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Combinatorics / *Combinatoire*

Refinements and Symmetries of the Morris identity for volumes of flow polytopes

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Dedicated to Doron Zeilberger in recognition of his 70th birthday.

Abstract. Flow polytopes are an important class of polytopes in combinatorics whose lattice points and volumes have interesting properties and relations. The Chan–Robbins–Yuen (CRY) polytope is a flow polytope with normalized volume equal to the product of consecutive Catalan numbers. Zeilberger proved this by evaluating the Morris constant term identity, but no combinatorial proof is known. There is a refinement of this formula that splits the largest Catalan number into Narayana numbers, which Mészáros gave an interpretation as the volume of a collection of flow polytopes. We introduce a new refinement of the Morris identity with combinatorial interpretations both in terms of lattice points and volumes of flow polytopes. Our results generalize Mészáros’s construction and a recent flow polytope interpretation of the Morris identity by Corteel–Kim–Mészáros. We prove the product formula of our refinement following the strategy of the Baldoni–Vergne proof of the Morris identity. Lastly, we study a symmetry of the Morris identity bijectively using the Danilov–Karzanov–Koshevoy triangulation of flow polytopes and a bijection of Mészáros–Morales–Striker.

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1. Introduction

1.1. Foreword

Flow polytopes play a fundamental role in combinatorial optimization through their relation to maximum matching and minimum cost problems (e.g. see [29, Chapter 13]). Flow polytopes have been used in various fields like toric geometry [14] and representation theory [2]. More recently, they have been related to geometric and algebraic combinatorics thanks to connections

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with Schubert polynomials [10], diagonal harmonics [21], Gelfand–Tsetlin polytopes [18], and generalized permutahedra [22].

Given a graph G with vertex set $\{0, 1, \dots, n, n+1\}$ and edges (i, j) oriented $i \rightarrow j$ if $i < j$, we associate with G a net flow vector $\mathbf{a} = (a_0, a_1, \dots, a_n, -\sum_{i=0}^n a_i)$ such that vertex i has net flow a_i for $i = 0, 1, \dots, n$. The set of all flows with net flow vector \mathbf{a} , called the *flow polytope*, is denoted by $\mathcal{F}_G(\mathbf{a})$. Define $K_G(\mathbf{a})$ as the number of lattice points (integer flows) of $\mathcal{F}_G(\mathbf{a})$, called the *Kostant vector partition function*. The name comes from the fact that for the complete graph k_{n+2} , $K_{k_{n+2}}(\mathbf{a})$ is a vector partition function studied by Kostant in the context of Lie algebras (e.g. [15]). The following theorem, which appears in unpublished work of Postnikov and Stanley and in the work of Baldoni–Vergne [2], relates the volume of a flow polytope to a Kostant partition function. See Section 3 for a new recursive proof of this result.

Theorem 1 (Postnikov–Stanley, Baldoni–Vergne [2]). *For a loopless digraph G with vertices $\{0, 1, \dots, n+1\}$ having unique source 0 and unique sink $n+1$,*

$$\text{vol } \mathcal{F}_G(1, 0, \dots, 0, -1) = K_G \left(0, d_1, \dots, d_n, -\sum_{i=1}^n d_i \right), \quad (1)$$

where $d_i = \text{indeg}_G(i) - 1$.

An important example of a flow polytope is the Chan–Robbins–Yuen (CRY) polytope [6], defined as $\text{CRY}_{n+1} := \mathcal{F}_{k_{n+2}}(1, 0, \dots, 0, -1)$. Zeilberger calculated the volume of CRY_{n+1} algebraically using the Morris constant term identity, equivalent to the famous Selberg integral formula (see [11]). For convenience, we use the term *volume* in this paper to refer to normalized volume.

Theorem 2 (Zeilberger’s Morris Identity [34]). *For positive integers n, a , and b , and nonnegative integer c , define the constant term*

$$M_n(a, b, c) := \text{CT}_x \prod_{i=1}^n (1 - x_i)^{-b} x_i^{-a+1} \prod_{1 \leq i < j \leq n} (x_j - x_i)^{-c},$$

where $\text{CT}_x := \text{CT}_{x_n} \cdots \text{CT}_{x_1}$. Then

$$M_n(a, b, c) = \prod_{j=0}^{n-1} \frac{\Gamma(a-1+b+(n-1+j)\frac{c}{2})\Gamma(\frac{c}{2}+1)}{\Gamma(a+j\frac{c}{2})\Gamma(b+j\frac{c}{2})\Gamma(\frac{c}{2}(j+1)+1)}. \quad (2)$$

By specializing this identity, Zeilberger proved that the volume of CRY_{n+1} is the product of the first $n-1$ Catalan numbers.

Theorem 3 (Zeilberger [34]). *The volume of the polytope CRY_{n+1} is given by $M_n(1, 1, 1) = \prod_{i=1}^{n-1} C_i$, where $C_i = \frac{1}{i+1} \binom{2i}{i}$ is the i^{th} Catalan number.*

Despite the numerous interpretations of C_n , no combinatorial proof of Theorem 3 is known. Corteel–Kim–Mészáros [7, Theorem 1.2] also showed that for any positive a, b , and c , $M_n(a, b, c)$ gives the volume of the flow polytope on the following graph. For positive integer n , let $k_{n+2}^{a,b,c}$ denote the graph on vertex set $\{0, 1, \dots, n+1\}$ with edge $(0, i)$, $i \in [1, n]$ appearing with multiplicity a , edge $(i, n+1)$, $i \in [1, n]$, appearing with multiplicity b , and (i, j) , $1 \leq i < j \leq n$, appearing with multiplicity c (see Figure 1). Note that $k_{n+2} = k_{n+2}^{1,1,1}$. Then they showed the following.

Theorem 4 (Corteel–Kim–Mészáros [7]). *Let n, a and b be positive integers, c be a nonnegative integer, and let $a_i = a - 1 + c(i-1)$ for $i = 1, \dots, n$. Then*

$$\text{vol } \mathcal{F}_{k_{n+2}^{a,b,c}}(1, 0, \dots, 0, -1) = K_{k_{n+2}^{a,b,c}} \left(0, a_1, \dots, a_n, -\sum_{i=1}^n a_i \right) = M_n(a, b, c). \quad (3)$$

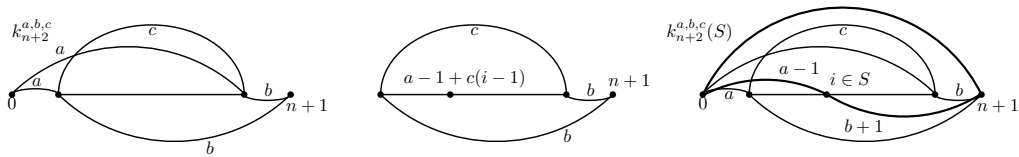


Figure 1. The graph $k_{n+2}^{a,b,c}$, the graph of the Kostant partition function corresponding to the volume of $\mathcal{F}_{k_{n+2}^{a,b,c}}$, and the graph of one of the polytopes corresponding the volume interpretation of $\Psi_n(k, a, b, c)$.

From the product formula in (2) it follows that $M_n(a, b, c)$ is symmetric in a and b . This is less clear from the volume and lattice point interpretation of $M_n(a, b, c)$ in (3).

In addition, there is an interesting refinement of the volume formula $M_n(1, 1, 1)$ of the CRY polytope. Namely, the following conjecture of Chan–Robbins–Yuen [6, Conj. 2] settled by Zeilberger [34], refines the product $C_n C_{n-1} \cdots C_1$ by splitting C_n into a sum of Narayana numbers $N(n, k) = \frac{1}{n} \binom{n}{k} \binom{n}{k-1}$. The original conjecture used the Kostant partition function interpretation and Mészáros [19, Thm. 11] then gave a geometric interpretation of this refinement by providing a collection of interior disjoint polytopes whose volumes equal $N(n, k) \prod_{i=1}^{n-1} C_i$. To state these interpretations, we introduce some notation. Given a graph $k_{n+2}^{a,b,c}$ as above and a k -element set $S \in \binom{[n]}{k}$, let $k_{n+2}^{a,b,c}(S)$ be the graph obtained from taking $k_{n+2}^{a,b,c}$, adding n edges $(0, n+1)$, and for each $i \in S$ deleting one of the a incoming edges $(0, i)$ and adding an outgoing edge $(i, n+1)$ (See Figure 1).

Theorem 5 (Zeilberger [34]; Mészáros [19]). *For a positive integer n and a nonnegative integer $k \leq n$, the product $N(n, k) \prod_{i=1}^{n-1} C_i$ equals the following:*

- (i) *The sum of Kostant partition functions $K_{k_{n+2}}(0, a_1, \dots, a_n, -\sum_{j=1}^n a_j)$ such that for $i \in [n]$, $a_i \leq i-1$, with $a_i = i-1$ holding for exactly k values of i .*
- (ii) *The volume of the interior disjoint polytopes $\{\mathcal{F}_{k_{n+2}(S)} \mid S \in \binom{[n]}{k}\}$.*

In [34], Zeilberger sketched the proof of Theorem 5 using Aomoto’s refinement of the Selberg integral [1], but no explicit refinement of $M_n(a, b, c)$ was given (see also [33]).

The aims of this paper are threefold: give such a refinement of $M_n(a, b, c)$, with a product formula that implies Theorems 2 and 3, provide geometric and lattice point interpretations of the refinement extending Theorems 5 and 4, and lastly study the symmetry and new relations of $M_n(a, b, c)$. We next describe our main results.

1.2. A new refinement of $M_n(a, b, c)$

Our refinement is inspired by a related refinement of $M_n(a, b, c)$ introduced by Baldoni–Vergne [3] to prove the Morris identity (Theorem 2), for which we extend a Kostant partition function interpretation (see Section 6) but which did not imply Theorem 5 as a special case.

To state our results, define the constant term

$$\Psi_n(k, a, b, c) := \text{CT}_x \left[t^k \right] \prod_{i=1}^n (1 - x_i)^{-b} x_i^{-a+1} \left(1 + t \frac{x_i}{1 - x_i} \right) \prod_{1 \leq i < j \leq n} (x_j - x_i)^{-c}. \quad (4)$$

In the case that $k = 0$, $\Psi_n(0, a, b, c) = M_n(a, b, c)$. We now give Kostant partition function and polytope volume interpretations for $\Psi_n(k, a, b, c)$, as well as an explicit product formula.

Theorem 6. *For positive integers n, a , and b , nonnegative integer c , and nonnegative integer $k \leq n$, the constant term $\Psi_n(k, a, b, c)$ equals the following:*

(i) the sum of Kostant partition functions of the form

$$K_{k_{n+2}}^{a,b,c} \left(0, a_1, \dots, a_n, -\sum_{j=1}^n a_j \right)$$

such that for $i \in [n]$, $a_i \leq a - 1 + c(i - 1)$, with $a_i = a - 1 + c(i - 1)$ holding for exactly $n - k$ values of i .

(ii) the volume of the interior disjoint polytopes $\{\mathcal{F}_{k_{n+2}}^{a,b,c}(S) \mid S \in \binom{[n]}{k}\}$. Thus,

$$\Psi_n(k, a, b, c) = \sum_{S \in \binom{[n]}{k}} \text{vol} \mathcal{F}_{k_{n+2}}^{a,b,c}(S).$$

We see that when $a = b = c = 1$, the Kostant partition function interpretation of $\Psi_n(k, a, b, c)$ reduces to Theorem 5, giving that $\Psi_n(k, 1, 1, 1) = N(n, k) \prod_{i=1}^{n-1} C_i$. As a corollary, our constant term $\Psi_n(k, a, b, c)$ refines the Morris constant term $M_n(a, b + 1, c)$.

Corollary 7. Let n, a , and b be positive integers, and let c be a nonnegative integer. Then

$$M_n(a, b + 1, c) = \sum_{k=0}^n \Psi_n(k, a, b, c). \quad (5)$$

We also compute the following explicit product formula for $\Psi_n(k, a, b, c)$ that completes our refinement and new proof of the Morris identity.

Theorem 8. For positive integers n, a , and b , nonnegative integer c , and nonnegative integer $k \leq n$, the constant term $\Psi_n(k, a, b, c)$ is given by

$$\Psi_n(k, a, b, c) = \binom{n}{k} M_n(a, b, c) \prod_{j=1}^k \frac{a - 1 + (n - j)\frac{c}{2}}{b + (j - 1)\frac{c}{2}}. \quad (6)$$

We show Theorem 8 by proving four recurrence relations satisfied by $\Psi_n(k, a, b, c)$, by proving these relations uniquely define $\Psi_n(k, a, b, c)$, and by proving the product formula also satisfies these relations. This closely follows the approach of Baldoni–Vergne [3, p. 8] in their proof of the Morris identity. However, our proofs are combinatorial rather than algebraic, with the notable exception of the proof of the relation (19), which after a reformulation states that for $1 \leq k \leq n$,

$$k(b + (k - 1)c/2) \cdot \Psi_n(k, a, b, c) = (n - k + 1)(a - 1 + (n - k)c/2) \cdot \Psi_n(n - k + 1, b + 1, a - 1, c). \quad (7)$$

We leave as an open problem to prove this relation combinatorially, which would then imply a combinatorial proof of the volume formula for the CRY polytope (Theorem 3).

1.3. A fundamental symmetry of $M_n(a, b, c)$

We also explain the symmetry $M_n(a, b, c) = M_n(b, a, c)$ with the volume and lattice point interpretations of (3). In particular, we use a triangulation of flow polytopes of Danikov–Karzanov–Koshevoy [8] and a correspondence from [23] to give a bijection between the lattice points of $\mathcal{F}_{k_{n+2}}^{a,b,c}(0, a_1, a_2, \dots, -\sum_i a_i)$ and $\mathcal{F}_{k_{n+2}}^{b,a,c}(0, b_1, b_2, \dots, -\sum_i b_i)$ where $a_i = a - 1 + c(i - 1)$ and $b_i = b - 1 + c(i - 1)$ for $i = 1, \dots, n$. The bijection holds for any graph G and as a special case we obtain a bijection of Postnikov [27] between lattice points of $(p - 1)\Delta^{q-1}$ and $(q - 1)\Delta^{p-1}$ further studied in [12].

1.4. Outline

The rest of this paper is structured as follows. In Section 2, we establish basic theory surrounding flow polytopes, Kostant partition functions, and the Morris constant term identity. This includes closed formulas and asymptotics for special cases of $M_n(a, b, c)$. Section 3 gives a new recursive proof of Theorem 1 by extending a well-known subdivision relation of flow polytopes to integer flows. In Section 4 we give three proofs of the symmetry $M_n(a, b, c) = M_n(b, a, c)$ including a bijection between lattice points of two flow polytopes. In Section 5, we prove our results for $\Psi_n(k, a, b, c)$, including Theorem 6, Corollary 7, and Theorem 8. In Section 6, we apply our methods for $\Psi_n(k, a, b, c)$ to the Baldoni–Vergne constant term and prove Theorem 49, and in Section 7 we provide final remarks and some open questions.

2. Background and Notation

2.1. Flow polytopes and their subdivisions

Given a loopless acyclic connected digraph G with vertex set $\{0, 1, \dots, n, n+1\}$ and m edges, we orient edge (i, j) from i to j if $i < j$. We can then represent each edge (i, j) by the positive type A_n root $\alpha(i, j) = e_i - e_j$. We also define M_G to be the $(n+2) \times m$ matrix whose columns are given by the multiset $\{\{\alpha(e)\}\}_{e \in E(G)}$.

Then given a net flow vector $\mathbf{a} = (a_0, a_1, \dots, a_n, -\sum_{i=0}^n a_i)$, where a_i represents the net flow at vertex i , we define an \mathbf{a} -flow \mathbf{f}_G as a vector $\mathbf{f}_G = (f(e))_{e \in E(G)}$ satisfying $M_G \mathbf{f}_G = \mathbf{a}$. We now define the *flow polytope* $\mathcal{F}_G(\mathbf{a})$ as the set of all \mathbf{a} -flows on G . More precisely, $\mathcal{F}_G(\mathbf{a}) := \{\mathbf{f}_G \in \mathbb{R}_{\geq 0}^m \mid M_G \mathbf{f}_G = \mathbf{a}\}$. In the absence of an explicit vector \mathbf{a} , it is implied that $\mathbf{a} = (1, 0, \dots, 0, -1)$. In other words, $\mathcal{F}_G := \mathcal{F}_G(1, 0, \dots, 0, -1)$. If G has a unique source 0 and sink $n+1$, then the *dimension* of \mathcal{F}_G is $m - n - 1$. The vertices of \mathcal{F}_G are given by unit flows along maximal directed paths from the source to the sink called *routes*.

Next we define a notion of equivalence for flow polytopes. Let $\text{aff}(\cdot)$ denote affine span. For two flow polytopes $P \subset \mathbb{R}^n$ and $Q \subset \mathbb{R}^m$, we say that P and Q are *integrally equivalent*, denoted $P \equiv Q$, if there exists an affine transformation $\varphi: \mathbb{R}^n \rightarrow \mathbb{R}^m$ that is a bijection both when restricted between P and Q and when restricted between $\text{aff}(P) \cap \mathbb{Z}^n$ and $\text{aff}(Q) \cap \mathbb{Z}^m$. Polytopes that are integrally equivalent share many similar properties, including the same volume and Ehrhart polynomials.

For a digraph G as above, we denote by G^r the digraph with the same vertices and edges $E(G^r) = \{(i, j) \mid (n+1-j, n+1-i) \in E(G)\}$. That is, the digraph obtained from G by reversing the edges and relabeling the vertices $i \mapsto n+1-i$. By reversing the flows, one shows that the flow polytopes of G and G^r with netflow $(1, 0, \dots, 0, -1)$ are integrally equivalent.

Lemma 9. *For a loopless digraph G with vertices $\{0, 1, \dots, n+1\}$ having a unique source 0 and unique sink $n+1$ then $\mathcal{F}_G \equiv \mathcal{F}_{G^r}$.*

We now give a recursive subdivision of flow polytopes used by Postnikov–Stanley in their unpublished work. See also [20, Section 4].

Let $G = (\{0, 1, \dots, n, n+1\}, E)$. We now repeatedly apply the following algorithmic step, called the *reduction rule*: starting with a graph G_0 on vertex set $\{0, 1, \dots, n, n+1\}$ and $(i, j), (j, k) \in E(G_0)$ for some $i < j < k$, we reduce G_0 to two graphs G_1 and G_2 with vertex set $\{0, 1, \dots, n, n+1\}$ and edge sets

$$E(G_1) := E(G_0) \setminus \{(j, k)\} \cup \{(i, k)\}, \quad (8)$$

$$E(G_2) := E(G_0) \setminus \{(i, j)\} \cup \{(i, k)\}. \quad (9)$$

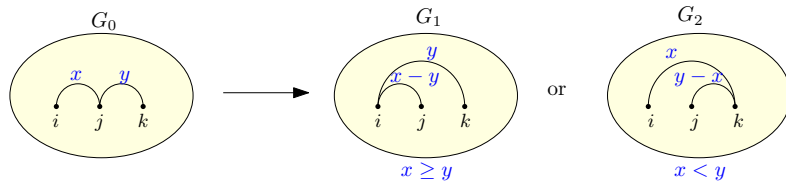


Figure 2. The subdivision lemma reduces a flow polytope to two interior disjoint polytopes whose union is integrally equivalent to the original flow polytope.

Proposition 10 (Subdivision Lemma, Postnikov, Stanley [30] (e.g. [19, Proposition 1])). *Given a graph G_0 on the vertex set $\{0, 1, \dots, n, n+1\}$ and $(i, j), (j, k) \in E(G_0)$ for arbitrary $i < j < k$, define G_1 and G_2 by the above reduction rule. Then we have*

$$\mathcal{F}_{G_0} \equiv \mathcal{F}_{G_1} \cup \mathcal{F}_{G_2}, \quad \mathcal{F}_{G_1}^\circ \cap \mathcal{F}_{G_2}^\circ = \emptyset,$$

where \mathcal{F}_G° denotes the interior of the polytope \mathcal{F}_G .

The subdivision lemma is illustrated in Figure 2. The proof can be found in [20]. Define a graph G to be *reducible* if we can apply the reduction rule to two of its edges (that is, there exists $(i, j), (j, k) \in E(G)$). Otherwise, the graph G is *irreducible*. We now define the reduction tree $\mathcal{T}(G)$ of a graph G . The root of $\mathcal{T}(G)$ is G , and each node G_0 has two children G_1 and G_2 described by the reduction rule. Each leaf of $\mathcal{T}(G)$ is hence irreducible. $\mathcal{T}(G)$ is not unique and depends on the order of reductions applied, but the number of leaves is always the same.

2.2. Kostant partition functions

We now examine the lattice points of $\mathcal{F}_G(\mathbf{a})$, i.e. the integer flows. For a graph G on vertex set $\{0, 1, \dots, n, n+1\}$ and (i, j) oriented $i \rightarrow j$ if $i < j$, denote by $\mathcal{F}_G^\mathbb{Z}(\mathbf{a})$ the set of lattice points of the flow polytope $\mathcal{F}_G(\mathbf{a})$, and define $K_G(\mathbf{a}) := \#\mathcal{F}_G^\mathbb{Z}(\mathbf{a})$ to be the number of such lattice points, called the *Kostant partition function*. The name comes from interpreting the function in the case of $G = k_{n+2}$ as giving the number of ways of writing \mathbf{a} as a \mathbb{N} -linear combination of the type A positive roots $e_i - e_j$, where e_i is the i th standard vector and $i < j$. In the theory of semisimple Lie algebras there are classical formulas for weight and tensor product multiplicities in terms of $K_{k_{n+2}}(\mathbf{a})$ (see [15, Section 24]).

The generating function of Kostant partition functions on G is given by

$$\sum_{\mathbf{a}} K_G(\mathbf{a}) \mathbf{x}^{\mathbf{a}} = \prod_{(i,j) \in E(G)} \left(1 - x_i x_j^{-1}\right)^{-1},$$

where the term $x_i x_j^{-1}$ represents a single flow along the edge (i, j) , and the number of flows with net flow of j at vertex i is represented by the coefficient of x_i^j . In particular, for the graph $k_{n+2}^{a,b,c}$, the generating function can be simplified to

$$\sum_{\mathbf{a}} K_{k_{n+2}^{a,b,c}}(\mathbf{a}) \mathbf{x}^{\mathbf{a}} = \prod_{i=1}^n (1 - x_0 x_i^{-1})^{-a} (1 - x_i x_{n+1}^{-1})^{-b} \prod_{1 \leq i < j \leq n} (1 - x_i x_j^{-1})^{-c}. \quad (10)$$

Theorem 1 relates the volume of a flow polytope to a Kostant partition function with a certain net flow vector. Using the generating function for Kostant partition functions, this has very useful implications, such as Theorem 4. To prove Theorem 4, first apply Theorem 1 for $k_{n+2}^{a,b,c}$. Since the net flow at the source is zero, we can ignore the term $\prod_{i=1}^n (1 - x_0 x_i^{-1})^{-a}$. Because the total flow is conserved, the flow at vertex $n+1$ is already determined, so we can simplify the product by setting $x_{n+1} = 1$. The result follows by extracting the appropriate coefficient in (10), and expressing it as a

constant term extraction (see [7, Theorem 1.2]). This approach thus gives a way to express Kostant partition functions as a constant term.

2.3. Catalan numbers, Narayana numbers, and Proctor's formula

The Catalan numbers satisfy the formula $C_n = \frac{1}{n+1} \binom{2n}{n}$, and are one of the most ubiquitous sequences in combinatorics. For instance, the Catalan number C_n counts more than 200 different combinatorial objects [31]. The Catalan numbers are refined by the Narayana numbers $N(n, k) = \frac{1}{n} \binom{n}{k} \binom{n}{k-1}$ such that

$$C_n = \sum_{k=1}^n N(n, k).$$

In analogy to the Catalan numbers, the Narayana number $N(n, k)$ counts the number of lattice paths from $(0, 0)$ to (n, n) that do not pass above the line $y = x$ and has $2k - 1$ turns. Notably, both Narayana and Catalan numbers appear in Theorem 5, where the Narayana refine the volume of the CRY polytope. Proctor's formula describes another form in which Catalan numbers appear. In [28], Proctor shows that

$$\prod_{1 \leq i < j \leq n} \frac{2(a-1) + i + j - 1}{i + j - 1} = \det [C_{n-2+i+j}]_{i,j=1}^{a-1}.$$

We will see Catalan numbers appear in several forms in Section 2.4 for special cases of the Morris identity, including through Proctor's formula.

2.4. The Morris constant term identity $M_n(a, b, c)$

We first formalize the notion of a constant term extraction. For a Laurent series $f(x_i)$, we denote the coefficient of x_i^j by $[x_i^j]f(x_i)$, and we denote the constant term in x_i by $\text{CT}_{x_i} f(x_i)$. Similarly, for a Laurent series $f(x_1, x_2, \dots, x_n)$, we denote the constant term by $\text{CT}_x f(x_1, x_2, \dots, x_n) := \text{CT}_{x_n} \cdots \text{CT}_{x_1} f(x_1, x_2, \dots, x_n)$.

Similarly, we define the *residue* of $f(x_i)$ with respect to x_i as the coefficient of x_i^{-1} . We denote this by $\text{Res}_{x_i} f(x_i) := [x_i^{-1}]f(x_i)$, and we also use the notation $\text{Res}_x f(x_1, \dots, x_n) := \text{Res}_{x_n} \cdots \text{Res}_{x_1} f(x_1, \dots, x_n)$. A useful property of residues is that for a meromorphic function $f(x_1, x_2, \dots, x_n)$, the residue of a partial derivative is always zero. That is,

$$\text{Res}_{x_i} \frac{\partial}{\partial x_i} f(x_1, x_2, \dots, x_n) = 0.$$

We now give some special properties and cases of the Morris constant term identity (2). Note that for $c > 0$, we can substitute $\Gamma(x+1) = x\Gamma(x)$ to obtain the following alternate form of Morris identity

$$M_n(a, b, c) = \frac{1}{n!} \prod_{j=0}^{n-1} \frac{\Gamma(a-1+b+(n-1+j)\frac{c}{2})\Gamma(\frac{c}{2})}{\Gamma(a+j\frac{c}{2})\Gamma(b+j\frac{c}{2})\Gamma(\frac{c}{2}(j+1))}.$$

This form of the Morris identity is used in most of our computational proofs.

Recall that $M_n(1, 1, 1)$ is a product of consecutive Catalan numbers. Interestingly, the case $M_n(a, 1, 1)$ strongly resembles $M_n(1, 1, 1)$, and is, by Proctor's formula, a product of Catalan numbers times a determinant of Catalan numbers.

Corollary 11. [2, 19] *The constant term $M_n(a, 1, 1)$ can be expressed as a product of consecutive Catalan numbers times a determinant of Catalan numbers.*

$$M_n(a, 1, 1) = C_1 C_2 \cdots C_{n-1} \prod_{1 \leq i < j \leq n} \frac{2(a-1) + i + n - 1}{i + n - 1} = C_1 C_2 \cdots C_{n-1} \cdot \det [C_{n-2+i+j}]_{i,j=1}^{a-1}.$$

Next, we list simplified identities for some other special cases of the Morris identities. Proofs of these formulas and other special cases, namely $M_n(a, b, 1)$ and $M_n(a, b, 2k)$, are rather computational and are hence provided in the Appendix. Intriguingly, the explicit formula for $M_n(a, b, 1)$ strongly resembles the formula for $M_n(a, 1, 1)$.

Corollary 12. *For positive integers n, a , and b , the constant term $M_n(a, b, 1)$ is given by*

$$M_{2n}(a, b, 1) = C_1 C_2 \cdots C_{2n-1} \prod_{1 \leq i < j \leq 2n} \frac{2(a+b-2) + i + j - 1}{i + j - 1} \prod_{1 \leq i \leq n} \frac{\binom{2a+2b+4i-2}{2a+2i-1}}{\binom{2a+2b+4i-2}{2i+1}}$$

$$M_{2n-1}(a, b, 1) = \binom{a+b-2}{a-1} C_1 C_2 \cdots C_{2n-2} \prod_{1 \leq i < j \leq 2n-1} \frac{2(a+b-2) + i + j - 1}{i + j - 1} \prod_{1 \leq i \leq n-1} \frac{\binom{2a+2b+4i-4}{2a+2i-2}}{\binom{2a+2b+4i-4}{2i}}.$$

By expressing the above special cases in terms of *superfactorials*, we also give in the Appendix asymptotic results for the following values of the Morris identity: $M_n(1, 1, 1)$, $M_n(n, 1, 1)$ and $M_n(n, n, 1)$.

Lastly, we also give a formula for $M_n(a, b, c)$ for even c , which curiously differs significantly from other computed special cases.

Corollary 13. *For positive integers n, a, b and k , the constant term $M_n(a, b, 2k)$ is given by the product*

$$M_n(a, b, 2k) = \prod_{i=1}^n \frac{(a+b-2 + (2i-3)k)!k!}{((i-2)k)!(ik)!} \binom{a+b-2 + (2i-2)k}{a-1 + (i-1)k}.$$

3. A recursive proof of Theorem 1

In this section we give a new recursive proof of Theorem 1 by introducing a subdivision map for the right-hand side of (1). To give our proof, we first show that all subdivisions reduce to a similar form.

Lemma 14. *Every connected directed graph G on vertex set $\{0, 1, \dots, n+1\}$ with unique source 0 and unique sink $n+1$ can be reduced to subdivisions G' with the same vertex set, unique source and sink and for $i \in [n]$, $\text{outdeg}_{G'}(i) = 1$.*

Proof. We apply the following algorithm:

- (1) Consider if graph G has a non-empty set S of vertices i such that $\text{indeg}_G(i) > 1$ and $\text{outdeg}_G(i) > 1$. Then we apply the reduction rule at any vertex in S .
- (2) Consider if graph G has a non-empty set T of vertices i such that $\text{indeg}_G(i) = 1$ and $\text{outdeg}_G(i) > 1$. Then we apply the reduction rule at any vertex in T .

We note the net flow for a vertex in T is zero, so the flow along the incoming edge must be at least the flow along any of the outgoing edges. Obtaining G_1 and G_2 as in (8) and (9), applying the map on flows in the subdivision as shown in Figure 2, gives that $\mathcal{F}_{G_1} = \emptyset$, and can be disregarded. Hence, we see that the uniqueness of the sink and source are also preserved in G_2 .

- (3) We continually apply steps (1) and (2) until $S = T = \emptyset$, at which point we conclude $\text{outdeg}_G(i) = 1$ for $i \in [n]$.

Since the graph is finite, we see the algorithm must terminate. \square

We now prove the following lemma, which establishes the base case for our induction.

Lemma 15 (Base Case). For a graph G on vertex set $\{0, 1, \dots, n+1\}$ with m edges, unique source 0 , unique sink $n+1$, and where $\text{outdeg}_G(i) = 1$ for $i \in [n]$, we have that

$$\text{vol } \mathcal{F}_G(1, 0, \dots, 0, -1) = K_G \left(0, d_1, \dots, d_n, -\sum_{i=1}^n d_i \right) = 1,$$

where $d_i = \text{indeg}_G(i) - 1$.

Proof. First we show $\text{vol } \mathcal{F}_G(1, 0, \dots, 0, -1) = 1$. Since $\text{outdeg}_G(i) = 1$ for $i \in [n]$ then the source has outdegree $m - n$, and the flows along these $m - n$ edges determines a unique flow on G . To see this, note that the flows of the outgoing edges of vertices in the set $\{0, 1, \dots, i\}$ for $i \in [n]$ determine recursively the outgoing flow at vertex $i + 1$. We see that \mathcal{F}_G is integrally equivalent to a $(m - n - 1)$ -dimensional simplex and has normalized volume 1.

Next we show that $K_G(0, d_1, \dots, d_n, -\sum_{i=1}^n d_i) = 1$. We recursively show that there is only one integer flow f with the desired net flow. Since the source has net flow zero, then $f(0, i) = 0$ for $i \in [n]$. Then the flows of the outgoing edges of vertices in the set $\{0, 1, \dots, i\}$ recursively determine the outgoing flow from vertex $i + 1$ since $\text{outdeg}_G(i + 1) = 1$. Thus, only a single integer flow f is possible. \square

We now define some notation. For a reducible graph G_0 on vertex set $\{0, 1, \dots, n, n+1\}$, let G_1 and G_2 be obtained by equations (8) and (9) for fixed $(i, j), (j, k) \in E(G_0)$. Let $d'_i = \text{indeg}_{G_1}(i) - 1$, and let $d''_i = \text{indeg}_{G_2}(i) - 1$. Also, let $\mathbf{d} = (0, d_2, \dots, d_n, -\sum_{i=2}^n d_i)$ and likewise define

$$\mathbf{d}_1 = \left(0, d'_2, \dots, d'_n, -\sum_{i=2}^n d'_i \right) \text{ and } \mathbf{d}_2 = \left(0, d''_2, \dots, d''_n, -\sum_{i=2}^n d''_i \right).$$

We prove that if G_1 and G_2 satisfy Theorem 1, so does G_0 . By the subdivision lemma, we have that

$$\text{vol } \mathcal{F}_{G_0} = \text{vol } \mathcal{F}_{G_1} + \text{vol } \mathcal{F}_{G_2}.$$

Hence, it suffices we show the following lemma.

Lemma 16 (Inductive Step). Let G_0, G_1, G_2 and $\mathbf{d}, \mathbf{d}_1, \mathbf{d}_2$ be as defined above. Then,

$$K_{G_0}(\mathbf{d}) = K_{G_1}(\mathbf{d}_1) + K_{G_2}(\mathbf{d}_2).$$

Proof. Note that since $\mathbf{d}_1 \neq \mathbf{d}_2$, we have that $\mathcal{F}_{G_1}^{\mathbb{Z}}(\mathbf{d}_1)$ and $\mathcal{F}_{G_2}^{\mathbb{Z}}(\mathbf{d}_2)$ are disjoint. We give a bijection

$$\varphi : \mathcal{F}_{G_0}^{\mathbb{Z}}(\mathbf{d}) \rightarrow \mathcal{F}_{G_1}^{\mathbb{Z}}(\mathbf{d}_1) \dot{\cup} \mathcal{F}_{G_2}^{\mathbb{Z}}(\mathbf{d}_2).$$

For an integer flow $f \in F_{G_0}^{\mathbb{Z}}(\mathbf{d})$, let $x = f(i, j)$, and let $y = f(j, k)$. Then we denote by $F_{G_0}^{\mathbb{Z}}(\mathbf{d}; y \leq x)$ the subset of $F_{G_0}^{\mathbb{Z}}(\mathbf{d})$ where $y \leq x$, and likewise let $F_{G_0}^{\mathbb{Z}}(\mathbf{d}; y > x)$ denote the subset of $F_{G_0}^{\mathbb{Z}}(\mathbf{d})$ where $y > x$.

We let φ_1 be restriction of φ to $F_{G_0}^{\mathbb{Z}}(\mathbf{d}; y \leq x)$, and φ_2 the restriction of φ to $F_{G_0}^{\mathbb{Z}}(\mathbf{d}; y > x)$. We now construct φ_1 and φ_2 as bijections with disjoint codomains where the union is the codomain of φ . We define φ_1 and φ_2 as illustrated in Figure 3.

More formally, we define

$$\varphi_1 : F_{G_0}^{\mathbb{Z}}(\mathbf{d}; y \leq x) \rightarrow \mathcal{F}_{G_1}^{\mathbb{Z}}(\mathbf{d}_1),$$

where $f \mapsto f'$ given by

$$f'(e) = \begin{cases} x - y, & e = (i, j) \\ y, & e = (i, k) \\ f(e), & e \in E(G_1) \setminus \{(i, j), (i, k)\}. \end{cases}$$

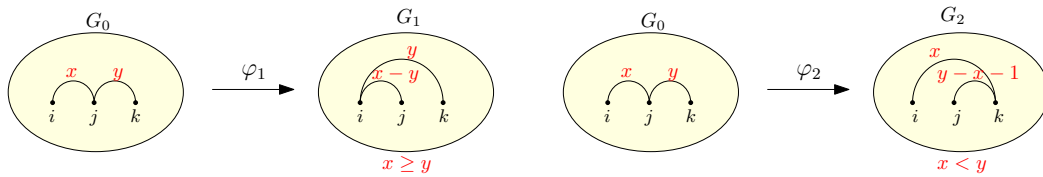


Figure 3. The maps φ_1 and φ_2 for integer flows of a subdivided graph.

Since the indegrees in G_0 and G_1 are the same, we see that the net flow vector is $\mathbf{d}_1 = \mathbf{d}$, so the map is well-defined. We now construct the inverse map φ_1^{-1} with $f \mapsto f'$ given by

$$f'(e) = \begin{cases} f(i, j) + f(i, k), & e = (i, j) \\ f(i, k), & e = (j, k) \\ f(e), & e \in E(G_0) \setminus \{(i, j), (j, k)\}. \end{cases}$$

The net flow vector is again unchanged, so the map is well-defined and therefore φ_1 is a bijection.

We now construct a second bijection

$$\varphi_2 : \mathcal{F}_{G_0}^{\mathbb{Z}}(\mathbf{d}; y > x) \rightarrow \mathcal{F}_{G_2}^{\mathbb{Z}}(\mathbf{d}_2),$$

with $f \mapsto f'$ given by

$$f'(e) = \begin{cases} y - x - 1, & e = (j, k) \\ x, & e = (i, k) \\ f(e), & e \in E(G_2) \setminus \{(j, k), (i, k)\}. \end{cases}$$

The only change in indegrees is that $d_j'' = d_j - 1$ and $d_k'' = d_k + 1$. However, the outgoing flow at vertex j also decreases by 1, whereas the incoming flow at vertex k also decreases by 1, so the net flow vector is indeed \mathbf{d}_2 . We similarly construct the inverse map φ_2^{-1} with $f \mapsto f'$ given by

$$f'(e) = \begin{cases} f(i, k), & e = (i, j) \\ f(j, k) + f(i, k) + 1, & e = (j, k) \\ f(e), & e \in E(G_0) \setminus \{(i, j), (j, k)\}. \end{cases}$$

The only indegrees that change are $d_j = d_j'' + 1$ and $d_k = d_k'' - 1$, but since the outgoing flow at vertex j increases by 1 and the incoming flow at vertex k decreases by 1, we see the graph is locally unchanged. Hence, the map is well-defined and φ_2 is a bijection as well.

Since $\mathcal{F}_{G_1}^{\mathbb{Z}}(\mathbf{d}_1)$ and $\mathcal{F}_{G_2}^{\mathbb{Z}}(\mathbf{d}_2)$ are disjoint, we have that φ is a bijection, and the result follows. \square

Recursive proof of Theorem 1. The proof follows from the base case given in Lemma 14 and Lemma 15, and the inductive step established in Lemma 16. \square

4. Symmetry of $M_n(a, b, c)$

This section is about a fundamental symmetry of $M_n(a, b, c)$, that is invariant under switching a and b . We give three proofs of this result with the three perspectives for $M_n(a, b, c)$ illustrated in Theorem 4: as a product formula, as the volume of a flow polytope, and as the number of certain integer flows. The last proof is bijective.

Corollary 17 (Symmetry of M_n). *For a positive integers n, a and b and nonnegative integers c we have that $M_n(a, b, c) = M_n(b, a, c)$.*

4.1. Three proofs of Corollary 17

First proof. This can directly be seen from symmetry of a and b in the product formula (2) of $M_n(a, b, c)$. \square

The second proof is based on Lemma 9.

Proof. The result is a Corollary of [20, Proposition 2.3] for netflow $\mathbf{a} = (1, 0, \dots, 0, -1)$. \square

Second proof of Corollary 17. Since the reverse of the graph $k_{n+2}^{a,b,c}$ is the graph $k_{n+2}^{b,a,c}$ then by Lemma 9 we have that $\mathcal{F}_{k_{n+2}^{a,b,c}} \equiv \mathcal{F}_{k_{n+2}^{b,a,c}}$. The volume of the two polytopes must be equal and the result follows by (3). \square

The third proof is based on the following lemma from [20]. The lemma was originally proved geometrically by combining Lemma 9 with Theorem 1. We give here a bijective proof of this result.

Lemma 18 ([20, Corollary 1.4]). *For a loopless digraph G with vertices $\{0, 1, \dots, n+1\}$ having a unique source 0 and unique sink $n+1$ we have*

$$K_G\left(0, d_1, \dots, d_n, -\sum_{i=1}^n d_i\right) = K_{G^r}\left(0, d_1^r, \dots, d_n^r, -\sum_{i=1}^n d_i^r\right),$$

where $d_i = \text{indeg}_G(i) - 1$ and $d_i^r = \text{indeg}_{G^r}(i) - 1$.

Remark 19. Lemma 18 states that two flow polytopes have the same number of lattice points. These polytopes can have different dimensions because of the different vertices with zero net-flow. That is, if $G|_{[n+1]}$ is the subgraph of G restricted to vertices $[n+1]$ then the equivalent polytopes $\mathcal{F}_{G|_{[n+1]}}(d_1, \dots, d_n, -\sum_i d_i)$ and $\mathcal{F}_{G^r|_{[n+1]}}(d_1^r, \dots, d_n^r, -\sum_i d_i^r)$ have dimensions $E(G|_{[n+1]}) - n - 2$ and $E(G^r|_{[n+1]}) - n - 2$, respectively.

Third proof of Corollary 17. Since the reverse of the graph $k_{n+2}^{a,b,c}$ is the graph $k_{n+2}^{b,a,c}$ then by Lemma 18 we have that

$$K_{k_{n+2}^{a,b,c}}\left(0, t_1, \dots, t_n, -\sum_{i=1}^n t_i\right) = K_{k_{n+2}^{b,a,c}}\left(0, s_1, \dots, s_n, -\sum_{i=1}^n s_i\right),$$

where $t_i = a - 1 + c(i - 1)$ and $s_i = b - 1 + c(i - 1)$ for $i = 1, \dots, n$. The result then follows by Theorem 4. \square

The rest of the section is devoted to the bijective proof of Lemma 18 using a bijection from [23, Section 7] and inspired by a bijection between lattice points of generalized permutahedra (see [27, Theorem 12.9] and [12]).

4.2. Bijection between lattice points of $\mathcal{F}_G(0, d_1, \dots)$ and $\mathcal{F}_{G^r}(0, d_1^r, \dots)$

First, we describe the Danilov, Karzanov, Koshevoy (DKK) triangulations of \mathcal{F}_G from [8]. Recall that the vertices of \mathcal{F}_G are given by unit flows along routes on G , i.e. directed paths in G from the source 0 to the sink $n+1$. Given a route R with vertex v , Rv and vR denote the subpaths ending and starting at v , respectively. A *framing* at an inner vertex v is a pair $(\leq_{\text{in}(v)}, \leq_{\text{out}(v)})$ of linear orderings on the set of incoming edges $\text{in}(v)$ to v and on the set of outgoing edges $\text{out}(v)$ from v . A *framed graph* (G, \leq) is a graph G with a framing \leq at each internal vertex.

Fix a framed graph (G, \leq) . For an internal vertex i , let $\text{In}(i)$ and $\text{Out}(i)$ be the sets of maximal paths ending and starting at i , respectively. Given a framed graph, we define an order $\leq_{\text{In}(i)}$ on $\text{In}(i)$ as follows. Given distinct paths R, Q in $\text{In}(i)$, let $j \leq i$ be the smallest vertex after which Ri and Qi coincide and let e_R be the edge of R entering i and e_Q be the edge of Q entering i . We have $R \leq_{\text{In}(i)} Q$ if and only if $e_R \leq_{\text{in}(i)} e_Q$. We define an order $\leq_{\text{Out}(i)}$ on $\text{Out}(i)$ analogously. We

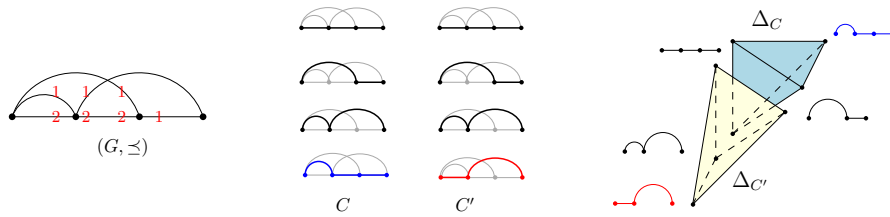


Figure 4. A framed graph (G, \leq) , the two maximal cliques in $\mathcal{C}_{G, \leq}^{\max}$ and the corresponding DKK triangulation of \mathcal{F}_G .

say that routes R and Q with a common inner vertex i are *coherent at i* whenever $Ri \leq_{In(i)} Qi$ if and only if $iR \leq_{Out(i)} iQ$. That is, if the paths iR and iQ are ordered the same as Ri and Qi . Routes R and Q are *coherent* if they are coherent at each internal vertex they have in common. A set of pairwise coherent routes is called a *clique*. Let $\mathcal{C}^{\max}(G, \leq)$ be the set of maximal cliques of (G, \leq) . For a maximal clique C , let Δ_C be the convex hull of the vertices of \mathcal{F}_G corresponding to the routes in C .

In [8, Theorems 1. & 2] show that given a framed graph (G, \leq) , the set $\{\Delta_C \mid C \in \mathcal{C}^{\max}(G, \leq)\}$ is the set of top dimensional simplices in a regular unimodular triangulation of \mathcal{F}_G . See Figure 4. As a corollary, we obtain another combinatorial object, maximal cliques, whose number gives the volume of \mathcal{F}_G .

Corollary 20. For a framed graph (G, \leq) where G is a loopless digraph with vertices $\{0, 1, \dots, n+1\}$ having a unique source 0 and sink $n+1$ then $K_G(0, d_1, \dots, d_n, -\sum_{i=1}^n d_i) = |\mathcal{C}^{\max}(G, \leq)|$.

In [23] the authors give an explicit bijection between the maximal cliques and the integer flows. The map from the cliques to the flows is as follows.

Definition 21 ([23, Section 7]). Given a framed graph (G, \leq) and $\mathbf{d} = (0, d_1, \dots, d_n, -\sum_i d_i)$, let $\Omega_{G, \leq} : \mathcal{C}^{\max}(G, \leq) \rightarrow \mathcal{F}_G^{\mathbb{Z}}(\mathbf{d})$ be defined as follows $\Omega_{G, \leq} : C \mapsto f$ where $f(e) = n(e) - 1$ where $n(i, j)$ is the number of times edge (i, j) appears in the set of prefixes $\{Rj \mid R \in C\}$.

Lemma 22 ([23, Lemma 7.9]). Given a framed graph (G, \leq) , the map $\Omega_{G, \leq}$ is a bijection between maximal cliques $\mathcal{C}^{\max}(G, \leq)$ and integer flows in $\mathcal{F}_G^{\mathbb{Z}}(0, d_1, \dots, d_n, -\sum_i d_i)$.

The inverse map $\Omega_{G, \leq}^{-1}$ can be found in [23, Section 7]) (denoted $\Lambda_{G, \leq}$).

A framing (G, \leq) induces the following framing on the reverse graph G^r : for an internal vertex i , let $\leq_{in(i)}^r := \leq_{out(i)}$ and $\leq_{out(i)}^r := \leq_{in(i)}$. We denote this induced framing by (G^r, \leq^r) .

Remark 23. Note that the framings (G, \leq) and (G^r, \leq^r) induce the same triangulation of $\mathcal{F}_G \equiv \mathcal{F}_{G^r}$. In other words up to reversing the routes, the cliques of (G, \leq) and (G^r, \leq^r) are the same. Thus $r : C \mapsto \{Q^r \mid Q \in C\}$ is a bijection between $\mathcal{C}^{\max}(G, \leq)$ and $\mathcal{C}^{\max}(G^r, \leq^r)$.

We are now ready to define the map that will give the desired bijection from the integer flows in G and the integer flows in G^r .

Definition 24. Given a framed graph (G, \leq) , let $\Theta_{G, \leq} := \Omega_{G^r, \leq^r} \circ r \circ \Omega_{G, \leq}^{-1}$ which is a map from $\mathcal{F}_G^{\mathbb{Z}}(0, d_1, \dots, d_n, -\sum_i d_i)$ to $\mathcal{F}_{G^r}^{\mathbb{Z}}(0, d_1^r, \dots, d_n^r, -\sum_i d_i^r)$. See Figure 5 for example.

Lemma 25. Given a framed graph (G, \leq) , the map $\Theta_{G, \leq}$ is a bijection between $\mathcal{F}_G^{\mathbb{Z}}(0, d_1, \dots, d_n, -\sum_i d_i)$ to $\mathcal{F}_{G^r}^{\mathbb{Z}}(0, d_1^r, \dots, d_n^r, -\sum_i d_i^r)$.

Proof. By Lemma 22 $\Omega_{G, \leq}^{-1}$ is a bijection between $\mathcal{F}_G^{\mathbb{Z}}(0, d_1, \dots, d_n, -\sum_i d_i)$ and $\mathcal{C}^{\max}(G, \leq)$. Next, by reversing the routes in the cliques, we identify C in $\mathcal{C}^{\max}(G, \leq)$ with $r(C)$ in $\mathcal{C}^{\max}(G^r, \leq^r)$. Finally, by Lemma 22 the latter is in bijection with $\mathcal{F}_{G^r}^{\mathbb{Z}}(0, d_1^r, \dots, d_n^r, -\sum_i d_i^r)$ via Ω_{G^r, \leq^r} . \square

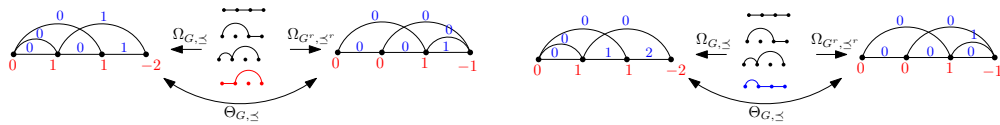


Figure 5. Example of the correspondence $\Theta_{G,\leq}$ between integer flows in $\mathcal{F}_G(0, d_1, d_2, \dots)$ and $\mathcal{F}_{G^r}(0, d_1^r, d_2^r, \dots)$.

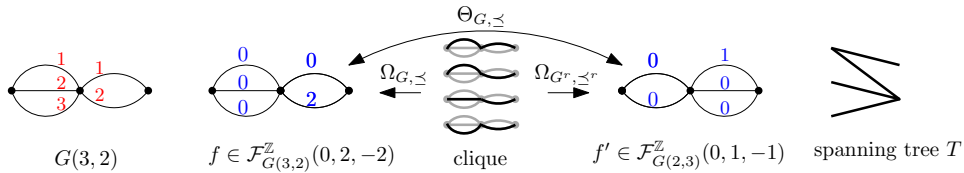


Figure 6. Example of correspondence $\Theta_{G,\leq}$ that gives the same bijection between the lattice points of $(p-1)\Delta^{q-1}$ and $(q-1)\Delta^{p-1}$ studied by Postnikov [27] in terms of spanning trees.

Finally, the correspondence $\Theta_{G,\leq}$ gives the bijective proof of Lemma 18.

Proof of Lemma 18. The result follows by Lemma 25. \square

Remark 26. The DKK triangulation of a flow polytope \mathcal{F}_G gives the bijection between the integer flows. Indeed, given a clique in $\mathcal{C}_{G,\leq}^{\max}$, $\Omega_{G,\leq} : C \mapsto f$ where $f(i, j) = n(i, j) - 1$ where $n(i, j)$ is the number of times (i, j) appears in the set of prefixes $\{jR \mid R \in C\}$ and $\Omega_{G^r,\leq^r} : C \mapsto f'$ where $f'(i, j) = n'(i, j) - 1$, where $n'(i, j)$ is the number of times edge $(n+1-j, n+1-i)$ appears in the set of suffixes $\{(n+1-j)R \mid R \in C\}$. See Figure 5.

Example 27. For positive integers p and q , let $G(p, q)$ be the graph with vertices $\{0, 1, 2\}$, p edges $(0, 1)$, and q edges $(1, 2)$. One readily sees that the flow polytope $\mathcal{F}_{G(p,q)}$ is the product of simplices $\Delta^{p-1} \times \Delta^{q-1}$. By Theorem 47 we have that

$$\text{vol } \mathcal{F}_{G(p,q)} = K_{G(p,q)}(0, p-1, -p+1) = \#((p-1)\Delta^{q-1} \cap \mathbb{Z}^q) = \binom{p+q-2}{p-1}.$$

Since $G(p, q)^r = G(q, p)$, then for each of the $p!q!$ framings of $G(p, q)$ we have that $\Theta^{G(p,q),\leq}$ is a bijection between the lattice points of $(p-1)\Delta^{q-1}$ and $(q-1)\Delta^{p-1}$. By a result of Postnikov [27, Lemma 12.5], if C is a maximal clique in $\mathcal{C}_{G(p,q)}^{\max}$, then the following subgraph of the complete bipartite graph on vertices $\{1, \dots, p\} \cup \{\bar{1}, \dots, \bar{q}\}$ is a spanning tree: the subgraph T with edges (i, \bar{j}) if C contains the route of $G(p, q)$ with the i th edge $(0, 1)$ and j th edge $(1, 2)$ in the framing (see Figure 6). Moreover, the coherent condition on the routes of the clique translate to the *compatibility* condition on spanning trees (see [27, Section 12] and [12, Section 2]), and the bijection $\Theta^{G(p,q),\leq}$ consists of mapping the right degrees minus one to the left degrees minus one of the spanning tree T [27, Theorem 12.9].

5. A new refinement of $M_n(a, b, c)$

We define the following constant term in order to refine $M_n(a, b, c)$.

Definition 28. Define the following constant term

$$\Psi_n(k, a, b, c) := \text{CT}_x \left[t^k \right] \prod_{i=1}^n (1 - x_i)^{-b} x_i^{-a+1} \left(1 + t \frac{x_i}{1 - x_i} \right) \prod_{1 \leq i < j \leq n} (x_j - x_i)^{-c}.$$

5.1. Volume and Kostant partition function interpretations for $\Psi_n(k, a, b, c)$

We prove both parts of Theorem 6.

Proof of Theorem 6 (i). We first prove the Kostant partition interpretation of $\Psi_n(k, a, b, c)$. We specialize the generating function in equation (10) as in the proof of Theorem 4 in Section 2.2. Note that compared with $M_n(a, b, c)$, $\Psi_n(k, a, b, c)$ has an extra term $[t^k] \prod_{i=1}^n (1 + t \frac{x_i}{1 - x_i})$. This term selects k values of $\{1, \dots, n\}$ and for each selected i multiplies the generating series by $\frac{x_i}{1 - x_i}$.

By linearity of constant term extraction,

$$\text{CT}_{x_i} \frac{x_i}{1 - x_i} f(x_1, \dots, x_n) = \sum_{r=1}^{\infty} \text{CT}_{x_i} x_i^r f(x_1, \dots, x_n) = \sum_{r=1}^{\infty} [x_i^{-r}] f(x_1, \dots, x_n). \quad (11)$$

We then substitute the generating function for $M_n(a, b, c)$ is substituted for $f(x_1, \dots, x_n)$ in the RHS of (11) and apply the Kostant partition function interpretation of $M_n(a, b, c)$. However, instead of taking the case where the net flow at vertex i is $a - 1 + c(i - 1)$, we sum the cases where the net flow at i is $a - 2 + c(i - 1)$, $a - 3 + c(i - 1)$, \dots . Hence, this is equivalent to strictly decreasing the net flow at vertex i in the Kostant partition function interpretation. Since we take the coefficient of t^k , there are exactly k vertices with net flow $a_i < a - 1 + c(i - 1)$, and $n - k$ vertices with net flow $a_i = a - 1 + c(i - 1)$. The result follows. \square

Example 29. The number $\Psi_2(k, a, b, c)$ for $k = 0, 1, 2$ counts the following sums of Kostant partition functions:

$$\begin{aligned} \Psi_2(0, a, b, c) &= K_{k_4^{a,b,c}}(0, a - 1, a + c - 1) \\ \Psi_2(1, a, b, c) &= \sum_{t_2 < a + c - 1} K_{k_4^{a,b,c}}(0, a - 1, t_2, -a - t_2 + 1) + \sum_{t_1 < a - 1} K_{k_4^{a,b,c}}(0, t_1, a + c - 1, -t_1 - a - c + 1) \\ \Psi_2(2, a, b, c) &= \sum_{t_1 < a - 1, t_2 < a + c - 1} K_{k_4^{a,b,c}}(0, t_1, t_2, -t_1 - t_2). \end{aligned}$$

We now prove the volume interpretation. To do so, we first define a modification of $k_{n+2}^{a,b,c}$.

Definition 30. For a set $S \subseteq [n]$, let $k_{n+2}^{a,b,c}(S)$ be the graph obtained from $k_{n+2}^{a,b,c}$ by adding n edges $(0, n + 1)$ and for each $i \in S$ we delete one of the a incoming edges $(0, i)$ and add an outgoing edge $(i, n + 1)$.

For a set $S \subseteq [n]$, define also the set $T(S)$ as the set of vectors $\mathbf{a} = (0, a_1, a_2, \dots, a_n, -\sum_{j=1}^n a_j)$ with $a_i < a - 1 + c(i - 1)$ for $i \in S$ and $a_i = a - 1 + c(i - 1)$ for $i \notin S$.

Proof of Theorem 6 (ii). First we show that

$$\sum_{\mathbf{a} \in T(S)} K_{k_{n+2}^{a,b,c}}(\mathbf{a}) = \text{vol } \mathcal{F}_{k_{n+2}^{a,b,c}(S)}. \quad (12)$$

Consider the Kostant partition functions on the left-hand side. For each vertex $i \in S$, we remove an incoming edge $(0, i)$ (decreasing a by 1) to create a weak inequality instead of a strict equality. We then add an outgoing edge $(i, n + 1)$ to carry the necessary flow to force the equality $a_i = \text{indeg}(i) - 1$. This process is shown in Figure 7a.

We note that if we were to add the edge $(0, n + 1)$ n times, the volume would not change. This is due to Theorem 1, which gives the volume as a Kostant partition function where the source has

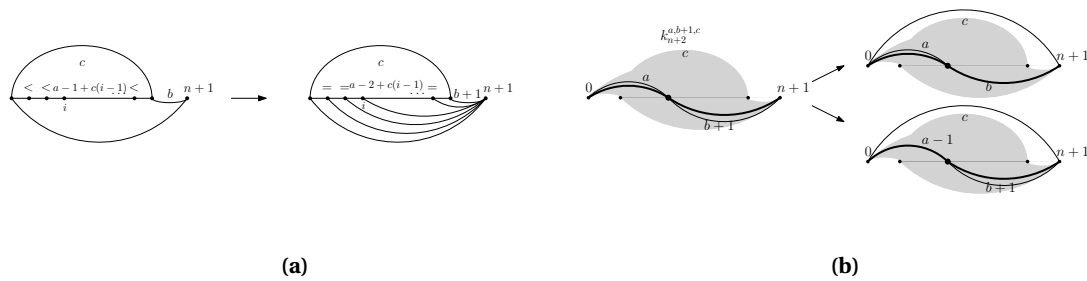


Figure 7. (A) For each strict inequality where $a_i < \text{indeg}(i) - 1$, the inequality can be weakened by decreasing a by 1 to obtain $a_i \leq \text{indeg}(i) - 1$. We then add an additional edge $(i, n+1)$ to carry the necessary flow such that $a_i = \text{indeg}(i) - 1$. (B) The graph $k_{n+2}^{a,b+1,c}$ is shown in gray on the left with some highlighted edges in black. The graphs on the right give the two subdivisions obtained from applying the subdivision lemma on a single internal vertex.

zero net flow. Since an edge $(0, n+1)$ also would not affect the indegree of any internal vertex, it has no effect on the Kostant partition function or volume. By adding the edge $(0, n+1)$ n times, the graph becomes $k_{n+2}^{a,b,c}(S)$.

Hence, since each internal vertex has net flow $a_i = \text{indeg}(i) - 1$, we apply Theorem 1 again to obtain that this Kostant partition function is equal to $\text{vol} \mathcal{F}_{k_{n+2}^{a,b,c}(S)}$, thus proving equation (12).

We can now sum both sides of equation (12) over $S \in \binom{[n]}{k}$. Since the Kostant partition function interpretation of $\Psi_n(k, a, b, c)$ counts all flows with exactly k strict inequalities $a_i < a - 1 + c(i-1)$, we see that the left-hand side is now $\Psi_n(k, a, b, c)$, and the result follows. \square

On a polytope level, the volume interpretation of Theorem 6(ii) translates to the following result.

Lemma 31. For $S \subseteq [n]$, the polytopes $\mathcal{F}_{k_{n+2}^{a,b,c}(S)}$ are interior disjoint and satisfy

$$\mathcal{F}_{k_{n+2}^{a,b+1,c}} \equiv \bigcup_{S \subseteq [n]} \mathcal{F}_{k_{n+2}^{a,b,c}(S)}.$$

Proof. We apply the subdivision lemma (8) and (9) at each internal vertex of $\mathcal{F}_{k_{n+2}^{a,b+1,c}}$ exactly once. For each internal vertex i , the edge $(0, n+1)$ is added, and either an incoming edge $(0, i)$ or outgoing edge $(i, n+1)$ is deleted. This is shown in Figure 7b.

That is, for each vertex i , one of two cases must hold:

- (i) Edge $(0, i)$ appears $a-1$ times and edge $(i, n+1)$ appears $b+1$ times.
- (ii) Edge $(0, i)$ appears a times and edge $(i, n+1)$ appears b times.

Each reduced graph is the polytope $\mathcal{F}_{k_{n+2}^{a,b,c}(S)}$, where S is the set of vertices satisfying case (i). Since the polytopes $\mathcal{F}_{k_{n+2}^{a,b,c}(S)}$ are obtained by applying the reduction rule on $\mathcal{F}_{k_{n+2}^{a,b+1,c}}$, they are interior disjoint by Proposition 10, and their union is integrally equivalent to $\mathcal{F}_{k_{n+2}^{a,b+1,c}}$. \square

As an application of these interpretations we now prove Corollary 7, which refines the product $M_n(a, b, c)$.

Example 32. The polytope $\mathcal{F}_{k_4^{a,b+1,c}}(1, 0, \dots, 0, -1)$ can be subdivided into four flow polytopes of the form $\mathcal{F}_{k_4^{a,b+1,c}(S)}(1, 0, \dots, 0, -1)$, which are grouped into three collections based on the size of S . The volume of the collection corresponding to $|S| = k$ is counted by $\Psi_2(k, a, b, c)$. See Figure 8.

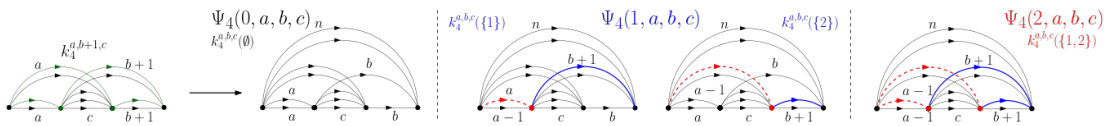


Figure 8. Three collections of flow polytopes $\mathcal{F}_{k_4^{a,b,c}}(S)$ of the subdivision of $\mathcal{F}_{k_4^{a,b,c}}$ whose volumes are given by $\Psi_2(k, a, b, c)$ for $k = 0, 1, 2$ respectively.

Proof of Corollary 7 via Kostant partition function. The sum on the right-hand side of (5) over k of $\Psi_n(k, a, b, c)$ is the sum of all Kostant partition functions $K_{k_{n+2}^{a,b,c}}(0, a_1, \dots, a_n, -\sum_{j=1}^n a_j)$ such that for $i \in [n]$, $a_i \leq a - 1 + c(i - 1)$. This is equivalent to adding another edge between each vertex i and the sink with flows such that each net flow satisfies $a_i = a - 1 + c(i - 1)$. Hence we see this sum is $M_n(a, b + 1, c)$. \square

Proof of Corollary 7 via volumes. Using Lemma 31 and computing the volume on both sides gives

$$\text{vol} \mathcal{F}_{k_{n+2}^{a,b+1,c}} = \sum_{S \subseteq [n]} \text{vol} \mathcal{F}_{k_{n+2}^{a,b,c}}(S).$$

The result follows by applying Theorem 4 to the left-hand side and Theorem 6 to the right-hand side. \square

5.2. Recurrence Relations of $\Psi_n(k, a, b, c)$

In this section we prove recurrence relations satisfied by $\Psi_n(k, a, b, c)$ and that are instrumental to our proof of Theorem 8. First we show two cases where $\Psi_n(k, a, b, c)$ is equivalent to the Morris identity.

Proposition 33. *Let n, a, b, c be positive integers. Then*

$$\Psi_n(0, a, b, c) = M_n(a, b, c) \quad (13)$$

$$\Psi_n(n, a, b, c) = M_n(a - 1, b + 1, c). \quad (14)$$

Proof. The first equation holds since n equalities implies the exact same constraints as those of $M_n(a, b, c)$. The second equation holds since 0 equalities implies the upper bound of the inequalities can be decreased by 1 (by decreasing a by 1) to make a weak inequality, and another edge from each vertex to the sink can be added with the necessary flow to force equality. This transformation gives a bijection with $M_n(a - 1, b + 1, c)$. \square

We now construct a bijection on the sets of integer flows of flow polytopes

$$F_{k_{n+2}^{1,b,c}} \left(0, 0, a_2, \dots, a_n, -\sum_{j=2}^n a_j \right) \quad \text{and} \quad \mathcal{F}_{k_{n+1}^{c+1,b,c}} \left(0, a_2, \dots, a_n, -\sum_{j=2}^n a_j \right)$$

by contracting an edge in the corresponding graph.

Proposition 34. *For a net flow vector*

$$\mathbf{a} = \left(0, 0, a_2, \dots, a_n, -\sum_{j=2}^n a_j \right), \quad \text{let} \quad \tilde{\mathbf{a}} := \left(0, a_2, \dots, a_n, -\sum_{j=2}^n a_j \right).$$

Then for positive integers b and n and nonnegative integer c ,

$$K_{k_{n+2}^{1,b,c}}(\mathbf{a}) = K_{k_{n+1}^{c+1,b,c}}(\tilde{\mathbf{a}}). \quad (15)$$

Proof. Define the map

$$\varphi : \mathcal{F}_{k_{n+2}}^{\mathbb{Z}}(\mathbf{a}) \rightarrow \mathcal{F}_{k_{n+1}}^{\mathbb{Z}}(\tilde{\mathbf{a}}),$$

where $f \mapsto f'$, with $f'(i, j)$ given by

$$f'(i, j) = \begin{cases} 0, & i = 0 \\ f(i+1, j+1), & 1 \leq i < j \leq n. \end{cases}$$

This map contracts edge $(0, 1)$ to create vertex 0 and relabels each vertex $i \in [2, n+1]$ by $i-1$. We see the c edges $(1, i)$ for $i \in [2, n]$ become identical with the edges of the form $(0, i)$, $i \in [2, n]$. Hence, the graph transforms to become $k_{n+1}^{c+1, b, c}$. We now show that φ is a bijection. It is sufficient we show φ has a well-defined inverse function for all $f \in \mathcal{F}_{k_{n+1}}^{c+1, b, c}(\tilde{\mathbf{a}})$. Define the inverse map

$$\varphi^{-1} : \mathcal{F}_{k_{n+1}}^{c+1, b, c}(\tilde{\mathbf{a}}) \rightarrow \mathcal{F}_{k_{n+2}}^{\mathbb{Z}}(\mathbf{a})$$

where $f \mapsto f'$, with $f'(i, j)$ given by

$$f'(i, j) = \begin{cases} 0, & 0 \leq i \leq 1 \\ f(i-1, j-1), & 2 \leq i < j \leq n+1. \end{cases}$$

Note that for

$$(i, j) \in E\left(\mathcal{F}_{k_{n+2}}^{\mathbb{Z}}(\mathbf{a})\right), f(0, j) = f(1, j) = 0$$

as the net flows at vertices 0 and 1 are both zero, so we see φ^{-1} is indeed our desired inverse function. Thus φ is a bijection, and the result follows. \square

We further strengthen this contraction identity to hold bijectively for $\Psi_n(k, 1, b, c)$.

Lemma 35 (Contraction Lemma). *For positive integers b and n and nonnegative integers c and $k \leq n$,*

$$\Psi_n(k, 1, b, c) = \Psi_{n-1}(k, c+1, b, c).$$

Proof. Recall $\Psi_n(k, 1, b, c)$ is the sum of Kostant partition functions of the form $K_{k_{n+2}}^{1, b, c}(\mathbf{a})$, with $\mathbf{a} = (0, 0, a_2, \dots, -\sum_{j=2}^n a_j)$ where for $i \in [2, n]$, $a_i < c(i-1)$ holds k times and $a_i = c(i-1)$ holds $n-1-k$ times (since the first internal vertex trivially satisfies this equality). Let A be the set of all such \mathbf{a} satisfying these conditions. That is,

$$\Psi_n(k, 1, b, c) = \sum_{\mathbf{a} \in A} K_{k_{n+2}}^{1, b, c}(\mathbf{a}).$$

Similarly, let A' be the set of all $\mathbf{a}' = (0, a_2, \dots, -\sum_{j=2}^n a_j)$ where for $i \in [2, n]$, $a_i < c(i-1)$ holds k times and $a_i = c(i-1)$ holds $n-1-k$ times. Then

$$\Psi_{n-1}(k, c+1, b, c) = \sum_{\mathbf{a}' \in A'} K_{k_{n+2}}^{c+1, b, c}(\mathbf{a}').$$

We see that the map

$$\varphi : A \rightarrow A', \left(0, 0, a_2, \dots, -\sum_{j=2}^n a_j\right) \mapsto \left(0, a_2, \dots, -\sum_{j=2}^n a_j\right)$$

is a bijection. By equating the Kostant partition functions using equation (15), the result follows. \square

As a result of this lemma, the following two corollaries are immediate.

Corollary 36. *For positive integers b and n and nonnegative integer c ,*

$$M_n(1, b, c) = M_{n-1}(c+1, b, c).$$

Proof. This is a result of Lemma 35 when $k = 0$. \square

Corollary 37. *For positive integers a and n , it holds that*

$$M_n(a, 1, 1) = \sum_{k=0}^n \Psi_{n-1}(k, a, 1, 1).$$

Proof. By Corollary 36 and Proposition 17, we see that $M_n(a, 1, 1) = M_{n-1}(a, 2, 1)$. Hence applying Lemma 7, the result follows. \square

Following the approach of Baldoni–Vergne in their proof of Theorem 47, we now give relations of $\Psi_n(k, a, b, c)$ that we later show uniquely determine this function.

Lemma 38. *For nonnegative integer c , positive integers a, b, n , and nonnegative integer $k \leq n$, the constant term $\Psi_n(k, a, b, c)$ satisfies the following identities:*

$$\Psi_n(n, a, b, c) = \Psi_n(0, a-1, b+1, c) \quad (16)$$

$$\Psi_n(n-1, 1, b, c) = \Psi_{n-1}(0, c, b+1, c) \quad (17)$$

$$\Psi_n(0, 1, b, 0) = 1 \quad (18)$$

$$k(b + (k-1)c/2) \cdot \Psi_n(k, a, b, c) = (n-k+1)(a-1 + (n-k)c/2) \cdot \Psi_n(k-1, a, b, c) \\ \text{for } 1 \leq k \leq n. \quad (19)$$

Proof. The first relation follows immediately from the bijections in Proposition 33. The second relation follows by applying Lemma 35 to the left-hand side, which turns the equation into $\Psi_{n-1}(n-1, c+1, b, c) = \Psi_{n-1}(0, c, b+1, c)$, which follows from the first relation. The third relation follows from applying the interpretation of Theorem 6 (i) to the left-hand side of (18), which gives $a_i = a-1 + c(n-1) = 0$ for all $i \in [n]$. Hence there is only one flow where the flow along each edge is zero. Lastly we prove the fourth relation. This is the sole relation we prove algebraically.

Let

$$U := \prod_{i=1}^n (1-x_i)^{-b} x_i^{-a} \prod_{1 \leq i < j \leq n} (x_j - x_i)^{-c},$$

and let

$$P_k := k!(n-k)! \left[t^k \right] \prod_{i=1}^k \left(1 + t \frac{x_i}{1-x_i} \right),$$

where

$$\left[t^k \right] \prod_{i=1}^k \left(1 + t \frac{x_i}{1-x_i} \right) = e_k \left(\frac{x_i}{1-x_i} \right)$$

is the k^{th} elementary symmetric function in $\frac{x_i}{1-x_i}$. We have that

$$\frac{\partial}{\partial x_1} (1-x_1) \frac{x_1}{1-x_1} \cdots \frac{x_k}{1-x_k} U \\ = \left(b \frac{x_1}{1-x_1} \cdots \frac{x_k}{1-x_k} + (1-a) \frac{x_2}{1-x_2} \cdots \frac{x_k}{1-x_k} + c(1-x_1) \frac{x_1}{1-x_1} \cdots \frac{x_k}{1-x_k} \sum_{j=2}^n \frac{1}{x_j - x_1} \right) U. \quad (20)$$

If c is odd, then U is antisymmetric. Anti-symmetrizing over \mathfrak{S}_n gives:

$$\sum_{w \in \mathfrak{S}_n} (-1)^w w \cdot \left(\frac{\partial}{\partial x_1} (1-x_1) \frac{x_1}{1-x_1} \frac{x_2}{1-x_2} \cdots \frac{x_k}{1-x_k} U \right) \\ = bP_k U + (1-a)P_{k-1} U + c \sum_{w \in \mathfrak{S}_n} w \cdot \left((1-x_1) \frac{x_1}{1-x_1} \frac{x_2}{1-x_2} \cdots \frac{x_k}{1-x_k} \sum_{j=2}^n \frac{1}{x_j - x_1} \right) U. \quad (21)$$

To evaluate the sum, we seek pairings of summands that reduce easily. Consider when w is the identity permutation. Then for each summand in $\sum_{j=2}^n \frac{1}{x_1 - x_j}$, for $2 \leq j \leq k$, we see that

$$\frac{(1-x_1)x_1x_j}{(x_j-x_1)(1-x_1)(1-x_j)} + \frac{(1-x_j)x_1x_j}{(x_1-x_j)(1-x_1)(1-x_j)} = \frac{x_1x_j}{(1-x_1)(1-x_j)}. \quad (22)$$

On the other hand, for $j > k$,

$$\frac{(1-x_1)x_1}{(x_j-x_1)(1-x_1)} + \frac{(1-x_j)x_j}{(x_1-x_j)(1-x_j)} = -1. \quad (23)$$

Thus for each $w \in \mathfrak{S}_n$ and $j \in [2, n]$, we pair the summand

$$w \cdot \left((1-x_1) \frac{x_1}{1-x_1} \frac{x_2}{1-x_2} \cdots \frac{x_k}{1-x_k} \cdot \frac{1}{x_j-x_1} \right)$$

with the summand obtained by taking w and transposing $w(1)$ and $w(j)$. Hence we duplicate the sum and simplify with equations (22) and (23):

$$\begin{aligned} 2 \sum_{w \in \mathfrak{S}_n} w \cdot \left((1-x_1) \frac{x_1}{1-x_1} \frac{x_2}{1-x_2} \cdots \frac{x_k}{1-x_k} \sum_{j=2}^n \frac{1}{x_j-x_1} \right) U \\ = \sum_{w \in \mathfrak{S}_n} w \cdot \left((k-1) \frac{x_1}{1-x_1} \frac{x_2}{1-x_2} \cdots \frac{x_k}{1-x_k} - (n-k) \frac{x_2}{1-x_2} \cdots \frac{x_k}{1-x_k} \right) U \\ = (k-1)P_k U - (n-k)P_{k-1} U. \end{aligned}$$

Thus, the expression in equation (21) simplifies to:

$$\begin{aligned} \sum_{w \in \mathfrak{S}_n} (-1)^w w \cdot \left(\frac{\partial}{\partial x_1} (1-x_1) \frac{x_1}{1-x_1} \frac{x_2}{1-x_2} \cdots \frac{x_k}{1-x_k} U \right) \\ = bP_k U + (1-a)P_{k-1} U - \frac{c}{2}(n-k)P_{k-1} U + \frac{c}{2}(k-1)P_k U. \quad (24) \end{aligned}$$

Since the residue of a partial derivative of an analytic function is always zero, taking the residues of the terms allows setting the equation to 0:

$$0 = b \operatorname{Res}_x P_k U + (1-a) \operatorname{Res}_x P_{k-1} U - \frac{c}{2}(n-k) \operatorname{Res}_x P_{k-1} U + \frac{c}{2}(k-1) \operatorname{Res}_x P_k U.$$

By definition of $\Psi(\cdot)$, we have that $\operatorname{Res}_x P_k U = k!(n-k)!\Psi_n(k, a, b, c)$, which gives:

$$\left(b + (k-1)\frac{c}{2} \right) k!(n-k)!\Psi_n(k, a, b, c) = \left(a - 1 + (n-k)\frac{c}{2} \right) (k-1)!(n-k+1)!\Psi_n(k-1, a, b, c).$$

Simplifying gives relation (19) for odd c . When c is even, U is symmetric, so symmetrizing over \mathfrak{S}_n gives:

$$\begin{aligned} \sum_{w \in \mathfrak{S}_n} w \cdot \left(\frac{\partial}{\partial x_1} (1-x_1) \frac{x_1}{1-x_1} \frac{x_2}{1-x_2} \cdots \frac{x_k}{1-x_k} U \right) \\ = bP_k U + (1-a)P_{k-1} U + c \sum_{w \in \mathfrak{S}_n} w \cdot \left((1-x_1) \frac{x_1}{1-x_1} \frac{x_2}{1-x_2} \cdots \frac{x_k}{1-x_k} \sum_{j=2}^n \frac{1}{x_j-x_1} \right) U, \end{aligned}$$

which is essentially identical to when c is odd, and the proof follows verbatim. \square

5.3. Closed Formula for $\Psi_n(k, a, b, c)$

Our proof for the closed formula of $\Psi_n(k, a, b, c)$ follows the recurrence approach used by Baldoni-Vergne [3, p. 8] (see also [21, Proposition 3.11]) using the recurrences proven in the previous section.

Lemma 39. *The relations (16)-(19) uniquely determine the function $\Psi_n(k, a, b, c)$.*

Proof.

Case 1. Consider if $c = 0, n \geq 1$, and $a \geq 1$. To compute $\Psi_n(k, a, b, 0)$, we repeatedly apply equation (19) to increment k until $k = n$, at which point we apply equation (16). Thus $\Psi_n(k, a, b, c)$ reduces to calculating $\Psi_n(0, a - 1, b + 1, c)$:

$$\Psi_n(k, a, b, 0) \xrightarrow{(19)} \Psi_n(k+1, a, b, 0) \xrightarrow{(19)^*} \Psi_n(n, a, b, 0) \xrightarrow{(16)} \Psi_n(0, a-1, b+1, 0).$$

By iterating this recursion, we see this is equivalent to calculating $\Psi_n(0, 1, a + b - 1, 0)$. By equation (18), this is equal to 1.

Case 2. Consider if $c \geq 1, n = 1$, and $a \geq 1$. Since $\prod_{1 \leq i < j \leq n} (x_j - x_i)^{-c}$ is the empty product, this is equivalent to when $c = 0$, which implies that

$$\Psi_1(k, c, a + b + c(n-2), c) = \Psi_1(k, c, a + b + c(n-2), 0).$$

This reduces to Case 1.

Case 3. Consider if $c \geq 1, n \geq 2$, and $a \geq 1$. Similar to in Case 2, to compute $\Psi_n(k, a, b, c)$ we repeatedly apply equation (19) to increment k until $k = n$, at which point we apply equation (16). Thus $\Psi_n(k, a, b, c)$ reduces to calculating $\Psi_n(0, a - 1, b + 1, c)$.

$$\Psi_n(k, a, b, c) \xrightarrow{(19)} \Psi_n(k+1, a, b, c) \xrightarrow{(19)^*} \Psi_n(n, a, b, c) \xrightarrow{(16)} \Psi_n(0, a-1, b+1, c).$$

We iterate this recursion until $a = 1$, at which point we reduce the calculation to finding $\Psi_n(0, 1, a + b - 1, c)$. Now we again increment k by (19) until $k = n - 1$. Applying equation (17) reduces the calculation to finding $\Psi_{n-1}(0, c, a + b, c)$.

$$\begin{aligned} \Psi_n(k, 1, a + b - 1, c) &\xrightarrow{(19)} \Psi_n(k+1, 1, a + b - 1, c) \\ &\xrightarrow{(19)^*} \Psi_n(n-1, 1, a + b - 1, c) \xrightarrow{(17)} \Psi_{n-1}(0, c, a + b, c). \end{aligned}$$

We now repeatedly apply the above two cycles until we reduce n to 1, in which case we reduce the computation to $\Psi_1(0, c, a + b + c(n-2), c)$. Since $n = 1$, this becomes Case 2.

Since all cases eventually reduce to case 1, the result follows. \square

Using the fact that the relations (16)-(19) uniquely define $\Psi_n(k, a, b, c)$, we now prove our explicit product formula.

Proof of Theorem 8. By Lemma 39, it is sufficient to show the product formula for $\Psi_n(k, a, b, c)$ in equation (6) satisfies the relations (16)-(19). To show equation (16), recall that:

$$M_n(a-1, b+1, c) = \prod_{j=0}^{n-1} \frac{\Gamma(a+b-1+(n-1+j)\frac{c}{2})\Gamma(\frac{c}{2}+1)}{\Gamma(a-1+j\frac{c}{2})\Gamma(b+1+j\frac{c}{2})\Gamma((j+1)\frac{c}{2}+1)}.$$

Recall that $\Gamma(x+1) = x\Gamma(x)$. Hence, $\Gamma(a-1+j\frac{c}{2}) = \Gamma(a+j\frac{c}{2})/(a-1+j\frac{c}{2})$, and $\Gamma(b+1+j\frac{c}{2}) = (b+j\frac{c}{2})\Gamma(b+j\frac{c}{2})$. Substituting gives

$$M_n(a-1, b+1, c) = M_n(a, b, c) \prod_{j=0}^{n-1} \frac{a-1+j\frac{c}{2}}{b+j\frac{c}{2}} = M_n(a, b, c) \prod_{j=1}^n \frac{a-1+(n-j)\frac{c}{2}}{b+(j-1)\frac{c}{2}}.$$

To show equation (17) from (16), it is sufficient to show the product formula satisfies Lemma 35.

First we show $M_n(1, b, c) = M_{n-1}(c+1, b, c)$. To do so, consider the ratio

$$\frac{M_n(1, b, c)}{M_{n-1}(c+1, b, c)} = \frac{1}{n} \cdot \frac{\Gamma(b+(n-1)\frac{c}{2})\Gamma(\frac{c}{2})}{\Gamma(b+(n-1)\frac{c}{2})\Gamma(\frac{c}{2}n)} \cdot \frac{\Gamma(\frac{c}{2}n+1)}{\Gamma(\frac{c}{2}+1)\Gamma(1)}$$

Since $\Gamma(x+1) = x\Gamma(x)$, the above ratio simplifies to 1, and the result follows.

Using the above equality, it is sufficient to show that $\binom{n}{k} \prod_{j=1}^k (n-j) = \binom{n-1}{k} \prod_{j=1}^k (n+1-j)$. Both sides of the equation simplify to $\prod_{j=1}^k (n+1-j)(n-j)/j$, thus proving relation the product formula satisfies Lemma 35 and hence equation (17).

To show equation (18), recall that since $k=0$, we have that $\binom{n}{k} \prod_{j=1}^k \frac{a-1+(n-j)\frac{c}{2}}{b+(j-1)\frac{c}{2}} = 1$. Then

$$\Psi_n(0, 1, b, 0) = M_n(1, b, 0) = 1.$$

To show equation (19), note that

$$\frac{\Psi_n(k, a, b, c)}{\Psi_n(k-1, a, b, c)} = \frac{\binom{n}{k}}{\binom{n}{k-1}} \cdot \frac{a-1+(n-k)\frac{c}{2}}{b+(k-1)\frac{c}{2}} = \frac{n-k+1}{k} \cdot \frac{a-1+(n-k)\frac{c}{2}}{b+(k-1)\frac{c}{2}}.$$

Rearranging gives the desired recurrence relation, and the result follows. \square

We also compute the following special cases of $\Psi_n(k, a, b, c)$, which generalize the special cases of $M_n(a, b, c)$ computed in Section 2.4.

Corollary 40. *The constant term $\Psi_n(k, a, b, c)$ satisfies the following:*

$$\Psi_n(k, a, 1, 1) = \frac{1}{n+2(a-1)} \binom{n}{k} \binom{n+2(a-1)}{k+1} M_n(a, 1, 1) \quad (25)$$

$$\Psi_n(k, 1, b, 1) = \binom{n-1}{k} \binom{n}{k} \binom{k+2b-1}{k}^{-1} M_n(1, b, 1) \quad (26)$$

$$\Psi_n(k, 1, 1, c) = N(n, k+1) \prod_{j=1}^k \frac{c(j+1)}{c(j-1)+2} M_n(1, 1, c). \quad (27)$$

Proof. We manipulate the product formula given in Theorem (8) to obtain the desired relations. Note that

$$\begin{aligned} \prod_{j=1}^k \frac{2(a-1)+(n-j)}{j+1} &= \frac{1}{n+2(a-1)} \binom{n+2(a-1)}{k+1} \\ \prod_{j=1}^k \frac{n-j}{j+2b-1} &= \frac{\binom{n-1}{k}}{\binom{k+2b-1}{k}} \\ \prod_{j=1}^k \frac{(n-j)c}{2+(j-1)c} &= \frac{1}{n} \binom{n}{k+1} \prod_{j=1}^k \frac{c(j-1)+2c}{c(j-1)+2}. \end{aligned}$$

The results follows from substitution. \square

Remark 41. Note that unlike $M_n(a, b, c)$ (see Proposition 17), in most cases $\Psi_n(k, a, b, c) \neq \Psi_n(k, b, a, c)$. For instance, by plugging into equations (25) and (26), we obtain $\Psi_2(1, 2, 1, 1) = 6$, whereas $\Psi_2(1, 1, 2, 1) = 1$. Instead, we have the following symmetry.

Proposition 42 (Symmetry of Ψ_n). *For positive integers a, b, n and nonnegative integers c and $k \leq n$,*

$$\Psi_n(k, a, b, c) = \Psi_n(n-k, b+1, a-1, c).$$

Proof. By Theorem 6 (ii), $\Psi_n(k, a, b, c)$ is the volume of the interior disjoint polytopes

$$\left\{ \mathcal{F}_{k_{n+2}^{a,b,c}(S)} \mid S \in \binom{[n]}{k} \right\}.$$

Note that the reverse graph of $k_{n+2}^{a,b,c}(S)$ is the graph $k_{n+2}^{b+1,a-1,c}([n] \setminus S)$. So by Lemma 9 we have that $\mathcal{F}_{k_{n+2}^{a,b,c}(S)} \equiv \mathcal{F}_{k_{n+2}^{b+1,a-1,c}([n] \setminus S)}$ and both polytopes have the same volume. Thus we have

$$\Psi_n(k, a, b, c) = \sum_{S \in \binom{[n]}{k}} \text{vol} \mathcal{F}_{k_{n+2}^{a,b,c}(S)} = \sum_{S \in \binom{[n]}{n-k}} \text{vol} \mathcal{F}_{k_{n+2}^{b+1,a-1,c}(S)} = \Psi_n(n-k, b+1, a-1, c).$$

□

Remark 43. Combining Proposition 42 with Corollary 7 yields $M_n(a, b+1, c) = M_n(b+1, a, c)$, recovering the symmetry of the Morris identity (Corollary 17).

As a corollary, we also have the following special case when $a = b = c = 1$.

Corollary 44. For positive integer n and nonnegative integer $k \leq n$,

$$\Psi_n(k, 1, 1, 1) = N(n, k+1)C_{n-1} \cdots C_1.$$

Proof. This follows from (25) when $a = 1$. □

Corollary 45. Theorem 6 and Theorem 8 imply Theorem 1.5.

Proof. Theorem 8 implies Corollary 44. Applying Corollary 37 and the Kostant partition function interpretation in Theorem 6 thus gives Theorem 1.5. □

Remark 46. Corollary 44 and Lemma 31 in the case that $a = b = c = 1$ give a coarser version of the refinement provided by Mészáros [19, Theorem 13].

6. The Baldoni–Vergne refinement of $M_n(a, b, c)$

To prove the Morris identity, Baldoni–Vergne defined the generating function

$$\phi'_n(k, a, b, c) := k!(n-k)!e_k \prod_{i=1}^n (1-x_i)^{-b} x_i^{-a+1} \prod_{1 \leq i < j \leq n} (x_j - x_i)^{-c},$$

where $e_k = [t^k] \prod_{i=1}^n (1+tx_i)$ is the k th elementary symmetric polynomial. They proved several recurrence relations that computed the constant term $\Phi'_n(k, a, b, c) := \text{CT}_x \phi'_n(k, a, b, c)$, which implies the Morris identity when $k = 0$.

Theorem 47 (Baldoni–Vergne [3, Corollary 10]). For positive integer n and nonnegative integer k, a, b, c , with $a+b \geq 2$, the constant term $\Phi'_n(k, a, b, c)$ is given by the formula

$$\Phi'_n(k, a, b, c) = n! \cdot M_n(a, b, c) \prod_{j=1}^k \frac{a-1+(n-j)\frac{c}{2}}{a+b-2+(2n-j-1)\frac{c}{2}}. \quad (28)$$

Interestingly, the Baldoni–Vergne constant term does not generalize the refinement of $M_n(1, 1, 1)$ given by Theorem 5 which helped motivate our new refinement of $M_n(a, b, c)$ in Section 5. To more naturally interpret this constant term with Kostant partition functions, we scale $\Phi'_n(k, a, b, c)$.

Definition 48. We define the following modification of the Baldoni–Vergne constant term:

$$\Phi_n(k, a, b, c) := \text{CT}_x \left[t^k \right] \prod_{i=1}^n (1-x_i)^{-b} x_i^{-a+1} (1+tx_i) \prod_{1 \leq i < j \leq n} (x_j - x_i)^{-c}. \quad (29)$$

Equivalently, we have $\Phi_n(k, a, b, c) := \Phi'_n(k, a, b, c) / (k!(n-k)!)$. We now show the main result of this section.

Theorem 49. $\Phi_n(k, a, b, c)$ is the sum of Kostant partition functions $K_{k, a, b, c}(0, a_1, \dots, a_n, -\sum_{j=1}^n a_j)$ such that $a-2+c(i-1) \leq a_i \leq a-1+c(i-1)$, with $a_i = a-1+c(i-1)$ for exactly $n-k$ values of $i \in [n]$.

Proof. Recall that specializing the generating function of equation (10) gives the Morris constant term

$$M_n(a, b, c) = \text{CT}_x \prod_{i=1}^n (1 - x_i)^{-b} x_i^{-a+1} \prod_{1 \leq i < j \leq n} (x_j - x_i)^{-c}.$$

We see that $\Phi_n(k, a, b, c)$ has an additional term $[t^k] \prod_{i=1}^n (1 + tx_i)$, which replaces CT_{x_i} with $[x_i^{-1}]$ for exactly k values of i , so this is equivalent to decreasing the net flow at these vertices by 1.

Recall from equation (3) that $M_n(a, b, c) = K_{k_{n+2}^{a,b,c}}(0, a_1, a_2, \dots, a_n, -\sum_{j=1}^n a_j)$, with $a_i = a - 1 + c(i - 1)$ for all $i \in [n]$. Decreasing the net flow by one at exactly k vertices gives the desired Kostant partition function interpretation.

There are $k!(n - k)!$ ways of distinguishing the vertices based on their net flow, we also obtain the combinatorial interpretation for $\Phi'_n(k, a, b, c)$, and the result follows. \square

Remark 50. We note that $\Phi_n(k, a, b, c)$ does not seem to have a refinement similar to Corollary 7. Summing $\Phi(\cdot)$ over k removes all restrictions on t terms from the expression, giving:

$$\sum_{k=0}^n \Phi_n(k, a, b, c) := \text{CT}_x \prod_{i=1}^n (1 - x_i)^{-b} x_i^{-a+1} (1 + x_i) \prod_{1 \leq i < j \leq n} (x_j - x_i)^{-c},$$

for which a simplified expression is not immediate.

We now give the recurrence relations used by Baldoni–Vergne [3] to prove Theorem 47. These relations served as the inspiration for the relations for $\Psi_n(k, a, b, c)$ in Section 5.

Proposition 51 (Baldoni–Vergne [3, Theorem 8]). *The constant term $\Phi'_n(k, a, b, c)$ is uniquely determined by the following relations:*

- (1) $\Phi'_n(n, a, b, c) = \Phi'_n(0, a - 1, b, c)$
- (2) $\Phi'_n(n - 1, 1, b, c) = \Phi'_{n-1}(0, c, b, c)$
- (3) $\Phi'_n(0, 1, b, 0) = r!$
- (4) $\Phi'_1(k, 0, b, c) = 0$
- (5) $(a + b - 2 + \frac{c}{2}(2n - k - 1))\Phi'_n(k, a, b, c) = (a - 1 + \frac{c}{2}(n - k))\Phi'_n(k - 1, a, b, c).$

Remark 52. One can give combinatorial proofs for all but the last relation in a nearly identical manner to our combinatorial proofs for $\Psi_n(k, a, b, c)$ in Lemma 38.

7. Final remarks

In this paper we investigated a symmetry and a refinement of the Morris identity with several combinatorial interpretations, including a certain sum of Kostant partition functions and the volume of a collection of polytopes. We demonstrated how these collections of polytopes subdivide the graph $k_{n+2}^{a,b,c}$, and proved a product formula for our refinement. We now give some possible avenues for future exploration.

7.1. The recurring appearance of Aomoto's integral

The original Morris constant term identity [26] strongly resembles the Selberg integral, and the two identities are known to be equivalent (see [26] and [11]). Interestingly, the product formula for the Baldoni–Vergne refinement of the Morris identity (equation (28)) greatly resembles Aomoto's integral [1]. However, the relationship between these two seemingly related identities is as of yet unclear. Intriguingly, Zeilberger also cites Aomoto's integral in his proof of Conjecture 2 of Chan–Robbins–Yuen, and while we did not see an immediate application of Aomoto's integral in our proof of the product formula of $\Psi_n(k, a, b, c)$, this seems to suggest these refinements of the Morris identity are in some way related to Aomoto's generalization of the Selberg integral. For a recent bijective proof of the Selberg integral see [13].

7.2. Combinatorial proof of the Morris identity

This paper provides multiple combinatorial proofs of recurrence relations for $\Psi_n(k, a, b, c)$ that could contribute to a combinatorial proof of the Morris constant term identity, and therefore, the volume formula for the Chan–Robbins–Yuen polytope. With the approach of this paper, the only remaining step is to give a combinatorial proof for equation (19). A combinatorialization of our algebraic proof of (19), or a new combinatorial proof altogether, would certainly be interesting. We also note that equation (19) can be rewritten as

$$\left(kb + \binom{k}{2}c\right) \cdot \Psi_n(k, a, b, c) = \left((n-k+1)(a-1) + \binom{n-k+1}{2}c\right) \cdot \Psi_n(k-1, a, b, c).$$

where the extra factors on the left-hand and right-hand sides appear to be selecting certain edges of the graph $k_{n+2}^{a,b,c}$. Applying the identity in Proposition 42 to the right-hand side, the expression becomes

$$\left(kb + \binom{k}{2}c\right) \cdot \Psi_n(k, a, b, c) = \left((n-k+1)(a-1) + \binom{n-k+1}{2}c\right) \cdot \Psi_n(n-k+1, b+1, a-1, c).$$

where both sides have very similar structures. Given that a combinatorial proof of the Morris identity has been elusive and would serve immediately as a combinatorial proof for the volume formula of the Chan–Robbins–Yuen polytope. See [5, 16, 32] for combinatorial proofs of volumes of flow polytopes \mathcal{F}_G for other graphs G .

7.3. Volume of polytopes with different net flow vectors

In Section 3, we presented a new recursive proof of Theorem 1. Generalizing Theorem 1 is the following theorem of Baldoni–Vergne–Lidskii.

Theorem 53 (Baldoni–Vergne–Lidskii [2]). *Let G be a connected digraph on vertex set $\{0, 1, \dots, n, n+1\}$ with m edges directed $i \rightarrow j$ if $i < j$ and such that for $i \in \{0, 1, \dots, n\}$, there is at least one outgoing edge at vertex i . Then for a fixed net flow vector*

$$\mathbf{a} = \left(\sum_{j=1}^n a_j, -a_1, -a_2, \dots, -a_{n+1}\right), \quad a_i \in \mathbb{Z}_{\geq 0},$$

it holds that

$$\text{vol } \mathcal{F}_G(\mathbf{a}) = \sum_{\mathbf{j}} \binom{m-n-1}{j_0, \dots, j_{n+1}} a_1^{j_1} \cdots a_{n+1}^{j_{n+1}} \cdot K_G(0, d_1 - j_1, \dots, d_{n+1} - j_{n+1}),$$

where $d_i = \text{indeg}_G(i) - 1$. and the sum is over compositions \mathbf{j} of $m - n - 1$ with $n + 1$ parts.

In our proof in Section 3 of Theorem 1, the map φ on Kostant partition functions that we introduced is not specific to flow polytopes with net flow vector $(1, 0, \dots, 0, -1)$. This means that the inductive step will not change significantly for a different net flow vector, and as such, it is worth investigating whether there is a simple recursive proof for Theorem 53 considering new base cases with net flow vector \mathbf{a} . Such a proof would provide a better understanding of how volumes of flow polytope and Kostant partition functions are refined by the subdivision lemma. See [17] for another recent proof of this more general volume formula.

7.4. Dual graph of triangulations of \mathcal{F}_G

In Section 4 we used the Danilov–Karzanov–Koshevoy (DKK) triangulation of flow polytopes in terms of cliques of routes. Given a triangulation of a polytope, it is of interest to study its dual graph. This is the graph whose vertices are the top-dimensional simplices connected by an edge if the pair of simplices have a common facet (see [9, Chapter 1]). In our context the number of vertices of such dual graph gives the volume of the polytope. In [4], the authors show that for certain planar graphs G and for two framings called *length* and *planar*, the dual graphs of the DKK triangulations of \mathcal{F}_G are isomorphic to a generalization of the *Tamari lattice (associahedron)* and certain principal order ideals in *Young’s lattice*, respectively. It would be of interest to study the dual graph with $M_n(a, b, c)$ vertices of the DKK triangulation of the flow polytopes $\mathcal{F}_{k_{n+2}^{a,b,c}}$ for the length and planar framing.

7.5. Triangulations of flow polytopes

In Section 4 we used the DKK triangulation of the flow polytope \mathcal{F}_G of a framed graph (G, \leq) to obtain a bijection $\Theta_{G, \leq}$ between the integer flows of $\mathcal{F}_G(0, d_1, \dots)$ and of $\mathcal{F}_{G^*}(0, d_1, \dots)$. This is related (see Example 27) and was motivated by work of Postnikov [27, Section 12], who showed that a triangulation τ of *root polytopes* Q_H for bipartite graphs H with vertices $\{1, \dots, p\} \cup \{\bar{1}, \dots, \bar{q}\}$ give a bijection ϕ_τ between lattice points of two *trimmed generalized permutahedra* P_H^- and $P_{H^*}^+$, where H^* is the obtained by flipping H . Galashin, Nenashev, Postnikov [12] studied the bijections ϕ_τ and showed that they uniquely specify the triangulation τ . It would be of interest to do a similar study of the bijections $\Theta_{G, \leq}$.

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Appendix

In this section, we give some computational proofs for Section 2. In multiple of the proofs below, we use the Legendre duplication formula:

$$\Gamma\left(x + \frac{1}{2}\right)\Gamma(x) = 2^{1-2x}\sqrt{\pi} \cdot \Gamma(2x). \quad (30)$$

We also use the following expression deducible from the Legendre duplication formula. For positive integers x and k ,

$$\Gamma(x + k + 1/2)\Gamma(x) = 2^{1-2x}\sqrt{\pi} \cdot \Gamma(2x) \prod_{j=0}^{k-1} \left(x + j + \frac{1}{2}\right). \quad (31)$$

Proof of Corollary 12. First, consider the ratio $M_n(a, b, 1)/M_{n-1}(a, b, 1)$. By (30),

$$\begin{aligned} \frac{M_n(a, b, 1)}{M_{n-1}(a, b, 1)} &= \frac{1}{n} \cdot \frac{\Gamma\left(a + b + n - \frac{5}{2}\right)\Gamma\left(a + b + n - 2\right)}{\Gamma\left(a + b - 2 + \frac{1}{2}n\right)} \cdot \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(a + \frac{n-1}{2}\right)\Gamma\left(b + \frac{n-1}{2}\right)\Gamma\left(\frac{n}{2}\right)} \\ &= \frac{1}{n} \cdot \frac{2^{6-2(a+b+n)}\Gamma(2(a+b+n)-5)\pi}{\Gamma\left(a + b - 2 + \frac{1}{2}n\right)\Gamma\left(a + \frac{n-1}{2}\right)\Gamma\left(b + \frac{n-1}{2}\right)\Gamma\left(\frac{n}{2}\right)}. \end{aligned}$$

Substitution with (31) then gives

$$\begin{aligned}\frac{M_n(a, b, 1)}{M_{n-1}(a, b, 1)} &= \frac{(2(a+b+n)-6)!}{n!(2a+n-2)!\prod_{j=0}^{b-3}(2a+n+2j)\prod_{j=0}^{b-2}(n+2j+1)} \\ &= \frac{(2(a+b+n)-6)!}{n!!(2a+n-3)!!(2b+n-3)!!(2a+2b+n-6)!!}.\end{aligned}$$

To cancel the double factorials, we instead consider the ratio $M_{n+1}(1, 1, 1)/M_{n-1}(1, 1, 1)$:

$$\begin{aligned}\frac{M_{n+1}(a, b, 1)}{M_{n-1}(a, b, 1)} &= \frac{(2(a+b+n)-4)!(2(a+b+n)-6)!}{(n+1)!(2a+n-2)!(2b+n-2)!(2a+2b+n-5)!} \\ &= \frac{\binom{2a+2b+2n-4}{2a+n-2}}{\binom{2a+2b+2n-4}{n}} C_{n-1} C_n \prod_{i=1}^{n-1} \frac{2(a+b-2)+n+i-1}{n+i-1} \prod_{i=1}^n \frac{2(a+b-2)+n+i}{n+i}.\end{aligned}$$

We establish the base cases $M_0(a, b, 1) = 1$, $M_1(a, b, 1) = \binom{a+b-2}{a-1}$, so by telescoping, we see that:

$$\begin{aligned}M_{2n}(a, b, 1) &= \prod_{i=1}^n \frac{M_{2i}(a, b, 1)}{M_{2i-2}(a, b, 1)} \\ M_{2n-1}(a, b, 1) &= \binom{a+b-2}{a-1} \prod_{i=1}^{n-1} \frac{M_{2i+1}(a, b, 1)}{M_{2i-1}(a, b, 1)}.\end{aligned}$$

Plugging in with (32) gives the desired result. \square

The remaining proofs in this section follow a similar scheme, using $M_0(a, b, c) = 1$.

Corollary 54. *For c even, we have that:*

$$M_n(1, 1, c) = \prod_{i=1}^n \left(\frac{(2i-3)\frac{c}{2}}{(i-1)\frac{c}{2}} \right) \left(\frac{(2i-2)\frac{c}{2}}{(i-1)\frac{c}{2}} \right) \left(\frac{i\frac{c}{2}}{(i-1)\frac{c}{2}} \right)^{-1}.$$

Proof. Again, consider $M_n(1, 1, c)/M_{n-1}(1, 1, c)$. We see that

$$\begin{aligned}\frac{M_n(1, 1, c)}{M_{n-1}(1, 1, c)} &= \frac{1}{n} \cdot \frac{((2n-3)\frac{c}{2})!((2n-2)\frac{c}{2})!}{((n-2)\frac{c}{2})!} \cdot \frac{(\frac{c}{2}-1)!}{((n-1)\frac{c}{2})!^2(\frac{c}{2}n-1)!} \\ &= \frac{((2n-3)\frac{c}{2})!((2n-2)\frac{c}{2})!}{((n-2)\frac{c}{2})!} \cdot \frac{(\frac{c}{2})!}{((n-1)\frac{c}{2})!^2(\frac{c}{2}n)!}.\end{aligned}$$

With some rearrangement, we get the equation

$$\frac{M_n(1, 1, c)}{M_{n-1}(1, 1, c)} = \left(\frac{n\frac{c}{2}}{(n-1)\frac{c}{2}} \right)^{-1} \left(\frac{(2n-3)\frac{c}{2}}{(n-1)\frac{c}{2}} \right) \left(\frac{(2n-2)\frac{c}{2}}{(n-1)\frac{c}{2}} \right).$$

Since $M_n(1, 1, c) = \prod_{i=1}^n M_i(1, 1, c)/M_{i-1}(1, 1, c)$, the result follows. \square

Corollary 55. *For c odd, we have that:*

$$M_n(1, 1, c) = \prod_{i=1}^n \frac{(1+(2i-3)c)!((i-1)c)!c!!}{((i-2)c)!!((i-1)c)!!^2(ic)!!((2i-3)\frac{c}{2}+\frac{1}{2})!}.$$

Proof. Let $k = \lfloor \frac{c}{2} \rfloor$ or $\frac{c}{2} = k + \frac{1}{2}$. We then simplify the ratio $M_n(1, 1, c)/M_{n-1}(1, 1, c)$ using (31).

$$\begin{aligned} \frac{M_n(1, 1, c)}{M_{n-1}(1, 1, c)} &= \frac{1}{n} \cdot \frac{\Gamma\left(1 + (2n-3)\frac{c}{2}\right) \Gamma\left(1 + (2n-2)\frac{c}{2}\right)}{\Gamma\left(1 + (n-2)\frac{c}{2}\right)} \cdot \frac{\Gamma\left(\frac{c}{2}\right)}{\Gamma\left(1 + (n-1)\frac{c}{2}\right)^2 \Gamma\left(\frac{c}{2}n\right)} \\ &= \frac{1}{n} \cdot \frac{2^{c(3-2n)-1} \Gamma(2 + (2n-3)c) \prod_{j=0}^{k-1} \left(1 + (2n-3)\frac{c}{2} + j + \frac{1}{2}\right)}{2^{c(2-n)-1} \Gamma(2 + (n-2)c) \prod_{j=0}^{k-1} \left(1 + (n-2)\frac{c}{2} + j + \frac{1}{2}\right)} \cdot \frac{\Gamma\left(\frac{c}{2}\right)}{\Gamma\left(1 + (n-1)\frac{c}{2}\right) \Gamma\left(\frac{c}{2}n\right)} \\ &= 2^{1-c} \cdot \frac{(1 + (2n-3)c)!c!!}{(1 + (n-2)c)!(1 + (n-1)c)!} \prod_{j=0}^{k-1} \frac{1 + (2n-3)\frac{c}{2} + j + \frac{1}{2}}{\left(1 + (n-2)\frac{c}{2} + j + \frac{1}{2}\right) \left(1 + (n-1)\frac{c}{2} + j + \frac{1}{2}\right)} \\ &= 2^{1-c+k} \cdot \frac{((2n-3)c)!!c!!((2n-2)c)!!}{((n-2)c)!!((n-1)c)!!^2(nc)!!} = \frac{(1 + (2n-3)c)!((n-1)c)!c!!}{((n-2)c)!!(((n-1)c)!!)^2(nc)!! \left((2n-3)\frac{c}{2} + \frac{1}{2}\right)!}. \end{aligned}$$

We have $M_n(1, 1, c) = \prod_{i=1}^n M_i(1, 1, c)/M_{i-1}(1, 1, c)$, and the result follows. \square

Proof of Corollary 13. We again compute the ratio $M_n(a, b, 2k)/M_{n-1}(a, b, 2k)$:

$$\begin{aligned} \frac{M_n(a, b, 2k)}{M_{n-1}(a, b, 2k)} &= \frac{1}{n} \cdot \frac{\Gamma(a+b-1 + (2n-3)k) \Gamma(a+b-1 + (2n-2)k)}{\Gamma(1 + (n-2)k)} \cdot \frac{\Gamma(k)}{\Gamma(a + (n-1)k) \Gamma(b + (n-1)k) \Gamma(kn)} \\ &= \frac{(a+b-2 + (2n-3)k)!(a+b-2 + (2n-2)k)!}{((n-2)k)!} \cdot \frac{k!}{((a-1) + (n-1)k)!((b-1) + (n-1)k)!(kn)!} \\ &= \frac{(a+b-2 + (2n-3)k)!k!}{((n-2)k)!(nk)!} \cdot \left(\frac{a+b-2 + (2n-2)k}{a-1 + (n-1)k} \right). \end{aligned}$$

As with the above proofs, $M_n(a, b, 2k) = \prod_{i=1}^n M_i(a, b, 2k)/M_{i-1}(a, b, 2k)$, so the result follows. \square

7.6. Asymptotic Analysis

In this subsection, we examine the asymptotics of the Morris identity product formula. We use the standard asymptotics notations $f \sim g$ and $f = O(g)$.

Lemma 56. We have the following asymptotic formula:

$$\log M_n(1, 1, 1) = n^2 \log 2 - \frac{3}{2} n \log n + O(n).$$

Proof. It is well known that $C_n \sim 4^n / (n^{3/2} \sqrt{\pi})$. Then

$$\log M_n(1, 1, 1) = \sum_{i=1}^{n-1} \log C_i \sim \sum_{i=1}^{n-1} i \log 4 - \frac{3}{2} \log i - \frac{1}{2} = \binom{n}{2} \log 4 - \frac{n-1}{2} - \frac{3}{2} \log((n-1)!)$$

By Stirling's formula and some manipulation, the result follows. \square

Proposition 57. For positive integer n , $\log M_n(n, 1, 1) = (9 \log 2 - \frac{9}{2} \log 3) n^2 + O(n \log n)$.

Proof. By [24], we have an alternative formulation of Proctor's formula

$$\prod_{1 \leq i < j \leq n} \frac{2(n-1) + i + j - 1}{i + j - 1} = \frac{\mathfrak{J}(2(n-1) + 2n) \mathfrak{J}(2(n-1) + 1) \Lambda(n)}{\Lambda(2(n-1) + n) \mathfrak{J}(2n)}$$

where $\Lambda(n) = 1! \cdot 2! \cdots (n-1)!$ and $\mathfrak{J}(n) = (n-2)!(n-4)! \cdots$. Then

$$\begin{aligned} \log \prod_{1 \leq i < j \leq n} \frac{2(n-1) + i + j - 1}{i + j - 1} &= \log \mathfrak{J}(4n-2) + \log \mathfrak{J}(2n-1) + \log \Lambda(n) - \log \Lambda(3n-2) - \log \mathfrak{J}(2n) \\ &= \log \mathfrak{J}(4n) + \log \Lambda(n) - \log \Lambda(3n) + O(n \log n). \end{aligned}$$

In [25], we have the asymptotics

$$\begin{aligned} \log \Lambda(n) &= \frac{1}{2} n^2 \log n - \frac{3}{4} n^2 + O(n \log n) \\ \log \mathfrak{J}(n) &= \frac{1}{4} n^2 \log n - \frac{3}{8} n^2 + O(n \log n). \end{aligned}$$

Then substituting with some manipulation gives that the above sum is equal to

$$\left(8 \log 2 - \frac{9}{2} \log 3\right) n^2 + O(n \log n).$$

The result follows by applying Lemma 56. \square

Proposition 58. For positive integer n and fixed positive integer a and b , we have that

$$\log M_n(n, n, 1) = 2n^2 \log n + \left(3 + 13 \log 2 + \frac{9}{2} \log 3 - \frac{25}{4} \log 5\right) n^2 + O(n \log n).$$

Proof. With some manipulation, we can show that

$$M_n(a, b, 1) \leq \binom{a+b-2}{a-1} \frac{\mathfrak{J}(n) \mathfrak{J}(2a+2b+n-4)}{\mathfrak{J}(2a+n-2) \mathfrak{J}(2b+n-2)} M_n(1, 1, 1) \prod_{1 \leq i < j \leq n-1} \frac{2(a+b-2) + j + i - 1}{j + i - 1},$$

where equality holds when n is odd. The lower bound occurs by removing the $\binom{a+b-2}{a-1}$ term. Thus

$$\begin{aligned} \log M_n(n, n, 1) &= \log \frac{\mathfrak{J}(n) \mathfrak{J}(5n-4)}{\mathfrak{J}(3n-2) \mathfrak{J}(3n-2)} + \log M_n(2n-1, 1, 1) + O(n \log n) \\ &= \log \frac{\mathfrak{J}(n) \mathfrak{J}(5n)}{\mathfrak{J}(3n)^2} + \log M_n(2n-1, 1, 1) + O(n \log n). \end{aligned}$$

The first term on the right-hand side is

$$\log \frac{\mathfrak{J}(n) \mathfrak{J}(5n)}{\mathfrak{J}(3n)^2} = 2n^2 \log n + \left(\frac{25}{4} \log 5 - \frac{9}{2} \log 3 + 3\right) n^2 + O(n \log n).$$

The second term is

$$\log \frac{\mathfrak{J}(6n) \mathfrak{J}(4n) \Lambda(n)}{\Lambda(5n) \mathfrak{J}(2n)} + \log M_n(1, 1, 1) + O(n \log n) = \left(13 \log 2 + 9 \log 3 - \frac{25}{2} \log 5\right) n^2 + O(n \log n).$$

The result follows from summing the two terms. \square

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