacher. More than 85% of the class time was spent in teacher lecture mode (TL), with students copying notes from the board. The teacher often posed a question, paused, then proceeded to answer himself. For example, on Day 2 of the observation, students were reviewing Riemann sums. The teacher wanted students to write the definition of the definite integral of a function on the interval \([0, 3]\).

*Teacher:* How do you get \( \Delta x = \frac{3}{4} \)?

*Teacher:* Well, formulas that are useful I will use boxes. This will always be:

\[
\Delta x = \frac{b-a}{n} \quad x_1 = a \quad x_i = x_1 + (i - 1)\Delta x
\]

How do you get \( b \) and \( a \)?

*Teacher:* Those are given, so we could do \( \Delta x = \frac{3-0}{4} = \frac{3}{4} \)

*Teacher:* How can we get closer to the integral?

*Teacher:* We take more rectangles.

During this episode, students were busy taking notes. They did not volunteer any answers. No gender issues arose, and female students asked a few clarification questions, which demonstrated that they felt safe and at ease in the class. However, student participation and engagement were at a minimum.

**Multiple Representations at CSNE.** The predominant representations used by Professor Smith were numerical, symbolic and verbal. Graphs were used sparingly as more of an add-on than as a meaning-making tool. A few applications were used in the
computation of the area of the definite integral, but context was not used to situate or launch problems. The class worked with few graphs, and no connections were made to prior student knowledge, even with familiar graphs. For example, on Day 2, the teacher asked students to find the area in the first quadrant under \( f(x) = \sqrt{9 - x^2} \). The teacher then said:

*Teacher:* I am going to draw this. This will actually be half or a quarter of a perfect circle.

He did not ask students to either draw the graph themselves or to say what the graphs is. The question presented was in the context of the Riemann sum definition of the integral and approximating the exact value by increasing the number of partitions. An exact answer, by integration, would have involved trigonometric substitution. Students could have predicted the answer by geometry because they knew the area of a circle or of a quarter circle, but they were not asked to do so.

The teacher showed students how to get the area using the midpoint approximation. He paused to make connections between the symbols and the numbers is in the problem, but other than asking questions like “Do you get this?” he did not ask for input from the class. The teacher mentioned that students would “need to practice the formulas” for approximating the definite integral and wrote the following on the board:

*Teacher:* Might as well use midpoint formula if not told otherwise \( \Delta x = \frac{b-a}{n} \)

So, \( \Delta x = \frac{3-0}{4} = \frac{3}{4} \) So:

\[
MPS: \sum_{i=1}^{5} f \left( \frac{x_i + x_{i+1}}{2} \right) \Delta x = \left( \frac{x_1 + x_2}{2} \right) \Delta x + f \left( \frac{x_2 + x_3}{2} \right) \Delta x + \left( \frac{x_3 + x_4}{2} \right) \Delta x + f \left( \frac{x_4 + x_5}{2} \right) \Delta x
\]

He then drew the picture below and commented:
Teacher: You should get in practice of writing it out. I am going to do one more step. I will factor the $\Delta x$ first

$$
\frac{3}{4} \sum \left[ f \left( \frac{0 + \frac{3}{4}}{2} \right) + f \left( \frac{\frac{3}{4} + \frac{3}{2}}{2} \right) + f \left( \frac{\frac{3}{2} + \frac{5}{4}}{2} \right) f \left( \frac{\frac{5}{4} + \frac{3}{2}}{2} \right) \right]
$$

Teacher: Theoretically, you could plug this into the calculator. There is a neat program. I put it in Moodle. You can just put in the sum and delta x: I will show you a trick you should try to follow. Go to List. 2 Stat. Go to Math. There’s something called Sequence.

The professor demonstrated how to enter the formula above in the calculator as

$$
\frac{3}{4} \sum \left( \text{seq} \left( \frac{3}{8}, \frac{19}{8}, \frac{3}{4} \right) \right) \approx 7.1638
$$

and added:

Teacher: Why do you do everything else? You don’t have to.

While the teacher lectured, students were busy entering the steps in the calculator or taking notes. When he paused, Jackie, a female student in the class asked about the 8 in the denominator.

Jackie: I do not understand the 8 in the denominator. I get the delta x being $\frac{3}{4}$

Teacher: The first 3/8 is the first midpoint, and the last is the last midpoint. If you add 5/4 and 3, you get 19/8. Does that work for you?

Jackie: Yes. I got it.
Most of the representations in Professor Smith’s class focused on numerical and symbolic concepts. When they are used, they were accompanied by the expectation that students go home and review what was done in class. Students did review the notes when they got home, since they on occasion came to class a little early and asked each other or the teacher about issues of which they were unsure. A symbolic approach followed by a numerical one was characteristic of the class.

The excerpt that follows, highlights the teacher talking about approximations to the definite integral, and making use of symbolic and numerical representations. The teacher explained how this approximation can be written symbolically with a summation notation, just like the left and right endpoint approximations could. There is no graphical representation to accompany this explanation, but several summands are then written out to make the notation more transparent.

**Teacher:** Last one I want to show you is the midpoint formula to approximate the integral using the midpoint formula for 6 divisions is:

$$Mid \ S = \sum_{i=1}^{6} f \left( \frac{x_i + x_{i+1}}{2} \right) \Delta x$$

Think about this. It’s

$$f \left( \frac{x_1 + x_2}{2} \right) \Delta x + f \left( \frac{x_2 + x_3}{2} \right) \Delta x + \cdots + f \left( \frac{x_6 + x_7}{2} \right) \Delta x$$

This is always a formula. Just like always we have the left-hand sum and right-hand sum equal to: $LHS = \sum_{i=1}^{n} f(x_i) \Delta x \hspace{1cm} RHS = \sum_{i=2}^{n+1} f(x_{i+1}) \Delta x$

The teacher completed the example making some comments about the exam and better approximations while students copied down notes.

Teachers often use probing sequences to highlight, clarify or to make explicit a particular strategy. Thus, they can position student thinking in relation to the mathematics in ways that can support student understanding. Students followed what the teacher showed them on the calculator. However, throughout the class, the teacher offered
students few opportunities to talk about or to grapple with mathematical concepts on their own.

**Background Questionnaire at CSNE.** Nineteen students were enrolled in Professor Smith’s class at CSNE. Most students at CSNE took Calculus for their major and a majority (all but two) responded that they were motivated to work hard by their desire to get good grades. Eight students (five out of twelve female and three out of seven male) students had taken Calculus in high school.

**Summary of the Enacted Curriculum at CSNE.** The CSNE class was characterized by almost no student–teacher interaction, and virtually no student–student interaction. There was no group work and little whole class discussion with the students. The class consisted almost exclusively of lecture. The teacher suggested that the students needed to practice the “formulas” a lot to be proficient.

The mathematical tasks as enacted in the CSNE curriculum focused on lower-level demand tasks that are mainly described as *memorization* (Stein, 2000), involving reproducing previously learned facts, rules, formulas or definitions or committing these to memory. Based on the classroom observations and working hypothesis, it was expected that students in this class would perform more strongly on numerical and symbolic problems than on graphical, contextual or verbal. Based on the indicators from the LOP, multiple representation instruction rated highest on the numerical representation, followed by the symbolical one. The numerical representation rated a 4 out of 7 because the teacher modeled using numerical approximations throughout the presentation of the FTC and because students practiced these approximations during class. While the teacher used analytic representations heavily, the symbolic
representation rated lower than the numerical one, because students did not have the chance to talk about this representation during class. The other three representations (verbal, graphical and contextual) were all rated lower because they were infrequently used by students. However, the teacher used verbal explanations extensively. The overall enacted MR score according to the proposed scheme in the analysis is 1.

Table 22. MR scores for CSNE.

<table>
<thead>
<tr>
<th>Class</th>
<th>Graphical</th>
<th>Numerical</th>
<th>Verbal</th>
<th>Contextual</th>
<th>Symbolic</th>
<th>Overall</th>
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<tr>
<td>CSNE</td>
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<td>4</td>
<td>3</td>
<td>1</td>
<td>3</td>
<td>1</td>
</tr>
</tbody>
</table>

Throughout the observation, there were only two instances of student initiated discourse (SID), and only two instances coded as FSL for female student learning, although both these instances involved clarification questions. There were no instances that coded as RF, CR, or JE because students in this class did not have the opportunity to engage with mathematical content meaningfully, but were rather tacit observers of a curriculum that unfolded before their eyes.

**Themes and Patterns in the Enacted FTC Curriculum**

After completing the classroom portraits, a list of themes and patterns in each class was made based on the portrait, the Lesson Observation Protocol and the coding scheme used. It is summarized in Table 23. Differences were observed in the level of student engagement, pedagogical approaches and the multiple representations used in the classroom. Classrooms were labeled by the predominant pedagogical method used in instruction as it emerged in the inductive analysis of the field notes and LOP ratings. RCC was labeled as Whole Group Interactive; HCCC as Small Group and Whole Class Interactive, and CSNE as Traditional Lecture and non-interactive.
Patterns Emerging from the Analysis of the Enacted Curriculum

In the exploration of the research questions regarding the nature of multiple representations in the Enacted Curriculum, similarities and differences in the use of multiple representations across the three sites were noted through the Lesson Observation Protocol and field notes. Although all sites used multiple representations of the FTC, these were reflected differently at the three locations. Although there were variations in the use of multiple representations by the teacher, the main difference was student participation in the classroom discourse. One of the patterns that emerged was the use of interaction as a gateway to student understanding with multiple representations, with students at RCC and HCCC being engaged in an interactive way, and students at CSNE being exposed to a lecture style classroom.

Student participation is essential to student understanding (NCTM, 2000, 2009). Learning is shaped by their opportunities to participate in mathematical practices (Yackel, Rasmussen, & King, 2000), and this opportunity is particularly important to those students from underrepresented groups, such as females (Rasmussen & Ellis, 2013). Two classes observed were of a highly interactive nature. At RCC, the teaching format was mostly large group interactive, with short periods of lecture. The teacher predominantly used the IRE pattern of discourse, but he called on various students to answer or explain reasoning. On the other hand, the HCCC instructor used a combination or small group interactive work, along with lecture and large group discussions in his implementation of the FTC. In the third class, the predominant method of instruction was lecture and very few interactions were present. Female students in the first two classes were active participants in the mathematics discourse. While female students are in
general less likely to take risks or veer away from the enacted curriculum of the instructor (Rasmussen, 2012; Boaler & Staples, 2004; Blackett & Tall, 1997), the class or small group discussions afforded them risk taking opportunities, and opportunities to learn or to expand their learning. For example, in Professor Brown’s class we see Cindy as a student, who grows in her understanding due to the supportive and collaborative environment that allows her to try out her conjectures. Other students are supportive as they encourage her with comments such as “You can do this”, or “You try now”, and Cindy herself says “I guess I got it. I could never do this in high school.”

Although participants were not randomly selected, and a baseline for student learning was not calculated, the background questionnaire controlled for prior knowledge. Students in all three classes, had similar prior experiences with calculus, and the percentages of female students, who had taken calculus in high school, were similar. All the students were motivated by wanting good grades or by the desire to understand. Students at RCC and HCCC were actively engaged in the classroom discourse, while students at CSNE were not invited to participate in mathematical discourse.

Another theme that emerged was that of the enacted multiple representations at each site. Instructors used a variety of multiple representations to talk about the Fundamental Theorem. The RCC curriculum included significant graphical, symbolic and verbal representations in the presentation of the FTC. Also, due to the nature of the implementation, students were encouraged to make significant use of graphical, symbolic and verbal representations in their class work. To a lesser extent, this was true of numeric and contextual representations, but the opportunities were not as frequent. Similarly, at HCCC, the teacher and students used contextual, graphical, verbal and symbolic
representations extensively while communicating about the FTC. At CSNE, the teacher made extensive use of symbolic and numerical representations, but there was no opportunity for student discourse and interaction.

According to the LOP, a rating of 6 or 7 corresponds to significant use of a representation, both by the instructor and the students, a rating a 1 or 2 indicates little or no significant use of that representation, while a mid-range rating is between 3 and 5. The class at CSNE was rated lower on all enacted representations because the representations were used by the teacher, but not actively used by students. These ratings were recorded in Table 23.

Table 23. Enacted curriculum LOP scores for all three schools.

<table>
<thead>
<tr>
<th>Site</th>
<th>Verbal</th>
<th>Graphical</th>
<th>Numerical</th>
<th>Contextual</th>
<th>Symbolic</th>
<th>Overall</th>
</tr>
</thead>
<tbody>
<tr>
<td>RCC</td>
<td>4</td>
<td>5</td>
<td>4</td>
<td>2</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>HCCC</td>
<td>4</td>
<td>5</td>
<td>3</td>
<td>6</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>CSNE</td>
<td>3</td>
<td>1</td>
<td>4</td>
<td>1</td>
<td>3</td>
<td>1</td>
</tr>
</tbody>
</table>

More importantly, a theme that surfaced was that in the interactive classes, students used more representations and the level of representations was deeper, according to the LOP protocol in Appendix A. Even if the teacher did not specify a specific representation, students connected several representations in their dialogue, leading to a better understanding. Because of the classroom interaction, there was a level of safety that permitted risk taking, which is particularly important in the mathematics classroom, and especially so for female students (Boaler, 2002, 2008). In addition, in the HCCC’s enactment of the curriculum, significant emphasis was placed on group work. While female students shy away from taking chances in their problem-solving, the groups afforded them the security they needed to decide to explore multiple representations. For
example, in the small group investigative task, Cindy was not afraid to ask questions or to think out loud and the other students included and encouraged her participation. A summary of the patterns is in the table below.

Table 24. Patterns and themes in the enacted curriculum at all three schools.

<table>
<thead>
<tr>
<th>Class</th>
<th>Patterns And Themes In The Enacted Curriculum</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>RCC</td>
</tr>
<tr>
<td>Class Style</td>
<td>Whole Group Interactive</td>
</tr>
<tr>
<td>Pedagogy</td>
<td>Teacher centered discussion</td>
</tr>
<tr>
<td></td>
<td>IRE Pattern</td>
</tr>
<tr>
<td></td>
<td>Teacher lecture</td>
</tr>
<tr>
<td>Student-Student Interactions</td>
<td>No group work</td>
</tr>
<tr>
<td>Classroom Discourse</td>
<td>Student participation in class discourse</td>
</tr>
<tr>
<td>Teacher-Student Interactions</td>
<td>Teaching is done in a large group interactive format where teacher poses questions and students answer</td>
</tr>
<tr>
<td>Perspective on Mathematics</td>
<td>Meaning making perspective towards mathematics</td>
</tr>
<tr>
<td>Multiple Representations</td>
<td>Multiple representation: Emphasis on graphical, verbal, symbolic representations with some contextual representations</td>
</tr>
<tr>
<td></td>
<td>HCCC</td>
</tr>
<tr>
<td></td>
<td>Small Group and Whole Group Interactive</td>
</tr>
<tr>
<td>Pedagogy</td>
<td>Teacher uses a variety of teaching methods including small group work, lecture, group discussions</td>
</tr>
<tr>
<td>Student-Student Interactions</td>
<td>Group work</td>
</tr>
<tr>
<td>Classroom Discourse</td>
<td>Students self-select themselves into groups and interact with each other</td>
</tr>
<tr>
<td>Teacher-Student Interactions</td>
<td>Teacher is not the ultimate source of knowledge in the classroom</td>
</tr>
<tr>
<td>Perspective on Mathematics</td>
<td>Recognition that mathematics is about ideas and meaning rather than procedural understanding</td>
</tr>
<tr>
<td>Multiple Representations</td>
<td>Ample use of multiple representations, especially context and graphical representations</td>
</tr>
<tr>
<td></td>
<td>CSNE</td>
</tr>
<tr>
<td></td>
<td>Traditional Lecture Whole Group Non-interactive</td>
</tr>
<tr>
<td>Pedagogy</td>
<td>Teacher centered discussion</td>
</tr>
<tr>
<td></td>
<td>Teacher does not ask students to work on problems in class</td>
</tr>
<tr>
<td>Student-Student Interactions</td>
<td>No group work</td>
</tr>
<tr>
<td>Classroom Discourse</td>
<td>No student participation in class discourse</td>
</tr>
<tr>
<td>Teacher-Student Interactions</td>
<td>Teacher is the only source of knowledge</td>
</tr>
<tr>
<td>Perspective on Mathematics</td>
<td>No meaning-making perspective toward mathematics</td>
</tr>
<tr>
<td>Multiple Representations</td>
<td>Multiple representations: No contextual problems given in class</td>
</tr>
</tbody>
</table>

The quantitative section presented in the next chapter analyzes the FTC assessments and *Think-Alouds*. These results are then linked to the enacted curriculum at each site to either confirm or revise the hypothesis of this study. To the extent possible,
the background questionnaire was used to rule out other factors influencing student understanding.

Chapter 5 reports results obtained under student understanding of the Fundamental Theorem of Calculus. This concept appears on the left side of the concept variable map in Figure 15 presented at the introduction of Chapter 3. The results on student understanding come from both the FTC assessment taken by all students and from the Think-Aloud Data collected from the nine participants selected for the follow up semi-structured interviews.
CHAPTER 5
RESULTS ON STUDENT UNDERSTANDING

Chapter Organization

The previous chapter described the Enacted Curriculum observed at the three research sites and the classroom portraits constructed based on field notes and the Lesson Observation Protocol. The purpose was to report the results on the Enacted Curriculum. To make connections between the enacted curriculum and students’ understanding of the FTC, this chapter will describe the results falling under Student Understanding. To measure students’ understanding of the FTC, the study used the following instruments introduced in Chapter 3: 1) Background Questionnaire, 2) Five FTC Problems, and 3) Think-Aloud Protocol. Field notes and LOP data were used to further triangulate findings.

The first part of the chapter discusses the results of the FTC Assessments, which include: 1) descriptive statistics, 2) ANOVA results comparing student scores on the FTC assessments, 3) regression results involving student scores, student cognitive preference and perception of their instruction with multiple representations, 4) ANOVA results involving gender and, 5) regression results involving gender, student scores, and site.

The second part of the chapter discusses the results of the Think-Alouds for the nine students who participated in the semi-structured interviews following the FTC assessments, and identifies themes across sites, then triangulates the results with written work from the FTC Assessment.

Student Understanding of the FTC

To characterize the distribution of scores on the FTC assessment at the three locations, descriptive statistics such as mean, median, mode, variance, range, were
calculated. The students’ individual overall assessment score (TS) and five representation scores (verbal, graphical, numerical, contextual and symbolic) from the *Five Problems Involving the FTC* were used to make comparisons between student understanding at the three locations, and between males and females. Boxplots of the assessment scores at each site were created. These statistics are reported below by site.

**Descriptive Statistics**

**RCC Assessments**

Twenty-three students (17 Male and 6 Female) from Riverside Community College completed in the *Five Problems Involving the FTC*.

Table 25. RCC scores.

<table>
<thead>
<tr>
<th>RCC</th>
<th>Male</th>
<th>Median</th>
<th>Range</th>
<th>Minimum</th>
<th>Maximum</th>
<th>Std. Dev.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total score</td>
<td>49.06</td>
<td>50.50</td>
<td>51.00</td>
<td>19.00</td>
<td>70.00</td>
<td>14.35</td>
</tr>
<tr>
<td></td>
<td>51.10</td>
<td>53.80</td>
<td>50.00</td>
<td>25.00</td>
<td>75.00</td>
<td>16.63</td>
</tr>
<tr>
<td></td>
<td>50.06</td>
<td>51.70</td>
<td>56.00</td>
<td>19.00</td>
<td>75.00</td>
<td>16.10</td>
</tr>
<tr>
<td>Graphical</td>
<td>58.81</td>
<td>60.00</td>
<td>60.00</td>
<td>27.00</td>
<td>87.00</td>
<td>16.24</td>
</tr>
<tr>
<td></td>
<td>58.67</td>
<td>60.00</td>
<td>40.00</td>
<td>33.00</td>
<td>73.00</td>
<td>15.51</td>
</tr>
<tr>
<td></td>
<td>58.78</td>
<td>60.00</td>
<td>60.00</td>
<td>27.00</td>
<td>87.00</td>
<td>15.72</td>
</tr>
<tr>
<td>Numerical</td>
<td>55.69</td>
<td>58.00</td>
<td>50.00</td>
<td>33.00</td>
<td>83.00</td>
<td>17.23</td>
</tr>
<tr>
<td></td>
<td>53.00</td>
<td>54.50</td>
<td>50.00</td>
<td>25.00</td>
<td>75.00</td>
<td>19.50</td>
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<tr>
<td></td>
<td>54.86</td>
<td>62.50</td>
<td>58.00</td>
<td>25.00</td>
<td>83.00</td>
<td>18.16</td>
</tr>
<tr>
<td>Verbal</td>
<td>46.75</td>
<td>56.00</td>
<td>78.00</td>
<td>0.00</td>
<td>78.00</td>
<td>23.56</td>
</tr>
<tr>
<td></td>
<td>55.83</td>
<td>56.00</td>
<td>23.00</td>
<td>44.00</td>
<td>67.00</td>
<td>7.28</td>
</tr>
<tr>
<td></td>
<td>50.21</td>
<td>56.00</td>
<td>78.00</td>
<td>0.00</td>
<td>78.00</td>
<td>20.95</td>
</tr>
<tr>
<td>Contextual</td>
<td>42.94</td>
<td>43.50</td>
<td>87.00</td>
<td>0.00</td>
<td>87.00</td>
<td>24.90</td>
</tr>
<tr>
<td></td>
<td>45.50</td>
<td>40.00</td>
<td>93.00</td>
<td>0.00</td>
<td>93.00</td>
<td>30.42</td>
</tr>
<tr>
<td></td>
<td>43.29</td>
<td>40.00</td>
<td>93.00</td>
<td>0.00</td>
<td>93.00</td>
<td>28.96</td>
</tr>
<tr>
<td>Symbolic</td>
<td>41.13</td>
<td>56.00</td>
<td>78.00</td>
<td>0.00</td>
<td>78.00</td>
<td>25.46</td>
</tr>
<tr>
<td></td>
<td>42.50</td>
<td>33.00</td>
<td>67.00</td>
<td>11.00</td>
<td>78.00</td>
<td>24.99</td>
</tr>
<tr>
<td></td>
<td>42.93</td>
<td>44.50</td>
<td>67.00</td>
<td>11.00</td>
<td>78.00</td>
<td>23.65</td>
</tr>
<tr>
<td>Participants</td>
<td>16 Male, 6 Female, 22 Overall</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
The maximum possible score was 100 points. The minimum score at RCC was 19.00, maximum 75.00, median 51.70, and standard deviation of 16.10. Female students had a minimum score of 25.00, maximum of 75.00, median score of 53.80, and standard deviation of 16.63. Male students had a minimum score of 19.00, maximum of 70.00, median score of 49.10, and standard deviation of 14.35. Students at RCC scored highest on the graphical representation, with a median score of 60.00, followed by the numerical representation with a score of median score of 62.50. They scored lowest on the symbolical and contextual representations with median scores of 44.40 and 40.00 respectively. Scores of their FTC assessment are illustrated in Table 25.

**HCCC Assessments**

At HCCC, 14 students (8 Male and 6 Female) took the FTC assessment. The minimum score at HCCC was 14.80, maximum 85.80, median 62.00, and standard deviation of 20.23. Female students had a minimum score of 34.00 (out of 100.00), a maximum of 85.80, a median score of 59.30, and standard deviation of 20.05. Male students had a minimum score of 14.80, a maximum of 77.40, a median score of 62.00, and standard deviation of 21.70. Scores for on the FTC assessment are illustrated in both table and box plot form. The scores for male and female students appeared similar. Students at HCCC scored highest on the context problem with a median score of 73.00 and the lowest on the symbolic problem with a median score of 50.00. HCCC summary statistics are reported in Table 26.
### Table 26. HCCC scores.

<table>
<thead>
<tr>
<th>HCCC</th>
<th>Mean</th>
<th>Median</th>
<th>Range</th>
<th>Minimum</th>
<th>Maximum</th>
<th>Std. Dev.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total score</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Male</td>
<td>55.73</td>
<td>62.00</td>
<td>62.60</td>
<td>14.80</td>
<td>77.40</td>
<td>21.70</td>
</tr>
<tr>
<td>Female</td>
<td>57.70</td>
<td>59.30</td>
<td>51.80</td>
<td>34.00</td>
<td>85.80</td>
<td>20.05</td>
</tr>
<tr>
<td>Overall</td>
<td>56.57</td>
<td>62.00</td>
<td>71.00</td>
<td>14.80</td>
<td>85.80</td>
<td>20.23</td>
</tr>
<tr>
<td>Graphical</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Male</td>
<td>62.38</td>
<td>66.50</td>
<td>53.00</td>
<td>27.00</td>
<td>80.00</td>
<td>18.05</td>
</tr>
<tr>
<td>Female</td>
<td>50.00</td>
<td>46.50</td>
<td>53.00</td>
<td>27.00</td>
<td>80.00</td>
<td>21.89</td>
</tr>
<tr>
<td>Overall</td>
<td>57.07</td>
<td>60.00</td>
<td>53.00</td>
<td>27.00</td>
<td>80.00</td>
<td>20.00</td>
</tr>
<tr>
<td>Numerical</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Male</td>
<td>56.25</td>
<td>62.50</td>
<td>75.00</td>
<td>0.00</td>
<td>75.00</td>
<td>24.33</td>
</tr>
<tr>
<td>Female</td>
<td>47.17</td>
<td>46.00</td>
<td>50.00</td>
<td>25.00</td>
<td>75.00</td>
<td>17.99</td>
</tr>
<tr>
<td>Overall</td>
<td>52.36</td>
<td>58.00</td>
<td>75.00</td>
<td>0.00</td>
<td>75.00</td>
<td>21.56</td>
</tr>
<tr>
<td>Verbal</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Male</td>
<td>50.13</td>
<td>61.50</td>
<td>78.00</td>
<td>0.00</td>
<td>78.00</td>
<td>26.74</td>
</tr>
<tr>
<td>Female</td>
<td>65.00</td>
<td>61.50</td>
<td>67.00</td>
<td>33.00</td>
<td>100.00</td>
<td>22.73</td>
</tr>
<tr>
<td>Overall</td>
<td>56.50</td>
<td>61.50</td>
<td>100.00</td>
<td>0.00</td>
<td>100.00</td>
<td>25.34</td>
</tr>
<tr>
<td>Contextual</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Male</td>
<td>66.75</td>
<td>73.00</td>
<td>60.00</td>
<td>27.00</td>
<td>87.00</td>
<td>20.38</td>
</tr>
<tr>
<td>Female</td>
<td>69.00</td>
<td>77.00</td>
<td>80.00</td>
<td>20.00</td>
<td>100.00</td>
<td>33.14</td>
</tr>
<tr>
<td>Overall</td>
<td>67.71</td>
<td>73.00</td>
<td>80.00</td>
<td>20.00</td>
<td>100.00</td>
<td>25.45</td>
</tr>
<tr>
<td>Symbolic</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Male</td>
<td>43.13</td>
<td>50.00</td>
<td>67.00</td>
<td>0.00</td>
<td>67.00</td>
<td>28.28</td>
</tr>
<tr>
<td>Female</td>
<td>57.33</td>
<td>55.50</td>
<td>78.00</td>
<td>22.00</td>
<td>100.00</td>
<td>26.88</td>
</tr>
<tr>
<td>Overall</td>
<td>49.21</td>
<td>50.00</td>
<td>100.00</td>
<td>0.00</td>
<td>100.00</td>
<td>27.60</td>
</tr>
</tbody>
</table>

**Participants** 8 Male, 6 Female, 14 Overall

### CSNE Assessments

At CSNE, seventeen students (12 Female and 5 Male) took the FTC assessment. Out of a possible 100 points the overall minimum score at CSNE was 0.00 (all answers were incorrect), maximum 77.60, median 42.80, and standard deviation of 20.00. Female students had a minimum score of 0.00, a maximum of 67.40, a median score of 37.90, and standard deviation of 19.83. Male students had a minimum score of 23.40, a maximum of 77.60, a median score of 51.80, and standard deviation of 16.8. Students at CSNE scored lowest on the contextual problem with a median score of 13.00, and the
highest on the numerical problem with the median score of 58.00. Further statistics are in Table 27 below.

Table 27. CSNE scores.

<table>
<thead>
<tr>
<th>CSNE</th>
<th>Mean</th>
<th>Median</th>
<th>Range</th>
<th>Minimum</th>
<th>Maximum</th>
<th>Std. Dev.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Male</td>
<td>Female</td>
<td>Overall</td>
<td>Male</td>
<td>Female</td>
<td>Overall</td>
</tr>
<tr>
<td>Total Score</td>
<td>52.60</td>
<td>35.25</td>
<td>41.60</td>
<td>51.80</td>
<td>37.90</td>
<td>42.80</td>
</tr>
<tr>
<td></td>
<td>54.20</td>
<td>67.40</td>
<td>77.60</td>
<td>23.40</td>
<td>67.60</td>
<td>77.60</td>
</tr>
<tr>
<td></td>
<td>77.60</td>
<td>16.80</td>
<td>19.83</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Graphical</td>
<td>48.40</td>
<td>33.42</td>
<td>38.90</td>
<td>60.00</td>
<td>27.00</td>
<td>40.00</td>
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<td></td>
<td>80.00</td>
<td>87.00</td>
<td>87.00</td>
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<td></td>
<td>80.00</td>
<td>31.60</td>
<td>27.99</td>
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<td></td>
</tr>
<tr>
<td>Numerical</td>
<td>68.00</td>
<td>45.75</td>
<td>53.90</td>
<td>60.00</td>
<td>46.00</td>
<td>58.00</td>
</tr>
<tr>
<td></td>
<td>80.00</td>
<td>83.00</td>
<td>83.00</td>
<td>0.00</td>
<td>83.00</td>
<td>83.00</td>
</tr>
<tr>
<td></td>
<td>80.00</td>
<td>23.94</td>
<td>31.60</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Verbal</td>
<td>51.10</td>
<td>53.83</td>
<td>52.80</td>
<td>60.00</td>
<td>56.00</td>
<td>60.00</td>
</tr>
<tr>
<td></td>
<td>80.00</td>
<td>156.00</td>
<td>56.00</td>
<td>0.00</td>
<td>156.00</td>
<td>56.00</td>
</tr>
<tr>
<td></td>
<td>80.00</td>
<td>31.60</td>
<td>41.97</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Contextual</td>
<td>38.10</td>
<td>21.00</td>
<td>27.30</td>
<td>60.00</td>
<td>13.00</td>
<td>13.00</td>
</tr>
<tr>
<td></td>
<td>80.00</td>
<td>53.00</td>
<td>73.00</td>
<td>0.00</td>
<td>73.00</td>
<td>73.00</td>
</tr>
<tr>
<td></td>
<td>80.00</td>
<td>31.60</td>
<td>25.00</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Symbolic</td>
<td>57.40</td>
<td>22.25</td>
<td>35.20</td>
<td>60.00</td>
<td>0.00</td>
<td>44.00</td>
</tr>
<tr>
<td></td>
<td>80.00</td>
<td>67.00</td>
<td>100.00</td>
<td>0.00</td>
<td>67.00</td>
<td>100.00</td>
</tr>
<tr>
<td></td>
<td>80.00</td>
<td>31.60</td>
<td>28.92</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Participant</td>
<td>7 Male, 12 Female, 19 Overall</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Boxplots for Student Understanding of the FTC**

Boxplots for the total score and individual representations by site are presented below in Figure 24. On the left top row, Total Scores at CSNE visually appear slightly lower than scores at HCCC and RCC. On the right side of the first row, the Graphical Scores at CSNE, which was the lecture curriculum and which did not include graphical representations are lower than corresponding scores in at the other two sites, which both
included these representations in the curriculum and student participation during instruction.

Comparable results appear in the bottom row on the left side, where students’ Contextual Scores at CSNE are lower than those at RCC and HCCC. The same figure shows that scores at HCCC, where Professor Brown and the students worked on contextual problems throughout the course, are higher than scores at the other two locations.

Symbolical scores at CSNE represented on the bottom left row are also lower than the corresponding scores at the other two locations. Despite the heavy use of symbolical representations on the part of the teacher, students in that course did not have the opportunity to engage with that representation on their own. Symbolical scores boxplots at all three locations appear lower in general than scores on other representations.

Numerical and Verbal Score boxplots presented in the second row of the figure appear similar for all three sites.

As hypothesized, there seems to be a connection between the enacted curriculum and student understanding within some representations. For example, as presented in Chapter 4, Enacted Curriculum scores for the contextual and graphical representations were both equal to 1. These will be explored further in Chapter 6.
Figure 24. Boxplots of the total and individual representation scores by site. From left to right: Total, Graphical, Numerical, Verbal, Contextual, and Symbolic scores.
Inferential Statistics on Student Understanding

Following descriptive statistics, quantitative methods were used to further confirm differences or similarities in student understanding as measured by the FTC assessments. The results of the inferential tests included in this chapter are analyzed in Chapter 6, in conjunction with the results on the enacted curriculum presented in Chapter 4, to help generate conjectures as to the relation between differences in student understanding observed, and the classroom experience.

ANOVA Results on Student Understanding

ANOVA tests for difference in means at the three sites for the Total Score and for Individual Representation scores for the students’ assessment scores indicate a close to significant difference in total score (TS) across sites \([F (2, 52) = 2.77, p = 0.07]\) and significant differences in mean scores across sites on the contextual and graphical representation scores. More specifically:

1. There was a significant difference by site in the Graphical Score (GS) at the three sites \([F (2, 52) = 3.2, p = 0.049]\). Students at CSNE performed significantly lower than students at RCC \((p = 0.044)\) and HCCC \((p = 0.037)\) as confirmed by a post hoc Fisher LSD test.

2. There was a significant difference in Contextual Score (CS) at the three sites \([F (2, 52) = 8.64, p = 0.0006]\). Students at CSNE performed significantly lower than students at RCC \((p = 0.008)\) and HCCC \((p = 0.0005)\) as confirmed by a post hoc Fisher LSD test.

These results were confirmed by a two-way ANOVA that analyzed differences in scores by site and by gender and their interaction. This analysis is included in answering
research question 2 regarding the effects of gender on student understanding and is addressed later in this chapter. Table 28 reports the results of the ANOVA tests.

Table 28. ANOVA results of student understanding.

<table>
<thead>
<tr>
<th>Score</th>
<th>$p$</th>
<th>Significance</th>
<th>$F$ (2, 52)</th>
<th>$Eta$-square</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total</td>
<td>0.07</td>
<td>Not Significant</td>
<td>2.77</td>
<td>0.31</td>
</tr>
<tr>
<td>Graphical</td>
<td>0.049</td>
<td>Significant</td>
<td>3.2</td>
<td>0.20</td>
</tr>
<tr>
<td>Contextual</td>
<td>0.0006</td>
<td>Significant</td>
<td>8.64</td>
<td>0.31</td>
</tr>
<tr>
<td>Verbal</td>
<td>0.070</td>
<td>Not significant</td>
<td>0.21</td>
<td>0.04</td>
</tr>
<tr>
<td>Numerical</td>
<td>0.77</td>
<td>Not Significant</td>
<td>0.27</td>
<td>0.11</td>
</tr>
<tr>
<td>Symbolic</td>
<td>0.50</td>
<td>Not Significant</td>
<td>0.61</td>
<td>0.2</td>
</tr>
</tbody>
</table>

Regression Results on Student Understanding of the FTC, Students’ Cognitive Preference and Perceived Representational Instruction

To address the role of other factors, such as students’ representational cognitive preference and their perceived representational instruction, in students’ understanding of the FTC, several regression analyses were pursued. Regression analysis on Total Score on the FTC assessment (TS) and of individual representation scores (GS, VS, CS, NS, SS) as a function of site, cognitive preference (CP), perceived representational instruction (PR), and accommodated preference was performed. Regression analysis was performed for each cognitive representation score (GCP, VCP, NCP, CCP, and SCP), and on each perceived representational instruction score (VPR, CPR, NPR, GPR, SPR). The models used in the regression are:

$$TS = F (GCP, VCP, NCP, CCP, SCP)$$

$$TS = F (VPR, CPR, NPR, GPR, SPR)$$
Overall, results suggest that student perception of the instruction was generally not significant, but their cognitive preference, particularly in some domains, yielded some significance, in the contextual, graphical and numerical domains.

**Cognitive Preference and Student Understanding**

The cognitive preference was measured on a Likert scale, with 1 representing low preference for that representation, and 4 high preference for the representation. Regression analysis suggests that graphical, and symbolical cognitive preference were significant in student understanding as measured by the Total Score (TS), but verbal, contextual and numerical were not. Numerical cognitive preference was close to significant.

The exact results appear in Table 29, and possible interpretations of these results provided in Chapter 6.

Table 29. Total Score (TS) as a function of Cognitive Preference (CP), \[F (5, 49) = 7.23; R\text{-square} = 0.43\].

<table>
<thead>
<tr>
<th>Cognitive Preference</th>
<th>(p\text{-value})</th>
<th>Significance</th>
<th>95% Confidence Interval</th>
<th>Eta-Square</th>
</tr>
</thead>
<tbody>
<tr>
<td>Graphical* (GCP)</td>
<td>0.018</td>
<td>Significant</td>
<td>(1.5, 14)</td>
<td>0.109</td>
</tr>
<tr>
<td>Symbolic* (SCP)</td>
<td>0.000</td>
<td>Significant</td>
<td>(6.4, 18.3)</td>
<td>0.263</td>
</tr>
<tr>
<td>Numerical* (NCP)</td>
<td>0.053</td>
<td>Close to significant</td>
<td></td>
<td>0.074</td>
</tr>
<tr>
<td>Contextual (CCP)</td>
<td>0.108</td>
<td>Not significant</td>
<td></td>
<td>0.052</td>
</tr>
<tr>
<td>Verbal (VCP)</td>
<td>0.133</td>
<td>Not significant</td>
<td></td>
<td>0.045</td>
</tr>
</tbody>
</table>

Graphical cognitive preference (GCP) had a significant effect on Total Score (\(p = 0.018\) \(df = 49\)).
Symbolic cognitive preference (SCP) is significant in Total Score \((p = 0.000, df = 49)\).

Numerical cognitive preference (NCP) is not significant (but was close) in Total Score \((p =0.053 \text{ } df=49)\).

Model obtained was: \(TS = 8.63 + 4.41 \text{ CCP} - 7.51 \text{ NCP} + 7.91 \text{ GCP} - 4.98 \text{ VCP} + 12.38 \text{ SCP}\). The \(R^2\) value for this model was 0.426, meaning that 42.6\% of the variability was accounted by this model. The regression table is presented in Table 30. Students’ preference for some representations appears to influence the Total Score representing student understanding of the FTC. A 1-point increase in Graphical cognitive preference increases the Total Score by approximately 8 points, and a 1-point increase in Symbolical cognitive preference increases the Total Score by approximately 12 points. As has been seen previously in the literature review, symbolical followed by graphical representations are the prominent representations in students’ mathematical experience, suggesting this as a possible reason as to why these representations are the significant ones in the Total Score dependent variable. More work needs to be done to fully understand the implications of this result.

Table 30. Regression of Total Score and Cognitive Preference.

<table>
<thead>
<tr>
<th>Source</th>
<th>SS</th>
<th>df</th>
<th>MS</th>
<th>Number of obs = 55</th>
<th>F(5, 49) = 7.23</th>
</tr>
</thead>
<tbody>
<tr>
<td>Model</td>
<td>8068.7896</td>
<td>5</td>
<td>1613.75792</td>
<td>Prob &gt; F = 0.0000</td>
<td>R-squared = 0.4246</td>
</tr>
<tr>
<td>Residual</td>
<td>10934.0977</td>
<td>49</td>
<td>223.14485</td>
<td>Adj R-squared = 0.3659</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>19002.8873</td>
<td>54</td>
<td>351.90532</td>
<td>Root MSE = 14.938</td>
<td></td>
</tr>
</tbody>
</table>

| Coef.    | Std. Err. | t     | P>|t|   | [95% Conf. Interval] |
|----------|-----------|-------|--------|----------------------|
| CCP      | 4.408892  | 2.690321 | 1.64 | 0.108                | -1.9975095 - 9.815294 |
| NCP      | -7.503267 | 3.78498 | -1.98 | 0.053 | -15.10947 - 9.9910 |
| GCP      | 7.914871  | 3.238839 | 2.44 | 0.018 | 1.406181 - 8.443746 |
| VCP      | -4.985929 | 3.262418 | -1.53 | 0.133 | -11.542 - 1.570146 |
| SCP      | 12.38227  | 2.960551 | 4.18 | 0.000 | 6.322824 - 18.33173 |
| _cons    | 8.631168  | 17.92772 | 0.48 | 0.632 | -27.39593 - 44.65826 |
Perceived Representational Instruction and Student Understanding

There was no significant difference from any of the representational preferences. The results appear in Table 31.

Table 31. Total Score (TS) as a function of Perceived Instruction (PR), \(F (5, 49) = 0.68\), R-square = 0.07.

<table>
<thead>
<tr>
<th>Perceived Instruction</th>
<th>p-value</th>
<th>Significance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Graphical (GPR)</td>
<td>0.649</td>
<td>Not Significant</td>
</tr>
<tr>
<td>Symbolic (SPR)</td>
<td>0.529</td>
<td>Not Significant</td>
</tr>
<tr>
<td>Numerical (NPR)</td>
<td>0.774</td>
<td>Not Significant</td>
</tr>
<tr>
<td>Contextual (CPR)</td>
<td>0.846</td>
<td>Not Significant</td>
</tr>
<tr>
<td>Verbal (VPR)</td>
<td>0.563</td>
<td>Not Significant</td>
</tr>
</tbody>
</table>

Needs Met: Accommodated Preference and Student Understanding

The relationship between accommodated cognitive preference and student understanding of the FTC was also examined. The variable measuring whether students’ cognitive preference was accommodated was named: “needs.” To measure needs met, this study used the difference between the standardized score provided by the researcher for the instruction on any of the five representations (GR, CR, NR, VR, SR) or average representational score (average of GR, PR, NR, VR, and SR) and the standardized scores for students’ cognitive preference for that representation (GPR, CPR, NPR, SPR, VPR), or average representational preference score (PR). For example, a researcher’s standardized score of 2 on the graphical representation, and a student’s score of 3 on the same representation, would result in a difference score of -1 on the graphical representation. In this context, a positive score indicated that the students’ needs were met, while a negative indicated the students’ needs were not met.
A summary table presents the results. At CSNE (CS), 17 out of 19 did not have their total representational needs met, while at RCC only three of 22 students did not have their representational needs met. HCCC distribution is in between. Frequency tables for “total needs” are illustrated in Table 32 below, where “zdiff” measures “needs”.

Table 32. Total Needs Met by Site.

<table>
<thead>
<tr>
<th>Total Needs Met</th>
<th>Site</th>
<th>HCCC</th>
<th>RCC</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Not Met</td>
<td>17</td>
<td>8</td>
<td>3</td>
<td>28</td>
</tr>
<tr>
<td>Met</td>
<td>2</td>
<td>6</td>
<td>19</td>
<td>27</td>
</tr>
<tr>
<td>Total</td>
<td>19</td>
<td>14</td>
<td>22</td>
<td>55</td>
</tr>
</tbody>
</table>

Regression models for student understanding as a function of “needs” included Total Score (TS) and individual representational scores (VS, GS, CS, NS, SS), as a function of needs on that representation, site and gender. The label used to denote “needs” is “zdiff” to indicate this score is the difference between the standardized overall score on cognitive preference reported by the student and the standardized score for overall multiple representations provided by the researcher. The suffixes zdiffn, zdiffc, zdiffg, zdiffs and zdiffv were used to refer to the needs for numerical, contextual, graphical, symbolical and verbal instruction, respectively. The needs ranged -2 to 2.

Analysis models included:

\[
\text{TS} = F (\text{zdiff}, \text{site}, \text{gender})
\]
\[
\text{CS} = F (\text{zdiffc}, \text{site}, \text{gender})
\]
\[
\text{GS} = F (\text{zdiffg}, \text{site}, \text{gender})
\]
\[
\text{VS} = F (\text{zdiffv}, \text{site}, \text{gender})
\]
\[
\text{NS} = F (\text{zdiffn}, \text{site}, \text{gender})
\]
SS = $F$ (zdiffs, site, gender)

Results from the regression analyses are shown in Table 33.

Table 33. Regression results for Scores, Needs, Site, and Gender.

<table>
<thead>
<tr>
<th>Score by Category</th>
<th>$P$-Value</th>
<th>Coefficient</th>
<th>Std Error</th>
<th>95% CI</th>
<th>$F$ (3,51), $\eta^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Total Score (TS)</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total needs(zdiff)</td>
<td>0.008*</td>
<td>-2.72</td>
<td>1.00</td>
<td>(-4.7, -0.8)</td>
<td>$F = 3.53$</td>
</tr>
<tr>
<td>Site</td>
<td>0.015*</td>
<td>9.46</td>
<td>3.76</td>
<td>(1.9, 17.0)</td>
<td>$\eta^2 = 0.17$</td>
</tr>
<tr>
<td>Gender</td>
<td>0.18</td>
<td>-6.79</td>
<td>5.06</td>
<td>Not sig.</td>
<td></td>
</tr>
<tr>
<td><strong>Graphical (GS)</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Graph need(zdiffg)</td>
<td>0.018*</td>
<td>-7.25</td>
<td>2.98</td>
<td>(-13.24, -1.3)</td>
<td>$F = 5.60$</td>
</tr>
<tr>
<td>Site</td>
<td>0.001</td>
<td>16.00</td>
<td>4.76</td>
<td>(6.44, 25.6)</td>
<td>$\eta^2 = 0.25$</td>
</tr>
<tr>
<td>Gender</td>
<td>0.233</td>
<td>-7.39</td>
<td>6.12</td>
<td>Not sig.</td>
<td></td>
</tr>
<tr>
<td><strong>Symbolic (SS)</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Symb.</td>
<td>0.842</td>
<td>-0.58</td>
<td>2.91</td>
<td>Not sig</td>
<td>$F = 0.50$</td>
</tr>
<tr>
<td>Needs(zdiffs)</td>
<td>0.786</td>
<td>1.31</td>
<td>4.81</td>
<td>Not sig</td>
<td>$\eta^2 = 0.03$</td>
</tr>
<tr>
<td>Site</td>
<td>0.304</td>
<td>-8.81</td>
<td>8.48</td>
<td>Not sig</td>
<td></td>
</tr>
<tr>
<td>Gender</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Numerical (NS)</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Num. needs (zdiffn)</td>
<td>0.24</td>
<td>2.31</td>
<td>1.95</td>
<td>Not sig</td>
<td>$F = 1.91$</td>
</tr>
<tr>
<td>Site</td>
<td>0.742</td>
<td>-1.07</td>
<td>3.22</td>
<td>Not sig</td>
<td>$\eta^2 = 0.10$</td>
</tr>
<tr>
<td>Gender</td>
<td>0.079</td>
<td>-10.16</td>
<td>5.67</td>
<td>Not sig</td>
<td></td>
</tr>
<tr>
<td><strong>Contextual (CS)</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Context needs</td>
<td>0.058**</td>
<td>6.57</td>
<td>3.39</td>
<td>Not sig</td>
<td>$F = 2.36$</td>
</tr>
<tr>
<td>Site</td>
<td>0.122</td>
<td>7.37</td>
<td>4.69</td>
<td>Not sig</td>
<td>$\eta^2 = 0.12$</td>
</tr>
<tr>
<td>Gender</td>
<td>0.348</td>
<td>-7.79</td>
<td>8.22</td>
<td>Not sig</td>
<td></td>
</tr>
<tr>
<td><strong>Verbal (VS)</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Verbal needs</td>
<td>0.467</td>
<td>-2.11</td>
<td>2.89</td>
<td>Not sig</td>
<td>$F = 0.39$</td>
</tr>
<tr>
<td>Site</td>
<td>0.462</td>
<td>3.45</td>
<td>4.65</td>
<td>Not sig</td>
<td>$\eta^2 = 0.02$</td>
</tr>
<tr>
<td>Gender</td>
<td>0.510</td>
<td>4.56</td>
<td>6.86</td>
<td>Not sig</td>
<td></td>
</tr>
</tbody>
</table>

*significant result
**close to a significant result
Note: $df = 51$ for each category

There was a significant difference in Total Score (TS) as a function of overall representational needs [$F (3,51) = 3.53$, $p = 0.008$]. Each increase of 1 point in needs accounted for a drop of 2.72 points in the total score on the average, with a 95% confidence.
There was also significant difference in **Total Score** by site \([F(3,51) = 3.53, p = 0.015]\).

There was a close to significant difference in **Contextual Score** (CS) as a function of needs met on the contextual representations \([F(3,51) = 2.36, p = 0.058]\).

There was a significant difference in **Graphical Score** (GS) as a function of student representational needs on the graphical representation \([F(3,51) = 5.60, p = 0.018]\).

Each one-point increase in needs corresponded to a 7.26-point drop in **Graphical Score** out of a possible 100 points, with a 95% confidence.

There was a significant difference in **Graphical Score** (GS) as a function of site \([F(3,51) = 5.60, p = 0.001]\).

The statistical results for the significant regressions are recorded in Table 34 and Table 35.

### Table 34. Regression for Total Score, Needs, Gender, and Site.

<table>
<thead>
<tr>
<th>Source</th>
<th>SS</th>
<th>df</th>
<th>MS</th>
<th>Number of obs</th>
<th>=</th>
<th>55</th>
<th>F(3, 51)</th>
<th>=</th>
<th>3.53</th>
</tr>
</thead>
<tbody>
<tr>
<td>Model</td>
<td>3268.52468</td>
<td>3</td>
<td>1089.50823</td>
<td>Prob &gt; F</td>
<td>=</td>
<td>0.0211</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Residual</td>
<td>15734.3626</td>
<td>51</td>
<td>308.516914</td>
<td>R-squared</td>
<td>=</td>
<td>0.1720</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Adj R-squared</td>
<td>=</td>
<td>0.1233</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>19002.8873</td>
<td>54</td>
<td>351.90532</td>
<td>Root MSE</td>
<td>=</td>
<td>17.565</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

| TS     | Coef.    | Std. Err. | t     | P>|t| | [95% Conf. Interval] |
|--------|----------|-----------|-------|------|---------------------|
| zdiff  | -2.724975| .9869537  | -2.76 | 0.008| -4.706367            | -.7435824 |
| Site   | 9.458622 | 3.755702  | 2.52  | 0.015| 1.918736            | 16.99851  |
| genderf1 | -6.787448 | 5.056546 | -1.34 | 0.185| -16.93889            | 3.363993  |
| _cons  | 41.62361 | 5.392218  | 7.72  | 0.000| 30.79828             | 52.44894  |
Table 35. Regression for Graphical Score, Needs, Gender, and Site.

```
. regress GS zdiffg Site genderf1
```

| Source     | SS          | df     | MS          | Number of obs = 55
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Model</td>
<td>7577.34404</td>
<td>3</td>
<td>2525.78135</td>
<td>F(3, 51) = 5.60</td>
</tr>
<tr>
<td>Residual</td>
<td>23000.4014</td>
<td>51</td>
<td>450.988263</td>
<td>Prob &gt; F(3, 51) = 0.0021</td>
</tr>
<tr>
<td>Total</td>
<td>30577.7455</td>
<td>54</td>
<td>566.254545</td>
<td>Adj R-squared = 0.2478</td>
</tr>
</tbody>
</table>

| GS         | Coef.       | Std. Err. | t     | P>|t|  | [95% Conf. Interval] |
|------------|-------------|-----------|-------|------|---------------------|
| zdiffg     | -7.258807   | 2.982428  | -2.43 | 0.018| -13.24628 -1.271333 |
| Site       | 15.99859    | 4.763124  | 3.36  | 0.001| 6.43622 25.56096  |
| genderf1   | -7.390337   | 6.116723  | -1.21 | 0.233| -19.67017 4.889498 |
| _cons      | 37.84454    | 6.982147  | 5.42  | 0.000| 23.8273 51.86179  |

**Effect of Prior Knowledge**

Chi-square tests on the proportions of students with little or with no prior knowledge showed that the classes were similar in background and experience with the FTC and formed a baseline for the study.

1. There was no difference in the proportion of students with no prior knowledge (NPK) and that of students with little prior knowledge (LPK) between the three classes, $\chi^2(2, N = 55) = 1.88, p = .41$.

2. There was no difference between the proportion of female students with no prior knowledge (NK.F) and that of female students with little prior knowledge (LK.F) in the three classes, although female students from CSNE had a somewhat higher percent of female students in the Little Prior Knowledge (LK) category, $\chi^2(2, N = 55) = 7.54, p = 0.183$. 

180
3. There was no difference between the proportion of male students with no prior knowledge and with low prior knowledge, in the three classes, $\chi^2(2, N = 55) = 0.089, p = .96$.

4. There was no difference between the proportions of female students and male students who had no prior knowledge (NK) or little prior knowledge, and the corresponding proportions of male students $\chi^2(2, N = 55) = 5.36, p = .07$.

After a baseline was established, the performance of students with little prior knowledge compared with that of students with no prior knowledge of the FTC was measured. A two sample one-tailed $t$-test for the effect of prior knowledge on student assessment showed that students with prior knowledge performed better than those students with no prior knowledge ($p = 0.004$) by at least 5.4 points. Perhaps this result is to be expected, but contrary to studies that indicate that Calculus should be left to colleges (Leitzelet al., 1987), the current study joins others who indicate that students who have some knowledge of this subject prior to college from their high school experience do better (Ferrini-Mundy & Gaudard, 1992; Tallman, Carlson, Bressoud, & Pearson, 2006; Bressoud, 2015) on Calculus concepts than those who did not take Calculus in high school.

**Gender and Student Understanding of the FTC**

To examine the relation between the use of multiple representations of the FTC in the classroom and female students understanding of the FTC, boxplots by gender and site were first created. Then, more quantitative methods were employed to find any statistically significant differences.
Female students at RCC had a median score of 53.8, a minimum of 25 and a maximum of 75. Summaries by class are listed in Table 36.

Table 36. Five-number summary for the Total Scores by Gender and overall.

<table>
<thead>
<tr>
<th></th>
<th>RCC</th>
<th></th>
<th>HCCC</th>
<th></th>
<th>CSNE</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Overall</td>
<td>Female</td>
<td>Male</td>
<td>Overall</td>
<td>Female</td>
<td>Male</td>
</tr>
<tr>
<td>Min</td>
<td>9.4</td>
<td>25</td>
<td>9.4</td>
<td>0</td>
<td>34</td>
<td>14.8</td>
</tr>
<tr>
<td>Q1</td>
<td>42</td>
<td>44.7</td>
<td>41.6</td>
<td>41.35</td>
<td>41.35</td>
<td>48.95</td>
</tr>
<tr>
<td>Median</td>
<td>51.6</td>
<td>53.8</td>
<td>49.2</td>
<td>62</td>
<td>59.3</td>
<td>62</td>
</tr>
<tr>
<td>Q3</td>
<td>56.4</td>
<td>56.45</td>
<td>56.2</td>
<td>69.85</td>
<td>69.15</td>
<td>70.45</td>
</tr>
<tr>
<td>Max</td>
<td>75</td>
<td>75</td>
<td>70</td>
<td>85.8</td>
<td>85.8</td>
<td>77.4</td>
</tr>
</tbody>
</table>

Boxplots by Gender

Boxplots for the total score and individual representations by gender are presented in Figure 25. On the left top row, Total Scores for female students visually appear slightly lower than scores those for male students. Differences between male and female students also appear in every representation other than the Verbal one: Graphical (top row right), Numerical (second row left), Contextual (bottom row left) and Symbolical (bottom row right) domains, with female scoring lower than their male counterparts. The Verbal Scores for male and female students appear similar.

Also, some scores show greater variability, especially in the Symbolical, Graphical, and Numerical domains. Inferential results presented later show that the difference in performance for male and female students was significant on the numerical domain, and that gender interacted with site on performance in the Symbolical domain.
Figure 25. Boxplots of the total score and gender. From left to right: Total, Graphical, Numerical, Verbal, Contextual, and Symbolic scores.
Boxplots by Gender and Site

Boxplots for the total score and individual representations by site and gender are presented in Figure 26. This figure vividly shows how female student scores in the CSNE curriculum, represented in on the boxplots on the left of each image, are lower than corresponding scores of all other groups in the Total Score (top left row), the Graphical Score (top right row), Contextual Score (bottom row left) and Symbolical Score (bottom row right). These are exactly the representations that the teacher did not employ in his enactment of the curriculum at CSNE, with scores for the enacted CSNE curriculum on these representations being 1 (Graphical), 1 (Contextual), and 3 (Symbolical) respectively.

Comparing the distribution of male and female scores at CSNE across representations with the other distributions, one notes that the male student distribution from CSNE is closer to the distribution of scores from the other groups, and higher than that for female students, despite the fact that all students at this site started with a similar FTC prior background. The only place where both male and female student scores appear lower for the CSNE site, is on the Contextual representations. Foreshadowing the discussion in Chapter 6, one conclusion suggested is that the female student performance, more closely aligned itself with the enactment of the FTC curriculum, than male student performance, as originally hypothesized.

Also of note is the fact that the HCCC scores for both female and male students are higher than scores of students in the other groups in the Contextual representation. As has been already presented in Chapter 4, the teacher of the HCCC Calculus course, used Contextual representations extensively in his instruction of the FTC, and the score
on the enactment of the FTC on the Contextual representations at HCCC was a 6 (out of 7) on the LOP instrument.

Figure 26. Boxplots for class, gender, and total score. From left to right: Total, Graphical, Numerical, Verbal, Contextual, and Symbolic scores.
Inferential Statistics on Student Understanding and Gender

This study was interested in understanding how the enacted curriculum at the three sites (RCCC, HCCC, and CSNE) influenced student understanding as measured by the Five Problems Involving the FTC. In particular, the study sought to understand whether the curriculum at the three sites had a different effect on female student achievement. This was a between-group unbalanced design. Two main effects were examined (gender and site) along with the gender and class interaction. There are two independent qualitative variables (gender and site) and one quantitative dependent variable (Total Score). Two-way ANOVA was used to examine the two main effects (student gender and instructional method) on the dependent variable (score) along with their interaction. Two-way ANOVA was appropriate because two are independent qualitative values, the dependent variable is quantitative and there are six groups (two groups for gender, and three for site, or teaching format). The variables in the research question are fully crossed. For the main effect of site there are three groups, so the method used for controlling the family-wise error rate at alpha = 0.05 was Fisher Least Significant Difference (LSD). For participants’ gender there are two groups, so the most appropriate method for comparing the scores between these groups, is a $t$-test of difference in means. For the Interaction Effects, Fisher LSD was also intended to be used. Results did not yield significance; however, there was an almost statistically significant interaction of gender across sites.
**(ANOVA Results on Student Understanding and Gender)**

**Total Score Results**

Using ANOVA to test the Omnibus hypothesis that the means of the Total Scores at the three sites were equal, did not find a significant difference in means at the three sites, although the \( p \)-Value is close to significance \( (p=0.071) \).

A \( t \)-test for differences in means between male and female score on the FTC assessment at alpha = 0.05, also was not significant \( (p >0.05) \).

To test for the interaction effects, whether the difference in male and female means differed by site, the Omnibus test once again came close to significance \( (p = 0.06) \), suggesting that gender interacts with the instructional site.

As the boxplots indicated, the effect of gender and the interaction effects across sites were plotted in

![Interaction Class and Gender](image)

**Figure 27.** Interaction, class and gender.
Although the twelve female students at CSNE started with a similar prior knowledge as students in other classes, gender interacted with the enacted curriculum across sites and their scores on the FTC assessment were lower than those of the other students. This result in itself was not enough to imply causation; however, it suggests something, either the CSNE curriculum or the participants’ background (though not their prior knowledge of the FTC) could explain this result. These issues will be explored in the next chapter.

**Individual Representational Results and Gender**

Gender interacted with site in achievement on the Symbolic domain \((p=0.035)\). This test was followed by a multiple comparison which found that the difference in means in female and male performance at CSNE differed significantly from the difference in means in female and male performance at the RCC \((p = .036)\) and at HCCC \((p = .026)\).

Table 37 reports the results of the two-way ANOVA for gender and site across all scores on the FTC assessment.

There was a significant difference in Numerical Scores by gender, with female students scoring significantly lower than male students \([F (5, 49) = 4.02, p = 0.05])

There were significant effects of Site on the Contextual and Graphical Scores. These results were already presented earlier.

Table 37. ANOVA for gender and site across all scores on FTC assessment.

<table>
<thead>
<tr>
<th>Dependent Variable</th>
<th>Independent Variable</th>
<th>p-Value</th>
<th>Significance</th>
<th>F value</th>
<th>Eta - Square</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total Score</td>
<td>Gender</td>
<td>.39</td>
<td>No</td>
<td>0.74</td>
<td>.015</td>
</tr>
<tr>
<td></td>
<td>Class</td>
<td>.1519</td>
<td>No</td>
<td>1.96</td>
<td>.074</td>
</tr>
<tr>
<td></td>
<td>Gender*Class</td>
<td>.2038</td>
<td>No</td>
<td>1.64</td>
<td>.062</td>
</tr>
</tbody>
</table>
### Regression Results on Student Understanding of the FTC, Students’ Cognitive Preference and Gender

To address the role of other factors, such as female students’ representational cognitive preference, in the ways female students understand the FTC, several regression analyses were pursued. Regression analysis on Total Score on the FTC assessment (TS) as a function of gender, site, cognitive preference (CP), and accommodated preference, was performed. The regression analysis model added gender and site to the previous model.

\[
TS = F(GCP, VCP, NCP, CCP, SCP, gender, site)
\]

#### Cognitive Preference and Student Understanding

There was a significant difference in Total Score (TS) as a function of symbolical cognitive preference, \((p = 0.000, df=54, R\text{-square} = 0.43)\).

There was also significant difference in Total Score as a function of graphical score \((p=0.028, df = 54, R\text{-square} = 0.43)\).

---

<table>
<thead>
<tr>
<th>Graphical Score</th>
<th>Gender</th>
<th>p-value</th>
<th>R-sq</th>
<th>Significance</th>
</tr>
</thead>
<tbody>
<tr>
<td>(F=2.45; p = 0.47; R-sq = 0.2)</td>
<td>.16</td>
<td>No</td>
<td>2.03</td>
<td>.040</td>
</tr>
<tr>
<td></td>
<td>.049</td>
<td>Significant</td>
<td>3.20</td>
<td>.120</td>
</tr>
<tr>
<td></td>
<td>.58</td>
<td>No</td>
<td>0.54</td>
<td>.020</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Numerical Score</th>
<th>Gender</th>
<th>p-value</th>
<th>R-sq</th>
</tr>
</thead>
<tbody>
<tr>
<td>(F = 1.33; p = 0.27; R-sq = 0.11)</td>
<td>.05</td>
<td>Significant</td>
<td>4.02</td>
</tr>
<tr>
<td></td>
<td>.77</td>
<td>No</td>
<td>0.27</td>
</tr>
<tr>
<td></td>
<td>.33</td>
<td>No</td>
<td>1.13</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Verbal Score</th>
<th>Gender</th>
<th>p-value</th>
<th>R-sq</th>
</tr>
</thead>
<tbody>
<tr>
<td>(F = 0.41; p = 0.84; R-sq = 0.05)</td>
<td>.27</td>
<td>No</td>
<td>1.20</td>
</tr>
<tr>
<td></td>
<td>.81</td>
<td>No</td>
<td>0.21</td>
</tr>
<tr>
<td></td>
<td>.83</td>
<td>No</td>
<td>0.18</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Contextual Score</th>
<th>Gender</th>
<th>p-value</th>
<th>R-sq</th>
</tr>
</thead>
<tbody>
<tr>
<td>(F = 4.40; p = 0.0022; R-sq = 0.31)</td>
<td>.58</td>
<td>No</td>
<td>0.31</td>
</tr>
<tr>
<td></td>
<td>.0006</td>
<td>Significant</td>
<td>8.64</td>
</tr>
<tr>
<td></td>
<td>.447</td>
<td>No</td>
<td>0.82</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Symbolic Score</th>
<th>Gender</th>
<th>p-value</th>
<th>R-sq</th>
</tr>
</thead>
<tbody>
<tr>
<td>(F = 2.08; p = 0.08; R-sq = 0.18)</td>
<td>.41</td>
<td>No</td>
<td>0.69</td>
</tr>
<tr>
<td></td>
<td>.55</td>
<td>No</td>
<td>0.61</td>
</tr>
<tr>
<td></td>
<td>.0354</td>
<td>Significant</td>
<td>3.58</td>
</tr>
</tbody>
</table>

---
The other results, though not statistically significant appear in Table 38.

Table 38. Regression results for Total Score as a function of Cognitive Preference, Site and Gender.

<table>
<thead>
<tr>
<th>Total Variable</th>
<th>Score</th>
<th>p-Value</th>
<th>95% CI</th>
<th>Eta square</th>
</tr>
</thead>
<tbody>
<tr>
<td>Graphical (GCP)</td>
<td>0.028*</td>
<td>(0.9, 14.4)</td>
<td>.098</td>
<td></td>
</tr>
<tr>
<td>Symbolic (SCP)</td>
<td>0.000*</td>
<td>(6.1, 19.0)</td>
<td>.247</td>
<td></td>
</tr>
<tr>
<td>Numerical (NCP)</td>
<td>0.109</td>
<td>Not significant</td>
<td>.054</td>
<td></td>
</tr>
<tr>
<td>Contextual (CCP)</td>
<td>0.159</td>
<td>Not significant</td>
<td>.042</td>
<td></td>
</tr>
<tr>
<td>Verbal (VCP)</td>
<td>0.185</td>
<td>Not significant</td>
<td>.037</td>
<td></td>
</tr>
<tr>
<td>Gender</td>
<td>0.714</td>
<td>Not significant</td>
<td>.003</td>
<td></td>
</tr>
<tr>
<td>Site</td>
<td>0.813</td>
<td>Not significant</td>
<td>.001</td>
<td></td>
</tr>
</tbody>
</table>

Regression models were also run separately for male and female students to see if the variability could be improved.

\[
\text{TS}_{\text{female}} = F(\text{GCP, VCP, NCP, CCP, SCP}, \text{site})
\]

\[
\text{TS}_{\text{male}} = F(\text{GCP, VCP, NCP, CCP, SCP}, \text{site})
\]

For female students, graphical cognitive preference was significant \((p = 0.016, df = 17, R \text{ square } = 0.61)\), while for male students, symbolical cognitive preference was significant \((p = 0.005, df = 30, R \text{-square} = 0.42)\). These results are included Table 39 and Table 40.

Table 39. Regression results for Total Score of female students as a function of Cognitive Preference, Site and Gender \((R \text{-square} = 0.61)\).

<table>
<thead>
<tr>
<th>Total Score Variable</th>
<th>p-value</th>
<th>95% CI</th>
<th>Eta Square</th>
</tr>
</thead>
<tbody>
<tr>
<td>Graphical (GCP)</td>
<td>0.016*</td>
<td>(3.5, 30)</td>
<td>.296</td>
</tr>
<tr>
<td>Symbolic (SCP)</td>
<td>0.232</td>
<td>Not Significant</td>
<td>.083</td>
</tr>
<tr>
<td>Numerical (NCP)</td>
<td>0.265</td>
<td>Not significant</td>
<td>.073</td>
</tr>
<tr>
<td>Contextual (CCP)</td>
<td>0.193</td>
<td>Not significant</td>
<td>.098</td>
</tr>
<tr>
<td>Verbal (VCP)</td>
<td>0.900</td>
<td>Not significant</td>
<td>.001</td>
</tr>
<tr>
<td>Site</td>
<td>0.69</td>
<td>Not significant</td>
<td>.027</td>
</tr>
</tbody>
</table>
Although not all the quantitative results were significant, some important points can be made. The three classes, who started on equal footing, show some significant differences in their understanding of the FTC across representations, and across gender. Gender differences do exist in achievement on the FTC assessment, either in Total Score or in individual representational scores. Finally, it is important to note the role of cognitive preference and of having students’ needs met in the understanding of the FTC. For female students, Graphical cognitive preference was significant in their Total Score, while for male students, Symbolical cognitive preference, had a significant effect on student performance as measured by the Total Score. If the enacted curriculum is lacking in opportunities for students to engage with graphical representations, it may inadvertently have a greater negative impact on women. These results will be analyzed in Chapter 6. The remainder of the current chapter will be devoted to the results of the Qualitative Think-Aloud Results.

In addition to the Five Problems involving the FTC, three students from each site (two female and one male) participated in a Think-Aloud problem session followed by a post interview. During the Think-Aloud, students were asked to pick one problem of their
choice from five problems (one for each representation) and to solve it out loud, explaining their thinking. The problems were very similar, but in different representations. At the completion of the *Think-Aloud*, each participant was asked a set of questions regarding their choice of problem, their solution, and their perception of their understanding of the FTC. *Think-Alouds* were videotaped showing the participants hands as they were working through the problem and their work was collected.

Of the students at RCC, one picked the contextual problem, one the graphical one, and the other one the symbolic problem. All the HCCC students picked the contextual problem. Two of the CSNE students picked the graphical problem and the third the symbolical problem. This section summarizes the results of the *Think-Alouds* by the problem chosen by students to solve, and then compares results across sites.

Below is an excerpt of the Problem A, as it was given to the students. Two students, Anne (CSNE), and Allison (CSNE) chose this problem.

**Problem A**

Consider the graph of $f(x)$. Let $F(x)$ be the anti-derivative of $f(x)$ with $F(0) = 0$. Answer the following questions.

1. Use an integral to define $F(x)$.
2. Please determine in each case the main features of the function $F(x)$ on the indicated interval, such as:
   (a) Regions where is $F(x)$ increasing and where is it decreasing.
   (b) Regions of concavity of $F(x)$.
   (c) Location of any maxima and/or minima.
3. In the space on the right, graph the function $F(x)$ that corresponds to $f(x)$ on the left.

![Figure 28. Graph for problem A from the Think-Aloud.](image)
**Anne’s Think-Aloud (CSNE)**

**Background and FTC assessments.** Anne is a student at CSNE. She had a cumulative grade of 14 on the FTC assessment. She scored highest on the numerical representation, where she received a grade of 33, and lowest on the symbolic problem, where her score was a 0.

The Post Interview and Background Questionnaire revealed Anne does not remember if she used multiple representations before, and that she is motivated to do well in math by grades. She has not had Calculus in high school. Anne does not feel like she has a good understanding of the FTC. “Definitely not. I am just trying to fit the pieces together. If we spent more time on it, maybe,” she said.

**Synopsis of Problem Solving.**

1. Anne starts her problem by writing a definition of $F(x)$, which involves a definite integral and the endpoints given for the problem. She confuses the anti-derivative with a definite integral, and appears not to understand the statement of the FTC. She begins by writing:

   $\int_{0}^{2} f(x)\,dt \rightarrow f(2) - f(0)$

2. She correctly identifies $f(x)$ from the picture given as $f(x) = |x - 1|$ and moves on to compute $F(x)$ using the power rule. She ignores the $-1$ and the absolute value and reasons that $F(x) = \frac{x^{n+1}}{n+1} = \frac{x^{1+1}}{1} = \frac{x^2}{1} = x^2$. Her written work is shown in Figure 29.
Anne then evaluates $F(x)$ using incorrect functional notation as $F(2)^2 - F(0)^2 = 4$. She then moves on to draw the graph of $F(x)$ as recorded in Figure 30.

For the second part of the problem which asks about the characteristics of $F(x)$, Anne uses her graph above to conclude that a) $F(x)$ increasing $(0, \infty)$, b) concave up always $(-\infty, \infty)$ and concave down never and that c) $F(x)$ has a min $(0,0)$.

**Anne’s Use of MR.** Anne’s use of multiple representations is symbolical, graphical and numerical. She does not make full connections among representations, or of the individual representations themselves. Although Anne uses symbols in much of her computations, the symbols are incorrectly written, and even the way functional notation
is used, shows little understanding of the meaning behind the symbols. While realizing that the graph drawn is \( f(x) = |x - 1| \), she does not translate this to mean that the slope of \( F(x) \) is positive and thus \( F(x) \) is increasing.

**Anne’s Understanding of the FTC.** Anne’s explanation of the FTC illustrates she confuses the definition of the integral, but she has some idea that it has to do with evaluating antiderivatives at definite points: “I watched a video on it, in addition to class lecture. If you have definite points with the antiderivative, then you find the space.” Her *Think-Aloud* also shows she confuses the antiderivative function with a definite integral.

Anne has an incomplete understanding of the first and second derivative, including the first and second derivative tests. She incorrectly decides that the antiderivative of \( f(x) = |x - 1| \) is \( F(x) = x^2 \) because she knows “something about the power rule” for antiderivatives. However, she does not appear to know that \( f(x) \) represents the slope of \( F(x) \), and is unable to make connections between the graphical or symbolical representations for \( f(x) \) and what these representations say about the shape of the graph of \( F(x) \). Instead, she draws her conclusions from \( F(x) = x^2 \) which she is unable to identify as in conflict with a positive derivative \( f(x) \). Likewise, she reasons that \( F(x) \) is concave up everywhere simply from the shape of \( y = x^2 \) not making any connections with the second derivative of \( F(x) \), or the first derivative of \( f(x) \).

**Allison’s Think-Aloud (CSNE)**

**Allison’s Background and FTC Assessments.** Allison is a first-year student at CSNE. Her cumulative score is 57.4. Allison scored highest on the numerical, verbal and symbolic problems, and lowest on the contextual one with a score of 33. Allison has taken Calculus before.
In her Post Interview and Background Questionnaire, she revealed that she worked on homework mostly by herself, and that her teacher used various representations: “he showed us things in different ways.” Allison chose the graphical problem because she feels “more comfortable looking at pictures. A piece of it is in front of me already and I can figure it out.” According to Allison, the Fundamental Theorem of Calculus involves using an integral to define a function and applying that toward concavity and regions of increase or decrease. She does not feel like she has a solid understanding of the FTC and hopes she will understand it better before the final.

**Synopsis of Allison’s Problem Solving.** After reading the first part out loud, Allison writes on her test sheet that regarding using the integral to define \( F(x) \), Allison writes down that \( F'(x) = f(x) \).

Allison is thinking graphically of the image of a parabola pointing up as the canonical representation for concave up graphis, but Allison also understands that the information is in conflict with an always positive derivative presented in the problem, which would indicate no regions of decrease. She then proceeds to write down that this in integral form as \( \int_0^2 f(x) \, dx \rightarrow \int_0^2 f(2) \, dx \) along with other information in the problem, as illustrated in

Figure 31, without evaluating or commenting on the meaning of her writing.

![Figure 31. Allison’s result for part defining F(x).](image)
She then moves on to part 2 of the problem and decides that $F(x)$ is increasing from 0 to 2, since the graph of the derivative is positive, and records this on her paper. The question about concavity stumps her. She decides that the graph should be concave up since “it is pointing up,” but she is not sure “how something can be concave up and still never decreasing,” and moves on to another part of the problem. She focuses on the intercepts, saying:

At the x intercepts something happens with the derivative, but I do not remember what that is. I think the derivative has zero and then the function a maximum or a min, I think, or it may be the opposite, so maybe the derivative has a max or a min and the function has a zero.

Going with her first guess she says “I guess it has to be concave up and it is increasing. It would have to be a minimum at (1,0),” but then changes her mind:

That does not make sense to me. It would look like the original. So, the derivative is positive, and so the function is increasing. So, can I have a minimum? Concavity has to do with whether the function is pointing up or down, and how it opens. And it has to do with the second derivative.

She changes the answer again “it is increasing from 1 to 2 and decreasing from 0 to 1. Then the maximum would be at $x = 2$ and min at $x = 0$.” She writes down max at $x = 2$ or $(2,1)$ and min at $x = 0$ or $(0, -1).” “That still does not make sense.”

She continues, as documented in

Figure 31: “If I had the equation, I may be able to figure out the concavity. These are just straight lines.” She then writes down each piece $y = -x + 1$ on $[0, 1]$ and $y = x - 1$ on $[1, 2]$ and computes the first derivative and the second derivative of each piece to be -1 and 1 for the first derivative. She concludes that –1 means something is decreasing and +1 means it is increasing but not sure what is increasing or decreasing.
She moves back to the original equation and rewrites the equations with the label $F'(x)$, but cannot remember the formulas for the antiderivative and asks, “maybe we can leave it at that”, as can be seen in Figure 32.

![Figure 32](image)

**Figure 32. Allison’s computation.**

**Allison’s Use of MR.** Allison uses graphical and symbolical representation in her solution. She initially starts with symbolical representations in her solution, but much of her reasoning is done using graphs although some of the conclusions she draws are incorrect. She can correctly read regions of increase and decrease, but has trouble visualizing functions that are concave up and never decreasing, probably thinking of a positive leading coefficient parabola as her leading model. Her symbolical work illustrates inexperience or a limited understanding or interpretation of what the symbols mean, or incorrect application of anti-differentiation rules.

**Allison’s Understanding of the FTC.** Allison has a partial understanding of the first and second derivative tests, but is not able to relate between the graphical features of
the antiderivative and the zeros of \( f(x) \) or the regions of increase or decrease on \( f(x) \). She understands \( F'(x) = f(x) \), but her understanding is procedural and rote. She is not able to fully move across various representations and hesitates or gives up when she perceives conflicting information. Also, she does not seem to understand the idea of the definite integral, either its representation as a signed area, or how to apply the FTC to find definite integrals. Although she can compute simple derivatives, such as derivative of \( x \) being equal to 1, she cannot go backwards to find antiderivatives for \(-x+1\) or \(x - 1\).

One student from RCC, Christine, chose to solve Problem C.

**Problem C**

Consider the linear function \( f(x) \) with y intercept 4 and slope -2 on the interval \([0,4]\). Consider now the absolute value of \( f(x) \), \( g(x) = |f(x)| \) on the interval \([0,4]\).

Let \( G(x) \) be the anti-derivative of \( g(x) \) with \( G(0) = 0 \)

1. Use the integral to define \( G(x) \).
2. Please determine in each case the main features of the function \( G(x) \) on \([0, 2]\) such as,
   (a) For \( G(x) \), where is this function increasing and where is it decreasing?
   (b) Regions of concavity of \( G(x) \).
   (c) Location of any maxima and/or minima.

**Christine’s Think-Aloud (RCC)**

**Christine’s Background and FTC Assessments.** Christine is a first-year student at RCC. She has taken the AP Calculus BC test in high school. Her average score was 75 on the FTC assessment, with highest score of 80 on the contextual problem and lowest score of 60 on the numerical problem. The Post Interview and Background Questionnaire reveal that when doing homework Christine gets help from “nobody other than the teacher this year.” She feels that her classroom experience helped a lot with “stuff like this.” She has a great deal of experience with multiple representations, including tables,
graphs, and formulas, but she likes the contextual representation the least. She chose problem C to solve because of avoidance or familiarity:

I was avoiding rate of change—or not rate of change, but problems where there is a moving object. I forgot stuff with average rate of change, and I saw a discontinuous function and was not going to do a discontinuous function. I usually do not use graphs, like in high school I would do anything not to use graphs...I used to do any kind of other algebra stuff just to get out of the graphs but I used it here because it is a linear function. And I guess my classroom experience helped because he’d always throw a graph at us in class.

Her perception of her understanding of the FTC is that she understands part of it, but “you just can’t always get it that fast.” When asked if she has a complete understanding of the FTC, Christine responds, “No, I do not, because I do not know how to prove it [the FTC].”

Christine is motivated by her “drive to be successful as a future actuary.”

**Synopsis of Christine’s Problem Solving.** Christine began reading aloud the problem and recording the information as $b = 4, m = -2$. For the first part of the question regarding the relationship between $G(x)$ and $g(x)$, she “derives” $G(x)$ to give her $G’(x) = g(x)$ and then “take(s) the integral” to give her $G(x) = \int g(x)dx$. She boxes this result as illustrated in Figure 33.

![Figure 33. Christine’s answer to the first question.](image)

Moving to the second part, regarding the features of $G(x)$ such as the regions of increase and decrease, she decides that since $f(x)$ is linear she “can write the equation for
it.” She records $f(x) = -2x+4$ and $g(x)=|\text{-}2x+4|$. She wonders out loud if she should “separate $g(x)$ in two functions”, but pauses “to find this [the anti-derivative] at least for function which does not have the absolute value first.” She computes the first and second derivatives as $f'(x) = -2$ and $f''(x) = 0$. She remarks that she knows that for part (b) regarding the regions of concavity, the “second derivative stays positive”, so “then there is no regions of concavity.” She goes on to consider the absolute value of $f(x)$, as specified in the problem, and draws the graph of $g(x)$ as featured in Figure 34 below.

Figure 34. Christine’s drawing of $g(x)$.

Pointing to the place where $x=2$ she comments “I know the derivative at a point like that is not defined”, but “by looking at the graph” $G(x)$ would be concave up. She methodically remarks that she is still thinking about part (a), and records her answer for part (b) as concave up on $[0, 4]$. She pauses at to think about part (c) of the problem asking her to find the maxima and the minima. Pointing to the place where $g(x)$ is zero, she sets $-2x+4$ equal to 0 to find that $x = 2$, and decides (incorrectly) this is the minimum. She does not look at the sign of $g(x)$ at the right and left of $x = 2$, and she does not find a $y$-value for her minimum.
Lastly, Christine goes back to part (a). She notes that \( G(x) = \int |(-2x + 4)|\,dx \) and uses the FTC to conclude that the derivative of \( G(x) \) is \(-2x+4\) which is positive:

“Since it is an absolute value it will always be increasing”, so “\( G(x) \) is increasing on the entire interval \([0, 4]\).” Her answer to this part is in

Figure 35.

![Figure 35. Christine’s work on the regions of increase or decrease.](image)

**Christine’s Use of MR.** Christine uses a combination of verbal, symbolical and graphical means to solve the problem. She has ease using and translating among all these representations, although she makes a few minor notational mistakes. She can comfortably transfer among representations, for most of the interview, but she does not realize some of her information is in conflict.

**Christine’s Understanding of the FTC.** Christine’s explanation of the FTC, starts off symbolically and writes:

\[
F'(x) = f(x) \quad \int_a^b f(x)\,dx = F(b) - F(a)
\]

She mumbles pointing to the “x” in the integrand: “I think I got in trouble for this. Is this ‘t’? Does it matter?”, she asks herself, then moves on to deconstruct its meaning:
“So, in my own words...” She does not complete her thought out loud but continues to think about the meaning:

“Sometimes, I think of it graphically but I am not exactly [sure] how this fits in, ‘cause I know integrals are just area under the curve. So, if I take that derivative, that is just the slope. So, the slope of the antiderivative is the original function.”

Reflecting on the second part of her writing she adds:

I don’t understand why you do $F(b)$ minus $F(a)$. It’s just the endpoints. It messes me up. I guess it gives the relationship between the area under the curve and its antiderivative. So, the Fundamental Theorem of Calculus, relates the area under curve to the antiderivative. But I guess I am just saying the same thing as I wrote.

I don’t completely understand how it came to be. How it was made. I don’t think I ever saw how it was derived.

Based on Christine’s reasoning on the *Think-Aloud*, her FTC Assessment, and her responses to the Post Interview Questions, Christine understands the symbolical form of the FTC, how to translate it and interpret it verbally, but she has some issues visualizing how $F(b) – F(a)$ would relate to the area “under the curve.” She has a solid understanding of the first and second derivative tests, and how to apply them and she methodically approaches the problem. She hesitates in her interpretation of the graph of the derivative of $g(x)$, where she confuses the minimum of $g(x)$ with the minimum of $G(x)$. She adds to this error when she concludes based on the second derivative test that $G(x)$ is increasing on $[0,4]$, but places its minimum in the middle of the interval at $x=2$. Despite her being unable to realize this area of conflict, Christine has a good to very good understanding of the FTC.

Below is an excerpt of the Problem D. Four students, Dan (RCC), Deliana (HCCC), Dave (HCCC), and Dawn (HCCC) completed this problem.

**Problem D**

A ladybug is crawling along a rod starting at point A.
The bug is traveling at a rate of \( v(t) = t - 5 \) inches per second from \( t = 0 \) to 5. Then, from \( t = 5 \) to \( t = 10 \) minutes, the bug travels at a rate equal to \( v(t) = 5 - t \) inches per second.

1. What is the relationship between the velocity of the ladybug and its distance from point A?
2. Determine the following about the distance from A.
   (a) What is the bug’s distance at \( t = 1 \)? At \( t = 2 \)? Can you determine what the distance is at any time \( t \)?
   (b) When does the bug change direction of travel or stop?
   (c) When (for what times) is the distance increasing and where is it decreasing?
   (d) When is the bug accelerating and when is she decelerating if at all?
   (e) What can be said the maximum and/or minimum distance from A?

Dan’s Think-Aloud (RCC)

Dan’s Background and FTC Assessments. Dan is a first-year student at RCC. He has taken Calculus online in high school. The FTC assessments indicate that Dan’s average score was 73 out of 100 possible points, and he scored highest on the Contextual problem with a score of 93, and lowest on the Verbal problem with a score of 56.

The Post Interview and Background Questionnaire reveal Dan likes to work alone, and does “all the problems in the book.” He occasionally emails his teacher if he has questions, and says knows a lot about multiple representations as his other teachers “all used tables, graphs, and formulas.” He feels like he has a “pretty good” understanding of the FTC. He chose to solve problem D because “that’s the only one with a story. There’s a ladybug. I wanted like word problems, analyze reality. Don’t want to just look at something scary. Here instead of saying find something you are taught to understand math.” He does not believe he has a complete understanding of the FTC.

Synopsis of Dan’s Problem-solving. Dan begins the problem by labeling the known quantities. He uses symbolical representations as he separates the velocity function in two components according to the absolute value into
\[ v_1(t) = \frac{(t-5) \text{ in}}{s} \quad \text{for} \ 0 < t < 5 \quad \text{and} \quad v_2(t) = \frac{(5-t) \text{ in}}{s} \quad \text{for} \ 5 < t < 10 \]

To find the distance at \( t = 1 \), he says “we need to find the antiderivative to find the distance traveled.” He offers a visual representation of the distance from A to B using arrows to point in the direction of travel. In doing so, he explains that for the first five seconds the bug “is traveling from A to B, and then back.” He illustrates this with the graphical schematic in Figure 36.

![Figure 36. Dan’s demonstration for the velocity graph.](image)

Then he computes \( S(t) = \int_0^5 (t - 5) \, dt \), and realizes, “that gives the whole distance for \( t \) less than 5”, and backtracks to find a general form for the antiderivative that he could plug numbers into to find the distance from point A at \( t = 1, t = 2 \), and so on. He worries about how to find “C”, but then decides that since this is distance from A, and the bug starts at A, C should be zero.

He computes:

\[
S(t) = \left( \frac{1}{2} t^2 - 5t \right) \\
S(1) = \left( \frac{1}{2} - 5 \right) = -\frac{9}{2} = -4.5 \\
S(2) = \left( \frac{1}{2} (4) - 5(2) \right) = 2 - 10 = -8
\]

![Figure 37. Dan’s computation of the distance traveled.](image)
As Dan solves the problem, he explains his reasoning and alters his original idea about the ladybug’s direction of travel: “Technically this is going in the negative direction since the distance is negative.”

After answering questions 1 and 2a, Dan moves on to explain that the bug changes direction when the velocity is zero and offers an alternate explanation, to the effect that since the bug traveled -12.5 feet in the first five seconds and 12.5 feet in the next five seconds, it must have changed direction at $t = 5$:

“It takes the same distance to travel in opposite directions. -12.5 at first, and then +12.5, so that is how I know it stopped,” he said.

In trying to find where the maximum and minimum distances are, Dan starts off by claiming that he “needs to make a graph.” Instead of doing that, he turns to the sign of the velocity to argue that the distance is decreasing.

$$V_1(t) = t - 50 < t < 5 \rightarrow \text{negative so distance is decreasing}$$

$$V_2(t) = 5 - t5 < t < 10 \rightarrow \text{negative so distance decreasing}$$

To find out when the bug is accelerating or decelerating, Dan says that the acceleration on the first five seconds is 1, and on the second set of five seconds is $-1$, connecting with the second derivative test: “If the second derivative is negative, the graph is concave down. If it is positive, it is concave up.” He further explains “when the bug is going from A to B, it is accelerating, and then when it is going from B to A, it is decelerating.”

The last part of the question asks him to find the maximum or minimum distance. Dan says that the critical point is at $t = 5$, so that is where the maximum or minimum
distance is, and concludes that that is also where the graph changes from concave up to down. His accompanying graph is below.

![Graph of distance traveled](image)

Figure 38. Dan’s graph of the distance traveled.

**Dan’s Use of MR.** Dan translated the problem to a symbolical domain and used a combination of contextual, symbolical, and numerical methods in his solution, but he predominantly used symbolical methods in the calculation. When graphical methods were used, they were used only at the beginning, or at the end, to help understand the problem or the solution.

**Dan’s Understanding of the FTC.** Based on Dan’s reasoning in the Think-Aloud, his FTC assessment scores, and his responses to the Post Interview, Dan understands the symbolical form of the FTC, and how to apply it in computations. He also understands how the symbolical form of the FTC relates to the contextual form. He knows that to get the distance function he must integrate the velocity and alludes to acceleration as derivative of position. He does not fully understand or make connections with the graphical representation.

Dan’s attitude toward mathematics is that mathematics must make sense. He backtracks when he gets a negative answer to say that “technically, the bug is moving in the negative direction,” and offers two explanations to the question about when the bug
changed direction, both numerical and contextual in nature—one, that the bug changed
direction when the velocity is zero, that is when $t = 5$, and the other having to do with
distance traveled. Reasoning that $S(5) = \int_0^5 (t - 5) \, dt = \left( \frac{1}{2} t^2 - 5t \right) |^5_0 = -12.5$, and $s$
(10) = $\left( 25 - \frac{1}{2} (25) \right) = 12.5$, he concludes that the bug has to turn around.

Dan realized that the velocity is always negative and he does make the connection
that the distance is decreasing, but he does not see the problem with his prior
computations of positive and negative distances.

$$V_1(t) = t - 5 \quad \text{for } 0 < t < 5 \rightarrow \text{negative so distance is decreasing}$$

$$V_2(t) = 5 - t \quad \text{for } 5 < t < 10 \rightarrow \text{negative so distance decreasing}$$

Also, when he decided that the distance has a local minimum at $t = 5$, he shows a
graphical representation that once again contradicts his original notion that the distance is
decreasing, since he argues that the minimum distance is at zero, for $t = 5$, and a
decreasing function would not have a minimum.

Dan correctly identifies that where the function is concave up, the bug is
accelerating and where the graph is concave down, it is decelerating, and gives correct
answers based on his velocity functions, but once again does not make the connection
with the graph he drew which is only concave up.

Dan applies the FTC correctly in finding anti-derivatives, but he confuses the
theorem with the idea of definite integrals. When asked what the FTC says in his words,
he responds: “Yes, I think that if you have a function $f(x)$ and you want to find the area
under that function, then that area is the integral of $f(x)$ from whatever points you have.”
Deliana’s *Think-Aloud* (HCCC)

**Deliana’s Background and FTC Assessments.** Deliana is a second-year student at HCCC. Her overall grade on the FTC Assessments was a 67. She scored lowest on the numerical problem and highest on the contextual problem, with scores of 50 and 100 respectively.

The Background Questionnaire and Post Interview reveal that classroom experience exposed her to a lot of graphs, charts and tables:

“Graphs, charts, tables, etc. were all used to help me learn and have better understanding of the material that was being covered. Depending what is being taught, I find that graphs are extremely helpful.”

When Deliana studies she uses a mix of methods to learn.

“My first instinct is to go to Google or YouTube to see if I can figure it out on my own, but if that fails I tend to go to fellow classmates or people who have taken the course before,” she says.

She chose the problem because “I like particle problems. It gave me a good opportunity to use graphs and sign charts.”

About her curriculum influence on her problem of choice, Deliana recalls that:

My teacher had a whole lesson on drawing graphs of antiderivatives from looking at various graphs of functions $f(x)$. It helped me build upon the foundation I had for derivatives and antiderivatives. I just made connections and it was another way of figuring out how to do a problem if no specific equations are given.

Deliana believes she has a good understanding of the FTC, “but most likely not a full understanding. There is always something to learn as a student.”

The FTC states that the integral from $a$ to $b$ of $f(x)dx = F(b) - F(a)$. $F(x)$ represents the antiderivative. It is used for continuous functions. It relates derivatives and antiderivatives, and it is used to compute the area under a curve.
**Synopsis of Deliana’s Problem Solving.** Deliana starts the problem by saying “What I like to do is to graph the velocity to visualize it.” Pointing to the graph in Figure 39 she produced, she adds, “This is the graph of the velocity. The intercept is at -5, where the time starts. At \( t = 10 \) it will have a velocity of 5 so it is moving at the constant rate.” Then she explains that the distance from A is the integral of the velocity function.

![Figure 39. Deliana's graph of velocity.](image)

Moving to part 2 of the problem she integrates the velocity function and sets the initial condition at that the distance from A at \( t = 0 \) is 0, so that \( s(0) = 0 \), as in Figure 40.

![Figure 40. Deliana's part 2.](image)
Deliana moves on to compute the rest of part 2(a) by substituting in the position function to find \( S(1) = \frac{1}{2}(1)^2 - 5(1) = \frac{1}{2} - 5 = -4.5 \) and \( S(2) = \frac{1}{2}(2)^2 - 5(2) = 2 - 10 = -8 \).

Then she moves to part 2(b) and finds where the bug is changing direction by setting the velocity equal to zero: \( V(t) = 0; \ 0 = t - 5 \) and obtaining \( 5 = t \).

She explains how she knows that this is a minimum using the first derivative test. She uses a sign chart to see where the velocity is positive and where it is negative and she explains how this connects to the graphs of \( s(t) \) as shown in Figure 41 below. When completing the sign chart on the left she points to the graph of \( v(t) \) from her first figure saying “All the values to the left of \( t = 5 \) are negative. And all the values to the right of \( t = 5 \) are positive. So, it is decreasing, and then it is increasing. That is how I know it is a minimum.”

![Figure 41. Deliana’s graph of the minimum.](image)

For part 2(d) Deliana finds that the bug has a constant acceleration equal to 1 by taking the derivative of the velocity.

Moving to the last part of the problem, to find the minimum distance, Deliana initially wants to set \( S(t) = 0 \) since she thinks that this is a minimum. She then realizes that she gets \( t = 0 \) and that at \( t = 0 \), the bug’s position was at 0, but she had found it to be
at negative values earlier, and had concluded that she had a critical point at $t = 5$. She decides to “backtrack” and finds the minimum to be $-25/2$ inches from point A at $t = 5$, and the maximum distance is $0$ at $t = 5$. Her image of the position and corresponding work is in

Figure 42 below.

![Figure 42](image)

**Figure 42. Deliana’s minimum and maximum distance computations.**

**Deliana’s use of MR.** True to her words Deliana uses multiple representations of various kinds and shows ease transferring among representations. She translates the symbolic and contextual representations to various graphical forms to and skillfully interprets the first derivative test. She uses numerical and symbolical representations to find the bug’s position, and explains her reasoning verbally relating everything to the context.

**Deliana’s Understanding of the FTC.** As demonstrated in the *Think-Aloud*, Post Interview, and FTC assessments, Deliana has a very good understanding of the FTC, and
of related Calculus concepts such as first and second derivative tests. Her application of the FTC is solid and shows ease of both symbolic manipulation and conceptual understanding, as she can transfer seamlessly from various representations or ideas. One area of difficulty arises when she ignores the second interval for the velocity given in the problem, so her solution is incomplete. Perhaps her experience was limited with piece-wise defined functions. In all other cases, Deliana has a very logical and reason driven attitude toward problem solving, using various explanations and stopping and re-evaluating. When she comes across areas of conflict, such as the conflict over the maximum and minimum distance, she backtracks to resolve it. Deliana understands the FTC in several ways; contextually, symbolically, and verbally and is able to navigate all the representations with a practitioner skill.

**Dave’s Think-Aloud (HCCC)**

**Dave’s Background and FTC Assessments.** Dave is a second-year student as HCCC. His FTC Assessments average a score of 73 out of 100 points. Dave scored highest on the contextual problem with a score of 100, and the rest of his scores are 67 on each problem.

The Background Questionnaire and Post Interview reveal that Dave does homework “pretty much alone” and occasionally he uses some sort of online resource. Dave chose the ladybug problem because he “started learning derivatives with moving particles, positioned on a number line.” It is a concept that he “was familiar with.” However, he shares that part of his solution was “more personal. That is something that I came up with. The classroom taught me that is if I was going to graph it then I should
convert it and graph the distance, and sometimes I would do that, but this time I had something else in my head to visualize it, and that is what I did.”

**Synopsis of Dave’s Problem Solving.** Dave begins the problem commenting on the relation between velocity, acceleration and position. He then explains that the distance is the integral of the velocity function. He originally writes the distance as a definite integral, from 0 to 5, but he changes his mind and erases the limits:

When I think if this (problem), we have the distance and the derivative of that is velocity. The derivative of that is acceleration. So, since we have velocity, we need the integral of velocity. For the time interval of zero to five to get our distance. We are not directly finding that, so we just need a function for that. Yeah, that just comes out to be in respect to t. (At this point, Dave erases the limits he had written as can be seen in Figure 43.) So, this would be the function is for our integral which would be our integral for distance. Distance equals that and that is part one.

Dave computes the integral of \( v(t) = 5 - t \) as \( \int (t - 5) dt = t^2 - 5t \), forgetting to divide by 2, and he does not discuss about an initial value, or maybe he tacitly knows that the initial distance from point A is 0. He then proceeds to part 2 of the problem and evaluates the integral for \( t = 1 \) and then the distance for any time \( t \).

![Figure 43. Dave’s calculation of the distance traveled by the ladybug.](image)

Following a transcript from the *Think-Aloud*:
So now we can just plug in our values we can evaluate this at zero and one since we started at zero which gives us one minus five minus zero. So, the distance is negative four, which is four in the negative direction along the rod B. When does the bug change direction; oh, and can you determine distance at any time? Which I said before, we can use this. By plugging in any values of zero and getting distance at any time and any value of $t$ would give us our distance for time of time.

To figure out where the ladybug changes direction, Dave continues by saying “to change direction, we would need to look at the acceleration. Since we have the velocity, we need to take another derivative of that to get what our acceleration would be. But it should be at $t = 5$, I think.”

Dave explains his calculations out loud and concludes that the acceleration for $t < 5$ is equal to 1, and thereafter is equal to $-1$. He reasons that the bug is initially decelerating in the negative direction, and then after $t = 5$, the bug is accelerating in the negative $t$ direction as illustrated in Figure 44.

![Figure 44](image.png)

Figure 44. Dave’s work around the bug’s acceleration.

Since Dave is now unsure of his reasoning, since it contradicts his prediction for the bug changing directions at $t = 5$. Surprised, Dave reevaluates and finds a graphical
solution by looking at the velocity vectors. His work, to which he refers in his Post Interview as his personal choice or representation is presented in Figure 45. The arrows pointing to the left show the magnitude of the velocity at various times. He explains:

I'm just going to backtrack a little bit and make a new line. So, this is A. For the first five seconds, it is going at t minus five inches speed. Which means its position is t squared minus five t. So, it is going in this direction starting off at a rate of negative five and after one second it is going at a rate of negative four. So, at a rate of five, it stops and is moving. And then at that point it turns around and goes at a rate of five minus t. but now t is already at five which means it is now going at after one more second, it's at five minus six which is negative one. So then, why am I confusing myself?

From his representation, Dave decides the bug does not change direction, but stops for an instant at $t = 0$.

![Figure 45. Dave's graph of the bug's velocity.](image)

For the next part of the problem, about the times for which the distance is increasing or decreasing, Dave uses his prior work on the acceleration to reason that the distance is always increasing.

Well, the bug started off in a negative direction and it is decelerating until it reaches a time of five. At a time of five it stopped and starts accelerating in a negative direction.
If we look at the five minus \( t \) for \( t \) is greater than five, any value of \( t \) that we can use, so if we take the integral of that we have five \( t \) minus \( t \) squared. Any value that we put in that is greater than five is negative. So, let's do six. That's already the integral. Would equal thirty minus thirty-six. So, we have a distance of negative six. We continue to move along, and the distance is getting greater from the start point at zero. So, distance is increasing.

Dave’s conclusion is that the minimum distance is at \( t = 0 \), and that the distance approaches infinity is illustrated in Figure 46.

![Figure 46](image)

Figure 46. Dave’s conclusion about the bug’s distance from A.

**Dave’s Use of MR.** Dave uses verbal, symbolical, numerical, contextual, and graphical representations in solving the problem. He starts off translating the problem to symbolical representations, but thoroughly explains the connections between the contextual problem, the velocity, acceleration, and position functions. He can transfer with ease among all representations.

**Dave’s Understanding of the FTC.** During the Post Interview, Dave offered a verbal interpretation for the FTC, which shows him to have a good grasp of the theorem.

It basically just says that the derivative and integral are reverses of each other. That if you do one and then the other, you wind up with what you started with. If
you take the derivative you get one thing and if you take the integral you go back to the original thing and vice versa.

His work around the FTC shows him using multiple representations effectively to reason, make connections, explain, and fully delve into the task at hand. He believes his understanding is good enough to get him through most problems and does not get flustered when faced with apparently contradictory ideas. His computational mistake in finding a simple antiderivative may be careless, but it does not take away from the strength of his understanding. Interestingly, his approach to graphing the velocity vector is unique, and he is confident enough to point out to this approach of being his own.

**Dawn’s Think-Aloud (HCCC)**

**Dawn’s Background and FTC Assessments.** Dawn is a first-year student at HCCC. Her overall grade on the FTC Assessments was an 85. She scored lowest on the numerical problem and highest on the verbal and symbolical problems, with scores of 67 out of 100 on the numerical problems and 100 (out of 100) on the other two.

The Background Questionnaire and Post Interview reveal that classroom experience exposed her to multiple representations: “We always used different ways to do problems in high school” and that she chose to solve the contextual problem because “I like physics and we did problems like this in class.” Dawn believes she has a “pretty good” understanding of the FTC since she is contemplating “being a math major.”

**Synopsis of Dawn’s Problem Solving.** Dawn begins the first part of the problem by writing down the information about the velocity and then explaining that she will be “drawing the velocity first.” She produces the graph in Figure 47, then voices her answer to part 1 of the problem as: “the distance is the integral of the velocity,”
Next, Dawn proceeds to find the distance at \( t = 1 \) and at \( t = 2 \) by integration, as -4.5 inches and -8 inches. Her calculations are illustrated in Figure 48.

**Figure 47.** Dawn’s representation of the velocity function.

**Figure 48.** Dawn’s solution to finding the distance.
Although, she does not interpret the negative sign, for the distance at any given time, she writes this distance as.

\[ d(t) = -\int |t - 5| \, dt \]

To figure out where the bug changes direction or stops, Dawn begins by “looking at the velocity graph.” Pointing to this graph, she explains graphically that the bug stops for an instant at \( t = 5 \) since then “the velocity is zero.” She remarks that “since the velocity is always negative, the bug does not stop. Well, I mean it stops here at \( t = 5 \) but then continues to move away from A.” Her answers are recorded in Figure 49.

Figure 49. Dawn’s conclusions about the bug’s direction of travel.

From this she concludes first that “the distance is always decreasing since the velocity is negative” and then adds:

From this velocity graph, I can easily draw the distance graph. It is first decreasing and concave up for \( t \) between 0 and 5, since the slope here (pointing to the velocity slope) is positive, and then it is decreasing and concave down, since the slope here (pointing to the second branch of the velocity graph) is negative. Then at the zero of the velocity, I have an inflection point and I can find it too.
As she is speaking, Dawn draws the graph shown in Figure 50 without putting coordinates for any special points yet. Then she continues her solution by deciding the bug is “accelerating from $t = 0$ to $t = 5$, and decelerating from $t = 5$ and $t = 10$.” To find the maximum distance, Dawn computes the coordinates of the inflection point to be (5, -25/2) by looking at the area under the x axis, and goes on to fill in the coordinates of the inflection point. Judging “by symmetry” that at $t = 10$, the bug would be at the point (10, -25) on the rod, 25 inches away from A, she adds these coordinates on her graph as well and concludes that “the maximum distance is negative 25, and that there is no minimum distance.”

![Figure 50. Dawn’s representation for the distance away from A.](image)

**Dawn’s Use of MR.** Dawn displays sophisticated use of various representations. During her problem solving, she translates the original contextual problem to graphical and symbolical representations. She uses numerical representations to compute the values of the integral, and interprets the distance traveled as the area “under the graph.” She also
shows a great ability to use verbal representations to communicate her ideas and justifications, referring and connection between the sign of the values of \( v(t) \), or the sign of the slope of \( v(t) \) and the features of the distance function correctly. Of all the students, she is the only one who favors a graphical representation of the solution, and her area interpretation is quite elegant.

**Dawn’s Understanding of the FTC.** As demonstrated in her *Think-Aloud*, Dawn understands the anti-differentiation rules, and how the definite integral of \( f(x) \) on an interval, gives change in the antiderivative on that interval. Dawn also understands how the FTC applies to the problem she was given, and as she works through her problem relating the velocity and the distance from point A, although she does not explicitly say that she chooses the antiderivative that passes through 0 at \( t = 0 \). During the post interview, Dawn says she understands the FTC pretty well, and although she confuses its statement with a definition of antiderivatives initially she is able to apply the FTC skillfully in the context of the story problem. According to Dawn, the FTC is “how you do integrals,” and “that integral of the rate of change gives the parent function,” or

" \int f(x)dx = F(x) + c. " Later she adds that:

The Fundamental Theorem of Calculus, lets you find integrals more efficiently. When we learned integrals, we used rectangles to find the area first and then took limits as the number (of subintervals) went to infinity. With the FTC, it’s a lot easier. All you have to do is to find the antiderivative and plug in the limits of integration. Yeah, I think that is how I would explain it.

Below is an excerpt of the Problem E, as it was given to the students. Two students, Emily (RCCC) and Emanuel (CSNE) chose this problem.

**Problem E**

Consider now the function \( g(x) = 2x - 4 \) on the interval \([0, 4]\).

Let \( G(x) = \int_0^x g(t)dt \)
1. What is the relationship between \( g(x) \) and \( G(x) \)?
2. Please determine in each case the main features of the function \( G(x) \), such as:
   (a) Where is this function increasing and where is it decreasing?
   (b) Regions of concavity.
   (c) Location of the maxima and/or minima.
   (d) Location of any inflection points.

**Emily’s Think-Aloud (RCC)**

**Emily’s Background and FTC assessments.** Emily is a first-year student at RCC. She is a math major. She has taken Honors Precalculus, but this is her first encounter with Calculus. Emily’s overall assessment score on the FTC assessment is 56.6. She scored highest on the graphical and numerical representations with a 67 score on each. She scored 33 on the symbolical problem.

The Post Interview and Background Questionnaire reveal that Emily usually works by herself by looking at the answers in the back of the book and working backwards. She also goes to her teacher or uses online services to clarify her thinking. She chose the symbolical representation because it was her “favorite” and because she felt it was “easiest” given her high school experience since “various classes have done it. It’s easiest for me to see how to connect it.” When asked how she feels about her understanding of the FTC, she says she does not feel like she knows it “at the moment.”

**Synopsis of Emily’s Problem Solving.** Emily chose problem E, the symbolical problem. She showed some confusion between a definite integral and antiderivatives as she writes \( G(x) = \int_0^4 (2x - 4) \, dx \). She also ignored the absolute value in her calculations. Her graphical representation of a parabola with correct roots shows she is indeed thinking of functions and this can be seen in her work below. Later, she integrates
to find $G(x) = x^2 - 4x$. She does not think about various initial conditions to see which antiderivative to take for $G(x)$.

Figure 51. Emily’s picture of $G(x)$.

Emily is proficient in the use of derivatives to find extrema, and she knows how to use and apply the first and second derivative tests. She correctly finds the minimum $x$-value, but does not give a $y$-value. To find maxima and minima and the concavity, she uses both symbolical and graphical representations, again not using the absolute value.

Figure 52. Emily's calculations.

She correctly identifies the regions of increase and decrease for the function she worked with as decreasing on $(-\infty, 2)$ and increasing on $(2, \infty)$. She uses the second derivative test to conclude that the function is always concave up.
Emily’s Use of MR. Emily’s problem-solving trajectory involves symbolical, graphical and numerical methods. She stays within the representation given except that she turns to other representations to do certain calculations. She ignores the symbolical nature of the absolute value and just treats it as not existent.

Emily’s Knowledge of FTC. Emily knows the FTC says, “something about integrals like \( \int_a^b f(x)dx = F(b) - F(a) \), but she admits she does not know what that means. She explains that it is “how you do integrals.” Her approach is procedural, but she does apply the FTC correctly reasoning that \( G'(x) = g(x) \).

Emanuel’s Think-Aloud (CSNE)

Background and FTC Assessments. Emanuel is a first-year student at CSNE. He had Calculus in high school. His FTC Assessment average 51.8, with highest scores of 67 on the symbolic and the numerical representations and lowest scores of 40 on the contextual and graphical representations. Emanuel is familiar with multiple representations more from high school. “We did more representations in high-school. In this class, less so. Mostly we used functions, and we just started graphs when you came to observe us.” He gets help on his math mostly by talking to the girl next to him, and he has gone twice to office hours. His classroom experience has influenced his choice, allowing him to feel “confident in with maxima and minima” and “more confident with functions.”

Synopsis of Emanuel’s Problem Solving. Emanuel starts by reading the question and by writing that \( G'(x) \) is equal to \( g(x) \), although he either ignores or does not realize there is an absolute value in the integrand.
Moving to the second part, he reasons:

Since this is the derivative of $G$ of $x$, I set that equal to 0 and get $x = 2$. So, that's going to be a max or a min. Because where the derivative $x$ equals zero of a function that is where it's a max or a min there. It's going from increasing to decreasing or decreasing to increasing. Then I plug in test points so I can do 0 and 3. And $f'(0)$ would be negative, so it is negative all throughout here. And $f'(3)$ is all positive throughout here, so it is increasing after 2, yes increasing on $(2, \infty)$ and decreasing on $(-\infty, 2)$. And from this you can find the max and min too. If it goes from negative to positive like this, that it is going to be the min and to find the exact point, you can plug two into the original equation. So, if the original equation is two $x$ plus four, the anti-derivative is $x$ squared minus four $x$, so... if you plug 2 into that you get four minus eight so negative four. So, it's a min at two, negative four.

Emanuel's work is illustrated below. He does not worry about the initial condition.

Figure 54. Emanuel's computations for the antiderivative $G(x)$ and its extrema.
To find the regions of concavity, Emanuel uses the second derivative test as shown in figure and concludes the function is always concave up.

Figure 55. Emanuel’s work on concavity.

**Emanuel’s Use of MR.** Emanuel uses predominantly symbolic and numerical approaches such as finding and evaluating integrals and figuring out signs of the first and second derivatives. While he draws a number line, he uses this graphical representation numerically mostly to keep track of his signs. Although he chose the symbolical representation because it was the most familiar to him, he misses the meaning of the absolute value, which would have made for a different graph for the derivative, and for an always increasing antiderivative $G(x)$.

**Emanuel’s Understanding of the FTC.** Emanuel is comfortable applying the FTC to both evaluate definite integrals and to differentiate functions defined by integrals. He does so efficiently, and only does the computations that are necessary to solve the problem. In his own words, which were delivered without hesitation, he demonstrates he is comfortable with the language of mathematics, and that he does indeed understand the FTC, at least in one of its forms. To him, the FTC “means finding the area under a function if you have the endpoints. Say it’s 0 to 1. If you have the antiderivative, and plug in 0 and 1 and then you subtract, that is your answer.” When asked about his perception
about his understanding he responds: “I know how to do the problems. I would not say that I have a complete understanding.”

**Summary of Results on the Think-Alouds**

The nine *Think-Alouds* presented here illustrate a few themes related to the research questions analyzed in this study, such as student problem solving trajectory, transfer among representations, dealing with areas of conflict, problem choice, and their connections with their classroom experience. Students in the study chose a variety of problems to solve for their *Think-Aloud* and shared various interpretations of the Fundamental Theorem of Calculus. In most cases, the students said they chose a problem based on either familiarity, or one they felt would be easier to solve, given their experience. Other reasons cited by students included avoidance—a particular problem was chosen because it made them less afraid. Attraction to a story problem was cited in only two cases. Four of the nine students, Dawn, Dan, Dave, and Deliana, chose to solve contextual problems, two chose the graphical problem, one chose the verbal problem, and two chose the symbolical problem. None of the students chose the numerical problem.

Students at HCCC, where the contextual representation had been a large part of the curriculum, all chose to solve the contextual problem. However, none students at CSNE, where the instructor had used the numerical representation regularly, attempted to solve the numerical problem. In all cases student choice was related to experience.

The problem-solving trajectory of the students and their use of multiple representations also varied. Students who chose to solve the contextual problem were naturally forced to make connections to other representations from what was presented as a “real life” situation. In doing so, the number of multiple representations they used was
greater than those used by students, who did not choose the contextual problem. Also, they did not appear to get stuck or ignore part of the problem, as other students did. Dave and Deliana, for example, specifically used the word “backtrack” when they went back to deal with areas that were in conflict; whereas, Christine, Dan, Allison, Emanuel and Ann all glossed over these difficulties. Dave explained his decision to solve the problem with story in it, the context helped them “understand the math,” and make sense of the problem. All four of these students were able to make some good progress in finding the distance from point A. However, none of the students distinguished between the distance from A, and the position of the ladybug on the number line. Dawn used the words distance and position interchangeably.

Almost all students had an idea that the FTC related the integral and the corresponding derivative. Some discussed it symbolically (Christine, and Dan), and some presented it verbally (Dawn, Christine, Dave, Emanuel, and Deliana), and it was clear that some students could apply it successfully. Two students did not know the FTC or confused it with the first or second derivative tests.

A summary of the students’ problem of choice, school, solution trajectory, researcher’s assessment of that students’ understanding of the FTC, and the students’ own assessment of their understanding is set forth in the conclusion of this chapter in Table 41. The summary sheds light on the patterns already noted.
Table 41. Summary of Student *Think-Alouds.*

<table>
<thead>
<tr>
<th>Problem No.</th>
<th>Problem A</th>
<th>Problem C</th>
<th>Problem D</th>
<th>Problem E</th>
</tr>
</thead>
<tbody>
<tr>
<td>Assessment</td>
<td>Graphical</td>
<td>Verbal</td>
<td>Contextual</td>
<td>Symbolic</td>
</tr>
<tr>
<td>Student</td>
<td></td>
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<tr>
<td>Ann</td>
<td>Graphical</td>
<td>Symbolical</td>
<td>Symbolical</td>
<td></td>
</tr>
<tr>
<td>Allison</td>
<td>Graphical</td>
<td>Symbolical</td>
<td>Symbolical</td>
<td></td>
</tr>
<tr>
<td>Christine</td>
<td>Graphical</td>
<td>Verbal</td>
<td>Contextual</td>
<td>Symbolic</td>
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<td>Dan</td>
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<td>Dave</td>
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<td>Deliana</td>
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<td>Dawn</td>
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<td>Emily</td>
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<td>Emanuel</td>
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<tr>
<td>School</td>
<td>CSNE</td>
<td>RCC</td>
<td>HCCC</td>
<td>RCC</td>
</tr>
<tr>
<td>Reason for Choice</td>
<td>In high school my student used a lot of graphs so I knew more</td>
<td>I don’t like word problems; It looked friendlier</td>
<td>I was avoiding rate of change and discontinuous functions; I know linear functions</td>
<td>It was the only one with a story.</td>
</tr>
<tr>
<td>Belief About Understanding of FTC</td>
<td>Definitely not. I am just trying to piece pieces together.</td>
<td>I don’t think I understand it.</td>
<td>I do not. I do not know how it came about. Because I do not know how to prove it</td>
<td>I know how to use it.</td>
</tr>
<tr>
<td>Researcher Assessment of Understanding of FTC</td>
<td>Novice Incomplete</td>
<td>Novice</td>
<td>Practitioner</td>
<td>Practitioner</td>
</tr>
<tr>
<td>Problem No.</td>
<td>Problem A</td>
<td>Problem C</td>
<td>Problem D</td>
<td>Problem E</td>
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<tr>
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<td>Dave</td>
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<tr>
<td>School</td>
<td>CSNE</td>
<td>CSNE</td>
<td>RCC</td>
<td>HCCC</td>
</tr>
<tr>
<td>Researcher</td>
<td>Anti-differentiation formulas, connections of ( f(x) ) and ( F(x) )</td>
<td>Confuses both FTC with first and second derivative test.</td>
<td>Correctly applies the FTC to compute antiderivatives</td>
<td>Is able to navigate through various representations of the FTC</td>
</tr>
<tr>
<td>Assessment</td>
<td>Cannot transfer among representations</td>
<td>Weak symbol manipulation</td>
<td>Gives correct symbolic statement of the FTC</td>
<td>Confuses the FTC with first and second derivative test.</td>
</tr>
<tr>
<td>Detail</td>
<td>Confuses FTC with FTC</td>
<td>Weak symbol manipulation</td>
<td>Gives verbal explanation of the FTC that is incomplete.</td>
<td>Ignores the FTC with FTC</td>
</tr>
<tr>
<td></td>
<td>Confuses functions and definite integrals</td>
<td>Ignores areas of conflict</td>
<td>Knows how to apply the FTC in finding integrals.</td>
<td>Backtracks when faced with areas of conflict</td>
</tr>
<tr>
<td></td>
<td>Ignores the absolute value</td>
<td>Ignores areas of conflict</td>
<td>Does not pick up on areas of conflict</td>
<td>Ignores the absolute value</td>
</tr>
<tr>
<td></td>
<td>FTC is how you find definite integrals</td>
<td>FTC means finding the area under a function if you have the endpoints.</td>
<td>FTC means finding the area under a function if you have the endpoints.</td>
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<td>FTC means finding the area under a function if you have the endpoints.</td>
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</tbody>
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**Triangulating Think-Alouds and Five Problems Involving the FTC Results**

As part of qualitative analysis, student work done in the *Think-Aloud* interviews, was compared with their results on the *Five Problems Involving the FTC* to the work on the corresponding problem to the students’ choices on the *Think-Aloud* to triangulate the results. For example, both the *Think-Aloud* and the FTC quantitative results indicated that Anne and Allison, the two female students at CSNE had a shallower understanding of the FTC. To further corroborate these findings, the researcher also examined written work on the corresponding problem, and noted that this work was consistent with work done by these students in the *Think-Alouds*. Triangulation was useful to provide for a more comprehensive data set, to support the validity of results, to eliminate inconsistencies, and to draw conclusions from the data.

Below is analysis of the problems in the FTC assessment for the nine students who participated in the *Think-Alouds* which were consistent with their work on the FTC Assessment.

**Problem A from FTC Assessment**

Consider the graph of \( f(x) \). Let \( F(x) \) be the anti-derivative of \( f(x) \) with \( F(0) = 0 \)

1. Use an integral to define \( F(x) \).
2. Determine in each case the main features of the function \( F(x) \) on the indicated interval, such as:
   (a) Regions where \( F(x) \) is increasing and where it decreasing.
   (b) Regions of concavity of \( F(x) \).
   (c) Location of any maxima and/or minima.
3. In the space on the right, graph the function \( F(x) \) that corresponds to \( f(x) \) on the left.
Anne and Allison are students at CSNE, who had chosen problem A for their Think-Aloud. The Think-Alouds by these students revealed that their understanding of the FTC was incomplete, procedural, and that they relied on Google or YouTube for much support in their learning. Although they chose to solve the graphical problem in their Think-Aloud, their solution was almost entirely symbolical, and neither Anne nor Allison exhibited a representational facility around the FTC, often confusing the integral and the derivative, or not fully transferring among various representations.

**Anne’s Problem Solving (CSNE)**

Anne’s work on Problem A on the *Five Problems Involving the FTC* received 53 points out of 100 possible points. Her rating according to the scoring rubric is of Apprentice to Practitioner. In her work on Problem A, Anne displays weak symbol manipulation, and possibly confuses the integral and the derivative, as her work illustrated when she writes, $f(x) = \int f(x) + F(0) \rightarrow \int f(x)$. As in her Think-Aloud, Anne does not give the correct placement for her anti-derivative graph, but recognizes the regions of increase and decrease correctly as (0,1) and (1,2) respectively.
Allison’s Problem Solving (CSNE)

In Allison’s work on Problem A in the Five Problems Involving the FTC assessment, Allison received a score of 13 out of 100 possible points. Allison confuses integral with derivative, and although she attempts to solve the problems symbolically, much of her symbolical work is mathematically lacking. Although Allison correctly writes that \( \int f(x) \, dx = F(x) - F(0) \), and although she recognizes correctly the graph of \( f(x) \), she goes on to compute that \( \int \! -x + 1 \, dx = -1 + c \). Similarly, Allison does not connect the slope of \( f(x) \) with the second derivative of \( F \), making incorrect conclusions about concavity. For example, she concludes that the graph of \( F(x) \) has “no concavity, that there is no maximum or minimum and she draws the horizontal line \( y = 0 \) as her answer to the graph of the antiderivative. For these reasons, her work is classified as Novice according to the scoring rubric.

Both Anne and Allison’s solutions on their written work on the FTC assessment corroborate the Think-Aloud and the Enacted Curriculum results. Allison and Anne have an incomplete understanding of the FTC, with a problem-solving trajectory that focuses on procedural and symbolic understanding, and make incorrect conclusions about the shape of the antiderivative graph, the region of concavity, and increase and decrease. Their curriculum was heavily based on symbolical manipulations, with little opportunity for students to think graphically or to participate in mathematical discourse.

**Problem C from FTC Assessment**

Consider the linear function \( f(x) \) with y intercept 4 and slope -2.
Let \( F(x) \) be the anti-derivative of \( f(x) \) with \( F(0) = 0 \)
1. Use the integral to define \( F(x) \).
2. Determine the main features of the function \( F(x) \) such as:
   (a) Where is \( F(x) \) increasing and where is it decreasing?
   (b) Regions of concavity.
3. What is the location of any maxima and/or minima if any?

(c) Draw a sketch of $F(x)$.

Christine’s Problem Solving (RCC)

Christine chose problem C on her Think-Aloud because she was “familiar” with linear functions. In her corresponding problem in the Five Problems Involving the FTC assessment, she received a score of 67 out of 100 points. Her understanding according to the scoring rubric was assessed as that of a Practitioner. Christine’s work on this problem appears similar to that on the Think-Aloud. She lists the relevant information as $b = 4$ and $m = -2$, and draws the graph of $f(x) = -2x + 4$.

Her solution concluded that “$F(x)$ is increasing for $x < 2$ and increasing for $x > 2$”, and is correct in applying the FTC and the first derivative test.

For part (b) of the problem Christine writes that “$f''(x) = -$, so concave down”. Here her notation is misleading, as she probably means $F''(x)$ is negative.

She correctly concludes that $F(x)$ has a maximum at $x = 2$ but does not provide a $y$ value for this maximum in part (c). Some of her answers are illustrated in Figure 57.

![Figure 57. Christine’s work on problem C.](image)
Christine uses similar methods to solve this problem as she has used in the *Think-Aloud*, including symbolical, graphical and verbal representations validating the interpretation of these results.

**Problem D from FTC Assessment**

A ladybug is crawling along a rod with starting point at A. The bug is traveling at a rate of \( v(t) = 5 - t \) in./min. between \( t = 0 \) and \( t = 10 \) minutes. Answer the following regarding the ladybug’s displacement or distance away from A.

1. What is the relationship between the velocity of the ladybug and its distance?
2. Labeling \( d(t) \) as the distance the bug is from A, determine in each case the following about this distance:
   a. What is the bug’s distance at \( t = 1? \) \( t = 5? \) \( t \) any time?
   b. When (for what times) is the distance increasing and where is it decreasing?
   c. When does the bug change direction of travel or stop?
   d. When is the bug accelerating and when is she decelerating if at all?
   e. What can be said about the maximum and/or minimum distance and when does it happen if at all?

**Dan’s Problem Solving (RCC)**

Dan received a score of 93 in the contextual problem D corresponding to his choice on the *Think-Aloud*. His work on this problem of the *Five Problems Involving the FTC* is mostly symbolical and numerical. His understanding is rated as Expert on this problem.

For part (a), Dan wrote that:

\[
s(t) = \int_0^t v(u) \, du = 5t - \frac{t^2}{2}
\]

\[
s(1) = 5 - \frac{1}{2} = 4.5
\]

\[
s(5) = 12.5
\]
Dan answers parts (c) and (d) of the problem regarding the places where the bug changes direction and where the bug is decelerating, correctly and provides reasons for his solution as that "v(t) = 0 at t = 5" and that "a(t) = -1", so the bug is decelerating.

Dan’s correct solution for the last part indicates that maximum distance is 12.5 units, and the “minimum distance is 0 at t = 0, and t = 10.”

As in his Think-Aloud, Dan appears to have translated the problem to the symbolical domain and to have used a combination of contextual, symbolical and numerical methods. This work is consistent with his work on the Think-Aloud.

**Deliana’s Problem Solving (HCCC)**

Deliana’s score on the contextual on the FTC assessment, problem D, is 100%, illustrating her choice of problem to solve in Think-Aloud matches her performance on the Five Problems Involving the FTC. Her solution pathway is on the problem on the FTC assessment is similar to the pathway undertaken on the Think-Aloud. Like in her Think-Aloud. For example, Figure 58 from her work, on problem D of the Five Problems Involving the FTC is similar to her Think-Aloud written work. As in her Think-Aloud, Deliana integrates and finds the constant of integration to be zero.

\[ s(t) = \int v(t)dt = 5t - (1/2)t^2 + c \]  
She concludes that c = 0.

The rest of her problem illustrates a good transfer from contextual representations to symbolical and numerical results as she answers parts c and d of the problem regarding the places where the bug changes direction or where the bug is decelerating. She concludes that “v(t) = 0 at t = 5 and that v’(t) = a(t) = -1, decelerating.” Deliana also
correctly also writes that the maximum distance is at the vertex of parabolas, and the minimum at $t = 0$, and $t = 10$.

Figure 58. Deliana’s work on problem D.

Deliana’s rating of understanding on of the FTC on this representation is rated using the scoring rubric in Appendix E is that of an Expert on this representation.

**Dave’s Problem Solving (HCCC)**

Dave received a score of 73 on the contextual problem D. His approach to solving this problem on the *Five Problems Involving the FTC* is mostly symbolical and numerical. His understanding is rated as Practitioner on this problem, mostly because of some incorrect usage of symbols, such as omitting the $dt$ symbol in the integrand, or not completing a full answer to the last part of the problem. Whereas in the *Think-Aloud*, Dave used an interesting graphical representation to show that the bug did not change direction, graphical work is absent in this problem.

For part (a) regarding the distance, Dan wrote:

$$\int_0^t (5 - t) - \frac{t^2}{2} \, dt \Bigg|_0^1 = 5 - \frac{1}{2} = 4.5 \text{ (for } t = 1), \quad \Bigg|_0^5 = 12.5 \text{ (for } t = 5)$$
He also wrote that the distance for any \( t \) equals \( 5t - \frac{t^2}{2} \).

For parts (b), (c), and (d), he gave correct answers, but he did not provide explanations. For the last part asking for the maximum and minimum distance, he did not provide a minimum distance but provided a good explanation for the maximum as he answered, “Max distance at directional change at \( t = 5 \), distance of 12.5”. His solution illustrates good transfer from contextual to symbolical results, good understanding of the FTC, although he did not always use the correct symbols.

![Figure 59. Dave's partial work on Problem D.](image)

**Dawn’s Problem Solving (HCCC)**

Dawn’s score on the contextual on the FTC assessment, Problem D, is 100%, illustrating her choice of problem to solve in *Think-Aloud* matches her performance on the *Five Problems Involving the FTC*. Her solution pathway is on the problem on the FTC assessment is similar to the pathway undertaken on the *Think-Aloud*.

While in her *Think-Aloud* Dawn used a graphical and symbolic approach, this approach is more symbolical although it still has graphical elements. Dawn evaluates that \( d(t) = \int v(t)dt = 5t - \frac{t^2}{2} \) and correctly finds \( d(1) \) and \( d(5) \). A graph of the velocity and of the distance illustrated on the right in Figure 60. Without seeing the actual problem-solving process, it is hard to know if Dawn used the graph after she evaluated the integral.
as a check or if she drew it before. The assumption is based on the fact that we write left to right, was that the graph was drawn after the symbolical computation.

Dawn correctly provides answers to the rest of the problem, namely that the bug stops at \( t = 5 \) but does not give a reason, aside the picture for \( v(t) \), that the bug is always decelerating since \( a(t) = -1 \), and that the maximum distance is at \( t = 5 \), \( d(5) = \frac{25}{2} \), and the minimum at \( t = 0 \) and \( t = 10 \), with minimum \( d(0) = d(10) = 0 \).

Her solution illustrates good transfer from contextual to symbolical, numerical, and graphical results. Dawn is rated as Expert on her understanding on of the FTC on this representation using the scoring rubric in Appendix E.

![Graph of \( V(t) = 5 - t \)
\( d(t) = \int_0^t (5 - t) \, dt = 5t - \frac{t^2}{2} \)

- a) \( d(4) = 4.5 \)
  b) increasing for \( v > 0 \) on \((0, 5)\)
  c) \( d(5) = \frac{25}{2} \)
  d) decreasing for \( v < 0 \) on \((5, 10)\)
- e) bug stops at \( t = 5 \)
- d) decelerating for all \( t \)
  a = -1
e) maximum at \( t = 5 \) \( d(5) = \frac{25}{2} \)
  min at \( t = 0, 10 \) \( d(0) = d(10) = 0 \)

Figure 60. Dawn’s work on problem D.

**Problem E from FTC Assessment**

Consider the function \( f(x) = 2x - 4 \). Let \( F(x) = \int_0^x f(t) \, dt \)

1. What is the relationship between \( f(x) \) and \( F(x) \)?
2. Please determine in each case the main features of the function \( F(x) \), such as:
   (a) Where is this function increasing and where is it decreasing?
(b) Regions of concavity of $F(x)$.
(c) Location of the maxima and/or minima of $F(x)$.

**Emily’s Problem Solving (RCC)**

Emily scored 33 out of 100 points on Problem E of the *Five Problems Involving the FTC* assessment. Emily’s work on this problem is inconsistent with her performance on the Think-Aloud, possibly due to a lack of time since this was the last problem on the assessment. Her other assessments on problems A, B, C, and D are scored at 67, 67, 56, and 60 respectively, suggesting that this may have been the case.

Emily’s work is symbolic and numerical when she writes: $F(x)$. In attempting to answer part (a) of the problem, Emily sets $2x - 4 = 0$ to find $x = 2$ as a critical point, but does not finish her answer parts (a) and (b) asking for the regions of increase and decrease and for the regions of concavity of $F(x)$. For her answer for part (c) she evaluates “$F(2) = 4 - 8 = -4$ minimum” but does not say anything about the maximum. For this reason, Emily’s understanding as scored on the scoring rubric was evaluated at the Apprentice level.

**Emanuel’s Problem Solving (CSNE)**

Emanuel scored 89 out of 100 points on Problem E in the *Five Problems Involving the FTC* assessment. Emanuel’s work on this problem for part is purely symbolic and numerical initially and he appears to be proficient at it as he writes $f(x) = 2x - 4; F(x) = x^2 - 4x$, with no other steps. There is also picture on the right side of the page of this parabola with a little note saying “$F(x)$ graph to help”, almost as if his note it to help the researcher understand his thinking. The vertex of the parabola is
indicated by a point that is 4 units down from the $x$-axis and 2 units to the right of the $y$-axis.

The only other notes are Emanuel’s answers:

(a) increasing: $(2, \infty)$, decreasing: $(-\infty, 2)$

(b) Concavity is up on the entire function $F(x)$

(c) Minimum at $(2, -4)$. No maximum

It appears that Emanuel is relying more on his algebra and knowledge of functions, rather on Calculus to solve this problem, including graphical representations of the parabola. He appears to have graphed this parabola either by hand by looking at the roots, or with the graphing calculator, and evaluated the value of the vertex. Unlike other solutions, he does not graph the derivative function $f(x)$, and nor does he appear to use the first or second derivative tests to find his answers. For this reason, his understanding as scored on the scoring rubric was evaluated at the Apprentice level. See Figure 61.

Figure 61. Emanuel’s work on problem E.
In summary, triangulation using the in-class assessment to corroborate and expand the results from the *Five Problems Involving the FTC* and from the *Think-Alouds* has been helpful in validating the interpretation of the results already obtained. Allison and Anne, similar to their *Think-Aloud*, showed a weak understanding of the FTC, and used similar strategies to solve the problem. Dave, Dawn, Deliana, Dan, Christine and Emanuel, also used similar strategies to those seen in their *Think-Alouds*. Although their solutions are less developed in their *FTC Assessment* than on the *Think-Aloud*, their thinking and their use of representations and transfer among them is consistent. Emily’s work seems incomplete, and possible reasons have been suggested. Overall, this process bolstered the study by increasing the overall credibility of the interpretation of the data.

Whereas Chapter 4 concentrated on the results on the Enacted Curriculum, using mostly quantitative means, Chapter 5 addressed Student Understanding of the FTC using multiple representations along with the effect of gender. Although not all results were significant, there were some quantitative differences by site and by gender, particularly with respect to graphical, contextual, and symbolical representations. *Think-Alouds* and student hand-written work revealed additional sources of information regarding student reasoning, problem solving, and general understanding of the FTC. Student cognitive preference in some domains also played a factor in student understanding. In the next chapter the results from Chapters 4 and 5 will be addressed together as a whole to answer the research questions of the study.
CHAPTER 6
CONCLUSIONS

The literature review outlined the status of collegiate Calculus teaching and learning in the U.S. and made the case for conducting a research study aimed at making explicit the connection between the use of multiple representations in the enacted curriculum and student understanding of the Fundamental Theorem of Calculus. In the current chapter, the results presented in Chapters 4 and 5 will be synthesized to answer the research questions. Then, attention will be given to limitations of the study and to further implication for mathematics education research. The chapter is organized as follows: 1) Overview, 2) Answering the research questions, 3) Limitations of the study, 3) Discussion, 4) Implications and new directions and 6) Recommendations.

Overview

As discussed in earlier chapters, the 2010-2015 MAA large scale survey of Calculus instruction at two-year and four-year colleges, indicated that Calculus students not only have a difficult time with Calculus, but that many choose to abandon STEM majors based on their experiences in their first-year Calculus class. Despite calls for reform by organizations such as the NCTM (2000), research in the theory of learning (Dubinsky and McDonald, 2001; Janvier 1987; Kaput, 1994), and documented learning difficulties of students taking Calculus (Aspinwall & Shaw, 2002; Haciomeroglu, Aspinwall & Presmeg, 2009, 2010), the MAA survey of more than 11,000 students revealed that Calculus instruction in this country is still conducted in a highly traditional manner and that Calculus instructors still believe that ”students learn best from lectures”
(Bressoud, 2013). The survey also suggested that women in STEM drop out at rates almost twice that of male students in similar programs.

Subsequent work has revealed characteristics of colleges deemed “successful” in the teaching of Calculus based on pass rate, retention and course satisfaction (Bressoud, Carlson, Mesa & Rasmussen, 2013). These characteristics include training of TA’s, coordination of the Calculus classes at the departmental level, advising, and active learning pedagogy (Bressoud, Carlson, Mesa & Rasmussen, 2013). The MAA study, however, did not make an explicit connection between what happens in the Calculus classroom setting and student understanding of the Calculus concepts they are learning. While active learning strategies have been identified to encourage student engagement and help them be less likely to drop out of STEM curricula (Laursen et. al, 2014; Freeman et.al, 2014). Research conducted thus far, by the MAA and others, has made no connection between the enacted curriculum and student understanding as measured by common instruments. Recent research has also not made connections between the enacted curriculum and gender. The research presented here extends the MAA results by making connections between the enacted curriculum with multiple representations and student understanding apparent, by using a common lesson observation protocol, and common assessments for student understanding as well as gender.

The researcher follows NCTM in believing that “mathematics is a living subject which seeks to understand patterns that permeate the world around us and the mind within us. It is important that…students move beyond rules to be able to express things in the language of mathematics” (NCTM, 2000). Knowing mathematics and understanding mathematics, as used in this work, are one and the same. The study adopted a situated
perspective in which learning occurs in terms of participation in a social community and in which mathematical reasoning is developed by communal practices. “Understanding refers to an integrated and functional grasp of mathematical ideas,” (NRC, 2001) in which students transfer easily among various representations. Student understanding of the FTC was measured from multiple sources, including FTC assessments, Think-Alouds, a Background Questionnaire and field notes. Although it can never be exactly determined what somebody else actually knows, a multiple source approach helped obtain a closer approximation to students’ true understanding of the FTC.

Multiple representations are important in creating meaning (Kaput, 1984), and in developing versatile thinking (Tall, 1997). As presented in the Conceptual Framework in Chapter 3, the study explored the idea that students’ understanding of the FTC would be related to their experience with multiple representations in the enacted curriculum and that students would tend to use the representations their teachers used. The study also hypothesized a greater alignment between student use of multiple representations and classroom experience for female students. Furthermore, the study anticipated that the classroom culture and discourse, along with the purposeful use multiple representations, would play a key role in student understanding of the FTC and provide access to deeper learning, particularly for female students. The results of the study confirmed that student understanding of the FTC is in alignment with the presence and depth of use of multiple representations in the curriculum. The study also revealed some new dimensions or patterns that relate student understanding of the FTC and the enacted curriculum. The next section is devoted to making this connection evident by answering the research questions.
Answering the Research Questions

Research Question 1

What is the nature of the relationship between students’ use of multiple representations in the enacted curriculum and student understanding of the FTC?

Student understanding or knowing of the FTC was viewed in this study as the ability to engage successfully and collaboratively with multiple representations of the FTC and to use critical thinking in communicating about the FTC with an emphasis on conceptual understanding rather than rote memorization.

Sub-question 1: In what ways do MR appear in the enacted curriculum?

Extensive studies show the importance of multiple representations in developing mathematical habits of mind and in developing versatile thinking (Tall, 2012; Janvier, 1987). For these representations to be meaningful, the representations need to be made explicit in the context of what the students already know (Kaput, 1994). Although the ability to understand and work with multiple representations is a trademark for mathematical success, MR are often missing or underutilized in our classrooms, including Calculus, due to lack of time, or a belief that they are not important, or lack of teacher experience in using multiple representations (Arcavi, 2003; Guzman, 2002; MAA, 2012). The results of the study resonate with this literature. Several representations were not supported adequately by the enacted curriculum at some of the sites. Some representations were missing entirely, or the opportunity for students to engage productively with these representations was not present in the curriculum.

A variety of multiple representations were used in teaching the FTC at the three observed locations. HCCC displayed the greatest variability in the types of
representations used in the classroom, with contextual, graphical, verbal and symbolic representations predominating but had fewer numerical representations. At RCC, the enacted curriculum emphasized graphical, verbal and symbolic representations, with few contextual representations. The CSNE curriculum emphasized symbolic, verbal and numerical representations, but was lacking in any contextual representations, and graphical representations were used only on a few occasions.

Each of these classes differed significantly in the way the curriculum was enacted. The RCC and HCCC courses were highly interactive, although in diverse ways, while the CSNE course was not. CSNE favored a predominantly lecture format, with little teacher–student or student–student interaction; RCC used whole class discussion on a regular basis, with the teacher employing an IRE pattern to engage the class in conversations about the mathematical concepts, and HCCC students were engaged a variety of learning methods including both small group work and large group discussion.

Finally, the three locations differed in the cognitive demand of the tasks involved. According to The Task Analysis Guide for Mathematics developed by Stein et. al. (Stein, M.K. et al., 2000), RCC and HCCC included high level demand tasks in the enacted FTC curriculum that were categorized as *procedures with connections* (students are guided for understanding for developing deeper levels of mathematical understanding, or *doing mathematics* (students are engaged in complex and non-algorithmic thinking, not explicitly suggested by the task, task instructions, or a worked or worked examples, and they explore and understand the nature of mathematical concepts). At HCCC especially and to a lesser extent at RCC, students engaged in complex problems where they had to justify answers, construct arguments, reason with patterns, and transfer among
representations. The HCCC curriculum focused on connections with real world problems, and with historical context; RCC emphasized connections with prior content such as algebra, geometry and precalculus. At CSNE, the enacted curriculum was categorized as it was of a low cognitive demand and it concentrated on memorizing procedures, with little opportunity for students to make sense of the concepts involved.

Sub-question 2: What is the nature of the relationship between the use of MR in the classroom and students’ overall understanding of the FTC?

Student understanding was compared to the results of the enacted curriculum captured in the classroom portraits, field notes and *Think-Alouds*. Student understanding was defined here as the ability to engage successfully and collaboratively with multiple representations of the FTC and to use critical thinking when they communicated about the FTC during class, on the FTC assessments, and during *Think-Alouds*. Students, though not randomly selected, all had a similar prior knowledge of Calculus, based on the *Background Questionnaire* results. So, for the purposes of the study, they were considered to have started on equal footing.

To understand the relation between student understanding and the nature of the curriculum, descriptive statistics, boxplots, along with ANOVA comparisons of student scores on the FTC assessments were compared to the results on the enacted curriculum described in sub-question 1. *Think-Alouds*, the *Background Questionnaire*, and written work from the subset of the participants interviewed in the *Think-Aloud* were used to corroborate or expand results. It was hypothesized that student understanding and use of multiple representations would align with the use of multiple representations in the enacted curriculum.
Learners’ knowledge or understanding of the FTC, including their use of multiple representations in the FTC Assessments and the Think-Alouds, the ways they communicate about the FTC during class or during the semi-structured interviews that followed the Think-Alouds, along with their problem-solving trajectories undertaken in the Think-Alouds, all converge to support the study hypothesis that student understanding of the FTC aligns itself with their class experience with multiple representations.

**Enacted Multiple Representations and Classroom Discourse**

The study sought not just to focus on the presence or absence of MR in the enacted curriculum, but rather to meaningfully note the quality, depth and student involvement with multiple representations during their classroom experience and connect these to what students understood about the FTC so that connections with student understanding could be drawn. RCC students routinely engaged in classroom discourse, albeit in a more teacher scaffolded environment. Their experience was of an IRE dialogue pattern where the teacher called on students to answer his or each other’s questions. The HCCC curriculum focused on group-work with relatively few teacher interventions, and where mathematics doing was the primary focus, and where students had the opportunity and were expected to grapple with sophisticated mathematical ideas daily to develop and solidify their knowledge. There were clear differences between the lecture-based CSNE curriculum and the two more active curricula at RCC and HCCC. At CSNE, about 90% of the class time was spent in lecture, with students listening to the instructor. By contrast, in the other two courses, a large part of the class time was spent with students engaged in mathematical ideas in student-centered activities, although the format varied.
Students at HCCC were frequently constructing arguments, puzzling over problems, analyzing data, and engaged in collaborative discourse with their peers.

The LOP observation protocol rated two curricula (RCC and HCCC) higher on the multiple representations section because the instructors in these classes used multiple representations in the class presentation in ways that built students’ engagement and opportunity to learn, which is in line with current mathematics education recommendations (NCTM, 2000; Cobb & Yackel, 1996). LOP scores rated student use of numerical, graphical, verbal contextual, or symbolic representations using the protocol’s 1 – 7 Likert Scale. The enacted multiple representations scores were in midrange for the active curricula (RCC and HCCC) and low range for the traditional curriculum (CSNE) as can be seen in Table 42.

Table 42. Researcher scores on the enacted curriculum.

<table>
<thead>
<tr>
<th>Student opportunity to use representations</th>
<th>RCC</th>
<th>HCCC</th>
<th>CSNE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Numerical</td>
<td>4</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>Graphical</td>
<td>5</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>Verbal</td>
<td>5</td>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td>Contextual</td>
<td>3</td>
<td>6</td>
<td>1</td>
</tr>
<tr>
<td>Symbolic</td>
<td>4</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>Average</td>
<td>20/5 = 4</td>
<td>23/5 = 4.6</td>
<td>11/5=2.2</td>
</tr>
</tbody>
</table>

Lower scores indicate low opportunity to learn using multiple representations, whereas higher scores indicate active use of multiple representations by both students and instructors. Students at CSNE had little opportunity to use contextual, symbolic and graphical representations, and limited engagement with the other two representations. RCC students used limited contextual and numerical representations with the remaining
representations being used in the mid-range to high mid-range. At HCCC only the numerical representation was rated as a 3, because other representations featuring more prominently in student classroom discourse.

**FTC Assessments and the Enacted Curriculum**

The scores in Table 42 agree with descriptive and inferential statistics from Chapter 5, supporting the study’s hypotheses. The Total Score (TS) on the FTC assessment and the enactment scores in Table 42 for the CSNE curriculum (traditional lecture based classroom) were generally lower than the scores at the other two locations (RCC, and HCCC) which involved students in classroom discourse. Although ANOVA results indicated no significant difference in total score (TS) across the sites, boxplots and other individual representation statistical tests brought to light a more-subtle picture.

In the boxplots in Figure 24, symbolical, graphical and contextual scores at CSNE appeared lower than the corresponding scores at the other two locations, and the difference was significant in the contextual and graphical domain. The students at CSNE had little or no exposure to contextual problems and had lower scores on the contextual problem. Students from HCCC had a great deal of exposure to word problems and had the greatest score on that representation. *Enacted Curriculum* scores for the contextual and graphical representations at CSNE were both equal to 1, as presented in Chapter 4 in Table 23, because this curriculum did not include any or did not include meaningful mathematics using these representations.

The correlation between the Total Score (TS) and the researcher’s scores for the Enacted Curriculum does not imply causation. However, all students had similar prior mathematical preparation as indicated by their prior Calculus experience. Had the CSNE
students started out weaker in their mathematical preparation, than the other two groups, they may have done worse no matter what happened in the enacted curriculum. Since students started out with similar preparation, results suggest that the way in that the CSNE curriculum was enacted may have contributed to the lower achievement on the FTC assessment. Classroom portraits of the Enacted Curriculum presented in Chapter 4, the results of the Think-Alouds from Chapter 5 and the student written work on the FTC assessment, corroborate and augment the quantitative results established in the FTC assessment.

NCTM’s (2000) Equity Principle states, “Excellence in mathematics education requires equity-high expectations and strong support for all students” (NCTM, 2000). Teachers’ expectations of their students have implications on student learning (Knapp, 1997). Students at CSNE had been exposed to a curriculum that focused on traditional teaching and featured lower cognitive demand tasks. This curriculum did not afford students with the opportunity to dialogue about mathematics only had a limited understanding of the FTC in all representations, but had developed fewer strategies for doing mathematics.

RCC students had a greater opportunity to involve themselves with the FTC. During classroom observations, they showed their evolving understanding through the use of explanations and by volunteering to come to the board to present solutions to problems. Although the course followed an IRE pattern, students were actively engaged and participated in class discourse. Their answers and thinking was encouraged and valued, and the teacher included everyone in the discussion, at times delaying some students from answering so that all students had the chance to think about his questions.
RCC did not often include connections to real world problems, but included connections to prior work and articulated students’ understanding during the whole group discussions. Of all five representations discussed by this research, contextual enacted representations were least present in the RCC curriculum, while they were prominent in the HCCC one. Correspondingly, though not statistically significant, boxplots of scores at RCC were lower than those at HCCC in this domain. Interestingly, RCC scores were also lower than scores at HCCC on the symbolical domain, even though the curriculum had included ample symbolical work. Possible reasons may involve the difference in the classroom discourse and in some of the curricular tasks at the two sites. The RCC curriculum, while engaging all students, did so using IRE dialogue, and had few contextual representations. The researcher took this to mean that students at HCCC may have had a greater opportunity to do heavy-mathematical lifting on their own, which may have allowed them to become more proficient in understanding the symbolical nature of the FTC, as well as the contextual one.

In this study, students at CSNE had lower assessment scores than students at RCC and HCCC. The differences were significant on the Contextual and Graphical representations, and they were particularly striking for female students, who performed lower than students from all other groups on every representation except the numerical and verbal ones, as illustrated in Figure 24, and the results in Chapter 5. Although not all differences were statistically significant, combined with the visual results from Figure 25, and with the results of the Enacted Curriculum, the Think-Alouds, and the FTC Assessments, the evidence suggests that these nuances are a consequence of the curriculum. The lecture style classroom at CSNE disproportionately hurt female
students. This result is consistent to other literature (Joiner, Malone, & Haimes 2002; Laursen et. al, 2014).

**Think-Alouds and the Enacted Curriculum**

The *Think-Aloud* portion of the analysis illustrated that the students in the active curricula were more able to translate among representations, had a richer understanding of the FTC, engaged in more complex thinking and were able to analyze and to backtrack when encountering areas of conflict. This was particularly true of the class that used small group work and contextual problems extensively. These results do not independently establish causation, but when combined with the qualitative results of the *Think-Alouds*, and the recognition that students in the three courses started on an equal footing, there is good evidence indicating that this may be the case. The results are in line with the MAA results, which indicate that students in active learning curricula are more likely to be successful in completing their Calculus course, and with the constructivist ideas adopted in this paper.

The *Think-Alouds* also suggest that student knowledge and problem solving matches the students’ opportunities with multiple representations that their curriculum had afforded them. For example, all students had been exposed to a great deal of analytical representations during the FTC unit. Consequently, most of the students interviewed (seven out of nine) could compute integrals proficiently. Of course, being able to manipulate symbols, does not equate to understanding. When students are taught a set of rules with little or no contextual or conceptual meaning, the practice encourages students to use the symbols without understanding of the principles (Schoenfeld, 1985; Skemp, 1987). The *Think-Alouds* revealed that despite computational facility, most
students (Anne, Allison, Emily, Emanuel, Dan) became confused about the meaning of the FTC, or about other Calculus concepts such as the first and second derivative tests (Ann, Allison), had a superficial understanding of the various representations of the FTC (Ann, Allison, Christine, Dan, Emily, Emanuel), and most them ignored areas of conflict (Ann, Allison, Emily, Emanuel, Christine, Dan). These results agree with many previous findings (Aspinwall & Shaw, 2002; Pantozzi, 2009; Tall, 2009, 2010).

Active learning, broadly defined as learning through activities, problem-solving, discussions, which emphasizes higher order thinking promotes students learning and engagement (Freeman, Eddy, McDounough, Smith, Okoroafor, Jordt, 2013). Students at HCCC were exposed to various investigative tasks during group work exercises. The curriculum enacted there gave all students the opportunity to do mathematics. The nature of the tasks chosen by Professor Brown allowed students to generate multiple solutions to problems, as revealed during the Think-Alouds from this site. Dawn, Deliana and Dave all chose different approaches to the same problem. Moreover, during group work students were able to clarify areas of confusion collaboratively and regularly.

Teaching practices necessary to promote deep learning of mathematics include implementing tasks that promote reasoning and problem solving, facilitating meaningful mathematical discourse, using and connecting mathematical representations, supporting productive struggle, and eliciting and using evidence of student thinking (NCTM, 2014). The emphasis is on students doing the mathematics.

None of the students at CSNE chose contextual problems to solve, and the two female students who chose the graphical problem (problem A) did so based on high school experience with graphs as they revealed in the Background Questionnaire and
their Think-Alouds. Their choice was not based on their experience in the Calculus class. Even though this experience was more recent, it may have been less relevant. During the Think-Alouds, these two female students worked mostly symbolically even when choosing the graphical problem, just like their curriculum had emphasized, but their symbolic manipulation was incorrect and generally lacked meaning. Areas of difficulty while solving this graphical problem involved interpreting graphical notions about the derivative and the FTC, confusion about the first and second derivative test, about meaning of the integral, or about the meaning of the rules for anti-differentiation.

“Students must learn mathematics with understanding, actively building new knowledge from experience and prior knowledge” (NCTM 2000). Rather than learning mathematics with understanding, CSNE students referenced remembering formulas during the problem-solving process, and were not able to support their reasoning with appropriate mathematical reasoning. During the Think-Alouds from this site, Allison and Anne attempted to remember the procedures they had learned, but more often than not, they could not do so. When they reached an area of conflict, they simply stopped.

The one departure from the researcher’s hypothesis had to do with student performance on symbolic representations as evident in Figure 24. Students’ scores on the symbolical representations problems were generally lower than scores on the other representations in all three classes, despite the time, which in many cases was substantial, that students spent on this representation in the class. In two of the three classes (HCCC and RCC) the mean scores were between 7 and 8 points on this problem than on the overall assessment respectively as can be seen in Table 24 - Table 26. For the third class (CSNE), where symbolic representations were used extensively and almost exclusively,
mean scores were still than 6 points lower on this problem than on the overall assessment score (Table 26). These results suggest that the symbolic representations may be conceptually more difficult for students to deal with and that teachers and students may need to spend more time collaboratively unpacking this representation.

David Tall’s three worlds of mathematics represents stages experienced in mathematical learning as conceptual embodied, proceptual symbolic, and axiomatic formal (Tall, 2004). Students working on the symbolic representation problem were still operating in the proceptual symbolic domain, a world of algebraic manipulation and processes, and had not fully developed theoretical notational systems used in abstract mathematics. Conclusions are also in agreement with the BAPOS (Base Objects, Actions, Processes, Objects, and Schema), espoused by Chae (2003), which posits learners perform Actions on Base Objects that then are coordinated into Processes and represented by symbols having meaning as mental Objects, within a wider Schema. Symbolical understanding of the FTC, occurs as a later stage of learning, and consequently students in this study were rated lower on their understanding of this mathematical representation.

The Think-Alouds revealed that in the two active classrooms, students made more use of multiple representations in solving problems. They were more able to take risks when thinking about problem solving strategies and to back track when they reached areas of conflict. In the lecture class (CSNE), and to a smaller extent in the RCC curriculum that favored IRE and whole group discussions, students were limited in the number of strategies they attempted and either did not realize that they were dealing with contradictory results or ignored them. Since the background of the students was the same, this was interpreted as an indication of a shallower understanding of the mathematics
involved. However, in the lecture-driven course there was a great evidence of students mimicking procedures without an understanding of the mathematics involved. Consistent with results of other researchers (Aspinwall & Shaw, 2002; Haciomeroglu, 2007; Tall & Gray, 1994), students in this course showed a great deal of confusion about mathematical concepts such as increasing and decreasing, concave up and down, definite and indefinite integrals, and functions and their derivatives.

**Sub-question 3: What roles do other factors, such as representational cognitive preference and perceived representational instruction, play in the ways students understand the FTC?**

If we posit that student understanding of the FTC is the ability to work with and to communicate about multiple representations of the FTC in a seamless manner, the Total Score (TS) that students received on the FTC Assessment and as augmented by Think-Alouds captures their understanding. Analysis in this study identifies students’ cognitive preference (CP), students’ accommodated needs (zdiff), and prior knowledge (PK) as factors influencing student understanding.

Student perceived representational instruction (PR) appears to have no bearing on student understanding. Regression analysis for the Total Score on the FTC assessment (TS) as a function of site, cognitive preference (CP), perceived representational instruction (PR), and accommodated preference indicated that student perception of the instruction was not a significant factor in student understanding.

**Effect of Cognitive Preference**

Student cognitive preference, on the other hand, was significant in student achievement, particularly for some representations. From the results in Chapter 5,
Graphical cognitive preference and Symbolical cognitive preference were both significant and accounted for 43% of the variability in total score. A one-point increase, on a scale of 4, on the graphical preference corresponded to a 1.4% – 14.4% increase in the Total Score, with a standard error of 3%. Each one-point increase in the symbolical preference corresponded to a 6.4% – 18.3% increase in Total Score with a standard error of 3%. It is natural to ask why were these two representations significant and not the others, and what the implications are for instructional practices. The first question is not easily tractable, but the researcher’s belief is that the answer may have to do with the fact that these two representations may feature more prominently in the students’ entire curricular experience in mathematics. Regarding the possible implications suggested by the results, these may be that students will learn more if they learn in their preferred representations, and that understanding the factors affecting these representational preferences, such as classroom experience, interactions with peers, collaboration with colleagues, situated learning, etc., and thus may provide a key to understanding how to better support our students in their learning.

A representation, shown in Figure 62, that may be useful is that of students’ average cognitive preference against the total score. The figure illustrates the total scores (TS) versus the average cognitive preference, with data points of different shapes and colors to indicate the different sites. The average cognitive preference is the average of the scores reported by students in the Background Questionnaire as their cognitive preference on each of the five representations considered in this study (graphical, numerical, contextual, verbal and symbolical). Figure 62 also indicates the line of best fit
for each of the three sites observed for this study. The figure suggests a similar relation between Total Score and average cognitive preference.

![Graph showing correlation between Total Score and average cognitive preference](image)

**Figure 62.** Average cognitive preference and total score by site.

In this graph, students with an average cognitive preference score of 4 scored approximately 60 points or more or 200% higher than those with an average score of 2, who averaged in the low 20’s. Interestingly, this relation is consistent across gender and site, and points to the fact that more work needs to be done to precisely understand the role of cognitive preference on student achievement. Cognitive preference is self-reported here, and it is part of a complex variable that may also partly refer to student familiarity or experience with representations over the course of their entire schooling. The kinds of representational opportunities that students are provided with, may very well play a role in their perceived preference. Students who had a higher average cognitive preference for varied representations, scored higher than those who did not. Preference for
representations may also indicate understanding of the mathematics (in this case the FTC) around those representations. Representational needs are just part of the many factors that affect student understanding. Cognitive representational needs also play a key role in student understanding of the FTC. At CSNE (CS), 17 out of 19 did not have their total representational needs met, while at RCC only three of 22 students did not have their needs met. HCCC distribution was in between. Students’ needs were significant factors in the Total Score (TS) and in students’ Graphical Score (GS), and were close to significant in Contextual Scores (CS). The results of the regression undertaken are presented in Chapter 5. To further deconstruct these results, each increase in representational needs (zdiff) of one point (on a six-point scale) resulted, on average, in a drop in score of 1–5 points (out of 100) on the FTC assessment, with a standard error of 1.0 point, and a 1.3–13.2 drop (out of 100) on the Graphical Score with a standard error of 2.9 points. To effectively reach our students requires continually looking out for their needs. A finer scale for cognitive preference and more data points to calculate cognitive preference also may help to make this investigation more robust. One limitation is that for this particular study, the cognitive preference indicated is self-reported, so it may be confounded with students’ feelings about what they thought they would do well on.

**Effect of Prior Knowledge**

Of other factors affecting student understanding, student prior knowledge of calculus had a statistically significant effect on Total Score as measured by the FTC assessments. A two sample one tailed $t$-test for the effect of prior knowledge on student assessment showed that students with prior knowledge performed better than those students with no prior knowledge ($p = 0.004$) by at least 5.4 points. Many mathematicians
believe Calculus should be left to colleges, because high-school calculus is "watered down...stressing manipulations but slighting subtle processes" (Leitzel et al., 1987) and thus cannot focus on the conceptual nuances that college Calculus can provide. Another concern about Calculus in high-school is that students will not have enough time to learn preparatory topics such as algebra. The current study contradicts this belief and indicates that students who have some knowledge of this subject prior to college did better on Calculus concepts than those who did not take Calculus. The result is in agreement with former studies (Ferrini-Mundy & Gaudard, 1992) and more recent results from the MAA large-scale study (Bressoud et. al, 2013).

Conclusions about Research Question 1

Each of the observed classrooms used multiple representations in presenting the FTC, but to various extents, and the level of student engagement differed. The LOP tool ratings for the active curricula (RCC and HCCC) was higher on the multiple representations section than the rating for the lecture based curriculum (CSNE), because the observation protocol included students’ engagement and opportunity to learn. Student understanding at these two institutions, as measured by both the FTC assessment and the Think Alouds was correspondingly higher at RCC and HCCC. Results were corroborated by written work on the FTC assessment. This study suggests that courses that support multiple representations with active learning strategies and where mathematical discourse is part of the enacted curriculum may be more effective in promoting student understanding than lecture based curricula, or those curricula where there is little student–student or teacher–student interaction. The result is in agreement with the conceptual framework developed for this study presented in Chapter 3. Even though all
instructors discussed at least several representations in their presentation of the FTC, deeper student understanding of various representations was found in the two institutions that used active learning strategies. Active learning is an added dimension, or catalyst, for effective understanding of multiple representations in the classroom.

The Think-Alouds gave insight into student problem choices, problem solving trajectory, their experience in the classroom, and their preferences for various representations. Eight of the nine students picked problems that they thought they would do better on because of the class experience, or because of their high school experience. One student said he picked the problem because it was the only one with a story in it. Some students also expressed a dislike for some of the other problems and used a process of elimination. Further questioning revealed that a lack of experience or confidence was a reason for choosing a problem to solve. All three students in the HCCC course that emphasized contextual representations chose to do contextual representations. Furthermore, the problem-solving trajectory of those students was much richer, included more representations, and allowed them to resolve, rather than ignore, areas of conflict. Thus, cognitive preference, active classroom discourse, and the use of substantial representations enacted using rich tasks, all are key factors in student understanding for all students.

**Research Question 2**

To what extent does students’ gender influence their use of MRs and their understanding of the FTC? The second research question in this study looked at the extent to which students’ gender influences their use of multiple representations and their understanding of the
FTC. The MAA’s study of Calculus I showed some differences between the men and the women in college calculus. The study uncovered differences in preparation and persistence and reasons for not continuing. Women were much more likely to question their ability to handle the course work, and feel depressed about their progress. They were also more likely to leave science because they found it too competitive \((p < 0.01)\).

The literature review of gender differences in classroom environments also showed that women perceive that they are more involved in the course and achieve significantly better in courses that use active learning strategies (Blakett & Tall, 1998; Boaler, 2002; Joiner, Malone, & Haimes 2002).

The background questionnaire employed in this study corroborated these MAA findings. Women in the three classes rated their experience, grade expectations, and feelings about their understanding lower than the male students. Female students thought they would get a grade of 73 on the average, while male students felt they could, on average, earn a grade of 82. This section of this chapter now turns to gender difference.

**Sub-question 1: What is the relation between the use of multiple representations of the FTC in the classroom and female students understanding of the FTC?**

Female students who were not exposed to multiple representations in a meaningful and participatory fashion had a weaker understanding of the FTC on those representations and overall, were more likely to ignore areas of conflict, were not able to successfully transfer among representations, and were generally confined to the representations presented in class. These gender specific trends are associated to differences in the enacted curriculum at the three sites.
There were several factors missing in the CSNE curricular enactment. This course contained essentially no student participation in classroom discourse. In addition, the curriculum included few graphical representations and no contextual ones. The lecture-style delivery and the absence of several representations from the course, may have disproportionately affected female students.

Female Students at CSNE performed on average lower than students at the other two sites across groups in all categories except in the verbal domain, as indicated by boxplots of student scores at the three sites (see Figure 26.) Although ANOVA resulted in no significant difference in Total Score (TS) across sites and between males and females, with results close to significance, gender interacted with the classroom, or teaching format (see Figure 27): Female students at CSNE performed significantly lower than female students at the other two sites, despite starting with the same baseline in prior knowledge.

**Comparisons of Female Groups at the Three Sites**

On the graphical problem, female students at CSNE performed on the average 16 out of 100 points lower than female students at HCCC and 25 out of 100 points lower than female students at RCC. On the contextual problem, they scored 24.50 points lower than female students at RCC and 48.00 points lower than female students at HCCC, and on the symbolical problem, 22.25 points lower than female students at RCC and 35.08 points lower than female students at HCCC as can be seen in Table 43 below and in
Figure 26 from Chapter 5. The standard deviation and other descriptive statistics were already reported in Chapter 5. Table 43 also shows HCCC female achievement on the contextual representation was on the average highest of all three groups of female
students. Professor Brown from HCCC had his class work in groups with contextual word problems extensively.

Table 43. Mean Scores for Female Students at the Three Sites

<table>
<thead>
<tr>
<th>Mean Score</th>
<th>Female-RCC</th>
<th>Female-HCCC</th>
<th>Female-CSNE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total Score (TS)</td>
<td>51.10</td>
<td>57.70</td>
<td>35.25</td>
</tr>
<tr>
<td>Graphical Score (GS)</td>
<td>58.68</td>
<td>50.00</td>
<td>33.42</td>
</tr>
<tr>
<td>Numerical Score (NS)</td>
<td>53.00</td>
<td>47.17</td>
<td>45.75</td>
</tr>
<tr>
<td>Verbal Score (VS)</td>
<td>55.81</td>
<td>65.00</td>
<td>53.83</td>
</tr>
<tr>
<td>Contextual Score (CS)</td>
<td>45.50</td>
<td>69.00</td>
<td>21.00</td>
</tr>
<tr>
<td>Symbolical Score (SS)</td>
<td>42.50</td>
<td>57.33</td>
<td>22.25</td>
</tr>
</tbody>
</table>

To reap the benefits of a fully participatory classroom, mathematics needs to be accessible to all learners (Rosser, 1993). Boaler made the case for teaching mathematics in context, so skills would appear naturally (Boaler, 1994; Boaler & Staples, 2008). Gender differences in the present study support results found by others that indicate female students may prefer collaboration and reform, active pedagogies to lecture (Boaler, 1997; Joiner, Malone & Haimes, 2002; Rasmussen & Ellis, 2013), and point to female students’ lack in achievement being related to an educational tradition that values work in isolation and encourages procedural learning over, or without, a connected, conceptual, contextualized understanding first. Current results by Laursen et. al, 2013, agree with this interpretation.

Relating this to the classroom experience, CSNE students experienced very few graphical problems in their classroom discourse, and these representations were not part of the student–student or teacher–student classroom practice, but just part of the lecture presentation. For these reasons, the LOP rating for the CSNE enacted graphical representation was a 3 out of 7, compared to 5 at HCCC and 6 at RCC. Both female
students at CSNE interviewed for the Think-Aloud (Allison and Ann) chose the graphical problem because it looked friendlier based on their high school experience. Yet in solving this problem, they demonstrated a lack of understanding of how the graphical problem related to the FTC. Their problem-solving trajectory began with a transfer to the symbolical representation with which they had more experience in the FTC curriculum, and they derived none of their information from the graphical statement of the problem, except initially. Although both students indicated a strong preference for graphs in the Background Questionnaire, and this preference, along with documented high-school experience, may have played a role in their choice, the CSNE curriculum had not accommodated their graphical needs. Their language in the solving process referred to mathematics as formulas that needed to be remembered instead of sense making.

**Comparison of Female and Male Students within Site**

Comparison of scores for Female students from CSNE had a mean score of 17 points out of 100 below that of their male counterparts on the overall test, as reported in Table 27 in Chapter 5 and in Table 44 below. Female students at CSNE also scored on average 15 points lower on the graphical problem, 26 points lower on the symbolical problem, and 27 points lower on the contextual problem than male students in the same course (see Table 27 and Table 44). These results are also visible in the boxplots portrayed in Figure 26. Male and female students at RCC had comparable achievements across all representations, and at HCCC female students had a higher mean than their male counterparts on the contextual and symbolical domains with mean differences of approximately 14 and 15 points respectively, but their mean scores were lower on the
graphical and numerical domains. Their Total Scores, however, were approximately equal to the Total Score of male students at HCCC.

Table 44. Mean Scores for Students at the Three Sites.

<table>
<thead>
<tr>
<th>Mean Score</th>
<th>Female- RCC</th>
<th>Male- RCC</th>
<th>Female- HCCC</th>
<th>Male- HCCC</th>
<th>Female- CSNE</th>
<th>Male- CSNE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total Score (TS)</td>
<td>51.10</td>
<td>49.06</td>
<td>57.70</td>
<td>55.73</td>
<td>35.25</td>
<td>52.60</td>
</tr>
<tr>
<td>Graphical Score (GS)</td>
<td>58.68</td>
<td>58.81</td>
<td>50.00</td>
<td>62.38</td>
<td>33.42</td>
<td>48.40</td>
</tr>
<tr>
<td>Numerical Score (NS)</td>
<td>53.00</td>
<td>55.69</td>
<td>47.17</td>
<td>56.25</td>
<td>45.75</td>
<td>68.00</td>
</tr>
<tr>
<td>Verbal Score (VS)</td>
<td>55.81</td>
<td>46.75</td>
<td>65.00</td>
<td>50.13</td>
<td>53.83</td>
<td>51.10</td>
</tr>
<tr>
<td>Contextual Score (CS)</td>
<td>45.50</td>
<td>42.94</td>
<td>69.00</td>
<td>66.75</td>
<td>21.00</td>
<td>38.10</td>
</tr>
<tr>
<td>Symbolical Score (SS)</td>
<td>42.50</td>
<td>41.13</td>
<td>57.33</td>
<td>43.13</td>
<td>22.25</td>
<td>54.40</td>
</tr>
</tbody>
</table>

Boaler argues that female lack of performance may be more linked to pedagogy because women “won’t accept a system that promotes rote learning” (1997), and the findings of the study agree with this research. Rasmussen and Ellis (2013) also found that female students who dropped out of STEM because of their calculus course reported that they were less likely to contribute to class discussion, and their instructors were less likely to engage them in the lesson. The study by Laursen et al. of over 100 courses and at multiple institutions suggests that inquiry-based learning (IBL) and active curricula may be particularly beneficial to female students both in terms of cognitive gains and affective gains (Laursen, Hassi, Kogan & Weston, 2014). The current study did not include attitudinal questions about the female students’ perception of their level of engagement in class, but it does suggest that female students’ learning may be more closely related to the whole classroom dynamic, including to the use of multiple representations in the classroom, and to the problem-solving strategies and to
opportunities to learn that occur in the enacted curriculum. The CSNE story suggests that not only do female students lacking the opportunity to engage meaningfully with the enacted curriculum learn less than their male counterparts, as hypothesized originally in the Conceptual Framework diagram presented in Chapter 3. The opposite appears to also to be supported. When, for example, the curriculum supports multiple representations in a meaningful way, female students have a deeper and more complete understanding. For example, the HCCC curriculum supported and modeled collaborative work and high use of contextual representations. Female students at HCCC performed slightly better than male students on that representation, as suggested by the disaggregated boxplots in Figure 26 and in Table 44. Female students at HCCC also performed slightly higher than male students on the symbolical representation, but surprisingly not on the graphical representation, where the scored lower. The class size at HCCC was the smallest, with 8 male students and only 6 female students, and this may account for the inconsistencies. Also, as has been documented in Chapter 4, the Enacted Curriculum at HCCC, relied both heavily on group work and on tasks that were categorized as having the highest cognitive demand, but on many occasions classes worked with concepts at a higher pace than at RCC and at CSNE and did not often include time for closure or wrap-up.

Of three sites, the RCC student performance for men and female students is the closest on all domains. Of the three sites, the RCC curriculum was the most scaffolded, with the teacher continuously asking students to justify their reasons and make meaning of the concepts they were learning.
Comparison of Male and Female Differences Across Sites

Across sites, female students performed lower than male students on the numerical representation (p=0.049) as reported in Table 13. Categories and codes in enacted curriculum. Research on the enacted curriculum suggested that numerical representations were present to a lesser extent in the RCC and HCCC curricula. The CSNE curriculum included teacher use of this representation, but was lacking in student engagement in the problem-solving process as part of the classroom discourse. All three curricula were rated 3 out of 7 on the LOP document. The results support suggestions that female students would more closely align themselves with the enacted representations in the curriculum. Although male and female students had been exposed to the same curricula and had similar prior knowledge of Calculus, male students outperformed female students, and this was a possible consequence of the curriculum.

Sub-question 2: What is the relation between the use of multiple representations in the classroom and female students’ use of multiple representations?

Female student use of multiple representations was analyzed qualitatively from the Think-Alouds, Background Questionnaire and FTC assessment written work. The multiple representations used by female students during the Think-Aloud, was closely related to the curricular experience. While CSNE female students, like students at the other two locations, chose graphical representations because these seemed easier based on their high school experience, their solution was almost entirely symbolical and their use of the symbolical representation was limited. Since the enacted curriculum at CSNE had exposed them to analytical representations more frequently than other
representations, this indicates a possible connection between their classroom experience and their problem-solving strategies. The Background Questionnaire and Post Interviews confirmed that experience had played a factor in their reasoning. During the Think-Alouds, the two students tried to reproduce what their teacher had presented. Ann, for example, remarked she “can’t remember what he (her teacher) said,” when trying to figure out the regions of increase or decrease of the antiderivative graph. Their general problem-solving trajectory involves symbolical representations, and shows weakness in both conceptual and procedural fluency, a pattern corroborated in their written work on the FTC assessments. Comparing these students’ symbolical and procedural work with the enacted curriculum at CSNE shows a general agreement with the symbolical and procedural approach to the material. When a student could not recall the prescribed algorithm, they were not able to continue in the problem-solving process by relying upon reasoning or other representational knowledge.

In contrast to the students at CSNE, both female students at HCCC chose contextual representations, and their Think-Alouds showed them to use all the five representations in communicating about the FTC. Their Think-Alouds suggested a great deal of mathematical experience, a richness of thinking and an ability to think across representations that was not present in students from the other sites, and that they had a different attitude about mathematics, viewing it as a sense making activity.

Although Dawn and Deliana both chose the same problem to solve, their pathways were quite different. Deliana converted the contextual problem to a symbolic domain and used graphical and numerical means as secondary tool in her analysis of the problem. Dawn, of the other hand, relied heavily on both graphical and symbolic
reasoning to solve the problem. Both these students, like Dave, in the same class, can be classified as harmonic thinkers in the sense of Kruteskii et al (1976). They were versatile thinkers who moved seamlessly through representations, supporting their work with verbal representations, justifications and thorough analysis, much like experts do (Tall & Gray, 2001). They could reason abstractly construct viable arguments, and to strategically use all the representations discussed in this work. They could self-critique and to attend to precision in their language and problem solving. When reaching areas of conflict, these students were able to backtrack and resolve the conflicts. The richness in tasks and the flexibility in enactment of the HCCC curriculum, in which students had ample opportunity to connect across big ideas around the FTC, was an effective teaching strategy for these women, who were ready to undertake complex tasks, and did not expect easy recipes.

Female students at RCC chose two different problems to solve. Christine chose the verbal representation, and Emily the symbolical one. Their problem solving involved a variety of representations, and although not as varied as the representations chosen by HCCC students, their representations were generally more complex than that of students at CSNE, in alignment with a curriculum where students had opportunity to engage with mathematics meaningfully, and in which each student was encouraged to participate. The two female students showed a more rigid and incomplete understanding of the FTC than female students at HCCC. One possible explanation may be that although the classroom discourse encouraged participation, the interaction was heavily teacher directed, and students in this class did not have the opportunity to work on developing those habits of mind that are so important in developing a profound understanding of mathematical
ideas, in particular the ability and the confidence to use different strategies in grappling with problems, and the disposition to dig deeper in trying to understand ideas or solve mathematical tasks.

The patterns show that students in the two courses that used more representations and that engaged student in the class discourse had a richer and deeper approach to their solution.

**Sub-question 3: What is the role of other factors, cognitive preference, and perceived representational instruction, and accommodated preference in female student understanding of the FTC?**

Gallagher (1998) suggested that female students tend to be more conservative in their strategies in solving mathematical problems and tend to adhere more than male students to the models learned when approaching problems. The lack of models for various representations may consequently cause gaps in achievement as noted in the CSNE curriculum, but the presence of rich tasks in a supportive environment did the opposite for female students at HCCC. One other connection between the enacted curriculum and female student understanding of the FTC comes from their choice of problem to solve.

**Effect of Enacted Curriculum on Female Students’ Choices**

Allison and Anne from CSNE choose a graphical problem based on the fact that is seemed easier or friendlier based on their high school (not college) experience. Emily and Christine from RCC also stated their choice in terms of ease or avoidance of other problems. Table 45 re-iterates some results Chapter 5, just for female students.
Table 45. Female student connection with the enacted curriculum.

<table>
<thead>
<tr>
<th>Site</th>
<th>RCC</th>
<th>HCCC</th>
<th>CSNE</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Enacted Curriculum</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Whole Group Discussion</td>
<td>IRE</td>
<td>Discussion</td>
<td>Lecture</td>
</tr>
<tr>
<td>• Verbal</td>
<td>Whole and small groups</td>
<td>• Context - lots</td>
<td>No discussion</td>
</tr>
<tr>
<td>• Symbolic</td>
<td>• Graphical</td>
<td>• Verbal</td>
<td>No small group</td>
</tr>
<tr>
<td>• Graphical</td>
<td>• Symbolic</td>
<td></td>
<td>• Symbolic</td>
</tr>
<tr>
<td>• Some context</td>
<td></td>
<td></td>
<td>• Numerical</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>• Little Graphical</td>
</tr>
<tr>
<td><strong>Assessment Choice</strong></td>
<td>1 Contextual</td>
<td>3 Contextual</td>
<td>2 Graphical</td>
</tr>
<tr>
<td>1 Verbal, 1 Symbolical</td>
<td></td>
<td></td>
<td>1 Symbolical</td>
</tr>
<tr>
<td><strong>Representations in Solution Trajectory</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Christine (Female)</td>
<td>Problem C-Verbal</td>
<td>Deliana (Female)</td>
<td>Allison (Female)</td>
</tr>
<tr>
<td>“I know linear functions”; I was avoiding discontinuous functions”</td>
<td></td>
<td></td>
<td>“It seems easiest to connect to. “</td>
</tr>
<tr>
<td><strong>Solution:</strong></td>
<td></td>
<td><strong>Solution:</strong></td>
<td><strong>Solution:</strong></td>
</tr>
<tr>
<td>• Verbal</td>
<td></td>
<td>• Graphical</td>
<td>• Graphical</td>
</tr>
<tr>
<td>• Symbolical</td>
<td></td>
<td>• Verbal</td>
<td>• Symbolic</td>
</tr>
<tr>
<td>• Graphical</td>
<td></td>
<td>• Contextual</td>
<td>• Numerical</td>
</tr>
<tr>
<td>---------------</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ignores the absolute value</td>
<td></td>
<td>Uses MRs with ease</td>
<td>Confuses first and second derivative test.</td>
</tr>
<tr>
<td>Leaves the antiderivative in integral form.</td>
<td></td>
<td></td>
<td>Weak symbolic</td>
</tr>
<tr>
<td>Gives both a verbal and a symbolic statement of the FTC.</td>
<td></td>
<td></td>
<td>manipulations.</td>
</tr>
<tr>
<td>Emily (Female)</td>
<td>Problem E-Symbolic</td>
<td>Dawn (Female) Problem</td>
<td>Anne (Female) Problem</td>
</tr>
<tr>
<td>“It seems easiest to connect to. “</td>
<td></td>
<td>D-Context</td>
<td>A – Graphical</td>
</tr>
<tr>
<td><strong>Solution:</strong></td>
<td></td>
<td><strong>Solution:</strong></td>
<td></td>
</tr>
<tr>
<td>• Graphical</td>
<td></td>
<td>• Graphical</td>
<td>• Symbolic</td>
</tr>
<tr>
<td>• Symbolic</td>
<td></td>
<td>• Contextual</td>
<td>• Graphical</td>
</tr>
<tr>
<td>• Numerical</td>
<td></td>
<td>• Symbolic</td>
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<td></td>
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<td>• Verbal</td>
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<td></td>
<td></td>
<td>• Numerical</td>
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<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>Ignores absolute value. Calculates the antiderivative symbolically and then graphs the result.</td>
<td></td>
<td>Takes into account the absolute value graphically and answers the problem correctly from the beginning.</td>
<td>Ignores the absolute value</td>
</tr>
<tr>
<td>“FTC says something about integrals like $\int_{a}^{b} f(x) , dx = F(b) - F(a)$” but not sure what it means. It is how you do integrals.</td>
<td></td>
<td></td>
<td>Confuses concepts such as increasing and decreasing.</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Has difficulty computing integrals.</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Does not remember the FTC.</td>
</tr>
</tbody>
</table>
By contrast, both Dawn and Deliana from HCCC speak to their choice as a function of their liking the problem or the challenge. These students have been served well by the course enactment, as it has nurtured a desire to understand, to tackle difficult problems and to be confident in their ability to succeed.

**Effect of Female Cognitive Preference**

Female students’ cognitive representational preference, gender and students’ accommodated needs all play a role in the way students use and understand the FTC.

Graphical cognitive preference is significant in female student achievement as measured by the FTC Assessment, while for male students’ symbolical cognitive preference is significant. The regression models described in Chapter 5 accounted for 62% (for females) and 42% (for male) of the variability in the Total Score. This indicates student preferences need to be considered when implementing curriculum. The CSNE curriculum was particularly devoid of graphical representations and this may have consequently affected female students more than the male students in that course, as their FTC assessment showed them to perform significantly lower on that representation.

A representation of Total Score and average cognitive preference that displays student gender and illustrates how the relative positive relation of cognitive preference and achievement holds across site and gender as displayed in Figure 63. Teaching to the cognitive preference of students positively affects student understanding for male and female students alike. In this graph, all groups of students with an average cognitive preference score of 4 scored approximately 60 points or more or 200% higher than those with an average score of 2, who averaged in the low 20’s. This relation was consistent across gender and site. Cognitive preference is self-reported here, and it is part of a
complex variable that may measure student familiarity or experience with representations over the course of their entire schooling.

Figure 63. Total Score and cognitive preference by site and gender.

In summary, from written assignments on the FTC assessment and from the FTC assessment statistics, there was evidence of weaker understanding of the FTC for female students at CSNE including weak symbolic manipulation, and weak representational understanding. This agrees with their work in the *Think-Alouds*. By contrast female students at HCCC and RCC have a good procedural fluency and show greater understanding and variability in the use of multiple representations and in problem solving. Female students at HCCC where the curriculum was rich in contextual and
symbolical representations perform better on these parts of the FTC assessment than female students in other groups, but their scores across representations are more varied.

**Conclusions about Research Question 2**

From the Think-Alouds, CSNE students solve problems symbolically, give reasons for their choice in terms of avoidance and easiness. Female students at RCC choose the problem based on experience, use several representations in their solutions, and none choose contextual problems to solve. At HCCC, female students’ choice of problem is reflected in terms of their interest, desire for challenge, or liking. Solutions presented by female students at HCCC are the richest in both use of MR and development. They are most limited for female students at CSNE.

From the Classroom Portrait, the enacted curriculum at CSNE relied heavily on symbolical representations, followed by numerical ones, in a format that was highly traditional, with little opportunity for students to be makers of mathematics. Students were urged to “memorize how to do” things. The RCC curriculum used all representations except contextual ones and constantly engaged students to justify answers or seek understanding, although the presentation was IRE structured not allowing students to collaborate with each other. The HCCC curriculum used many representations and in particular contextual problems, and focused on collaborative learning and ill structured problem solving. Of all curricula, it was the least teacher-directed, had the highest cognitive demand tasks, and did not always include time for closure at the end of the lesson.

Given all these patterns, evidence suggests a strong connection between classroom practices and use of MR and female student understanding. In the Conceptual
Framework in Chapter 3, classroom discourse, along with multiple representations were presented as strong factors affecting female student understanding. Productive classroom practices include the use of collaborative learning, active participation by all students, rich cognitive tasks, and including time for wrap-up at the end of the lesson. When the curricular practices support the use of multiple representations by all students in the classroom discourse, there is a deeper understanding of the FTC, by all students, and in particular by female students, as seen in Dawn, Deliana, from HCCC, and to a lesser extent in Christine and Emily at RCC. There is also an excitement and a risk-taking attitude toward the challenges and joys that learning and figuring out mathematics brings. When the enacted curriculum is lacking in varied multiple representations and in classroom discourse, there is a shallow understanding of the FTC, avoidance of mathematical challenges, and a view of mathematics as procedural, particularly for female students like Anne and Allison.

**Limitations of the Study**

Other factors influence students’ understanding of the FTC, apart from teacher practices, student in-class activities, and assessments. These factors, which include homework, outside help, student life, etc., may be influencing variables in the results of the study, and to the extent to which this is possible, observations of these factors were used to corroborate, expand, or further explain results. Homework discussed during the lesson instruction, for example, was included in the field notes and used to expand the classroom portrait of enacted curriculum at each site. Some questions around the students’ prior mathematical background and outside class help were also included in the *Background Questionnaire*. 
One of the limitations of the study was the size of the groups, with the number of participants, who completed the FTC Assessment and the Background Questionnaire, ranging from 14–22 students per class. These numbers could not be controlled, since instructors had been invited to participate in the study early into the semester, but the FTC is a topic that appears toward the end of the course.

Another possible drawback was the rubric, designed for assessing the FTC assessments, included researcher anticipated responses based on her years of experience teaching this subject. Although the FTC assessment was tested with volunteers to ensure that the problems were clear and sufficient time was allotted, and the rubric design was tested with another researcher, the categories for potential responses selected in the rubric may have needed to be finer to capture nuances in student responses. Although this grading method was consistent across all tests, a subtler instrument may have help give a finer approximation for student understanding of the FTC.

In a future study, a more robust pre-assessment could be considered as a baseline for student prior knowledge. In the current study, the baseline was the students’ prior background in Calculus, and a test of equality of proportions was performed to conclude that the students’ background in the three courses was consistent.

Finally, cognitive preference and perceived representational instruction were on a Likert Scale of 1 to 4. Possibly a finer grade scale may have allowed for a wider spread of responses. The cognitive preference was self-reported by students, but no other instrument was used to measure student preference.
Discussion

The research questions addressed the nature of the relationship between students’ use of multiple representations in the enacted curriculum and student understanding of the FTC. The results of the study indicate that not only are multiple representations crucial to student understanding of the FTC, as reported in the literature by researchers such as Tall, Kaput, Presmeg and Sfard (Kaput, 1991; Sfard, 1991; Tall, 2004; Presmeg, 2006) asserted, but that classroom involvement is equally important. The result of this study augments the MAA findings that illustrate active learning is one of the characteristics of successful Calculus programs in terms of student retention and successful completion of Calculus I, by adding a student understanding dimension to the MAA results. The work presented here posited a connection between students’ understanding of Calculus concepts and the enacted curriculum. The study suggested that students in active curricula develop a better understanding of the FTC, develop more strategies for solving unusual problems, and are better equipped to deal with areas of conflict while problem solving. In addition, students’ engagement with contextual representations helps them generate more multiple representations while problem solving, and more than students who are exposed to other types of representations.

The results of the study indicate that not only are multiple representations crucial to student understanding of the FTC, but that classroom discourse is equally important. Namely, it suggests there is a relationship between the use of multiple representations in the classroom and student understanding of the FTC, as originally hypothesized. But it further suggests that this relationship is mediated by, and dependent on, student participation in the classroom discourse and activities. The results are consistent with and
add to the MAA study that identifies active learning as one of the crucial factors contributing to student satisfaction and retention in the mathematics classroom, by correlating the student engagement with student understanding.

This work also supports work of Boaler & Staples (2008), Gallagher (1998), and Joiner, Malone & Haimes (2002), by identifying that traditional teaching may be ineffective for female students, and points to collaborative learning as being better suited to their learning styles. The study adds to the existing body of work by identifying the possible role of cognitive preference on student achievement, which may be different for female and male students.

**Implications and New Directions**

Multiple representations are an essential component of a complete understanding of mathematical concepts, including the FTC.

Results of the study make the case for the enacted Calculus curriculum to include ample opportunities for students to engage with the mathematical discourse around multiple representations of the FTC in an active and meaningful way. Students who are exposed in their classroom experience to multiple representations of Calculus concepts, such as the FTC, in a variety of ways develop a more complete understanding of these concepts. They are more able to make sense of problems and persevere in solving them, reason abstractly and quantitatively and construct mathematical arguments. Experience with contextual representations seems to be particularly useful, since students have the freedom to choose a representation of their choice for their solution, and in the process, they develop more critical thinking and problem-solving skills, as was the case in the HCCC curriculum.
In this study many students used analytical reasoning as the predominant solution method, regardless of their cognitive preference and choice of problem to solve. For many students, this was their problem-solving trajectory, regardless of their self-reported cognitive preference or choice of problem to solve. Most students’ problem-solving trajectory involves transfer from the given representation to the symbolical one and analysis in that symbolical domain. These results corroborate results of Haciomeroglu, Aspinwall, and Presmeg (2010) and Tall (2004).

Female students’ learning or complete understanding of the FTC in its multiple representations formats may be hindered in classes where the curricular experience does not support explicit engagement with certain representations.

Although all classes relied heavily on symbolical representations, students in all of classes scored lowest on this representation. A possible explanation is that the symbolical representation is of a higher cognitive demand, and students may need more time and more explicit instruction to deconstruct its meaning. Students in this study were not generally able to make connections between this representation and others and many students concentrated only on procedural use, showed incorrect notations, and lacked understanding.

Female student’s use of multiple representations of calculus concepts aligns itself more with the types of representation used in the classroom, and lack of models of how to work with representations in the enacted curriculum may affect female student achievement or understanding more than male students contributing to an achievement gap as suggested by Gallagher (1998).
Results of this study may be potentially useful to curriculum developers. The study suggests that purposeful development within and between representations needs to be encouraged and maintained throughout the Calculus curriculum. Not only are these representations essential to creating an understanding of the mathematics (Kaput, 1991; NCTM, 2000), but the active modeling is especially necessary to female student success in the curriculum. Furthermore, the study suggests that collaboration, and classroom discourse may be key components to providing access to a deeper understanding of the FTC, especially for female students. In a field that is still male dominated and where discourse can be confrontational (Kirkman, Maxwell & Rosse, 2004), collaboration can afford all students, including female students with opportunities for learners to do mathematics, i.e., to have a common discourse, and to evaluate and re-evaluate their personal ideas about the FTC, just as the HCCC curriculum had demonstrated, through its careful attention to group work. This study did not evaluate how the group functioned and whether all or not all the students in a particular group learned.

A new dimension emerging from this study is the role of worthwhile tasks involving representations or of tasks of higher cognitive demand to motivate student learning and to promote the development of mathematical habits of mind as their understanding of the FTC is evolving. The role of cognitive preference and of accommodated student needs and how these concepts relate to student understanding was also discussed.

The study identified some possible differences the effect of various cognitive preferences between female students and male students on student understanding, with
female students’ scores being impacted mostly by their graphical cognitive preference, and male students’ achievement by the symbolical preference.

The study also suggests that the use of contextual representations in a collaborative setting may be an effective means to help all students generate and develop a multi representational understanding of the FTC. Finally, the study also suggests that symbolical representations, although used predominantly, are still something students struggle with, and that they need to be more thoughtfully deconstructed for students for them to create meaning.

**Recommendations**

In conclusion, this study urges colleges and instructors alike to be purposeful about instruction, particularly by including multiple representations in the curriculum and using collaborative learning practices centered around meaningful tasks. There is more work to be done to better understand the relation between multiple representations and the enacted Calculus curriculum, and student understanding and gender. This would include studying the impact on student understanding and achievement of cognitive representational preferences, on the varying cognitive demands of different representational tasks, on productive classroom discourse involving multiple representations, and on the need to accommodate the different representational needs of students. Although this work is devoted to the Fundamental Theorem of Calculus, it would be interesting to see how results extend to other areas of mathematics.
APPENDIX A

MODIFIED SAMPI (LOP)

Code Number: __________
SAMPI—Western Michigan University (Modified for NOYCE/UMASS NOV. 2010; Modified by Vasu, 2015)

HIGHER ED CALCULUS LESSON OBSERVATION DEBRIEFING FORM--Version B (Snapshot)

Complete this form using the observer's notes and information from the pre- and post-observation interviews. Use the “Definitions of Indicators” tool as a reference. Complete as soon after the observation session as possible.

DATE OF OBSERVATION ________________________ OBSERVER ______________

TIME OF OBSERVATION: Start ____ End ____ No. of Students ___

INFORMATION ABOUT THE LESSON AND CLASSROOM

1. Institution: □ 2-year college □ 4-year college
   □ Other. Please describe: ________________________________

2. In a few sentences, describe the lesson observed—objectives, primary activities, where this lesson fits in the overall unit of study.
   Description:

3. Indicate MAJOR ways that student activities were conducted over the entire course of the lesson.
   □ Whole group activity □ Small group activity □ Pairs of students □ As individuals

4. Rate the arrangement of the room relative to how well it facilitates student interactions.

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</table>
   Inhibits interactions Facilitates interactions

5. Indicate the primary general purpose(s)—not specific objectives—of this lesson based on the pre- and post-observation interviews and what's learned during the observation.

   □ Identify prior student knowledge □ Show how a concept applies in the real-world
   □ Introduce new concepts □ Develop appreciation for the core ideas of the
   □ Develop understanding of concepts subject matter of the lesson
   □ Review concepts □ Develop awareness of contributions of experts in the
   □ Learn processes/skills related to the subject subject matter from diverse backgrounds
   □ Learn vocabulary/specific facts □ Other. Describe: ________________________________

6. Briefly describe the instructional materials used in the lesson (e.g., textbooks, modules, kits, software, web-based materials, equipment/supplies, audio-visuals). Give specific names/publishers of materials being used.
KEY ELEMENTS OF THE LESSON

In this section, rate each of the indicators or answer the questions in four areas: planning/organization, implementation, content, and classroom culture. Note that any single lesson may not provide enough evidence for every indicator or question. In that case, check the DON'T KNOW box (but only as a last resort). Note any other indicators you consider important to the lesson. Use the "Definitions of the Indicators" tool for clarification.

## PLANNING/ORGANIZATION OF THE LESSON

1. Does the lesson come directly from a pre-packaged program (i.e., kit, text series, modules, web-based program) with very few teacher modifications?  
   - Yes
   - No
   - Don't Know
   
   If yes, name of program and specific lesson.

2. Rate the adequacy of classroom resources (supplies, equipment) to support the lesson.  
   
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<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Few resources</td>
<td>Many resources</td>
<td></td>
<td></td>
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<td></td>
</tr>
</tbody>
</table>
   
   Supporting evidence for rating:

3. Was the lesson organized to provide substantive teacher-student interactions?  
   - Yes
   - No
   - Don't Know
   
   If yes, what is the evidence?

4. Was the lesson organized to provide substantive student-student interactions?  
   - Yes
   - No
   - Don't Know
   
   If yes, what is the evidence?

5. Were investigative tasks essential elements of the lesson plan (e.g., manipulation of information to help make sense of content, elements of problem-solving situations, connection to real-world experiences)?  
   - Yes
   - No
   - Don't Know
   
   If yes, what is the evidence?

6. Was the lesson organized so that it appropriately addressed students' experiences, developmental levels, preparedness, and/or learning styles regardless of gender?  
   - Yes
   - No
   - Don't Know
   
   If yes, what is the evidence?
Planning/Organization Continued…

7. Was the lesson organized so that it appropriately addressed issues of access, equity, and/or diversity?

<table>
<thead>
<tr>
<th>Yes</th>
<th>No</th>
<th>Don't Know</th>
</tr>
</thead>
</table>

If yes, what is the evidence?

8. Did the lesson incorporate student and/or teacher use of technology (i.e., computers, video/digital cameras, monitoring equipment, calculators)?

<table>
<thead>
<tr>
<th>Yes</th>
<th>No</th>
<th>Don't Know</th>
</tr>
</thead>
</table>

Note: If incorporation of technology was a major part of the lesson, complete the TECHNOLOGY TO SUPPORT THE LESSON section on PAGE 12 of this form.

9. Other comments about lesson planning/organization or other indicators of importance.
IMPLEMENTATION OF THE LESSON

1. The students appeared confident of their understanding of the lesson.

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<tbody>
<tr>
<td>Limited confidence</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Great confidence</td>
</tr>
</tbody>
</table>

Supporting evidence for rating: 

Don’t Know

2. Periods of teacher-student interaction were probing and substantive (questioning and dialog emphasized higher-order thinking and deep understanding and exposed students' prior knowledge).

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<th>4</th>
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</tr>
</thead>
<tbody>
<tr>
<td>Weak student-teacher interaction</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Strong student-teacher interaction</td>
</tr>
</tbody>
</table>

Supporting evidence for rating: 

Don’t Know

3. Classroom management was effective in engaging all students in the lesson.

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<tr>
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</thead>
<tbody>
<tr>
<td>Limited effectiveness</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Very effective</td>
</tr>
</tbody>
</table>

Supporting evidence for rating: 

Don’t Know

4. The pace of the lesson was appropriate for the developmental levels of the students.

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</tr>
</thead>
<tbody>
<tr>
<td>Poorly paced</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Well paced</td>
</tr>
</tbody>
</table>

Supporting evidence for rating: 

Don’t Know

5. Periods of student-student interaction were focused on pertinent lesson content and enhanced individual understanding of it.

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<thead>
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<th>4</th>
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</thead>
<tbody>
<tr>
<td>Interaction not productive</td>
<td></td>
<td></td>
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<td></td>
<td>Interaction very productive</td>
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</tbody>
</table>

Supporting evidence for rating: 

Don’t Know

6. The lesson was organized so there was adequate time for students and/or the teacher to reflect on the lesson and its content.

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<tbody>
<tr>
<td>Little or no time devoted to reflection</td>
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<td></td>
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<td>Considerable time devoted to reflection</td>
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</table>

Supporting evidence for rating: 

Don’t Know
7. The lesson was organized so there was adequate time for wrap-up and closure of the lesson.

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<tbody>
<tr>
<td></td>
<td>Little or no time devoted to closure</td>
<td>Considerable time devoted to closure</td>
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</table>

Supporting evidence for rating: Don’t Know

8. Teacher makes connections between the content and the students’ culture, community and families.

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<tbody>
<tr>
<td></td>
<td>Little or no connections to students’ culture</td>
<td>Strong connections to students’ culture</td>
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Supporting evidence for rating: Don’t Know

9. The teacher communicates high expectations for all students, challenging all students to engage in problem solving, question and the generation of knowledge.

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<tbody>
<tr>
<td></td>
<td>Expectation not sufficiently high</td>
<td>High expectations for all students</td>
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</table>

Supporting evidence for rating: Don’t Know

10. Female students were engaged in sense-making of this lesson.

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<tbody>
<tr>
<td></td>
<td>Low level of engagement</td>
<td>High level of engagement</td>
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Supporting evidence for rating: Don’t Know

11. Teacher is sensitive to issues of gender when facilitating this lesson.

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<tbody>
<tr>
<td></td>
<td>Limited sensitivity</td>
<td>Ample sensitivity</td>
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Supporting evidence for rating: Don’t Know
Implementation Continued…

12. Students regardless of gender were given equal attention.

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<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
</tr>
<tr>
<td>Few students given attention</td>
<td>All students given attention</td>
<td></td>
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</tr>
</tbody>
</table>

Supporting evidence for rating:

Don’t Know

13. Other comments about lesson implementation or other indicators of importance.

OVERALL RATING FOR IMPLEMENTATION OF THE LESSON

The overall rating represents the observer’s best summary judgment of the appropriateness and quality of the lesson IMPLEMENTATION. Overall ratings are not necessarily intended to be the numerical average of the ratings of the indicators for Implementation of the Lesson. There may be other factors that influence an overall rating.

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<td>6</td>
<td>7</td>
</tr>
<tr>
<td>Implementation of the lesson not at all consistent with best practice in standards-based inquiry-oriented teaching and learning</td>
<td>Implementation of the lesson very consistent with best practice in standards-based inquiry-oriented teaching and learning</td>
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</table>
### CONTENT OF THE LESSON

1. The content of the lesson was important and worthwhile.

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<th>7</th>
</tr>
</thead>
</table>
   Trivial content | Important content |

   Supporting evidence for rating:

   - [ ] Don’t Know

2. Students were intellectually engaged with important ideas related to the focus of the lesson.

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<tr>
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<th>6</th>
<th>7</th>
</tr>
</thead>
</table>
   Limited engagement | Significant engagement |

   Supporting evidence for rating:

   - [ ] Don’t Know

3. The subject matter was portrayed as a dynamic body of knowledge enriched by conjecture, investigation, analysis, and/or proof/justification.

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<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
</table>
   Limited portrayal | Strong portrayal |

   Supporting evidence for rating:

   - [ ] Don’t Know

4. The students showed an understanding of the concepts and content that were the focus of the lesson and the topical/conceptual area being addressed by the lesson.

<table>
<thead>
<tr>
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<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
</table>
   Limited understanding | Strong understanding |

   Supporting evidence for rating:

   - [ ] Don’t Know

5. The lesson included connections between concepts/content in this lesson and/or previous or future lessons in the overall unit or topic being addressed.

<table>
<thead>
<tr>
<th>1</th>
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<th>6</th>
<th>7</th>
</tr>
</thead>
</table>
   Weak connections | Strong connections |

   Supporting evidence for rating:

   - [ ] Don’t Know

6. The lesson included connections between this lesson and/or other areas of the same subject and/or other subjects.

<table>
<thead>
<tr>
<th>1</th>
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<th>7</th>
</tr>
</thead>
</table>
   Limited connections | Strong connections |

   Supporting evidence for rating:

   - [ ] Don’t Know
Content Continued…

7. The lesson incorporated applications of the content/concepts of the lesson to real-world situations.

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<th>4</th>
<th>5</th>
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<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Limited applications</td>
<td>Strong applications</td>
<td></td>
<td></td>
<td></td>
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</tr>
</tbody>
</table>

Supporting evidence for rating:

Don’t Know

8. The lesson included abstractions (theories and models) as appropriate.

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<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Few abstractions</td>
<td>Many abstractions</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Supporting evidence for rating:

Don’t Know

9. The lesson included the following representations (check all that apply):

- Numerical
- Verbal
- Graphical
- Contextual
- Analytical/Formulas

10. The students responded positively to learning the concepts and the content of the lesson.

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<tr>
<th>1</th>
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<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Negative response</td>
<td>Positive response</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Supporting evidence for rating:

Don’t Know

11. Other comments about lesson content or other indicators of importance.

OVERALL RATING FOR CONTENT OF THE LESSON

The overall rating represents the observer’s best summary judgment of the appropriateness and quality of the lesson CONTENT. Overall ratings are not necessarily intended to be the numerical average of the ratings of the indicators for Content of the Lesson. There may be other factors that influence an overall rating.

<table>
<thead>
<tr>
<th>1</th>
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<th>4</th>
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<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Insignificant or trivial content addressed in the lesson</td>
<td>Significant content consistent with standards and benchmarks addressed in this lesson</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
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</tbody>
</table>
CLASSROOM CULTURE IN WHICH THE LESSON WAS CONDUCTED

1. Active participation of all students was encouraged and valued.

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<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Participation not encouraged/valued</td>
<td>Participation strongly encouraged/valued</td>
<td></td>
<td></td>
<td></td>
<td></td>
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</tr>
</tbody>
</table>

Supporting evidence for rating: Don’t Know

2. The teacher showed respect for and valued students’ ideas, questions, and/or contributions to the lesson regardless of gender.

<table>
<thead>
<tr>
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<th>5</th>
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<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Limited respect/value</td>
<td>Great respect/value</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Supporting evidence for rating: Don’t Know

3. Students showed respect for and valued each other’s ideas, questions, and/or contributions to the lesson.

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<thead>
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</tr>
</thead>
<tbody>
<tr>
<td>Limited respect/value</td>
<td>Great respect/value</td>
<td></td>
<td></td>
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</tbody>
</table>

Supporting evidence for rating: Don’t Know

4. The classroom climate for the lesson encouraged students to generate ideas, questions, conjectures, and/or propositions.

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</tr>
</thead>
<tbody>
<tr>
<td>Climate discouraged students</td>
<td>Climate encouraged students</td>
<td></td>
<td></td>
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</tr>
</tbody>
</table>

Supporting evidence for rating: Don’t Know

5. Student-student interactions reflected collaborative working relationships.

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<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Limited collaborative relationships</td>
<td>Strong collaborative relationships</td>
<td></td>
<td></td>
<td></td>
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</tr>
</tbody>
</table>

Supporting evidence for rating: Don’t Know
Classroom Culture Continued…

6. Teacher-female student interactions reflected collaborative working relationships.

<table>
<thead>
<tr>
<th>1</th>
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<th>5</th>
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<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Limited collaborative relationships</td>
<td>Strong collaborative relationships</td>
<td></td>
<td></td>
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</tr>
</tbody>
</table>

Supporting evidence for rating:  

Don’t Know

7. The teacher's language and behavior showed sensitivity to issues of gender, race/ethnicity, special needs, and/or socio-economic status.

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<tr>
<th>1</th>
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<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Little sensitivity</td>
<td>Strong sensitivity</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Supporting evidence for rating:  

Don’t Know

8. Teacher-student interactions reflect teacher knowledge of and appreciation for students’ lives outside of the classroom including knowledge of family, culture and the life of the community.

<table>
<thead>
<tr>
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<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Little knowledge and appreciation</td>
<td>Strong knowledge and appreciation</td>
<td></td>
<td></td>
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<td></td>
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</tr>
</tbody>
</table>

Supporting evidence for rating:  

Don’t Know

9. Female students asserted themselves with confidence.

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
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<th>4</th>
<th>5</th>
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<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Little confidence</td>
<td>Ample confidence</td>
<td></td>
<td></td>
<td></td>
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</tr>
</tbody>
</table>

Supporting evidence for rating:  

Don’t Know

10. All students have the opportunity to participate in the lesson regardless of gender.

<table>
<thead>
<tr>
<th>1</th>
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<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Limited female participation</td>
<td>Strong female participation</td>
<td></td>
<td></td>
<td></td>
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<td></td>
</tr>
</tbody>
</table>

Supporting evidence for rating:  

Don’t Know
Classroom Culture Continued…

11. Other comments about classroom culture or other indicators of importance.

OVERALL RATING FOR CLASSROOM CULTURE

The overall rating represents the observer’s best summary judgment of the appropriateness and quality of the CLASSROOM CULTURE. Overall ratings are not necessarily intended to be the numerical average of the ratings of the indicators for Classroom Culture. There may be other factors that influence an overall rating.

<table>
<thead>
<tr>
<th>1</th>
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<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Classroom culture not supportive of student learning</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Classroom culture very supportive of student learning</td>
</tr>
</tbody>
</table>
USE OF TECHNOLOGY TO SUPPORT THE LESSON

Complete this section when information and other electronic technology are used in a major way to support the lesson being observed.

1. List the major types(s) of technology hardware being used by the teacher and students to support the lesson.

   Teacher:

   Students:

2. List the major type(s) of software or programs being used to support this lesson (such as word processing, spreadsheets, mapping software, desktop publishing, PowerPoint, video production, dynamic geometry, dynamic statistics). Be as specific as possible about the software version being used.

3. Student technology use arrangement:

   Computers
   - Whole group activity (i.e., all students in lab setting).
   - Students per computer? ____
   - In groups of 2-4 at classroom computers.
   - Individual activity (single student working at computer or students taking turns)

   Other (video camera, video editor, Palms, etc.)
   - Whole group activity.
   - Small groups working together with equipment.
   - Individual activity (single student using equipment or students taking turns)

4. Indicate the primary intended purpose(s) for which technology was used.
   - Production: Students create a product (publication, web page, presentation, video, model, maps, etc.)
   - Presentation: Teacher and/or students present (PowerPoint, video, music, publication)
   - Communication: Students use Internet/email to communicate with peers, experts, and other audiences.
   - Internet Research: Use the Internet to gather information.
   - Original Research: Use monitoring or recording devices to gather data.
   - Data Compilation/Analysis: Use technology to organize and analyze data.
   - Visualization: Use graphing calculators or visualization software to see or manipulate relationships or objects.
   - Other. Describe: ____________________

5. If this lesson is part of a curriculum unit or series of lessons, is technology used to support other lessons in the unit or series? Yes No

6. In using the technology and/or accessing information through technology, were students limited to specific procedures or sources devised by the teacher or directed by the instructional materials? (Note: This may vary by grade or student skill level.)

   Yes | No | Don't Know
Use of Technology Continued…

7. Technology resources were adequate to support the lesson.

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<thead>
<tr>
<th>1</th>
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</tr>
</thead>
<tbody>
<tr>
<td>Inadequate resources</td>
<td>Many resources</td>
<td></td>
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</tbody>
</table>

Supporting evidence for rating:

Don’t Know

8. Technology use was effectively integrated into this lesson (not an “add-on” or novelty).

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</thead>
<tbody>
<tr>
<td>Poor integration</td>
<td>Very effective integration</td>
<td></td>
<td></td>
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</tbody>
</table>

Supporting evidence for rating:

Don’t Know

9. The use of technology enhanced student learning of the lesson’s core concepts/content.

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<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Did little to enhance learning</td>
<td>Strongly enhanced learning</td>
<td></td>
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</tbody>
</table>

Supporting evidence for rating:

Don’t Know

10. The use of technology supported real-world application of the lesson concepts/content.

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<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Did little to support</td>
<td>Strongly supported</td>
<td></td>
<td></td>
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</tr>
</tbody>
</table>

Supporting evidence for rating:

Don’t Know

11. Technology use enhanced the ability of students to collaborate with each other.

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<tr>
<th>1</th>
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<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Did little to enhance</td>
<td>Strongly enhanced</td>
<td></td>
<td></td>
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</tbody>
</table>

Supporting evidence for rating:

Don’t Know

12. Classroom management was effective in engaging female students in the use of the technology.

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</tr>
</thead>
<tbody>
<tr>
<td>Limited effectiveness</td>
<td>Very effective</td>
<td></td>
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</tbody>
</table>

Supporting evidence for rating:

Don’t Know
Use of Technology Continued…

13. The teacher shows skills and ability in using technology to support student learning (consider both technical skills and lesson design).

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</tr>
</thead>
<tbody>
<tr>
<td>Limited skills/ability</td>
<td></td>
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<td></td>
<td></td>
<td></td>
<td>Strong skills/ability</td>
</tr>
</tbody>
</table>

Supporting evidence for rating:

14. Other comments about use of technology or other indicators of importance.

**OVERALL RATING FOR USE OF TECHNOLOGY TO SUPPORT THE LESSON**

The overall rating represents the observer’s best summary judgment of the appropriateness and quality of the USE OF TECHNOLOGY TO SUPPORT THE LESSON. Overall ratings are not necessarily intended to be the numerical average of the ratings of the indicators for Use of Technology to Support the Lesson. There may be other factors that influence an overall rating.

<table>
<thead>
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<th>7</th>
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</thead>
<tbody>
<tr>
<td>Use of technology has little effect on teaching and learning in this lesson</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Use of technology greatly enhances teaching and learning in this lesson</td>
</tr>
</tbody>
</table>
CHARACTER OF MULTIPLE REPRESENTATIONS IN TO SUPPORT THE LESSON

**Numerical Representations.**

N1. The lesson included Numerical Representations

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<td>6</td>
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</tbody>
</table>

Superficial use of Numerical Representations | Significant use of Numerical representations

Supporting evidence for rating:

Don’t Know

N2. Students were encouraged to use numerical representations

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<td>6</td>
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</tbody>
</table>

Limited encouragement | Significant encouragement

Supporting evidence for rating:

Don’t Know

N3. Students use numerical representations to reason make conjectures, analyze or justify their solutions.

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<td>6</td>
</tr>
</tbody>
</table>

Limited use | Strong use

Supporting evidence for rating:

Don’t Know

**OVERALL RATING FOR NUMERICAL REPRESENTATIONS**

The overall rating represents the observer’s best summary judgment of the appropriateness and quality of the lesson’s GRAPHICAL REPRESENTATIONS. Overall ratings are not necessarily intended to be the numerical average of the ratings of the indicators for Implementation of the Lesson. There may be other factors that influence an overall rating.

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</tbody>
</table>

NR in the lesson not at all consistent with best practice in standards-based inquiry-oriented teaching and learning

NR in the lesson very consistent with best practice in standards-based inquiry-oriented teaching and learning
Graphical Representations.
G1. The lesson included Numerical Representations

<table>
<thead>
<tr>
<th>1</th>
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<th>7</th>
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</thead>
</table>
Superficial use of Numerical Representations | Significant use of Numerical representations

Supporting evidence for rating: [ ] Don’t Know

G2. Students were encouraged to use numerical representations

<table>
<thead>
<tr>
<th>1</th>
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</thead>
</table>
Limited encouragement | Significant encouragement

Supporting evidence for rating: [ ] Don’t Know

G3. Students use numerical representations to reason make conjectures, analyze or justify their solutions.

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<tr>
<th>1</th>
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<th>4</th>
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<th>7</th>
</tr>
</thead>
</table>
Limited use | Strong use

Supporting evidence for rating: [ ] Don’t Know

OVERALL RATING FOR GRAPHICAL REPRESENTATIONS IN THE LESSON
The overall rating represents the observer’s best summary judgment of the appropriateness and quality of the lesson’s GRAPHICAL REPRESENTATIONS. Overall ratings are not necessarily intended to be the numerical average of the ratings of the indicators for Implementation of the Lesson. There may be other factors that influence an overall rating.

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</thead>
</table>
GR in the lesson not at all consistent with best practice in standards-based inquiry-oriented teaching and learning | GR in the lesson very consistent with best practice in standards-based inquiry-oriented teaching and learning
**Verbal Representations.**

V1. The lesson included Verbal Representations

<table>
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<tr>
<th>1</th>
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<th>7</th>
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</thead>
</table>

Superficial use of Verbal Representations  Significant use of Verbal representations

Supporting evidence for rating:  

Don’t Know

V2. Students were encouraged to use Verbal representations

<table>
<thead>
<tr>
<th>1</th>
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<th>7</th>
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</thead>
</table>

Limited encouragement  Significant encouragement

Supporting evidence for rating:  

Don’t Know

V3. Students use Verbal representations to reason make conjectures, analyze or justify their solutions.

<table>
<thead>
<tr>
<th>1</th>
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<th>4</th>
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<th>7</th>
</tr>
</thead>
</table>

Limited use  Strong use

Supporting evidence for rating:  

Don’t Know

**OVERALL RATING FOR VERBAL REPRESENTATIONS IN THE LESSON**

The overall rating represents the observer’s best summary judgment of the appropriateness and quality of the lesson’s VERBAL REPRESENTATIONS, Overall ratings are not necessarily intended to be the numerical average of the ratings of the indicators for Implementation of the Lesson. There may be other factors that influence an overall rating.

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<thead>
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VR in the lesson not at all consistent with best practice in standards-based inquiry-oriented teaching and learning  

VR in the lesson very consistent with best practice in standards-based inquiry-oriented teaching and learning

303
Contextual Representations.

C1. The lesson included Contextual Representations

| 1 | 2 | 3 | 4 | 5 | 6 | 7 |

Superficial use of Contextual Representations

Significant use of Contextual representations

Supporting evidence for rating:

C2. Students were encouraged to use Contextual representations

| 1 | 2 | 3 | 4 | 5 | 6 | 7 |

Limited encouragement

Significant encouragement

Supporting evidence for rating:

C3. Students use numerical representations to reason make conjectures, analyze or justify their solutions.

| 1 | 2 | 3 | 4 | 5 | 6 | 7 |

Limited use

Strong use

Supporting evidence for rating:

OVERALL RATING FOR CONTEXTUAL REPRESENTATIONS IN THE LESSON

The overall rating represents the observer’s best summary judgment of the appropriateness and quality of the lesson’s CONTEXTUAL REPRESENTATIONS. Overall ratings are not necessarily intended to be the numerical average of the ratings of the indicators for Implementation of the Lesson. There may be other factors that influence an overall rating.

| 1 | 2 | 3 | 4 | 5 | 6 | 7 |

CR in the lesson not at all consistent with best practice in standards-based inquiry-oriented teaching and learning

CR in the lesson very consistent with best practice in standards-based inquiry-oriented teaching and learning

Don’t Know
Symbolic Representations.

S1. The lesson included Symbolic Representations

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Superficial use of Symbolic Representations  Significant use of symbolic representations

Supporting evidence for rating: Don’t Know

S2. Students were encouraged to use symbolic representations

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Limited encouragement  Significant encouragement

Supporting evidence for rating: Don’t Know

S3. Students use symbolic representations to reason make conjectures, analyze or justify their solutions.

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</tr>
</tbody>
</table>

Limited use  Strong use

Supporting evidence for rating: Don’t Know

OVERALL RATING FOR SYMBOLIC REPRESENTATIONS IN THE LESSON

The overall rating represents the observer’s best summary judgment of the appropriateness and quality of the lesson SYMBOLIC REPRESENTATIONS. Overall ratings are not necessarily intended to be the numerical average of the ratings of the indicators for Implementation of the Lesson. There may be other factors that influence an overall rating.

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</tbody>
</table>

SR in the lesson not at all consistent with best practice in standards-based inquiry-oriented teaching and learning  SR in the lesson very consistent with best practice in standards-based inquiry-oriented teaching and learning
**OPTIONAL SUMMARY RATING OF THE LESSON**

Depending on how the data from the observation of lessons are going to be used, the observer may want to do a summary rating of the entire lesson, based on the ratings of the four major elements (five elements, if the technology support material is used). If the purpose of the set of observations is to get an overview of the nature and quality of lessons being conducted, the summary rating can be useful. However, unless the number of the set of lessons is fairly large (an adequate proportion of the classrooms being sampled and selected randomly) generalizing from the summary ratings of the sample to the entire set of classrooms is problematic. The summary rating is useful for looking at change over time among all the classrooms, as long as the sampling is credible.

The summary rating represents the observer’s best judgment of the quality of the lesson. This rating is not necessarily intended to be the numerical average of the ratings of the indicators for the four elements: planning/organization, implementation, content, and classroom culture. There may be other factors that influence the summary rating.

**SUMMARY RATING OF THE LESSON**

<table>
<thead>
<tr>
<th></th>
<th>1</th>
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<tbody>
<tr>
<td>Overall, the lesson was</td>
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<tr>
<td>not at all reflective of a</td>
<td>standards-based inquiry-</td>
<td>an excellent example of a high-</td>
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<td>an excellent example of a high-</td>
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<tr>
<td>oriented lesson</td>
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</tr>
</tbody>
</table>

Note: Modifications to this instrument by UMass/Noyce Team include:

1) Removed references to Social Studies and Language Arts;
2) Added two indicators to Lesson Implementation related to culturally responsive teaching (#8 & 9);
3) Added one indicator to Classroom Culture related to culturally responsive teaching (#8); and
4) Added dynamic geometry and dynamic statistics software to the list of technologies in the Technology section (#2); and
5) Minor formatting changes

Modifications to the instrument by Vasu, 2015 include:

1) Change K-12 to Higher Education Calculus;
2) Removed references to Grade Level;
3) Changed core subjects to higher education institution category;
4) Added one indicator to Lesson Planning/Organization to reflect female gender specific (#6)
5) Added four indicators to Lesson Implementation to reflect student engagement and understanding (#1 & #10) and teacher’s attention (#11 & #12);
6) Added three indicators to Content of Lesson to reflect multiple representations (#9) and student affect (#10);
7) Added four indicators to Classroom Culture to reflect female gender specific orientation (#2 & #6) and level of confidence (#9 & #10);
8) Added one indicator to Use of Technology to reflect female gender specific orientation (#12);
9) Added *Character of Multiple Representations to Support the Lesson* key element
10) Minor formatting changes
<table>
<thead>
<tr>
<th>Indicator</th>
<th>Focus Questions &amp; Statements</th>
<th>Examples</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Numerical Representations</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>N1. The lesson included Numerical Representations</td>
<td>Did the lesson contain/incorporate the use of numerical representations to develop or to explain mathematical concepts?</td>
<td>Teacher makes use of tables of numbers, ordered pairs, numerical sequences and patterns to develop mathematical ideas.</td>
</tr>
<tr>
<td>N2. Rate the extent to which the lesson encouraged student use of numerical representations.</td>
<td>The teacher actively encourages the use of numerical representations to support ideas/concepts presented in the lesson.</td>
<td>All students use encouraged to numerical representation to communicate about their mathematical ideas. Lesson encourages students to make conjectures, debate, reason and justify their thinking</td>
</tr>
<tr>
<td>N3. Rate the extent to which students use numerical representations to reason, make conjectures, investigate analyze and justify their solutions.</td>
<td>To what extent do students use numerical representations to reason, make conjectures, investigate analyze and justify their solutions?</td>
<td>All students are actively using numerical representations to reason, make conjectures, investigate analyze and justify their solutions.</td>
</tr>
<tr>
<td><strong>Graphical Representations</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>G1. The lesson included Graphical Representations</td>
<td>Did the lesson contain/incorporate the use of graphical representations to develop or to explain mathematical concepts?</td>
<td>Teacher makes use of graphs and their properties to develop mathematical ideas. Explanations and other teacher moves include graphs and their features in a meaningful way.</td>
</tr>
<tr>
<td>G2. Rate the extent to which the lesson encouraged student use of graphical representations.</td>
<td>The teacher actively encourages the use of graphical representations to support ideas/concepts presented in the lesson.</td>
<td>All students are encouraged to use graphs and their properties to communicate about their mathematical ideas. Lesson encourages students to make conjectures, debate, reason and justify their thinking</td>
</tr>
<tr>
<td>G3. Rate the extent to which students use graphical representations to reason, make conjectures, investigate analyze and justify their solutions.</td>
<td>To what extent do students use graphical representations to reason, make conjectures, investigate analyze and justify their solutions?</td>
<td>All students are actively using graphical representations to reason, make conjectures, investigate analyze and justify their solutions.</td>
</tr>
<tr>
<td><strong>Verbal Representations</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>V1. The lesson included Verbal Representations</td>
<td>Did the use of verbal representations, mathematical language and words to develop or to explain mathematical concepts?</td>
<td>Teacher makes use of words and verbal explanations to develop mathematical ideas</td>
</tr>
<tr>
<td>G2. Rate the extent to which the lesson encouraged student use of verbal representations.</td>
<td>The teacher actively encourages the use of verbal representations to support ideas/concepts presented in the lesson.</td>
<td>All students are encouraged to use verbal representations to communicate their mathematical ideas. Lesson encourages students to make conjectures, debate, reason and justify their thinking</td>
</tr>
<tr>
<td>Indicator</td>
<td>Focus Questions &amp; Statements</td>
<td>Examples</td>
</tr>
<tr>
<td>--------------------------------------------------------------------------</td>
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<td>-------------------------------------------------------------------------</td>
</tr>
<tr>
<td>G3. Rate the extent to which students use verbal representations to reason, make conjectures, investigate analyze and justify their solutions.</td>
<td>To what extent do students use verbal representations to reason, make conjectures, investigate analyze and justify their solutions?</td>
<td>All students are actively using verbal explanations, language, and verbal representations to reason, make conjectures, investigate analyze and justify their solutions.</td>
</tr>
<tr>
<td><strong>Contextual Representations</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>C1. The lesson included Contextual Representations</td>
<td>Did the lesson contain/incorporate the use of contextual representations to develop or to explain mathematical concepts?</td>
<td>Teacher makes use of context, stories, real life situations and concrete models to develop mathematical ideas</td>
</tr>
<tr>
<td>C2. Rate the extent to which the lesson encouraged student use of contextual representations.</td>
<td>The teacher actively encourages the use of contextual representations to support ideas/concepts presented in the lesson.</td>
<td>All students are encouraged to use contextual representations, stories, concrete examples and real-life situations to communicate about their mathematical ideas. Lesson encourages students to make conjectures, debate, reason and justify their thinking.</td>
</tr>
<tr>
<td>C3. Rate the extent to which students use numerical representations to reason, make conjectures, investigate analyze and justify their solutions.</td>
<td>To what extent do students use contextual representations to reason, make conjectures, investigate analyze and justify their solutions?</td>
<td>All students are actively using concrete examples, real life situations and contextual representations to reason, make conjectures, investigate analyze and justify their solutions.</td>
</tr>
<tr>
<td><strong>Symbolic Representations</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>S1. The lesson included Symbolic Representations</td>
<td>Did the lesson contain/incorporate the use of symbolic representations to develop or to explain mathematical concepts?</td>
<td>Teacher makes use of symbols, formulas, and other symbolic representations to develop mathematical ideas</td>
</tr>
<tr>
<td>S2. Rate the extent to which the lesson encouraged student use of symbolic representations.</td>
<td>The teacher actively encourages the use of symbolic representations to support ideas/concepts presented in the lesson.</td>
<td>All students are encouraged to use symbolic representation to communicate about their mathematical ideas. Lesson encourages students to make conjectures, debate, reason and justify their thinking.</td>
</tr>
<tr>
<td>S3. Rate the extent to which students use symbolic representations to reason, make conjectures, investigate analyze and justify their solutions.</td>
<td>To what extent do students use symbolic representations to reason, make conjectures, investigate analyze and justify their solutions?</td>
<td>All students are actively using symbolic language representations to reason, make conjectures, investigate analyze and justify their solutions.</td>
</tr>
</tbody>
</table>
## ADDITIONAL INDICATORS FOR MODIFIED SAMPI (LOP) QUESTIONS

<table>
<thead>
<tr>
<th>Indicator</th>
<th>Focus Questions and Statements</th>
<th>Examples</th>
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</thead>
<tbody>
<tr>
<td><strong>Planning and Organization</strong></td>
<td></td>
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<tr>
<td>6. Was the lesson organized so that it</td>
<td>Were the instructional strategies appropriate for the developmental level of the students regardless of gender? Did the lesson build on students’ prior experiences regardless of gender? Was it designed as part of a sequence and did it build on previous activities?</td>
<td>Teacher reminds students regardless of gender to remember relevant experiences. Students regardless of gender are called on to share special knowledge or experience. Lessons include as appropriate activities involving individuals, pairs, small groups and the whole group. A variety of teaching formats are included</td>
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<tr>
<td><strong>Implementation of the lesson</strong></td>
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<tr>
<td>10. Female students were engaged in sense</td>
<td>Were female students engaged in application, analysis, deep understanding?</td>
<td>Female students are called on to answer substantive questions regarding the lesson. Female students discuss mathematics</td>
</tr>
<tr>
<td>making of this lesson</td>
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<tr>
<td>11. Teacher is sensitive to issues of</td>
<td>Are the lesson objectives and implementation sensitive to gender?</td>
<td>Teacher and lesson is relevant to female students’ prior experience.</td>
</tr>
<tr>
<td>gender when facilitating this lesson</td>
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<tr>
<td>12. Students regardless of gender were</td>
<td>Are all students regardless of gender given attention?</td>
<td>Teacher interacts, calls on, or responds to both male and female students equally. All students are given equal attention</td>
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<td>given regardless attention.</td>
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<tr>
<td><strong>Content of the lesson</strong></td>
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</tr>
<tr>
<td>9. The lesson included numerical, verbal,</td>
<td>Does the enacted curriculum include multiple representations?</td>
<td>Teacher uses graphs, symbols, tables, contexts and language to accomplish the lesson objectives. Students use multiple representations in their discussion.</td>
</tr>
<tr>
<td>contextual, graphical, contextual,</td>
<td></td>
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<tr>
<td>analytical representations?</td>
<td></td>
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</tr>
<tr>
<td><strong>Classroom Culture</strong></td>
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</tr>
<tr>
<td>2. The teacher showed respect for and</td>
<td>The teacher accepts ideas from male and female students equally and without making judgments or until there is no more discussion. Students regardless of gender are encouraged to offer alternative solutions. Students are expected to make a case for their ideas.</td>
<td>The teacher solicits ideas from all students regardless of gender, accepting them without judging them immediately. All students, regardless of gender, are expected to discuss and to defend ideas. The teacher values discussion and encourages conversation among students about the ideas</td>
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<tr>
<td>valued students’ idea, questions, and/or</td>
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<tr>
<td>contributions to the lesson regardless of</td>
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<tr>
<td>gender.</td>
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<tr>
<td>6. Teacher-female student interactions</td>
<td>The teacher and female students work together to solve problems and seek answers to questions as</td>
<td>Teacher and female students coordinate their efforts interacting in open and non-judgmental ways</td>
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<td>reflected</td>
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<tr>
<td>collaborative working relationships.</td>
<td>they develop conceptual understanding.</td>
<td>and accepting each other’s ideas. Teacher supports female students in their work (but does not do their work).</td>
</tr>
<tr>
<td>9. Female students asserted themselves with confidence.</td>
<td>Female students participate in class discussions and are confident.</td>
<td>Female students answer and ask questions, volunteer to solve problems, and are able to defend their thinking with confidence.</td>
</tr>
<tr>
<td>10. All students have the opportunity to participate in the lesson regardless of gender.</td>
<td>Are all students given the opportunity to participate in the lesson?</td>
<td>Teacher pays attention to all students regardless of gender and tries to engage all students in the class.</td>
</tr>
<tr>
<td>Technology</td>
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<tr>
<td>12. Classroom management was effective in engaging female students in the use of technology.</td>
<td>Did teacher manage classroom activities with technology in ways that engaged female students with technology?</td>
<td>Female students understand how to use technology and proceed to use it as intended. If turns need to be taken, all students are engaged in meaningful tasks while waiting.</td>
</tr>
</tbody>
</table>
APPENDIX B

FIVE PROBLEMS INVOLVING THE FTC

(All students at all sites complete. Scored using rubric in appendix E)
- Please solve the problems below with sufficient details for someone to be able to follow your reasoning.
- Write solutions on the separate blank paper provided
- Use the graph space below for the graph in problem A

Problem A
Consider the graph of \( f(x) \). Let \( F(x) \) be the anti-derivative of \( f(x) \) with \( F(0) = 0 \)
1. Use an integral to define \( F(x) \).
2. Determine in each case the main features of the function \( F(x) \) on the indicated interval, such as:
   (a) Regions where \( F(x) \) is increasing and where is it decreasing.
   (b) Regions of concavity of \( F(x) \).
   (c) Location of any maxima and/or minima.
3. In the space on the right, graph the function \( F(x) \) that corresponds to \( f(x) \) on the left

Problem Set B
Consider the numerical values for \( f(x) \) below and assume \( f(x) \) is continuous and piece-wise linear on the interval from \([0, 4]\). Let \( F(x) \) be the anti-derivative of \( f(x) \) with \( F(0) = 0 \)

<table>
<thead>
<tr>
<th>X</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f'(x) = f(x) )</td>
<td>-2</td>
<td>-1</td>
<td>0</td>
<td>3</td>
<td>2</td>
</tr>
</tbody>
</table>

1. (a) Sketch the points in the table above. (b) Use an integral to define \( F(x) \).
2. Determine in each case the following about \( F(x) \):
   (a) Find \( F(1) \). On what intervals is the \( F(x) \) increasing and where is it decreasing?
   (b) Regions of concavity of \( F(x) \)
   (c) Location of any maxima and/or minima.
Problem C

Consider the linear function \( f(x) \) with y-intercept 4 and slope -2. Let \( F(x) \) be the anti-derivative of \( f(x) \) with \( F(0) = 0 \).

1. Use the integral to define \( F(x) \).
2. Determine the main features of the function \( F(x) \) such as:
   (a) Where is \( F(x) \) increasing and where is it decreasing?
   (b) Regions of concavity.
   (c) What is the location of any maxima and/or minima if any?
3. Draw a sketch of \( F(x) \).

Problem D

A ladybug is crawling along a rod with starting point at A. The bug is traveling at a rate of \( v(t) = 5 - t \) inches/minute between \( t = 0 \) and \( t = 10 \) minutes. Answer the following regarding the ladybug’s displacement or distance away from A.

1. What is the relationship between the velocity of the ladybug and its distance?
2. Labeling \( d(t) \) as the distance the bug is from A, determine in each case the following about this distance:
   (a) What is the bug’s distance at \( t = 1 \)? At \( t = 5 \)? At any time \( t \)?
   (b) When (for what times) is the distance increasing and where is it decreasing?
   (c) When does the bug change direction of travel or stop?
   (d) When is the bug accelerating and when is she decelerating if at all?
   (e) What can be said about the maximum and/or minimum distance and when does it happen if at all?

Problem E

Consider the function \( f(x) = 2x - 4 \). Let \( F(x) = \int_0^x f(t) \, dt \)

1. What is the relationship between \( f(x) \) and \( F(x) \)?
2. Please determine in each case the main features of the function \( F(x) \), such as:
   (a) Where is this function increasing and where is it decreasing?
   (b) Regions of concavity of \( F(x) \).
   (c) Location of the maxima and/or minima of \( F(x) \).
APPENDIX C

THINK-ALOUD PROBLEMS AND PROTOCOL

Use the following coding scheme to design a five-element identity code that may be used in future surveys. Neither the researcher nor your teacher will attempt to connect your code with your identity. CODING SCHEME: Last letter of first name; two numbers for day of birth (e.g. 01 for 1st, 02 for 2nd, etc.), second letter of last name and first letter of birth month. For example: Ileana Vasu, born on May 2nd, would have the code A02AM

CODE: ____________________________ GENDER: ____________

Below are five problems in different representations (verbal, graphical, numerical, contextual, verbal.) Choose the problem that most appeals to you and solve the problem. Please explain your reasoning out loud.

**Problem A**
Consider the graph of \( f(x) \). Let \( F(x) \) be the anti-derivative of \( f(x) \) with \( F(0) = 0 \). Answer the following questions.
1. Use an integral to define \( F(x) \).
2. Please determine in each case the main features of the function \( F'(x) \) on the indicated interval, such as:
   (a) Regions where \( F(x) \) is increasing and where is it decreasing.
   (b) Regions of concavity of \( F(x) \).
   (c) Location of any maxima and/or minima.
3. In the space on the right, graph the function \( F(x) \) that corresponds to \( f(x) \) on the left.

![Graph of f(x)](image)

**Problem B**
Consider the numerical values for \( f(x) \) below and assume \( f(x) \) is continuous and piece-wise linear on \([0, 4]\). Let \( F(x) \) be the anti-derivative of \( f(x) \) with \( F(0) = 0 \). Answer the following questions, first for the function in the table below

<table>
<thead>
<tr>
<th>X</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F'(x) = f(x) )</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>
1. Use the integral to define $F(x)$.
2. Determine in each case the following about $F(x)$:
   (a) Find $F(1)$ and $F(2)$ as closely as you can. Can you say anything else about $F(x)$?
   (b) On what intervals is the $F(x)$ increasing and where is it decreasing?
   (b) Regions of concavity of $F(x)$?
   (c) Location of any maxima and/or minima.

**Problem C**
Consider the linear function $f(x)$ with y intercept 4 and slope -2 on the interval [0, 4]. Consider now the absolute value of $f(x)$, $g(x) = |f(x)|$ on the interval [0, 4]. Let $G(x)$ be the anti-derivative of $g(x)$ with $G(0) = 0$

1. Use the integral to define $G(x)$.
2. Please determine in each case the main features of the function $G(x)$ on [0, 2] such as
   (a) For $G(x)$, where is this function increasing and where is it decreasing?
   (b) Regions of concavity of $G(x)$.
   (c) Location of any maxima and/or minima.

**Problem Set D**
A ladybug is crawling along a rod starting at point A. The bug is traveling at a rate of $v(t) = t - 5$ inches per second from $t = 0$ to $t = 5$. Then, from $t = 5$ to $t = 10$ minutes, the bug travels at a rate equal to $v(t) = 5 - t$ inches per second.

1. What is the relationship between the velocity of the ladybug and its distance from point A?
2. Determine the following about the distance from A.
   (a) What is the bug’s distance at $t = 1$? At $t = 2$? Can you determine what the distance is at any time $t$?
   (b) When does the bug change direction of travel or stop?
   (c) When (for what times) is the distance increasing and where is it decreasing?
   (d) When is the bug accelerating and when is she decelerating if at all?
   (e) What can be said the maximum and/or minimum distance from A?

**Problem E**
Consider now the function $g(x) = 2x - 4$ on the interval [0, 4]. Let $G(x) = \int_{0}^{x} g(t)dt$

1. What is the relationship between $g(x)$ and $G(x)$?
2. Please determine in each case the main features of the function $G(x)$, such as:
   (a) Where is this function increasing and where is it decreasing?
   (b) Regions of concavity.
   (c) Location of the maxima and/or minima.
   (d) Location of any inflection points.

PROTOCOL FOR THINK ALOUD

Purpose:

To examine how students’ use of multiple representations relates to their understanding of the Fundamental Theorem of Calculus.

I will quickly talk through the protocol below. Students will separately fill out a consent form.

Protocol:

Hello. My name is Ileana Vasu, and I am trying to understand how multiple representations feature in students’ learning of the Fundamental Theorem of Calculus, which you learned this semester. I am very interested in hearing students’ voice and thinking about this theorem. I want to thank you for taking time to participate in this project.

This interview has two parts. First, I will ask you to solve a Calculus problem of your choice and walk me through your thinking. I will ask you to tell me what you are thinking as you are solving this problem. At the end there will be a brief set of questions about your process. If there is a question you do not want to answer, you do not have to. Responses will only be reviewed by the researcher, and your identity will not be disclosed to anyone. Do you have any questions?
THINK ALOUD POST OBSERVATION QUESTIONS
A Study of Students’ Understanding of the FTC Multiple Representations

All students at the three sites work will have to work one of five problems involving the FTC which are included in appendix B. The five problems are similar but they are in various representations. Students will be asked to do one problem of their choice based on what they think they could do better on. Students will not be told they were given the very similar mathematical problems in different representations. These questions are to be answered by the nine students selected to be interviewed. There may be sub-questions depending on whether the students switch representations or not.

POST INTERVIEW QUESTIONS
1) Have you used multiple representations in the classroom?
2) Who helps you with the homework?
3) Why did you choose particular problem to solve?
4) How did your classroom experience influence the choice of representations for your solutions?
5) Tell me in your own words what the Fundamental Theorem of Calculus means.
6) Do you think you have a full understanding of the Fundamental Theorem of Calculus?

Thank you very much.
APPENDIX D

BACKGROUND QUESTIONNAIRE

Fundamental Theorem of Calculus, Multiple Representations and Gender

Use the following coding scheme to design a five-element identity code that may be used in future surveys. Neither the researcher nor your teacher will attempt to connect your code with your identity. CODING SCHEME: Last letter of first name; two numbers for day of birth (e.g. 01 for 1st, 02 for 2nd, etc.), second letter of last name and first letter of birth month. For example: Ileana Vasu, born on May 2nd, would have the code A02AM

CODE: ___________________________  GENDER: ____________

1. Did you take Calculus in high school? If so, when, where and what course? (E.g. HS, AP, Honors)

2. Have you been exposed to the FTC before? What sort of representations do you recall using at that time?

3. What motivates you to work hard in math?

4. What is your best estimate for a grade in this class – please use a scale from 1-100.

5. Who helps you with learning mathematics?

6. In your prior mathematics classes, to what extent did you find multiple representations of mathematical concepts utilized effectively?

Check the box that best reflects your position in regard to your learning of the FTC in this class.

<table>
<thead>
<tr>
<th>Statement</th>
<th>Strongly Agree</th>
<th>Agree</th>
<th>Disagree</th>
<th>Strongly Disagree</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 The lessons included graphs, charts, pictures and drawings in presenting the Fundamental Theorem of Calculus.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2 The lessons included numerical (tables of data, sequences) in presenting the Fundamental Theorem of Calculus.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3 The lessons used verbal presentation (mathematical explanations of concepts) and words like “integrals”, “anti-derivatives” in teaching the Fundamental Theorem of Calculus.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4 The lessons used stories, word problems and real contexts in presenting the Fundamental Theorem of Calculus.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5 The lesson used mathematical symbols (algebraic formulas, integrals and derivatives) in presenting the Fundamental Theorem of Calculus.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6 I like problems or ideas presented in graphical ways.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7 I like when problems and ideas in Calculus are presented using tables of values</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8 I like when problems and ideas in Calculus are presented through stories and real-life contexts.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9 I like when problems and ideas in Calculus are presented using symbolical means such as formulas, integrals and derivative symbols.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10 I like when problems and ideas in Calculus are presented through verbal mathematical explanations</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
## APPENDIX E

### MATH RUBRIC FOR THE FIVE FTC PROBLEMS

<table>
<thead>
<tr>
<th>Level</th>
<th>Points</th>
<th>Problem Solving Strategy</th>
</tr>
</thead>
<tbody>
<tr>
<td>Novice</td>
<td>0</td>
<td>Incorrect solutions. No strategy is chosen, or a strategy is chosen that will not lead to a solution. No correct reasoning nor justification for reasoning is present.</td>
</tr>
<tr>
<td>Apprentice</td>
<td>1</td>
<td>A partially correct strategy is chosen, or a correct strategy for only solving part of the task is chosen. Evidence of drawing on some relevant previous knowledge is present, showing some relevant engagement in the task.</td>
</tr>
<tr>
<td>Practitioner</td>
<td>2</td>
<td>A correct strategy is chosen based on the mathematical situation in the task. Achieve the correct solution. Some minor technical details may be missing.</td>
</tr>
<tr>
<td>Expert</td>
<td>3</td>
<td>A correct strategy is chosen based on the mathematical situation in the task. Achieve the correct solution. An efficient strategy is chosen and progress towards a solution is evaluated. Evidence is used to justify and support decisions; the argument is supported by mathematical properties.</td>
</tr>
</tbody>
</table>
### Problem A

<table>
<thead>
<tr>
<th>Problem A Score</th>
<th>Question 1</th>
<th>Q2a</th>
<th>Q2b</th>
<th>Q2c</th>
<th>Q3</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Expert</strong> 3 points</td>
<td>$f(t)dt = F(x) \vee \int_0^x (1 - t)dt = f(x)$</td>
<td>$F(x)$ is increasing when $F''(x)&gt;0$ so on ${0,1}$ and decreasing on $(1,2)$, when $F''(x)=f(x)&lt;0$</td>
<td>CU if $F'=f&gt;0$ so nowhere</td>
<td>Max at $x=1$ since $f(x)=F'(x)=0$. Max value is $\frac{1}{2}$ since the area under $f(x)$ between 0 and 1 is $\frac{1}{2}$</td>
<td><strong>F(x)</strong></td>
</tr>
<tr>
<td>Practitioner 2 points</td>
<td>$\int_0^x f(x)dx = F(x)$ or $\int_0^x (1 - x)dx$</td>
<td>$F(x)$ is increasing when on ${0,1}$ and decreasing on $(1,2)$</td>
<td>CU nowhere</td>
<td>Max at $x=1$ and $y=1/2$</td>
<td>Quadratic down, incorrect placement or vertex</td>
</tr>
</tbody>
</table>

Possible Inflection when $F''=0$ and changing sign so where $F'=0$. Since $f''(x)=-1$ for all $x$, there are no inflection points.

Max value is $\frac{1}{2}$ since the area under $f(x)$ between 0 and 1 is $\frac{1}{2}$.
<table>
<thead>
<tr>
<th>Problem A Score</th>
<th>Question 1</th>
<th>Q2a</th>
<th>Q2b</th>
<th>Q2c</th>
<th>Q3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Apprentice</td>
<td>$\int_0^2 f(x) , dx$ &lt;br&gt; $\int_0^2 (1 - x) , dx$</td>
<td>Increasing nowhere and decreasing everywhere because slope is negative</td>
<td>No concavity Inflection when the second derivative is zero but draw no other conclusions</td>
<td>$x = 1$ (no $y$ given)</td>
<td>Quadratic up, other placements</td>
</tr>
<tr>
<td>1 point</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Novice</td>
<td>Blank or other answers that do not include the integral of $f(x)$ at all.</td>
<td>Other incorrect answers.</td>
<td>$x = 0$ or blank Other Inaccurate conclusions drawn</td>
<td>Blank or incorrect Incorrect answer</td>
<td>Blank, or linear</td>
</tr>
<tr>
<td>0 points</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
# ANTICIPATED TYPICAL ANSWERS TO GUIDE SCORING THE FUNDAMENTAL THEOREM OF CALCULUS PROBLEMS

**Problem B**

<table>
<thead>
<tr>
<th>Problem B Score</th>
<th>Question 1</th>
<th>Q2a</th>
<th>Q2b</th>
<th>Q2c</th>
</tr>
</thead>
<tbody>
<tr>
<td>Expert 3 points</td>
<td>$\int_0^x f(t)dt = F(x)$ or $\int_0^x (t-2)dx$</td>
<td>Graph of points to see the relationship may be present</td>
<td>$\int_0^1 (x-2)dx = \frac{x^2}{2} - 2x \mid_0^1 = -1.5$</td>
<td>Concave up since $F(1) = -1.5$</td>
</tr>
<tr>
<td>Practitioner 2 points</td>
<td>$\int_0^x f(x)dx = F(x)$ or $\int_0^x (x-2)dx$</td>
<td>Correct answer but no justifications</td>
<td>$F(1) = -1.5$</td>
<td>Correct answer but no justifications</td>
</tr>
<tr>
<td>Apprentice 1 point</td>
<td>$\int_0^4 (x-2)dx$ or $\int_0^4 f(x)dx$</td>
<td>Incorrect computation of integral or area but correct reasoning</td>
<td>$F'(x) = 0$</td>
<td>No concavity since $f'' = 0$ (confusing $F''$ and $f''$)</td>
</tr>
<tr>
<td>Novice 0 points</td>
<td>Blank or $F(4) = 4 - 2 = 2$</td>
<td>$F(1) = 1 - 2 = -1$ Confusing $F$ and $f$</td>
<td>Other incorrect answers</td>
<td>Max is at $x=4$</td>
</tr>
</tbody>
</table>
**Problem C**

<table>
<thead>
<tr>
<th>Problem C Score</th>
<th>Question 1</th>
<th>Q2a</th>
<th>Q2b</th>
<th>Q2c</th>
<th>Q3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Expert 3 points</td>
<td>$f(t),dt = F(x)$ $\vee$ $\int_0^x \int_0^x (-2t + 4),dt = F(x)$ or $\int_0^x (-2t + 4),dt = F(x) = -x^2 + 4x$</td>
<td>$F(x)$ is increasing when $F'(x)&gt;0$ so on $(2, 4)$ and decreasing on $(0, 2)$, when $F'(x)=f(x) &lt; 0$ or Graph $y = -x^2 + 4x$ to make same argument</td>
<td>CU if $F''=f' &gt; 0$ so nowhere CD on for all $x$ since the second derivative is positive</td>
<td>Maximum at $x = 2$ since $f(x)=F'(x)=0$ there and changes sign to the left and right. Maximum value is 4 since the signed area under $f(x)$ between 0 and 2 is 4 or at: $\int_0^2 (f(x)),dx = 4$</td>
<td><img src="Image" alt="Graph" /></td>
</tr>
<tr>
<td>Practitioner 2 points</td>
<td>$\int_0^x (-2x + 4),dt = F(x)$</td>
<td>$F(x)$ is increasing when on ${0,2}$ and decreasing on $(2,4)$</td>
<td>CU on for $x$ CD on $[0,4]$</td>
<td>Max at $x = 2$ and $y = 4$. Some justification like above, but some arguments not clearly given</td>
<td>No max labeled</td>
</tr>
</tbody>
</table>

Table continues…
<table>
<thead>
<tr>
<th>Problem C Score</th>
<th>Question 1</th>
<th>Q2a</th>
<th>Q2b</th>
<th>Q2c</th>
<th>Q3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Apprentice 1 point</td>
<td>$\int f(x)dx$</td>
<td>Increasing nowhere and decreasing everywhere because slope is negative</td>
<td>No concavity Inflection when the second derivative is zero. Inaccurate conclusions drawn</td>
<td>$x = 2$ (no $y$ given)</td>
<td>Incorrect placement or incorrect max value</td>
</tr>
<tr>
<td>Novice 0 points</td>
<td>Blank or other answers that do not include the integral of $f(x)$ at all.</td>
<td>Other incorrect answers.</td>
<td>$x = 0$ or blank</td>
<td>Blank or incorrect Incorrect answer</td>
<td>Blank or linear $y = -2$ or $y = -2x + 4$</td>
</tr>
</tbody>
</table>
### ANTICIPATED TYPICAL ANSWERS TO GUIDE SCORING THE FUNDAMENTAL THEOREM OF CALCULUS PROBLEMS

**Problem D**

<table>
<thead>
<tr>
<th>Problem D Score</th>
<th>Question 1</th>
<th>Q2a</th>
<th>Q2b</th>
<th>Q2c</th>
<th>Q2d</th>
<th>Q2e</th>
</tr>
</thead>
<tbody>
<tr>
<td>Expert</td>
<td>Position is the integral of velocity or x</td>
<td>$d = \int_{0}^{1} (5 - t) , dt$</td>
<td>Distance increasing if $5 - t &gt; 0$</td>
<td>Change direction at $t = 5$</td>
<td>Always decelerating $a(t) = v'(t) = -1$</td>
<td>max distance at $t = 5$ of 12.5 in</td>
</tr>
<tr>
<td>3 points</td>
<td>$d = \int_{0}^{1} (5 - t) , dt$</td>
<td>$= 5t - \frac{t^2}{2}$ = 4.5 in</td>
<td>So, if $t &lt; 5$ decreasing at $t &gt; 5$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td><em>evaluated at 1 minus at 0 this is the symbol I can’t get to work</em> or area under $y=5-t$ at $t=5$, $d = 12.5$</td>
<td><em>evaluated at 1 minus at 0 this is the symbol I can’t get to work</em> or area under $y=5-t$ at $t=5$, $d = 12.5$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Practitioner</td>
<td>$\int_{0}^{10} v(t) , dt$</td>
<td>Correct answer no justification</td>
<td>if $t &lt; 5$ increasing if $t &gt; 5$ decreasing</td>
<td>N/A</td>
<td>Correct answer no justification</td>
<td>Just max distance or just time for max distance</td>
</tr>
<tr>
<td>2 points</td>
<td>$\int_{0}^{10} v(t) , dt$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Apprentice</td>
<td>$v'(t) = d(t)$</td>
<td>Correct idea</td>
<td>$\int_{0}^{1} (5 - t) , dt$</td>
<td>N/A</td>
<td>Never $f'' = -1$ confusing $F$ and $f''$</td>
<td>Accelerating at $t &lt; 5$ decelerating at $t &lt; 5$ confusing $v(t)$ and $a(t)$</td>
</tr>
<tr>
<td>1 point</td>
<td>$v'(t) = d(t)$</td>
<td></td>
<td>$\int_{0}^{1} (5 - t) , dt$</td>
<td>N/A</td>
<td></td>
<td>N/A</td>
</tr>
<tr>
<td>Novice</td>
<td>Blank and other answers</td>
<td>$d(t) = f(t)$</td>
<td>Not increasing since $f' = -1 &lt; 0$</td>
<td>Blank and other answers</td>
<td>Blank and other answers</td>
<td>Blank and other answers</td>
</tr>
<tr>
<td>0 points</td>
<td>$d(1) = f(1) = 5 - 1 = 4$ Blank and other answers</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
## ANTICIPATED TYPICAL ANSWERS TO GUIDE SCORING THE FUNDAMENTAL THEOREM OF CALCULUS PROBLEMS

**Problem E**

<table>
<thead>
<tr>
<th>Problem E Score</th>
<th>Question 1</th>
<th>Q2a</th>
<th>Q2b</th>
<th>Q2c</th>
</tr>
</thead>
<tbody>
<tr>
<td>Expert 3 points</td>
<td>$F(x) = \int_a^x (2t - 4),dt$ and or $x^2 - 4x$</td>
<td>Increasing if $2t - 4 &gt; 0$ so if $t &gt; 2$ decreasing if $2t - 4 &lt; 0$ so if $t &lt; 2$</td>
<td>Concave up for all $t$ since $F'' &gt; 0$</td>
<td>Min when $t=2$ min value $F(2) = -4$ or by graph or calculator</td>
</tr>
<tr>
<td>Practitioner 2 points</td>
<td>$F(x) = \int (2x - 4),dx$</td>
<td>Correct answer and if $t &gt; 2, t &lt; 2$ no justification or graph of $y = x^2 - 4x$</td>
<td>Concave up no justifications</td>
<td>Min value at $t=2$ equal to $-4$ no answer (for max?)</td>
</tr>
<tr>
<td>Apprentice 1 point</td>
<td>N/A</td>
<td>Always since $f'(x) = 2 &gt; 0$ confusing $f'$ and $F'$</td>
<td>No concavity $f'' = 0$</td>
<td>Min at $t=2$ no max</td>
</tr>
<tr>
<td>Novice 0 points</td>
<td>Other answers or blank</td>
<td>Other answers or blank</td>
<td>Other answers or blank</td>
<td>Other answers or blank</td>
</tr>
</tbody>
</table>
REFERENCES


