

2015

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## Recommended Citation

Kevrekidis, Panayotis G.; Stefanov, Atanas G.; and Xu, Haitao, "Traveling Waves for the Mass in Mass Model of Granular Chains" (2015). *Mathematics and Statistics Department Faculty Publication Series*. 1222.

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# TRAVELING WAVES FOR THE MASS IN MASS MODEL OF GRANULAR CHAINS

PANAYOTIS G. KEVREKIDIS, ATANAS G. STEFANOV, AND HAITAO XU

**ABSTRACT.** In the present work, we consider the mass in mass (or mass with mass) system of granular chains, namely a granular chain involving additionally an internal resonator. For these chains, we rigorously establish that under suitable “anti-resonance” conditions connecting the mass of the resonator and the speed of the wave, bell-shaped traveling wave solutions continue to exist in the system, in a way reminiscent of the results proven for the standard granular chain of elastic Hertzian contacts. We also numerically touch upon settings where the conditions do not hold, illustrating, in line also with recent experimental work, that non-monotonic waves bearing non-vanishing tails may exist in the latter case.

## 1. INTRODUCTION

A topic that has been progressively gaining more attention over the last two decades within the nonlinear dynamical systems of the famous Fermi-Pasta-Ulam type [9], concerns the study of the so-called granular crystals [25, 28, 15, 32]. The latter consist of chains of elastically interacting (through the so-called Hertzian contacts) beads that are not only very experimentally accessible, but also extensively tunable and controllable, as regards their materials, geometry, heterogeneity, etc. Another reason for the considerable appeal of such simple lattice structure, which can be tailored to be one-, two- or even three-dimensional, is the wealth of nonlinear excitations that have arisen in such settings, and which include robust traveling waves, bright and dark breather structures, and shock waves among others, as summarized in the above reviews. Finally, yet another element promoting the interest in such setups concerns their potential relevance to a broad range of applications such as actuating devices [17], acoustic lenses [29], mechanical diodes [22, 21, 2], logic gates [20] and sound scramblers [7, 26].

Arguably, the most significant and well-studied excitation, the principal workhorse around which many theoretical analyses and experimental results have been centered in such granular systems is the traveling wave. This structure was originally established in the pioneering work of Nesterenko in [27] (see also the review of [25]). While it was originally

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*Date:* February 9, 2015.

*2000 Mathematics Subject Classification.* 37L60, 35C15, 35Q51.

*Key words and phrases.* Solitary Waves; Granular Chains; Nonlinear Lattices; Traveling Waves; Calculus of Variations.

Stefanov’s research is supported in part by nsf-dms 1313107. Kevrekidis acknowledges support from the National Science Foundation under grant DMS-1312856, from ERC and FP7-People under grant 605096, from the US-AFOSR under grant FA9550-12-10332, and from the Binational (US-Israel) Science Foundation through grant 2010239. P.G.K.’s work at Los Alamos is supported in part by the U.S. Department of Energy.

believed to be genuinely compact, asymptotic analysis illustrated its doubly exponential nature (in the granular system without precompression; in the presence of precompression the decay of the wave becomes exponential) [6, 1]. From a mathematically rigorous perspective, the work of [10] enabled as a special case example application the proof of existence of such a traveling wave in the work of [24], while later the framework of [8] (earlier also discussed in the physical/computational literature by [12]) enabled the rigorous identification of the bell-shaped nature of the wave without [30] or with [31] precompression.

More recently, variants of the standard granular system in which internal oscillators or resonators are present in each of the lattice nodes (i.e., for each of the beads) have been proposed theoretically and some of them have also been realized experimentally. Arguably, one of the earliest examples of this type (which, however, has not yet been experimentally implemented, to the best of our knowledge) is the so-called cradle system [13, 14], where a local, linear oscillator was added to each bead, enabling the observation of some intriguing dynamical features. More recently, another variant of the granular chain, namely the locally resonant granular crystals or so-called mass-in-mass or mass-with-mass systems have been proposed and experimentally realized, respectively in [3] and [11]. The former setting involves an internal resonator within the chain (e.g. a second bead embedded in the principal one), while in the latter case, the resonator is external to the bead (see e.g. the prototype also put forth in [16]).

The recent experiment of [18] is expected to provide a significant boost to studies along this direction, as it offered a different so-called woodpile configuration constructed out of orthogonally-stacked rods (with every second rod aligned, in this alternating 0-90 degree configuration) for which it was demonstrated that the MiM/MwM description of a granular chain with an internal resonator can be tuned to be the relevant one [19]. In fact, such a system allows, depending on the parametric regime and the vibrational modes of the rods, to create settings where controllably two- or more internal resonator modes are relevant in the mathematical model description. Moreover, this experiment reported the observation of traveling waves for this system with persistent tails.

The focus of the present study is to provide a rigorous analysis of the traveling waves in such a MiM/MwM system bearing an internal resonator. In particular, the surprising finding that our analysis will rigorously establish concerns the fact that under special (i.e., non-generic or isolated) so-called anti-resonance conditions, the traveling waves of this chain will *still* bear a bell-shaped structure, monotonically decaying on either side. That is to say, for such isolated combinations of the resonator mass/wave speed, the tails of the wave will be absent. On the contrary, the generic setting will, in fact, involve such a resonant excitation through the passing the wave of the internal resonator and as such, it will lead to the formation of the weakly nonlocal solitary waves (nanoptera) reported in [18]. Our presentation of these results will be structured as follows. In section 2, we will provide the general setup of the problem and an integral (single component) reformulation thereof. In section 3, the main result will be presented and proved. Finally, in section 4, some supporting numerical computations will be provided in order to corroborate the relevant results.

## 2. ANALYSIS OF THE PROBLEM: THE MODEL AND ITS INTEGRAL REFORMULATION

Following the earlier MiM/MwM studies [3, 11, 16, 18, 19], we consider the prototypical mathematical model of the form:

$$(1) \quad \begin{cases} \partial_{tt}X_i = [(X_{i-1} - X_i)_+^p - (X_i - X_{i+1})_+^p] + \tilde{k}(x_i - X_i) \\ \nu\partial_{tt}x_i = -\tilde{k}(x_i - X_i) \end{cases}$$

where  $p > 1$ . Here  $X_i$  represent the displacements of the granular beads and  $x_i$  those of the internal (or external) resonators.  $\tilde{k}$  corresponds to the (normalized) elastic constant of the bead-resonator coupling, while  $\nu$  is the (again, normalized to the mass of the beads) mass of the resonator particles. In the strain variables  $Y_i := X_{i-1} - X_i$ ,  $y_i = x_{i-1} - x_i$ , the equations take the form

$$(2) \quad \begin{cases} \partial_{tt}Y_i = (Y_{i+1}^p)_+ - 2(Y_i^p)_+ + (Y_{i-1}^p)_+ + \tilde{k}(y_i - Y_i) \\ \nu\partial_{tt}y_i = -k(y_i - Y_i) \end{cases}$$

We first transform the problem to a setting which allows us to use the methods of calculus of variations. This is similar to the approach that we took in the earlier works [30, 31] to treat the case of monomer chains (in the absence of resonators).

**2.1. Reduction to a single equation for  $\Phi$ .** We are now looking for traveling wave solutions of (2) in the form  $Y_i(t) = \Phi(i - ct)$  and  $y_i(t) = \Psi(i - ct)$ , where we assume that  $\Phi$  will be a positive function and we set hereafter for simplicity  $\tilde{k} = 1$ . Plugging this ansatz in the system (2) yields the following system of advance-delay differential equations

$$(3) \quad \begin{cases} c^2\Phi'' = \Delta_{discr.}(\Phi^p) + (\Psi - \Phi) \\ \nu c^2\Psi'' = -(\Psi - \Phi) \end{cases}$$

where, we have introduced the discrete Laplacian  $\Delta_{discr}f(x) = f(x+1) - 2f(x) + f(x-1)$ .

In order to restate the problem in its equivalent Fourier variable form, we introduce the Fourier transform and its inverse as

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x \xi} dx; \quad f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi)e^{2\pi i x \xi} d\xi$$

The second derivative operator  $\partial_x^2$  has the following representation

$$\widehat{\partial_x^2 f}(\xi) = -4\pi^2 \xi^2 \hat{f}(\xi).$$

whereas

$$\widehat{\Delta_{discr} f}(\xi) = -4 \sin^2(\pi \xi) \hat{f}(\xi).$$

Thus, taking Fourier transform in the second equation of (3) allows us to express

$$(4) \quad \widehat{\Psi}(\xi) = \frac{1}{1 - 4\pi^2 \nu c^2 \xi^2} \widehat{\Phi}(\xi).$$

Plugging this last formula in the first equation of (3) then yields the following equation for  $\widehat{\Phi}$

$$-4\pi^2 c^2 \xi^2 \widehat{\Phi}(\xi) = -4 \sin^2(\pi \xi) \widehat{\Phi^p}(\xi) + \frac{4\pi^2 \nu c^2 \xi^2}{1 - 4\pi^2 \nu c^2 \xi^2} \widehat{\Phi}(\xi).$$

Solving for  $\widehat{\Phi}$ , we obtain

$$(5) \quad \widehat{\Phi}(\xi) = \frac{1 - 4\pi^2 c^2 \nu \xi^2}{1 + \nu - 4\pi^2 c^2 \nu \xi^2} \frac{\sin^2(\pi \xi)}{c^2 \pi^2 \xi^2} \widehat{\Phi^p}(\xi)$$

**2.2. An integral equation for  $\Phi$ .** With the assignment  $A = A_\nu := \sqrt{1 + \frac{1}{\nu}}$ , we can represent

$$\frac{1 - 4\pi^2 c^2 \nu \xi^2}{1 + \nu - 4\pi^2 c^2 \nu \xi^2} = 1 - \frac{\nu}{1 + \nu - 4\pi^2 c^2 \nu \xi^2} = 1 + \frac{1}{2A} \left( \frac{1}{2\pi c \xi - A} - \frac{1}{2\pi c \xi + A} \right),$$

which allows us to further rewrite (5) as follows

$$(6) \quad \widehat{\Phi}(\xi) = \frac{\sin^2(\pi \xi)}{c^2 \pi^2 \xi^2} \widehat{\Phi^p}(\xi) + \frac{1}{2A} \left( \frac{1}{2\pi c \xi - A} - \frac{1}{2\pi c \xi + A} \right) \frac{\sin^2(\pi \xi)}{c^2 \pi^2 \xi^2} \widehat{\Phi^p}(\xi).$$

Recall that (see [8] and also [30], [31]) taking the ‘‘tent’’ function,  $\Lambda(x) = (1 - |x|)_+$  or

$$\Lambda(x) = \begin{cases} 1 - |x| & |x| \leq 1, \\ 0 & |x| > 1. \end{cases}$$

we have  $\widehat{\Lambda}(\xi) = \frac{\sin^2(\pi \xi)}{\pi^2 \xi^2}$ .

Next, we compute the inverse Fourier transform of  $\frac{1}{2A} \left( \frac{1}{2\pi c \xi - A} - \frac{1}{2\pi c \xi + A} \right)$ . We have

$$\int \frac{1}{2\pi c \xi - A} e^{2\pi i \xi x} d\xi = \frac{e^{i \frac{A}{c} x}}{2\pi c} i \int \frac{\sin(zx)}{z} dz = \frac{e^{i \frac{A}{c} x}}{2c} i \operatorname{sgn}(x),$$

whence

$$\mathcal{F}^{-1} \left[ \frac{1}{2A} \left( \frac{1}{2\pi c \xi - A} - \frac{1}{2\pi c \xi + A} \right) \right] = -\frac{1}{2Ac} \sin \left( \frac{A}{c} x \right) \operatorname{sgn}(x).$$

Thus, taking inverse Fourier transform in (6), we obtain the analogous to [8] (see also [12]) representation for the solution of the form:

$$(7) \quad c^2 \Phi = \Lambda * \Phi^p - \frac{1}{2Ac} \sin \left( \frac{A}{c} x \right) \operatorname{sgn}(x) * \Lambda * \Phi^p.$$

The next task is to compute the kernel

$$-\frac{1}{2Ac} \sin \left( \frac{A}{c} x \right) \operatorname{sgn}(x) * \Lambda.$$

Clearly, this is an even function, being the convolution of two even functions. Thus, we only need to compute it for  $x > 0$  and then we can take an even extension across zero. A direct computation shows

$$-\frac{1}{2Ac} \sin \left( \frac{A}{c} x \right) \operatorname{sgn}(x) * \Lambda = \begin{cases} \frac{x-1}{A^2} + \frac{c}{A^3} \left( \sin\left(\frac{A}{c}\right) \cos\left(\frac{A}{c}x\right) - \sin\left(\frac{A}{c}x\right) \right) & x \in (0, 1). \\ \frac{c \sin\left(\frac{A}{c}\right) (\cos\left(\frac{A}{c}\right) - 1)}{A^3} & x \geq 1 \end{cases}$$

At this point, we introduce a few more notations in order to rewrite (7) in more compact form. Namely, let

$$\mu := \frac{A}{c} > 0, \quad G(x) = G_{\mu,A}(x) := \begin{cases} \frac{\sin(\mu) \cos(\mu x) - \sin(\mu x) \operatorname{sgn}(x)}{\sin(\mu x) (\cos(\mu) - 1)} & x \in (-1, 1). \\ \frac{\sin(\mu x) (\cos(\mu) - 1)}{\mu} \operatorname{sgn}(x) & |x| \geq 1. \end{cases}$$

Note that the function  $G$  is compactly supported only if  $\mu = 2n\pi$ .

We can now rewrite (7) in the form

$$(8) \quad A^2 c^2 \Phi = (A^2 - 1) \Lambda * \Phi^p + G * \Phi^p.$$

For positive solutions of (8), we can take the transformation  $Z = \Phi^p$ , which leads us to

$$(9) \quad A^2 c^2 Z^{1/p} = (A^2 - 1) \Lambda * Z + G * Z = ((A^2 - 1) \Lambda + G) * Z.$$

Denote

$$K(x) := (A^2 - 1) \Lambda + G(x).$$

Note that  $K = K(A, c; x)$  and the problem (11) now reads

$$(10) \quad A^2 c^2 Z^{1/p} = K_{A,c} * Z.$$

**2.3. The anti-resonance condition.** Hereafter, we restrict ourselves to the case of compactly supported kernel  $K$  in the convolution problem (9). That is, we require

$$(11) \quad \mu = 2n\pi, n \in \mathbf{N}$$

in order to achieve that the function  $G$  (and hence  $K$ ) is supported in  $(-1, 1)$ . We refer to (11) as the **anti-resonance condition** for the parameters. Notice that this is a condition that physically connects the mass of the resonator (or effectively the ratio of its mass to that of the principal bead) to the speed of the wave. We can then obtain the following

**Lemma 1.** *Let  $n \in \mathbf{N}, \mu = 2\pi n$ . There exists an irrational number  $A_0 = 1.10328\dots$ , so that for  $A \geq A_0$ , the kernel*

$$K_{A,c}(x) = (A^2 - 1) \Lambda(x) - \frac{\sin(\mu x)}{\mu} \operatorname{sgn}(x) \chi_{(-1,1)}(x),$$

is positive in  $(0, 1)$ . For  $A > \sqrt{2}$ , the function  $K$  is decreasing in  $(0, 1)$ .

*Proof.* Since  $K$  is obviously an even function, it suffices to consider the case  $0 \leq x \leq 1$ . Taking the derivative of  $K$ , we see that

$$K'(x) = -\cos(\mu x) - (A^2 - 1) \leq 1 - (A^2 - 1) \leq 2 - A^2 \leq 0$$

so long as  $A \geq \sqrt{2}$ . Thus,  $K$  is decreasing in  $(0, 1)$  for  $A \geq \sqrt{2}$ .

Next, we study the positivity of  $K$ . Again, it suffices to consider  $x \in (0, 1)$ . We have

$$K(x) = (A^2 - 1)(1 - x) - \frac{\sin(\mu x)}{\mu} = (1 - x) \left[ A^2 - 1 - \frac{\sin(\mu x)}{\mu(1 - x)} \right].$$

Since  $\mu = 2n\pi$ , we can rewrite the last expression as follows

$$K(x) = (1 - x) \left[ A^2 - 1 + \frac{\sin(2\pi n(1 - x))}{2\pi n(1 - x)} \right].$$

Now, since

$$-a_0 = \inf_{z \in (0, 2\pi)} \frac{\sin z}{z} = \inf_{z \in (0, \infty)} \frac{\sin z}{z} = -0.217234\dots,$$

we conclude that  $K(x) \geq 0$ , provided  $A \geq A_0 = \sqrt{1 + a_0} = 1.10328\dots$

□

We can now proceed to establish our main result.

### 3. MAIN RESULT: BELL-SHAPED TRAVELING WAVES PERSIST UNDER SUITABLE ANTI-RESONANCE CONDITIONS

We have the following main result.

**Theorem 1.** *Let  $\frac{A}{c} = 2n\pi$  for some integer  $n = 1, 2, \dots$*

*Then, for  $A \geq \sqrt{2}$ , the equation (10) (and hence (5)) has bell-shaped solution  $\Phi$ . That is, there is  $\Phi : \mathbf{R} \rightarrow \mathbf{R}_+$  a positive, even,  $C^\infty$  smooth function, so that  $\Phi$  is decaying in  $(0, \infty)$ . In addition,  $\Phi$  is doubly exponentially decaying, just as the travelling waves in the classical monomer case.*

*For  $\sqrt{2} > A \geq A_0 = 1.10328\dots$ , the problem (10) (and hence (5)) has a positive solution  $\Phi$ , which we cannot guarantee to be bell-shaped <sup>1</sup>.*

We perform the proof in several steps, under the different assumptions on  $A$ . Following the idea in [30, 31], we let  $q = 1 + \frac{1}{p}$  and we set up the following maximization problem

$$(12) \quad \begin{cases} \langle K * z, z \rangle \rightarrow \max \\ \int_{\mathbf{R}^1} |z(x)|^q dx = 1 \end{cases}$$

Note that  $q \in (1, 2)$ . The plan of actions is as follows: we first show that (12) has a solution with the desired properties. After that, we derive its Euler-Lagrange equation. This is of course closely related to (10), except for the Lagrange multipliers, which need to be adjusted.

**3.1. Existence for the maximizers.** We will show that the following lemma holds.

**Lemma 2.** *Let  $A \geq A_0$ . Then, the maximization problem (12) has a solution. Moreover, this solution is positive. If in addition  $A \geq \sqrt{2}$ , then it is also bell-shaped.*

First, we show that the quantity  $\langle K * z, z \rangle$  is bounded, if  $z$  satisfies the constraint. Indeed, by Hölder's and Young's inequality<sup>2</sup>

$$|\langle K * z, z \rangle| \leq \|K * z\|_{L^{q'}} \|z\|_{L^q} \leq \|K\|_{L^{\frac{q}{2q-2}}} \|z\|_{L^q}^2,$$

But  $K$  is a bounded function with support in  $(-1, 1)$ , hence  $K \in L^{\frac{q}{2q-2}}$ . Next, we show that an eventual solution to (12) is necessarily positive. Indeed, for any function  $z$ , we have that the function  $w := |z|$  satisfies the constraint and moreover

$$\langle K * z, z \rangle = \int \int K(x-y) z(x) z(y) dx dy \leq \int \int K(x-y) w(x) w(y) dx dy = \langle K * w, w \rangle.$$

<sup>1</sup>The latter feature will, however, be illustrated in our case example numerical computations of the next section.

<sup>2</sup>Note that since  $q = 1 + \frac{1}{p} < 2$ , we have that  $\frac{q}{2q-2} > 1$ .

Thus, the supremum in (12) may be taken over all  $z \geq 0$ . In fact, one can see that if  $z$  is not non-negative, then  $w = |z|$  provides a bigger value and hence  $z$  may not be a solution to (12). Denote

$$J^{\max} = \sup_{\|z\|_{L^q}=1} \langle K * z, z \rangle.$$

Clearly,  $J^{\max} > 0$ . Pick a maximizing sequence, say  $z_n : z_n \geq 0$ . That is  $\|z_n\|_{L^q} = 1$  and

$$\langle K * z_n, z_n \rangle \rightarrow J^{\max}.$$

We will show that (an appropriate translate of)  $z_n$  converges strongly (in  $L^q$ ) to a solution  $z_0$ . To that end, we apply the concentration compactness method of Lions, [23]. The outcome is that (after we pick a subsequence, which we call again  $z_n$ ), one of three scenarios occur:

- (tightness) There exists  $y_k \in \mathbf{R}^1$ , so that for each  $\epsilon > 0$ , there is  $R = R(\epsilon)$ , so that

$$\int_{y_k-R}^{y_k+R} z_k^q(x) dx > 1 - \epsilon$$

- (vanishing) For every  $R > 0$ , there is

$$\limsup_{k \rightarrow \infty} \sup_y \int_{y-R}^{y+R} z_k^q(x) dx = 0.$$

- (dichotomy) There exists  $\alpha \in (0, 1)$ , so that for any  $\epsilon > 0$ , there are  $R = R(\epsilon)$  and  $R_k \rightarrow \infty$ ,  $y_k \in \mathbf{R}^1$ , so that for all large enough  $k$ ,

$$\left| \int_{y_k-R}^{y_k+R} z_k^q(x) dx - \alpha \right| < \epsilon^q, \quad \left| \int_{|y-y_k|>R_k} z_k^q(x) dx - (1-\alpha) \right| < \epsilon^q$$

We will proceed to show that vanishing and dichotomy may not occur, which will leave us with tightness. In the tightness scenario, we will easily show that a translate of  $z_k$  will converge strongly to  $z_0$ .

**Vanishing does not occur:**

Assume that it does. Let  $\epsilon > 0$ ,  $R = 10$  and  $k_0 = k_0(\epsilon)$  is so large that

$$(13) \quad \sup_y \int_{y-10}^{y+10} z_k^q(x) dx < \epsilon^q.$$

for all  $k > k_0$ . We have for each  $y_0 \in \mathbf{R}^1$ ,

$$\int_{y_0-5}^{y_0+5} K * z_k(y) z_k(y) dy \leq \|z_k\|_{L^q(y_0-5, y_0+5)} \|K * z_k\|_{L^{q'}(y_0-5, y_0+5)}.$$

Since  $\text{supp}K \subset (-1, 1)$ , and the integration is over  $(y_0 - 5, y_0 + 5)$ , it follows that

$$\chi_{(y_0-5, y_0+5)} K * z_k(y) = \chi_{(y_0-5, y_0+5)} K * [z_k \chi_{(y_0-6, y_0+6)}].$$

We have by Young's inequality that

$$\|K * z_k\|_{L^{q'}(y_0-5, y_0+5)} \leq \|K\|_{L^{\frac{q}{2q-2}}} \|z_k\|_{L^q(y_0-6, y_0+6)}.$$



It follows that

$$\int_{y_0-5}^{y_0+5} K * z_k(y) z_k(y) dy \leq C \|z_k\|_{L^q(y_0-10, y_0+10)}^2 \leq C \epsilon^{2-q} \|z_k\|_{L^q(y_0-10, y_0+10)}^q,$$

where in the last inequality, we have used (13). Summing the last inequality over  $y_0 = 0, \pm 1, \dots$  yields

$$\langle K * z_k, z_k \rangle \leq \sum_{y_0 \in \mathcal{Z}} \int_{y_0-5}^{y_0+5} K * z_k(y) z_k(y) dy \leq C_1 \epsilon^{2-q} \|z_k\|_{L^q} = C_1 \epsilon^{2-q},$$

for all large enough  $k$  and for all  $\epsilon > 0$ . On the other hand  $\langle K * z_k, z_k \rangle \rightarrow J^{\max}$ , which will be a contradiction, if we have had selected  $\epsilon$  small enough. Note that again, we have used  $q < 2$ . Thus, vanishing may not occur.

**Dichotomy does not occur:**

Assume that it does. Let  $\epsilon > 0$  and denote

$$z_k^1(y) := z_k(y) \chi_{(y_k-R, y_k+R)}(y); \quad z_k^2(y) = z_k(y) \chi_{|y-y_k| > R_k}(y),$$

so that  $\|z_k - z_k^1 - z_k^2\|_{L^q} = O(\epsilon)$ . From support considerations and the Young's estimates that we have provided earlier, it is clear that for all large enough  $k$

$$\langle K * z_k, z_k \rangle = \langle K * z_k^1, z_k^1 \rangle + \langle K * z_k^2, z_k^2 \rangle + O(\epsilon).$$

Now, from the definition of  $J^{\max}$

$$\langle K * z_k^1, z_k^1 \rangle = \|z_k^1\|_q^2 \langle K * \frac{z_k^1}{\|z_k^1\|_q}, \frac{z_k^1}{\|z_k^1\|_q} \rangle \leq J^{\max} \|z_k^1\|_q^2 = J^{\max} (\alpha^2 + O(\epsilon^q)).$$

Similarly,

$$\langle K * z_k^2, z_k^2 \rangle \leq J^{\max} ((1 - \alpha)^2 + O(\epsilon^q)).$$

Putting everything together yields

$$\langle K * z_k, z_k \rangle \leq J^{\max} (\alpha^2 + (1 - \alpha)^2 + O(\epsilon^q)) + O(\epsilon).$$

This is again produces a contradiction (with a judiciously small choice of  $\epsilon$ ), since  $\langle K * z_k, z_k \rangle \rightarrow J^{\max}$ ,  $\alpha^2 + (1 - \alpha)^2 < 1$ . Thus, dichotomy does not occur either.

Thus, tightness is the only alternative. Now, consider  $\tilde{z}_n(y) := z_n(y - y_n)$ . Note that  $\tilde{z}_n$  satisfies the constraint and also  $\langle K * \tilde{z}_n, \tilde{z}_n \rangle = \langle K * z_n, z_n \rangle \rightarrow J^{\max}$ . We have that for all  $\epsilon > 0$ , there exists  $R = R(\epsilon)$ , so that

$$(14) \quad \int_{-R}^R \tilde{z}_n^q(y) dy > 1 - \epsilon.$$

for all  $n$ . Now, by weak compactness in  $L^q$ , it follows that  $\tilde{z}_n$  has a weakly convergent subsequence (denoted again by  $\tilde{z}_n$  for convenience), with a limit say  $z_0$ .

We now claim that the sequence  $\{K * \tilde{z}_n\}_n$  is strongly pre-compact in  $L^{q'}$ . Indeed, by the Young's inequality, we have for all  $n$

$$\|K * \tilde{z}_n\|_{W^{1, q'}} \leq \|K * \tilde{z}_n\|_{L^{q'}} + \|K' * \tilde{z}_n\|_{L^{q'}} \leq C \|K\|_{W^{1, \frac{q}{2q-2}}} \|z_n\|_{L^q} = C \|K\|_{W^{1, \frac{q}{2q-2}}},$$

where by inspection  $K \in W^{1, \frac{q}{2q-2}}$ . Also

$$\int_{|y| > R+1} |K * \tilde{z}_n(y)|^{q'} dy \leq \|K\|_{L^{\frac{q}{2q-2}}}^{q'} \|\tilde{z}_n\|_{L^q(|y| > R)}^{q'} \leq C\epsilon^{q'/q}.$$

By the compactness criteria in  $L^{q'}$  spaces (i.e. the Riesz-Tamarkin condition), it follows that  $\{K * \tilde{z}_n\}_n$  is pre-compact. This, together with the pointwise limit  $K * \tilde{z}_n \rightarrow K * z_0$ , which follows from the weak convergence of  $\tilde{z}_k$  implies that for some subsequence

$$\|K * \tilde{z}_{n_l} - K * z_0\|_{L^{q'}} \rightarrow 0.$$

We now have by Hölder's

$$\begin{aligned} |\langle K * \tilde{z}_{n_l}, \tilde{z}_{n_l} \rangle - \langle K * z_0, z_0 \rangle| &\leq |\langle K * \tilde{z}_{n_l} - K * z_0, \tilde{z}_{n_l} \rangle| + |\langle z_0, K * \tilde{z}_{n_l} - K * z_0 \rangle| \\ &\leq \|K * \tilde{z}_{n_l} - K * z_0\|_{L^{q'}} (\|z_0\|_{L^q} + \|\tilde{z}_{n_l}\|_{L^q}) \rightarrow 0. \end{aligned}$$

Thus,

$$\langle K * z_0, z_0 \rangle = J^{\max}.$$

On the other hand, by the lower semicontinuity of the norm with respect to the weak convergence  $\|z_0\|_{L^q} \leq 1$ . But then  $\|z_0\|_{L^q} = 1$ , since otherwise

$$J^{\max} = \langle K * z_0, z_0 \rangle = \|z_0\|_{L^q}^2 \langle K * \frac{z_0}{\|z_0\|_{L^q}}, \frac{z_0}{\|z_0\|_{L^q}} \rangle \leq J^{\max} \|z_0\|_{L^q}^2 < J^{\max},$$

a contradiction. Thus, we have shown that the limit  $z_0$  is indeed a solution to the maximization problem (12).

**3.2. Bell-shapedness of the solution in the case  $A \geq \sqrt{2}$ .** In this case, we have that the kernel  $K$  is bell-shaped. We show now that the solution  $z_0$  is in addition bell-shaped. In order to explain the setup, we need a few definitions.

For a function  $f : \mathbf{R}^1 \rightarrow \mathbf{R}^1$ , one defines the distribution function

$$d_f(\alpha) := \text{meas}\{x : |f(x)| > \alpha\}.$$

Clearly, the function  $d_f(\alpha)$  is non-increasing, whence one can define its “inverse” as follows

$$f^*(t) = \inf\{s : d_f(s) \leq t\}.$$

We call  $f^* : \mathbf{R}_+^1 \rightarrow \mathbf{R}_+^1$  the *non-increasing rearrangement* of  $f$ . Note that the two functions have the same distribution function, that is  $d_f(\alpha) = d_{f^*}(\alpha)$ , so in particular  $\|f\|_{L^p} = \|f^*\|_{L^p} : 0 < p \leq \infty$ . Finally, define the even function  $f^\#(t) = f^*(2|t|)$ , which also satisfies  $\|f\|_{L^p} = \|f^\#\|_{L^p} : 0 < p \leq \infty$ . Clearly, a function is bell-shaped if and only if  $f^\# = f$ . In this setting, we have the Riesz rearrangement inequality<sup>3</sup>, [4], , which states

$$\int_{\mathbf{R}^1} \int_{\mathbf{R}^1} f(x-y)g(y)h(x)dx dy \leq \int_{\mathbf{R}^1} \int_{\mathbf{R}^1} f^\#(x-y)g^\#(y)h^\#(x)dx dy.$$

Using that fact and since  $K$  is bell-shaped, it is clear that in the maximization problem (12), we can restrict our attention to bell-shaped entries  $z$ . Indeed, taking an arbitrary

<sup>3</sup>The original inequality appeared implicitly in the work of Riesz.

function  $z$ , note that  $z^\#$  would satisfy the constraint  $\|z^\#\|_{L^q} = \|z\|_{L^q} = 1$  and moreover, by the Riesz rearrangement inequality

$$\langle K * z, z \rangle = \int_{\mathbf{R}^2} K(x-y)z(y)z(x)dydx \leq \int_{\mathbf{R}^2} K(x-y)z^\#(y)z^\#(x)dx dy = \langle K * z^\#, z^\# \rangle.$$

Based on the above formulation, one may derive the existence of  $z_0$  the same way as before. Note however that one may completely circumvent the Lions concentration compactness arguments in the bell-shaped case. Indeed, assuming that  $z$  is bell-shaped and satisfies the constraint  $\|z\|_{L^q} = 1$ , we have for every  $x > 0$ ,

$$|z(x)|^q x \leq \int_0^x |z(y)|^q dy \leq \int_0^\infty |z(y)|^q dy = 1/2.$$

Hence  $|z(x)| \leq 2^{-q}x^{-1/q}$ . Thus, any maximizing sequence  $\{z_n\}$  of bell-shaped functions will produce a pre-compact in  $L^{q'}$  sequence  $K * z_n$ . Indeed, we have, similarly to the arguments above that  $K * z_n \in W^{1,q'}$ . In addition, by support considerations for any  $R > 10$ , we have for every  $n$

$$\int_{|x|>R} |K * z_n(x)|^{q'} dx \leq C \int_{|x|>R-1} |z(x)|^{q'} dx \leq C \int_{|x|>R-1} \frac{1}{|x|^{q'/q}} dx \leq CR^{1-\frac{q'}{q}} \rightarrow 0,$$

as  $R \rightarrow \infty$ , since  $q < 2 < q'$ . Thus, the elementary pointwise bounds obtained from bell-shapedness help us verify immediately the Riesz-Tamarkin criteria and hence the pre-compactness of  $K * z_n$  in  $L^{q'}$ , without having to resort to the Lions' theory. After that, we finish by standard arguments as before.

**3.3. The Euler-Lagrange equation for (12).** The next step is to derive the Euler-Lagrange equation that the solution  $z_0$  of (12) satisfies. We proceed as follows. Take  $\epsilon \in \mathbf{R}^1$  and a test function  $h$ . Since  $z_0$  is a maximizer for (12), it must be that

$$\left\langle K * \frac{z_0 + \epsilon h}{\|z_0 + \epsilon h\|_{L^q}}, \frac{z_0 + \epsilon h}{\|z_0 + \epsilon h\|_{L^q}} \right\rangle \leq J^{\max}$$

for all  $\epsilon, h$ . We can rewrite this as follows

$$(15) \quad \langle K * (z_0 + \epsilon h), z_0 + \epsilon h \rangle \leq J^{\max} \|z_0 + \epsilon h\|_{L^q}^2$$

Taking Taylor expansions in  $\epsilon$ , we find

$$\begin{aligned} \langle K * (z_0 + \epsilon h), z_0 + \epsilon h \rangle &= \langle K * z_0, z_0 \rangle + 2\epsilon \langle K * z_0, h \rangle + O(\epsilon^2) = \\ &= J^{\max} + 2\epsilon \langle K * z_0, h \rangle + O(\epsilon^2), \\ \|z_0 + \epsilon h\|_{L^q}^2 &= (1 + \epsilon q \int z_0^{q-1}(x)h(x)dx + O(\epsilon^2))^2 = 1 + 2\epsilon q \langle z_0^{q-1}, h \rangle + O(\epsilon^2). \end{aligned}$$

Plugging in this in (15), we obtain

$$2\langle K * z_0, h \rangle = 2J^{\max} q \langle z_0^{q-1}, h \rangle + O(\epsilon),$$

which is valid for all  $\epsilon, h$ . Taking limit as  $\epsilon \rightarrow 0$ , we obtain

$$\langle K * z_0 - qz_0^{q-1}, h \rangle = 0.$$

Since this is satisfied for all test functions  $h$  and  $q - 1 = \frac{1}{p}$ , we obtain  $K * z_0 = qz_0^{\frac{1}{p}}$ . If we now take  $z_0 = \mu Z$ ,

$$K * Z = q\mu^{\frac{1}{p}-1} Z^{\frac{1}{p}}.$$

Clearly, with an appropriate choice of  $\mu$ , we can arrange so that  $Z$  indeed satisfies (10).

We now proceed to show that  $Z$  and  $\Phi = Z^p$  are both  $C^\infty$ . First, let us see that from the equation (10), the function  $Z$  never vanishes. Indeed, assume the opposite,  $Z(x_0) = 0$ . Then,

$$0 = A^2 c^2 Z^{1/p}(x_0) = \int_{-1}^1 K(y) Z(x-y) dy.$$

Since  $K(y) > 0$  in  $(-1, 1)$ , it would follow that  $Z|_{(x_0-1, x_0+1)} = 0$ . Iterating this argument yields that  $Z(x) = 0$  for all  $x$ , a contradiction. Now, we have from (10) that  $\Phi$  is differentiable and moreover

$$A^2 c^2 \Phi' = K' * \Phi^p.$$

It follows that  $\Phi'$  is continuous. By induction, we can show

$$A^2 c^2 \Phi^{(l)} = K' * (\Phi^p)^{(l-1)},$$

and hence  $\Phi^{(l)}$  is continuous, since  $\Phi^{(j)}$ ,  $j < l$  are all continuous and  $\Phi \neq 0$ . For integer values of  $p$ , this is obvious. Even for non-integer values of  $p$ ,  $\Phi^p$  is as smooth as  $\Phi$ , since  $\Phi(x) > 0$ .

Finally, let us establish the double exponential rate of decay for the bell-shaped solutions of (5). So, assuming  $A \geq \sqrt{2}$ , we have that  $\Phi$  is bell-shaped and  $K$  is decaying for  $x > 0$ . We have

$$A^2 c^2 \Phi(x) = \int_{x-1}^{x+1} K(y) \Phi^p(x-y) dy.$$

Denote  $\tilde{\Phi} := \left( \frac{2\|K\|_{L^\infty}}{A^2 c^2} \right)^{\frac{1}{p-1}} \Phi$ , so that

$$\tilde{\Phi}(x) = \frac{1}{2\|K\|_{L^\infty}} \int_{x-1}^{x+1} K(y) \tilde{\Phi}^p(x-y) dy.$$

Using the bell-shapedness of both  $K$  and  $\Phi$  we conclude

$$\tilde{\Phi}(x) \leq \tilde{\Phi}^p(x-1).$$

Iterating this inequality for  $n < x$ ,

$$\tilde{\Phi}(x) \leq \tilde{\Phi}^{p^n}(x-n).$$

which yields the double exponential rate of decay for  $\tilde{\Phi}$  and hence for  $\Phi$ . This completes the proof of Theorem 1.

In order to corroborate the above conclusions, as well as to examine the cases which do not adhere to the conditions of the above theorem (both cases of anti-resonance, but with  $A < \sqrt{2}$ , as well as cases where the anti-resonance condition is not satisfied), we turn to some case example numerical computations in the next section.

## 4. NUMERICAL RESULTS

4.1. **Anti-resonance scenario** ( $\mu = 2\pi n$ ). When  $\mu = 2\pi n$ ,  $n \in \mathbb{N}$ , we obtained the expression of kernel  $K_{A,c}(x)$  (as shown in Lemma 1) in the convolution equation:

$$A^2 c^2 \Phi = K_{A,c} * (\Phi)^p.$$

When a solution of  $\Phi$  is obtained, one can similarly find  $\Psi$  from:

$$(16) \quad \widehat{\Psi}(\xi) = \frac{1}{1 + \nu - 4\pi^2 c^2 \nu \xi^2} \frac{\sin^2(\pi \xi)}{c^2 \pi^2 \xi^2} \widehat{\Phi^p}(\xi)$$

and its corresponding integral equation

$$(17) \quad \Psi = M_{A,c} * \Phi^p$$

where

$$M_{A,c}(x) = \frac{1}{c^2 \nu A^2} \left[ (1 - |x|) + \frac{\sin(\mu x)}{\mu} \operatorname{sgn}(x) \right] \chi_{(-1,1)}(x).$$

Using the above expressions, the solution for both the bead and the resonator strains can be not only theoretically analyzed but also numerically computed.

Due to the similarity between  $M_{A,c}$  and  $K_{A,c}$ , we will mainly discuss the properties of  $K_{A,c}$  and those of  $M_{A,c}$  directly follow. Since  $K_{A,c}$  is an even function and has finite support  $(-1, 1)$ , we only need to consider its properties on  $(0, 1)$ . Noting that  $K'_{A,c}(0^+) = K'_{A,c}(1^-) = -A^2$ ,  $K'_{A,c}(0^-) = K'_{A,c}(-1^+) = A^2$ , we appreciated that  $K_{A,c}$  is continuous everywhere but not continuously differentiable at  $x = -1, 0, 1$ . We now separate the different cases of interest.

- (1) The case analyzed fully in the previous section (i.e., the “best case scenario”): when  $A > \sqrt{2}$ ,  $K'_{A,c}(x) < 0$  for  $x \in (0, 1)$  so that  $K_{A,c}$  is increasing on  $[-1, 0]$  and decreasing on  $[0, 1]$ . This regime also implies the positivity of  $K_{A,c}$  since  $K_{A,c}(-1) = K_{A,c}(1) = 0$ . Due to these good features of  $K_{A,c}(x)$ , and as a by-product of the proof of the previous section, our numerical computations confirm that the system bears bell-shaped solutions for  $\Phi$  and  $\Psi$ , as illustrated in the typical results shown in Fig. 1.
- (2) The “intermediate” case: when  $1.10328 \approx A_0 < A < \sqrt{2}$ ,  $K_{A,c}(x) \geq 0$  still holds for all  $x$  but the kernel is no longer monotone on  $(-1, 0)$  or  $(0, 1)$ . Although in this category  $K_{A,c}(x)$  has non-monotonic variations (with the number of local minima being  $n$ ), the corresponding solution of  $\Phi$  still features a bell-shaped profile as it did in the previous case (see Fig. 2 for such examples in the case of different anti-resonances). The difference between the (non-monotonic) kernel and the (monotonic) solution is, apparently, caused by the smoothing effect of convolution.
- (3) Turning now to cases for which our analytical arguments can no longer be used (even in a qualitative way), we first consider:  $1 < A_{1,n} < A < A_0$ . Here,  $K_{A,c}(x)$  is neither necessarily positive, nor decreasing on  $(0, 1)$ . However, as Fig. 3 suggests, we notice from the numerical results that a bell-shaped solution of  $\Phi$  and  $\Psi$  could still exist as long as  $A$  is not too small, i.e.  $A > A_{1,n} > 1$ . Here  $A_{1,n}$  decreases over  $n$  and it is the threshold above which our numerical method is able to converge to a solution, as shown in the right panel of Fig. 4.

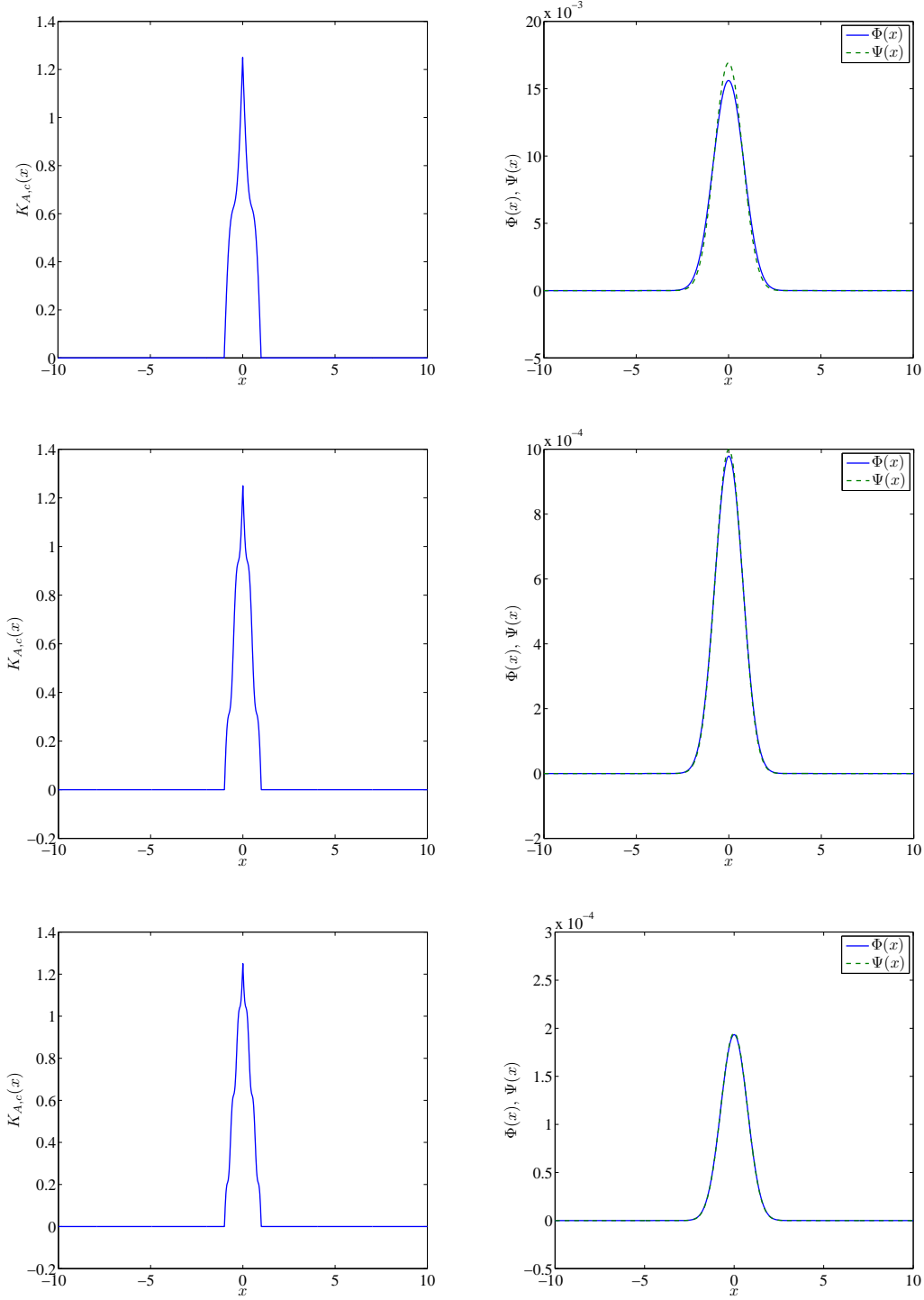


FIGURE 1. The top panels show the kernel of the convolution equation  $K_{A,c}(x)$  and corresponding solution  $\Phi$  and  $\Psi$  with  $A = 1.5$  and  $\mu = 2\pi$ . In the middle panels and bottom panels,  $\mu$  is set as  $4\pi$  and  $6\pi$ , respectively, corresponding to higher anti-resonances.

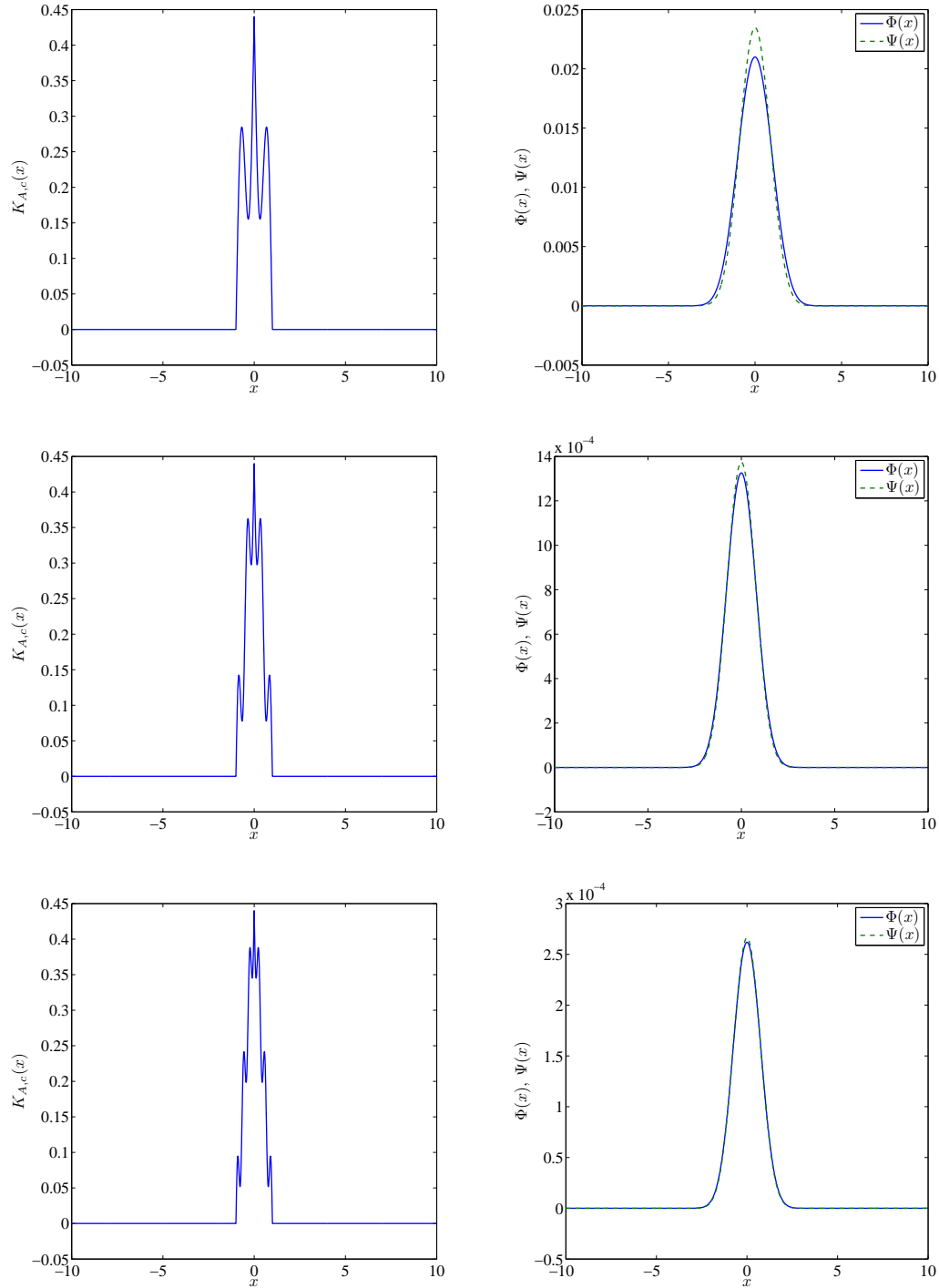


FIGURE 2. This figure is structured similarly as Fig. 1 but  $A$  has changed to 1.2 instead of 1.5 here. The top panels show the kernel of the convolution equation  $K_{A,c}(x)$  and corresponding solution  $\Phi$  and  $\Psi$  with  $A = 1.2$  and  $\mu = 2\pi$ . In the middle panels and bottom panels,  $\mu$  is set as  $4\pi$  and  $6\pi$ , respectively.

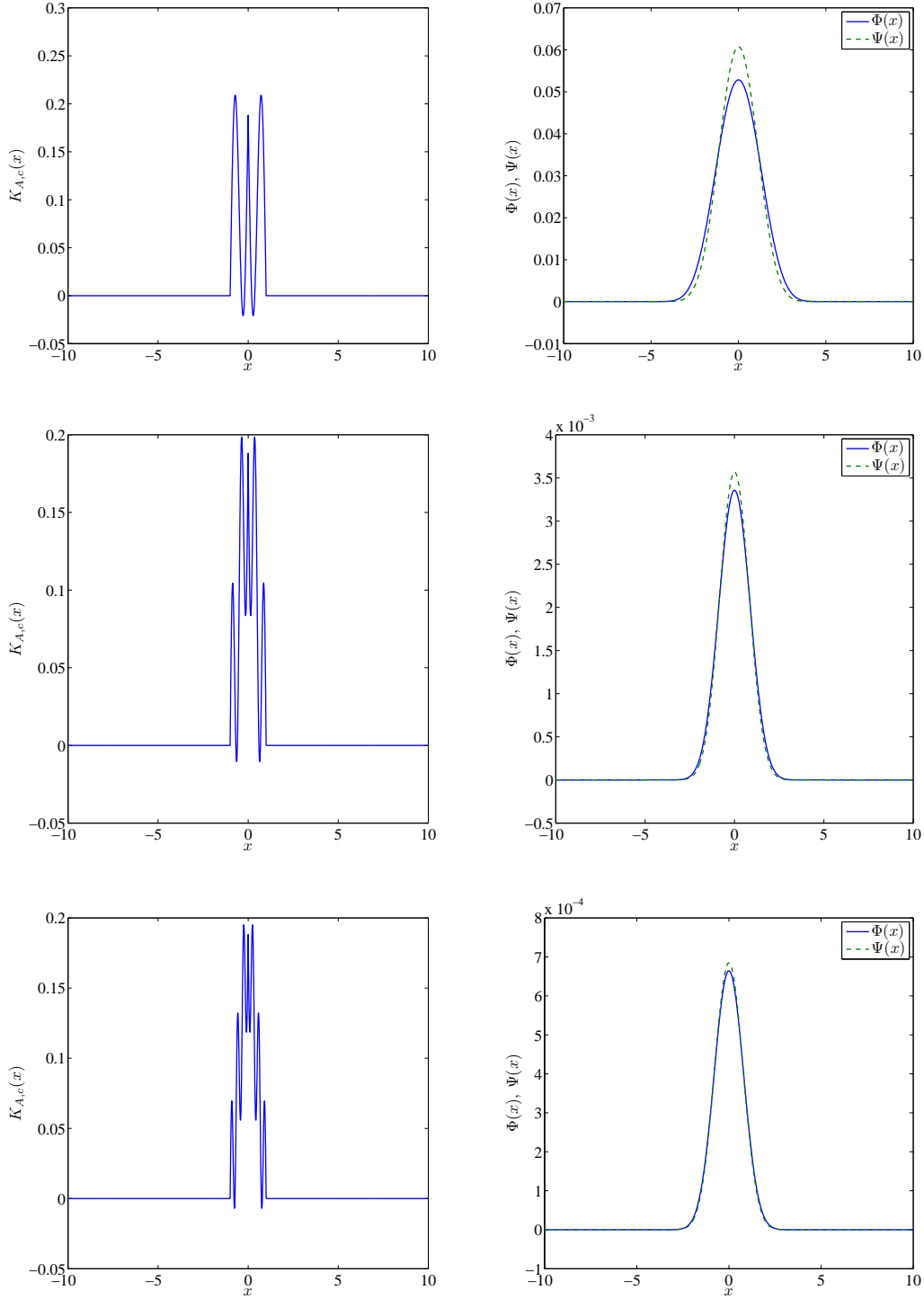


FIGURE 3. This figure is similar to the above figures and the only difference is that  $A = 1.09$  here. The top panels show the kernel of the convolution equation  $K_{A,c}(x)$  and corresponding solution  $\Phi$  and  $\Psi$  with  $A = 1.09$  and  $\mu = 2\pi$ . In the middle panels and bottom panels,  $\mu$  is set as  $4\pi$  and  $6\pi$ , respectively.



- (4) When  $1 < A < A_{1,n}$ , the substantial intervals of negative values of  $K_{A,c}(x)$  lead our numerical scheme to failure of convergence to any solution for  $\Phi$  and  $\Psi$ . The plot of a typical example of  $K_{A,c}(x)$  within this category is given in the left panel of Fig. 4. The dependence of  $A_{1,n}$  on  $n$  is shown in the right panel of the figure, as indicated above.

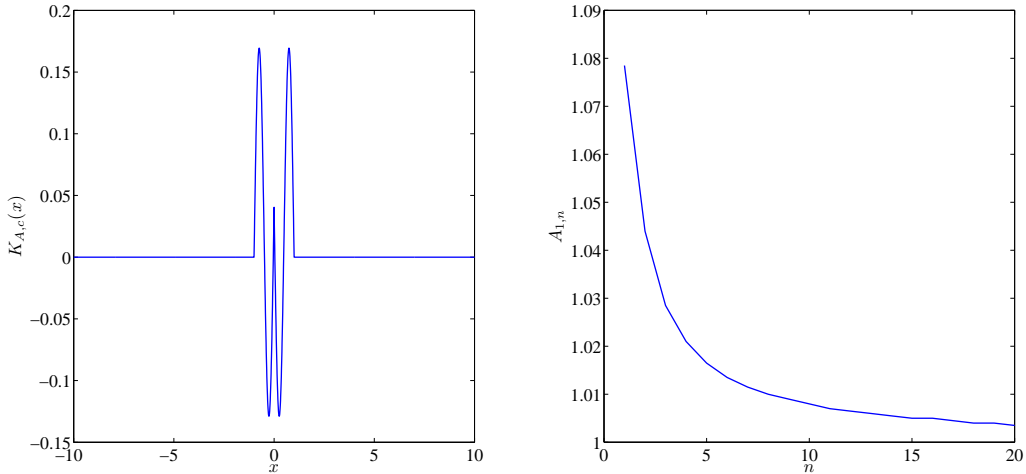


FIGURE 4. The left panels shows the kernel of the convolution equation  $K_{A,c}(x)$  with  $A = 1.02$  and  $\mu = 2\pi$  while how  $A_{1,n}$  decreases over  $n$  is revealed in the right panel.

- (5) Resonance case with  $\mu \neq 2\pi n$ . In this case, the conditions for calculating the Fourier transform on  $\Phi$  and  $\Psi$  fail. By utilizing Fourier series instead (on a finite computational interval), we obtain that  $K_{A,c} = (A^2 - 1) \max(1 - |x|, 0) + G(x)$  has non-decaying oscillatory tails on both wings. Moreover, despite the fact that  $K_{A,c}$  is neither increasing nor non-negative on  $(-\infty, 0)$ , we are able to obtain numerical solutions for  $\Phi$  and  $\Psi$  from either the convolution equations (i.e. the integral equations) or the system of advance-delay differential equations, as shown in Fig. 5. It is interesting that the solution also has a bell shape in the center, but it possesses oscillatory tails on both wings. The computations of this resonant case and their subtleties (from a numerical perspective), as well as the different methods utilized to obtain the solutions are analyzed in detail elsewhere [33]. The rigorous analysis which is the main emphasis of the present work cannot, unfortunately, presently provide definitive insights about the latter case. This remains an important open problem for future study.

## 5. CONCLUSIONS & FUTURE CHALLENGES

In the present work, we have explored systems in the form of Mass in Mass or Mass with Mass dynamical lattices possessing an internal resonator, studying their traveling waves. In particular, we could explore the case of so-called anti-resonances whereby for

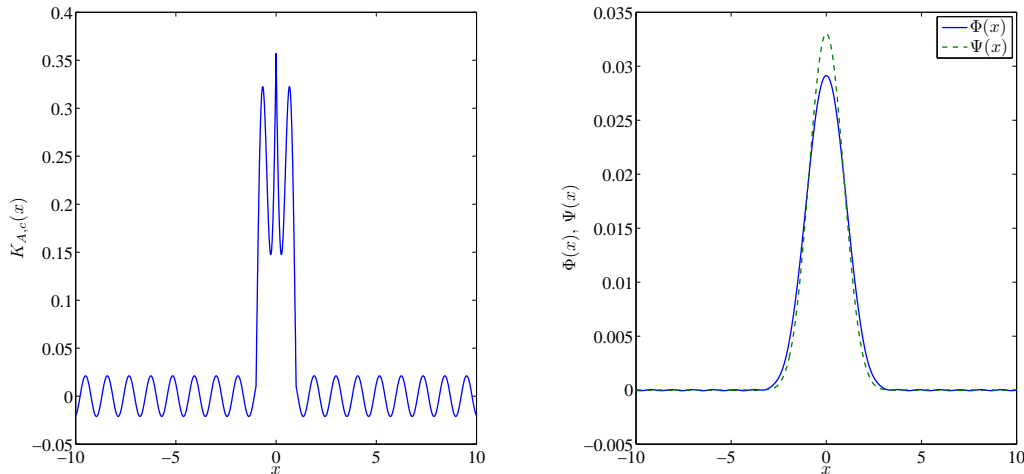


FIGURE 5. The left panel shows the kernel of the convolution equation  $K_{A,c}(x)$  and the right shows the corresponding solution  $\Phi$  and  $\Psi$  with  $A = 1.2$  and  $\mu = 2\pi - 0.5$ .

a particular set of “quantized” relations between the resonator mass and the wave speed (and for suitably small resonator masses), a bell-shaped traveling wave could be rigorously proven to exist. Interestingly, the bell-shaped waves were numerically found to exist when the anti-resonance condition applies, even beyond the mass threshold for which our proof holds. However, a secondary threshold was identified for sufficiently large resonator masses, beyond which our iterative scheme for the numerical identification of the waves no longer converged. Finally, when the anti-resonance condition is not upheld, typical findings suggest the existence of a wave of non-vanishing tail, in line also with recent experimental observations [18].

Naturally, many directions of future research emerge from the present (and recent) considerations of this novel class of systems. From a rigorous perspective, understanding the phenomenon of resonances and the formation of traveling waves with tails would be especially interesting. Equally interesting from an experimental perspective appears to be the actual experimental setting whereby an initial excitation of the chain, by construction, produces such tails which are arising only on one wing but not the other. Additionally, as indicated in [18], a distinct experimental possibility is that of bearing multiple resonators rather than one in the context of the woodpile configuration. Understanding how one vs. more such resonators may affect the observed phenomenology is another important direction for both theoretical and experimental future studies.

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