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Jennie D'Ambroise  
*Amherst College*

Panayotis G. Kevrekidis  
*University of Massachusetts Amherst, kevrekid@math.umass.edu*

B. A. Malomed  
*Tel Aviv University*

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Staggered parity-time-symmetric ladders with cubic nonlinearity

Jennie D’Ambroise, P. G. Kevrekidis, and Boris A. Malomed

1Department of Mathematics and Statistics, Amherst College, Amherst, Massachusetts 01002-5000, USA
2Department of Mathematics and Statistics, University of Massachusetts, Amherst, Massachusetts 01003-9305, USA
3Department of Physical Electronics, School of Electrical Engineering, Faculty of Engineering, Tel Aviv University, Tel Aviv 69978, Israel

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We introduce a ladder-shaped chain with each rung carrying a parity-time- (\(PT\)-) symmetric gain-loss dimer. The polarity of the dimers is staggered along the chain, meaning alternation of gain-loss and loss-gain runs. This structure, which can be implemented as an optical waveguide array, is the simplest one which renders the system \(PT\)-symmetric in both horizontal and vertical directions. The system is governed by a pair of linearly coupled discrete nonlinear Schrödinger equations with self-focusing or defocusing cubic onsite nonlinearity. Starting from the analytically tractable anticontinuum limit of uncoupled rungs and using the Newton’s method for continuation of the solutions with the increase of the inter-rung coupling, we construct families of \(PT\)-symmetric discrete solitons and identify their stability regions. Waveforms stemming from a single excited rung and double ones are identified. Dynamics of unstable solitons is investigated too.

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I. INTRODUCTION

A vast research area, often called discrete nonlinear optics, deals with evanescently coupled arrayed waveguides featuring material nonlinearity [1]. Discrete arrays of optical waveguides have drawn a great deal of interest not only because they introduce a vast phenomenology of the nonlinear light propagation, such as, e.g., the prediction [2] and experimental creation [3] of discrete vortex solitons, but also due to the fact that they offer a unique platform for emulating the transmission of electric signals in solid-state devices, which is obviously interesting for both fundamental studies and applications [1,4]. Furthermore, the flexibility of techniques used for the creation of virtual (photoinduced) [5] and permanently written [6] guiding arrays enables the exploration of effects which can be difficult to directly observe in other physical settings, such as Anderson localization [7].

Another field in which arrays of quasicrystalline waveguides find a natural application is the realization of the optical \(PT\) (parity-time) symmetry [8]. On the one hand, a pair of coupled nonlinear waveguides, which carry mutually balanced gain and loss, make it possible to realize \(PT\)-symmetric spatial or temporal solitons (if the waveguides are planar ones or fibers, respectively), which admit an exact analytical solution, including their stability analysis [9]. On the other hand, a \(PT\)-symmetric dimer, i.e., the balanced pair of gain and loss nodes, can be embedded, as a defect, into a regular guiding array, with the objective to study the scattering of incident waves on the dimer [10,11,13]. We note here in passing that sometimes, also the term “dipoles” may be used for describing such dimers; however, we will not make use of it here, to avoid an overlap in terminology with classical dipoles in electrodynamics as discussed, e.g., in [12]. Discrete solitons pinned to a nonlinear \(PT\) -symmetric defect have been reported too [13]. Such systems, although governed by discrete nonlinear Schrödinger (DNLS) equations corresponding to non-Hermitian Hamiltonians, may generate real eigenvalue spectra (at the linear level), provided that the gain-loss strength does not exceed a critical value, above which the \(PT\) symmetry is broken [14] (self-defocusing nonlinearity with the local strength growing, in a one-dimensional (1D) system, from the center to periphery at any rate faster that the distance from the center, gives rise to stable fundamental and higher-order solitons with unbreakable \(PT\)-symmetry [15]).

One- and two-dimensional (1D and 2D) lattices, built of \(PT\) dimers, were introduced in Refs. [16,17] and [18], respectively. Discrete solitons, both quiescent and moving ones, were found in these systems [16,18]. In the continuum limit, those solitons go over into those in the above-mentioned \(PT\)-symmetric coupler [9]. Accordingly, a part of the soliton family is stable, and another part is unstable. Pairs of parallel and antiparallel coupled dimers, in the form of \(PT\)-symmetric plaquettes (which may be further used as building blocks for 2D chains), were investigated too [19,20].

The objective of this work is to introduce a staggered chain of \(PT\)-symmetric dimers, with the orientations of the dimers alternating between adjacent sites of the chain. This can also be thought of as an extension of a plaquette from Refs. [19,20] towards a lattice. While this ladder-structured lattice is not a full 2D one, it belongs to a class of chain systems which may be considered as 1.5D models [21].

As shown in Sec. II, where the model is introduced, the fundamental difference from the previously studied ones is the fact that such a system, although being nearly one dimensional, actually realizes the \(PT\) symmetry in the 2D form, with respect to both horizontal and vertical directions. In Sec. III, we start the analysis from the solvable anticontinuum limit (ACL) [22], in which the rungs of the ladder are uncoupled (in the opposite continuum limit, the ladder degenerates into a single NLS equation). Using parametric continuation from this limit makes it possible to construct families of discrete solitons in a numerical form. Such solution branches are initiated, in the ACL, by a single excited rung, as well as by the excitation confined to several rungs. The soliton stability is systematically analyzed in Sec. III too and, if the modes are identified as unstable, their evolution is examined to observe the instability development. The paper is concluded by Sec. IV, where also some directions for future study are presented.
**II. THE MODEL**

We consider the ladder configuration governed by the DNLS system with intersite coupling constant $C$,

\[
\begin{align*}
\frac{i}{\hbar} & \frac{d\Psi_n}{dt} + \frac{C}{2} (\Phi_{n+1} + \Phi_{n-1} - 2\Psi_n) + \sigma |\Psi_n|^2 \Psi_n = i\gamma\Psi_n - \kappa \Phi_n, \\
\frac{i}{\hbar} & \frac{d\Phi_n}{dt} + \frac{C}{2} (\Psi_{n+1} + \Psi_{n-1} - 2\Phi_n) + \sigma |\Phi_n|^2 \Phi_n = -i\gamma\Phi_n - \kappa \Psi_n,
\end{align*}
\]

(1)

where evolution variable $i$ is the propagation distance, in terms of the optical realizations. Coefficients $+i\gamma$ and $-i\gamma$ with $\gamma > 0$ represent $\mathcal{PT}$-symmetric gain-loss dimers, whose orientation is staggered (alternates) along the ladder, the sites carrying gain and loss being represented by amplitudes $\Psi_n(t)$ and $\Phi_n(t)$, respectively. Cubic nonlinearity with coefficient $\sigma$ is present at every site, and $\kappa > 0$ accounts for the vertical coupling along the ladder’s rungs, each representing a $\mathcal{PT}$-symmetric dimer. The system is displayed in Fig. 1. As seen in the figure, the nearly 1D ladder realizes the $\mathcal{PT}$-symmetry in the 2D form, with respect to the horizontal axis, running between the top and bottom strings, and, simultaneously, with respect to any vertical axis drawn between rungs. By means of obvious rescaling, we can fix $|\sigma| = 1$, hence, the nonlinearity coefficient takes only two distinct values, which correspond, respectively, to the self-focusing and defocusing onsite nonlinearity, $\sigma = +1$ and $-1$. The usual DNLS equation admits the sign reversal of $\sigma$ by means of the well-known staggering transformation [22]. However, once we fix $\gamma > 0$ (and also $\kappa > 0$) in Eq. (1), this transformation cannot be applied, as it would also invert the signs of $\gamma$ and $\kappa$.

The single self-consistent continuum limit of system (1), corresponding to $C \rightarrow \infty$, is possible for the fields related by $\Psi = e^{i\delta} \Phi$, with phase shift $\delta = \gamma/C$. Replacing, in this limit, the finite-difference derivative by the one with respect to the continuous coordinate $x = n/\sqrt{C}$ yields the standard NLS equation

\[
\begin{align*}
\frac{i}{\hbar} & \frac{d\Psi}{dt} + \frac{\partial^2 \Psi}{\partial x^2} + \sigma |\Psi|^2 \Psi = -\kappa \Psi.
\end{align*}
\]

(2)

Given its “standard” nature, leading to a full mutual cancellation of the gain and loss terms, we will not pursue this limit further. Instead, as shown in the following, we will use as a natural starting point for examining nontrivial localized modes in the discrete system (1) the opposite ACL, which corresponds to $C \rightarrow 0$, i.e., the set of uncoupled rungs.

Stationary solutions to Eqs. (1) with real propagation constant $\Lambda$ are sought in the usual form $\Psi_n = e^{i\Lambda t} u_n$ and $\Phi_n = e^{i\Lambda t} v_n$, where functions $u_n$ and $v_n$ obey the stationary equations

\[
\begin{align*}
-\Lambda u_n + \frac{C}{2} (u_{n+1} + u_{n-1} - 2u_n) + \sigma |u_n|^2 u_n &= i\gamma u_n - \kappa v_n, \\
-\Lambda v_n + \frac{C}{2} (v_{n+1} + v_{n-1} - 2v_n) + \sigma |v_n|^2 v_n &= -i\gamma v_n - \kappa u_n.
\end{align*}
\]

(3)

Numerical solutions of these equations for discrete solitons are produced in the next section. To analyze the stability of the solutions, we add perturbations with an infinitesimal amplitude $\epsilon$ and frequencies $\omega$:

\[
\begin{align*}
\Psi_n(t) &= [u_n + \epsilon(a_n e^{i\omega t} + b_n e^{-i\omega t})] e^{i\Lambda t}, \\
\Phi_n(t) &= [v_n + \epsilon(c_n e^{i\omega t} + d_n e^{-i\omega t})] e^{i\Lambda t}.
\end{align*}
\]

(4)

The linearization of Eq. (1) with respect to the small perturbations leads to the eigenvalue problem

\[
\begin{align*}
M \begin{bmatrix} a_n \\
b^*_n \\
c_n \\
d^*_n \end{bmatrix} &= \omega \begin{bmatrix} a_n \\
b^*_n \\
c_n \\
d^*_n \end{bmatrix},
\end{align*}
\]

(5)

where $M$ is a $4N \times 4N$ matrix for the ladder of length $N$. Using standard indexing, $N \times N$ submatrices of $M$ are defined as

\[
\begin{align*}
M_{11} &= \text{diag}(p_n^* - \Lambda - C), \\
M_{12} &= -M_{21} = \text{diag}(\sigma u_n^2), \\
M_{13} &= -M_{31} = \text{diag}(\sigma v_n^2), \\
M_{14} &= -M_{41} = \text{diag}(\sigma u_n^2), \\
M_{22} &= \text{diag}(\Lambda + C - p_n), \\
M_{23} &= \text{diag}(q_n - \Lambda - C), \\
M_{24} &= \text{diag}(\Lambda + C - q_n), \\
M_{32} &= -M_{23} = \text{diag}(\sigma u_n^2), \\
M_{33} &= -M_{34} = \text{diag}(\sigma v_n^2), \\
M_{34} &= -M_{43} = \text{diag}(\sigma v_n^2), \\
M_{42} &= -M_{24} = -M_{42} = \frac{C}{2} G + \text{diag}(\kappa),
\end{align*}
\]

(6)

where $G$ is an $N \times N$ matrix of zero elements, except for the superdiagonals and subdiagonals that contain all ones.

For the zero solution of the stationary equation (3), $u_n = v_n = 0$, matrix $M$ has constant coefficients, hence perturbation eigenmodes can be sought for as $a_n = Ae^{i\kappa n}$, $b_n = 0$, $c_n = Be^{i\kappa n}$, $d_n = 0$. Then, Eq. (5) becomes a $2 \times 2$ system, whose eigenvalues can be found explicitly:

\[
\omega = -\left(\Lambda + C\right) \pm \sqrt{\left(C \cos k + \kappa^2\right)^2 - \gamma^2},
\]

(8)

so that $\omega$ is real only for $C \leq \kappa - \gamma$. In other words, the $\mathcal{PT}$ symmetry is broken, with $i\omega$ acquiring a positive real part, which drives the exponential growth of the perturbations, at $\gamma > \gamma_{\text{crit}}(C) \equiv \kappa - C$.

(9)
It is interesting to observe here that the coupling between the rungs decreases the size of the interval of the unbroken $PT$ symmetry of the single dimer [8,14].

In the stability region, Eq. (8) demonstrates that real perturbation frequencies take values in the following intervals:

$$-(\Lambda + C) - \sqrt{(k + C)^2 - \gamma^2} < \omega$$

$$< -(\Lambda + C) - \sqrt{(k - C)^2 - \gamma^2},$$

$$-(\Lambda + C) + \sqrt{(k - C)^2 - \gamma^2} < \omega$$

$$< -(\Lambda + C) + \sqrt{(k + C)^2 - \gamma^2}. \quad (10)$$

Similarly, for the perturbations in the form of $a_n = 0$, $b_n = A_0 e^{i\kappa n}$, $c_n = 0$, $d_n = B_0 e^{i\kappa n}$ the negatives of expressions (8) are also eigenvalues of the zero stationary solution, and at $\gamma < \gamma^{(1)}_C(C = 0)$, they fall into the negatives of intervals (10).

Simultaneously, Eq. (8) and its negative counterpart give the dispersion relation for plane waves ("phonons") in the linearized version of Eq. (1). Accordingly, intervals (10), along with their negative counterparts, represent phonon bands of the linearized system.

In Sec. III, we produce stationary solutions in the form of discrete solitons. This computation begins by finding exact solutions for the ACL, $C = 0$, and then continuing the solutions numerically to $C > 0$, by means of the Newton’s method for each $C$ (i.e., utilizing the converging solution obtained for a previous value of $C$ as an initial seed for the Newton’s algorithm with $C \to C + \Delta C$). As suggested by Eq. (9), we restrict the analysis to $0 < \gamma < \kappa$, so as to remain within the $PT$-symmetric region for $C = 0$. Subsequently, the stability interval of the so constructed solutions is identified, in a numerical form too.

III. DISCRETE SOLITONS AND THEIR STABILITY

A. Anticontinuum limit (ACL) $C = 0$

To construct stationary localized solutions of Eqs. (1) at $C = 0$, when individual rungs are decoupled, we substitute

$$u_n = e^{i\kappa n} v_n \quad (11)$$

with real $\delta_n$ in Eq. (3), which yields relations

$$\gamma = -\kappa \sin \delta_n, \quad \sigma |v_n|^2 = -\kappa \cos \delta_n + \Lambda. \quad (12)$$

For the uncoupled ladder, one can specify either a single-rung solution, with fields at all sites set equal to zero except for $u_1$ satisfying Eq. (12), or a double-rung solution with nonzero fields $u_1$ and $u_2$ satisfying the same equations. We focus on these two possibilities in the ACL (although larger-size solutions are obviously possible too). These are the direct counterparts of the single-node and two-node solutions that have been extensively studied in 1D and 2D DNLS models [22].

We take parameters satisfying constraints

$$\sigma > 0, \quad \Lambda > \kappa \quad (13)$$

to make the second equation (12) self-consistent. Then, two solution branches for $\delta_n$ are possible. The first branch satisfies $-\pi/2 \leq \delta_{in} \equiv \arcsin(-\gamma/\kappa) \leq 0$ and $\cos(\delta_{in}) > 0$. Choosing a solution with $\delta_n = \delta_{in}$ in the rung carrying nonzero fields, we name it an in-phase rung, as the phase shift between the gain and loss poles of the respective dimer is smaller than $\pi/2$, namely, $|\arg uu^*| \in [0, \pi/2]$. The second branch satisfies $-\pi \leq \delta_{out} \equiv \arcsin(-\gamma/\kappa) \leq -\pi/2$ and $\cos(\delta_{out}) \leq 0$. The rung carrying the solution with $\delta_n = \delta_{out}$ is called an out-of-phase one, as the respective phase shift between the elements exceeds $\pi/2$, viz., $|\arg uu^*| \in [\pi/2, \pi]$. The two branches meet and disappear at $\gamma = \kappa$, when $\delta_{in} = \delta_{out} = -\pi/2$. Recall that $\gamma = \kappa = \gamma^{(1)}_C(C = 0)$ [see Eq. (9)] is the boundary of the $PT$-symmetric region for $C = 0$. These branches can be also considered as stemming from the Hamiltonian limit of $\gamma = 0$, where $\delta_{in}$ and $\delta_{out} = \pi$ correspond, respectively, to the usual definitions of the in- and out-of-phase Hamiltonian dimers.

The stability eigenfrequencies for stationary solitons at $C = 0$ can be readily calculated analytically in the ACL [14]. In this case, $M$ has the same eigenvalues as submatrices

$$m_0 = \begin{pmatrix} -\Lambda - i\gamma & 0 & \kappa & 0 \\ 0 & -\Lambda - i\gamma & 0 & -\kappa \\ \kappa & 0 & -\Lambda + i\gamma & 0 \\ 0 & -\kappa & 0 & -\Lambda + i\gamma \end{pmatrix},$$

which is associated with zero-amplitude (unexcited) rungs, and

$$m_n = \begin{pmatrix} -\Lambda + p_n^* & \sigma u_n^2 & \kappa & 0 \\ -\sigma(u_n)^2 & -\Lambda + p_n & 0 & -\kappa \\ \kappa & 0 & -\Lambda + q_n & \sigma v_n^2 \\ 0 & -\kappa & -\sigma(u_n)^2 & -\Lambda + q_n \end{pmatrix},$$

associated with the excited ones, which carry nonzero stationary fields, with $v_n, u_n$ taken as per Eqs. (11) and (12). In other words, each of the four eigenvalues of $m_0$,

$$\omega = \pm \Lambda \pm \sqrt{\kappa^2 - \gamma^2}, \quad (14)$$

is an eigenvalue of $M$ with multiplicity equal to the number of zero-amplitude rungs, while each of the four eigenvalues of $m_n$,

$$\omega = \pm 0, \pm 2\Lambda \sqrt{2\sigma^2 - \alpha_e}, \quad (15)$$

appears as an eigenvalue of $M$ with multiplicity equal to the number of excited rungs. Here, $\alpha_e = \alpha_{in} = \kappa \cos(\delta_{in})/\Lambda \equiv \sqrt{(\kappa^2 - \gamma^2)/\Lambda^2}$, and $\alpha_u = \alpha_{out} = \kappa \cos(\delta_{out})/\Lambda \equiv -\sqrt{(\kappa^2 - \gamma^2)/\Lambda^2}$ for an in- and out-of-phase rung, respectively.

Equation (15) shows that the out-of-phase excited rung is always stable, as it has $\Re(\omega) = 0$. Similarly, the in-phase excited rung is stable for $\kappa^2 - \gamma^2 > \Lambda^2/4$, and unstable for $0 < \kappa^2 - \gamma^2 < \Lambda^2/4$. Thus, for solutions that contain an excited in-phase rung in the initial configuration at $C = 0$, there are the two critical values, viz., $\gamma^{(1)}_C(C = 0) = \kappa$ given by Eq. (9), and the additional one, which designates the instability area for the uncoupled in-phase rungs:

$$\gamma > \gamma^{(2)}_C(C = 0) = \sqrt{\kappa^2 - \Lambda^2/4}. \quad (16)$$

A choice alternative to Eq. (13) is

$$\sigma < 0, \quad \Lambda < -\kappa. \quad (17)$$

In this case, the analysis differs only in that the sign of $\alpha_e$ in Eq. (15) is switched. That is, the in-phase rung
is now associated to negative $\alpha_s = \alpha_{in} = \kappa \cos(\delta_m)/\Lambda = -\sqrt{(\kappa^2 - \gamma^2)/\Lambda^2}$, while the out-of-phase one to positive $\alpha_s = \alpha_{out} = \kappa \cos(\delta_m)/\Lambda \equiv \sqrt{(\kappa^2 - \gamma^2)/\Lambda^2}$. In this case, the in-phase rung is always stable, while its out-of-phase counterpart is unstable at $\gamma > \gamma_\ast^\square(C = 0)$ [see Eq. (16)].

**B. Discrete solitons at $C > 0$**

To construct soliton solutions for coupling constant $C$ increasing in steps of $\Delta C$, we write Eq. (3) as a system of $4N$ equations for $4N$ real unknowns $w_n, x_n, y_n, z_n$, with $u_n \equiv w_n + i x_n, v_n \equiv y_n + i z_n$. Then, we apply the Newton’s method with the initial guess at each step taken as the soliton solution found at the previous value of $C$, as mentioned above. Thus, the initial guess at $C = \Delta C$ is the analytical solution for $C = 0$ given by Eqs. (11) and (12) with parameters taken according to either Eq. (13) or (17).

Figure 2 shows $|u_n|^2$ for the solutions identified by this process on a (base 10) logarithmic scale as a function of $C$ for parameters taken as per Eq. (13). The logarithmic scale is chosen, as it yields a clearer picture of the variation of the solution’s spatial width, as $C$ varies. The different solutions displayed in Fig. 2 include those seeded by the single excited in- and out-of-phase rungs (the top row), and two-rung excitations for which there are three possibilities: in-phase in

![FIG. 2.](image)

![FIG. 3.](image)

![FIG. 4.](image)
sites. The individual phases of a phase difference that develops around the initially excited spatial expansion (across rungs) respective diagnostic, 

\[ w(C) = \sqrt{\sum_n n^2 |u_n|^2 / \sum_n |u_n|^2} \]  

versus \( C \) for the solutions shown in Fig. 2. It is relevant to point out that the variation of this width-measuring quantity is fairly weak in the case of the out-of-phase solutions and mixed ones, while it is more significant in the case of the single and double in-phase excited rungs.

In Fig. 5, the absolute value of the phase difference between fields \( u_n \) and \( v_n \) at two sides of the ladder is shown. In other words, this figure shows whether each rung of the ladder belongs to the in-phase or out-of-phase type, as a function of \( C \). This figure reveals that, as \( C \) increases, there is a progressive spatial expansion (across \( n \)) in the number of sites supporting a phase difference that develops around the initially excited sites. The individual phases of \( u_n \) and \( v_n \) are shown in Fig. 6.

We show in the bottom two plots of Fig. 6 that two different types of phase profiles can arise: one type has phase with the same sign on both the left and right sides of the outer portion of the ladder, and the second type has phases that are of opposite sign on the left and right sides of the outer portion of the ladder. We address this point more in the next section where we discuss stability.

Figures 7–11 are similar to their counterparts 2–6, respectively, but with the parameters taken as per Eq. (17) instead of Eq. (13). Comparing Figs. 5 and 10 shows that \( \sigma = +1 \) favors the solutions with in-phase rungs as \( C \) increases, while \( \sigma = -1 \) favors the out-of-phase rungs. In other words, the progressively expanding soliton keeps the in- and out-of-phase structures, in the case of the self-focusing (\( \sigma = +1 \)) and defocusing (\( \sigma = -1 \)) onsite nonlinearity, respectively, in agreement with the well-known principle that discrete solitons feature a staggered pattern in the case of the self-defocusing [22]. Also, according to Eq. (12), for \( \sigma = -1 \) the asymmetry of the mixed-phase solution is switched in comparison to the \( \sigma = +1 \) case, lending the in-phase rung a larger magnitude of the fields than in the out-of-phase one.

It is relevant to stress that the discrete solitons seeded in the ACL by double rungs feature a bidimer structure which does not carry a topological charge [20], i.e., the solitons cannot take the form of vortices, according to our numerical results (contrary to what is the case, e.g., for a ring containing a single \( PT \)-symmetric dipole [24]).

**C. Stability of the discrete solitons**

Figure 12 shows two-parameter stability diagrams for the solitons by plotting the largest instability growth rate (if different from zero) \( \max[\text{Re}(i\omega)] \) as a function of \( C \) and \( \gamma \) for parameter values taken per Eq. (13), and Fig. 13 shows the same as per Eq. (17). The respective stability boundaries are
FIG. 7. (Color online) The same as Fig. 2, but for $\sigma = -1$. Common parameters are $\kappa = 2$ and $\gamma = 1$. The initial values of $\delta_1$ and $\delta_2$ at $C = 0$ follow the pattern of Fig. 2. Other parameters are $\Lambda = 5$, $N = 80$, $\Delta C = 0.001$ on the top left, $\Lambda = 3.5$, $N = 80$, $\Delta C = 0.001$ on the top right, $\Lambda = 5$, $N = 80$, $\Delta C = 0.001$ on the middle left, $\Lambda = 3.5$, $N = 40$, $\Delta C = 0.001$ on the middle right, and finally $\Lambda = 3.085$, $N = 40$, $\Delta C = 0.0001$ on the bottom center. As $C$ increases, small amplitudes appear at adjacent sites, and the soliton gains width, as shown by means of $w$ in Fig. 9. The stability of the solitons shown here is predicted by the eigenvalue plots in Fig. 13 at $\gamma = 1$.

shown by green lines (white, in the black-and-white version of the figures). Some comments are relevant here. Recall that Eqs. (9) and (16) impose stability limitations, respectively, from the point of view of the zero-background solution in the former case, and the single-site excitation in the latter case. The former background-stability condition indicates that the line of $\gamma = \kappa - C$ (parallel to the antidiagonal cyan line in Fig. 12) poses an upper bound on the potential stability core part of the solution are possible too, and, as observed in these panels, they somewhat deform the resultant stability region. The additional instabilities stemming from the excited in-phase rungs in Fig. 12, and their out-of-phase counterparts in Fig. 13, are separately observed in the left panels of the

FIG. 8. (Color online) Profiles of the discrete solitons at $C = 0.4$, for $\sigma = -1$. The configurations of the initial $C = 0$ solution and parameters follow the same pattern as in Fig. 7.

FIG. 9. (Color online) The width diagnostic of the discrete solitons, defined as per Eq. (18), corresponding to each of the plots in Fig. 7. constructed. It can be seen in both Figs. 12 and 13, especially in the right panels of the former and left panels of the latter [where the instability defined by Eq. (16) is less relevant], that the background-instability threshold given by Eq. (9) is an essential stability boundary for the family of the discrete solitons. Of course, additional instabilities due to the localized core part of the solution are possible too, and, as observed in these panels, they somewhat deform the resultant stability region. The additional instabilities stemming from the excited

FIG. 10. (Color online) The same as Fig. 5 but for $\sigma = -1$, i.e., for the discrete solitons presented in Fig. 7. For the top left and middle left plots, the soliton’s field is nonzero at one or two in-phase rung(s) when $C = 0$, and as $C$ increases these rungs remain in phase, while all the others are out of phase. For the top right and middle right plots, the soliton’s field at $C = 0$ is nonzero and out of phase at one or two central rungs, and all rungs remain out of phase with the increase of $C$. In the bottom plot, only the $n = 1$ rung remains in phase, while all others are out of phase.
former figure and right panels of the latter one. Given that this critical point was found in the framework of the ACL, it features no $C$ dependence, but it clearly contributes to delimiting the stability boundaries of the discrete solitons; sometimes, this effect is fairly dramatic, as in the middle-row left and right panels of Figs. 12 and 13, respectively, i.e., the two-site, same-phase excitations may be susceptible to this instability mechanism. Although the precise stability thresholds may be fairly complex, arising from the interplay of localized and extended modes in the nonlinear ladder system, a general conclusion is that the above-mentioned instabilities play a critical role for the stability of the localized states in this system (see also the discussion below). Another essential conclusion is that the higher the coupling ($C$), the less robust the corresponding solutions are likely to be, the destabilization caused by the increase of $C$ being sometimes fairly dramatic.

The values of $i\omega$ whose maximum real part is represented in Figs. 12 and 13 were computed with the help of an appropriate numerical eigenvalue solver. At $C = 0$, the eigenvalues agree with Eqs. (14) and (15). As shown in Figs. 14 and 15, following the variation of $C$ and $\gamma$, eigenvalues (14), associated with the empty (zero-value) sites, vary in accordance with the prediction of Eq. (8), and eigenvalues (15), associated with excited rungs, also shift in the complex plane upon variation of $C, \gamma$. In the case of the mixed-phase solutions with asymmetric amplitude (seen in the bottom-most plot of Fig. 3), there is a stable region for low values of the parameters $C$ and $\gamma$. For larger values of $\gamma$, there are parametric intervals (across $C > 0$ for fixed $\gamma$) in which discrete solitons with phase profiles different from those initialized in the ACL of $C = 0$ have been identified; see the bottom two plots in Fig. 6. These distinct branches of the unstable solutions give rise to “streaks” observed in the bottom middle panel of Fig. 12. The amplitude profiles of such alternate solutions are similar to those shown in the bottom plot of Fig. 3, and the gain in width function defined in (18) as a function of $C$ is similar to the examples shown in the bottom plots of Figs. 2 and 4. Mechanisms by which solutions become unstable for these alternate solutions are outlined below.

The most obvious type of the instability is associated with initializing a solution at $C = 0$ from a single unstable rung, i.e., at $\gamma > \gamma_{\text{cr}}^2(C = 0)$ in (16). Eigenvalues for this type of the instability are shown in the top two panels of Fig. 15. There are three other scenarios of destabilization of the discrete solitons with the increase of $C$, each corresponding to a particular type of a critical point (transition to the instability). These transitions are demonstrated in Figs. 14 and 15. The first type occurs when eigenvalue $\omega$ associated with an excited rung collides with one of the intervals in Eq. (10). This weak instability generates an eigenfrequency quartet and is represented in Figs. 12 and 13, where the green boundary deviates (as $C$ increases from 0) from the threshold given by Eq. (16). Figures 14 and 15 illustrate this type of transition in more detail by plotting the eigenvalues directly in the complex plane.
The second type of the transition occurs when the intervals in Eq. (10) come to overlap at \( \gamma = \psi_{\text{c1}}(C) \) [see Eq. (9)]. This is the background instability at empty sites, as shown in Figs. 12 and 13 by bright spots originating from the corners of the diagrams, where \( \gamma = \kappa = \psi_{\text{c1}}(C = 0) \). A more detailed plot of these eigenvalues and the corresponding collisions in the complex eigenvalue plane is displayed in Fig. 14.

The third type of the instability onset occurs for essentially all values of \( C \) in the case of two in-phase rungs at \( \sigma > 0 \), or two out-of-phase ones at \( \sigma < 0 \). It may be thought of as a localized instability due to the simultaneous presence of two potentially unstable elements, due to the instability determined by Eq. (16). At \( C > 0 \), it is seen as the bright spots in Figs. 12 and 13 originating from \( \psi_{\text{c2}}(C = 0) = \sqrt{\kappa^2 - \Lambda^2}/4 \). The eigenvalues emerge from the corresponding zero eigenvalues at \( C = 0 \). That is, in the middle-row left plot of Fig. 12 at \( C = 0 \) for \( \gamma < \psi_{\text{c2}}(C = 0) \) there are four zero eigenvalues; as \( C \) increases, two of the four eigenvalues move from zero onto the real axis in the complex plane. A similar effect is observed at \( \gamma < \psi_{\text{c2}}(C = 0) \) in the middle-row right plot of Fig. 13.

Finally, it is worth making one more observation in connection, e.g., to Fig. 15 and the associated jagged lines in the top right panel of Fig. 13. Notice that, as \( C \) increases, initial stabilization of the mode unstable due to the criterion given by Eq. (16) takes place, but then a collision with the continuous spectrum on the imaginary axis provides destabilization anew. It is this cascade of events that accounts for the jaggedness of the curve in the top right of Fig. 13 and in similar occurrences.

![Figure 13](image1.png)

**Fig. 13.** (Color online) The same as in Fig. 12, but for parameter values from Fig. 7. Here, the cyan line is drawn on the top two plots in the left column, and the second cyan dot on the \( C = 0 \) axis appears only in the case where the out-of-phase excited rung is present at \( C = 0 \).

![Figure 14](image2.png)

**Fig. 14.** (Color online) Stability eigenvalues \( i\omega \) in the complex plane, for parameters chosen in accordance with the topmost left panel of Fig. 12 with \( \gamma = 0.5 \). For \( C = 0 \), in the top left plot we show the agreement of the numerically found eigenvalues (blue circles) with results produced by Eqs. (14) (green filled circles) and (15) (red filled). For \( C = 0.3 \), in the top right panel we show that the eigenvalues associated with the zero solution indeed lie within the predicted intervals (10), the boundaries of which are shown by dashed lines. Next, for \( C = 1 \), in the bottom left plot we observe that values of \( i\omega \) associated with the excited state have previously (at smaller \( C \)) merged with the dashed intervals, and now an unstable quartet has emerged from the axis. For \( C = 1.5 \), in the bottom right panel the critical point corresponding to Eq. (9) is represented, where unstable eigenvalues emerge from the axis at the values of \( \pm(\Lambda + C) \), as the intervals in Eq. (10) merge. Comparing plots in the bottom row, we conclude that the critical point of the latter type gives rise, in general, to a stronger instability than the former one.

![Figure 15](image3.png)

**Fig. 15.** (Color online) The same as in Fig. 14, but for parameters chosen in accordance with the top right panel of Fig. 13, with \( \gamma = 1.2 \). At \( C = 0 \), in the top left plot we show the agreement of the numerically found eigenvalues (blue circles) with Eqs. (14) (green filled circles) and (15) (red filled). At \( C = 0.275 \), in the top right we see that eigenvalues associated to the red x’s have moved inward towards zero. Next, for \( C = 0.3 \), in the bottom left panel we observe that, after merging with zero, the eigenvalues now emerge from zero on the imaginary axis. Finally, at \( C = 0.7 \) in the bottom right panel, we observe that, after the eigenvalues merge with the dashed-line intervals, an unstable quartet emerges from the axis.
We demonstrate that this instability leads to the growth of solution amplitudes and oscillations at the central rung. The corresponding (chiefly localized, although with a weakly side associated with gain) eigenmode is delocalized. It is shown in Fig. 17 that the collision of eigenvalues in intervals (10), which are all eigenfrequencies, while the bottom panels show how the evolution of each of the three solutions in the framework of Eq. (1) by means of the standard Runge-Kutta fourth-order integration scheme. In Figs. 16–18, we display examples of the evolution of the discrete soliton whose instability is predicted in the topmost left panel of Fig. 12 for $C = 1.5$ and $\gamma = 0.5$. The complex plane of all the eigenvalues for this solution is shown in the bottom right plot of Fig. 14. The top right plot has the same meaning as in Fig. 16, where here $|a_n|^2, |d_n|^2$ are mostly zero while $|b_n|^2, |c_n|^2$ have nonzero amplitudes. In the course of the evolution, the soliton does not maintain its shape. In particular, the solution profiles at $t = 22$ are shown in the left panel at the bottom bearing the apparent signature of the delocalized, unstable eigenmode; the delocalization is stronger in the $\Psi$ solution associated with gain. Similar to Fig. 16, we plot $D_1(t), D_2(t)$ in the bottom right plot. This plot shows that the central node experiences oscillations similar to Fig. 16, but the oscillatory effect is dominated by the delocalization seen in the bottom left plot, which grows and exceeds past the shorter peaks in the center.

**FIG. 16.** (Color online) The evolution of the soliton whose instability is predicted in the topmost left panel of Fig. 12 for $C = 1$ and $\gamma = 0.5$. The complex plane of all the eigenvalues for this solution is shown in the bottom left plot of Fig. 14. The top right plot shows the squared absolute values of perturbation amplitudes $a_0, c_0$ (higher amplitudes) and $b_0, d_0$ (lower amplitudes), defined in Eq. (4). The top left plot shows the solution at $t = 0$, and the bottom left plot shows the solution at $t = 122$ with $|\Psi_1(t = 122)|^2$ in blue and $|\Phi_1(t = 122)|^2$ in green. In the course of the evolution, the soliton maintains its shape, while the amplitude at the central rung ($n = 1$) grows with oscillations; the growth on the gain side, associated to $\Psi_1$, is ultimately dominant. Quantities $D_1(t) \equiv |\Psi_1(t)|^2 - |\Psi_1(0)|^2$ and $D_2(t) \equiv |\Phi_1(t)|^2 - |\Phi_1(0)|^2$ are shown in the bottom right plot, in order to better demonstrate the growing oscillations.

(e.g., in the top left plot of Fig. 12). We add this explanation to the set of possible instabilities discussed above to explain the complex form of the stability boundaries featured by our two-dimensional plots.

**D. Evolution of discrete solitons**

To verify the above predictions for the stability of the discrete solitons, we simulated evolution of the perturbed solutions in the framework of Eq. (1) by means of the standard Runge-Kutta fourth-order integration scheme. In Figs. 16–18, we display examples of the evolution of each of the three instability types which were identified in Sec. III B.

For the first type, when the instability arises from the collision of eigenvalues associated with the excited and empty rungs, the corresponding unstable eigenmode arises in the form of a quartet of eigenfrequencies. In Fig. 16, we demonstrate that this instability leads to the growth of the solution amplitudes and oscillations at the central rung. The corresponding (chiefly localized, although with a weakly decaying tail) instability eigenvectors are shown in the top panels of the figure, while the bottom panels show how the initial conditions evolve in time through the oscillatory growth, in accordance with the presence of the unstable complex eigenfrequencies.

For the second type of the instability, which arises from the collision of eigenvalues in intervals (10), which are all associated with empty rungs, the corresponding unstable eigenmode is delocalized. It is shown in Fig. 17 that the corresponding unstable soliton does not preserve its shape.

**FIG. 17.** (Color online) The evolution of the discrete soliton whose instability is predicted in the topmost left panel of Fig. 12 for $C = 1.5$ and $\gamma = 0.5$. The complex plane of all the eigenvalues for this solution is shown in the bottom right plot of Fig. 14. The top right plot has the same meaning as in Fig. 16, where here $|a_n|^2, |d_n|^2$ are mostly zero while $|b_n|^2, |c_n|^2$ have nonzero amplitudes. In the course of the evolution, the soliton does not maintain its shape. In particular, the solution profiles at $t = 22$ are shown in the left panel at the bottom bearing the apparent signature of the delocalized, unstable eigenmode; the delocalization is stronger in the $\Psi$ solution associated with gain. Similar to Fig. 16, we plot $D_1(t), D_2(t)$ in the bottom right plot. This plot shows that the central node experiences oscillations similar to Fig. 16, but the oscillatory effect is dominated by the delocalization seen in the bottom left plot, which grows and exceeds past the shorter peaks in the center.

**FIG. 18.** (Color online) The evolution of the solitary wave whose instability is predicted is in the middle left panel of Fig. 12 for $C = 0.5$ and $\gamma = 1.5$. The complex plane including all the eigenvalues for this solution is similar to the top right plot in Fig. 15. The top right plot here shows the squared absolute values of perturbation amplitudes $a_0, b_0$ (lower amplitudes) and $c_0, d_0$ (higher amplitudes) defined in Eq. (4). The top left plot shows the solution at $t = 0$, and the bottom left plot shows the solution at $t = 6.5$ with $|\Psi_0(6.5)|^2$ in blue and $|\Phi_0(6.5)|^2$ in green. In the course of the evolution, the soliton maintains its shape, while the amplitude at the rung $n = 2$ grows with weak oscillations and with higher growth on the $\Psi$ side associated with gain. Quantities $D_1(t) \equiv |\Psi_1(t)|^2 - |\Psi_1(0)|^2$ (blue line), $D_2(t) \equiv |\Phi_1(t)|^2 - |\Phi_1(0)|^2$ (green line), $E_1(t) \equiv |\Psi_2(t)|^2 - |\Psi_2(0)|^2$ (cyan line), and $E_2(t) \equiv |\Phi_2(t)|^2 - |\Phi_2(0)|^2$ (red line) are shown in the bottom right plot to highlight the growth.
Instead, the instability causes delocalization of the solution, which acquires a tail reminiscent of the spatial profile of the corresponding unstable eigenvector.

Lastly, the third type of the instability is shown in Fig. 18. It displays the case of two excited in-phase rungs at \( \sigma = 1 \). Other examples of the same type are similar, e.g., with two out-of-phase excited rungs at \( \sigma = -1 \). The instability has a localized manifestation with the amplitudes growing at the gain nodes of each rung and decaying at the loss ones.

IV. CONCLUSIONS

We have introduced the lattice of the ladder type with staggered pairs of mutually compensated gain and loss elements at each rung, and the usual onsite cubic nonlinearity, self-focusing or defocusing. This nearly-one-dimensional system is the simplest one which features two-dimensional \( \mathcal{PT} \)-symmetry. It may be realized in optics as a waveguide array. We have constructed families of discrete stationary solitons seeded by a single excited rung, or a pair of adjacent ones, in the anticontinuum limit of uncoupled rungs. The seed excitations may have the in-phase or out-of-phase structure in the vertical direction (between the gain and loss poles). The double seed with the in- and out-of-phase structures in the two rungs naturally features an asymmetric amplitude profile. We have identified the stability of the discrete solitons via the calculation of eigenfrequencies for small perturbations, across the system’s parameter space. A part of the soliton families are found to be dynamically stable, while unstable solitons exhibit three distinct scenarios of the evolution. The different scenarios stem, roughly, from interactions of localized modes with extended ones, from extended modes alone, or from localized modes alone.

A natural extension of the work may be the consideration of mobility of kicked discrete solitons in the present ladder system. It may also be interesting to seek nonstationary solitons with periodic intrinsic switching (cf. Ref. [25]). A challenging perspective is the development of a 2D extension of the system. Effectively, this would entail adding further alternating ladder pairs along the transverse direction and examining 2D discrete configurations. It may be relevant in such 2D extensions to consider different lattice settings that support not only solutions in the form of discrete solitary waves, but also ones built as discrete vortices, similarly to what has been earlier done in the DNLS system [22], and recently in another 2D \( \mathcal{PT} \)-symmetric system [18].

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