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Article

## Effective Congruences for Mock Theta Functions

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**Abstract:** Let  $M(q) = \sum c(n)q^n$  be one of Ramanujan's mock theta functions. We establish the existence of infinitely many linear congruences of the form:

$$c(An + B) \equiv 0 \pmod{\ell^j}$$

where  $A$  is a multiple of  $\ell$  and an auxiliary prime,  $p$ . Moreover, we give an effectively computable upper bound on the smallest such  $p$  for which these congruences hold. The effective nature of our results is based on the prior works of Lichtenstein [1] and Treneer [2].

**Keywords:** mock theta functions; congruences; harmonic weak Maass forms

**Classification:** MSC 11P83; 11F37

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## 1. Introduction and Statement of the Results

A *partition* of a positive integer,  $n$ , is a non-increasing sequence of positive integers that sum to  $n$ . Define  $p(n)$  to be the number of partitions of a non-negative integer,  $n$ . Ramanujan [3] proved the linear congruences:

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5} \\ p(7n+5) &\equiv 0 \pmod{7} \\ p(11n+6) &\equiv 0 \pmod{11} \end{aligned}$$

which were later extended by Atkin [4] and Watson [5] to include powers of five, seven and 11. Later, Atkin [6] developed a method to identify congruence modulo larger primes, such as:

$$\begin{aligned} p(17303n+237) &\equiv 0 \pmod{13} \\ p(1977147619n+815655) &\equiv 0 \pmod{19} \\ p(4063467631n+30064597) &\equiv 0 \pmod{31} \end{aligned}$$

Ahlgren and Ono [7–9] have shown that linear congruences for  $p(n)$  exist for all moduli,  $m$ , coprime to six.

These congruences arise from studying the arithmetic properties of the generating function:

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n} = 1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + \dots$$

which can also be written in Eulerian form:

$$\sum_{n=0}^{\infty} p(n)q^n = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q; q)_n^2}$$

where  $(x; q)_n := (1-x)(1-xq)(1-xq^2) \cdots (1-xq^{n-1})$ . By the change of sign,  $(q; q)_n \mapsto (-q; q)_n$ , we obtain one of Ramanujan's third-order mock theta functions:

$$f(q) = \sum_{n=0}^{\infty} a_f(n)q^n := 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(-q; q)_n^2} = 1 + q - 2q^2 + 3q^3 - 3q^4 + 3q^5 + \dots$$

The coefficients,  $a_f(n)$ , of  $f(q)$ , can be used to determine the number of partitions of  $n$  of even rank and of odd rank [10].

The function  $f(q)$  is one of Ramanujan's seventeen original mock theta functions, which are strange  $q$ -series that often have combinatorial interpretations (see [11] for a comprehensive survey of mock theta functions). These functions have been the source of much recent study. In [12–17], congruences for the coefficients of various mock theta functions are established. For example, in their investigation of

strongly unimodal sequences, Bryson, Ono, Pitman and Rhoades [14] prove the existence of congruences for the coefficients of Ramanujan's mock theta function:

$$\Psi(q) = \sum_{n=1}^{\infty} a_{\Psi}(n)q^n := \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q; q^2)_n} = q + q^2 + q^3 + 2q^4 + 2q^5 + \dots$$

In particular, they establish the congruence:

$$a_{\Psi}(11^4 \cdot 5n + 721) \equiv 0 \pmod{5} \quad (1.1)$$

In [17], Waldherr shows that Ramanujan's mock theta function:

$$\omega(q) = \sum_{n=0}^{\infty} a_{\omega}(n)q^n := \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(q; q^2)_n^2} = 1 + 2q^4 + 2q^5 + 2q^6 + \dots$$

satisfies:

$$a_{\omega}(40n + 27) \equiv a_{\omega}(40n + 35) \equiv 0 \pmod{5}$$

Congruences like the examples above have also been proven for other mock theta functions, such as Ramanujan's  $\phi(q)$  function [13,15] and a mock theta function associated with the Mathieu group,  $M_{24}$  [18]. It is natural to ask if a general theory of such congruences exists. In this paper, we build on the approaches of these previous works to establish the existence of linear congruences for all of Ramanujan's mock theta functions.

If  $M(q)$  is one of Ramanujan's mock theta functions, let  $q := e^{2\pi iz}$ , and let  $\delta$  and  $\tau$  be integers for which:

$$F(z) = \sum b(n)q^n := q^{\tau} M(q^{\delta}) \quad (1.2)$$

is the holomorphic part of a weight of  $1/2$  harmonic weak Maass form (to be defined in Section 2). We obtain congruences for the coefficients of  $M(q)$  as in Equation (1.1) by obtaining them for  $F(z)$ .

**Theorem 1.** *Let  $M(q)$  be one of Ramanujan's mock theta functions with  $F(z)$ , as in Equation (1.2). Let  $N$  be the level of  $F$ , and let  $\ell^j$  be a prime power with  $(\ell, N) = 1$ . Then, there is a prime,  $Q$ , and infinitely many primes,  $p$ , such that, for some  $m$ ,  $B \in \mathbb{N}$ , we have:*

$$b(p^4 \ell^m Qn + B) \equiv 0 \pmod{\ell^j}$$

Furthermore, the smallest such  $p$  satisfies  $p \leq C$ , where  $C$  is an effectively computable constant that depends on  $\ell^j$ ,  $N$  and other computable parameters.

**Remarks 1.** 1. Theorem 1 is a special case of Theorem 5 in Section 3, a more general result that applies to a weight of  $1/2$  harmonic Maass forms, whose holomorphic parts have algebraic coefficients and whose non-holomorphic parts are period integrals of a weight of  $3/2$  unary theta series. The next section will set up all the notation and preliminary results to state and prove the general theorem, as well as how Theorem 1 follows from it.

2. Theorem 1 has already been established for a few specific mock theta functions. For example, see [10] for  $f(q)$  and [14] for  $\Psi(q)$ .
3. The other computable parameters will be described toward the end of Section 3. Briefly, they involve computing the level of a certain half-integral weight modular form from the work of Treeneer [2] as well as the order of vanishing at the cusps; the constants from the results of Lichtenstein [1]; and, if we do not assume the Generalized Riemann Hypothesis, the constant of Lagarias, Montgomery and Odlyzko [19].

## 2. Nuts and Bolts

We shall utilize several important concepts from the theory of modular forms and harmonic Maass forms, and in this section, we summarize those topics.

### 2.1. Harmonic Maass Forms

Ramanujan's mock theta functions are essentially the holomorphic parts of a certain weight of  $1/2$  harmonic Maass forms. To begin, we define half integral weight harmonic weak Maass forms. Here, "harmonic" refers to the fact that these functions vanish under the weight,  $k$ , and hyperbolic Laplacian,  $\Delta_k$ ,

$$\Delta_k := -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) \quad (2.1)$$

for  $z = x + iy \in \mathbb{H}$ .

If  $N$  is a positive integer with  $4|N$  and  $\chi$ , a Dirichlet character modulo,  $N$ , a weight of  $k \in \frac{1}{2}\mathbb{Z}$  harmonic weak Maass form for a congruence subgroup,  $\Gamma \in \{\Gamma_1(N), \Gamma_0(N)\}$ , with nebentypus,  $\chi$ , is any smooth function,  $f : \mathbb{H} \rightarrow \mathbb{C}$ , satisfying:

1. For every  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ , we have:

$$f\left(\frac{az+b}{cz+d}\right) = \left(\frac{c}{d}\right)^{2k} \varepsilon_d^{-2k} \chi(d) (cz+d)^k f(z)$$

where:

$$\varepsilon_d := \begin{cases} 1 & \text{if } d \equiv 1 \pmod{4} \\ i & \text{if } d \equiv 3 \pmod{4} \end{cases}$$

2. We have  $\Delta_k f = 0$ .
3. There is a polynomial,  $P_f = \sum_{n \geq 0} c^+(n) q^n \in [q^{-1}]$ , such that:  $f(z) - P_f(z) = O(e^{-\epsilon y})$  as  $y \rightarrow +\infty$  for some  $\epsilon > 0$ . Analogous conditions are required at all cusps.

The term "weak" refers to the relaxed growth condition at the cusps described by (3). For convenience, we will refer to these harmonic weak Maass forms simply as harmonic Maass forms.

We adopt the following notation: if  $\chi$  is a Dirichlet character modulo,  $N$ , let  $S_k(N, \chi)$  (respectively,  $M_k(N, \chi)$ ,  $M_k^!(N, \chi)$ ,  $H_k(N, \chi)$ ) denote the space of cusp forms (respectively, holomorphic modular

forms, weakly holomorphic modular forms and harmonic Maass forms) of weight  $k$  on  $\Gamma_0(N)$  with Nebentypus  $\chi$ .

For  $f \in H_{2-k}(N, \chi)$  and  $1 < k \in \frac{1}{2}\mathbb{Z}$ , there is the unique decomposition,  $f = f^+ + f^-$  of Bruinier and Funke [20], where, following Ono in [21]:

$$f^+(z) := \sum_{n \gg -\infty} a^+(n)q^n$$

is referred to as the *holomorphic part* and:

$$f^-(z) := \sum_{n < 0} a^-(n)\Gamma(k-1, 4\pi|n|y)q^n$$

the *non-holomorphic part* and where  $\Gamma(a, b) := \int_b^\infty e^{-t}t^{a-1} dt$  is the incomplete Gamma-function.

If  $M(q)$  is one of Ramanujan's mock theta functions with  $F(z)$  as in (1.2), then by the work of Zwegers [22],  $f^+ = F$  is the holomorphic part of a weight of  $1/2$  harmonic weak Maass form, whose non-holomorphic part,  $f^-$ , is a period integral of a weight of  $3/2$  unary theta series. As a consequence, there exist integers,  $\delta_1, \dots, \delta_h$ , such that the coefficients,  $a^-(n)$ , are supported on exponents of the form,  $-\delta_i m^2$ .

As stated in Section 1, Theorem 1 is a special case of our general theorem in Section 3, which applies to a weight of  $1/2$  harmonic Maass forms with algebraic coefficients, whose non-holomorphic parts are period integrals of a weight of  $3/2$  unary theta series. Essentially, these congruences are obtained from the annihilation of a cusp form  $g(z)$ , related to  $f(z)$ , by the Hecke operators,  $T(p^2)$ . The cusp form,  $g(z)$ , is determined by a result of Treneer [2]. Moreover, the work of Lichtenstein [1] allows us to bound the first prime,  $p$ , such that  $T(p^2)$  annihilates  $g(z)$ . The details of the construction of  $g(z)$  follow.

## 2.2. Elements of the Proof

In the proof, we shall obtain a weakly holomorphic modular form,  $\widehat{f}(z)$ , from  $f$  by applying quadratic twists to annihilate the non-holomorphic part of  $f(z)$ . If  $Q$  is an odd prime, define  $\psi_Q := \left(\frac{\bullet}{Q}\right)$  and:  $G := \sum_{\lambda=1}^{Q-1} \psi_Q(\lambda)e^{2\pi i\lambda/Q}$ . Then, the  $Q$ -quadratic twist of  $f$  is defined as:

$$f \otimes \psi_Q := \frac{G}{Q} \sum_{\lambda=1}^{Q-1} \psi_Q(\lambda) f|_{\frac{1}{2}} \begin{pmatrix} 1 & -\frac{\lambda}{Q} \\ 0 & 1 \end{pmatrix}$$

**Remarks 2.** The definition of  $f \otimes \psi_Q$  given in ([23], III, Proposition 17) applies to modular forms, but this definition also makes sense for  $f \in H_{2-k}(N, \chi)$ , since the transformation,  $z \mapsto z - \lambda/Q$ , only affects the real part of  $z$  (the  $\Gamma$ -factor in  $f^-$  remains unchanged). As in the modular case (see [23], III, Proposition 17), the  $n$ th coefficient of  $f \otimes \psi_Q$  is  $\left(\frac{n}{Q}\right)$  times the  $n$ th coefficient of  $f$ .

The following lemma describes how twisting  $f$  affects the level:

**Lemma 2.** Suppose  $f$  satisfies the transformation:  $f|_k \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \chi(d)f$  for all  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$  and for some character,  $\chi$ , mod  $N$ . Let  $\psi$  be a character mod  $M$ , and let  $N' = \text{lcm}(NM, M^2)$ . Then:

$$(f \otimes \psi)|_k \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \chi(d)\psi^2(d)(f \otimes \psi)$$

for all  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N')$ .

*Proof.* Let  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N')$ . For each  $\lambda$  with  $0 \leq \lambda < M$ , let  $\lambda'$  denote the smallest nonnegative integer satisfying  $\lambda'a \equiv \lambda d \pmod{M}$ . Then, we have:

$$\begin{pmatrix} 1 & -\lambda/M \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & -\lambda/M \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} a - \frac{c\lambda}{M} & b + \frac{a\lambda' - d\lambda}{M} - \frac{c\lambda\lambda'}{M^2} \\ c & d + \frac{c\lambda'}{M} \end{pmatrix} \in \Gamma_0(N')$$

The lemma now follows from a standard argument (see [23], Proposition III.17(b)).  $\square$

We will see that for  $\widehat{f}$ ,  $\widehat{a}(n)$  is  $a^+(n)$ , if  $\left(\frac{-n}{Q}\right) = -1$ , and is zero, otherwise. We will then require a cusp form,  $g(z)$ , with the property that  $\widehat{f}(z) \equiv g(z) \pmod{\ell^j}$ . The existence of such a cusp form is guaranteed by the work of Treneer [2]. We first fix some notation. For  $f \in M_k^!(N, \chi)$  and a prime,  $\ell$ , define  $\alpha = \alpha(f, \ell)$  and  $\beta = \beta(f, \ell)$  to be the smallest nonnegative integers satisfying:

$$-\ell^\alpha < 4 \min_{\ell^2 | c} \{\text{ord}_{\frac{a}{c}} \widehat{f}\}, \quad -\ell^\beta < \min_{\ell^2 \nmid c} \{\text{ord}_{\frac{a}{c}} \widehat{f}\} \quad (2.2)$$

where  $\frac{a}{c}$  runs over a set of representatives for the cusps of  $\Gamma_0(N)$ . Theorem 3 below follows from Theorems 1.1 and 3.1 of [2], along with the proof of Theorem 3.1 of [2].

**Theorem 3.** ([2], Theorem 3.1) Let  $\ell^j$  be an odd prime power, and let  $N$  be a positive integer with  $(\ell, N) = 1$ . Suppose that  $f(z) = \sum a(n)q^n \in M_{\frac{k}{2}}^!(4N, \chi)$  has algebraic integer coefficients. If  $\alpha = \alpha(f, \ell)$  and  $\beta = \beta(f, \ell)$ , as in Equation (2.2), then there is a cusp form:

$$g(z) \in S_{\frac{k}{2} + \frac{\ell\beta(\ell^2-1)}{2}}(\Gamma_0(N\ell^2), \chi\psi_\ell^{k\alpha})$$

such that:

$$g(z) \equiv \sum_{\ell \nmid n} a(\ell^\alpha n)q^n \pmod{\ell^j}$$

Further, a positive proportion of primes,  $p \equiv -1 \pmod{4N\ell^j}$ , satisfy:

$$a(p^3 \ell^\alpha n) \equiv 0 \pmod{\ell^j}$$

for all  $n$  coprime to  $\ell p$ .

We end this section by recalling the action of Hecke operators,  $T(p^2)$ , on half integral weight cusp forms and stating a result of Lichtenstein [1], which will allow us to bound the smallest prime,  $p$ , such

that  $g(z) \mid T(p^2) \equiv 0 \pmod{\ell^j}$ . If  $\chi$  is a quadratic character,  $g(z) = \sum a(n)q^n \in M_{\lambda+1/2}(4N, \chi)$ , and  $(p, 4N) = 1$ , then:

$$g(z) \mid T(p^2) := \sum \left( a(p^2 n) + \left( \frac{(-1)^{\lambda n}}{p} \right) \chi(p) p^{\lambda-1} a(n) + p^{2\lambda-1} a(n/p^2) \right) q^n$$

To state Lichtenstein's result, we require the following notation. Let  $E$  be the smallest number field containing the coefficients of  $g(z)$ , and let  $\ell\mathcal{O}_E$  factor as  $\ell\mathcal{O}_E = \prod_m \lambda_m^{e_m}$ , where the  $\lambda_m$  are prime ideals of  $\mathcal{O}_E$ . Let  $S := S_k(\Gamma_0(2N))$ , and set  $d := \dim_{\mathbb{C}} S$ . Let:

$$s := \frac{k}{12} N \prod_{p|N} \left( 1 + \frac{1}{p} \right)$$

denote the Sturm bound for  $S$ . Let  $\{f_1, \dots, f_d\}$  be a basis for  $S$  consisting of normalized Hecke eigenforms, and let  $K$  be a number field containing the coefficients of all the  $f_i$ . Choose primes,  $\mu_m$ , of  $\mathcal{O}_{KE}$ , lying above each  $\lambda_m$ , such that the largest  $\mu_m$ -adic valuation of the first  $s$  coefficients of all the  $f_i$  is a minimum—let  $v_m$  denote this largest valuation. Let  $r_m := [\mathcal{O}_{KE}/\mu_m : \mathbb{F}_{\ell}]$ . Define:

$$B(S, \ell^j) := \prod_m \ell^{4dr_m(dv_m + \alpha_m)}$$

and  $L(S, \ell) := \ell \cdot \prod_{p|N} p$ .

**Theorem 4.** ([1], Theorem 1.2) *With the notation above, let  $B := B(S, \ell^j)$  and  $L := L(S, \ell)$ . There is an effectively computable constant,  $A_1$  (defined in [19]), such that for some prime,  $p \equiv -1 \pmod{2N\ell^j}$ , satisfying:*

$$p \leq 2(L^{(B-1)} B^B)^{A_1}$$

*we have  $g \mid T(p^2) \equiv 0 \pmod{\ell^j}$ . Assuming the Generalized Riemann Hypothesis, the prime,  $p$ , satisfies:*

$$p \leq 280B^2(\log B + \log L)^2$$

**Remarks 3.** *In particular, if the coefficients of  $f$  are rational integers, i.e.,  $E = \mathbb{Q}$ , then taking  $\mu = \mu_1$ ,  $v = v_1$  and  $r = r_1$ , we see that  $B = \ell^{4dr(dv+j)}$ .*

**Remarks 4.** *The quantity,  $B$ , defined in Theorem 4, arises from the elementary bound:*

$$\#\mathrm{GL}_2(\mathcal{O}_K/\mu^{dv+j}) \leq \ell^{4r(dv+j)}$$

*which, in certain cases, is easy to compute and is much smaller (see [1], Example 4.3).*

### 3. Statement of the General Theorem and Its Proof

Here, we state our general result, of which Theorem 1 is a special case. For ease of notation, we state it for harmonic Maass forms with holomorphic parts whose coefficients lie in  $\mathbb{Q}$ , but an analogous result holds for such forms with algebraic coefficients.



**Theorem 5.** Suppose  $f \in H_{\frac{1}{2}}(\Gamma_0(4N), \chi)$  has holomorphic part:  $f^+ = \sum_{n \geq n_0} a^+(n)q^n$  with  $a^+(n) \in \mathbb{Q}$  and non-holomorphic part:

$$f^- = \sum_{i=1}^h \sum_{n>0} a^-(\delta_i n^2) \Gamma\left(\frac{1}{2}, 4\pi\delta_i n^2\right) q^{-\delta_i n^2}$$

for some finite set of square-free  $\delta_i \in \mathbb{N}$ . Let  $\ell^j$  be a prime power with  $(4N, \ell) = 1$ .

(i) Let  $Q$  be an odd prime with  $\left(\frac{\delta_i}{Q}\right) = 1$  for  $1 \leq i \leq h$ . Then, we have:

$$\widehat{f} = \sum \widehat{a}(n)q^n := \sum_{\left(\frac{-n}{Q}\right)=-1} a^+(n)q^n \in M_{\frac{1}{2}}^1(\Gamma_0(4NQ^3), \chi)$$

(ii) Define  $\alpha = \alpha(\widehat{f}, \ell)$  and  $\beta = \beta(\widehat{f}, \ell)$ , as in Equation (2.2). Then, there exists a cusp form:

$$g \in S_{\frac{1}{2}+\ell\beta\left(\frac{\ell^2-1}{2}\right)}(\Gamma_0(4NQ^3\ell^2), \chi\psi_\ell^\alpha)$$

with the property that:

$$g \equiv \sum_{\ell \nmid n} \widehat{a}(\ell^\alpha n)q^n \pmod{\ell^j}$$

Further, a positive proportion of the primes,  $p \equiv -1 \pmod{4NQ^3\ell^j}$ , have:

$$\widehat{a}(p^3\ell^\alpha n) \equiv 0 \pmod{\ell^j} \quad (3.1)$$

for all  $n$  coprime to  $\ell p$ .

(iii) Define:  $S := S_{\ell\beta(\ell^2-1)}(\Gamma_0(2NQ^3\ell^2))$ , and let  $B = B(S, \ell^j)$  and  $L = L(S, \ell)$ , as given in Theorem 4 above. Then, the smallest prime,  $p$ , for which Equation (3.1) holds satisfies:

$$p \leq 2(L^{B-1}B^B)^{A_1}$$

Assuming GRH, this prime,  $p$ , satisfies:

$$p \leq 280B^2(\log B + \log L)^2$$

*Proof of Theorem 5.* To prove (i), we use  $Q$ -quadratic twists to annihilate the non-holomorphic part of  $f$ . We have:

$$\begin{aligned} f^- - \left(\frac{-1}{Q}\right) f^- \otimes \psi_Q &= \sum_{j=1}^s \sum_{n=1}^{\infty} \left(1 - \left(\frac{-1}{Q}\right) \left(\frac{-\delta_j n^2}{Q}\right)\right) a^-(\delta_j n^2) \Gamma(1/2, 4\pi\delta_j n^2 y) q^{-\delta_j n^2} \\ &= \sum_{j=1}^s \sum_{Q|n} a^-(\delta_j n^2) \Gamma(1/2, 4\pi\delta_j n^2 y) q^{-\delta_j n^2} \end{aligned}$$

Let  $\tilde{f} := f - \left(\frac{-1}{Q}\right) f \otimes \psi_Q$  and  $\hat{f} := -\frac{1}{2} \left(\frac{-1}{Q}\right) \tilde{f} \otimes \psi_Q$ . Recalling from Remarks 2 that the  $n$ th coefficient of  $f \otimes \psi_Q$  (respectively,  $\tilde{f} \otimes \psi_Q$ ) is  $\left(\frac{n}{Q}\right)$  times the  $n$ th coefficient of  $f$  (respectively,  $\tilde{f}$ ), since the non-holomorphic part of  $\tilde{f}$  is supported on exponents of the form,  $-\delta_j Q^2 n^2$ , the non-holomorphic part of the harmonic Maass form,  $\hat{f}$ , has no non-holomorphic part. It follows that:

$$\hat{f} \in M_{\frac{1}{2}}^!(\Gamma_0(4NQ^3, \chi))$$

and the  $n$ th coefficient of  $\hat{f}$  is:

$$-\frac{1}{2} \left(\frac{-1}{Q}\right) \left(\frac{n}{Q}\right) \left(1 - \left(\frac{-1}{Q}\right) \left(\frac{n}{Q}\right)\right) a^+(n) = \hat{a}(n)$$

That is,  $\hat{a}(n)$  is  $a^+(n)$ , if  $\left(\frac{-n}{Q}\right) = -1$ , and is zero otherwise.

To prove (ii), apply Theorem 3 to the weakly holomorphic modular form,  $\hat{f}$ .

Statement (iii) is immediate from Theorem 4. □

Theorem 1 now follows quickly from Theorem 5.

*Proof of Theorem 1.* Using Theorem 5 and taking  $m = \alpha$ , we obtain congruences of the form:

$$\hat{a}(p^3 \ell^m n) \equiv 0 \pmod{\ell^j}$$

for all  $n$  coprime to  $\ell p$ . We may choose  $Q$  in Theorem 5 to also be coprime to  $\ell$ . Then, since  $\hat{a}(k) = b(n)$ , when  $\left(\frac{-k}{Q}\right) = -1$ , we can take any integer,  $A$ , satisfying  $(A, p\ell) = 1$  and:

$$\left(\frac{-p^3 \ell^m A}{Q}\right) = -1$$

so that, replacing  $n$  by  $p\ell Qn + A$ , we obtain the congruences:

$$b(p^4 \ell^{m+1} Qn + B) \equiv 0 \pmod{\ell^j}$$

where  $B = p^3 \ell^m A$ . □

**Remarks 5.** For the case where  $\alpha = 0$ , Theorem 1 can be improved. Namely, we can have  $Q = \ell$ , and with  $A$  chosen as above, we can similarly obtain congruences  $b(p^4 \ell n + B) \equiv 0 \pmod{\ell^j}$ , where  $B = p^3 A$ .

**Remarks 6.** Congruences for a Ramanujan mock theta function,  $M(q) = \sum c(n)q^n$ , follow from Theorem 1 in the form,  $c(p^4 \ell^{\alpha+1} Qn + B') \equiv 0 \pmod{\ell^j}$ , where  $B' = (B - \tau)/\delta \in \mathbb{Q}$ . If  $(\delta, pQ\ell) = 1$ ,  $A$  may be chosen as above to also satisfy  $A \equiv \tau \ell^{-\alpha} p^{-3} \pmod{\delta}$  to get  $B' \in \mathbb{N}$ .

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## Conflicts of Interest

The authors declare no conflicts of interest.

## References

1. Lichtenstein, S. The effective Chebotarev density theorem and modular forms modulo  $m$ . *Proc. Am. Math. Soc.* **2008**, *136*, 3419–3428.
2. Treneer, S. Congruences for the coefficients of weakly holomorphic modular forms. *Proc. Lond. Math. Soc.* **2006**, *93*, 304–324.
3. Ramanujan, S. Congruence properties of partitions. *Math. Z.* **1921**, *9*, 147–153.
4. Atkin, A.O.L. Proof of a conjecture of Ramanujan. *Glasg. Math. J.* **1967**, *8*, 14–32.
5. Watson, G.N. Ramanujan's vermutung über zerfallungsanzahlen. *J. Reine Angew. Math.* **1938**, *179*, 97–128.
6. Atkin, A.O.L. Multiplicative congruence properties and density problems for  $p(n)$ . *Proc. Lond. Math. Soc.* **1968**, *18*, 563–576.
7. Ahlgren, S. The partition function modulo composite integers  $M$ . *Math. Ann.* **2000**, *318*, 793–803.
8. Ahlgren, S.; Ono, K. Congruence properties for the partition function. *Proc. Natl. Acad. Sci. USA* **2001**, *98*, 12882–12884.
9. Ono, K. Distribution of the partition function modulo  $m$ . *Ann. Math.* **2000**, *151*, 293–307.
10. Bringmann, K.; Ono, K. The  $f(q)$  mock theta function conjecture and partition ranks. *Invent. Math.* **2006**, *165*, 243–266.
11. Gordon, B.; McIntosh, R.J. A Survey of The Classical Mock Theta Functions. In *Partitions,  $q$ -Series, and Modular Forms*; Series Developments in Mathematics; Springer-Verlag: New York, NY, USA, 2012; Volume 23, pp. 95–244.
12. Alfes, C. Congruences for Ramanujan's  $\omega(q)$ . *Ramanujan J.* **2010**, *22*, 163–169.
13. Bruinier, J.; Ono, K. Identities and congruences for Ramanujan's  $\phi$  function. *Ramanujan J.* **2010**, *23*, 151–157.
14. Bryson, J.; Ono, K.; Pitman, S.; Rhoades, R. Unimodal sequences and quantum and mock modular forms. *Proc. Natl. Acad. Sci. USA* **2012**, *109*, 16063–16067.
15. Chan, S. Congruences for Ramanujan's  $\phi$  function. *Acta Arith.* **2012**, *153*, 161–189.
16. Garthwaite, S. The coefficients of the  $\omega(q)$  mock theta function. *Int. J. Number Theory* **2008**, *4*, 1027–1042.
17. Waldherr, M. On certain explicit congruences for mock theta functions. *Proc. Am. Math. Soc.* **2011**, *139*, 865–879.

18. Miezaki, T. On the Mathieu mock theta function. *Proc. Jpn. Acad. Ser. A Math. Sci.* **2012**, *88*, 28–30.
19. Lagarias, J.C.; Montgomery, H.L.; Odlyzko, A.M. A bound for the least prime ideal in the Chebotarev density theorem. *Invent. Math.* **1979**, *54*, 271–296.
20. Bruinier, J.H.; Funke, J. On two geometric theta lifts. *Duke Math. J.* **2004**, *125*, 45–90.
21. Ono, K. Unearthing the Visions of a Master: Harmonic Maass Forms and Number Theory. In *Current Developments in Mathematics, 2008*; International Press: Somerville, MA, USA, 2009; pp. 347–454.
22. Zwegers, S. Mock Theta Functions. Ph.D. Thesis, Universiteit Utrecht, Utrecht, The Netherlands, 2002.
23. Koblitz, N. *Introduction to Elliptic Curves and Modular Forms*, 2nd ed.; Series Graduate Texts in Mathematics; Springer-Verlag: New York, NY, USA, 1993; Volume 97, pp. x+248.

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