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WELL-POSEDNESS FOR THE CUBIC NONLINEAR SCHRÖDINGER EQUATIONS ON TORI

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DEDICATION

In memory of my father, Zhenmin Yue.

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ABSTRACT

WELL-POSEDNESS FOR THE CUBIC NONLINEAR SCHRÖDINGER EQUATIONS ON TORI

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This thesis studies the cubic nonlinear Schr̆odinger equation (NLS) on tori both from the deterministic and probabilistic viewpoints.

In Part I of this thesis, we prove global-in-time well-posedness of the Cauchy initial value problem for the defocusing cubic NLS on 4-dimensional tori and with initial data in the energy-critical space H^1. Furthermore, in the focusing case we prove that if a maximal-lifespan solution of the cubic NLS $u : I \times \mathbb{T}^4 \to \mathbb{C}$ satisfies $\sup_{t\in I} \|u(t)\|_{\dot{H}^1(\mathbb{T}^4)} < \|W\|_{\dot{H}^1(\mathbb{R}^4)}$, then it is a global-in-time solution; that is $I = \mathbb{R}$. Here W denotes the ground state on Euclidean space, which is a stationary solution of the corresponding focusing equation in $\mathbb{R}^4$.

In Part II of this thesis, we prove almost sure local-in-time well-posedness in both atomic spaces X^s and Fourier restriction spaces $X^{s,b}$ for the cubic NLS on $\mathbb{T}^d$ ($d \geq 3$) in the super-critical regime.

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CHAPTER 1

INTRODUCTION

1.1 Overview

This thesis contains two parts.

- In Part I we study the Cauchy initial value problem for the cubic nonlinear Schrödinger equation (NLS) on tori in dimensions four. We prove two results. The first result is for the defocusing problem and establishes large data global-in-time existence, uniqueness and continuous dependence of the flow map on the initial data. The problems that meet the above three criteria are said to be *globally (in time) well-posed*. The second result pertains to the focusing problem and establishes global-in-time well-posedness under suitable threshold conditions.

- In Part II we study the random data Cauchy initial value problem for the cubic NLS on tori in dimensions three and above, and establish local-in-time well-posedness in high probability in the super-critical regime.

The cubic NLS is given by

$$\begin{cases} i\partial_t u + \Delta u = \mu |u|^2 u, \quad (t,x) \in I \times \mathcal{M}^d, \\ u(0,x) = u_0(x), \end{cases} \tag{1.1.1}$$

where u is a complex-valued function on $I \times \mathcal{M}^d$. Here I is a time interval and $\mathcal{M}^d$ is a d-dimensional manifold. The constant μ is ± 1. In this thesis we especially focus on two types of manifolds: d-dimensional Euclidean spaces $\mathbb{R}^d$ and the d-dimensional tori $\mathbb{T}^d$. Of course functions on the latter domain ($\mathbb{T}^d$) can be viewed as periodic functions on the former domain ($\mathbb{R}^d$).

Two important conserved quantities of the solutions of (1.1.1) are the mass of u:

$$M(u)(t) := \int_{\mathcal{M}^d} |u(t)|^2 \, dx = M(u)(0) \tag{1.1.2}$$

and the energy (Hamiltonian) of u:

$$E(u)(t) := \frac{1}{2} \int_{\mathcal{M}^d} |\nabla u(t)|^2 \, dx + \frac{1}{4}\mu \int_{\mathcal{M}^d} |u(t)|^4 \, dx = E(u)(0). \tag{1.1.3}$$

If $\mu = 1$ the mass (3.1.2) and energy (3.1.3) are positive definite and give global-in-time bounds for the solution $u(t)$; the equations are called *defocusing* in this case. On the other hand, if $\mu = -1$ the energy could be negative and blow-up may occur; the equations are then called *focusing*.

1.1.1 Dispersion and the linear Schrödinger equation

The NLS is a fundamental example of a nonlinear dispersive partial differential equation (PDE) and arises as one of the universal models in nonlinear wave phenomena. In this context, *dispersion*[1] means that waves of different wavelengths travel at different phase velocities; specifically waves of short wavelength propagate faster than ones of long wavelength.

Both NLS (1.1.1) and the linear Schrödinger equation:

$$\begin{cases} i\partial_t u + \Delta u = 0, \quad (t, x) \in \mathbb{R} \times \mathcal{M}^d, \\ u(0, x) = u_0(x). \end{cases} \tag{1.1.4}$$

are dispersive equations. Heuristically, dispersion of the solution to (1.1.1) can be viewed as an effect which comes from (1.1.4), the linear part of NLS (1.1.1). Consider plane

[1] A classical dispersion phenomenon is the so-called chromatic dispersion, which describes that in a dispersive prism material dispersion causes different colors to refract at different angles, splitting white light into a spectrum.

wave solutions of the linear Schrödinger equation (1.1.4) in the form:

$$u(t,x) = e^{i(k\cdot x - \omega t)},$$

where k and ω are the wave number (spatial frequency) and the temporal frequency of the plane wave, respectively. Then, by (1.1.4), we hold a relation between spatial and temporal frequencies which is called the dispersion relation:

$$\omega = |k|^2.$$

Plane wave solutions can then be written as $u(t,x) = e^{ik\cdot(x-kt)}$. We see that plane wave solutions of wave number k travel at phase velocity k, and that the phase velocity goes to infinity as the wave number $k \to \infty$. This implies the dispersion of the linear Schrödinger equation (1.1.4).

The dispersion of the linear Schrödinger equation (1.1.4) plays an essential role in the local-in-time existence theory of the NLS, and dispersive properties yield an extremely useful set of estimates, known as Strichartz estimates [99][74][55][106].

Generally speaking, the linear equation is much better understood than the nonlinear equation. One way to approach the study of the nonlinear equation (1.1.1) is to treat the nonlinearity $\mu|u|^2u$ in (1.1.1) as 'negligible'; that is, to view the nonlinear equation (1.1.1) as a perturbation of the linear Schrödinger equation (1.1.4). This perturbation idea works well if the size of u is small (so that the nonlinearity $\mu|u|^2u$ is also small) or if we solve (1.1.1) only on a small time interval (so that the nonlinearity does not influence the solution too much cumulatively in time).

In order to study the long time behavior of the solution to (1.1.1) with *large data*, however, the approach above is not sufficient since the problem is no longer perturbative. We need some extra ideas such as an *energy induction argument* and other conserved quantities to fight off the long time influence of the nonlinearity.

The solution $u(x,t)$ to the linear problem (1.1.4) can be written down explicitly by using the Fourier transform. Indeed, we have that $u(t,x) := e^{it\Delta}u_0(x)$. The semigroup operator $e^{it\Delta}$ is the *linear Schrödinger propagator* defined by

$$\widehat{e^{it\Delta}u_0}(t,k) = e^{-it|k|^2}\widehat{u_0}(k), \text{ for any frequency } k \in \begin{cases} \mathbb{R}^d, \text{ if } \mathcal{M}^d = \mathbb{R}^d \\ \mathbb{Z}^d, \text{ if } \mathcal{M}^d = \mathbb{T}^d \end{cases} \tag{1.1.5}$$

acting on the initial data u_0. Here $\widehat{f}(k)$ means the Fourier transform of the function $f(x)$ at frequency k. Let us recall that a basic property of Fourier transform is that $\widehat{\Delta f}(k) = ik \cdot \widehat{\nabla f}(k) = -|k|^2\widehat{f}(k)$. See Chapter 2 for more details.

1.1.2 Scaling and other symmetries on $\mathbb{R}^d$

In Euclidean spaces $\mathbb{R}^d$, the semilinear NLS with power nonlinearity (p-NLS):

$$i\partial_t u + \Delta u = \mu|u|^{p-1}u, \qquad p > 1 \tag{1.1.6}$$

enjoys many symmetries:

- Space-time translation: If $u(t,x)$ solves (1.1.6), then for any $t_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}^d$ we have $u(t-t_0, x-x_0)$ solves (1.1.6).

- Phase rotation: If $u(t,x)$ solves (1.1.6), then for any $\theta_0 \in \mathbb{R}$ we have $e^{i\theta_0}u(t,x)$ solves (1.1.6).

- Galilean transformation: If $u(t,x)$ solves (1.1.6), then for any $v_0 \in \mathbb{R}^d$ we have $e^{iv_0\cdot x}e^{-it|v|^2}u(t, x-vt)$ solves (1.1.6).

- Time reversal: If $u(t,x)$ solves (1.1.6), then we have $\overline{u(-t,x)}$ solves (1.1.6).

- Scaling: If $u(t,x)$ solves (1.1.6), then for any dilation factor $\lambda > 0$ we have

  $$\lambda^{-2/(p-1)}u(\frac{t}{\lambda^2}, \frac{x}{\lambda}) \tag{1.1.7}$$

  solves (1.1.6).

Among all the symmetries the scaling symmetry (1.1.7) is particularly important, as it shows a relation between time of existence and regularity of initial data. Let us introduce the homogeneous Sobolev norm $\|\cdot\|_{\dot{H}^s(\mathbb{R}^d)}$ for functions defined in $\mathbb{R}^d$ with the regularity index s, which indicates that the $\|\cdot\|_{\dot{H}^s(\mathbb{R}^d)}$ norm measures the size of s-th order derivatives of functions in the L^2 sense (hence in this sense the regularity index s can be understood as a smoothness index). The scaling transform (1.1.7) preserves the homogeneous Sobolev norm of the initial data, $\|u_0\|_{\dot{H}^{s_c}(\mathbb{R}^d)}$ where $s_c := \frac{d}{2} - \frac{2}{p-1}$ is called the *critical regularity*.

From (1.1.7) it is clear that if we take $\lambda \to +\infty$ then:

- If $s > s_c$ (*sub-critical case*) the $\dot{H}^s$ norm of solutions can be scaled to be small while making the time interval of existence longer, which intuitively helps in establishing the local-in-time well-posedness theory.

- If $s = s_c$ (*critical case*) the norm is invariant while the interval of time is made longer. The well-posedness problem in this case is more difficult than the one in the sub-critical case.

- If $s < s_c$ the $\dot{H}^s$ norm of solutions grows as the time interval gets longer. Intuitively, scaling is against the well-posedness.

The above 'scaling argument' can be used as a formal guideline in the study of the well-posedness problem. Initial data in $\dot{H}^s$ with $s > s_c$ (*sub-critical regime*) is the best possible setting for well-posedness. Indeed, local-in-time well-posedness of (1.1.6) was proven by Cazenave-Weissler in [27].

For $\dot{H}^s$ data with $s = s_c$ (*critical regime*) the well-posedness problem is more difficult than the one in the sub-critical regime. In fact, the well-posedness in the sub-critical regime can be obtained from the well-posedness in the critical regime by a persistence of regularity argument. Bourgain [9] first proved the large data global-in-time well-posedness and scattering for the defocusing energy-critical ($s_c = 1$) NLS in $\mathbb{R}^3$ with radially symmetric initial data in $\dot{H}^1$ by introducing an induction method on the size of energy and a refined Morawetz inequality. A different proof of the same result was given by Grillakis in [57]. A breakthrough was made by Colliander-Keel-Staffilani-Takaoka-Tao in [31]. Their work extended the results of Bourgain [9] and Grillakis [57]. They proved global-in-time well-posedness and scattering of the energy-critical problem in $\mathbb{R}^3$ for general large data in $\dot{H}^1$. Similar results were then proven by Ryckman-Vişan [98] and Vişan [105] on the higher dimension $\mathbb{R}^d$ spaces. Furthermore, Dodson proved mass-critical ($s_c = 0$) global-in-time wellposedness results for $\mathbb{R}^d$ in his series of papers [40, 42, 43].

For $\dot{H}^s$ with $s < s_c$ (*super-critical regime*) the initial data u_0 is rougher than the critical regularity. Intuitively, scaling is against the well-posedness. This intuition was verified for example in [29][30], where it is shown that the initial value problem for NLS in $\mathbb{R}^d$ with the super-critical data leads to ill-posedness. More precisely, the solutions, whose the $\dot{H}^s$ norms approach arbitrary large with arbitrary small initial data in arbitrary small time, can be constructed. These solutions, which is called norm inflation, contradict the continuously dependence on the initial data.

1.1.3 $\mathbb{R}^d$ vs $\mathbb{T}^d$

One of the interesting derivations of NLS is Bose-Einstein condensation. A Bose-Einstein condensate is the state of matter of a gas of weakly interacting bosonic atoms confined by an external potential and cooled to temperatures very near absolute zero (0 Kelvin). S.N Bose [6] and A. Einstein [48] predicted Bose-Einstein condensation phenomena in 1924-1925. In 1995 Bose-Einstein condensation was produced by the experiments of Cornell-Wieman [1] and of W.S. Ketterle [33].

In Bose-Einstein condensation phenomena the majority of bosons are in the same quantum state for a sufficiently low temperature, and thus we can use a single wave function $u(t, x)$ to describe all bosons. The interactions between bosons lead to a nonlinear version of the Schrödinger equation. Taking the binary interactions between the bosons into account we can see that u satisfies a cubic NLS, which is also called the *Gross-Pitaevski equation* after work by Gross [58] and by Pitaevski [96]. Physically, it makes sense to study the problem both in the periodic setting ($x \in \mathbb{T}^3$: many-body quantum bosonic atoms in a three-dimensional cube) and the non-periodic setting ($x \in \mathbb{R}^3$).

Both the nonlinear and linear Schrödinger equations, however, have very different behaviors on $\mathbb{T}^d$ and $\mathbb{R}^d$, since *dispersion* is weaker on periodic domains. In the Euclidean space $\mathbb{R}^d$, the dispersive properties of the linear Schrödinger propagator $e^{it\Delta}$ play an essential role in estimating the nonlinear term. The decay estimates implied by dispersion yield an extremely useful set of estimates, known as *Strichartz estimates* [55][106][74]. For example, on $\mathbb{R}^4$, a useful Strichartz estimate is:

$$\|e^{it\Delta}u_0\|_{L^3_{t,x}(\mathbb{R}\times\mathbb{R}^4)} \lesssim \|u_0\|_{L^2_x(\mathbb{R}^4)}. \tag{1.1.8}$$

However, on the d-dimensional torus $\mathbb{T}^d$ we do not have similar strong dispersion estimates as in $\mathbb{R}^d$. In fact, the lack of strong dispersion estimates makes the periodic NLS harder than the Euclidean one. Instead, we only have weaker versions of the Strichartz estimate for frequency localized functions [7][17]. For example on $\mathbb{T}^4$, instead of (1.1.8), we have the following estimate:

$$\|e^{it\Delta}P_C u_0\|_{L^3_{t,x}([0,1]\times\mathbb{T}^4)} \lesssim N^{\varepsilon}\|u_0\|_{L^2_x(\mathbb{T}^4)}, \tag{1.1.9}$$

where ε could be any small positive number and P_C is a projection onto a cube of size N in the frequency space. The weaker Strichartz estimates on tori (c.f. Proposition 4.4.1) make it more difficult to prove well-posedness of NLS. Thus, a natural question to ask is:

- On $\mathbb{T}^d$ can we prove analogous well-posedness results to those in $\mathbb{R}^d$ mentioned above in the *sub-critical, critical, super-critical* regimes?

In the *sub-critical* regime, the local-in-time existence of solution to the p-NLS on $\mathbb{T}^d$ (1.1.6) can be established by using the Strichartz estimates on $\mathbb{T}^d$ in [7][17]. The Strichartz estimates on rational tori $\mathbb{T}^d$, which prove the local well-posedness of the periodic NLS, were initially developed by Bourgain [7]. In [7], number theoretical related lattice counting arguments, which only applied to rational tori, were used in the proof of the key L^p

estimate for the Strichartz estimates. Recently Bourgain-Demeter [17] proved the optimal Strichartz estimates on both rational and irrational tori via a totally different approach which doesn't depend on lattice counting but rather on the ℓ^2-decoupling argument for the truncated paraboloid. Other important references for Strichartz estimates for NLS on tori and the global well-posedness of the Cauchy problem in the sub-critical regime include [10, 34, 26, 11, 59, 82, 35, 39, 50]. Furthermore, on general compact manifolds, Burq-Gerard-Tzvetkov derived Strichartz-type estimates and applied these estimates to obtain global well-posedness of NLS in a series of papers [18, 19, 20, 21]. We also refer the reader to [112, 54, 61, 62] and references therein for other results of global existence for sub-critical NLS on compact manifolds.

In the *critical* regime, we can only prove global well-posedness results for the energy-critical ($s_c = 1$) NLS. To the best of our knowledge, however, the mass-critical ($s_c = 0$) case on tori is still an open problem, since the Strichartz estimates in [17] are not enough to obtain the local-in-time well-posedness theory. In the critical regime, Herr-Tataru-Tzvetkov [67] studied the global existence of the energy-critical NLS on $\mathbb{T}^3$ and first proved the global well-posedness with small initial data in $H^1(\mathbb{T}^3)$. They used crucial trilinear Strichartz-type estimates in the context of the critical atomic spaces U^p and V^p. These atomic spaces were systematically formalized by Hadac-Herr-Koch [60] (see also [83][68]) and now the atomic spaces U^p and V^p are widely used to reach the *critical* well-posedness theory for nonlinear dispersive equations. The large data global well-posedness for the energy-critical NLS on rational $\mathbb{T}^3$ was proven by Ionescu-Pausader [71]. This was the first large data critical global well-posedness result of NLS on a compact manifold. In a series of papers, Ionescu-Pausader [71, 72] and Ionescu-Pausader-Staffilani [73] developed a robust method to obtain energy-critical large data global well-posedness in more general manifolds ($\mathbb{T}^3$, $\mathbb{T}^3 \times \mathbb{R}$, and $\mathbb{H}^3$) based on the corresponding results in the Euclidean spaces of the same dimension. So far their method has been successfully applied to other manifolds [95, 100, 110, 111]. In particular, based on recent developments on the large data global well-posedness theory in product spaces $\mathbb{R}^m \times \mathbb{T}^d$ $(m, d \geq 1)$, many authors ([103, 63, 104, 64, 56, 28, 110, 111, 84]) studied the long time asymptotic behavior (scattering and modified scattering) of solutions to NLS.

In **Part I** of this thesis we prove energy-critical large data global-in-time well-posedness on $\mathbb{T}^4$ in the defocusing case using a similar strategy as in Ionescu-Pausader [71][72]. Furthermore, in the focusing case we also address the large data global in-time wellposedness below the ground state threshold.

In the *super-critical* regime, similar to the $\mathbb{R}^d$ case, scaling symmetry is against the well-posedness of NLS. A similar behavior[2] hold in the $\mathbb{T}^d$ case (see for example, Oh-Wang [94] for a form of ill-posedness of the cubic NLS on $\mathbb{T}$). However, ill-posedness in some cases can be circumvented by considering the evolution of suitably randomized initial data in some probability space and using an appropriate probabilistic methods. In the other words, one may hope to establish almost sure local-in-time well-posedness with respect to a certain probability random data space. This random data approach to local-in-time well-posedness first appeared in Bourgain's work on the invariance of the Gibbs measure associated to the cubic NLS on $\mathbb{T}^2$ [13]. Once this probabilistic local well-posedness was in place, Bourgain used similar arguments as in his 1D paper [8] to show almost sure global-in-time well-posedness and invariance of the Gibbs measure. Later, Burq-Tzvetkov [22] obtained similar random data local-in-time well-posedness in the context of the cubic nonlinear wave equation (NLW) on a three dimensional compact Riemannian manifold and almost sure global well-posedness and invariance of the Gibbs measure in the radial case [23].

In **Part II** of this thesis, we study the cubic NLS in the super-critical regime on tori $\mathbb{T}^d$ $(d \geq 3)$ via the probabilistic approach. After Bourgain's first two papers [8][13] on $\mathbb{T}^1$ and $\mathbb{T}^2$, Nahmod-Staffilani [91] proved an almost sure local-in-time well-posedness result for the quintic NLS on $\mathbb{T}^3$ in the super-critical regime. Part II should be viewed as a

[2] Namely some form of ill-posedness; i.e. a failure of one of the three properties defining local in time well-posedness.

natural continuation of these works. More precisely, in Part II we consider the random data Cauchy initial value problem for the cubic NLS on $\mathbb{T}^d, d \geq 3$ in the adapted atomic spaces X^s and show it is locally-in-time well-posed in high probability for super-critical data. We follow a similar strategy as in [13][91] coupled with a new lemma which modifies the "transfer principle" on atomic spaces applied to forms to obtain some specific time smoothing estimates on small time intervals. See Section 4.3 for details.

1.2 Structure of the Thesis

In Chapter 2, we introduce some preliminaries we will use in this thesis. We prove global-in-time well-posedness of the cubic NLS on $\mathbb{T}^4$ both in the defocusing and focusing cases in Chapter 3. In Chapter 4, we show local-in-time well-posedness in high probability in the super-critical regime. In Appendix A, we prove the profile decomposition for the linear Schrödinger propagator on $\mathbb{T}^4$ which is used in Chapter 3.

The results in this thesis are contained in the author's preprints [107], [108].

CHAPTER 2

PRELIMINARIES

In this chapter we introduce some preliminaries that we will use in the rest of this thesis. The first section is devoted to the mathematical notions used in this thesis. The Littlewood-Paley decomposition on $\mathbb{T}^d$, one of most useful techniques in both Part I and Part II of this thesis, which allows us to analyze the interaction between different frequencies, is presented in Section 2.2. In Section 2.3, we state the Strichartz estimates on $\mathbb{T}^d$.

2.1 Notation

Throughout this thesis, we write $A \lesssim B$ when $A \leq CB$ for some constant C. We write $A \sim B$ if $A \lesssim B$ and $B \lesssim A$. Also we use $A \lesssim_\sigma B$ when $A \leq C_\sigma B$ where C_σ is a constant depending on σ.

We define the Fourier transform on $\mathbb{T}^d$ by

$$\widehat{f}(n) := \int_{\mathbb{T}^d} f(x) e^{in\cdot x}\, dx, \text{ for } n \in \mathbb{Z}^d.$$

and use the classical function spaces L^p for $p \geq 1$, C^1, $\cdots$, C^k and C^∞. We also use the homogeneous Sobolev spaces $\dot{H}^s(\mathbb{T}^d)$ equipped with the norm

$$\|f\|_{\dot{H}^s(\mathbb{T}^d)} := \left(\sum_{k\in\mathbb{Z}^d} |k|^{2s} |\widehat{f}(k)|^2 \right)^{\frac{1}{2}},$$

and the inhomogeneous Sobolev spaces $H^s(\mathbb{T}^d)$ which are equipped with the norm

$$\|f\|_{H^s(\mathbb{T}^d)} := \left(\sum_{k \in \mathbb{Z}^d} (1 + |k|^2)^s |\widehat{f}(k)|^2 \right)^{\frac{1}{2}} .$$

Given a function $u = u(t, x)$ depending on time variable t and space variable x, the space-time Lebesgue spaces $L_t^q L_x^r$ are the Banach spaces of functions $u(x, t)$ equipped with the norm

$$\|u\|_{L_t^q L_x^r(I \times \mathbb{T}^d)} := \left(\int_I \|u(t, x)\|_{L_x^r(\mathbb{T}^d)}^q \, dt \right)^{\frac{1}{q}} ,$$

where I is a fixed time interval. Similarly we denote by $\|u\|_{L_{t,x}^p(I \times \mathbb{T}^d)} := \|u\|_{L_t^p L_x^p(I \times \mathbb{T}^d)}$.

The spaces $C(I : H^s(\mathbb{T}^d))$ (resp. $L^\infty(I : H^s(\mathbb{T}^d))$) denote the space of functions which are continuous in t (resp. in L^∞ in t) with values in the inhomogeneous Sobolev space $H^s(\mathbb{T}^d)$.

For a Hilbert space $\mathcal{H}$, we denote by $\langle \cdot, \cdot \rangle_{\mathcal{H} \times \mathcal{H}}$ the inner product of the Hilbert space $\mathcal{H}$.

2.2 Littlewood-Paley decomposition on $\mathbb{T}^d$

Definition 2.2.1 (Littlewood-Paley decomposition on $\mathbb{T}^d$)**.** For $N > 1$ a dyadic number, we denote by $P_{\leq N}$ the Fourier multiplier:

$$P_{\leq N} f = \sum_{n \in \mathbb{Z}^d : |n| \leq N} \hat{f}(n) e^{in \cdot x}.$$

Then $P_N = P_{\leq N} - P_{\leq N/2}$ and hence $f = \sum_{N \text{ is dyadic}} P_N f$. Moreover, if C is a subset of $\mathbb{Z}^d$, then the Fourier projection operator onto C is defined by P_C

$$P_C f = \sum_{n \in \mathbb{Z}^d : n \in C} \hat{f}(n) e^{in \cdot x}.$$

Like all Fourier multipliers, the Littlewood-Paley operators commute with the linear Schrödinger propagator $e^{it\Delta}$, as well as with differential operators.

Lemma 2.2.2. *Given a Schwartz function f, let*

$$S(f)(x) := \left(\sum_{N \text{ is dyadic}} |P_N f(x)|^2 \right)^{1/2}$$

denote the Littlewood-Paley square function. For $1 < p < \infty$,

$$\|S(f)\|_{L^p(\mathbb{T}^d)} \sim \|f\|_{L^p(\mathbb{T}^d)}.$$

2.3 Strichartz estimates on $\mathbb{T}^d$

In [7], Bourgain stated the Strichartz estimates in Lemma 2.3.1 below for the linear Schrödinger operator on tori as a general conjecture, and proved parts of this conjecture in the same paper [7]. Recently, in [17] Bourgain-Demeter established the full conjecture.

Lemma 2.3.1 (Strichartz-type estimates [7][17])**.** *If $p > \frac{2(d+2)}{d}$ and $d \geq 1$, then*

$$\|P_N e^{it\Delta} f\|_{L^p_{t,x}([-1,1]\times\mathbb{T}^d)} \lesssim_p N^{\frac{d}{2} - \frac{d+2}{p}} \|f\|_{L^2_x}$$

and

$$\|P_C e^{it\Delta} f\|_{L^p_{t,x}([-1,1]\times\mathbb{T}^d)} \lesssim_p N^{\frac{d}{2} - \frac{d+2}{p}} \|f\|_{L^2_x}$$

where C is a cube of side length N in frequency space and $f \in L^2(\mathbb{T}^d)$.

CHAPTER 3

§I: GLOBAL WELL-POSEDNESS OF THE CUBIC NLS ON $\mathbb{T}^4$

In this chapter we prove the results described above for Part I of this thesis. Namely, in this chapter, we study the global-in-time well-posedness of the cubic NLS on $\mathbb{T}^4$ both in the defocusing and focusing cases. This chapter is devoted to the proofs of Theorem 3.1.1 (defocusing) and Theorem 3.1.2 (focusing).

3.1 Introduction

We consider the cubic nonlinear Schrödinger equation (NLS),

$$(i\partial_t + \Delta)u = \mu u|u|^2, \tag{3.1.1}$$

in the periodic setting $x \in \mathbb{T}^4_\lambda$, where $\mu = \pm 1$ ($+1$: the defocusing case, -1: the focusing case). Here $u : \mathbb{R} \times \mathbb{T}^4_\lambda \to \mathbb{C}$ is a complex-valued function of time $t \in \mathbb{R}$ and space $x \in \mathbb{T}^4_\lambda$, which a general rectangler tori, i.e.

$$\mathbb{T}^4_\lambda := \mathbb{R}^4/(\prod_{i=1}^{4} \lambda_i \mathbb{Z}), \qquad \lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4),$$

where $\lambda_i \in (0, \infty)$ for $i = 1, 2, 3, 4$. If the ratio of any two $\lambda_i' s$ in $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ is an irrational number then $\mathbb{T}^4_\lambda$ is called an irrational torus, otherwise $\mathbb{T}^4_\lambda$ is called a rational torus. We mention this because the two cases have historically required different methods. Our proof, however, applies to both rational and irrational tori, so for convenience we use $\mathbb{T}^d := \mathbb{T}^4_\lambda$ to denote any torus.

Solutions of (3.1.1) conserve in both the mass of u:

$$M(u)(t) := \int_{\mathbb{T}^4} |u(t)|^2 \, dx \tag{3.1.2}$$

and the energy of u:

$$E(u)(t) := \frac{1}{2} \int_{\mathbb{T}^4} |\nabla u(t)|^2 \, dx + \frac{1}{4} \mu \int_{\mathbb{T}^4} |u(t)|^4 \, dx. \tag{3.1.3}$$

3.1.1 The defocusing case ($\mu = +1$)

In the defocusing case, our main theorem is global well-posedness of (3.1.1) with $H^1(\mathbb{T}^4)$ initial data. Here $H^1(\mathbb{T}^4)$ is the inhomogeneous Sobolev space defined in Section 2.

Theorem 3.1.1 (GWP of the defocusing NLS)**.** *If $u_0 \in H^1(\mathbb{T}^4)$, for any $T \in [0, \infty)$, there exists a unique global solution $u \in X^1([-T, T])$ of the initial value problem*

$$(i\partial_t + \Delta)u = u|u|^2, \qquad u(0) = u_0. \tag{3.1.4}$$

In addition, the mapping $u_0 \to u$ extends to a continuous mapping from $H^1(\mathbb{T}^4)$ to $X^1([-T, T])$ and $M(u)$ and $E(u)$ defined in (3.1.2) and (3.1.3) are conserved along the flow.

Here the space $X^1(I) \subset C(I : H^1(\mathbb{T}^4))$ is the adapted atomic space (see Definition 3.3.5).

In this chapter we prove the large data global well-posedness result of defocusing energy-critical NLS on the both rational and irrational tori in dimension 4. Our proof is closely related to the strategy developed by Ionescu-Pausader [71][72]. Compared to the $\mathbb{T}^3 \times \mathbb{R}$ case, in Ionescu-Pausader [72], there is less dispersion of the Schrödinger operator in the compact manifold $\mathbb{T}^4$. This means that it is more difficult to obtain sharp enough Strichartz-type estimates akin to Proposition 2.1 in [72]. We use the sharp Strichartz-type estimates (Lemma 2.3.1) recently proven by Bourgain-Demeter [17] in our proof.

Moreover, since Lemma 2.3.1 works both for rational or irrational tori, we can prove the result on both rational and irrational tori.

The main parts in the proof of Theorem 3.1.1 will follow the concentration-compactness framework of Kenig-Merle [75], which is a deep and broad road map to deal with critical problems (see also in [76][77]). Our first step is to obtain the critical local well-posedness theory and the stability theory of (3.1.1) in $\mathbb{T}^4$. For that purpose, we follow Herr-Tataru-Tzvetkov's idea [67][68] and introduce the adapted critical spaces X^s and Y^s, which are frequency localized modifications of atomic spaces U^p and V^p, as our solution spaces and nonlinear spaces. Applying Proposition 2.3.1 and the strip decomposition technique in [67, 68] to the atomic spaces in space-time frequency space, we obtain a crucial bilinear estimate and then the local well-posedness of (3.1.1). We then measure the solution in a weaker critical space-time norm Z, which plays a similar role as the $L^{10}_{x,t}$ norm in [31]. On the one hand, equipped with the Z-norm, we obtain the refined bilinear estimate (Lemma 4.6.3) and hence it is proven that the solution stays regular as long as the Z-norm stays finite (i.e. global well-posedness with a priori Z-norm bound). On the other hand, we show that concentration of a large amount of the Z -norm in finite time is self-defeating. The reason is as follows: concentration of a large amount of the Z-norm in finite time can only happen around a space-time point, which can be considered as a Euclidean-like solution. To implement this, arguing by contradiction, we construct a sequence of initial data which imply a sequence of solutions and lead the Z-norm towards infinity. Then following the profile decompositon idea (firstly by Gerard [52] in Sobolev embedding and Merle-Vega [88] in the Schrödinger equation), we perform a linear profile decomposition of the sequence of initial data with one Scale-1-profile and a series of Euclidean profiles that concentrate at space-time points. We get nonlinear profiles by running the linear profiles along the nonlinear Schrödinger flow as initial data. By the contradiction condition, the scattering properties of nonlinear Euclidean profiles and the defect of interaction between different profiles show that there is actually at most one profile which is the Euclidean profile. The corresponding nonlinear Euclidean profile is just the Euclidean-like solution we want. Euclidean-like solutions can be interpreted in some sense as solutions in the Euclidean space $\mathbb{R}^4$, however, this kind of concentration as a Euclidean-like solution is prevented by the global well-posedness theory on the Euclidean space $\mathbb{R}^4$ in Vişan-Ryckman [98] and Vişan [105]'s papers.

3.1.2 The focusing case ($\mu = -1$)

In the focusing case, we prove global well-posedness when both the modified energy and kinetic energy of the initial data is less than the energy and the kinetic energy of the ground state W in $\mathbb{R}^4$. Moreover,

$$W(x) = W(x,t) = \frac{1}{1+\frac{|x|^2}{8}} \qquad \text{in } \dot{H}^1(\mathbb{R}^4) \tag{3.1.5}$$

which is a stationary solution of the focusing case of (3.1.1) and also solves the elliptic equation

$$\Delta W + |W|^2 W = 0 \tag{3.1.6}$$

in $\mathbb{R}^4$. Then we define a constant C_4 by using the stationary solution W.

$$\|W\|^2_{\dot{H}^1(\mathbb{R}^4)} = \|W\|^4_{L^4(\mathbb{R}^4)} := \frac{1}{C_4^4} \qquad \text{and then} \qquad E_{\mathbb{R}^4}(W) = \frac{1}{4C_4^4}, \tag{3.1.7}$$

where $E_{\mathbb{R}^4}(W)$ is the energy of W in the Euclidean space $\mathbb{R}^4$:

$$E_{\mathbb{R}^4}(W) := \frac{1}{2}\int_{\mathbb{R}^4} |\nabla W(x)|^2\, dx - \frac{1}{4}\int_{\mathbb{R}^4} |W(x)|^4\, dx. \tag{3.1.8}$$

Note that C_4 is the best constant in Sobolev embedding (see Remark 3.2.2).

Theorem 3.1.2 (GWP of the focusing NLS)**.** *Assume $u_0 \in H^1(\mathbb{T}^4)$ and that u is a maximal-lifespan solution $u : I \times \mathbb{T}^4 \to \mathbb{C}$ satisfying*

$$\sup_{t\in I} \|u(t)\|_{\dot{H}^1(\mathbb{T}^4)} < \|W\|_{\dot{H}^1(\mathbb{R}^4)}. \tag{3.1.9}$$

Then for any $T \in [0, +\infty)$, $u \in X^1([-T,T])$ is a solution of the initial value problem

$$(i\partial_t + \Delta)u = -u|u|^2, \qquad u(0) = u_0. \tag{3.1.10}$$

For a technical reason, in the focusing case we should introduce two modified energies of u:

$$E_*(u)(t) := \frac{1}{2}(\|u(t)\|^2_{\dot{H}^1(\mathbb{T}^4)} + c_*\|u(t)\|^2_{L^2(\mathbb{T}^4)}) - \frac{1}{4}\|u(t)\|^4_{L^4(\mathbb{T}^4)}, \tag{3.1.11}$$

and

$$E_{**}(u)(t) := \frac{1}{2}(\|u(t)\|^2_{\dot{H}^1(\mathbb{T}^4)} + c_*\|u(t)\|^2_{L^2(\mathbb{T}^4)}) - \frac{1}{4}\|u(t)\|^4_{L^4(\mathbb{T}^4)} + \frac{c_*^2 C_4^4}{4}\|u(t)\|^4_{L^2(\mathbb{T}^4)}, \tag{3.1.12}$$

where c_* is a fixed constant determined by the Sobolev embedding on $\mathbb{T}^4$ (Lemma 3.2.1). By the definitions (3.1.11)(3.1.12), $E_*(u)(t)$ and $E_{**}(u)(t)$ are conserved in time.

We also introduce $\|u\|_{H^1_*(\mathbb{T}^4)}$ as a modified inhomogenous Sobolev norm:

$$\|u\|^2_{H^1_*(\mathbb{T}^4)} = \|u\|^2_{\dot{H}^1(\mathbb{T}^4)} + c_*\|u\|^2_{L^2(\mathbb{T}^4)} \tag{3.1.13}$$

Obviously, the $H^1_*(\mathbb{T}^4)$-norm and $H^1(\mathbb{T}^4)$-norm are two comparable norms ($\|u\|_{H^1_*(\mathbb{T}^4)} \simeq \|u\|_{H^1(\mathbb{T}^4)}$).

Corollary 3.1.3. *Assume that $u_0 \in H^1(\mathbb{T}^4)$ satisfies*

$$\|u_0\|_{H^1_*(\mathbb{T}^4)} < \|W\|_{\dot{H}^1(\mathbb{R}^4)}, \quad E_*(u_0) < E_{\mathbb{R}^4}(W); \tag{3.1.14}$$

OR

$$\|u_0\|_{\dot{H}^1(\mathbb{T}^4)} < \|W\|_{\dot{H}^1(\mathbb{R}^4)}, \quad E_{**}(u_0) < E_{\mathbb{R}^4}(W), \tag{3.1.15}$$

where $E_*(u_0)$ and $E_{**}(u_0)$ are two modified Energies defined in (3.1.11) and (3.1.12), and $E_{\mathbb{R}^4}(W)$ is the Energy in the Euclidean space defined in (3.1.8). Then for any $T \in [0,\infty)$, there exists a unique global solution $u \in X^1([-T,T])$ of the initial value problem (3.1.10). In addition, the mapping $u_0 \to u$ extends to a continuous mapping from $H^1(\mathbb{T}^4)$ to $X^1([-T,T])$ for any $T \in [0,\infty)$.

Remark 3.1.4. By the the energy trapping lemma (Theorem 2.5) in Section 2, either (3.1.14) or (3.1.15) implies the condition (3.1.9) in Theorem 3.1.2.

In the focusing case the global well-posedness result usually doesn't hold for arbitary data. For the energy-critical focusing NLS on $\mathbb{R}^d$, Kenig-Merle [75] first proved the global well-posedness and scattering with initial data below a ground state threshold ($E_{\mathbb{R}^d}(u_0) < E_{\mathbb{R}^d}(W)$ and $\|u_0\|_{\dot{H}^1} < \|W\|_{\dot{H}^1}$) in the radial case ($d \geq 3$). The corresponding results without the radial conditions were proven by Killp-Vişan [81] ($d \geq 5$) and Dodson [41] ($d = 4$). We also refer to [46, 70, 51, 47, 87, 44, 45] for other focusing NLS results.

In this chapter, we also prove a similar global well-posedness result for the energy-critical focusing NLS on $\mathbb{T}^4$ below the ground state threshold. As in the defocusing case, we follow the idea from Ionescu-Pausader [71][72] and use the focusing global well-posedness result [41] in $\mathbb{R}^4$ as a black box. It is known that the conditions in $\mathbb{R}^d$ are $E_{\mathbb{R}^d}(u_0) < E_{\mathbb{R}^d}(W)$ and $\|u_0\|_{\dot{H}^1} < \|W\|_{\dot{H}^1}$, which are tightly related to the Sobolev embedding with the best constant in $\mathbb{R}^d$. However the sharp version of Sobolev embedding (Lemma 3.2.1) is quite different. Thus, compared to the conditions for initial data in Eulidean space $\mathbb{R}^d$, the conditions in Corollary 3.1.3 should be different. A similar case is the focusing NLS on the hyperbolic space, which doesn't share the sharp version of the Sobolev embedding either. On the hyperbolic space, Fan-Kleinhenz [49] give a minimal ratio between energy and L^2 norm under the condition $E_{\mathbb{H}^3}(u_0) < E_{\mathbb{R}^3}(W)$ and $\|u_0\|_{\dot{H}^1} < \|W\|_{\dot{H}^1}$ in the radial case. Also, in Banica-Duyckaerts [3]'s paper about the focusing NLS on the hyperbolic space, they modified the Sobolev norm and energy by subtracting a multiple of the L^2 norm. On $\mathbb{T}^d$, based on the best constants of Sobolev embedding (Lemma 3.2.1) on $\mathbb{T}^d$, we should also modify the energy and Sobolev norm by adding terms related to the L^2 norm , so that the modified conditions together with Sobolev embedding derive the energy trapping property which controls the Sobolev norm globally in time. In Section 2, we will discuss the Sobolev embedding and energy trapping lemma in detail.

3.2 Energy trapping for the focusing NLS

Before proceeding to the proofs of the main theorems (Theorem 3.1.1 and Theorem 3.1.2), we explain how Theorem 3.1.2 implies Corollary 3.1.3 in the focusing case by using the energy trapping argument. In this section, we'll prove the energy trapping argument in $\mathbb{T}^4$ which is different from the energy trapping argument (Theorem 3.9 in [75]) in $\mathbb{R}^4$.

Lemma 3.2.1 (Sobolev embedding with best constants by [2][66][65])**.** *Let $f \in H^1(\mathbb{T}^4)$, then there exists a positive constant c_*, such that*

$$\|f\|^2_{L^4(\mathbb{T}^4)} \leq C_4^2(\|f\|^2_{\dot{H}^1(\mathbb{T}^4)} + c_*\|f\|^2_{L^2(\mathbb{T}^4)}). \tag{3.2.1}$$

where C_4 is the best constant of this inequality.

Remark 3.2.2. C_4 is the same constant as expressed in (3.1.7), because C_4 is also the best constant of the Sobolev embedding in $\mathbb{R}^4$, $\|f\|^2_{L^4(\mathbb{R}^4)} \leq C_4^2\|f\|^2_{\dot{H}^1(\mathbb{R}^4)}$, and the function $W(x)$ satisfies the Sobolev embedding with the best constant C_4 in $\mathbb{R}^4$.

Remark 3.2.3. Since $\|u\|^2_{H^1_*(\mathbb{T}^4)} = \|u\|^2_{\dot{H}^1(\mathbb{T}^4)} + c_*\|u\|^2_{L^2(\mathbb{T}^4)}$, the Sobolev embedding (Lemma 3.2.1) can be also written in the form:

$$\|f\|^2_{L^4(\mathbb{T}^4)} \leq C_4^2\|f\|^2_{H^1_*(\mathbb{T}^4)}.$$

Suppose $c_{opt} := \inf\{c_* : c_* \text{ holds (3.2.1)}\}$. By taking $f = 1$, it's easy to check that $c_{opt} \geq \frac{1}{C_4^2 Vol(\mathbb{T}^4)^{1/2}}$.

Lemma 3.2.4. *(i) Suppose $f \in H^1(\mathbb{T}^4)$ and $\delta_0 > 0$ satisfying*

$$\|f\|_{H^1_*(\mathbb{T}^4)} < \|W\|_{\dot{H}^1(\mathbb{R}^4)} \qquad \text{and} \qquad E_*(f) < (1-\delta_0)E_{\mathbb{R}^4}(W), \tag{3.2.2}$$

then there exists $\bar{\delta} = \bar{\delta}(\delta_0) > 0$ such that

$$\|f\|^2_{H^1_*(\mathbb{T}^4)} < (1-\bar{\delta})\|W\|^2_{\dot{H}^1(\mathbb{R}^4)} \tag{3.2.3}$$

$$\|f\|^2_{H^1_*(\mathbb{T}^4)} - \|f\|^4_{L^4(\mathbb{T}^4)} \geq \bar{\delta}\|f\|^2_{H^1_*(\mathbb{T}^4)}, \tag{3.2.4}$$

and in particular

$$E_*(f) \geq \frac{1}{4}(1+\bar{\delta})\|f\|^2_{H^1_*(\mathbb{T}^4)}. \tag{3.2.5}$$

(ii) Suppose $f \in H^1(\mathbb{T}^4)$ and $\delta_0 > 0$ satisfy

$$\|f\|_{\dot{H}^1(\mathbb{T}^4)} < \|W\|_{\dot{H}^1(\mathbb{R}^4)} \qquad \textit{and} \qquad E_{**}(f) < (1-\delta_0)E_{\mathbb{R}^4}(W), \tag{3.2.6}$$

then there exists $\bar{\delta} = \bar{\delta}(\delta_0) > 0$ such that

$$\|f\|^2_{\dot{H}^1(\mathbb{T}^4)} < (1-\bar{\delta})\|W\|^2_{\dot{H}^1(\mathbb{R}^4)} \tag{3.2.7}$$

$$\|f\|^2_{\dot{H}^1(\mathbb{T}^4)} - \|f\|^4_{L^4(\mathbb{T}^4)} + 2c_*\|f\|_{L^2(\mathbb{T}^4)} + c_*^2 C_4^4\|f\|^4_{L^2(\mathbb{T}^4)} \geq \bar{\delta}\|f\|^2_{\dot{H}^1(\mathbb{T}^4)}, \tag{3.2.8}$$

and in particular

$$E_{**}(f) \geq \frac{1}{4}(1+\bar{\delta})\|f\|^2_{\dot{H}^1(\mathbb{T}^4)}. \tag{3.2.9}$$

Proof. In the proof of part (i), we almost identically follow the proof of Lemma 3.4 in Kenig-Merle's paper [75], but use the $H^1_*(\mathbb{T}^4)$-norm instead of the $\dot{H}^1(\mathbb{T}^4)$-norm. Consider a quadratic function $g_1 = \frac{1}{2}y - \frac{C_4^4}{4}y^2$, and plug in $\|f\|^2_{H^1_*(\mathbb{T}^4)}$. By Sobolev embedding (Lemma 3.2.1) and the assumption (3.2.2), we have that

$$\begin{aligned} g_1(\|f\|^2_{H^1_*}) &= \frac{1}{2}\|f\|^2_{H^1_*} - \frac{C_4^4}{4}\|f\|^4_{H^1_*} \\ &\leq \frac{1}{2}\|f\|^2_{H^1_*} - \frac{1}{4}\|f\|^4_{L^4} = E_*(f) \\ &< (1-\delta_0)E_{\mathbb{R}^4}(W) = (1-\delta_0)g_1(\|W\|^2_{\dot{H}^1(\mathbb{R}^4)}). \end{aligned} \tag{3.2.10}$$

It is easy to see that $\|f\|^2_{H^1_*(\mathbb{T}^4)} < (1-\bar{\delta})\|W\|^2_{\dot{H}^1(\mathbb{R}^4)}$, from (3.2.10) and the property of quadratic function g_1, where $\bar{\delta} \sim \delta_0^{\frac{1}{2}}$.

Then choose $g_2(y) = y - C_4^4 y^2$. Plugging in $\|f\|^2_{H^1_*(\mathbb{T}^4)}$, by Sobolev embedding (Lemma 3.2.1), we have that

$$g_2(\|f\|^2_{H^1_*(\mathbb{T}^4)}) = \|f\|^2_{H^1_*(\mathbb{T}^4)} - C_4^4\|f\|^4_{H^1_*(\mathbb{T}^4)} \leq \|f\|^2_{H^1_*(\mathbb{T}^4)} - \|f\|^4_{L^4(\mathbb{T}^4)}. \tag{3.2.11}$$

Since $g_2(0) = 0$, $g_2''(y) = -2C_4^4 < 0$ and $\|f\|^2_{H^1_*(\mathbb{T}^4)} < (1-\bar{\delta})\|W\|^2_{\dot{H}^1(\mathbb{R}^4)}$, by Jensen's inequality and (3.1.7),

$$g_2(\|f\|^2_{H^1_*(\mathbb{T}^4)}) > g_2((1-\bar{\delta})\|W\|^2_{\dot{H}^1(\mathbb{R}^4)})\frac{\|f\|^2_{H^1_*(\mathbb{T}^4)}}{(1-\bar{\delta})\|W\|^2_{\dot{H}^1(\mathbb{R}^4)}} = \bar{\delta}\|f\|^2_{H^1_*(\mathbb{T}^4)}. \tag{3.2.12}$$

Together (3.2.11) and (3.2.12) imply (3.2.4).

By (3.2.4), we get (3.2.5)

$$E_*(f) = \frac{1}{4}\|f\|^2_{H^1_*(\mathbb{T}^4)} + \frac{1}{4}(\|f\|^2_{H^1_*(\mathbb{T}^4)} - \|f\|^4_{L^4(\mathbb{T}^4)}) \geq \frac{1}{4}(1+\bar{\delta})\|f\|^2_{H^1_*(\mathbb{T}^4)}.$$

The proof of part (ii) would be similar to part (i). Under the assumptions (3.2.6) of part (ii), by squaring the Sobolev embedding (Lemma 3.2.1) we have that

$$C_4^4\|f\|^4_{\dot{H}^1(\mathbb{T}^4)} \geq \|f\|^4_{L^4} - 2c_*\|f\|^2_{L^2(\mathbb{T}^4)} - c_*^2 C_4^4\|f\|^4_{L^2(\mathbb{T}^4)} \tag{3.2.13}$$

Plugging $\|f\|^2_{\dot{H}^1(\mathbb{T}^4)}$ into g_1, by (3.2.13), we hold that

$$\begin{aligned} g_1(\|f\|^2_{\dot{H}^1}) &= \frac{1}{2}\|f\|^2_{\dot{H}^1} - \frac{C_4^4}{4}\|f\|^4_{\dot{H}^1} \\ &\leq \frac{1}{2}\|f\|^2_{\dot{H}^1} - \frac{1}{4}\|f\|^4_{L^4} + \frac{c_*}{2}\|f\|^2_{L^2(\mathbb{T}^4)} + \frac{c_*^2C_4^4}{4}\|f\|^4_{L^2(\mathbb{T}^4)} = E_{**}(f) \\ &< (1-\delta_0)E_{\mathbb{R}^4}(W) = (1-\delta_0)g_1(\|W\|^2_{\dot{H}^1(\mathbb{R}^4)}). \end{aligned} \tag{3.2.14}$$

It is easy to know $\|f\|^2_{\dot{H}^1(\mathbb{T}^4)} < (1-\bar{\delta})\|W\|^2_{\dot{H}^1(\mathbb{R}^4)}$, from (3.2.14) and the property of quadratic function g_1, where $\bar{\delta} \sim \delta_0^{\frac{1}{2}}$. Similarly, we can also hold (3.2.8)(3.2.9) under the assumption (3.2.6). □

Theorem 3.2.5 (Energy trapping)**.** *(i) Let u be a solution of IVP (3.1.10), such that for $\delta_0 > 0$*

$$\|u_0\|_{H^1_*(\mathbb{T}^4)} < \|W\|_{\dot{H}^1(\mathbb{R}^4)}, \quad E_*(u_0) < (1-\delta_0)E_{\mathbb{R}^4}(W); \tag{3.2.15}$$

Let $I \ni 0$ be the maximal interval of existence. Then there exists $\bar{\delta} = \bar{\delta}(\delta_0) > 0$ such that for all $t \in I$

$$\|u(t)\|^2_{H^1_*(\mathbb{T}^4)} < (1-\bar{\delta})\|W\|_{\dot{H}^1(\mathbb{R}^4)}, \tag{3.2.16}$$

$$\|u(t)\|^2_{H^1_*(\mathbb{T}^4)} - \|u(t)\|^4_{L^4(\mathbb{T}^4)} \geq \bar{\delta}\|u(t)\|^2_{H^1_*(\mathbb{T}^4)}, \tag{3.2.17}$$

and in particular

$$E_*(u)(t) \geq \frac{1}{4}(1+\bar{\delta})\|u(t)\|^2_{H^1_*(\mathbb{T}^4)}. \tag{3.2.18}$$

(ii) Let u be a solution of IVP (3.1.10), such that, for $\delta_0 > 0$

$$\|u_0\|_{\dot{H}^1(\mathbb{T}^4)} < \|W\|_{\dot{H}^1(\mathbb{R}^4)}, \quad E_{**}(u_0) < (1-\delta_0)E_{\mathbb{R}^4}(W); \tag{3.2.19}$$

Let $I \ni 0$ be the maximal interval of existence. Then there exists $\bar{\delta} = \bar{\delta}(\delta_0) > 0$ such that for all $t \in I$

$$\|u(t)\|^2_{\dot{H}^1(\mathbb{T}^4)} < (1-\bar{\delta})\|W\|_{\dot{H}^1(\mathbb{R}^4)}, \tag{3.2.20}$$

$$\|u(t)\|^2_{\dot{H}^1(\mathbb{T}^4)} - \|u(t)\|^4_{L^4(\mathbb{T}^4)} + 2c_*\|u(t)\|^2_{L^2(\mathbb{T}^4)} + c_*^2 C_4^4\|u(t)\|^4_{L^2(\mathbb{T}^4)} \geq \bar{\delta}\|u(t)\|^2_{\dot{H}^1(\mathbb{T}^4)}, \tag{3.2.21}$$

and in particular

$$E_{**}(u)(t) \geq \frac{1}{4}(1+\bar{\delta})\|u(t)\|^2_{\dot{H}^1(\mathbb{T}^4)}. \tag{3.2.22}$$

Proof. By conservation of energy and mass, this theorem follows directly from Lemma 3.2.4 by the continuity argument. □

Remark 3.2.6. The energy trapping lemma (Theorem 3.2.5) shows that if the initial data satisfies the condition (3.1.14) or (3.1.15) then the solution $u(t)$ satisfies $\|u(t)\|_{\dot{H}^1(\mathbb{T}^4)} < \|W\|_{\dot{H}^1(\mathbb{R}^4)}$ for all t in the lifespan of the solution. Thus, Theorem 3.1.2 implies Corollary 3.1.3. In particular, we also obtain that $E_*(u)(t) \simeq \|u(t)\|^2_{H^1_*(\mathbb{T}^4)}$ under the assumption (3.2.15) and $E_{**}(u)(t) \simeq \|u(t)\|^2_{\dot{H}^1(\mathbb{T}^4)}$ under the assumption (3.2.19) by Theorem 3.2.5.

3.3 Adapted function spaces

In this section, we introduce X^s and Y^s spaces which are based on the atomic spaces U^p and V^p which were originally developed in unpublished work on wave maps by Tararu and then were applied to PDEs in [60][67][68], while we'll use the X^s and Y^s spaces in the proof of the defocusing and focusing global wellposedness. $\mathcal{H}$ is a separable Hilbert space on $\mathbb{C}$ and $\mathcal{Z}$ denotes the set of finite partitions $-\infty = t_0 < t_1 < ... < t_K = \infty$ of the real line, with the convention that $v(\infty) := 0$ for any function $v : \mathbb{R} \to \mathcal{H}$.

Definition 3.3.1 (Definition 2.1 in [67]). Let $1 \leq p < \infty$. For $\{t_k\}_{k=0}^K \in \mathcal{Z}$ and $\{\phi_k\}_{k=0}^{K-1} \subset \mathcal{H}$ with $\sum_{k=0}^K \|\phi_k\|_{\mathcal{H}}^p = 1$ and $\phi_0 = 0$. A U^p-atom is a piecewise defined function $a : \mathbb{R} \to \mathcal{H}$ of the form

$$a = \sum_{k=1}^{K} \mathbb{1}_{[t_{k-1},t_k)}\phi_{k-1}.$$

The atomic Banach space $U^p(\mathbb{R}, \mathcal{H})$ is then defined to be the set of all functions $u : \mathbb{R} \to \mathcal{H}$ such that

$$u = \sum_{j=1}^{\infty} \lambda_j a_j, \qquad \text{for } U^p\text{-atoms } a_j, \qquad \{\lambda_j\}_j \in \ell^1, \quad \|u\|_{U^p} < \infty,$$

where

$$\|u\|_{U^p} := \inf\{\sum_{j=1}^{\infty} |\lambda_j| : u = \sum_{j=1}^{\infty} \lambda_j a_j,\ \lambda_j \in \mathbb{C} \text{ and } a_j \text{ an } U^p \text{ atom}\}.$$

Here $\mathbb{1}_I$ denotes the indicator function over the time interval I.

Definition 3.3.2 (Definition 2.2 in [67]). Let $1 \leq p < \infty$. The Banach space $V^p(\mathbb{R}, \mathcal{H})$ is defined to be the set of all functions $v : \mathbb{R} \to \mathcal{H}$ with $v(\infty) := 0$ and $v(-\infty) := \lim_{t\to-\infty} v(t)$ exists, such that

$$\|v\|_{V^p} := \sup_{\{t_k\}_{k=0}^K \in \mathcal{Z}} (\sum_{k=1}^{K} \|v(t_k) - v(t_{k-1})\|_{\mathcal{H}}^p)^{\frac{1}{p}} \quad \text{is finite.}$$

Likewise, let V_-^p denote the closed subspace of all $v \in V^p$ with $\lim_{t\to-\infty} v(t) = 0$. $V_{-,rc}^p$ means all right-continuous V_-^p functions.

Remark 3.3.3 (Some embeding properties). Note that for $1 \leq p \leq q < \infty$,

$$U^p(\mathbb{R}, \mathcal{H}) \hookrightarrow U^q(\mathbb{R}, \mathcal{H}) \hookrightarrow L^\infty(\mathbb{R}, \mathcal{H}), \tag{3.3.1}$$

and functions in $U^p(\mathbb{R}, \mathcal{H})$ are right continuous, and $\lim_{t\to-\infty} u(t) = 0$ for each $u \in U^p(\mathbb{R}, \mathcal{H})$. Also note that,

$$U^p(\mathbb{R}, \mathcal{H}) \hookrightarrow V_{-,rc}^p(\mathbb{R}, \mathcal{H}) \hookrightarrow U^q(\mathbb{R}, \mathcal{H}). \tag{3.3.2}$$

Definition 3.3.4 (Definition 2.5 in [67])**.** For $s \in \mathbb{R}$, we let $U^p_\Delta H^s$, respectively $V^p_\Delta H^s$, be the space of all functions $u : \mathbb{R} \to H^s(\mathbb{T}^d)$ such that $t \mapsto e^{-it\Delta}u(t)$ is in $U^p(\mathbb{R}, H^s)$, respectively in $V^p(\mathbb{R}, H^s)$ with norm

$$\|u\|_{U^p_\Delta(\mathbb{R},H^s)} := \|e^{-it\Delta}u(t)\|_{U^p(\mathbb{R},H^s)}, \quad \|u\|_{V^p_\Delta(\mathbb{R},H^s)} := \|e^{-it\Delta}u(t)\|_{V^p(\mathbb{R},H^s)}.$$

Definition 3.3.5 (Definition 2.6 in [67])**.** For $s \in \mathbb{R}$, we define X^s as the space of all functions $u : \mathbb{R} \to H^s(\mathbb{T}^d)$ such that for every $n \in \mathbb{Z}^d$, the map $t \mapsto e^{it|n|^2}\widehat{u(t)}(n)$ is in $U^2(\mathbb{R}, \mathcal{C})$, and with the norm

$$\|u\|_{X^s} := (\sum_{n\in\mathbb{Z}^d} \langle n\rangle^{2s} \|e^{it|n|^2}\widehat{u(t)}(n)\|^2_{U^2_t})^{\frac{1}{2}} \quad \text{is finite.} \tag{3.3.3}$$

Definition 3.3.6 (Definition 2.7 in [67])**.** For $s \in \mathbb{R}$, we define Y^s as the space of all functions $u : \mathbb{R} \to H^s(\mathbb{T}^d)$ such that for every $n \in \mathbb{Z}^d$, the map $t \mapsto e^{it|n|^2}\widehat{u(t)}(n)$ is in $V^2_{rc}(\mathbb{R}, \mathcal{C})$, and with the norm

$$\|u\|_{Y^s} := (\sum_{n\in\mathbb{Z}^d} \langle n\rangle^{2s} \|e^{it|n|^2}\widehat{u(t)}(n)\|^2_{V^2_t})^{\frac{1}{2}} \quad \text{is finite.} \tag{3.3.4}$$

Note that

$$U^2_\Delta H^s \hookrightarrow X^s \hookrightarrow Y^s \hookrightarrow V^2_\Delta H^s. \tag{3.3.5}$$

Proposition 3.3.7 (Proposition 2.10 in [60])**.** *Suppose $u := e^{it\Delta}\phi$ which is a linear Schrödinger solution, then for any $T > 0$ we obtain that*

$$\|u\|_{X^s([0,T])} \leq \|\phi\|_{H^s}.$$

Proof. Since $u := e^{it\Delta}\phi$, then $\|u\|_{X^s} = (\sum_{n\in\mathbb{Z}^d} \langle n\rangle^{2s} \|\widehat{\phi}(n)\|^2_{U^2_t})^{\frac{1}{2}} \leq \|\phi\|_{H^s}$. □

Remark 3.3.8. Compared with Bourgain's $X^{s,b}$ first introduced in Bourgain,

$$\|v\|_{X^{s,b}} = \|e^{-it\Delta}v\|_{H^b_t H^s_x},$$

$$\|v\|_{U^p_\Delta H^s} = \|e^{-it\Delta}v\|_{U^p_t H^s_x},$$

$$\|v\|_{V^p_\Delta H^s} = \|e^{-it\Delta}v\|_{V^p_t H^s_x}.$$

Also later we will see that the atomic spaces enjoy the similar duality and transfer principle properties with $X^{s,b}$.

Remark 3.3.9. Follow the definitions, it's easy to check the following embedding property:

$$U^2_\Delta H^s \hookrightarrow X^s \hookrightarrow Y^s \hookrightarrow V^2_\Delta H^s \hookrightarrow L^\infty(\mathbb{R}, H^s). \tag{3.3.6}$$

Definition 3.3.10 (X^s and Y^s restricted to a time interval I)**.** For intervals $I \subset \mathbb{R}$, we define $X^s(I)$ and $Y^s(I)$ as following

$$X^s(I) := \{v \in C(I : H^s) : \|v\|_{X^s(I)} := \sup_{J\subset I,\ |J|\leq 1} \inf_{\tilde{v}|_J = v} \|\tilde{v}\|_{X^s} < \infty\},$$

and

$$Y^s(I) := \{v \in C(I : H^s) : \|v\|_{Y^s(I)} := \sup_{J\subset I,\ |J|\leq 1} \inf_{\tilde{v}|_J = v} \|\tilde{v}\|_{Y^s} < \infty\}.$$

We will consider our solution in $X^1(I)$ spaces, and then let's introduce nonlinear norm $N(I)$.

Definition 3.3.11 (Nonlinear norm $N(I)$)**.** Let $I = [0, T]$, then

$$\|f\|_{N(I)} := \left\| \int_0^t e^{i(t-t')\Delta} f(t')dt' \right\|_{X^1(I)}$$

Proposition 3.3.12 (Proposition 2.11 in [68])**.** *Let $s > 0$. For $f \in L^1(I, H^1(\mathbb{T}^4))$ we have*

$$\|f\|_{N(I)} \leq \sup_{v\in Y^{-1}(I)} \left| \int_I \int_{\mathbb{T}^4} f(t,x)\overline{v(t,x)}dxdt \right|. \tag{3.3.7}$$

Now, we will need a weaker norm Z, which plays a similar role as $L^{10}_{t,x}$ norm in [31].

Definition 3.3.13.

$$\|v\|_{Z(I)} := \sup_{J\subset I,\ |J|\leq 1} \left(\sum_{N\in 2^{\mathbb{Z}}} N^2 \|P_N v\|^4_{L^4(\mathbb{T}^4\times J)} \right)^{\frac{1}{4}}.$$

Remark 3.3.14. $\|v\|_{Z(I)}$ actually can be considered as

$$\sum_{p\in\{p_1,p_2,\cdots,p_k\}} \sup_{J\subset I,\ |J|\leq 1} \left(\sum_{N\in 2^{\mathbb{Z}}} N^{6-p} \|P_N v\|^p_{L^p(\mathbb{T}^4\times J)} \right)^{\frac{1}{p}},$$

where $\{p_1, p_2, \cdots, p_k\}$ should be the L^p estimates that we need to use in the proof of nonlinear estimate. In our case, we only need $\|P_N u\|_{L^4(\mathbb{T}^4\times I)} \lesssim \|P_N u\|_{Z(I)}$ in the proof of the nonlinear estimates, so we choose $\{p_1, p_2, \cdots, p_k\} = \{4\}$.

The following property shows us that $Z(I)$ is a weaker norm than $X^1(I)$.

Proposition 3.3.15.

$$\|v\|_{Z(I)} \lesssim \|v\|_{X^1(I)}$$

.

Proof. By the definition of $Z(I)$ and the following Strichartz-type estimates Proposition 3.3.17, we obtain that

$$\begin{aligned}\sup_{J\subset I,\ |J|\leq 1}\left(\sum_{N \text{ dydic number}} N^2\|P_N v\|^4_{L^4(\mathbb{T}^4\times J)}\right)^{\frac{1}{4}} &\lesssim \sup_{J\subset I,\ |J|\leq 1}\left(\sum_{N \text{ dydic number}} N^4\|P_N v\|^4_{X^0(J)}\right)^{\frac{1}{4}}\\ &\lesssim \|v\|_{X^1(I)}.\end{aligned}$$

$\square$

Proposition 3.3.16 (Proposition 2.19 in [60])**.** *Let $T_0 : L^2\times\cdots\times L^2 \to L^1_{loc}$ be a m-linear operator. Assume that for some $1 \leq p,\ q \leq \infty$*

$$\|T_0(e^{it\Delta}\phi_1, \cdots, e^{it\Delta}\phi_m)\|_{L^p(\mathbb{R},L^q_x(\mathbb{T}^d))} \lesssim \prod_{i=1}^{m}\|\phi_i\|_{L^2(\mathbb{T}^d)}. \tag{3.3.8}$$

Then, there exists an extension $T : U^p_\Delta \times\cdots\times U^p_\Delta \to L^p(\mathbb{R}, L^q(\mathbb{T}^d))$ satisfying

$$\|T(u_1, \cdots, u_m)\|_{L^p(\mathbb{R},L^q(\mathbb{T}^d))} \lesssim \prod_{i=1}^{m}\|u_i\|_{U^p_\Delta}, \tag{3.3.9}$$

such that $T(u_1, \cdots, u_m)(t, \cdot) = T_0(u_1(t), \cdots, u_m(t))(\cdot)$, a.e.

By the transfer principle proposition (Proposition 3.3.16) and Strichartz-type estimate Lemma 2.3.1, we obtain the following corollary:

Corollary 3.3.17. *If $p > 3$, for any $v \in U^p_\Delta([-1, 1])$,*

$$\|P_N v\|_{L^p([-1,1]\times\mathbb{T}^4)} \lesssim_p N^{2-\frac{6}{p}}\|v\|_{U^p_\Delta([-1,1])},$$

and

$$\|P_C v\|_{L^p([-1,1]\times\mathbb{T}^4)} \lesssim_p N^{2-\frac{6}{p}}\|v\|_{U^p_\Delta([-1,1])},$$

where C is a cube of side length N.

3.4 Local well-posedness and Stability theory

In this section, we present large-data local well-poesdness and stability results. Although Herr, Tataru and Tzvetkov's idea [68] together with Bourgain and Demeter's result [17] gives the local well-posedness of (3.1.4), to obtain the stability results, we need a refined nonlinear estimate and the local well-posedness result.

Definition 3.4.1 (Definition of solutions)**.** Given an interval $I \subseteq \mathbb{R}$, we call $u \in C(I : H^1(\mathbb{T}^4))$ a strong solution of (3.1.4) if $u \in X^1(I)$ and u satisfies that for all $t, s \in I$,

$$u(t) = e^{i(t-s)\Delta}u(s) - i\mu\int_s^t e^{i(t-t')\Delta}u(t')|u(t')|^2 dt'.$$

First, we need to introduce

$$\|u\|_{Z'(I)} := \|u\|^{\frac{3}{4}}_{Z(I)}\|u\|^{\frac{1}{4}}_{X^1(I)}. \tag{3.4.1}$$

Lemma 3.4.2 (Bilinear estimates in [68])**.** *Assuming $|I| \leq 1$ and $N_1 \geq N_2$, then we hold that*

$$\|P_{N_1}u_1 P_{N_2}u_2\|_{L^2_{x,t}(\mathbb{T}^4\times I)} \lesssim \left(\frac{N_2}{N_1} + \frac{1}{N_2}\right)^\kappa \|P_{N_1}u_1\|_{Y^0(I)}\|P_{N_2}u_2\|_{Y^1(I)} \tag{3.4.2}$$

for some $\kappa > 0$.

Remark 3.4.3. This Bilinear estimate is Proposition 2.8 in [68]. The proof of Lemma 4.6.3 relies on L^p estimates in Corolla 2.3.1 (for some $p < 4$). In the proof not only we need the decoupling properties for spatial frequency, but also we need further strip partitions to apply the decoupling properties for time frequency.

Let's introduce a refined nonlinear estimate.

Proposition 3.4.4 (Refined nonlinear estimate)**.** *For $u_k \in X^1(I)$, $k = 1, 2, 3$, $|I| \leq 1$, we hold the estimate*

$$\| \prod_{k=1}^{3} \widetilde{u_k} \|_{N(I)} \lesssim \sum_{\{i,j,k\}=\{1,2,3\}} \|u_i\|_{X^1(I)} \|u_j\|_{Z'(I)} \|u_k\|_{Z'(I)} \tag{3.4.3}$$

where $\widetilde{u_k} = u_k$ or $\widetilde{u_k} = \overline{u_k}$ for $k = 1, 2, 3$.

In particular, if there exist constants $A, B > 0$, such that $u_1 = P_{>A} u_1$, $u_2 = P_{>A} u_2$ and $u_3 = P_{<B} u_3$, then we obtain that

$$\| \prod_{k=1}^{3} \widetilde{u_k} \|_{N(I)} \lesssim \|u_1\|_{X^1(I)} \|u_2\|_{Z'(I)} \|u_3\|_{Z'(I)} + \|u_2\|_{X^1(I)} \|u_1\|_{Z'(I)} \|u_3\|_{Z'(I)}. \tag{3.4.4}$$

Proof. Suppose N_0, N_1, N_2, N_3 are dyadic and WLOG we assume $N_1 \geq N_2 \geq N_3$. By the Proposition 3.3.12, we obtain that

$$\begin{aligned} \| \prod_{k=1}^{3} \widetilde{u_k} \|_{N(I)} &\lesssim \sup_{\|u_0\|_{Y^{-1}}} \left| \int_{\mathbb{T}^4 \times I} \overline{u_0} \prod_{k=1}^{3} \widetilde{u_k} \, dxdt \right| \\ &\leq \sup_{\|u_0\|_{Y^{-1}}} \sum_{N_0, N_1 \geq N_2 \geq N_3} \left| \int_{\mathbb{T}^4 \times I} \overline{P_{N_0} u_0} \prod_{k=1}^{3} P_{N_k} \widetilde{u_k} \, dxdt \right| \end{aligned}$$

Then we know that $N_1 \sim \max(N_2, N_0)$ by the spatial frequency orthogonality. There are two cases:

1. $N_0 \sim N_1 \geq N_2 \geq N_3$;

2. $N_0 \leq N_2 \sim N_1 \geq N_3$.

Case 1: $N_0 \sim N_1 \geq N_2 \geq N_3$

By Cauchy-Schwartz inequality and Lemma 4.6.3, we have that

$$\begin{aligned} &\left| \int \overline{P_{N_0} u_0} P_{N_1} \widetilde{u_1} P_{N_2} \widetilde{u_2} P_{N_3} \widetilde{u_3} \, dxdt \right| \leq \|P_{N_0} u_0 P_{N_2} u_2\|_{L^2_{x,t}} \|P_{N_1} u_1 P_{N_3} u_3\|_{L^2_{x,t}} \\ \lesssim &\left(\frac{N_3}{N_1} + \frac{1}{N_3}\right)^{\kappa} \left(\frac{N_2}{N_0} + \frac{1}{N_2}\right)^{\kappa} \|P_{N_0} u_0\|_{Y^0(I)} \|P_{N_1} u_1\|_{Y^0(I)} \|P_{N_2} u_2\|_{X^1(I)} \|P_{N_3} u_3\|_{X^1(I)} \end{aligned} \tag{3.4.5}$$

Assume $\{C_j\}$ is a cube partition of size N_2, and $\{C_k\}$ is a cube partition of size N_3. By $\{P_{C_j} P_{N_0} u_0 P_{N_2} u_2\}_j$ and $\{P_{C_k} P_{N_1} u_1 P_{N_3} u_3\}_k$ are both almost orthogonal, Corollary

3.3.17 and definition of Z norm, we obtain that

$$
\begin{aligned}
&|\int \overline{P_{N_0}u_0} P_{N_1}\widetilde{u_1} P_{N_2}\widetilde{u_2} P_{N_3}\widetilde{u_3}\, dxdt| \leq \|P_{N_0}u_0 P_{N_2}u_2\|_{L^2_{x,t}} \|P_{N_1}u_1 P_{N_3}u_3\|_{L^2_{x,t}}\\
\lesssim&(\sum_{C_j} \|P_{C_j}P_{N_0}u_0 P_{N_2}u_2\|^2_{L^2_{x,t}})^{\frac{1}{2}} (\sum_{C_k} \|P_{C_k}P_{N_1}u_1 P_{N_3}u_3\|^2)^{\frac{1}{2}}\\
\lesssim&(\sum_{C_j} \|P_{C_j}P_{N_0}u_0\|^2_{L^4_{x,t}} \|P_{N_2}u_2\|^2_{L^4_{x,t}})^{\frac{1}{2}} (\sum_{C_k} \|P_{C_k}P_{N_1}u_1\|^2_{L^4_{x,t}} \|P_{N_3}u_3\|^2_{L^4_{x,t}})^{\frac{1}{2}}\\
\lesssim&(\sum_{C_j} \|P_{C_j}P_{N_0}u_0\|^2_{Y^0(I)} (N_2^{\frac{1}{2}}\|P_{N_2}u_2\|_{L^4_{x,t}})^2)^{\frac{1}{2}} (\sum_{C_k} \|P_{C_k}P_{N_1}u_1\|^2_{Y^0(I)} (N_3^{\frac{1}{2}}\|P_{N_3}u_3\|_{L^4_{x,t}})^2)^{\frac{1}{2}}\\
\lesssim&\|P_{N_0}u_0\|_{Y^0(I)} \|P_{N_1}u_1\|_{Y^0(I)} \|P_{N_2}u_2\|_{Z(I)} \|P_{N_2}u_2\|_{Z(I)}.
\end{aligned}
\tag{3.4.6}
$$

Interpolating (4.6.3) with (4.6.4) we obtain that

$$
\begin{aligned}
&|\int \overline{P_{N_0}u_0} P_{N_1}\widetilde{u_1} P_{N_2}\widetilde{u_2} P_{N_3}\widetilde{u_3}\, dxdt|\\
\lesssim&(\frac{N_3}{N_1}+\frac{1}{N_3})^{\kappa_1}(\frac{N_2}{N_0}+\frac{1}{N_2})^{\kappa_1} \|P_{N_0}u_0\|_{Y^{-1}(I)} \|P_{N_1}u_1\|_{X^1(I)} \|P_{N_2}u_2\|_{Z'(I)} \|P_{N_2}u_2\|_{Z'(I)}.
\end{aligned}
\tag{3.4.7}
$$

Then we sum (4.6.5) over all $N_0 \sim N_1 \geq N_2 \geq N_3$,

$$
\begin{aligned}
&\sum_{N_0\sim N_1\geq N_2\geq N_3} (\frac{N_3}{N_1}+\frac{1}{N_3})^{\kappa_1}(\frac{N_2}{N_0}+\frac{1}{N_2})^{\kappa_1} \|P_{N_0}u_0\|_{Y^{-1}(I)} \|P_{N_1}u_1\|_{X^1(I)} \|P_{N_2}u_2\|_{Z'(I)} \|P_{N_2}u_2\|_{Z'(I)}\\
\lesssim&\|u_0\|_{Y^{-1}(I)} \|u_1\|_{X^1 I} \|u_2\|_{Z'(I)} \|u_3\|_{Z'(I)}.
\end{aligned}
$$

Case 2: $N_0 \leq N_2 \sim N_1 \geq N_3$

Similarly we have that

$$
\begin{aligned}
&|\int \overline{P_{N_0}u_0} P_{N_1}\widetilde{u_1} P_{N_2}\widetilde{u_2} P_{N_3}\widetilde{u_3}\, dxdt|\\
\lesssim&(\frac{N_3}{N_1}+\frac{1}{N_3})^{\kappa}(\frac{N_0}{N_2}+\frac{1}{N_0})^{\kappa} \|P_{N_0}u_0\|_{Y^0(I)} \|P_{N_1}u_1\|_{Y^0(I)} \|P_{N_2}u_2\|_{X^1(I)} \|P_{N_3}u_3\|_{X^1(I)}.
\end{aligned}
\tag{3.4.8}
$$

Similar with (4.6.4), we obtain that:

$$
\begin{aligned}
&|\int \overline{P_{N_0}u_0} P_{N_1}\widetilde{u_1} P_{N_2}\widetilde{u_2} P_{N_3}\widetilde{u_3}\, dxdt| \\
\lesssim &\|P_{N_0}u_0\|_{Y^0(I)}\|P_{N_1}u_1\|_{Y^0(I)}\|P_{N_2}u_2\|_{Z(I)}\|P_{N_3}u_3\|_{Z(I)}.
\end{aligned}
\tag{3.4.9}
$$

We interpolate (4.6.6) with (4.6.7) and sum over $N_0 \leq N_2 \sim N_1 \geq N_3$. Then we have that

$$
\begin{aligned}
&\sum_{N_0 \leq N_2 \sim N_1 \geq N_3} |\int \overline{P_{N_0}u_0} P_{N_1}\widetilde{u_1} P_{N_2}\widetilde{u_2} P_{N_3}\widetilde{u_3}\, dxdt| \\
\lesssim &\|P_{N_0}u_0\|_{Y^{-1}(I)}\|P_{N_1}u_1\|_{X^1(I)}\|P_{N_2}u_2\|_{Z'(I)}\|P_{N_3}u_3\|_{Z'(I)}.
\end{aligned}
$$

Next we summarize these two cases and similarly consider $N_1 \geq N_3 \geq N_2$, $N_2 \geq N_1 \geq N_3$, $N_2 \geq N_3 \geq N_1$, $N_3 \geq N_1 \geq N_2$, and $N_3 \geq N_2 \geq N_1$, we can get the desired estimate (4.6.2).

In particular, if there exist constants $A, B > 0$ such that $u_1 = P_{>A}u_1$, $u_2 = P_{>A}u_2$ and $u_3 = P_{<B}u_3$, then we only consider the sum when $N_1 \geq N_2 \gtrsim N_3$ and $N_2 \geq N_1 \gtrsim N_3$. So we get the estimate (3.4.4). $\square$

Proposition 3.4.5 (Local Wellposedness)**.** *Assume that $E > 0$ is fixed. There exists $\delta_0 = \delta_0(E)$ such that if*

$$
\|e^{it\Delta}u_0\|_{Z'(I)} < \delta
$$

for some $\delta \leq \delta_0$, some interval $0 \in I$ with $|I| \leq 1$ and some function $u_0 \in H^1(\mathbb{T}^4)$ satisfying $\|u_0\|_{H^1} \leq E$, then there exists a unique strong solution to (3.1.1) $u \in X^1(I)$ such that $u(0) = u_0$. Besides we also have

$$
\|u - e^{it\Delta}u_0\|_{X^1(I)} \leq \delta^{\frac{5}{3}}. \tag{3.4.10}
$$

Proof. First, we consider the set

$$
S = \{u \in X^1(I) : \|u\|_{X^1(I)} \leq 2E, \qquad \|u\|_{Z'(I)} \leq a\},
$$

and the mapping

$$
\Phi(v) = e^{it\Delta}u_0 - i\mu \int_0^t e^{i(t-s)\Delta} v(s)|v(s)|^2\, ds.
$$

For u, $v \in S$, by Proposition 3.4.4, there exists a constant $C > 0$, we have that

$$
\begin{aligned}
&\|\Phi(u) - \Phi(v)\|_{X^1(I)} \\
\leq &C \left(\|u\|_{X^1(I)} + \|v\|_{X^1(I)}\right) \left(\|u\|_{Z'(I)} + \|v\|_{Z'(I)}\right) \|u - v\|_{X^1(I)} \\
\leq &CEa\|u - v\|_{X^1(I)}
\end{aligned}
$$

Similarly, using Proposition 3.3.7 and nonlinear estimate Proposition 3.4.4, we also obtain that

$$
\begin{aligned}
\|\Phi(u)\|_{X^1(I)} &\leq \|\Phi(0)\|_{X^1(I)} + \|\Phi(u) - \Phi(0)\|_{X^1(I)} \\
&\leq \|u_0\|_{H^1} + CEa^2
\end{aligned}
$$

and

$$
\begin{aligned}
\|\Phi(u)\|_{Z'(I)} &\leq \|\Phi(0)\|_{Z'(I)} + \|\Phi(u) - \Phi(0)\|_{Z'(I)} \\
&\leq \delta + CEa^2.
\end{aligned}
$$

Now, we choose $a = 2\delta$ and we let $\delta_0 = \delta_0(E)$ be small enough. We see that Φ is a contraction on S, so we have a fixed point u. And it's easy to check (3.4.10) and uniqueness in $X^1(I)$. $\square$

By a similar idea of Herr-Tataru-Tzvetkov [67], we can easily prove the global well-posedness result with small initial data by using Theorem 3.4.5.

Proposition 3.4.6 (Small data global wellposedness)**.** *If $\|\phi\|_{H^1(\mathbb{T}^4)} = \delta \leq \delta_0$, then the unique strong solution with initial data ϕ is global and satisfies*

$$\|u\|_{X^1([-1,1])} \leq 2\delta$$

and moreover

$$\|u - e^{it\Delta}\phi\|_{X^1([-1,1])} \lesssim \delta^2.$$

Lemma 3.4.7 (Z-norm controls the global existence)**.** *Assume that $I \subseteq \mathbb{R}$ is a bounded open interval.*

1. *If E is a nonnegative finite number, that u is a strong solution of (3.1.1) and*

$$\|u\|_{L_t^\infty(I,H^1)} \leq E.$$

Then, if

$$\|u\|_{Z(I)} < +\infty$$

there exists an open interval J with $\bar{I} \subset J$ such that u can be extended to a strong solution of (3.1.1) on J, besides

$$\|u\|_{X^1(I)} \leq C(E, \|u\|_{Z(I)}).$$

2. *(GWP with a priori bound)Assume C is some positive finite number and we have a priori bound $\|u\|_{Z(I)} < C$, for any solution u of (3.1.1) in the interval I, then this IVP (3.1.4) is well-posedness on I. (In particular, if u blows up in finite time, then u blows up in the Z-norm.)*

Proof. Consider the case $I = (0,T)$.

1. By the continuity arguments of $h(s) = \|e^{i(t-T_1)\Delta}u(T_1)\|_{Z'(T_1,T_1+s)}$ where $T_1 \geq T-1$ such that $\|u\|_{Z(T_1,T)} \leq \varepsilon$.

2. Combined (1) and Proposition 3.4.5, it's trivial to know.

□

Remark 3.4.8. This proof determines the ratio of X^1 norm and Z norm in the definition of Z' norm 3.4.1, and the portion of Z norm in Z' norm can be any number strictly between $\frac{1}{2}$ and 1.

Proposition 3.4.9 (Stability)**.** *Assume I is an open bounded interval, $\mu \in [-1,1]$, and $\tilde{u} \in X^1(I)$ satisfies the approximate Schrödinger equation*

$$(i\partial_t + \Delta)\tilde{u} = \mu\tilde{u}|\tilde{u}|^2 + e, \qquad on\ \mathbb{T}^4 \times I. \tag{3.4.11}$$

Assume in addition that

$$\|\tilde{u}\|_{Z(I)} + \|\tilde{u}\|_{L_t^\infty(I,H^1(\mathbb{T}^4))} \leq M, \tag{3.4.12}$$

for some $M \in [1,\infty]$. *Assume* $t_0 \in I$ *and* $u_0 \in H^1(\mathbb{T}^4)$ *is such that the smallness condition:*

$$\|u_0 - \tilde{u}(0)\|_{H^1(\mathbb{T}^4)} + \|e\|_{N(I)} \leq \varepsilon \tag{3.4.13}$$

holds for some $0 < \varepsilon < \varepsilon_1$, *where* $\varepsilon_1 \leq 1$. $\varepsilon_1 = \varepsilon_1(M) > 0$ *is a small constant.*

Then there exists a strong solution $u \in X^1(I)$ *of the NLS*

$$(i\partial_t + \Delta)u = \mu u|u|^2,$$

such that $u(t_0) = u_0$ *and*

$$\begin{aligned} \|u\|_{X^1(I)} + \|\tilde{u}\|_{X^1(I)} &\leq C(M), \\ \|u - \tilde{u}\|_{X^1(I)} &\leq C(M)\varepsilon. \end{aligned} \tag{3.4.14}$$

Proof. First, we need to show the short time Stability, which follows a similar proof as the proof of Proposition 3.4.5. Then, by using Lemma 3.4.7, we extend to the entire time interval. □

3.5 Euclidean profiles

In this section, we introduce the Euclidean profiles which are linear and nonlinear Schödinger solutions on $\mathbb{T}^4$ concentrated at a point. The Euclidean profiles perform similar with the solutions in the Euclidean space $\mathbb{R}^4$ and hence Euclidean profiles hold some similar well-posedness and scattering properties by using the theory for the NLS in Euclidean space $\mathbb{R}^4$, which is proven by Ryckman and Vişan [98][105] (the defocusing case) and Dodson [41] (the focusing case), as a black box. This is an analogue in 4 dimensions of the section 4 in [71], we follows closely the argument in the section 4 of [71].

We fix a spherically symmetric function $\eta \in C_0^\infty(\mathbb{R}^4)$ supported in the ball of radius 2 and equal to 1 in the ball of radius 1.

Given $\phi \in \dot{H}^1(\mathbb{R}^4)$ and a real number $N \geq 1$ we define

$$\begin{aligned} Q_N\phi \in H^1(\mathbb{R}^4), &\qquad (Q_N\phi)(x) = \eta(x/N^{\frac{1}{2}})\phi(x), \\ \phi_N \in H^1(\mathbb{R}^4), &\qquad \phi_N(x) = N(Q_N\phi)(Nx), \\ f_N \in H^1(\mathbb{T}^4), &\qquad f_N(y) = \phi_N(\Psi^{-1}(y)), \end{aligned} \tag{3.5.1}$$

where $\Psi : \{x \in \mathbb{R}^4 : |x| < 1\} \to O_0 \subseteq \mathbb{T}^4, \quad \Psi(x) = x.$

The cutoff function $\eta(\frac{x}{N^{1/2}})$ is useful to concentrate our focus on the range of a point, and the choice of the order 1/2 actually can be chosen any number between 1/2 and 1.

Thus $Q_N\phi$ is a compactly supported modification of profile ϕ. ϕ_N is a $\dot{H}^1$-invariant rescaling of $Q_N\phi$, and f_N is the function obtained by transferring ϕ_N to a neighborhood of 0 in $\mathbb{T}^4$.

Theorem 3.5.1 (GWP of the defocusing cubic NLS in $\mathbb{R}^4$ [98][105])**.** *Assume $\phi \in \dot{H}^1(\mathbb{R}^4)$ then there is an unique global solution $v \in C(\mathbb{R} : \dot{H}^1(\mathbb{R}^4))$ of the initial-value problem*

$$(i\partial_t + \Delta)v = v|v|^2, \qquad v(0) = \phi, \tag{3.5.2}$$

and

$$\|\nabla_{\mathbb{R}^4} v\|_{(L_t^\infty L_x^2 \cap L_t^2 L_x^4)(\mathbb{R}\times\mathbb{R}^4)} \leq C(E_{\mathbb{R}^4}(\phi)) < +\infty. \tag{3.5.3}$$

Moreover, this solution scatters in the sense that there exists $\phi^{\pm\infty} \in \dot{H}^1(\mathbb{R}^4)$, such that

$$\|v(t) - e^{it\Delta}\phi^{\pm\infty}\|_{\dot{H}^1(\mathbb{R}^4)} \to 0, \text{ as } t \to \pm\infty. \tag{3.5.4}$$

Besides, if $\phi \in H^5(\mathbb{R}^4)$ then $v \in C(\mathbb{R} : H^5(\mathbb{R}^4))$ and

$$\sup_{t\in\mathbb{R}} \|v(t)\|_{H^5(\mathbb{R}^4)} \lesssim_{\|\phi\|_{H^5(\mathbb{R}^4)}} 1. \tag{3.5.5}$$

Theorem 3.5.2 (GWP of the focusing cubic NLS in $\mathbb{R}^4$ [41]). *Assume $\phi \in \dot{H}^1(\mathbb{R}^4)$, under the assumption that*

$$\sup_{t \in \text{lifespan of } v} \|v(t)\|_{\dot{H}^1(\mathbb{R}^4)} < \|W\|_{\dot{H}^1(\mathbb{R}^4)},$$

then there is an unique global solution $v \in C(\mathbb{R} : \dot{H}^1(\mathbb{R}^4))$ of the initial-value problem

$$(i\partial_t + \Delta)v = -v|v|^2, \qquad v(0) = \phi, \tag{3.5.6}$$

and

$$\|\nabla_{\mathbb{R}^4} v\|_{(L_t^\infty L_x^2 \cap L_t^2 L_x^4)(\mathbb{R}\times\mathbb{R}^4)} \leq C(\|\phi\|_{\dot{H}^1(\mathbb{R}^4)}, E_{\mathbb{R}^4}(\phi)) < +\infty. \tag{3.5.7}$$

Moreover, this solution scatters in the sense that there exists $\phi^{\pm\infty} \in \dot{H}^1(\mathbb{R}^4)$, such that

$$\|v(t) - e^{it\Delta}\phi^{\pm\infty}\|_{\dot{H}^1(\mathbb{R}^4)} \to 0, \text{ as } t \to \pm\infty. \tag{3.5.8}$$

Besides, if $\phi \in H^5(\mathbb{R}^4)$ then $v \in C(\mathbb{R} : H^5(\mathbb{R}^4))$ and

$$\sup_{t\in\mathbb{R}} \|v(t)\|_{H^5(\mathbb{R}^4)} \lesssim_{\|\phi\|_{H^5(\mathbb{R}^4)}} 1. \tag{3.5.9}$$

Remark 3.5.3 (Persistence of regularity). Consider $\phi \in H^5(\mathbb{R}^4)$, and $v \in C(\mathbb{R} : \dot{H}^1(\mathbb{R}^4))$ is the solution of (3.1.1) with $v(0) = \phi$ and satisfying

$$\|\nabla_{\mathbb{R}^4} v\|_{(L_t^\infty L_x^2 \cap L_t^2 L_x^4)(\mathbb{R}\times\mathbb{R}^4)} < +\infty.$$

So we can have a finite partition $\{I_k\}_{k=1}^K$ of $\mathbb{R}$, ($I_k = [t_{k-1}, t_k)$, where $t_k = \infty$.) s.t. $\|\nabla_{\mathbb{R}^4} v\|_{L_t^4 L_x^{8/3}} < \frac{1}{2}$, for each k,

$$\begin{aligned}
\|v(t)\|_{L_t^\infty(I_k:H^5(\mathbb{R}^4))} &\leq \|e^{i(t-t_{k-1})\Delta} v(t_{k-1})\|_{H^5} + \|\langle\nabla\rangle^5 |v(t)|^2 v(t)\|_{L_t^2 L_x^{4/3}(I_k)} \\
\leq &\|v(t_{k-1})\|_{H_x^5} + \|\langle\nabla\rangle^5 v\|_{L_t^\infty L_x^2(I_k)} \|v(t)\|^2_{L_t^4 L_x^8(I_k)} \\
\leq &\|v(t_{k_1})\|_{H^5} + \frac{1}{4}\|v\|_{L_t^\infty(I_k:H^5(\mathbb{R}^4))}
\end{aligned}$$

which implies $\|v(t)\|_{L_t^\infty(I_k:H^5(\mathbb{R}^4))} \leq \frac{4}{3}\|v(t_{k-1})\|_{H^5}$ for each $1 \leq k \leq K$, so $\|v(t)\|_{L_t^\infty(\mathbb{R}:H_x^5(\mathbb{R}^4))} < \infty$.

Theorem 3.5.4. *Assume $T_0 \in (0,\infty)$, and $\mu \in \{-1,0,1\}$ are given, and define f_N as (3.5.1) above. Suppose*

$$\|\phi\|_{\dot H^1(\mathbb{R}^4)} < +\infty, \qquad \text{when } \mu \in \{0,1\};$$

or under the assumption that if v is a solution of (3.5.6) then v satisfies

$$\sup_{t\in \text{lifespan of } v} \|v(t)\|_{\dot H^1(\mathbb{R}^4)} < \|W\|_{\dot H^1(\mathbb{R}^4)},, \qquad \text{when } \mu \in \{-1\}.$$

Then the following conclusions hold:

1. *There is $N_0 = N_0(\phi, T_0)$ sufficiently large such that for any $N \geq N_0$ there is an unique solution $U_N \in C((-T_0N^{-2}, T_0N^{-2}) : H^1(\mathbb{T}^4))$ of the initial value problem*

$$(i\partial_t + \Delta)U_N = \mu U_N|U_N|^2, \qquad U_N(0) = f_N. \tag{3.5.10}$$

Moreover, for any $N \geq N_0$,

$$\|U_N\|_{X^1(-T_0N^{-2},T_0N^{-2})} \lesssim_{E_{\mathbb{R}^4}(\phi),\, \|\phi\|_{\dot H^1(\mathbb{R}^4)}} 1. \tag{3.5.11}$$

2. *Assume $\varepsilon_1 \in (0,1]$ is sufficiently small (depending on only $E_{\mathbb{R}^4}\phi$), $\phi' \in H^5(\mathbb{R}^4)$, and $\|\phi - \phi'\|_{\dot H^1(\mathbb{R}^4)} \leq \varepsilon_1$. Let $v' \in C(\mathbb{R} : H^5(\mathbb{R}^4))$ denote the solution of the initial value problem*

$$(i\partial_t + \Delta)v' = \mu v'|v'|^2, \qquad v'(0) = \phi'. \tag{3.5.12}$$

For R, $N \geq 1$, we define

$$\begin{aligned} v'_R(x,t) &= \eta(x/R)v'(x,t), \qquad (x,t) \in \mathbb{R}^4 \times (-T_0, T_0) \\ v'_{R,N}(x,t) &= Nv'_R(Nx, N^2t), \qquad (x,t) \in \mathbb{R}^4 \times (-T_0N^{-2}, T_0N^{-2}) \\ V_{R,N}(y,t) &= v'_{R,N}(\Psi^{-1}(y), t), \qquad (y,t) \in \mathbb{T}^4 \times (-T_0N^{-2}, T_0N^{-2}). \end{aligned} \tag{3.5.13}$$

Then there is $R_0 \geq 1$(depending on T_0, ϕ' and ε_1), for any $R \geq R_0$, we obtain that

$$\limsup_{N\to\infty} \|U_N - V_{R,N}\|_{X^1(-T_0N^{-2},T_0N^{-2})} \lesssim_{E_{\mathbb{R}^4}(\phi),\, \|\phi\|_{\dot H^1(\mathbb{R}^4)}} \varepsilon_1. \tag{3.5.14}$$

$V_{R,N}$ can be considered as solve NLS firstly, then cutoff and scaling, while U_N can be considered as cutoff and scaling firstly, then solve NLS.

Proof. We show Part(1) and Part(2) together, by Proposition 3.4.9 (stability).

Using Theorem 3.5.1 and Theorem 3.5.2, we know v' globally exists and satisfying

$$\|\nabla_{\mathbb{R}^4} v'\|_{(L_t^\infty L_x^2 \cap L_t^2 L_x^4)(\mathbb{R}\times\mathbb{R}^4)} \lesssim 1,$$

and

$$\sup_{t\in\mathbb{R}} \|v'(t)\|_{H^5(\mathbb{R}^4)} \lesssim_{\|\phi'\|_{H^5(\mathbb{R}^4)}} 1. \tag{3.5.15}$$

Let's consider $v'_R(x,t) = \eta(x/R)v'(x,t)$.

$$\begin{aligned}&(i\partial_t + \Delta_{\mathbb{R}^4})v'_R = (i\partial_t + \Delta_{\mathbb{R}^4})(\eta(x/R)v'(x,t))\\ =&\eta(x/R)(i\partial_t + \Delta_{\mathbb{R}^4})v'(x,t) + R^{-2}v'(x,t)(\Delta_{\mathbb{R}^4}\eta)(x/R) + 2R^{-1}\sum_{j=1}^{4}\partial_j v'(x,t)\partial_j\eta(x/R).\end{aligned}$$

which implies

$$(i\partial_t + \Delta_{\mathbb{R}^4})v'_R = \lambda|v'_R|^2 v'_R + e_R(x,t),$$

where $e_R(x,t) = \mu(\eta(x/R)-\eta^3(x/R))v'|v'|^2+R^{-2}v'(x,t)(\Delta_{\mathbb{R}^4}\eta)(x/R)+2R^{-1}\sum_{j=1}^4 \partial_j v'(x,t)\partial_j\eta(x/R)$. After scaling, we get

$$(i\partial_t + \Delta_{\mathbb{R}^4})v'_{R,N} = \mu|v'_{R,N}|^2 v'_R + e_{R,N}(x,t),$$

where $e_{R,N}(x,t) = N^3 e_R(Nx, N^2 t)$. with $V_{R,N}(y,t) = v'_{R,N}(\Phi^{-1}(y),t)$ and taking $N \geq 10R$, we obtain that

$$(i\partial_t + \Delta_{\mathbb{R}^4})V_{R,N}(y,t) = \mu|V_{R,N}|^2 V_{R,N} + E_{R,N}(y,t), \tag{3.5.16}$$

where $E_{R,N}(y,t) = e_{R,N}(\Phi^{-1}(y),t)$.

By Proposition 3.4.9, we need following conditions:

1. $\|V_{R,N}\|_{L_t^\infty([-T_0N^{-2},T_0N^{-2}]:H^1(\mathbb{T}^4))} + \|V_{R,N}\|_{Z([-T_0N^{-2},T_0N^{-2}])} \leq M$;

2. $\|f_N - V_{R,N}(0)\|_{H^1(\mathbb{T}^4)} \leq \varepsilon$;

3. $\|E_{R,N}\|_{N([-T_0N^{-2},T_0N^{-2}])} \leq \varepsilon$.

We will prove all 3 condition above:

Case 1: $\|V_{R,N}\|_{L_t^\infty([-T_0N^{-2},T_0N^{-2}]:H^1(\mathbb{T}^4))} + \|V_{R,N}\|_{Z([-T_0N^{-2},T_0N^{-2}])} \leq M$.

Since $v'(x,t)$ globally exists, $V_{R,N}(y,t)$ also globally exists. Given $T_0 \in (0,\infty)$,

$$
\begin{aligned}
&\sup_{t\in[-T_0N^{-2},T_0N^{-2}]} \|V_{R,N}(t)\|_{H^1(\mathbb{T}^4)} \leq \sup_{t\in[-T_0N^{-2},T_0N^{-2}]} \|v'_{R,N}(t)\|_{H^1(\mathbb{R}^4)} \\
=& \sup_{t\in[-T_0N^{-2},T_0N^{-2}]} \|Nv'_R(Nx,N^2t)\|_{H^1(\mathbb{R}^4)} \\
=& \sup_{t\in[-T_0N^{-2},T_0N^{-2}]} \frac{1}{N}\|v'_R(N^2t)\|_{L^2(\mathbb{R}^4)} + \|v'_R(N^2t)\|_{\dot{H}^1(\mathbb{R}^4)} \\
\leq& \sup_{t\in[-T_0,T_0]} \|v'_R\|_{H^1(\mathbb{R}^4)} = \sup_{t\in[-T_0,T_0]} \|\eta(x/R)v'(x,t)\|_{H^1(\mathbb{R}^4)} \\
\leq& \sup_{t\in[-T_0,T_0]} \|\eta(x/R)v'(x,t)\|_{L^2(\mathbb{R}^4)} + \|\nabla\eta(x/R)v'(x,t)\|_{L^2(\mathbb{R}^4)} + \|\eta(x/R)\nabla v'(x,t)\|_{L^2(\mathbb{R}^4)} \\
\leq& 2\|v'(x,t)\|_{H^1(\mathbb{R}^4)} \leq 2\|\phi'(t)\|_{H^5(\mathbb{R}^4)}.
\end{aligned}
$$

By Littlewood-Paley theorem and Sobolev embedding, we obtain that

$$
\begin{aligned}
&\|V_{R,N}\|_{Z([-T_0N^{-2},T_0N^{-2}])} = \sup_{J\subset[-T_0N^{-2},T_0N^{-2}]} (\sum_{M dyadic} M^2\|P_MV_{R,N}\|^4_{L^4(J\times\mathbb{T}^4)})^{\frac{1}{4}} \\
=& \sup_{J\subset[-T_0N^{-2},T_0N^{-2}]} \|(\sum_M(\langle 1-\Delta\rangle^{\frac{1}{4}}P_MV_{R,N})^4)^{\frac{1}{4}}\|_{L^4(J\times\mathbb{T}^4)} \\
\leq& \sup_{J\subset[-T_0N^{-2},T_0N^{-2}]} \|(\sum_M(\|\langle 1-\Delta\rangle^{\frac{1}{4}}P_MV_{R,N})^2)^{\frac{1}{2}}\|_{L^4(J\times\mathbb{T}^4)} \\
\lesssim& \sup_{J\subset[-T_0N^{-2},T_0N^{-2}]} \|\langle 1-\Delta\rangle^{\frac{1}{4}}V_{R,N}\|_{L^4(J\times\mathbb{T}^4)} \\
\leq& \sup_{J\subset[-T_0N^{-2},T_0N^{-2}]} \|\langle 1-\Delta\rangle^{\frac{1}{2}}V_{R,N}\|_{L_t^4(J)L_x^{\frac{8}{3}}(\mathbb{T}^4)} \\
\lesssim& \||v'_{R,N}| + |\nabla_{\mathbb{R}^4}v'_{R,N}|\|_{L_t^4L_x^{\frac{8}{3}}([-T_0N^{-2},T_0N^{-2}]\times\mathbb{R}^4)} \\
\lesssim& \|v'_R\|_{L_t^4L_x^{\frac{8}{3}}([-T_0,T_0]\times\mathbb{R}^4)} + \||\nabla_{\mathbb{R}^4}v'_R|\|_{L_t^4L_x^{\frac{8}{3}}([-T_0,T_0]\times\mathbb{R}^4)}.
\end{aligned}
$$

Since $\|v'_R\|_{L_t^4L_x^{\frac{8}{3}}([-T_0,T_0]\times\mathbb{R}^4)} + \||\nabla_{\mathbb{R}^4}v'_R|\|_{L_t^4L_x^{\frac{8}{3}}([-T_0,T_0]\times\mathbb{R}^4)} \lesssim \sup_t\|v'(t)\|_{H^5}$, by (3.5.15) we obtain $\|V_{R,N}\|_{Z([-T_0N^{-2},T_0N^{-2}])} \lesssim_{\|\phi'\|_{H^5(\mathbb{R}^4)}} 1$.

Case 2: $\|f_N - V_{R,N}(0)\|_{H^1(\mathbb{T}^4)} \leq \varepsilon$.

By Hölder inequality, we obtain that

$$\begin{aligned}
\|f_N - V_{R,N}(0)\|_{H^1(\mathbb{T}^4)} &\leq \|\phi_N(\Psi^{-1}(y)) - \phi'_{R,N}(\Psi^{-1}(y))\|_{\dot{H}^1(\mathbb{T}^4)} \\
&\leq \|\phi_N - \phi'_{R,N}\|_{\dot{H}^1(\mathbb{R}^4)} = \|Q_N\phi - \phi'_R\|_{\dot{H}^1(\mathbb{R}^4)} \\
&= \|\eta(\frac{x}{N^{\frac{1}{2}}})\phi(x) - \eta(\frac{x}{N^{\frac{1}{2}}})\phi'(x)\|_{\dot{H}^1(\mathbb{R}^4)} \\
&\leq \|\eta(\frac{x}{N^{\frac{1}{2}}})\phi(x) - \phi(x)\|_{\dot{H}^1(\mathbb{R}^4)} + \|\phi - \phi'\|_{\dot{H}^1(\mathbb{R}^4)} + \|\eta(\frac{x}{N^{\frac{1}{2}}})\phi'(x) - \phi'(x)\|_{\dot{H}^1(\mathbb{R}^4)}.
\end{aligned}$$

With $N \geq 10R$, and $R > R_0$, R_0 large enough, we have that

$$\|f_N - V_{R,N}(0)\|_{H^1(\mathbb{T}^4)} \leq 2\varepsilon_1.$$

Case 3: $\|E_{R,N}\|_{N([-T_0N^{-2},T_0N^{-2}])} \leq \varepsilon$.

Next, by Proposition 3.3.12 and scaling invariance, we obtain that

$$\begin{aligned}
\|E_{R,N}\|_{N([-T_0N^{-2},T_0N^{-2}])} &= \|\int_0^t e^{i(t-s)\Delta}E_{R,N}(s)\,ds\|_{X^1([-T_0N^{-2},T_0N^{-2}])} \\
&\leq \sup_{\|u_0\|_{Y^{-1}}=1} \left|\int_{\mathbb{T}^4\times[-T_0N^{-2},T_0N^{-2}]} \overline{u_0}\cdot E_{R,N}\,dxdt\right| \\
&\leq \sup_{\|u_0\|_{Y^{-1}}=1} \||\nabla|^{-1}u_0\|_{L_t^\infty L_x^2}\||\nabla|E_{R,N}\|_{L_t^1L_x^2} \\
&\leq \sup_{\|u_0\|_{Y^{-1}}=1} \|u_0\|_{Y^{-1}}\||\nabla|E_{R,N}\|_{L_t^1L_x^2([-T_0N^{-2},T_0N^{-2}]\times\mathbb{T}^4)} \\
&\leq \|\nabla_{\mathbb{R}^4}\,e_{R,N}\|_{L_t^1L_x^2([-T_0N^{-2},T_0N^{-2}]\times\mathbb{R}^4)} \\
&= \|\nabla_{\mathbb{R}^4}\,e_R\|_{L_t^1L_x^2([-T_0,T_0]\times\mathbb{R}^4)}.
\end{aligned}$$

$$|\nabla_{\mathbb{R}^4} e_R(x,t)| = |\nabla_{\mathbb{R}^4}(\mu(\eta(x/R) - \eta^3(x/R)))v'(x,t)|v'(x,t)|^2$$

$$+R^{-2}v'(x,t)(\Delta_{\mathbb{R}^4}\eta)(\frac{x}{R}) + 2R^{-1}(\sum_{j=1}^{4}\partial_j v'(x,t)\partial_j\eta(x/R)|$$

$$\leq |\nabla_{\mathbb{R}^4}(\eta(\frac{x}{R}) - \eta(\frac{x}{R})^3)v'(x,t)|v'(x,t)|^2| + 3|(\eta(\frac{x}{R}) - \eta(\frac{x}{R})^3)\nabla_{\mathbb{R}^4}v'(x,t)|v'(x,t)|^2|$$

$$+R^{-3}|v'(x,t)\nabla_{\mathbb{R}^4}\Delta_{\mathbb{R}^4}\eta(\frac{x}{R})| + R^{-2}|\nabla_{\mathbb{R}^4}v'(x,t)(\Delta_{\mathbb{R}^4}\eta)(\frac{x}{R})| + R^{-1}|\Delta_{\mathbb{R}^4}v'(x,t)\nabla_{\mathbb{R}^4}\eta(\frac{x}{R})|$$

$$\lesssim_{\|\phi'\|_{H^5(\mathbb{R}^4)}} \mathbb{1}_{[R,2R]}(|x|)\left(|v'(x,t)| + |\nabla_{\mathbb{R}^4}v'(x,t)|\right) + \frac{1}{R}\left(|\langle\nabla_{\mathbb{R}^4}\rangle^2 v'(x,t)|\right).$$

Since $\|\nabla^2_{\mathbb{R}^4}v'(x,t)\|_{L^\infty_x} \lesssim_{\|\phi'\|_{H^5}} 1$, $\|\nabla_{\mathbb{R}^4}v'(x,t)\|_{L^\infty_x} \lesssim_{\|\phi'\|_{H^5}} 1$, and $\|v'(x,t)\|_{L^\infty_x} \lesssim_{\|\phi'\|_{H^5}} 1$ (by Sobolev embedding), we obtain that

$$\|\nabla_{\mathbb{R}^4} e_R\|_{L^1_t L^2_x([-T_0,T_0]\times\mathbb{R}^4)} = \int_{-T_0}^{T_0}(\int_{\mathbb{R}^4}|\nabla_{\mathbb{R}^4} e_R|\,dx)^{\frac{1}{2}}dt$$

$$\leq \int_{-T_0}^{T_0}\left(\int_{\mathbb{R}^4}\mathbb{1}_{[R,2R]}(|x|)(|v'(x,t)|^2 + |\nabla_{\mathbb{R}^4}v'(x,t)|^2)\,dx + \frac{1}{R^2}\int_{\mathbb{R}^4}|\langle\nabla_{\mathbb{R}^4}\rangle^2 v'(x,t)|^2\,dx\right)^{\frac{1}{2}}dt$$

$$\lesssim_{\|\phi'\|_{H^5}} 2T_0\left(\int_{\mathbb{R}^4}\mathbb{1}_{[R,2R]}(|x|)\langle\nabla_{\mathbb{R}^4}\rangle^2 v'(x,t)|^2\,dx\right)^{\frac{1}{2}} + \frac{1}{R} \to 0, \text{ as } R \to \infty.$$

So we can obtain that

$$\|\nabla E_{R,N}\|_{L^1_t L^2_x([-T_0N^{-2},T_0N^{-2}]\times\mathbb{T}^4)} < \varepsilon_1,$$

where $R > R_0$, and R_0 large enough.

By checking all there conditions above, we have the desired result.

$\square$

Next, we prove a extinction lemma as Ionescu and Pausader [71] did in their paper about energy critical NLS in $\mathbb{T}^3$. The extinction lemma is the essential part why we prove the GWP result in $\mathbb{T}^4$.

Lemma 3.5.5 (Extinction Lemma)**.** *Let $\phi \in \dot{H}^1(\mathbb{R}^4)$, and define f_N as in (3.5.1). For any $\varepsilon > 0$, there exist $T = T(\phi, \varepsilon)$ and $N_0(\phi, \varepsilon)$ such that for all $N \geq N_0$, there holds that*

$$\|e^{it\Delta} f_N\|_{Z([TN^{-2}, T^{-1}])} \lesssim \varepsilon.$$

Proof. For $M \geq 1$, we define

$$K_M(x,t) = \sum_{\xi \in \mathbb{Z}^4} e^{-i[t|\xi|^2 + x\cdot\xi]\eta(\xi/M)} = e^{it\Delta} P_{\leq M}\delta_0.$$

We know from [Lemma 3.18, Bourgain[7]] that K_M satisfies

$$|K_M(x,t)| \lesssim \prod_{i=1}^{4} \left(\frac{M}{\sqrt{q_i}(1 + M|t/(\lambda_i) - a_i/q_i|^{\frac{1}{2}})} \right), \tag{3.5.17}$$

if a_i and q_i satisfying $\frac{t}{\lambda_i} = \frac{a_i}{q_i} + \beta_i$, where $q_i \in \{1, \cdots, M\}$, $a_i \in \mathbb{Z}$, $(a_i, q_i) = 1$ and $|\beta_i| \leq (Mq_i)^{-1}$ for all $i = 1, 2, 3, 4$.

From this, we conclude that for any $1 \leq S \leq M$,

$$\|K_M(x,t)\|_{L^\infty_{x,t}(\mathbb{T}^4 \times [SM^{-2}, S^{-1}])} \lesssim S^{-2}M^4. \tag{3.5.18}$$

This follows directly from (3.5.17) and Dirichlet's approximation lemma which is stated as following: *For any real numbers α, and any positive integer N, there exists integers p and q such* $1 \leq q \leq N$ *and* $|q\alpha - p| < \frac{1}{N}$.

Assume that $|t| \leq \frac{1}{S}$. $\frac{t}{\lambda_i} = \frac{a_i}{q_i} + \beta_i$ and $|\beta_i| \leq \frac{1}{Mq_i} \leq \frac{1}{M} \leq \frac{1}{S}$. So we obtain that

$$\left|\frac{a_i}{q_i}\right| \leq \frac{2}{S} \quad \Longrightarrow \quad q_i \geq \frac{a_i}{2}.$$

Therefore either $q_i \geq \frac{1}{2}S \quad (a_i \geq 1)$ or $a_i = 0$ for each i. If $q_i \geq \frac{1}{2}S \quad (a_i \geq 1)$, then

$$\frac{M}{\sqrt{q_i}(1 + M|t/(\lambda_i) - a_i/q_i|^{\frac{1}{2}})} \lesssim \frac{M}{\sqrt{q}} \lesssim S^{-\frac{1}{2}}M.$$

If $a_i = 0$, then

$$\frac{M}{\sqrt{q_i}(1 + M|t/(\lambda_i) - a_i/q_i|^{\frac{1}{2}})} \lesssim \frac{M}{\sqrt{q_i} + M|t|^{1/2}} \lesssim |t|^{-\frac{1}{2}} \leq S^{-\frac{1}{2}}M.$$

So we have that $|K_M(x,t)| \lesssim S^{-2}M^4$.

By the definition as in (3.5.1), to prove the extinction lemma, we may assume that $\phi \in C_0^\infty(\mathbb{R}^4)$, we claim that

$$\begin{aligned} \|f_N\|_{L^1(\mathbb{T}^4)} &\lesssim_\phi N^{-3} \\ \|P_K f_N\|_{L^2(\mathbb{T}^4)} &\lesssim_\phi \left(1+\frac{K}{N}\right)^{-10} N^{-1}. \end{aligned} \tag{3.5.19}$$

Let's consider the bound of $\|f_N\|_{L^1(\mathbb{T}^4)}$:

$$\begin{aligned} \|f_N\|_{L^1(\mathbb{T}^4)} &= \|\phi_N(\Psi^{-1}(y))\|_{L^1(\mathbb{T}^4)} = \|\phi_N(x)\|_{L^1(\mathbb{R}^4)} \\ &= \int_{\mathbb{R}^4} |N(Q_N\phi)(Nx)| \, dx \\ &= \frac{1}{N^3}\int_{\mathbb{R}^4} |Q_N\phi|(x)\, dx = \frac{1}{N^3}\int_{\mathbb{R}^4} |\eta(\frac{x}{N^{1/2}})\phi(x)|\, dx \\ &\leq \frac{1}{N^3}\|\phi\|_{L^1(\mathbb{R}^4)}. \end{aligned}$$

Let's consider the bound of $\|P_K f_N\|_{L^2(\mathbb{T}^4)}$:

$$\begin{aligned} \|P_K f_N\|_{L^2(\mathbb{T}^4)} &= \|P_K \phi_N\|_{L^2(\mathbb{R}^4)} \\ &= \|P_K N(Q_N\phi)(Nx)\|_{L^2(\mathbb{R}^4)} \\ &= \|N(P_{\frac{K}{N}} Q_N\phi)(Nx)\|_{L^2(\mathbb{R}^4)} = \frac{1}{N}\|P_{\frac{K}{N}} Q_N \phi\|_{L^2(\mathbb{R}^4)} \\ &= \frac{1}{N}\|P_{\frac{K}{N}}(\eta(\frac{x}{N^{\frac{1}{2}}})\phi(x))\|_{L^2(\mathbb{R}^4)} \\ &\leq \frac{1}{N}\left(1+\frac{K}{N}\right)^{-10} \|\eta(\frac{x}{N^{\frac{1}{2}}})\phi(x)\|_{H^{10}(\mathbb{R}^4)} \\ &\leq \frac{1}{N}\left(1+\frac{K}{N}\right)^{-10} \|\phi\|_{H^{10}}. \end{aligned}$$

By Proposition 2.3.1, for $p > 3$ we obtain that

$$\|e^{it\Delta}P_K f_N\|_{L^p_{x,t}(\mathbb{T}^4\times[-1,1])} \leq K^{2-\frac{6}{p}}\left(1+\frac{K}{N}\right)^{-10} N^{-1}\|\phi\|_{H^{10}}. \tag{3.5.20}$$

Then let's estimate $\|e^{it\Delta}f_N\|_{Z([TN^{-2},T^{-1}])}$. We know that

$$\|e^{it\Delta}f_N\|_{Z([TN^{-2},T^{-1}])} = \sup_{J\subset[TN^{-2},T^{-1}]}\left(\sum_K K^2\|P_K e^{it\Delta}f_N\|^4_{L^4(J\times\mathbb{T}^4)}\right)^{\frac{1}{4}}$$

To estimate it, we decompose the sum above into three part:

$$\left(\sum_{K\leq NT^{-\frac{1}{100}}}+\sum_{K\geq NT^{\frac{1}{100}}}+\sum_{NT^{-\frac{1}{100}}\leq K\leq NT^{\frac{1}{100}}}\right)K^2\|P_Ke^{it\Delta}f_N\|^4_{L^4([TN^{-2},T^{-1}]\times\mathbb{T}^4)}$$

.

Case 1: $K\leq NT^{-\frac{1}{100}}$:

By (3.5.20), we obtain that

$$\begin{aligned}&\sum_{K\leq NT^{-\frac{1}{100}}}K^2\|P_Ke^{it\Delta}f_N\|^4_{L^4([TN^{-2},T^{-1}]\times\mathbb{T}^4)}\\ \leq&\sum_{K\leq NT^{-\frac{1}{100}}}K^4\left(1+\frac{K}{N}\right)^{-40}N^{-4}\|\phi\|_{H^{10}}\\ \lesssim_\phi&(NT^{-\frac{1}{100}})^4N^{-4}=T^{-\frac{1}{25}}.\end{aligned}$$

Case 2: $K\geq NT^{\frac{1}{100}}$:

By (3.5.20), we obtain that

$$\begin{aligned}&\sum_{K\geq NT^{\frac{1}{100}}}K^2\|P_Ke^{it\Delta}f_N\|^4_{L^4([TN^{-2},T^{-1}]\times\mathbb{T}^4)}\\ \leq&\sum_{K\geq NT^{\frac{1}{100}}}K^4\left(1+\frac{K}{N}\right)^{-40}N^{-4}\|\phi\|_{H^{10}}\\ \leq&\sum_{K\geq NT^{\frac{1}{100}}}K^{-36}N^{36}\|\phi\|_{H^10}\\ \lesssim_\phi&T^{-\frac{4}{100}}.\end{aligned}$$

Case 3: $NT^{-\frac{1}{100}}\leq K\leq NT^{\frac{1}{100}}$:

Let's consider $K\in[NT^{-\frac{1}{100}},NT^{\frac{1}{100}}]$ and set $M\sim\max(K,N)$ and $S\sim T$.

$$\begin{aligned}\|e^{it\Delta}P_Kf_N\|_{L^\infty_{x,t}(\mathbb{T}^4\times[TN^{-2},T^{-1}])}&=\|K_M*f_N\|_{L^\infty_{x,t}(\mathbb{T}^4\times[TN^{-2},T^{-1}])}\\ &\leq\|K_M\|_{L^\infty_{x,t}(\mathbb{T}^4\times[TN^{-2},T^{-1}])}\|f_N\|_{L^1_{x,t}(\mathbb{T}^4)}\\ &\lesssim_\phi T^{-2}K^4N^{-3}\leq T^{-2+\frac{1}{25}}N.\end{aligned}\tag{3.5.21}$$

$$\begin{aligned}\|e^{it\Delta}P_N f_N\|_{L^3_{x,t}(\mathbb{T}^4\times[TN^{-2},T^{-1}])} &\lesssim_\phi K^\varepsilon\left(1+\frac{K}{N}\right)^{-10}N^{-1}\\ &\le N^{-1+\varepsilon}T^{-\frac{\varepsilon}{100}}.\end{aligned} \tag{3.5.22}$$

Interpolating (3.5.21) with (3.5.22), we have that

$$\|e^{it\Delta}P_K f_N\|_{L^4_{x,t}([TN^{-2},T^{-1}])} \lesssim_\phi \left(N^{-1+\varepsilon}T^{\frac{\varepsilon}{100}}\right)^{\frac{3}{4}}\left(T^{-2+\frac{1}{25}}N\right)^{\frac{1}{4}} \le N^{-\frac{1}{4}}T^{-\frac{1}{100}}. \tag{3.5.23}$$

Summing $K^2\|P_K e^{it\Delta}f_N\|^4_{L^4([TN^{-2},T^{-1}]\times\mathbb{T}^4)}$ over K, we obtain that

$$\begin{aligned}\sum_{NT^{-\frac{1}{100}}\le K\le NT^{\frac{1}{100}}} K^2\|P_K e^{it\Delta}f_N\|^4_{L^4([TN^{-2},T^{-1}]\times\mathbb{T}^4)} &\le \sum_{NT^{-\frac{1}{100}}\le K\le NT^{\frac{1}{100}}} K^2(N^{-\frac{1}{4}}T^{-\frac{1}{100}})^4\\ &\le \sum_{NT^{-\frac{1}{100}}\le K\le NT^{\frac{1}{100}}} K^2N^{-2}T^{-\frac{1}{25}}\\ &\le T^{-\frac{1}{50}}.\end{aligned}$$

Summarizing all three cases by setting T large enough, we hold the estimate. $\square$

Let's now consider $f\in L^2(\mathbb{T}^4)$, $t_0\in\mathbb{R}$ and $x_0\in\mathbb{T}^4$,

$$(\pi_{x_0}f)(x) := f(x-x_0),$$
$$(\Pi_{t_0,x_0})f(x) := (\pi_{x_0}e^{-it_0\Delta}f)(x).$$

As in (3.5.1), given $\phi\in\dot H^1(\mathbb{R}^4)$ and $N\ge 1$, we define

$$T_N\phi(x) := N\tilde\phi(N\Psi^{-1}(x)),\ \text{where}\ \tilde\phi(y) := \eta(y/N^{\frac{1}{2}})\phi(y)$$

and claim that $T_N:\dot H^1(\mathbb{R}^4)\to H^1(\mathbb{R}^4)$ is a linear operator with $\|T_N\phi\|_{H^1(\mathbb{T}^4)}\lesssim\|\phi\|_{\dot H^1(\mathbb{R}^4)}$.

Remark 3.5.6. To show $\|T_N\phi\|_{H^1(\mathbb{T}^4)} \lesssim \|\phi\|_{\dot{H}^1(\mathbb{R})}$.

$$\begin{aligned}
\|T_N\phi(x)\|_{H^1(\mathbb{T}^4)} &\lesssim \|T_N\phi(x)\|_{\dot{H}^1(\mathbb{T}^4)} \\
&= \|\nabla(N\eta(N^{\frac{1}{2}})\phi(Ny))\|_{L^2(\mathbb{R}^4)} \\
&\leq \|N^{\frac{3}{2}}(\nabla\eta)(N^{\frac{1}{2}}y)\phi(Ny)\|_{L^2(\mathbb{R}^4)} + \|N^2\eta(N^{\frac{1}{2}})\nabla\phi(Ny)\|_{L^2(\mathbb{R}^4)} \\
&\leq \|\mathbb{1}_{[0,N^{\frac{1}{2}}]}\|_{L^4(\mathbb{R}^4)}\|N^{\frac{3}{2}}|(\nabla\eta)(N^{\frac{1}{2}}y)\phi(Ny)|\|_{L^4_{\mathbb{R}^4}} + \|\phi\|_{\dot{H}^1(\mathbb{R}^4)} \\
&\leq \|N\phi(Ny)\|_{\dot{H}^1(\mathbb{R}^4)} + \|\phi\|_{\dot{H}^1(\mathbb{R}^4)} \\
&\leq 2\|\phi\|_{\dot{H}^1(\mathbb{R}^4)}.
\end{aligned}$$

Definition 3.5.7. Let $\widetilde{\mathcal{F}_e}$ denote the set of renormalized Euclidean frames

$$\begin{gathered}
\widetilde{\mathcal{F}_e} := \{(N_k, t_k, x_k)_{k\geq 1} : N_k \in [1,\infty),\ t_k \to 0,\ x_k \in \mathbb{T}^4,\ N_k \to \infty \\
\text{and either } t_k = 0 \text{ for any } k \geq 1 \text{ or } \lim_{k\to\infty} N_k^2|t_k| = \infty\}.
\end{gathered}$$

Proposition 3.5.8 (Euclidean profiles)**.** *Assume that $\mathcal{O} = (N_k, t_k, x_k)_k \in \widetilde{\mathcal{F}_e}$ and $\mu \in \{-1, 0, 1\}$. $\phi \in \dot{H}^1(\mathbb{R}^4)$. Suppose*

$$\|\phi\|_{\dot{H}^1(\mathbb{R}^4)} < +\infty, \qquad \textit{when } \mu \in \{0,1\};$$

or under the assumption that if v is a solution of (3.5.6) with $v(0) = \phi$ then v satisfies

$$\sup_{t\in \textit{lifespan of } v} \|v(t)\|_{\dot{H}^1(\mathbb{R}^4)} < \|W\|_{\dot{H}^1(\mathbb{R}^4)},, \qquad \textit{when } \mu \in \{-1\}.$$

Then

1. *there exists $\tau = \tau(\phi)$ such that for k large enough (depending only on ϕ and $\mathcal{O}$) there is a nonlinear solution $U_k \in X^1(-\tau,\tau)$ of the initial value problem (3.1.1) with initial data $U_k(0) = \Pi_{t_k,0}(T_{N_k}\phi)$ and*

$$\|U_k\|_{X^1(-\tau,\tau)} \lesssim_{E_{\mathbb{R}^4}(\phi),\, \|\phi\|_{\dot{H}^1(\mathbb{R}^4)}} 1; \tag{3.5.24}$$

2. *there exists an Euclidean solution $u \in C(\mathbb{R} : \dot{H}^1(\mathbb{R}^4))$ of*

$$(i\partial_t + \Delta_{\mathbb{R}^4})u = \mu u|u|^2$$

with scattering data $\phi^{\pm\infty}$ defined as Theorem 3.5.1 such that the following holds, up to a subsequence: for any $\varepsilon > 0$, there exists $T(\phi, \varepsilon)$ such that for all $T \geq T(\phi, \varepsilon)$, there exists $R(\phi, \varepsilon, T)$ such that for all $R \geq R(\phi, \varepsilon, T)$, there holds that

$$\|U_k - \widetilde{u_k}\|_{X^1(\{|t-t_k| \leq TN_k^{-2}\} \cap \{|t| < T^{-1}\})} \leq \varepsilon, \tag{3.5.25}$$

for k large enough, where

$$(\pi_{-x_k}\widetilde{u_k})(x,t) = N_k \eta(N_k \Psi^{-1}(x)/R) u(N_k \Psi^{-1}(x), N_k^2(t - t_k)). \tag{3.5.26}$$

In addition, up to a subsequence,

$$\|U_k(t) - \Pi_{t_k - t, x_k} T_{N_k} \phi^{\pm\infty}\|_{X^1(\{\pm(t-t_k) \geq \pm TN_k^{-2}\} \cap \{|t| < T^{-1}\})}, \leq \varepsilon, \tag{3.5.27}$$

for k large enough (depending on ϕ, ε, T, and R).

Proof. By the statement, it is equivalent to prove the case when $x_k = 0$.

Part (1): First, for k large enough, we can make

$$\|\phi - \eta(\frac{x}{N^{\frac{1}{2}}})\phi\|_{\dot{H}^1(\mathbb{R}^4)} \leq \varepsilon_1.$$

For each N_k, we choose $T_{0,N_k} = \tau N_k^2$(T_{0,N_k} is the coefficient in Lemma 3.5.4). For each T_{0,N_k}, we make R_k large enough to make Theorem 3.5.4 work. (Note: in this case, R_k determined by T_{0,N_k} as in the proof of Theorem 3.5.4.)

Part(2): Let's consider first case in Euclidean frame: $t_k = 0$ for all k. (3.5.24) is directly from Theorem 3.5.4, by choosing k, R for any fixed T large enough.

To prove (3.5.25), we need to choose $T(\phi, \delta)$ large enough, to make sure

$$\|\nabla_{\mathbb{R}^4} u\|_{L^3_{x,t}(\mathbb{R}^4 \times \{|t| > T(\phi,\delta)\})} \leq \delta.$$

By Theorem 3.5.1, we obtain that

$$\|u(\pm T(\phi, \delta)) - e^{\pm iT(\phi,\delta)\Delta}\phi^{\pm\infty}\|_{\dot{H}^1(\mathbb{R}^4)} \leq \delta,$$

which implies

$$\|U_{N_k}(\pm TN_k^{-2}) - \Pi_{-\pm T, x_k} T_{N_k} \phi^{\pm\infty}\|_{H^1(\mathbb{T}^4)} \leq \delta. \tag{3.5.28}$$

By Proposition 4.6.2 and Proposition 3.3.7, we have

$$\|e^{it\Delta}\left(U_{N_k}(\pm TN_k^{-2}) - \Pi_{-\pm T,x_k} T_{N_k}\phi^{\pm\infty}\right)\|_{X^1(|t|<T^{-1})}. \tag{3.5.29}$$

By Proposition 3.4.5, we obtain that

$$\|U_{N_k} - e^{it\Delta}U_{N_k}(\pm TN_k^{-2})\|_{X^1} \leq \delta, \tag{3.5.30}$$

and combining (3.5.29) and (3.5.30), we have

$$\|U_{N_k} - \Pi_{-t,x_k}T_{N_k}\phi^{\pm\infty}\|_{X^1(\{\pm t\geq \pm TN_k^{-2}\}\cap\{|t|<T^{-1}\})} \leq \varepsilon.$$

when we choose δ small enough.

The second case: $N_k^2|t_k| \to \infty$.

$$\begin{aligned} U_k(0) &= \Pi_{t_k,0}(T_{N_k}\phi) \\ &= e^{-it_k\Delta}\left(N_k^{\frac{1}{2}}\widetilde{\phi}(N_k\Psi^{-1}(x))\right) \\ &= e^{-it_k\Delta}\left(N_k^{\frac{1}{2}}\eta(N_k^{\frac{1}{2}}\Psi^{-1}(x))\phi(N_k\Psi^{-1}(x))\right). \end{aligned}$$

By existence of wave operator of NLS, we know the following initial value problem is global well-posed, so there exists v satisfying:

$$\begin{cases} (i\partial_t + \Delta_{\mathbb{R}^4})v = \mu v|v|^2, \\ \lim_{t\to-\infty}\|v(t) - e^{it\Delta}\phi\|_{\dot{H}^1(\mathbb{R}^4)} = 0. \end{cases} \tag{3.5.31}$$

We set

$$\widetilde{v_k}(t) = N_k^{\frac{1}{2}}\eta(N_k\Psi^{-1}(x)/R)v(N_k\Psi^{-1}(x), N_k^2 t),$$

so we have $\widetilde{v_k}(-t_k) = N_k^{\frac{1}{2}}\eta(N_k\Psi^{-1}(x)/R)v(N_k\Psi^{-1}(x), -N_k^2 t_k)$.

For k and R large enough,

$$\begin{aligned} &\|\widetilde{v_k}(-t_k) - e^{-it_k\Delta}N_k^{\frac{1}{2}}\eta(N_k^{\frac{1}{2}}\Psi^{-1}(x))\phi(N_k\Psi^{-1}(x))\|_{\dot{H}^1(\mathbb{T}^4)} \\ \leq &\|\eta(\frac{x}{N_k^{\frac{1}{2}}})v(x, -N_k^2 t_k) - e^{it_k N_k^2\Delta}\eta(\frac{x}{N_k^{\frac{1}{2}}})\phi(x)\|_{\dot{H}^1(\mathbb{R}^4)} \\ \leq &\varepsilon. \end{aligned}$$

So $V_k(t)$ solves initial value problem (3.1.4) in $\mathbb{T}^4$, with initial data $V_k(0) = \widetilde{V_k}(0)$, which implies $V_k(t)$ exists in $[-\delta, \delta]$, and $\|V_k(t) - \widetilde{V_k}(t)\|_{X^1([-\delta,dd])} \lesssim \varepsilon$.

By the stability property (Proposition 3.4.9), $\|U_k - V_k\|_{X^1([-\delta,\delta])} \to 0$, as $k \to \infty$. $\square$

The following corollary (Corollary 3.5.9) decompose the nonlinear Euclidean profiles U_k defined in the Proposition 3.5.8. This corollary follows closely in a part of the proof of Lemma 6.2 in [71]. I state it here as a corollary because the almost orthogonality of nonlinear profiles (Lemma 3.6.6) heavily relies on this decomposition lemma (Corollary 3.5.9).

Corollary 3.5.9 (Decomposition of the nonlinear Euclidean profiles U_k)**.** *Consider U_k is the nonlinear Euclidean profiles w.r.p.t. $\mathcal{O} = (N_k, t_k, x_k)_k \in \widetilde{\mathcal{F}_e}$ defined above. For any $\theta > 0$, there exist T_θ^0 sufficiently large such that for all $T_\theta \geq T_\theta^0$ and R_θ sufficiently large such that for all k large enough (depending on R_θ) we can decompose U_k as following:*

$$\mathbb{1}_{(-T_\theta^{-1}, T_\theta^{-1})}(t) U_k = \omega_k^{\theta,-\infty} + \omega_k^{\theta,+\infty} + \omega_k^\theta + \rho_k^\theta,$$

and $\omega_k^{\theta,\pm\infty}$, ω_k^θ, and ρ_k^θ satisfy the following conditions:

$$\begin{aligned}
\|\omega_k^{\theta,\pm\infty}\|_{Z'(-T_\theta^{-1},T_\theta^{-1})} + \|\rho_k^\theta\|_{X^1(-T_\theta^{-1},T_\theta^{-1})} &\leq \theta, \\
\|\omega_k^{\theta,\pm\infty}\|_{X^1(-T_\theta^{-1},T_\theta^{-1})} + \|\omega_k^\theta\|_{X^1(-T_\theta^{-1},T_\theta^{-1})} &\lesssim 1, \\
\omega_k^{\theta,\pm\infty} &= P_{\leq R_\theta N_k} \omega_k^{\theta,\pm\infty} \\
|\nabla_x^m \omega_k^\theta| + (N_k)^{-2} \mathbb{1}_{S_k^\theta} |\partial_t \nabla_x^m \omega_k^\theta| &\leq R_\theta (N_k)^{|m|+1} \mathbb{1}_{S_k^\theta}, \ 0 \leq |m| \leq 10,
\end{aligned} \tag{3.5.32}$$

where

$$S_k^\theta := \{(x,t) \in \mathbb{T}^4 \times (-T_\theta, T_\theta) : |t - t_k| < T_\theta (N_k)^{-2}, \ |x - x_k| \leq R_\theta (N_k)^{-1}\}.$$

Proof. By Proposition 3.5.8, there exists $T(\phi, \frac{\theta}{4})$, such that for all $T \geq T(\phi, \frac{\theta}{4})$, there exists $R(\phi, \frac{\theta}{4}, T)$ such that for all $R \geq R(\phi, \frac{\theta}{2}, T)$, there holds that

$$\|U_k - \widetilde{u_k}\|_{X^1(\{|t-t_k| \leq T(N_k)^{-2}\} \cap \{|t| < T^{-1}\})} \leq \frac{\theta}{2},$$

for k large enough, where

$$(\pi_{-x_k}\widetilde{u_k})(x,t) = N_k\eta(N_k\Psi^{-1}(x)/R)u(N_k\Psi^{-1}(x), N_k^2(t-t_k)),$$

where u is a solution of (3.1.1) with scattering data $\phi^{\pm\infty}$.

In addition, up to subsequence,

$$\|U_k - \Pi_{t_k-t,x_k}T_{N_k}\phi^{\pm\infty}\|_{X^1(\{\pm(t-t_k)\geq T(N_k)^{-2}\}\cap\{|t|\leq T^{-1}\})} \leq \frac{\theta}{4},$$

for k large enough (depending on ϕ, θ, T, and R).

Choose a sufficiently large $T_\theta > T(\phi, \frac{\theta}{4})$ based on the extinction lemma(Lemma 3.5.5), such that

$$\|e^{it\Delta}\Pi_{t_k,x_k}T_{N_k}\phi^{\pm\infty}\|_{Z(T_\theta(N_k)^{-2},T_\theta^{-1})} \leq \frac{\theta}{4}$$

when k large enough.

And then we choose $R_\theta = R(\phi, \frac{\theta}{2}, T_\theta)$.

Denote:

1. $\omega_k^{\theta,\pm\infty} := \mathbb{1}_{\{\pm(t-t_k)\geq T_\theta(N_k)^{-2},|t|\leq T_\theta^{-1}\}}\left(\Pi_{t_k-t,x_k}T_{N_k}\phi^{\theta,\pm\infty}\right)$,

   where

   $$\|\phi^{\theta,\pm\infty}\|_{\dot{H}^1(\mathbb{R}^4)} \lesssim 1,\ \phi^{\theta,\pm\infty} = P_{\leq R_\theta}(\phi^{\theta,\pm\infty}),$$

   which implies $\omega_k^{\theta,\pm\infty} = P_{\leq R_\theta N_\theta}\omega_k^{\theta,\pm\infty}$.

2. $\omega_k^\theta := \widetilde{u_k}\cdot\mathbb{1}_{S_k^\theta}$, where $S_k^\theta := \{(x,t)\in\mathbb{T}^4\times(-T_\theta,T_\theta) : |t-t_k| < T_\theta(N_k)^{-2},\ |x-x_k| \leq R_\theta(N_k)^{-1}\}$.

   By the stability property (Proposition 3.4.9) and Theorem 3.5.4, we can adjust ω_k^θ and $\omega_k^{\theta,\pm\infty}$, with an acceptable error, to make

   $$|\nabla_x^m\omega_k^\theta| + (N_k)^{-2}\mathbb{S}_{\mathbb{k}}^{\alpha,\theta}|\partial_t\nabla_x^m\omega_k^\theta| \leq R_\theta(N_k)^{|m|+1}\mathbb{1}_{S_k^\theta},\ 0\leq|m|\leq 10.$$

3. $\rho_k := \mathbb{1}_{(-T_\theta^{-1},T_\theta^{-1})}(t)U_k^\alpha - \omega_k^\theta - \omega^{\theta,+\infty} - \omega^{\theta,-\infty}$.

By (3.5.25) and (3.5.27), we obtain that

$$\|\rho_k^\theta\|_{X^1(\{|t|<T_\theta^{-1}\})} \leq \frac{\theta}{2}.$$

and then we have

$$\|\omega_k^{\theta,\pm\infty}\|_{Z'(-T_\theta^{-1},T_\theta^{-1})} + \|\rho_k^\theta\|_{X^1(-T_\theta^{-1},T_\theta^{-1})} \leq \theta,$$
$$\|\omega_k^{\theta,\pm\infty}\|_{X^1(-T_\theta^{-1},T_\theta^{-1})} + \|\omega_k^\theta\|_{X^1(-T_\theta^{-1},T_\theta^{-1})} \lesssim 1.$$

□

3.6 Profile decomposition

In this section, we construct the profile decomposition on $\mathbb{T}^4$ for linear Schrödinger equations. The arguments and propositions in this section is similar to those in the Section 5 of [72], except for one more lemma (Lemma 3.6.6) about almost orthogonality of nonlinear profiles which is useful in the focusing case.

As in the previous section, given $f \in L^2(\mathbb{R}^4)$, $t_0 \in \mathbb{R}$, and $x_0 \in \mathbb{T}^4$, we define:

$$(\Pi_{t_0,x_0})f(x) := (e^{-it_0\Delta}f)(x-x_0)$$
$$T_N\phi(x) := N\widetilde{\phi}(N\Psi^{-1}(x)),$$

where $\widetilde{\phi}(y) := \eta(\frac{y}{N^{\frac{1}{2}}})\phi(y)$.

Observe that $T_N : \dot{H}^1(\mathbb{R}^4) \to H^1(\mathbb{T}^4)$ is a linear operator with $\|T_N\phi\|_{H^1(\mathbb{T}^4)} \lesssim \|\phi\|_{\dot{H}^1(\mathbb{R}^4)}$.

Definition 3.6.1 (Euclidean frames). 1. We define a Euclidean frame to be a sequence $\mathcal{F}_e = (N_k, t_k, x_k)_k$ with $N_k \geq 1$, $N_k \to +\infty$, $t_k \in \mathbb{R}$, $t_k \to 0$, $x_k \in \mathbb{T}^4$. We say that two frames, $(N_k, t_k, x_k)_k$ and $(M_k, s_k, y_k)_k$ are orthogonal if

$$\lim_{k\to+\infty}\left(\ln\left|\frac{N_k}{M_k}\right| + N_k^2|t_k - s_k| + N_k|x_k - y_k|\right) = \infty.$$

Two frames that are not orthogonal are called equivalent.

2. If $\mathcal{O} = (N_k, t_k, x_k)_k$ is a Euclidean frame and if $\phi \in \dot{H}^1(\mathbb{R}^4)$, we define the Euclidean profile associated to $(\phi, \mathcal{O})$ as the sequence $\widetilde{\phi}_{\mathcal{O}_k}$:

$$\widetilde{\phi}_{\mathcal{O}_k} := \Pi_{t_k, x_k}(T_{N_k}\phi).$$

Proposition 3.6.2 (Equivalence of frames [72]). *(1) If $\mathcal{O}$ and $\mathcal{O}'$ are equivalent Euclidean frames, then there exists an isometry $T : \dot{H}^1(\mathbb{R}^4) \to \dot{H}^1(\mathbb{R}^4)$ such that for any profile $\widetilde{\phi}_{\mathcal{O}'_k}$, up to a subsequence there holds that*

$$\limsup_{k\to\infty} \|\widetilde{T\phi}_{\mathcal{O}_k} - \widetilde{\phi}_{\mathcal{O}'_k}\|_{H^1(\mathbb{T}^4)} = 0.$$

(2) If $\mathcal{O}$ and $\mathcal{O}'$ are orthogonal Euclidean frames and $\widetilde{\phi}_{\mathcal{O}_k}$, $\widetilde{\phi}_{\mathcal{O}'_k}$ are corresponding profiles, then, up to a subsequence:

$$\lim_{k\to\infty} \langle \widetilde{\phi}_{\mathcal{O}_k}, \widetilde{\phi}_{\mathcal{O}'_k} \rangle_{H^1 \times H^1(\mathbb{T}^4)} = 0; \tag{3.6.1}$$

$$\lim_{k\to\infty} \langle |\widetilde{\phi}_{\mathcal{O}_k}|^2, |\widetilde{\phi}_{\mathcal{O}'_k}|^2 \rangle_{L^2 \times L^2(\mathbb{T}^4)} = 0. \tag{3.6.2}$$

Proposition 3.6.3 (Profile decompositions). *Consider $\{f_k\}_k$ a sequence of functions in $H^1(\mathbb{T}^4)$ and $0 < A < \infty$ satisfying*

$$\limsup_{k\to+\infty} \|f_k\|_{H^1(\mathbb{T}^4)} \leq A$$

and a sequence of intervals $I_k = (-T_k, T^k)$ such that $T_k, T^k \to 0$ as $k \to \infty$. Up to passing to a subsequence, assume that $f_k \rightharpoonup g \in H^1(\mathbb{T}^4)$. There exists $J^* \in \{0, 1, ...\} \cup \{\infty\}$, and a sequence of profile $\widetilde{\psi}_k^\alpha := \widetilde{\psi}_{\mathcal{O}_k^\alpha}^\alpha$ associated to pairwise orthogonal Euclidean frames $\mathcal{O}^\alpha$ and $\psi^\alpha \in H^1(\mathbb{R}^4)$ such that extracting a subsequence, for every $0 \leq J \leq J^*$, we have

$$f_k = g + \sum_{1\leq\alpha\leq J} \widetilde{\psi}_k^\alpha + R_k^J \tag{3.6.3}$$

where R_k^J is small in the sense that

$$\limsup_{J\to J^*} \limsup_{k\to\infty} \|e^{it\Delta} R_k^J\|_{Z(I_k)} = 0. \tag{3.6.4}$$

Besides, we also have the following orthogonality relations:

$$
\begin{aligned}
\|f_k\|_{L^2}^2 &= \|g\|_{L^2}^2 + \|R_k^J\|_{L^2}^2 + o_k(1). \\
\|\nabla f_k\|_{L^2}^2 &= \|\nabla g\|_{L^2}^2 + \sum_{\alpha \leq J} \|\nabla_{\mathbb{R}^4} \psi^\alpha\|_{L^2(\mathbb{R}^4)}^2 + \|\nabla R_k^J\|_{L^2}^2 + o_k(1). \\
\lim_{J \to J^*} \limsup_{k \to \infty} &\left| \|f_k\|_{L^4}^4 - \|g\|_{L^4}^4 - \sum_{\alpha \leq J} \|\widetilde{\psi_k^\alpha}\|_{L^4}^4 \right| = 0.
\end{aligned}
\tag{3.6.5}
$$

The proof of Proposition 3.6.3 is in Appendix A.

Remark 3.6.4. g and $\widetilde{\psi}_k^\alpha$ for all α are called profiles. In addition, we call g is Scale-1-profile, and $\widetilde{\psi}_k^\alpha$ are called Euclidean profiles.

Remark 3.6.5 (Almost orthogonality of the energy). By (3.5.1), we have that $\|\widetilde{\psi}_k^\alpha\|_{L^2(\mathbb{T}^4)} \leq \frac{1}{N_k}\|\psi^\alpha\|_{L^2(\mathbb{R}^4)} \to 0$ as $k \to \infty$ and $\|\widetilde{\psi}_k^\alpha\|_{\dot{H}^1(\mathbb{T}^4)}^2 = \frac{1}{N_k}\|\nabla \eta(\frac{\cdot}{N_k^{\frac{1}{2}}})\psi^\alpha\|_{L^2(\mathbb{R}^4)}^2 + \|\eta(\frac{\cdot}{N_k^{\frac{1}{2}}})\psi^\alpha\|_{\dot{H}^1(\mathbb{R}^4)}^2$. Then above and (3.6.5), we know that

$$
\lim_{J \to J^*} \lim_{k \to \infty} \left(\sum_{1 \leq \alpha \leq J} E(\widetilde{\psi}_k^\alpha) + E(R_k^J) + E(g) - E(f_k) \right) = 0.
$$

Lemma 3.6.6 (Almost orthogonality of nonlinear profiles)**.** *Define U_k^α, U_k^β as the maximal life-span I_k solutions of (3.1.1) with initial data $U_k^\alpha(0) = \widetilde{\psi}_{\mathcal{O}_k^\alpha}^\alpha$, $U_k^\beta(0) = \widetilde{\psi}_{\mathcal{O}_k^\beta}^\beta$, where $\mathcal{O}^\alpha$ and $\mathcal{O}^\beta$ are orthogonal. And define G to be the solution of the maximal lifespan I_0 of (3.1.1) with initial data $G(0) = g$. And $0 \in I_k$ and $\lim_{k \to \infty} |I_k| = 0$. Then*

$$
\lim_{k \to \infty} \sup_{t \in I_k} \langle U_k^\alpha(t), U_k^\beta(t) \rangle_{\dot{H}^1 \times \dot{H}^1} = 0, \quad \lim_{k \to \infty} \sup_{t \in I_k \cap I_0} \langle U_k^\alpha(t), G(t) \rangle_{\dot{H}^1 \times \dot{H}^1} = 0. \tag{3.6.6}
$$

Proof. Set $U_k^0(0) = g$ and $U_k^0 = G$ for all k, such that U_k^0 can be considered as a nonlinear profile with a trivial frame $\mathcal{O} = (1, 0, 0)_k$.

For any $\theta > 0$, by the decomposition of the nonlinear profiles U^α and U^β (Corollary 3.5.9), there exist $T_{\theta,\alpha}$, $R_{\theta,\alpha}$, $T_{\theta,\beta}$, $R_{\theta,\beta}$ sufficiently large

$$
\begin{aligned}
U_k^\alpha &= \omega_k^{\alpha,\theta,-\infty} + \omega_k^{\alpha,\theta,+\infty} + \omega_k^{\alpha,\theta} + \rho_k^{\alpha,\theta}, \\
U_k^\beta &= \omega_k^{\beta,\theta,-\infty} + \omega_k^{\beta,\theta,+\infty} + \omega_k^{\beta,\theta} + \rho_k^{\beta,\theta}.
\end{aligned}
$$

For U_k^0, set $U_k^0 := \omega_k^{\alpha,\theta,-\infty} + \omega_k^{\alpha,\theta,+\infty} + \omega_k^{\alpha,\theta} + \rho_k^{\alpha,\theta}$ where $\rho_k^{0,\theta} = \omega_k^{0,\theta} = 0$ and $\omega^{0,\theta,+\infty} = \omega^{0,\theta,-\infty} = \frac{1}{2}G$. And by taking $T_{\theta,0}$ large, it is easy to make $\|G\|_{Z'(-T_{\theta,0},T_{\theta,0})} \leq \theta$. So $\langle U_k^\alpha(t), G(t)\rangle_{\dot{H}^1\times\dot{H}^1}$ can be considered as a special case of $\langle U_k^\alpha(t), U_k^\beta(t)\rangle_{\dot{H}^1\times\dot{H}^1}$ when $\beta = 0$.

Since $\rho_k^{\alpha,\theta}$, $\rho_k^{\beta,\theta}$ are the small term with the X^1-norm less then θ, for any fixed $t \in I_k$, it will suffice to consider the following three terms:

1. $\langle \omega_k^{\alpha,\theta,\pm\infty}, \omega_k^{\beta,\theta,\pm\infty}\rangle_{\dot{H}^1\times\dot{H}^1}$;
2. $\langle \omega_k^{\alpha,\theta,\pm\infty}, \omega_k^{\beta,\theta}\rangle_{\dot{H}^1\times\dot{H}^1}$;
3. $\langle \omega_k^{\alpha,\theta}, \omega_k^{\beta,\theta}\rangle_{\dot{H}^1\times\dot{H}^1}$.

Case (1): $\langle \omega_k^{\alpha,\theta,\pm\infty}, \omega_k^{\beta,\theta,\pm\infty}\rangle_{\dot{H}^1\times\dot{H}^1}$.

By the constructions of $\omega_k^{\alpha,\theta,\pm\infty}, \omega_k^{\beta,\theta,\pm\infty}$ in the proof of Lemma 3.5.9, we obtain that

$$\omega_k^{\alpha,\theta,\pm\infty} := \mathbb{1}_{\{\pm(t-t_k^\alpha)\geq T_{\alpha,\theta}(N_k^\alpha)^{-2}, |t|\leq T_{\alpha,\theta}^{-1}\}} \left(\Pi_{t_k^\alpha - t, x_k^\alpha} T_{N_k^\alpha} \phi^{\alpha,\theta,\pm\infty}\right), \tag{3.6.7}$$

$$\omega_k^{\alpha,\theta,\pm\infty} := \mathbb{1}_{\{\pm(t-t_k^\beta)\geq T_{\beta,\theta}(N_k^\beta)^{-2}, |t|\leq T_{\beta,\theta}^{-1}\}} \left(\Pi_{t_k^\beta - t, x_k^\beta} T_{N_k^\beta} \phi^{\beta,\theta,\pm\infty}\right). \tag{3.6.8}$$

For any fixed $t \in I_k$, we obtain that

$$\langle \omega_k^{\alpha,\theta,\pm\infty}(t), \omega_k^{\beta,\theta,\pm\infty}(t)\rangle_{\dot{H}^1\times\dot{H}^1} = \langle \phi_{\mathcal{O}_k^\alpha}^{\alpha,\theta,\pm\infty}, \phi_{\mathcal{O}_k^\beta}^{\beta,\theta,\pm\infty}\rangle_{\dot{H}^1\times\dot{H}^1}.$$

By (3.6.1) of Proposition 3.6.2, we obtain that

$$\lim_{k\to\infty} \sup_t \langle \omega_k^{\alpha,\theta,\pm\infty}(t), \omega_k^{\beta,\theta,\pm\infty}(t)\rangle_{\dot{H}^1\times\dot{H}^1} = 0.$$

Case (2): $\langle \omega_k^{\alpha,\theta,\pm\infty}, \omega_k^{\beta,\theta}\rangle_{\dot{H}^1\times\dot{H}^1}$.

By the constructions of $\omega_k^{\alpha,\theta,\pm\infty}, \omega_k^{\beta,\theta,\pm\infty}$ in the proof of Lemma 3.5.9, we obtain that

$$\omega_k^{\beta,\theta} := \widetilde{u_k}^\beta \cdot \mathbb{1}_{S_k^{\beta,\theta}},$$

where $S_k^{\beta,\theta} := \{(x,t) \in \mathbb{T}^4 \times (-T_{\beta,\theta}, T_{\beta,\theta}) : |t - t_k^\beta| < T_{\beta,\theta}(N_k^\beta)^{-2},\ |x - x_k^\beta| \leq R_{\beta,\theta}(N_k^\beta)^{-1}\}$ and $\widetilde{u_k}^\beta$ is defined in (2). Following a similar proof of the **Case 4** in the proof of (3.8.1) in Lemma 3.7.4, we have that $\lim_{k\to\infty} \sup_t \langle \omega_k^{\alpha,\theta,\pm\infty}, \omega_k^{\beta,\theta}\rangle_{\dot{H}^1\times\dot{H}^1} = 0$.

Case (3): $\langle \omega_k^{\alpha,\theta}, \omega_k^{\beta,\theta} \rangle_{\dot{H}^1\times\dot{H}^1}$.

For $\varepsilon > 0$ small.

If $N_k^\alpha/N^\beta k + N_k^\beta/N_k^\alpha \leq \varepsilon^{-1000}$ and k is large enough then $S_k^{\alpha,\theta} \cap S_k^{\beta,\theta} = \emptyset$. (By the definition of orthogonality of frames, $N_k^\alpha/N_k^\beta + N_k^\beta/N_k^\alpha \leq \varepsilon^{-1000}$ implies $(N_k^\alpha)^2|t_k^\alpha - t_k^\beta| \to \infty$ or $N_k^\alpha|x_k^\alpha - x_k^\beta| \to \infty$, so $S_k^{\alpha,\theta} \cap S_k^{\beta,\theta} = \emptyset$.) In this case, $\omega_k^{\alpha,\theta}\, \omega_k^{\beta,\theta} \equiv 0$.

If $N_k^\alpha/N_k^\beta \geq \varepsilon^{-1000}/2$. Denote that

$$\omega_k^{\alpha,\theta}\omega_k^{\beta,\theta} = \omega_k^{\alpha,\theta}\widetilde{\omega}_k^{\beta,\theta} := \omega_k^{\alpha,\theta} \cdot (\omega_k^{\beta,\theta}\mathbb{1}_{(t_k^\alpha - T_{\alpha,\theta}(N_k^\alpha)^{-2}, t_k^\alpha + T_{\alpha,\theta}(N_k^\alpha)^{-2})}(t)).$$

By $\varepsilon^{10}N_k^\alpha >> \varepsilon^{-10}N_k^\beta$ and the Claim † in the proof of Lemma 3.8.2, we obtain that

$$\begin{aligned}
\langle \omega_k^{\alpha,\theta}, \omega_k^{\beta,\theta} \rangle_{\dot{H}^1\times\dot{H}^1} &\leq \langle P_{\leq \varepsilon^{10}N_k^\alpha}\omega_k^{\alpha,\theta}, \omega_k^{\beta,\theta} \rangle_{\dot{H}^1\times\dot{H}^1} + \langle P_{>\varepsilon^{10}N_k^\alpha}\omega_k^{\alpha,\theta}, P_{>\varepsilon^{-10}N_k^\beta}\omega_k^{\beta,\theta} \rangle_{\dot{H}^1\times\dot{H}^1} \\
&\quad + \langle P_{>\varepsilon^{10}N_k^\alpha}\omega_k^{\alpha,\theta}, \omega_k^{\beta,\theta} \rangle_{\dot{H}^1\times\dot{H}^1} \\
&\lesssim \varepsilon.
\end{aligned}$$

□

3.7 Proof of the main theorems

It suffice to prove the solutions remain bounded in Z-norm on intervals of length at most 1. To obtain this, we run the induction on the $E(u) + M(u)$ ($\mu = +1$) and $\|u\|_{L_t^\infty \dot{H}^1}$ ($\mu = -1$).

Definition 3.7.1. Define

$$\Lambda(L,\tau) = \begin{cases} \sup_{\substack{\text{u is a solution}\\ \text{of (3.1.1)}}} \{\|u\|_{Z(I)} : E(u) + M(u) \leq L,\ |I| \leq \tau\} & \text{if } \mu = +1 \\ \sup_{\substack{\text{u is a solution}\\ \text{of (3.1.1)}}} \{\|u\|_{Z(I)} : \sup_{t\in I} \|u(t)\|_{\dot{H}^1(\mathbb{T}^4)}^2 < L,\ |I| \leq \tau\} & \text{if } \mu = -1 \end{cases}$$

where u is any strong solution of (3.1.1) with initial data u_0 in interval I of length $|I| \leq \tau$.

It is easy to see that Λ is an increasing function of both L and τ, and moreover, by the definition we have the sublinearity of Λ in τ: $\Lambda(L, \tau + \sigma) \leq \Lambda(L,\tau) + \Lambda(L,\sigma)$. Hence we define

$$\Lambda_0(L) = \lim_{\tau \to 0} \Lambda(L, \tau),$$

and for all τ, we have that $\Lambda(L,\tau) < +\infty \Leftrightarrow \Lambda_0(L) < +\infty$. Finally, we define

$$E_{max} = \sup\{L : \Lambda_0(L) < +\infty\}.$$

Theorem 3.7.2. *Consider E_{max} defined above, if $\mu = +1$ (the defocusing case), then $E_{max} = +\infty$; if $\mu = -1$ (the focusing case), then $E_{max} \geq \|W\|^2_{\dot{H}^1(\mathbb{R}^4)}$.*

Corollary 3.7.3. *Suppose u is a solution of (3.1.1) in some time interval with the initial data $u_0 \in H^1(\mathbb{T}^4)$.*

1. *(the defocusing case) If $\mu = +1$ and $\|u_0\|_{H^1(\mathbb{T}^4)} < +\infty$, then u is a global solution.*
2. *(the focusing case) If $\mu = -1$ and under the assumption that*

$$\sup_t \|u(t)\|_{\dot{H}^1(\mathbb{T}^4)} < \|W\|_{\dot{H}^1(\mathbb{R}^4)},$$

then u is a global solution.

Proof of Theorem 3.7.2. Suppose for contradiction that $E_{max} < +\infty$ (if $\mu = +1$), or $E_{max} < \|W\|_{\dot{H}^1(\mathbb{R}^4)}$ (if $\mu = -1$). By the definition of E_{max}, there exists a sequence of solutions u_k such that

$$\begin{cases} E(u_k) + M(u_k), \text{ (if } \mu = +1) \\ \sup_{t\in[-T_k,T^k]} \|u(t)\|_{\dot{H}^1(\mathbb{T}^4)}, \text{ (if } \mu = -1) \end{cases} \to E_{max}, \quad \|u_k\|_{Z(-T_k,0)},\ \|u_k\|_{Z(0,T^k)} \to +\infty. \tag{3.7.1}$$

for some $T_k,\ T^k \to 0$ as $k \to +\infty$. For the simplicity of notations, set

$$L(\phi) := \begin{cases} E(\phi) + M(\phi), \text{ (if } \mu = +1), \\ \sup_{t\in[-T_k,T^k]} \|u_\phi(t)\|^2_{\dot{H}^1(\mathbb{T}^4)}, \text{ (if } \mu = -1), \end{cases}$$

where $u_\phi(t)$ is the solution of (3.1.1) with initial data $u_\phi(0) = \phi$. By the Proposition 3.6.3, after extracting a subsequence, (3.7.1) gives a sequence of profiles $\widetilde{\psi}^\alpha_k$, where $\alpha, k =$

$1, 2, \cdots$, and a decomposition

$$u_k(0) = g + \sum_{1 \leq \alpha \leq J} \widetilde{\psi}_k^\alpha + R_k^J.$$

satisfying

$$\limsup_{J \to \infty} \limsup_{k \to \infty} \| e^{it\Delta} R_k^J \|_{Z(I_k)} = 0. \tag{3.7.2}$$

And moreover the almost orthogonality in the Proposition 3.6.3 and the almost orthogonality of nonlinear profiles (Lemma 3.6.6), we obtain that

$$\begin{aligned} L(\alpha) := \lim_{k \to +\infty} L(\widetilde{\psi}^\alpha_{\mathcal{O}_k^\alpha}) \in [0, E_{max}], \\ \lim_{J \to J^*} \left(\sum_{1 \leq \alpha \leq J} L(\alpha) + \lim_{k \to \infty} L(R_k^J) \right) + L(g) = E_{max}, \end{aligned} \tag{3.7.3}$$

Case 1: $g \neq 0$ and no any Euclidean profiles.

There is no any Euclidean profiles, and by Remark 3.2.6, $\|g\|_{H^1(\mathbb{T}^4)} \lesssim L(g) \leq E_{max}$. Then, by $I_k \to 0$ as $k \to \infty$, there exist, $\eta > 0$, s.t. for k large enough

$$\| e^{it\Delta} u_k(0) \|_{Z(-T_k, T^k)} \leq \| e^{it\Delta} g \|_{Z(-\eta, \eta)} + \varepsilon \leq \delta_0$$

where δ_0 is given by the local theory in Proposition 3.4.5. In this case. we conclude that $\|u_k\|_{Z(-T_k, T^k)} \lesssim 2\delta_0$ which contradicts (3.7.1).

Case 2: $g = 0$ and only one Euclidean profile $\widetilde{\psi}_k^1$ such that $L(1) = E_{max}$.

By Remark 3.6.5 and (3.7.3), we obtain that $L(\widetilde{\psi}_k^1) \leq E_{max}$ which implies $\|\psi\|_{\dot{H}^1(\mathbb{R}^4)} < \infty$ (if $\mu = +1$) or $\sup_t \|u_\psi\|_{\dot{H}^1(\mathbb{R}^4)} < \|W\|_{\dot{H}^1(\mathbb{R}^4)}$ (if $\mu = -1$). Denote U_k^1 is the solution of (3.1.1) with $U_k^1(0) = \widetilde{\psi}_k^1$. In this case, we use the part (1) of Proposition 3.5.8 and Remark 3.2.6, Given some $\epsilon > 0$, for k large enough, we have that

$$\|U_k^1\|_{X^1(-T_k, T^k)} \leq \|U_k^1\|_{X^1(-\delta, \delta)} \lesssim 1, \qquad \text{and} \qquad \|U_k^1(0) - u_k(0)\|_{H^1(\mathbb{T}^4)} \leq \epsilon. \tag{3.7.4}$$

By (3.7.4) and Proposition 3.4.9, we obtain that

$$\|u_k\|_{Z(I_k)} \lesssim \|u_k\|_{X^1(I_k)} \lesssim 1,$$

which contradicts (3.7.1).

Case 3: At least two of all profiles are nonzero

By (3.7.3), $L(g) < E_{max}$ and $L(\alpha) < E_{max}$ for any $\alpha = 1, 2, \cdots$ By almost orthogonality and relabeling the profiles, we can assume that for all α,

$$L(\alpha) \leq L(1) < E_{max} - \eta,\ L(g) < E_{max} - \eta,\ \text{for some } \eta > 0.$$

Define U_k^α as the maximal life-span solution of (3.1.1) with initial data $U_k^\alpha(0) = \widetilde{\psi}_k^\alpha$ and G to be the maximal life-span solution of (3.1.1) with initial data $G(0) = g$.

By the definition of Λ and the hypothesis $E_{max} < \infty$ (if $\mu = +1$) and $E_{max} < E_W$ (if $\mu = -1$), we have

$$\|G\|_{Z(-1,1)} + \lim_{k\to\infty} \|U_k^\alpha\|_{Z(-1,1)} \leq 2\Lambda(E_{max} - \eta/2, 2) \lesssim 1.$$

By Proposition 3.4.7, it follows that for any α and any $k > k_0(\alpha)$ sufficient large,

$$\|G\|_{X^1(-1,1)} + \|U_k^\alpha\|_{X^1(-1,1)} \lesssim 1.$$

For $J, k \geq 1$, we define

$$U_{prof,k}^J := G + \sum_{\alpha=1}^{J} U_k^\alpha = \sum_{\alpha=0}^{J} U_k^\alpha.$$

where we set that $U_k^0 := G$.

Claim *that there is a constant Q such that*

$$\|U_{prof,k}^J\|_{X^1(-1,1)}^2 + \sum_{\alpha=0}^{J} \|U_k^\alpha\|_{X^1(-1,1)}^2 \leq Q^2, \tag{3.7.5}$$

uniformly on J.

From (3.7.2) we know that there are only finite many profiles such that $L(\alpha) \geq \frac{\delta_0}{2}$. We may assume that for all $\alpha \geq A$, $L(\alpha) \leq \delta_0$. Consider U_k^α for $k \geq A$, by small data

GWP result (Proposition 3.4.6), we have that

$$
\begin{aligned}
&\|U^J_{prof,k}\|_{X^1(-1,1)} = \|\sum_{0\le\alpha\le J} U^\alpha_k\|_{X^1(-1,1)}\\
\le &\sum_{0\le\alpha\le A}\|U^\alpha_k\|_{X^1(-1,1)} + \|\sum_{A\le\alpha\le J}(U^\alpha_k - e^{it\Delta}U^\alpha_k(0))\|_{X^1(-1,1)} + \|e^{it\Delta}\sum_{A\le\alpha\le J}U^\alpha_k(0)\|_{X^1(-1,1)}\\
\lesssim &(A+1) + \sum_{A\le\alpha\le J}\|U^\alpha_k(0)\|^2_{H^1} + \|\sum_{A\le\alpha\le J}U^\alpha_k(0)\|_{H^1}\\
\lesssim &(A+1) + \sum_{A\le\alpha\le J} L(\alpha) + E^{\frac{1}{2}}_{max}\\
\lesssim &1.
\end{aligned}
$$

And also similarly, we have that

$$
\begin{aligned}
\sum_{\alpha=0}^{J}\|U^\alpha_k\|^2_{X^1(-1,1)} &= \sum_{\alpha=0}^{A-1}\|U^\alpha_k\|^2_{X^1(-1,1)} + \sum_{A\le\alpha\le J}\|U^\alpha_k\|^2_{X^1(-1,1)}\\
&\lesssim A + \sum_{A\le\alpha\le J} L(\alpha)\\
&\lesssim 1.
\end{aligned}
$$

We denote that

$$U^J_{app,k} = \sum_{0\le\alpha\le J} U^\alpha_k + e^{it\Delta}R^J_k$$

is a solution of the approximation equation (3.4.11) with the error term:

$$
\begin{aligned}
e &= (i\partial_t + \Delta)U^J_{app,k} - F(U^J_{app,k})\\
&= \sum_{0\le\alpha\le J} F(U^\alpha_k) - F(\sum_{0\le\alpha\le J} U^\alpha_k + e^{it\Delta}R^J_k),
\end{aligned}
$$

where $F(u) = u|u|^2$.

From (3.7.5) we know $\|U^J_{app,k}\|_{X^1(-1,1)} \le Q$

By Lemma 3.7.4 (proven later), we obtain that

$$\limsup_{k\to\infty}\|e\|_{N(I_k)} \le \varepsilon/2, \text{ for } J \ge J_0(\varepsilon).$$

We use the stability proposition (Proposition 3.4.9) to conclude that u_k satisfies

$$\|u_k\|_{X^1(I_k)} \lesssim \|U^J_{app,k}\|_{X^1(I_k)} \le \|U^J_{prof,k}\|_{X^1(-1,1)}\|e^{it\Delta}R^J_k\|_{X^1(-1,1)} \lesssim 1.$$

which contradicts (3.7.1). $\square$

Lemma 3.7.4. *With the same notation, we obtain that*

$$\limsup_{J\to\infty}\limsup_{k\to\infty}\|\sum_{0\le\alpha\le J}F(U_k^\alpha)-F(\sum_{0\le\alpha\le J}U_k^\alpha+e^{it\Delta}R_k^J)\|_{N(I_k)}=0. \tag{3.7.6}$$

3.8 Proof of Lemma 3.7.4

Consider

$$\begin{aligned}&\|\sum_{0\le\alpha\le J}F(U_k^\alpha)-F(U_{prof,k}^J+e^{it\Delta}R_k^J)\|_{N(I_k)}\\ \le&\|F(U_{prof,k}^J+e^{it\Delta}R_k^J))-F(U_{prof,k}^J)\|_{N(I_k)}+\|F(U_{prof,k}^J)-\sum_{0\le\alpha\le J}F(U_k^\alpha)\|_{N(I_k)}.\end{aligned}$$

It will suffice that we can prove

$$\limsup_{J\to\infty}\limsup_{k\to\infty}\|F(U_{prof,k}^J+e^{it\Delta}R_k^J))-F(U_{prof,k}^J)\|_{N(I_k)}=0, \tag{3.8.1}$$

and

$$\limsup_{J\to\infty}\limsup_{k\to\infty}\|F(U_{prof,k}^J)-\sum_{0\le\alpha\le J}F(U_k^\alpha)\|_{N(I_k)}=0. \tag{3.8.2}$$

Before prove (3.8.1) and (3.8.2), we need several lemma.

Denote that $\mathfrak{D}_{p,q}(a,b)$ stands for a $p+q$ - linear expression with p factors consisting of either $\overline{a}$ or a and q factors consisting of either $\overline{b}$ or b.

Lemma 3.8.1 (a high-frequency linear solution does not interact significantly with a low-frequency profile)**.** *Assume that $B,N\ge 2$, and dyadic numbers, and assume that $\omega:\mathbb{T}^4\times(-1,1)\to\mathbb{C}$ is a function satisfying*

$$|\nabla^j\omega|\le N^{j+1}\mathbb{1}_{|x|\le N^{-1},|t|\le N^{-2}},\ j=0,1.$$

Then we hold that

$$\|\mathfrak{D}_{2,1}(\omega,e^{it\Delta}P_{>BN}f)\|_{N(-1,1)}\lesssim(B^{-1/200}+N^{-1/200})\|f\|_{H^1(\mathbb{T}^4)}.$$

Proof. We may assume that $\|f\|_{H^1(\mathbb{T}^4)} = 1$ and $f = P_{>BN} f$. By Proposition 3.3.12, we obtain that

$$
\begin{aligned}
&\|\mathfrak{D}_{2,1}(\omega, e^{it\Delta} P_{>BN} f)\|_{N(I)} \\
\leq &\|\mathfrak{D}_{2,1}(\omega, e^{it\Delta} P_{>BN} f)\|_{L^1((-1,1),H^1)} \\
\lesssim &\|\mathfrak{D}_{2,1}(\omega, \nabla e^{it\Delta} f)\|_{L^1((-1,1),L^2)} + \|e^{it\Delta} f\|_{L_t^\infty L_x^2} \|\omega\|_{L_t^2 L_x^\infty} \||\nabla \omega| + |\omega|\|_{L_t^2 L_x^\infty} \\
\lesssim &\|\mathfrak{D}_{2,1}(\omega, \nabla e^{it\Delta} f)\|_{L^1((-1,1),L^2)} + B^{-1}.
\end{aligned}
$$

(It's easy to check that $\|\omega\|_{L_t^2 L_x^\infty} \leq \left(\int_{-N^{-2}}^{N^{-2}} (N)^2 \, dt\right)^{\frac{1}{2}} =, \|\nabla \omega\|_{L_t^2 L_x^\infty} \leq \left(\int_{-N^{-2}}^{N^{-2}} N^4 \, dt\right)^{\frac{1}{2}} = N$, and $\|P_{>BN} f\|_{L^2} \leq \frac{1}{BN} \|f\|_{H^1}$.)

Now we let $W(x,t) := N^4 \eta_{\mathbb{R}^4}(N \Psi^{-1}(x)) \eta_{\mathbb{R}}(N^2 t)$,

$$
\begin{aligned}
&\|\mathfrak{D}_{2,1}(\omega, \nabla e^{it\Delta} f)\|_{L^1((-1,1),L^2)}^2 \\
= &\left(\int_{-1}^{1} \|\mathfrak{D}_{2,1}(\omega, \nabla e^{it\Delta} f)\| \, dt\right)^2 \\
\leq &\|\omega\|_{L_t^4 L_x^\infty}^4 \|\frac{1}{N^2} W^{\frac{1}{2}} \nabla e^{it\Delta} f\|_{L^2(\mathbb{T}^4 \times [-1,1])} \\
\lesssim &N^{-2} \|W^{\frac{1}{2}} \nabla e^{it\Delta} f\|_{L^2(\mathbb{T}^4 \times [-1,1])}^2 \\
\lesssim &\sum_{j=1}^{4} \langle e^{it\Delta} \partial_j f, W e^{it\Delta} \partial_j f \rangle_{L^2 \times L^2} \, dt \\
\lesssim &\sum_{j=1}^{4} \int_{-1}^{1} \langle \partial_j f, [\int_{-1}^{1} e^{-it\Delta} W e^{it\Delta} \, dt] \rangle_{L^2 \times L^2}.
\end{aligned}
$$

It remains to prove that

$$
\|K\|_{L^2(\mathbb{T}^4) \to L^2(\mathbb{T}^4)} \lesssim N^2 (B^{-\frac{1}{100}} + N^{-\frac{1}{100}}),
$$

where $K = P_{>BN} \int_{\mathbb{R}} e^{-it\Delta} W e^{it\Delta} P_{>BN} \, dt$.

We compute the Fourier coefficients of K as follows:

$$\begin{aligned}
c_{p,q} &= \langle e^{ipx}, Ke^{iqx}\rangle \\
&= \int_{\mathbb{T}^4} \overline{P_{>BN}e^{ipx}} \int_{\mathbb{R}} e^{-it\Delta} W e^{it\Delta} P_{>BN}\, dtdx \\
&= (1-\eta_{\mathbb{R}^4})(p/BN)(1-\eta_{\mathbb{R}^4})(q/BN) \int_{\mathbb{T}^4} \overline{e^{ipx}} \int_{\mathbb{R}} e^{-it\Delta} W e^{it\Delta} e^{iqx}\, dtdx \\
&= (1-\eta_{\mathbb{R}^4})(p/BN)(1-\eta_{\mathbb{R}^4})(q/BN) \int_{\mathbb{T}^4\times[-1,1]} \overline{e^{-it|p|^2+ipx}} W(t,x) e^{-it|q|^2+iqx}\, dxdt \\
&= (1-\eta_{\mathbb{R}^4})(p/BN)(1-\eta_{\mathbb{R}^4})(q/BN) C_{\mathcal{F}_{x,t}}(p-q, |q|^2-|p|^2).
\end{aligned}$$

Hence, we obtain that

$$|c_{p,q}| \lesssim N^{-2}\left(1+\frac{\left||p|^2-|q|^2\right|}{N^2}\right)^{-10}\left(1+\frac{|p-q|}{N}\right)^{-10} \mathbb{1}_{\{|p|\geq BN\}}\mathbb{1}_{\{|q|\geq BN\}}.$$

Using Schur's lemma.

$$\|K\|_{L^2(\mathbb{T}^4)\to L^2(\mathbb{T}^4)} \lesssim \sup_{p\in\mathbb{Z}^4}\sum_{q\in\mathbb{Z}^4}|c_{p,q}| + \sup_{q\in\mathbb{Z}^4}\sum_{p\in\mathbb{Z}^4}|c_{p,q}|.$$

It suffices to prove that

$$N^{-4}\sup_{|p|\geq BN}\sum_{v\in\mathbb{Z}^4}\left(1+\frac{\left||p|^2-|p+v|^2\right|}{N^2}\right)^{-10}\left(1+\frac{|v|}{N}\right)^{-10} \lesssim B^{-\frac{1}{100}}+N^{-\frac{1}{100}} \tag{3.8.3}$$

Consider (3.8.3) in the following 3 cases.

Case 1:

$$\begin{aligned}
&\sum_{|v|\geq NB^{\frac{1}{100}}}\left(1+\frac{\left||p|^2-|p+v|^2\right|}{N^2}\right)^{-10}\left(1+\frac{|v|}{N}\right)^{-10} \\
\lesssim &\sum_{|v|\geq NB^{\frac{1}{100}}}\left(1+\frac{|v|}{N}\right)^{-10} \\
\lesssim &\int_{|v|\geq N,\, v\in\mathbb{R}^4}\left(1+\frac{|v|}{N}\right)^{-10} dv \\
\lesssim &\left(1+\frac{NB^{\frac{1}{100}}}{N}\right)^{-6} \\
\lesssim &B^{-\frac{6}{100}}.
\end{aligned}$$

Case 2:

$$
\begin{aligned}
&\sum_{\substack{|v|\le NB^{\frac{1}{100}}\\ |v\cdot p|\ge N^2B^{\frac{1}{10}}}} \left(1+\frac{\big||p|^2-|p+v|^2\big|}{N^2}\right)^{-10}\left(1+\frac{|v|}{N}\right)^{-10}\\
\lesssim &\sum_{\substack{|v|\le NB^{\frac{1}{100}}\\ |v\cdot p|\ge N^2B^{\frac{1}{10}}}} \left(1+\frac{2|v\cdot p|}{N^2}\right)^{-10}\\
\lesssim &(1+B^{\frac{1}{10}})^{-6}\lesssim B^{-\frac{6}{10}}.
\end{aligned}
$$

Case 3:

Denote $\hat{p}=\frac{p}{|p|}$

$$
\begin{aligned}
&N^{-4}\sup_{|p|\ge BN}\sum_{\substack{|v|\le NB^{\frac{1}{100}}\\ |p\cdot v|\le N^2B^{\frac{1}{10}}}} \left(1+\frac{\big||p|^2-|p+v|^2\big|}{N^2}\right)^{-10}\left(1+\frac{|v|}{N}\right)^{-10}\\
\le &N^{-4}\sup_{|p|\ge BN}\sum_{\substack{|v|\le NB^{\frac{1}{100}}\\ |\hat{p}\cdot v|\le NB^{-\frac{9}{10}}}} \left(1+\frac{\big||p|^2-|p+v|^2\big|}{N^2}\right)^{-10}\left(1+\frac{|v|}{N}\right)^{-10}\\
\le &N^{-4}\sup_{|p|\ge BN}\sum_{\substack{|v|\le NB^{\frac{1}{100}}\\ |\hat{p}\cdot v|\le NB^{-\frac{9}{10}}}} 1\\
\le &N^{-4}\sup_{|p|\ge BN}\#\{v:|v|\le NB^{\frac{1}{100}},\ |\hat{p}\cdot v|\le NB^{-\frac{9}{10}}\}\\
= &N^{-4}(NB^{\frac{1}{100}})^3NB^{-\frac{9}{10}}\\
\le &B^{-\frac{87}{100}}.
\end{aligned}
$$

□

Lemma 3.8.2. *Assume that $\mathfrak{O}_\alpha=(N_{k,\alpha},t_{k,\alpha},x_{k,\alpha})_k\in\mathcal{F}_e,\ \alpha\in\{1,2\}$, are two orthogonal frames, $I\subseteq(-1,1)$ is a fixed open interval, $0\in I$, and $T_1,\ T_2,\ R\in[1,\infty)$ are fixed numbers, $R\ge T_1+T_2$. For k large enough, for $\alpha\in\{1,2\}$*

$$
|\nabla_x^m\omega_k^{\alpha,\theta}|+(N_{k,\alpha})^{-2}\mathbb{1}_{S_k^{\alpha,\theta}}|\partial_t\nabla_x^m\omega_k^{\alpha,\theta}|\le R_{\theta,\alpha}(N_k^\alpha)^{|m|+1}\mathbb{1}_{S_k^{\alpha,\theta}},\ 0\le|m|\le 10,
$$

where

$$S_k^{\alpha,\theta} := \{(x,t) \in \mathbb{T}^4 \times I : |t - t_{k,\alpha}| < T_\alpha (N_{k,\alpha})^{-2},\ |x - x_{k,\alpha}| \le R(N_{k,\alpha})^{-1}\}.$$

and assume that $(\omega_{k,1}, w_{k,2}, f_k)_k$ *are 3 sequences of functions with properties* $\|f_k\|_{X^1(I)} \le 1$ *for all* k *large enough, then*

$$\limsup_{k\to\infty} \|\omega_{k,1}\, \omega_{k,2}\, f_k\|_{N(I)} = 0$$

Proof. For $\varepsilon > 0$ small.

If $N_{k,1}/N_{k,2} + N_{k,2}/N_{k,1} \le \varepsilon^{-1000}$ and k is large enough then $S_{k,1} \cap S_{k,2} = \emptyset$. (By the definition of orthogonality of frames, $N_{k,1}/N_{k,2} + N_{k,2}/N_{k,1} \le \varepsilon^{-1000}$ implies $N_{k,1}^2 |t_{k,1} - t_{k,2}| \to \infty$ or $N_{k,1}|x_{k,1} - x_{k,2}| \to \infty$, so $S_{k,1} \cap S_{k,2} = \emptyset$.) In this case, $\omega_{k,1}\, \omega_{k,2}\, f_k \equiv 0$.

If $N_{k,1}/N_{k,2} \ge \varepsilon^{-1000}/2$. Denote that

$$\omega_{k,1}\omega_{k,2} = \omega_{k,1}\widetilde{\omega}_{k,2} := \omega_{k,1} \cdot (w_{k,2} \mathbb{1}_{(t_{k,1} - T_1 N_{k,1}^{-2}, t_{k,1} + T_1 N_{k,1}^{-2})}(t)).$$

Claim † *For* k *large enough,*

1. $\|\widetilde{\omega}_{k,2}\|_{X^1(I)} \lesssim_R 1$;

2. $\|P_{>\varepsilon^{-10} N_{k,2}} \widetilde{\omega}_{k,2}\|_{X^1(I)} \lesssim_R \varepsilon$;

3. $\|\widetilde{\omega}_{k,2}\|_{Z(I)} \lesssim_R \varepsilon$;

4. $\|\omega_{k,1}\|_{X^1(I)} \lesssim_R 1$;

5. $\|P_{\le \varepsilon^{10}} \omega_{k,1}\|_{X^1(I)} \lesssim_R \varepsilon$.

By this *Claim* †, Proposition 3.4.4, and $\varepsilon^{10} N^1 >> \varepsilon^{-10} N_2$ we obtain that

$$\begin{aligned}\|\omega_{k,1}\omega_{k,2} f_k\|_{N(I)} \le & \|(P_{\le \varepsilon^{10} N_{k,1}} \omega_{k,1})(\widetilde{\omega}_{k,2}) f_k\|_{N(I)} + \|(P_{>\varepsilon^{10} N_{k,1}} \omega_{k,1})(P_{>\varepsilon^{-10} N_{k,2}} \widetilde{\omega}_{k,2}) f_k\|_{N(I)} \\ & + \|(P_{>\varepsilon^{10} N_{k,1}} \omega_{k,1})(P_{\le \varepsilon^{-10} N_{k,2}} \widetilde{\omega}_{k,2}) f_k\|_{N(I)} \\ \lesssim_R & \varepsilon.\end{aligned}$$

More detail about the *Claim* †:

(1) Consider $\widetilde{\omega}_{k,2} w_{k,2} \mathbb{1}_{(t_{k,1}-T_1 N_{k,1}^{-2}, t_{k,1}+T_1 N_{k,1}^{-2})}(t)$.

$$\begin{aligned}\|\widetilde{\omega}_{k,2}\|_{X^1(I)} &\lesssim \|\widetilde{\omega}_{k,2}(0)\|_{H^1} + \left(\sum_N \|P_N(i\partial_t+\Delta)\widetilde{\omega}_{k,2}\|^2_{L^1_t([0,1],H^1)}\right)^{\frac{1}{2}} \\ &\lesssim \left(\int_{|x-x_{k,2}|\le RN_{k,2}^{-1}} |\langle\nabla\rangle\widetilde{\omega}_{k,2}(0)|^2\,dx\right)^{\frac{1}{2}} \\ &+ \left(\sum_N \left(\int_I dt \|P_N(\partial_t\widetilde{\omega}_{k,2})\|_{H^1} + \|P_N\Delta\widetilde{\omega}_{k,2}\|_{H^1}\right)^2\right)^{\frac{1}{2}} \\ &\lesssim (R^2N_{k,2}^4R^4N_{k,2}^{-4})^{\frac{1}{2}} + \int_I (\|\partial_t\widetilde{\omega}_{\alpha,k}\|_{H^1} + \|\Delta\widetilde{\omega}_{k,2}\|_{H^1})\,dt \\ &\lesssim 1.\end{aligned}$$

(2) Consider the high frequency part of $\widetilde{\omega}_{k,2}$.

$$\begin{aligned}&\|P_{>\varepsilon^{-10}N_{k,2}}\widetilde{\omega}_{k,2}\|_{X^1(I)} \\ \lesssim&\|P_{>\varepsilon^{-10}}\widetilde{\omega}_{k,2}(0)\|_{H^1} + \left(\sum_{N>\varepsilon^{-10}N_{k,2}} \|P_N(i\partial_t+\Delta)\widetilde{\omega}_{k,2}\|^2_{L^1_t([0,1],H^1)}\right)^{\frac{1}{2}} \\ \le&\left(\int_{|x-x_{k,2}|\le RN_{k,2}^{-1}} |P_{>\varepsilon^{-10}N_{k,2}}\langle\nabla\rangle\widetilde{\omega}_{k,2}(0)|^2\,dx\right)^{\frac{1}{2}} + \int \|P_{>\varepsilon^{-10}N_{k,2}}(i\partial_t+\Delta)\widetilde{\omega}_{k,2}\|_{H^1}\,dt \\ \le&\left(\int_{|x-x_{k,2}|\le RN_{k,2}^{-1}} \left(\frac{\varepsilon^{10}}{N_{k,2}}\right)^2 |P_{>\varepsilon^{-10}N_{k,2}}\langle\nabla\rangle^2\widetilde{\omega}_{k,2}(0)|^2\,dx\right)^{\frac{1}{2}} + \int_{|t-t_{k,2}|<N_{k,2}^{-2}R} \frac{\varepsilon^{10}}{N_{k,2}}\|(i\partial_t+\Delta)\widetilde{\omega}_{k,2}\|_{H^2}\,dt \\ \le&\varepsilon^{10}R^3 + N_{k,2}^{-2}R\frac{\varepsilon^{10}}{N_{k,2}}(R^4N_{k,2}^{-2}R^2N_{k,2}^{1}0)^{\frac{1}{2}} \\ \lesssim&\varepsilon^{10}R^4.\end{aligned}$$

(3): Consider the Z-norm of $\widetilde{\omega}_{k,2}$.

$$\begin{aligned}
\|\widetilde{\omega}_{k,2}\|_{Z(I)} &\leq \left(\sum_N N^2\|P_N\widetilde{\omega}_{k,2}\|^4_{L^4(\mathbb{T}^4\times(t_{k,1}-RN_{k,1}^{-2},t_{k,1}+RN_{k,1}^{-2}))}\right)^{1/4}\\
&\leq \left(\sum_N \|\nabla^{\frac{1}{2}}P_N\widetilde{\omega}_{k,2}\|^4_{L^4_{t,x}}\right)^{1/4}\\
&\lesssim \|\left(\sum_N |\nabla^{1/2}P_N\widetilde{\omega}_{k,2}|^4\right)^{1/4}\|_{L^4}\\
&\lesssim \|\left(\sum_N |\nabla^{1/2}P_N\widetilde{\omega}_{k,2}|^2\right)^{1/2}\|_{L^4}\\
&\lesssim \|\nabla^{1/2}\widetilde{\omega}_{k,2}\|_{L^4(\mathbb{T}^4\times(t_{k,1}-RN_{k,1}^{-2},t_{k,1}+RN_{k,1}^{-2}))}\\
&\lesssim R^{\frac{9}{4}}\left(\frac{N_{k,2}}{N_{k,1}}\right)^{\frac{1}{2}}\\
&\leq R^{\frac{9}{4}}\varepsilon^{500}.
\end{aligned}$$

(4): Similar with (1).

(5):

$$\begin{aligned}
&\|P_{\leq\varepsilon^{10}N_{k,1}}\omega_{k,1}\|_{X^1(I)}\\
\leq&\|P_{\leq\varepsilon^{10}N_{k,1}}\omega_{k,1}(0)\|_{H^1}+\int\|P_{\leq\varepsilon^{10}N_{k,1}}(i\partial_t+\Delta)\omega_{k,1}\|_{H^1}\,dt\\
\lesssim&\varepsilon^{10}N_{k,1}\left(\|P_{\leq\varepsilon^{10}N_{k,1}}\omega_{k,1}(0)\|_{L^2}+\int\|P_{\leq\varepsilon^{10}N_{k,1}}(i\partial_t+\Delta)\omega_{k,1}\|_{L^2}\,dt\right)\\
\lesssim&\varepsilon^{10}R^4.
\end{aligned} \tag{3.8.4}$$

□

Proof of (3.8.1).

$$\begin{aligned}
&F(U^J_{prof,k}+e^{it\Delta}R^J_k)-F(U^J_{prof,k})\\
=&\mathfrak{D}_{2,1}(U^J_{prof,k},e^{it\Delta}R^J_k)+\mathfrak{D}_{1,2}(U^J_{prof,k},e^{it\Delta}R^J_k)+|e^{it\Delta}R^J_k|^2e^{it\Delta}R^J_k
\end{aligned}$$

First, by the nonlinear estimate (Proposition 3.4.4), we have

$$\begin{aligned}
&\||e^{it\Delta}R^J_k|^2e^{it\Delta}R^J_k\|_{N(I_k)}\\
\lesssim&\|e^{it\Delta}R^J_k\|^2_{Z'(I_k)}\|e^{it\Delta}R^J_k\|_{X^1(I_k)}
\end{aligned}$$

Since $\|e^{it\Delta}R_k^J\|_{Z'(I_k)} \to 0$ as $J, k \to \infty$, and $\|e^{it\Delta}R_k^J\|_{X^1(I_k)} \lesssim 1$,

$$\limsup_{J\to\infty}\limsup_{k\to\infty} \||e^{it\Delta}R_k^J|^2 e^{it\Delta}R_k^J\|_{N(I_k)} = 0.$$

Second, also by the nonlinear estimate Proposition 3.4.4 and Proposition 4.6.2,

$$\begin{aligned}
&\|\mathfrak{D}_{1,2}(U^J_{prof,k}, e^{it\Delta}R_k^J)\|_{N(I_k)} \\
\lesssim&\|U^J_{prof,k}\|_{X^1(I_k)}\|e^{it\Delta}R_k^J\|_{X^1(I_k)}\|e^{it\Delta}R_k^J\|_{Z'(I_k)} \to 0,
\end{aligned}$$

as $k, J \to \infty$.

Third, consider

$$\|\mathfrak{D}_{2,1}(U^J_{prof,k}, e^{it\Delta}R_k^J)\|_{N(I_k)},$$

assume $\varepsilon > 0$ is fixed, there exists $A = A(\varepsilon)$ sufficiently large, such that for all $J \geq A$ and $k \geq k_0(J)$

$$\|U^J_{prof,k} - U^A_{prof,k}\|_{X^1(-1,1)} \leq \varepsilon.$$

Then

$$\begin{aligned}
&\|\mathfrak{D}_{2,1}(U^J_{prof,k}, e^{it\Delta}R_k^J)\|_{N(I_k)} \\
\leq&\|\mathfrak{D}_{2,1}(U^A_{prof,k}, e^{it\Delta}R_k^J)\|_{N(I_k)} + \|\mathfrak{D}_{1,1,1}(U^A_{prof,k}, U^J_{prof,k} - U^A_{prof,k}, e^{itDD}R_k^J)\|_{N(I_k)} \\
&+ \|\mathfrak{D}_{2,1}(U^J_{prof,k} - U^A_{prof,k}, e^{it\Delta}R_k^J)\|_{N(I_k)} \to \|\mathfrak{D}_{2,1}(U^A_{prof,k}, e^{it\Delta}R_k^J)\|_{N(I_k)} + \varepsilon,
\end{aligned}$$

as $k, J \to \infty$.

It remains to prove that

$$\limsup_{J\to\infty}\limsup_{k\to\infty} \|\mathfrak{D}_{2,1}(U^A_{prof,k}, e^{it\Delta}R_k^J)\|_{N(I_k)} \lesssim \varepsilon.$$

By the definition of $U^A_{prof,k}$, it suffices to prove that for any $\alpha_1, \alpha_2 \in \{0, 1, \cdots, A\}$,

Fix $\theta = \varepsilon A^{-2}/10$, apply the decomposition in Lemma 3.5.9 to all nonlinear profiles U_k^α, $\alpha = 1, 2, \cdots, A$. We assume that

$$T_{\theta,\alpha} = T_\theta, \quad \text{and } R_{\theta,\alpha} = R_\theta,$$

for any $\alpha = 1, 2, \cdots, A$.

$$\limsup_{J\to\infty} \limsup_{k\to\infty} \|\mathfrak{D}_{1,1,1}(U_k^{\alpha_1}, U_k^{\alpha_2}, e^{it\Delta}R_k^J)\|_{N(I_k)} \lesssim \varepsilon A^{-2}. \tag{3.8.5}$$

Case 1: $\alpha_1 = 0$ or $\alpha_2 = 0$.

Without loss of generality, suppose $\alpha_2 = 0$.

Since $\|U_k^0\|_{X^1(-1,1)} = \|G\|_{X^1(-1,1)} \lesssim 1$, for any k large enough such that $\|G\|_{Z'(I_k)} \lesssim \varepsilon A^{-2}$, and $\|G\|_{X^1(I_k)} \lesssim 1$.

By the nonlinear estimate Proposition 3.4.4 and Proposition 4.6.2,

$$\begin{aligned}
&\|\mathfrak{D}_{1,1,1}(G, U_k^{\alpha_2}, e^{it\Delta}R_k^J)\|_{N(I_k)} \\
\lesssim &\|G\|_{Z'(I_k)}\|U_k^{\alpha_2}\|_{Z'(I_k)}\|e^{it\Delta}R_k^J\|_{X^1(I_k)} + \|G\|_{Z'(I_k)}\|U_k^{\alpha_2}\|_{X^1(I_k)}\|e^{it\Delta}R_k^J\|_{Z'(I_k)} \\
&+ \|G\|_{X^1(I_k)}\|U_k^{\alpha_2}\|_{Z'(I_k)}\|e^{it\Delta}R_k^J\|_{Z'(I_k)} \\
\lesssim &\varepsilon A^{-2},
\end{aligned}$$

when taking k, J large enough.

Case 2: $\alpha_1 \neq 0$, $\alpha_2 \neq 0$ and $\alpha_1 = \alpha_2$.

Taking k large enough, we have $I_k \subset (-T_\theta^{-1}, T_\theta^{-1})$

$$\mathbb{1}_{I_k}(t)U_k^\alpha = \omega_k^{\alpha,\theta,-\infty} + \omega_k^{\alpha,\theta,+\infty} + \omega_k^{\alpha,\theta} + \rho_k^{\alpha,\theta}.$$

By the nonlinear estimate Proposition 3.4.4, (3.5.32) and Lemma 3.8.2 (since $\|e^{it\Delta}R_k^J\|_{X^1(I_k)} \lesssim 1$ uniformly for both k and J), we obtain that

$$\begin{aligned}
\|\mathfrak{D}_{1,1,1}(U_k^{\alpha_1}, U_k^{\alpha_2}, e^{it\Delta}R_k^J)\|_{N(I_k)} &\lesssim \frac{1}{2}A^{-2}\varepsilon + \|\mathfrak{D}_{1,1,1}(\omega_k^{\alpha_1,\theta,+\infty}, \omega_k^{\alpha_1,\theta,-\infty}, e^{it\Delta}R_k^J)\|_{N(I_k)} \\
&\lesssim A^{-2}\varepsilon,
\end{aligned}$$

when k large enough.

Case 3: $\alpha_1 \neq 0$, $\alpha_2 \neq 0$ and $\alpha_1 \neq \alpha_2$.

Using Lemma 3.8.1, and set B sufficiently large and k sufficiently large, we obtain that,

$$\begin{aligned}\|\mathfrak{D}_{2,1}(\omega_k^{\alpha,\theta}, P_{>BN_{k,\alpha}} e^{it\Delta} R_k^J)\|_{N(I_k)} &\lesssim (\frac{1}{B^{1/200}} + \frac{1}{N_{k,\alpha}^{1/200}})\|R_k^J\|_{H^1} \\ &\lesssim \frac{\varepsilon}{4} A^{-2}.\end{aligned} \tag{3.8.6}$$

We may also assume that B is sufficiently large such that, for k large enough, by a similar estimate as (3.8.4), we obtain that

$$\|P_{\leq B^{-1}N_{k,\alpha}} \omega_k^{\alpha,\theta}\|_{X^1(I_k)} \leq \frac{\varepsilon}{4} A^{-2}. \tag{3.8.7}$$

Using the modified nonlinear estimate (3.4.4) of Lemma 3.4.4 and bounds (3.8.6) (3.8.7), it remains to prove that

$$\limsup_{J\to\infty} \limsup_{k\to\infty} \|\mathfrak{D}_{2,1}(P_{>B^{-1}N_{k,\alpha}} \omega_k^{\alpha,\theta}, P_{\leq BN_{k,\alpha}} e^{it\Delta} R_k^J)\|_{N(I_k)} = 0.$$

$\square$

Proof of (3.8.2).

$$F(U_{prof,k}^J) - \sum_{0\leq\alpha\leq J} F(U_k^\alpha) = \sum_{\substack{0\leq\alpha_1,\alpha_2,\alpha_3\leq J \\ \alpha_1\neq\alpha_2 \text{ or } \alpha_1\neq\alpha_3 \text{ or } \alpha_2\neq\alpha_3}} \mathfrak{D}_{1,1,1}(U_k^{\alpha_1}, U_k^{\alpha_2}, U_k^{\alpha_3})$$

By (3.7.5), we choose $A(\theta)$ large enough, such that $\sum_{A\leq\alpha\leq J} \|U_\alpha\|_{X^1(-1,1)}^2 \leq \theta$.

So we have

$$\begin{aligned}&\|\sum_{\substack{0\leq\alpha_1,\alpha_2,\alpha_3\leq J \\ \alpha_1\neq\alpha_2 \text{ or } \alpha_1\neq\alpha_3 \text{ or } \alpha_2\neq\alpha_3}} \mathfrak{D}_{1,1,1}(U_k^{\alpha_1}, U_k^{\alpha_2}, U_k^{\alpha_3})\|_{N(I_k)} \\ \leq&\|\sum_{\substack{0\leq\alpha_1,\alpha_2,\alpha_3\leq A \\ \alpha_1\neq\alpha_2 \text{ or } \alpha_1\neq\alpha_3 \text{ or } \alpha_2\neq\alpha_3}} \mathfrak{D}_{1,1,1}(U_k^{\alpha_1}, U_k^{\alpha_2}, U_k^{\alpha_3})\|_{N(I_k)} + \theta.\end{aligned}$$

Using Lemma 3.5.9,

$$\begin{aligned}&\|\sum_{\substack{0\leq\alpha_1,\alpha_2,\alpha_3\leq A \\ \alpha_1\neq\alpha_2 \text{ or } \alpha_1\neq\alpha_3 \text{ or } \alpha_2\neq\alpha_3}} \mathfrak{D}_{1,1,1}(U_k^{\alpha_1}, U_k^{\alpha_2}, U_k^{\alpha_3})\|_{N(I_k)} \\ \leq&\|\sum_{F} \mathfrak{D}_{1,1,1}(W_k^1, W_k^2, W_k^3)\|_{N(I_k)},\end{aligned}$$

where

$$\begin{aligned}F :=&\{(W_k^1, W_k^2, W_k^3) : W_k^i \in \{\omega_k^{\alpha,\theta,+\infty}, \omega_k^{\alpha,\theta,-\infty}, \omega_k^{\alpha,\theta}, \rho_k^{\alpha,\theta}\},\\ &\text{for } 0 \le \alpha \le A, \text{ and each } i, \text{at least two different } \alpha\}\end{aligned}$$

and $\#F < A^3$

Consider the following several cases:

Case 1: the terms containing one error component $\rho_k^{\alpha,\theta}$.

By the nonlinear estimate (Proposition 3.4.4),

$$\|\mathfrak{D}_{1,1,1}(W_k^1, W_k^2, \rho_k^{\alpha,\theta})\|_{N(I_k)} \le \|\rho_k^{\alpha,\theta}\|_{X^1(I_k)} \|W_k^1\|_{X^1(I_k)} \|W_k^2\|_{X^1(I_k)} \lesssim \theta,$$

for k large enough.

Case 2: the terms containing two scattering components $\omega_k^{\alpha,\theta,\pm\infty}$ and $\omega_k^{\beta,\theta,\pm\infty}$(maybe $\alpha = \beta$ or not).

$$\begin{aligned}&\|\mathfrak{D}_{1,1,1}(\omega_k^{\alpha,\theta,\pm\infty}, \omega_k^{\beta,\theta,\pm\infty}, W_k^3)\|_{N(I_k)}\\ \le&\|W_k^3\|_{X^1(I_k)}(\|\omega_k^{\alpha,\theta,\pm\infty}\|_{X^1(I_k)} + \|\omega_k^{\beta,\theta,\pm\infty}\|_{X^1(I_k)})(\|\omega_k^{\alpha,\theta,\pm\infty}\|_{Z'(I_k)} + \|\omega_k^{\beta,\theta,\pm\infty}\|_{Z'(I_k)})\\ \lesssim&\theta,\end{aligned}$$

for k large enough.

Case 3: the terms containing two different cores $\omega_k^{\alpha,\theta}$ and $\omega_k^{\beta,\theta}$ with $\alpha \ne \beta$.

By Lemma 3.8.2, for k large enough, we obtain that

$$\|\mathfrak{D}_{1,1,1}(\omega_k^{\alpha,\theta}, \omega_k^{\beta,\theta}, W_k^3)\|_{N(I_k)} \lesssim \theta.$$

Case 4: the others: $\mathfrak{D}_{2,1}(\omega_k^{\alpha,\theta}, \omega_k^{\beta,\theta,\pm\infty})$ with $\alpha \ne \beta$.

Case 4.1: $\limsup_{k\to\infty} \frac{N_{k,\beta}}{N_{k,\alpha}} = +\infty$.

By Lemma 3.8.1, and choosing B and k large enough,

$$\|\mathfrak{D}_{2,1}(\omega^{\alpha,\theta}, P_{>BN_{k,\alpha}}\omega^{\beta,\theta,\pm\infty})\|_{N(I_k)} \lesssim (B^{-1/200} + N_{k,\alpha}^{-1/200}) \lesssim \theta. \tag{3.8.8}$$

And for the other part,

$$\begin{aligned}\|P_{\leq BN_{k,\alpha}}\omega_k^{\beta,\theta,\pm\infty}\|_{X^1(I_k)} &= \|P_{\leq BN_{k,\beta}\frac{N_{k,\alpha}}{N_{k,\beta}}}\omega_k^{\beta,\theta,\pm\infty}\|_{X^1(I_k)} \\ &= \|P_{\leq BN_{k,\beta}\frac{N_{k,\alpha}}{N_{k,\beta}}}\pi_{x_k^\beta}T_{N_{k,\beta}}(\phi^{\beta,\theta,\pm\infty})\|_{H^1(\mathbb{T}^4)} \\ &= \|P_{\leq B\frac{N_{k,\alpha}}{N_{k,\beta}}}\phi^{\beta,\theta,\pm\infty}\|_{\dot{H}^1(\mathbb{R}^4)} \to 0, \text{ as } k\to\infty.\end{aligned}$$

So for k large enough, we obtain that

$$\|\mathfrak{D}_{2,1}(\omega_k^{\alpha,\theta}, P_{\leq BN_{k,\alpha}}\omega_k^{\beta,\theta,\pm\infty})\|_{N(I_k)} \lesssim \|P_{\leq BN_{k,\alpha}}\omega_k^{\beta,\theta,\pm\infty}\|_{X^1(I_k)}\|\omega_k^{\alpha,\theta}\|_{X^1(I_k)}^2 \lesssim \theta.$$

Case 4.2: $\limsup_{k\to\infty}\frac{N_{k,\alpha}}{N_{k,\beta}} = +\infty$.

We assume that B is sufficiently large such that for k large, by a similar estimate as (3.8.8), we obtain that

$$\|\mathfrak{D}_{2,1}(\omega_k^{\alpha,\theta}, P_{>BN_{k,\beta}}\omega_k^{\beta,\theta,\pm\infty})\|_{N(I_k)} \lesssim (B^{-1/200} + N_{k,\beta}^{-1/200}) \lesssim \theta.$$

And by the similar estimate as (3.8.4), for k large enough, we obtain that

$$\|P_{\leq N_{k,\beta}}\omega_k^{\alpha,\theta}\|_{X^1(I_k)} = \|P_{\leq N_{k,\alpha}\frac{N_{k,\beta}}{N_{k,\alpha}}}\omega_k^{\alpha,\theta}\|_{X^1(I_k)} \lesssim \theta.$$

and $\|P_{>N_{k,\beta}}\omega_k^{\alpha,\theta}\|_{X^1(I_k)} \lesssim 1$.

Consider the remaining part, by the nonlinear estimate (4.6.2) and (3.4.4),

$$\begin{aligned}&\|\mathfrak{D}_{2,1}(\omega_k^{\alpha,\theta}, P_{\leq BN_{k,\beta}}\omega_k^{\beta,\theta,\pm\infty})\|_{N(I_k)} \\ \lesssim&\|\mathfrak{D}_{2,1}(P_{\leq N_{k,\beta}}\omega_k^{\alpha,\theta}, P_{\leq BN_{k,\beta}}\omega_k^{\beta,\theta,\pm\infty})\|_{N(I_k)} + \|\mathfrak{D}_{2,1}(P_{>N_{k,\beta}}\omega_k^{\alpha,\theta}, P_{\leq BN_{k,\beta}}\omega_k^{\beta,\theta,\pm\infty})\|_{N(I_k)} \\ &+ \|\mathfrak{D}_{1,1,1}(P_{>N_{k,\beta}}\omega_k^{\alpha,\theta}, P_{\leq N_{k,\beta}}\omega_k^{\alpha,\theta}, P_{\leq BN_{k,\beta}}\omega_k^{\beta,\theta,\pm\infty})\|_{N(I_k)} \\ \lesssim&\|P_{\leq N_{k,\beta}}\omega_k^{\alpha,\theta}\|_{X^1(I_k)} + \|\mathfrak{D}_{2,1}(P_{>N_{k,\beta}}\omega_k^{\alpha,\theta}, P_{\leq BN_{k,\beta}}\omega_k^{\beta,\theta,\pm\infty})\|_{N(I_k)} \\ \lesssim&\theta + \|P_{\leq BN_{k,\beta}}\omega_k^{\beta,\theta,\pm\infty}\|_{Z'(I_k)} \\ \lesssim&\theta,\end{aligned}$$

where $\limsup_{J\to\infty}\limsup_{k\to\infty}\|P_{\le BN_{k,\beta}}\omega_k^{\beta,\theta,\pm\infty}\|_{Z'(I_k)} = 0$(by a similar estimate with (3.5.32) from extinction lemma.)

Case 4.3: $N_{k,\alpha}\approx N_{k,\beta}$ and $N_{k,\alpha}|x_k^\alpha - x_k^\beta|\to\infty$ as $k\to\infty$.

From Proposition 3.6.2, we can use an equivalent frame of $\mathcal{O}^\alpha$ to adjust $N_{k,\alpha}$ and t_k^α such that $N_{k,\alpha}=N_{k,\beta}$ and $t_k^\alpha = t_k^\beta$.

By the definition of $\omega_k^{\alpha,\theta}$ and $\omega_k^{\beta,\theta,\pm\infty}$, for k large enough, we obtain that $\omega_k^{\beta,\theta}\omega_k^{\alpha,\theta,\pm\infty}\equiv 0$.

Case 4.4: $N_{k,\alpha}\approx N_{k,\beta}$ and $N_{k,\alpha}^2|t_k^\alpha - t_k^\beta|\to\infty$ as $k\to\infty$.

By Proposition 3.6.2, we can adjust $N_{k,\alpha}$ such that $N_{k,\alpha}=N_{k,\beta}:=N_k$.

By the definition of $\omega_k^{\alpha,\theta}$ and $\omega_k^{\beta,\theta,\pm\infty}$, taking k large enough and $N_k^2|t_k^\alpha - t_k^\beta|>T_\theta$, we obtain that

$$\omega_k^{\alpha,\theta}\omega_k^{\beta,\theta,\pm\infty} = \mathbb{1}_{[t_\alpha-\frac{T_\theta}{N_k^2},t_\alpha+\frac{T_\theta}{N_k^2}]}\omega_k^{\alpha,\theta}\omega_k^{\beta,\theta,\pm\infty},$$

and also $\omega_k^{\alpha,\theta,\pm\infty} = P_{\le R_\theta N_k}\omega_k^{\alpha,\theta,\pm\infty}$.

By (3.5.19) and (3.5.21), for any $T\le N_k$, we obtain that

$$\begin{aligned}\|\omega_k^{\beta,\theta,\pm\infty}\|_{L^2(\mathbb{T}^4)} &= \|P_{\le R_\theta N_k}\omega_k^{\beta,\theta,\pm\infty}\|_{L^2(\mathbb{T}^4)}\\ &\lesssim (1+R_\theta)^{-10}\frac{1}{N_k},\end{aligned}\tag{3.8.9}$$

and

$$\sup_{|t-t_k^\beta|\in[TN_k^{-2},T^{-1}]}\|w_k^{\beta,\theta,\pm\infty}\|_{L^\infty(\mathbb{T}^4)}\lesssim T^{-2}R_\theta^4 N_k.\tag{3.8.10}$$

Interpolate (3.8.9) and (3.8.10), we can obtain that

$$\sup_{|t-t_k^\beta|\in[TN_k^{-2},T^{-1}]}\|w_k^{\beta,\theta,\pm\infty}\|_{L^p(\mathbb{T}^4)}\lesssim_{R_\theta} T^{\frac{4}{p}-2}N_k^{1-\frac{4}{p}}.\tag{3.8.11}$$

By choosing $T_k = N_k|t_k^\alpha - t_k^\beta|^{\frac12}\to\infty$ as $k\to\infty$ and using (3.8.11), we obtain that

$$\sup_{t\in[t_k^\alpha-\frac{T_\theta}{N_k^2},t_k^\alpha+\frac{T_\theta}{N_k^2}]}\|\omega_k^{\beta,\theta,\pm\infty}\|_{L^\infty(\mathbb{T}^4)}\lesssim_{R_\theta} T_k^{-2}N_k,\tag{3.8.12}$$

and

$$\sup_{t\in[t_k^\alpha-\frac{T_\theta}{N_k^2},t_k^\alpha+\frac{T_\theta}{N_k^2}]}\|\langle\nabla\rangle\omega_k^{\beta,\theta,\pm\infty}\|_{L^4(\mathbb{T}^4)}\lesssim_{R_\theta} T_k^{-1}N_k. \tag{3.8.13}$$

So by using of Leibniz rule, (3.5.32) (3.8.13) and (3.8.12), we obtain that

$$\begin{aligned}
&\|\mathfrak{D}_{2,1}(\omega_k^{\alpha,\theta},\omega_k^{\beta,\theta,\pm\infty})\|_{N([t_k^\alpha-\frac{T_\theta}{N_k^2},t_k^\alpha+\frac{T_\theta}{N_k^2}])}\\
\lesssim&\|\mathfrak{D}_{2,1}(\omega_k^{\alpha,\theta},\omega_k^{\beta,\theta,\pm\infty})\|_{L^1([t_k^\alpha-\frac{T_\theta}{N_k^2},t_k^\alpha+\frac{T_\theta}{N_k^2}],H^1(\mathbb{T}^4))}\\
\lesssim&\int_{t_k^\alpha-\frac{T_\theta}{N_k^2}}^{t_k^\alpha+\frac{T_\theta}{N_k^2}}\left(\|\mathfrak{D}_2(\langle\nabla\rangle\omega_k^{\alpha,\theta})\|_{L^2(\mathbb{T}^4)}\|\omega_k^{\beta,\theta,\pm\infty}\|_{L^\infty(\mathbb{T}^4)}+\|\mathfrak{D}_2(\omega_k^{\alpha,\theta})\|_{L^4(\mathbb{T}^4)}\|\langle\nabla\rangle\omega_k^{\beta,\theta,\pm\infty}\|_{L^4(\mathbb{T}^4)}\right)dt\\
\lesssim&\int_{t_k^\alpha-\frac{T_\theta}{N_k^2}}^{t_k^\alpha+\frac{T_\theta}{N_k^2}}\left(N_k^2T_k^{-2}R_\theta^8+N_k^2T_k^{-1}R_\theta^3\right)dt\\
\lesssim&T_k^{-1}T_\theta R_\theta^8\to 0 \text{ as } k\to\infty.
\end{aligned}$$

□

CHAPTER 4

§II: ALMOST SURE WELL-POSEDNESS FOR THE CUBIC NLS IN THE SUPER-CRITICAL REGIME ON $\mathbb{T}^d$, $d \geq 3$

In this chapter we prove the results described above for Part II of this thesis. Namely, in this chapter, we study the random data Cauchy problem of the cubic NLS on $\mathbb{T}^d$, $d \geq 3$ in the super-critical regime. The probabilistic approach helps us construct local-in-time well-posedness for the cubic NLS local-in-time well-posedness in high probability with super-critical random initial data.

4.1 Introduction

We consider the Cauchy initial value problem,

$$\begin{cases} iu_t + \Delta u = \rho u|u|^2, \qquad \rho = \pm 1, \qquad x \in \mathbb{T}^d \ (d \geq 3) \\ u(0, x) = \phi^\omega(x), \end{cases} \tag{4.1.1}$$

where the randomized initial data is

$$\phi^\omega(x) = \sum_{n \in \mathbb{Z}^d} \frac{g_n(\omega)}{\langle n \rangle^{d-1-\alpha}} e^{in \cdot x}, \text{ where } \langle n \rangle = \sqrt{1 + |n|^2}, \tag{4.1.2}$$

and $(g_n(\omega))_{n \in \mathbb{Z}^d}$ is a sequence of complex i.i.d. centered Gaussian random variables on a probability space $(\Omega, A, \mathbb{P})$.

This random data approach to LWP first appeared in Bourgain's work on the invariance of the Gibbs measure associated to the cubic NLS on $\mathbb{T}^2$. Once this probabilistic local well-posedness was established, Bourgain used similar arguments as in his 1D paper [8] to further show almost sure global well-posednes and invariance of the Gibbs measure. Later, Burq-Tzvetkov [22] obtained similar random data local well-posedness in the context of the cubic nonlinear wave equation (NLW) on a three dimensional compact Riemannian manifold and almost sure global well-posedness and invariance of the Gibbs measure in the radial case [23].

The random data approach to local and global well-posedness has now been used to tackle various nonlinear dispersive and wave equations (for references, NLS: [89, 14, 15, 16, 37, 32, 91, 69, 5, 79, 92]; NLW: [113, 24, 86, 85, 101, 93, 97]; Navier-Stokes equations: [109, 36, 90]; Benjamin-Ono equations: [38]) in different manifolds ($\mathbb{R}^d$, $\mathbb{T}^4$ or $\mathbb{S}^d$ etc.) and obtained probabilistic LWP or almost sure GWP results.

Remark 4.1.1. Consider initial data $\phi \in H^{s_c-\alpha-\epsilon}(\mathbb{T}^4)$ for any $\epsilon > 0$ of the form

$$\phi(x) = \sum_{n\in\mathbb{Z}^d} \frac{1}{\langle n\rangle^{d-1-\alpha}} e^{in\cdot x}. \tag{4.1.3}$$

If we replace the Fourier coefficients of (4.1.3) with randomized coefficients $\frac{g_n(\omega)}{\langle n\rangle^{d-1-\alpha}}$, then the randomization of (4.1.3) becomes the random initial data (4.1.2) of (4.1.1).

Our main result can be stated as follows,

Theorem 4.1.2 (Main Theorem)**.** *Suppose $d \geq 3$ and*

$$s_r(d) = \begin{cases} \frac{1}{7} & d = 3 \\ \frac{4}{19} & d = 4 \\ \frac{1}{4} & d \geq 5. \end{cases} \tag{4.1.4}$$

Let $0 \leq \alpha < s_r(d)$, $s \in [s_c, s_c + s_r(d) - \alpha)$. Then there exists $\delta_0 > 0$ and $r = r(s,\alpha) > 0$ such that for any $0 < \delta < \delta_0$, there exists $\Omega_\delta \in A$ with

$$\mathbb{P}(\Omega_\delta^c) < e^{-\frac{1}{\delta^r}},$$

and for each $\omega \in \Omega_\delta$ there exists a unique solution u of (4.1.1) in the space

$$S(t)\phi^\omega + X^s([0,\delta])_{dist},$$

where $S(t)\phi^\omega$ is the linear evolution of the initial data ϕ^ω given by (4.1.2).

Here we denoted by $X^s([0,\delta])_{\text{dist}}$ the metric space $(X^s([0,\delta]), \text{dist})$ where dist is the metric defined by (4.1.7) and $X^s([0,\delta])$ is the adapted atomic space introduced in the Definition 3.3.5.

Remark 4.1.3. We also prove the analog of Main Theorem in $X^{s,b}$ (Theorem 4.7.1) instead of the atomic space X^s in the Section 7, but we hold the theorem in $X^{s,b}$ only when $s \in (s_c, s_c + s_r(d) - \alpha]$ (the proof of Theorem 4.7.1 fails when $s = s_c$). If we only consider the statement of theorems, for some $s > s_c$, the solution space $S(t)\phi^\omega + X^{s,b}([0,\delta])_{\text{dist}}$ is indeed in the space $S(t)\phi^\omega + X^{s_c,b}([0,\delta])_{\text{dist}}$. However, the proof of $s = s_c$ case is still important in the sense that we obtain the nonlinear estimate at the regularity of s_c. Especially in the case of $s_c = 1$, the nonlinear estimate at the regularity of s_c would be necessary if we try to control the energy in a long-time term.

To prove Theorem 4.1.2, first we consider the initial value problem below,

$$\begin{cases} iv_t + \Delta v = \mathcal{N}(v), \qquad \rho = \pm 1, \qquad x \in \mathbb{T}^d \\ v(0,x) = \phi^\omega(x), \end{cases} \tag{4.1.5}$$

where

$$\mathcal{N}(v_1, v_2, v_3) := \rho(v_1 v_2 v_3 - 2v_1 \int_{\mathbb{T}^d} v_2 v_3 dx) = \mathcal{N}_1(v_1, v_2, v_3) + \mathcal{N}_2(v_1, v_2, v_3), \tag{4.1.6}$$

and set $\mathcal{N}(v) := \mathcal{N}(v, \overline{v}, v)$. The nonlinearity $\mathcal{N}(v)$ in (4.1.5) is the same as in Bourgain's periodic 2D cubic NLS paper [13] and that the $N(v)$ there is the *Wick ordered nonlinearity.*

Suppose $\beta_v(t) = 2\int_{\mathbb{T}^d} |v|^2 dx$ and define $u(t,x) := e^{-i\rho\beta_v(s)ds} v(t,x)$. We observe that u solve IVP (4.1.1). Now suppose that one obtains well-posedness for the IVP (4.1.5) in a certain Banach space $(X, \|\cdot\|)$ then one can transfer those results to the IVP (4.1.1) by using a metric space $X_{\text{dist}} := (X, \text{dist})$ where

$$d(u,v) := \|e^{i\rho\beta_u(s)ds} u(t,x) - e^{i\rho\beta_v(s)ds} v(t,x)\|. \tag{4.1.7}$$

We define

$$v_0^\omega = S(t)\phi^\omega(x), \tag{4.1.8}$$

and $w(x,t)$ solves the following the IVP (4.1.9), then we know that $v = v_0^\omega + w$ solves the IVP (4.1.5) which is the gauged NLS we want to solve.

$$\begin{cases} iw_t + \Delta w = \mathcal{N}(w + v_0^\omega), & x \in \mathbb{T}^d \\ w(0,x) = 0, \end{cases} \tag{4.1.9}$$

where $\mathcal{N}(\cdot)$ was defined in (4.1.6).

We are now ready to state the almost sure well-posedness result for the IVP (4.1.9) which implies the main theorem (Theorem 4.1.2).

Theorem 4.1.4. *Suppose $d \geq 3$ and $s_r(d)$ is defined as (4.1.4). Let $0 \leq \alpha < s_r(d)$, $s \in [s_c, s_c + s_r(d) - \alpha)$. Then there exists $\delta_0 > 0$ and $r = r(s,\alpha) > 0$ such that for any $0 < \delta < \delta_0$, there exists $\Omega_\delta \in A$ with*

$$\mathbb{P}(\Omega_\delta^c) < e^{-\frac{1}{\delta^r}},$$

and for each $\omega \in \Omega_\delta$ there exists a unique solution w of (4.1.9) in the space $X^s([0,\delta]) \cap C([0,\delta], H^s(\mathbb{T}^4))$.

4.2 Probabilistic set up

Lemma 4.2.1. *Let $\{g_n(\omega)\}_{n\in\mathbb{Z}^d}$ be a sequence of complex i.i.d. mean zero Gaussian random variables on a probability space $(\Omega, A, \mathbb{P})$. Then given $\epsilon, \delta > 0$, there exists a subset $\Omega_\delta \subset \Omega$ satisfying $\mathbb{P}(\Omega_\delta^c) \leq e^{-\frac{1}{\delta^\epsilon}}$, such that*

$$|g_n(\omega)| \lesssim \frac{1}{\delta^\epsilon} \log(\langle n \rangle + 1).$$

Proof. For each n and a small $\epsilon > 0$, we have a constant C,

$$\mathbb{E}e^{|g_n(\omega)|} \leq C.$$

Set $M = \frac{1}{\delta^\epsilon}$, and the we have

$$\mathbb{E}\left|\frac{e^{|g_n(\omega)|}}{e^M}\right| \leq Ce^{-\frac{1}{\delta^\epsilon}}$$

Then we obtain,

$$Ce^{-\frac{1}{\delta^\epsilon}} > \mathbb{E}\left|\frac{e^{|g_n(\omega)|}}{e^M}\right| \geq \sum_{j\in\mathbb{Z}^d} \mathbb{P}(e^{|g_j(\omega)|} \geq e^M \langle j\rangle^d) = \sum_{j\in\mathbb{Z}^d} \mathbb{P}(|g_j(\omega)| \geq \frac{1}{\delta^\epsilon} + d\log\langle j\rangle).$$

Exclude $\Omega_\delta^c := \cup_j\{|g_j(\omega)| \geq \frac{1}{\delta^\epsilon} + d\log\langle j\rangle\}$ from Ω, for all $\omega \in \Omega_\delta$, we have

$$|g_n(\omega)| \leq \frac{1}{\delta^\epsilon} + d\log\langle n\rangle \lesssim \frac{1}{\delta^\epsilon}\log(\langle n\rangle + 1), \text{for } n \in \mathbb{Z}^d.$$

with $\mathbb{P}(\Omega_\delta^c) < Ce^{-\frac{1}{\delta^\epsilon}}$.

□

Lemma 4.2.2 (Lemma 3.1 in [91]). *Let $\{g_n(\omega)\}_n$ be a sequence of complex i.i.d. mean zero Gaussian random variables on a probability space $(\Omega, A, \mathbb{P})$ and $(c_n) \in \ell^2$. Define*

$$F(\omega) := \sum_n c_n g_n(\omega). \tag{4.2.1}$$

Then there exists $C > 0$ such that for every $\lambda > 0$ we have

$$\mathbb{P}(\{\omega : |F(\omega)| > \lambda\}) \leq \exp(\frac{-C\lambda^2}{\|F(\omega)\|_{L^2(\Omega)}^2}). \tag{4.2.2}$$

As a consequence there exists $C > 0$ such that for every $q \geq 2$ and every $(c_n) \in \ell^2$,

$$\|\sum_n c_n g_n(\omega)\|_{L^q(\Omega)} \leq C\sqrt{q}(\sum_n |c_n|^2)^{\frac{1}{2}}.$$

Lemma 4.2.3 (Lemma 3.5 in [31]). *Let $f^\omega(x,t) = \sum c_n g_n(\omega)e^{i(n\cdot x+|n|^2t|)}$. Then, for $p, q \geq 2$, there exists $\delta_0, c, C > 0$ such that*

$$\mathbb{P}(\|f^\omega\|_{L_t^pL_x^q(\mathbb{T}^4\times[0,\delta])} > \lambda) < C\exp(-\frac{c\lambda^2}{\delta^{\frac{2}{p}}\|c_n\|_{l_n^2}^2}) \tag{4.2.3}$$

for $\delta < \delta_0$.

Proof. By Lemma 4.2.2, there exists $C > 0$ such that

$$\|\sum_n c_n g_n(\omega)\|_{L^r(\Omega)} \leq C\sqrt{r}(\sum_n |c_n|^2)^{\frac{1}{2}},$$

for every $r \geq 2$. By Minkowski integral inequality, we have

$$\begin{aligned}\mathbb{E}(\|f^\omega\|^r_{L^p_t L^q_x(\mathbb{T}^4\times[0,\delta])})^{\frac{1}{r}} &\leq \big\|\|f^\omega\|_{L^r(\Omega)}\big\|_{L^p_t L^q_x(\mathbb{T}^4\times[0,\delta])} \\ &\leq C\sqrt{r}\big\|\|c_n\|_{l^2_n}\big\|_{L^p_t L^q_x(\mathbb{T}^4\times[0,\delta])} \\ &\leq C\sqrt{r}\delta^{\frac{1}{p}}\|c_n\|_{l^2_n}\end{aligned}$$

for $r \geq p$. By Chebyshev's Inequality, we have

$$\mathbb{P}(\|f^\omega\|_{L^p_t L^q_x(\mathbb{T}^4\times[0,\delta])} > \lambda) < C^r\lambda^{-r}r^{\frac{r}{2}}\delta^{\frac{r}{p}}\|c_n\|^r_{l^2_n}. \tag{4.2.4}$$

If $\lambda < \sqrt{p}C\delta^{-\frac{1}{p}}e\|c_n\|_{l^2_n}$, then (4.2.3) easily holds.

If $\lambda \geq \sqrt{p}C\delta^{\frac{1}{p}}e\|c_n\|_{l^2_n}$, then we set

$$r = [\frac{\lambda}{Ce\delta^{\frac{1}{p}}\|c_n\|_{l^2_n}}]^2 \quad (\geq p).$$

So that (4.2.4) yields (4.2.3). □

Corollary 4.2.4. *Let $p, q \geq 2$, and $P_N R = \sum_{|n|\sim N}\frac{g_n(\omega)}{\langle n\rangle^{d-1-\alpha}}e^{i(n\cdot x+|n|^2t|)}$, where N is a dyadic coordinate. There exists $A \subset \Omega$, C and $c > 0$, with $\mathbb{P}(A) < Ce^{-\frac{1}{\delta^c}}$, such that for each $\omega \in A^c$ and each dyadic coordinate N, we have*

$$\|P_N R\|_{L^p_t L^q_x([0,\delta]\times\mathbb{T}^d)} \leq \delta^c\frac{\log N}{N^{s_c-\alpha}}.$$

for $\delta < \delta_0$.

Proof. By Lemma 4.2.3, for each dyadic coordinate N, set $\lambda = \delta^{\frac{1}{2p}}\log N\|P_N R\|_{L^2_x}$, there exists $A_N \subset \Omega$, such that for $\omega \in A^c_N$, with $\mathbb{P}(A_N) < C\exp(-\frac{\log N}{\delta^{\frac{1}{p}}})$ we obtain that

$$\|P_N R\|_{L^p_t L^q_x([0,\delta]\times\mathbb{T}^d)} \leq \delta^{\frac{1}{2p}}\frac{\log N}{N^{s_c-\alpha}},$$

since $\|P_N R\|_{L^2_x} \sim \frac{1}{N^{s_c-\alpha}}$.

Set $A = \cup_N A_N$, and $c = \frac{1}{p}$, then we have

$$\begin{aligned}
\mathbb{P}(A) &\leq \sum_N \mathbb{P}(A_N) \leq \sum_N C\exp(-\frac{(\log N)^2}{\delta^c}) \\
&\leq \sum_{k=1}^{\infty} Ce^{-\frac{k^2}{\delta^c}} \leq \sum_{k=1}^{\infty} Ce^{-\frac{k}{\delta^{\frac{1}{p}}}} \\
&= \frac{Ce^{-\frac{1}{\delta^c}}}{1-e^{-\frac{1}{\delta^c}}} < 2Ce^{-\frac{1}{\delta^c}}.
\end{aligned}$$

when δ is small enough. $\square$

Lemma 4.2.5 (Proposition 3.1 in [91]). *For fixed $n \in \mathbb{Z}^d$, let*

$$D(n) = \{(n_1, n_2, n_3) \in \mathbb{Z}^d \times \mathbb{Z}^d \times \mathbb{Z}^d : n = n_1 - n_2 + n_3,\ n_2 \neq n_1,\ n_2 \neq n_3,\ n_1 \neq n_3\}.$$

Given $\{c_{n_1,n_2,n_3}\}_{l_2(D(n))}$, define F_n by

$$F_n := \sum_{D(n)} c_{n_1,n_2,n_3} g_{n_1}(\omega)\overline{(g_{n_2})}(\omega) g_{n_3}(\omega).$$

Then there exists $C > 0$ such that for every $\lambda > 0$ we have

$$\mathbb{P}(\{\omega : |F_n(\omega)| > \lambda\}) \leq \exp\left(\frac{-C\lambda^{2/3}}{\|F_n(\omega)\|_{L^2(\Omega)}^{2/3}}\right).$$

4.3 Adapted function spaces and transfer principle

In this section, we first recall the definitions of adapted atomic functions which we already define in Chapter 2: U^p (Definition 3.3.1), V^p (Definition 3.3.2), U^p_Δ and V^p_Δ (Definition 3.3.4), X^s (Definition 3.3.5) and Y^s (Definition 3.3.6). Here X^s is the function space where the solution exists. Also we hold the embedding properties Remark 3.3.3 and (3.3.5).

Furthermore, we define the time localized version of the function spaces: $U^p_\Delta(I)$, $V^p_\Delta(I)$, $X^s(I)$ and $Y^s(I)$.

Definition 4.3.1 (The corresponding restriction spaces to a time interval I)**.** For $p \geq 1$ and a bounded time interval I. Define $U^p(I)$, $V^p(I)$, $X^s(I)$ and $Y^s(I)$ with the restriction norms:

$$\|u\|_{U^p(I)} = \inf\{\|\widetilde{u}\|_{U^p} : \widetilde{u}(t) = u(t), t \in I\} \text{ and } \|u\|_{V^p(I)} = \inf\{\|\widetilde{u}\|_{V^p} : \widetilde{u}(t) = u(t), t \in I\};$$

$$\|u\|_{X^s(I)} = \inf\{\|\widetilde{u}\|_{X^s} : \widetilde{u}(t) = u(t), t \in I\} \text{ and } \|u\|_{Y^s(I)} = \inf\{\|\widetilde{u}\|_{Y^s} : \widetilde{u}(t) = u(t), t \in I\}.$$

Proposition 4.3.2 (Proposition 2.19 in [60])**.** *Let $T_0 : L_x^2 \times \cdots \times L_x^2 \to L^1_{x,loc}(\mathbb{T}^d)$ be an m-linear operator. Assume that for some $1 \leq p \leq \infty$*

$$\|T_0(e^{it\Delta}\phi_1, \cdots, e^{it\Delta}\phi_m)\|_{L^p(\mathbb{R}\times\mathbb{T}^d)} \lesssim \prod_{i=1}^m \|\phi_i\|_{L^2(\mathbb{T}^d)}. \tag{4.3.1}$$

Then, there exists an extension $T : U^p_\Delta \times \cdots \times U^p_\Delta \to L^p(\mathbb{R} \times \mathbb{T}^d)$ satisfying

$$\|T(u_1, \cdots, u_m)\|_{L^p(\mathbb{R}\times\mathbb{T}^d)} \lesssim \prod_{i=1}^m \|u_i\|_{U^p_\Delta}; \tag{4.3.2}$$

and such that $T(u_1, \cdots, u_m)(t, \cdot) = T_0(u_1(t), \cdots, u_m(t))(\cdot)$, a.e.

Proposition 4.3.3. *Let $T_0 : L_x^2 \times \cdots \times L_x^2 \to L^1_{x,loc}(\mathbb{T}^d)$ be m-linear operator. Assume that for some bounded time interval $I \subset \mathbb{R}$, and $1 < q \leq \infty$*

$$\left|\int_J \int_{\mathbb{T}^d} T_0(e^{it\Delta}\phi_1, \cdots, e^{it\Delta}\phi_m)\, dxdt\right| \lesssim |J|^{\frac{1}{q}} \prod_{i=1}^m \|\phi_i\|_{L^2(\mathbb{T}^d)}, \qquad \textit{for any } J \subset I. \tag{4.3.3}$$

Then, for $1 \leq p < \infty$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$, there exists an extension $T : U^p_\Delta \times \cdots \times U^p_\Delta \to L^1_{x,t,loc}(I \times \mathbb{T}^d)$ satisfying

$$\left|\int_I \int_{\mathbb{T}^d} T(u_1, \cdots, u_m)\, dxdt\right| \lesssim |I|^{\frac{1}{q}} \prod_{i=1}^m \|u_i\|_{U^p_\Delta(I)}; \tag{4.3.4}$$

and such that $T(u_1, \cdots, u_m)(t, \cdot) = T_0(u_1(t), \cdots, u_m(t))(\cdot)$, a.e.

Remark 4.3.4. In Hadac-Herr-Koch's paper [60], they derived a "transfer principle" as Proposition 4.3.2, which consider the L^p norm of the multilinear operator T over the whole time space $\mathbb{R}$, while Proposition 4.3.3 focus on the integral in time (or actually L^1 norm is also fine) on a finite time interval I. By a stronger assumption (which gives some better estimates on each small intervals J), Proposition 4.3.3 somehow takes advantage of the finite time interval to improve the bounds from U^1 norm to U^p. In the following proof of Proposition 4.5.1, the **Case B** heavily relies on Proposition 4.3.3.

Proof. By multi-linearity of T_0 and definition of U^p norm, it will suffice to show that (4.3.4) is true for all U^p_Δ-atoms u_i. Let $a_1, \cdots, a_m$ be U^p_Δ-atoms given as

$$a_i = \sum_{k_i=1}^{K_i} \mathbb{1}_{I_{k_i,i}} e^{it\Delta}\phi_{k_i-1,i}, \quad \text{for } i = 1, \cdots, m.$$

where $I_{k_i,i} = [t_{k_i-1,i}, t_{k_i,i})$, and such that

$$\sum_{k_i=1}^{K_i} \|\phi_{k_i-1,i}\|^p_{L^2_x} = 1. \tag{4.3.5}$$

Then, by (4.3.3), Cauchy-Schwarz inequality and by induction,

$$\begin{aligned}
&\left|\int_I \int_{\mathbb{T}^d} T(a_1, \cdots, a_m)(t)\, dxdt\right| \\
\leq& \sum_{\substack{1\leq k_1\leq K_1 \\ \cdots \\ 1\leq k_m\leq K_m}} \left|\int_{\cap_{i=1}^m I_{k_i,i}} \int_{\mathbb{T}^d} T_0(e^{it\Delta}\phi_{k_1-1,1}, \cdots, e^{it\Delta}\phi_{k_m-1,m})\, dxdt\right| \\
\leq& \sum_{\substack{1\leq k_1\leq K_1 \\ \cdots \\ 1\leq k_m\leq K_m}} |\cap_{i=1}^m I_{k_i,i}|^{\frac{1}{q}} \prod_{i=1}^m \|\phi_{k_i-1,i}\|_{L^2_x} \qquad (4.3.6) \\
\leq& \sum_{\substack{1\leq k_2\leq K_2 \\ \cdots \\ 1\leq k_m\leq K_m}} \prod_{i=2}^m \|\phi_{k_i-1,i}\|_{L^2_x} \left(\sum_{1\leq k_1\leq K_1} |\cap_{i=1}^m I_{k_i,i}|\right)^{\frac{1}{q}} \left(\sum_{1\leq k_1\leq K_1} \|\phi_{k_1-1,1}\|^p_{L^2_x}\right)^{\frac{1}{p}} \qquad (4.3.7)
\end{aligned}$$

For fixed $k_2, k_3, \cdots, k_m$, since

$$I_{k_2,2} \cap \cdots \cap I_{k_m,m} = \cup_{1\leq k_1\leq K_1} \left(I_{k_1,1} \cap I_{k_2,2} \cap \cdots \cap I_{k_m,m}\right),$$

we have

$$\left(\sum_{1\leq k_1\leq K_1} |\cap_{i=1}^m I_{k_i,i}|\right)^{\frac{1}{q}} = |\cap_{i=2}^m I_{k_i,i}|^{\frac{1}{q}}. \tag{4.3.8}$$

Based on (4.3.5) (4.3.8) (4.3.7), we obtain that

$$
\begin{aligned}
&|\int_I\int_{\mathbb{T}^d} T(a_1,\cdots,a_m)(t)\,dxdt| \\
\le& \sum_{\substack{1\le k_2\le K_2\\ \cdots\\ 1\le k_m\le K_m}} \prod_{i=2}^m \|\phi_{k_i-1,i}\|_{L^2_x} \left(\sum_{1\le k_1\le K_1} |\cap_{i=1}^m I_{k_i,i}|\right)^{\frac{1}{q}} \left(\sum_{1\le k_1\le K_1} \|\phi_{k_1-1,1}\|_{L^2_x}^p\right)^{\frac{1}{p}} \\
\le& \sum_{\substack{1\le k_2\le K_2\\ \cdots\\ 1\le k_m\le K_m}} |\cap_{i=2}^m I_{k_i,i}|^{\frac{1}{q}} \prod_{i=2}^m \|\phi_{k_i-1,i}\|_{L^2_x}
\end{aligned}
$$

If we iterate (4.3.6) (4.3.7) on $k_2, k_3, \cdots, k_m$, finally we obtain that

$$
\begin{aligned}
&|\int_I\int_{\mathbb{T}^d} T(a_1,\cdots,a_m)(t)\,dxdt| \\
\le& \sum_{\substack{1\le k_2\le K_2\\ \cdots\\ 1\le k_m\le K_m}} |\cap_{i=2}^m I_{k_i,i}|^{\frac{1}{q}} \prod_{i=2}^m \|\phi_{k_i-1,i}\|_{L^2_x} \\
\le& \sum_{\substack{1\le k_3\le K_3\\ \cdots\\ 1\le k_m\le K_m}} |\cap_{i=3}^m I_{k_i,i}|^{\frac{1}{q}} \prod_{i=3}^m \|\phi_{k_i-1,i}\|_{L^2_x} \\
&\cdots \\
\le& |I|^{\frac{1}{q}}.
\end{aligned}
$$

So we obtain (4.3.4). □

Proposition 4.3.5 (Proposition 2.20 in [60]). *Let $q_1,...,q_m > 2$ ($m \in \mathbb{N}$), E be a Banach space and $T : U^{q_1}_\Delta \times \cdots \times U^{q_m}_\Delta \to E$ be a bounded m-linear operator with*

$$\|T(u_1,\cdots,u_m)\|_E \le C\prod_{i=1}^m \|u_i\|_{U^{q_i}_\Delta}. \tag{4.3.9}$$

And also assume there exists $0 < C_2 < C$ such that we hold,

$$\|T(u_1,\cdots,u_m)\|_E \le C_2\prod_{i=1}^m \|u_i\|_{U^2_\Delta}. \tag{4.3.10}$$

Then, T satisfies the estimate

$$\|T(u_1,\cdots,u_m)\|_E \le C_2(\log\frac{C}{C_2}+1)\prod_{i=1}^m \|u_i\|_{V^2_\Delta}, \quad u_i \in V^2_{rc},\ i=1,...,m. \tag{4.3.11}$$

To make the proposition 4.3.5 suitable for the following nonlinear estimates, we also need to introduce a similar interpolation proposition for the integral of T over a time interval I as following:

Proposition 4.3.6. *Let $q_1, ..., q_m > 2$ ($m \in \mathbb{N}$), and $T : U_\Delta^{q_1} \times \cdots \times U_\Delta^{q_m} \to L^1_{x,t,loc}(I \times \mathbb{T}^d)$ be a m-linear operator with*

$$\left| \int_I \int_{\mathbb{T}^d} T(u_1, \cdots, u_m)\, dxdt \right| \leq C \prod_{i=1}^{m} \|u_i\|_{U_\Delta^{q_i}}. \tag{4.3.12}$$

And also assume there exists $0 < C_2 < C$ such that we hold,

$$\left| \int_I \int_{\mathbb{T}^d} T(u_1, \cdots, u_m)\, dxdt \right| \leq C_2 \prod_{i=1}^{m} \|u_i\|_{U_\Delta^2}. \tag{4.3.13}$$

Then, T satisfies the estimate

$$\left| \int_I \int_{\mathbb{T}^d} T(u_1, \cdots, u_m)\, dxdt \right| \leq C_2 (\log \frac{C}{C_2} + 1) \prod_{i=1}^{m} \|u_i\|_{V_\Delta^2}, \quad u_i \in V^2_{rc,\Delta},\ i = 1, ..., m. \tag{4.3.14}$$

Proof. The proof is almost the same as that of Proposition 2.20 in [60], since $\left|\int_I \int_{\mathbb{T}^d} T(u_1, \cdots, u_m)\, dxdt\right|$ is m-sublinear for $u_1, \cdots, u_m$ as $\|T(u_1, \cdots, u_m)\|_E$ in Prop 4.3.5. □

Definition 4.3.7 (Duhamel operator)**.** Let $f \in L^1_{loc}([0, \infty), L^2(\mathbb{T}^4))$, and we define the Duhamel operator $\mathcal{I}$

$$\mathcal{I}(f)(t) := \int_0^t e^{i(t-t')\Delta} f(t')dt', \tag{4.3.15}$$

for $t > 0$ and $\mathcal{I}(f)(t) := 0$ otherwise.

Proposition 4.3.8 (Proposition 2.11 in [67])**.** *Let $s > 0$, and a time interval $I = [0, \delta]$. For $f \in L^1(I, H^s(\mathbb{T}^4))$ we have $\mathcal{I}(f) \in X^s(I)$ and*

$$\|\mathcal{I}(f)\|_{X^s(I)} \leq \sup_{\|v\|_{Y^{-s}(I)}=1} \left| \int_0^\delta \int_{\mathbb{T}^d} f(t,x)\overline{v(t,x)} dxdt \right|. \tag{4.3.16}$$

4.4 Auxiliary lemmata

In Chapter 2, we already state the Strichartz estimates (Lemma 2.3.1) on $\mathbb{T}^4$. Let us now state the Strichartz estimates on d-dimensional tori $\mathbb{T}^d$:

Proposition 4.4.1 (Strichartz estimate[7][17]). *Let $p > p_c$, where $p_c = \frac{2(d+2)}{d}$. For all $N \geq 1$ we have*

$$\|P_N e^{it\Delta}\phi\|_{L^p_{x,t}(\mathbb{T}\times\mathbb{T}^d)} \lesssim N^{\frac{d}{2}-\frac{d+2}{p}}\|P_N\phi\|_{L^2_x(\mathbb{T}^d)}, \tag{4.4.1}$$

$$\|P_C e^{it\Delta}\phi\|_{L^p_{x,t}(\mathbb{T}\times\mathbb{T}^d)} \lesssim N^{\frac{d}{2}-\frac{d+2}{p}}\|P_C\phi\|_{L^2_x(\mathbb{T}^d)}, \tag{4.4.2}$$

$$\|P_N u\|_{L^p_{x,t}(\mathbb{T}\times\mathbb{T}^d)} \lesssim N^{\frac{d}{2}-\frac{d+2}{p}}\|P_C u\|_{U^p_\Delta L^2}, \tag{4.4.3}$$

$$\|P_C u\|_{L^p_{x,t}(\mathbb{T}\times\mathbb{T}^d)} \lesssim N^{\frac{d}{2}-\frac{d+2}{p}}\|P_C u\|_{U^p_\Delta L^2}, \tag{4.4.4}$$

where C is a cube in $\mathbb{Z}^d$ with sides parallel to the axis of side length N.

Note that the last inequality (4.7.6)(4.7.7) follows (4.4.1)(4.4.2) and Proposition 4.3.2.

Lemma 4.4.2 (Integer lattice counting estimates [7]). *Denote the number of set $\{(X_1, \cdots, X_d) \in \mathbb{Z}^d : X_1^2 + \cdots + X_d^2 = A\}$ by $C_{d,A}$. Then $C_{d,A}$ can be bounded by*

$$\begin{array}{ll} A^{\epsilon} & (d=2) \\ A^{\frac{1}{2}+\epsilon} & (d=3) \\ A^{1+\epsilon} & (d=4) \\ A^{\frac{d-2}{2}} & (d>4) \end{array}.$$

where ϵ is an arbitrary small positive number.

By Lemma 4.4.2, it's easy to obtain the following lattice counting lemmas.

Lemma 4.4.3. *Let S_R be a sphere of radius R, B_r be a ball of radius r, and $\mathcal{P}$ be a plane in $\mathbb{R}^d$ for $d \geq 3$. Then*

$$|\mathbb{Z}^d \cap S_R| \leq R^{d-2+\epsilon}, \tag{4.4.5}$$

$$|\mathbb{Z}^d \cap B_r \cap \mathcal{P}| \leq r^{d-1}, \tag{4.4.6}$$

where $|\cdot|$ denotes cardinality and ϵ is an arbitrary small positive number.

Lemma 4.4.4. *Consider the set*

$$S = \{(n_1, n_2, n_3) \in \mathbb{Z}^d \times \mathbb{Z}^d \times \mathbb{Z}^d : n_2 \neq n_1, n_3, |n_i| \sim N_i \text{ for } i = 1, 2, 3, \text{ and } \langle n_2 - n_1, n_2 - n_3 \rangle = \mu\}.$$

For a fixed n_2, $|S(n_2)| \lesssim N_1^{d-1} N_3^{d-1} \min\{N_1, N_3\}^\epsilon$, where $|\cdot|$ denotes cardinality and ϵ is an arbitrary small positive number.

Lemma 4.4.5 (Bounds of Fourier coefficients of Characteristic function)**.** *Consider $\mathbb{1}_{[a,b]}(t)$ as a function in $L^2([0, 2\pi])$ where $a, b \in [0, 2\pi]$, then the Fourier coefficients $|\mathcal{F}(\mathbb{1}_{[a,b]})(k)| \leq \frac{2}{|k|}$ for all $k \in \mathbb{Z}$.*

Proof. $|\mathcal{F}(\mathbb{1}_{[a,b]})(k)| = |\frac{e^{ikb} - e^{ika}}{ik}| \leq \frac{2}{|k|}$. □

Lemma 4.4.6 (Lemma 6.3 in [91])**.** *Let $\mathcal{A} = (A_{ik})_{\substack{1 \leq i \leq N \\ 1 \leq k \leq M}}$ be an $N \times M$ matrix. Then*

$$\|\mathcal{A}\mathcal{A}^*\| \leq \max_{1 \leq j \leq N} \sum_{k=1}^{M} |A_{jk}|^2 + \left(\sum_{i \neq j} \left| \sum_{k=1}^{M} A_{ik} \overline{A_{jk}} \right|^2 \right)^{\frac{1}{2}}$$

where $\|\cdot\|$ is the 2-norm.

4.5 Estimate for nonlinear term

To estimate $\|\mathcal{I}(\mathcal{N}(w + v_0^\omega))\|_{X^s([0,\delta])}$, by Prop 4.3.8 the we just need to bound the integral $\int_0^\delta \int_{\mathbb{T}^4} \mathcal{N}(w + v_0^\omega)\overline{u^{(0)}} dxdt$, where $\delta < 1$. This section will focus on estimating this integral.

Proposition 4.5.1. *Suppose $d \geq 3$ and $s_r(d)$ is given in (4.1.4). Let $0 \leq \alpha < s_r(d)$, $s \in [s_c, s_c + s_r(d) - \alpha)$, $r > 0$, $0 \leq \delta < 1$, and $I = [0, \delta]$. There exist $\Omega_\delta \subset \Omega$ with $\mathbb{P}(\Omega_\delta^c) < e^{-1/\delta^r}$, and $c > 0$, such that we obtain that*

$$\begin{aligned}
&\left| \int_0^\delta \int_{\mathbb{T}^d} \mathcal{N}(w^{(1)} + v_0^\omega, \overline{w^{(2)} + v_0^\omega}, w^{(3)} + v_0^\omega)\overline{u^{(0)}}\, dxdt \right| \\
\lesssim &\|u^{(0)}\|_{Y^{-s}(I)} \left(\delta^{c \min\{1, s - s_c\}} \|w^{(1)}\|_{X^s(I)} \|w^{(2)}\|_{X^s(I)} \|w^{(3)}\|_{X^s(I)} + \delta^c \sum_{\substack{S_J \subset \{1,2,3\} \\ J \neq \{1,2,3\}}} \prod_{j \in S_J} \|w^{(j)}\|_{X^s(I)} \right),
\end{aligned}$$

where v_0^ω is defined (4.1.8), $u^{(0)} \in Y^{-s}(I)$ and $w^{(i)} \in X^s(I)$ for $i = 1, 2, 3$. (when the subset $S_J = \emptyset$, $\prod_{j \in S_J} \|w^{(j)}\|_{X^s(I)} = 1$.)

To show Proposition 4.5.1, it is clear that $\mathcal{N}(w + v_0^\omega)$ can be expressed as

$$\sum_{u^{(i)} \in \{w, v_0^\omega\}, i=1,2,3} \mathcal{N}(u^{(1)}, \overline{u^{(2)}}, u^{(3)}). \tag{4.5.1}$$

We dyadically decompose

$$u_i = P_{N_i} u^{(i)}, \text{ where } i \in \{0, 1, 2, 3\}.$$

By the symmetry, in the following of this chapter we suppose that $N_1 \geq N_2 \geq N_3$, and we need to estimate the following integral case by case,

$$\int_0^\delta \int_{\mathbb{T}^d} \mathcal{N}(\widetilde{u}_1, \widetilde{u}_2, \widetilde{u}_3) \overline{u}_0 \, dxdt, \tag{4.5.2}$$

where $\widetilde{u_i} = u_i$ or $\overline{u_i}$ and only one of $\widetilde{u}_i$ can be $\overline{u_i}$.

Remark 4.5.2. To make the integral $\int_0^\delta \int_{\mathbb{T}^d} \widetilde{u}_1 \widetilde{u}_2 \widetilde{u}_3 \overline{u}_0 \, dxdt$ (which is the main term of (4.5.2)) nontrivial, the two highest frequencies must be comparable, which means $N_1 \sim \max\{N_0, N_2\}$ ($\frac{1}{4} N_1 \leq \max\{N_0, N_2\} \leq 4 N_1$). It is easy to show if $\frac{1}{4} N_1 > \max\{N_0, N_2\}$ or $\max\{N_0, N_2\} > 4 N_1$, then the integral $\int_0^\delta \int_{\mathbb{T}^d} \widetilde{u}_1 \widetilde{u}_2 \widetilde{u}_3 \overline{u}_0 \, dxdt$ is zero. Then the following two cases need to considered:

- $N_0 \sim N_1 \geq N_2$;
- $N_0 < N_2 \sim N_1$.

Now let's summarize all cases of (u_1, u_2, u_3) we should consider. Denote

$$R_i = P_{N_i} v_0^\omega \text{ and } D_i = P_{N_i} w \text{ for } i \in \{1, 2, 3\}. \tag{4.5.3}$$

The list of all cases of (u_1, u_2, u_3) is below:

A. $u_1 = D_1$:

(a) (D_1, D_2, D_3);

(b) (D_1, D_2, R_3);

(c) (D_1, R_2, D_3);

(d) (D_1, R_2, R_3);

B. $u_1 = R_1$:

(a) (R_1, R_2, R_3);

(b) (R_1, R_2, D_3);

(c) (R_1, D_2, R_3);

(d) (R_1, D_2, D_3).

4.5.1 Case A (a)

We consider the all deterministic case $u_i = D_i$ for all $i \in \{1, 2, 3\}$. It's directly the local well-posed result for the critical data following the strichartz estimates Proposition 4.4.1 (the case $d = 4$ is in [80]).

Proposition 4.5.3. *Assume N_i, $i = 0, 1, 2, 3$, are dyadic numbers and $N_1 \geq N_2 \geq N_3$, and $0 \leq \delta \leq 1$. For $s \geq s_c$, there exists $c > 0$, so that we can bound the integral:*

$$\left| \int_0^\delta \int_{\mathbb{T}^d} \mathcal{N}(\widetilde{D}_1, \widetilde{D}_2, \widetilde{D}_3) \overline{u}_0 \, dx dt \right| \lesssim \delta^{c \min\{1, s - s_c\}} \left(\frac{N_3 \min\{N_0, N_2\}}{N_2^2} \right)^c \frac{1}{N_2^{s - s_c}} \|u_0\|_{Y^{-s}} \|D_1\|_{X^s} \|D_2\|_{X^s} \|D_3\|_{X^s}$$

where u_0, $\widetilde{D}_1$, $\widetilde{D}_2$, and $\widetilde{D}_3$ is defined as (4.5.3).

Proof. We decompose $\mathbb{R}^d = \cup_j C_j$, where each C_j is a cube of side-length N_2. Let P_{C_j} denote the family of Fourier projections onto the cube C_j. We write $C_j \sim C_k$ if the sum set $\{c_1 + c_2 : c_1 \in C_j, c_2 \in C_k\}$ overlaps the Fourier support of $P_{\leq 2N_2}$. Observe that given C_k there are a bounded number of $C_j \sim C_k$. If $N_0 \sim N_1 \geq N_2$, and we decompose u_0 and D_1 with Fourier projections onto the small cubes of size N_2. If $N_0 < N_2 \sim N_1$, and we also decompose u_0 and D_1 with Fourier projections onto the cubes of size N_2, however the frequency of u_0 has Fourier support of $P_{\leq N_0}$ which is only in one cube of size N_2. For the case of $N_0 < N_2 \sim N_1$, the cube decomposition doesn't help, but for simplicity of notations, we use the same cube decomposition.

(1) Case: $d = 3$

First, let's consider $\mathcal{N}_1(\widetilde{D}_1, \widetilde{D}_2, \widetilde{D}_3) = \pm \widetilde{D}_1 \widetilde{D}_2 \widetilde{D}_3$. Set $\frac{11}{2}^+$ satisfying

$$\frac{1}{\frac{11}{2}^+} = \frac{2}{11} - c_1, \tag{4.5.4}$$

where $c_1 = \min\{\frac{2}{11} - \epsilon, \frac{2}{5}(s - s_c)\}$. (In this chapter, we always use ϵ as a small positive number which can be chosen arbitrarily small, and ϵ may be different in the different positions.)

By the cube decomposition, and Hölder inequality, we obtain that

$$\begin{aligned}
&\left| \int_0^\delta \int_{\mathbb{T}^3} \overline{u}_0 \widetilde{D}_1 \widetilde{D}_2 \widetilde{D}_3 \, dxdt \right| &(4.5.5)\\
\leq &\sum_{C_j \sim C_k} \left| \int_0^\delta \int_{\mathbb{T}^3} (P_{C_j} \overline{u}_0)(P_{C_k} \widetilde{D}_1) \widetilde{D}_2 \widetilde{D}_3 \, dxdt \right| \\
\lesssim &\sum_{C_j \sim C_k} \|P_{C_j} u_0\|_{L^{\frac{11}{3}}_{t,x}} \|P_{C_k} D_1\|_{L^{\frac{11}{3}}_{t,x}} \|D_2\|_{L^{\frac{11}{3}}_{t,x}} \|D_3\|_{L^{\frac{11}{2}}_{t,x}} \\
\leq &\delta^{c_1} \sum_{C_j \sim C_k} \|P_{C_j} u_0\|_{L^{\frac{11}{3}}_{t,x}} \|P_{C_k} D_1\|_{L^{\frac{11}{3}}_{t,x}} \|D_2\|_{L^{\frac{11}{3}}_{t,x}} \|D_3\|_{L^{\frac{11}{2}+}_{t,x}}. &(4.5.6)
\end{aligned}$$

By Strichartz estimates (Lemma 4.4.1) and (4.5.6), we obtain that

$$\begin{aligned}
(4.5.5) \leq &\delta^{c_1} \sum_{C_j \sim C_k} \min\{N_0, N_2\}^{\frac{3}{22}} N_2^{\frac{6}{22}} N_3^{\frac{13}{22}+5c_1} \|P_{C_j} u_0\|_{Y^0} \|P_{C_k} D_1\|_{X^0} \|D_2\|_{X^0} \|D_3\|_{X^0} \\
\lesssim &\delta^{c_1} (\frac{\min\{N_0, N_2\}}{N_2})^{\frac{3}{22}} (\frac{N_3}{N_2})^{\frac{1}{11}} N_3^{2(s-s_c)} \sum_{C_j \sim C_k} \|P_{C_j} u_0\|_{Y^{-s}} \|P_{C_k} D_1\|_{X^s} \|D_2\|_{X^{s_c}} \|D_3\|_{X^{s_c}} \\
\lesssim &\delta^{c \min\{1, (s-s_c)\}} (\frac{N_3 \min\{N_0, N_2\}}{N_2^2})^c \frac{1}{N_2^{s-s_c}} \|u_0\|_{Y^{-s}} \|D_1\|_{X^s} \|D_2\|_{X^s} \|D_3\|_{X^s},
\end{aligned}$$

where $c = \frac{1}{11}$.

Second, let's consider $\mathcal{N}_2(\widetilde{D}_1, \widetilde{D}_2, \widetilde{D}_3) = \pm\widetilde{D}_1 \int_{\mathbb{T}^3} \widetilde{D}_2\widetilde{D}_3 dx$.

$$
\begin{aligned}
&\left|\int_0^\delta \int_{\mathbb{T}^3} \overline{u}_0 \widetilde{D}_1 dx \int_{\mathbb{T}^3} \widetilde{D}_2\widetilde{D}_3 dxdt\right| \\
\leq &\sum_{C_j\sim C_k} \left|\int_0^\delta \int_{\mathbb{T}^3} (P_{C_j}\overline{u}_0)(P_{C_k}\widetilde{D}_1) dx \int_{\mathbb{T}^3} \widetilde{D}_2\widetilde{D}_3 dxdt\right| \\
\lesssim &\sum_{C_j\sim C_k} \|P_{C_j}u_0\|_{L_t^{\frac{11}{3}} L_x^2} \|P_{C_k}D_1\|_{L_t^{\frac{11}{3}} L_x^2} \|D_2\|_{L_t^{\frac{11}{3}} L_x^2} \|D_3\|_{L_t^{\frac{11}{2}} L_x^2} \\
\lesssim &\sum_{C_j\sim C_k} \|P_{C_j}u_0\|_{L_{t,x}^{\frac{11}{3}}} \|P_{C_k}D_1\|_{L_{t,x}^{\frac{11}{3}}} \|D_2\|_{L_{t,x}^{\frac{11}{3}}} \|D_3\|_{L_{t,x}^{\frac{11}{2}}}.
\end{aligned}
$$

Then we follow the same approach for $\mathcal{N}_1$ term, we can hold the same bound of $\mathcal{N}_2$.

(2) Case: $d \geq 4$

First, let's consider $\mathcal{N}_1(\widetilde{D}_1, \widetilde{D}_2, \widetilde{D}_3) = \pm\widetilde{D}_1\widetilde{D}_2\widetilde{D}_3$. Set 3^+ and ∞^- satisfying the following conditions:

$$
\frac{1}{\infty^-} = c_2, \qquad \frac{1}{3^+} = \frac{1}{3} - \frac{c_2}{3}. \tag{4.5.7}
$$

where $c_2 = \frac{2}{d+2}\min\{\frac{1}{4}, s - s_c\} + c_3$, $c_3 = \frac{1}{d+2}\min\{s - s_c, \frac{1}{3}\}$.

By the cube decomposition, Hölder inequality, and Lemma 4.4.1, we obtain that

$$
\begin{aligned}
&\left|\int_0^\delta \int_{\mathbb{T}^d} \overline{u}_0 \widetilde{D}_1\widetilde{D}_2\widetilde{D}_3 \, dxdt\right| \\
\leq &\sum_{C_j\sim C_k} \left|\int_0^\delta \int_{\mathbb{T}^d} (P_{C_j}\overline{u}_0)(P_{C_k}\widetilde{D}_1)\widetilde{D}_2\widetilde{D}_3 \, dxdt\right| \\
\lesssim &\sum_{C_j\sim C_k} \|P_{C_j}u_0\|_{L_{t,x}^{3+}} \|P_{C_k}D_1\|_{L_{t,x}^{3+}} \|D_2\|_{L_{t,x}^{3+}} \|D_3\|_{L_{t,x}^{\infty-}} \\
\leq &\delta^{c_3} \sum_{C_j\sim C_k} \min\{N_0, N_2\}^{\frac{d}{6}-\frac{2}{3}+\frac{(d+2)c_2}{3}} N_2^{\frac{d}{3}-\frac{4}{3}+\frac{2(d+2)c_2}{3}} N_3^{\frac{d}{2}-(d+2)(c_2-c_3)} \\
&\times \|P_{C_j}u_0\|_{Y^0} \|P_{C_k}D_1\|_{X^0} \|D_2\|_{X^0} \|D_3\|_{X^0} \\
\lesssim &\delta^{c_3} \left(\frac{\min\{N_0, N_2\}}{N_2}\right)^{\frac{d}{6}-\frac{2}{3}+\frac{(d+2)c_2}{3}} \left(\frac{N_3}{N_2}\right)^{\max\{\frac{1}{2}, 1-2(s-s_c)\}-\min\{s-s_c,\frac{1}{3}\}} N_3^{\min\{s-s_c,\frac{1}{3}\}} \\
&\times \sum_{C_j\sim C_k} \|P_{C_j}u_0\|_{Y^{-s}} \|P_{C_k}D_1\|_{X^s} \|D_2\|_{X^{s_c}} \|D_3\|_{X^{s_c}} \\
\lesssim &\delta^{c\min\{1,s-s_c\}} \left(\frac{N_3\min\{N_0, N_2\}}{N_2^2}\right)^c \frac{1}{N_2^{s-s_c}} \|u_0\|_{Y^{-s}} \|D_1\|_{X^s} \|D_2\|_{X^s} \|D_3\|_{X^s},
\end{aligned}
$$

where $c = \frac{1}{3(d+2)}$ (it's easy to chech that $\max\{\frac{1}{2}, 1 - 2(s - s_c)\} - \min\{s - s_c, \frac{1}{3}\} \geq \frac{1}{6} > c$).

Second, let's consider $\mathcal{N}_2(\widetilde{D}_1, \widetilde{D}_2, \widetilde{D}_3) = \pm\widetilde{D}_1 \int_{\mathbb{T}^d} \widetilde{D}_2\widetilde{D}_3 dx$.

$$
\begin{aligned}
&\left|\int_0^\delta \int_{\mathbb{T}^d} \overline{u}_0 \widetilde{D}_1 dx \int_{\mathbb{T}^d} \widetilde{D}_2\widetilde{D}_3 dxdt\right| \\
\leq &\sum_{C_j\sim C_k} \left|\int_0^\delta \int_{\mathbb{T}^d} (P_{C_j}\overline{u}_0)(P_{C_k}\widetilde{D}_1) dx \int_{\mathbb{T}^d} \widetilde{D}_2\widetilde{D}_3 dxdt\right| \\
\lesssim &\sum_{C_j\sim C_k} \|P_{C_j}u_0\|_{L_t^{3+}L_x^2} \|P_{C_k}D_1\|_{L_t^{3+}L_x^2} \|D_2\|_{L_t^{3+}L_x^2} \|D_3\|_{L_t^{\infty-}L_x^2} \\
\lesssim &\sum_{C_j\sim C_k} \|P_{C_j}u_0\|_{L_{t,x}^{3+}} \|P_{C_k}D_1\|_{L_{t,x}^{3+}} \|D_2\|_{L_{t,x}^{3+}} \|D_3\|_{L_{t,x}^{\infty-}}.
\end{aligned}
$$

Then following the same approach for $\mathcal{N}_1$ term, we can hold the bound of $\mathcal{N}_2$.

□

4.5.2 Case A (b)

We consider the case $(u_1, u_2, u_3) = (D_1, D_2, R_3)$.

Proposition 4.5.4. *Assume N_i, $i = 0, 1, 2, 3$, are dyadic numbers and $N_1 \geq N_2 \geq N_3$, and $0 \leq \delta \leq 1$. For $s \geq s_c$ and $0 \leq \alpha < s_c$, there exist $c, r > 0$ and subset $\Omega_\delta \subset \Omega$ with $\mathbb{P}(\Omega_\delta^c) \leq e^{-1/\delta^r}$, so that for all $\omega \in \Omega_\delta$ and all $N_1 \geq N_2 \geq N_3$, we can bound the integral:*

$$
\left|\int_0^\delta \int_{\mathbb{T}^d} \mathcal{N}(\widetilde{D}_1, \widetilde{D}_2, \widetilde{R}_3)\overline{u}_0 dxdt\right| \lesssim \delta^c \left(\frac{1}{N_2N_3}\right)^c \|u_0\|_{Y^{-s}} \|D_1\|_{X^s} \|D_2\|_{X^s},
$$

where u_0, $\widetilde{D}_1$, $\widetilde{D}_2$, and $\widetilde{R}_3$ is defined as (4.5.3).

Proof. Let P_{C_j} denote the family of Fourier projections onto the cube C_j of size N_2. We write $C_j \sim C_k$ if the sum set overlaps the Fourier support of P_{2N_2}.

(1) Case: $d = 3$

First, let's consider $\mathcal{N}_1(\widetilde{D}_1, \widetilde{D}_2, \widetilde{R}_3) = \pm\widetilde{D}_1\widetilde{D}_2\widetilde{R}_3$.

By Corollary 4.2.4, there exists Ω_δ with $\mathbb{P}(\Omega_\delta^c) < e^{-1/\delta^r}$ and $c' > 0$, such that for all N_3 and $\omega \in \Omega_\delta$, we obtain that

$$
\|R_3\|_{L_{t,x}^{\frac{11}{2}}([0,\delta]\times\mathbb{T}^3)} \leq \delta^{c'} \frac{\log N_3}{N_3^{s_c-\alpha}}. \tag{4.5.8}
$$

By Lemma 4.4.1, Cauchy-Schwarz inequality and (4.5.8),

$$
\begin{aligned}
&\left|\int_0^\delta \int_{\mathbb{T}^3} \overline{u}_0 \widetilde{D}_1 \widetilde{D}_2 \widetilde{R}_3 dxdt\right| \\
\leq & \sum_{C_j \sim C_k} \left|\int_0^\delta \int_{\mathbb{T}^3} (P_{C_j}\overline{u}_0)(P_{C_k}\widetilde{D}_1)\widetilde{D}_2 \widetilde{R}_3 dxdt\right| \\
\lesssim & \sum_{C_j \sim C_k} \|P_{C_j} u_0\|_{L^{\frac{11}{3}}_{t,x}} \|P_{C_k} D_1\|_{L^{\frac{11}{3}}_{t,x}} \|D_2\|_{L^{\frac{11}{3}}_{t,x}} \|R_3\|_{L^{\frac{11}{2}}_{t,x}} \\
\lesssim & \sum_{C_j \sim C_k} \min\{N_0, N_2\}^{\frac{3}{22}} N_2^{\frac{6}{22}} \|P_{C_j} u_0\|_{Y^0} \|P_{C_k} D_1\|_{X^0} \|D_2\|_{X^0} \|R_3\|_{L^{\frac{11}{2}}_{t,x}} \\
\leq & \delta^{c'} \frac{\log N_3}{N_3^{s_c-\alpha}} \frac{1}{N_2^{\frac{1}{11}+s-s_c}} \sum_{C_j \sim C_k} \|P_{C_j} u_0\|_{Y^{-s}} \|P_{C_k} D_1\|_{X^s} \|D_2\|_{X^s} \\
\leq & \delta^c (\frac{1}{N_2 N_3})^c \|P_{C_j} u_0\|_{Y^{-s}} \|D_1\|_{X^s} \|D_2\|_{X^s},
\end{aligned}
$$

where $c = min(c', s_c - \alpha - \epsilon, \frac{1}{11})$.

Second, $\mathcal{N}_2(\widetilde{D}_1, \widetilde{D}_2, \widetilde{R}_3) = \pm \widetilde{D}_1 \int_{\mathbb{T}^3} \widetilde{D}_2 \widetilde{R}_3 dx$. We can bound $|\int_0^\delta \int_{\mathbb{T}^3} \mathcal{N}_2(\widetilde{D}_1, \widetilde{D}_2, \widetilde{R}_3)\overline{u}_0 dxdt|$ by $\sum_{C_j \sim C_k} \|P_{C_j} u_0\|_{L^{\frac{11}{3}}_{t,x}} \|P_{C_k} D_1\|_{L^{\frac{11}{3}}_{t,x}} \|D_2\|_{L^{\frac{11}{3}}_{t,x}} \|R_3\|_{L^{\frac{11}{2}}_{t,x}}$, using Hölder inequality. Then we can bound the second part via the same way.

(2) Case: $d \geq 4$

First, let's consider $\mathcal{N}_1(\widetilde{D}_1, \widetilde{D}_2, \widetilde{R}_3) = \pm \widetilde{D}_1 \widetilde{D}_2 \widetilde{R}_3$.

Set 3^{++} and ∞^{--} as following:

$$\frac{1}{3^{++}} = \frac{1}{3} - \frac{1}{6(d+2)}, \quad \frac{1}{\infty^{--}} = \frac{1}{2(d+2)}. \tag{4.5.9}$$

By Corollary 4.2.4, there exists Ω_δ with $\mathbb{P}(\Omega_\delta^c) < e^{-1/\delta^r}$ and $c' > 0$, such that for all N_3 and $\omega \in \Omega_\delta$, we obtain that

$$\|R_3\|_{L^{\infty^{--}}_{t,x}([0,\delta]\times\mathbb{T}^d)} \leq \delta^{c'} \frac{\log N_3}{N_3^{s_c-\alpha}}. \tag{4.5.10}$$

By Lemma 4.4.1, Cauchy-Schwarz inequality and (4.5.10),

$$\begin{aligned}
&\left|\int_0^\delta \int_{\mathbb{T}^d} \overline{u}_0 \widetilde{D}_1 \widetilde{D}_2 \widetilde{R}_3 dxdt\right| \\
\leq &\sum_{C_j \sim C_k} \left|\int_0^\delta \int_{\mathbb{T}^d} (P_{C_j}\overline{u}_0)(P_{C_k}\widetilde{D}_1) \widetilde{D}_2 \widetilde{R}_3 dxdt\right| \\
\lesssim &\sum_{C_j \sim C_k} \|P_{C_j} u_0\|_{L^{3++}_{t,x}} \|P_{C_k} D_1\|_{L^{3++}_{t,x}} \|D_2\|_{L^{3++}_{t,x}} \|R_3\|_{L^{\infty--}_{t,x}} \\
\lesssim &\sum_{C_j \sim C_k} \min\{N_0, N_2\}^{\frac{d}{6}-\frac{1}{2}} N_2^{\frac{d}{3}-1} \|P_{C_j} u_0\|_{Y^0} \|P_{C_k} D_1\|_{X^0} \|D_2\|_{X^0} \|R_3\|_{L^{\infty--}_{t,x}} \\
\leq &\delta^{c'} \frac{\log N_3}{N_3^{s_c-\alpha}} \frac{1}{N_2^{\frac{1}{2}+s-s_c}} \sum_{C_j \sim C_k} \|P_{C_j} u_0\|_{Y^{-s}} \|P_{C_k} D_1\|_{X^s} \|D_2\|_{X^s} \\
\leq &\delta^c (\frac{1}{N_2 N_3})^c \|P_{C_j} u_0\|_{Y^{-s}} \|D_1\|_{X^s} \|D_2\|_{X^s},
\end{aligned}$$

where $c = min(c', s_c - \alpha - \epsilon, \frac{1}{2})$.

Second, $\mathcal{N}_2(\widetilde{D}_1, \widetilde{D}_2, \widetilde{R}_3) = \pm \widetilde{D}_1 \int_{\mathbb{T}^d} \widetilde{D}_2 \widetilde{R}_3 dx$. We can bound $|\int_0^\delta \int_{\mathbb{T}^d} \mathcal{N}_2(\widetilde{D}_1, \widetilde{D}_2, \widetilde{R}_3)\overline{u}_0 dxdt|$ by $\sum_{C_j \sim C_k} \|P_{C_j} u_0\|_{L^{3++}_{t,x}} \|P_{C_k} D_1\|_{L^{3++}_{t,x}} \|D_2\|_{L^{3++}_{t,x}} \|R_3\|_{L^{\infty--}_{t,x}}$, using Hölder inequality. Then we can bound the second part via the same way.

□

4.5.3 Case A (c)

We consider the case $(u_1, u_2, u_3) = (D_1, R_2, D_3)$.

Proposition 4.5.5. *Assume N_i, $i = 0, 1, 2, 3$, are dyadic numbers and $N_1 \geq N_2 \geq N_3$, and $0 \leq \delta \leq s_c$. For $s \geq s_c$ and $0 \leq \alpha < \frac{d}{6}$, there exists $c, r > 0$ and subset $\Omega_\delta \subset \Omega$ with $\mathbb{P}(\Omega_\delta^c) \leq e^{-1/\delta^r}$, so that for all $\omega \in \Omega_\delta$ and all $N_1 \geq N_2 \geq N_3$, we can bound the integral:*

$$\left|\int_0^\delta \int_{\mathbb{T}^4} \mathcal{N}(\widetilde{D}_1, \widetilde{R}_2, \widetilde{D}_3)\overline{u}_0 dxdt\right| \lesssim \delta^c (\frac{1}{N_2 N_3})^c \|u_0\|_{Y^{-s}} \|D_1\|_{X^s} \|D_3\|_{X^s},$$

where u_0, $\widetilde{D}_1$, $\widetilde{R}_2$, and $\widetilde{D}_3$ is defined as (4.5.3).

Proof. Let P_{C_j} denote the family of Fourier projections onto the cube C_j of size N_2. We write $Cj \sim Ck$ if the sum set overlaps the Fourier support of P_{2N_2}.

(1) Case: $d = 3$

First, let's consider $\mathcal{N}_1(\widetilde{D}_1, \widetilde{R}_2, \widetilde{D}_3) = \pm \widetilde{D}_1 \widetilde{R}_2 \widetilde{D}_3$.

By Corollary 4.2.4, there exists Ω_δ with $\mathbb{P}(\Omega_\delta^c) < e^{-1/\delta^r}$ and $c' > 0$, such that for all N_2 and $\omega \in \Omega_\delta$, we obtain that

$$\|R_2\|_{L_{t,x}^{\frac{11}{2}}([0,\delta]\times\mathbb{T}^3)} \leq \delta^{c'} \frac{\log N_2}{N_2^{s_c-\alpha}}. \tag{4.5.11}$$

By Lemma 4.4.1, Hölder inequality and (4.5.11),

$$\begin{aligned}
&\left| \int_0^\delta \int_{\mathbb{T}^3} \overline{u}_0 \widetilde{D}_1 \widetilde{R}_2 \widetilde{D}_3 dxdt \right| \\
\leq & \sum_{C_j \sim C_k} \left| \int_0^\delta \int_{\mathbb{T}^3} (P_{C_j} \overline{u}_0)(P_{C_k} \widetilde{D}_1) \widetilde{R}_2 \widetilde{D}_3 dxdt \right| \\
\lesssim & \sum_{C_j \sim C_k} \|P_{C_j} u_0\|_{L_{t,x}^{\frac{11}{3}}} \|P_{C_k} D_1\|_{L_{t,x}^{\frac{11}{3}}} \|R_2\|_{L_{t,x}^{\frac{11}{2}}} \|D_3\|_{L_{t,x}^{\frac{11}{3}}} \\
\lesssim & \min\{N_0, N_2\}^{\frac{3}{22}} N_2^{\frac{3}{22}} N_3^{\frac{3}{22}} \sum_{C_j \sim C_k} N_2^{3\epsilon} \|P_{C_j} u_0\|_{Y^0} \|P_{C_k} D_1\|_{X^0} \|D_3\|_{X^0} \|R_2\|_{L_{t,x}^{\frac{11}{2}}} \\
\leq & \delta^{c'} \frac{\log(N_2)}{N_2^{s_c-\alpha-\frac{3}{11}} N_3^{s-\frac{3}{22}}} \sum_{C_j \sim C_k} \|P_{C_j} u_0\|_{Y^{-s}} \|P_{C_k} D_1\|_{X^s} \|D_3\|_{X^s} \\
\leq & \delta^c \left(\frac{1}{N_2 N_3}\right)^c \|P_{C_j} u_0\|_{Y^{-s}} \|D_1\|_{X^s} \|D_3\|_{X^s},
\end{aligned}$$

where $c = min(c', s_c - \alpha - \frac{3}{11} - \epsilon, s_c - \frac{3}{22})$.

Second, $\mathcal{N}_2(\widetilde{D}_1, \widetilde{R}_2, \widetilde{D}_3) = \pm \widetilde{D}_1 \int_{\mathbb{T}^3} \widetilde{R}_2 \widetilde{D}_3 dx$. We can bound $|\int_0^\delta \int_{\mathbb{T}^3} \mathcal{N}_2(\widetilde{D}_1, \widetilde{R}_2, \widetilde{D}_3) \overline{u}_0 dxdt|$ by $\sum_{C_j \sim C_k} \|P_{C_j} u_0\|_{L_{t,x}^{\frac{11}{3}}} \|P_{C_k} D_1\|_{L_{t,x}^{\frac{11}{3}}} \|D_3\|_{L_{t,x}^{\frac{11}{3}}} \|R_2\|_{L_{t,x}^{\frac{11}{2}}}$, using Hölder inequality. Then we can bound the second part via the same way.

(2) Case: $d \geq 4$

First, let's consider $\mathcal{N}_1(\widetilde{D}_1, \widetilde{R}_2, \widetilde{D}_3) = \pm \widetilde{D}_1 \widetilde{R}_2 \widetilde{D}_3$.

Set 3^{++} and ∞^{--} as (4.5.9). By Corollary 4.2.4, there exists Ω_δ with $\mathbb{P}(\Omega_\delta^c) < e^{-1/\delta^r}$ and $c' > 0$, such that for all N_2 and $\omega \in \Omega_\delta$, we obtain that

$$\|R_2\|_{L_{t,x}^{\infty^{--}}([0,\delta]\times\mathbb{T}^d)} \leq \delta^{c'} \frac{\log N_2}{N_2^{s_c-\alpha}}. \tag{4.5.12}$$

By Lemma 4.4.1, Cauchy-Schwarz inequality and (4.5.12),

$$
\begin{aligned}
&\left|\int_0^\delta \int_{\mathbb{T}^d} \overline{u}_0 \widetilde{D}_1 \widetilde{R}_2 \widetilde{D}_3 dxdt\right| \\
\leq &\sum_{C_j \sim C_k} \left|\int_0^\delta \int_{\mathbb{T}^d} (P_{C_j}\overline{u}_0)(P_{C_k}\widetilde{D}_1) \widetilde{R}_2 \widetilde{D}_3 dxdt\right| \\
\lesssim &\sum_{C_j \sim C_k} \|P_{C_j} u_0\|_{L^{3++}_{t,x}} \|P_{C_k} D_1\|_{L^{3++}_{t,x}} \|D_3\|_{L^{3++}_{t,x}} \|R_2\|_{L^{\infty--}_{t,x}} \\
\lesssim &\sum_{C_j \sim C_k} \min\{N_0, N_2\}^{\frac{d}{6}-\frac{1}{2}} N_2^{\frac{d}{6}-\frac{1}{2}} N_3^{\frac{d}{6}-\frac{1}{2}} \|P_{C_j} u_0\|_{Y^0} \|P_{C_k} D_1\|_{X^0} \|D_3\|_{X^0} \|R_2\|_{L^{\infty--}_{t,x}} \\
\leq &\delta^{c'} \frac{\log N_2}{N_2^{\frac{d}{6}-\alpha}} \frac{1}{N_3^{\frac{d}{3}-\frac{1}{2}+s-s_c}} \sum_{C_j \sim C_k} \|P_{C_j} u_0\|_{Y^{-s}} \|P_{C_k} D_1\|_{X^s} \|D_3\|_{X^s} \\
\leq &\delta^c \left(\frac{1}{N_2 N_3}\right)^c \|P_{C_j} u_0\|_{Y^{-s}} \|D_1\|_{X^s} \|D_3\|_{X^s},
\end{aligned}
$$

where $c = min(c', \frac{d}{6} - \alpha - \epsilon, \frac{d}{3} - \frac{1}{2})$.

Second, $\mathcal{N}_2(\widetilde{D}_1, \widetilde{D}_2, \widetilde{R}_3) = \pm \widetilde{D}_1 \int_{\mathbb{T}^d} \widetilde{D}_2 \widetilde{R}_3 dx$. We can bound $|\int_0^\delta \int_{\mathbb{T}^d} \mathcal{N}_2(\widetilde{D}_1, \widetilde{D}_2, \widetilde{R}_3)\overline{u}_0 dxdt|$ by $\sum_{C_j \sim C_k} \|P_{C_j} u_0\|_{L^{3++}_{t,x}} \|P_{C_k} D_1\|_{L^{3++}_{t,x}} \|D_2\|_{L^{3++}_{t,x}} \|R_3\|_{L^{\infty--}_{t,x}}$, using Hölder inequality. Then we can bound the second part via the same way.

$\square$

4.5.4 Case A (d)

We consider the case $(u_1, u_2, u_3) = (D_1, R_2, R_3)$.

Proposition 4.5.6. *Assume N_i, $i = 0, 1, 2, 3$, are dyadic numbers and $N_1 \geq N_2 \geq N_3$, and $0 \leq \delta \leq 1$. For $s \geq s_c$ and $0 \leq \alpha < s_c$, there exist $c, r > 0$ and subset $\Omega_\delta \subset \Omega$ with $\mathbb{P}(\Omega_\delta^c) \leq e^{-1/\delta^r}$, so that for all $\omega \in \Omega_\delta$ and all $N_1 \geq N_2 \geq N_3$, we can bound the integral:*

$$
\left|\int_0^\delta \int_{\mathbb{T}^4} \mathcal{N}(\widetilde{D}_1, \widetilde{R}_2, \widetilde{R}_3)\overline{u}_0 dxdt\right| \lesssim \delta^c \left(\frac{1}{N_2 N_3}\right)^c \|u_0\|_{Y^{-s}} \|D_1\|_{X^s},
$$

where u_0, $\widetilde{D}_1$, $\widetilde{R}_2$, and $\widetilde{R}_3$ is defined as (4.5.3).

Proof. Let P_{C_j} denote the family of Fourier projections onto the cube C_j of size N_2. We write $Cj \sim Ck$ if the sum set overlaps the Fourier support of P_{2N_2}.

First, let's consider $\mathcal{N}_1(\widetilde{D}_1, \widetilde{R}_2, \widetilde{R}_3) = \pm \widetilde{D}_1 \widetilde{R}_2 \widetilde{R}_3$.

Set p_c^+, q as

$$\frac{1}{p_c^+} = \frac{d}{2(d+2)} - \epsilon \text{ and } \frac{1}{q} = \frac{1}{2} - \frac{1}{p_c^+}. \tag{4.5.13}$$

By Corollary 4.2.4, there exists Ω_δ with $\mathbb{P}(\Omega_\delta^c) < e^{-1/\delta^r}$ and $c' > 0$, such that for all N and $\omega \in \Omega_\delta$, we obtain that

$$\|P_N v_0^\omega\|_{L^q_{t,x}([0,\delta]\times\mathbb{T}^4)} \leq \delta^{c'} \frac{\log N}{N^{s_c-\alpha}}. \tag{4.5.14}$$

By Lemma 4.4.1, Cauchy-Schwaz inequality and (4.5.14),

$$\begin{aligned}
&\left|\int_0^\delta \int_{\mathbb{T}^4} \overline{u}_0 \widetilde{D}_1 \widetilde{R}_2 \widetilde{R}_3 dx dt\right| \\
\leq &\sum_{C_j \sim C_k} \left|\int_0^\delta \int_{\mathbb{T}^4} (P_{C_j}\overline{u}_0)(P_{C_k}\widetilde{D}_1)\widetilde{R}_2\widetilde{R}_3 dx dt\right| \\
\lesssim &\sum_{C_j \sim C_k} \|P_{C_j} u_0\|_{L^{p_c^+}_{t,x}} \|P_{C_k} D_1\|_{L^{p_c^+}_{t,x}} \|R_2\|_{L^q_{t,x}} \|R_3\|_{L^q_{t,x}} \\
\lesssim &\sum_{C_j \sim C_k} N_2^{2\epsilon} \|P_{C_j} u_0\|_{Y^0} \|P_{C_k} D_1\|_{X^0} \|R_2\|_{L^q_{t,x}} \|R_3\|_{L^q_{t,x}} \\
\leq &\delta^{2c'} \frac{\log N_2}{N_2^{s_c-\alpha}} \frac{\log N_3}{N_3^{s_c-\alpha}} N_2^{2\epsilon} \sum_{C_j \sim C_k} \|P_{C_j} u_0\|_{Y^{-s}} \|P_{C_k} D_1\|_{X^s} \\
\leq &\delta^c \left(\frac{1}{N_2 N_3}\right)^c \|P_{C_j} u_0\|_{Y^{-s}} \|u_1\|_{X^s},
\end{aligned}$$

where $c = min(2c', s_c - \alpha - \epsilon)$.

Second, $\mathcal{N}_2(\widetilde{D}_1, \widetilde{R}_2, \widetilde{R}_3) = \pm \widetilde{D}_1 \int_{\mathbb{T}^4} \widetilde{R}_2 \widetilde{R}_3 dx$. We can bound $|\int_0^\delta \int_{\mathbb{T}^4} \mathcal{N}_2(\widetilde{D}_1, \widetilde{R}_2, \widetilde{R}_3)\overline{u}_0 dx dt|$ by $\sum_{C_j \sim C_k} \|P_{C_j} u_0\|_{L^{p_c^+}_{t,x}} \|P_{C_k} D_1\|_{L^{p_c^+}_{t,x}} \|R_2\|_{L^q_{t,x}} \|R_3\|_{L^q_{t,x}}$, using Hölder inequality. Then we can bound the second part via the same way. $\square$

In **Case B**, the top frequency is random term, so that the approach in **Case A** fails. In the following proofs of subcases of **Case B**, it will suffice to focus on the frequencies satisfying $N_0 \sim N_1 \geq N_2$, since if $N_0 < N_2 \sim N_1$, then **Case B** can be treated as **Case A** which the top frequency is deterministic term.

4.5.5 Case B (a)

We consider the all random case $(u_1, u_2, u_3) = (R_1, R_2, R_3)$.

Proposition 4.5.7. *Assume N_i, $i = 0, 1, 2, 3$, are dyadic numbers and for any N_1, N_2, N_3, satisfying $N_1 \geq N_2 \geq N_3$, and $0 \leq \delta \leq 1$. For $\alpha < \frac{1}{4}$ and $s_c \leq s < s_c + \frac{1}{4} - \alpha$, there exist $c, r > 0$ and subset $\Omega_\delta \subset \Omega$ with $\mathbb{P}(\Omega_\delta^c) \leq e^{-1/\delta^r}$, so that for all $\omega \in \Omega_\delta$ and all $N_1 \geq N_2 \geq N_3$, we can bound the integral:*

$$\left| \int_0^\delta \int_{\mathbb{T}^d} \mathcal{N}(\widetilde{R}_1, \widetilde{R}_2, \widetilde{R}_3) \overline{u}_0 dx dt \right| \lesssim \delta^c (\frac{1}{N_1})^c \|u_0\|_{Y^{-s}},$$

where $\widetilde{R}_1$, $\widetilde{R}_2$, and $\widetilde{R}_3$ is defined as (4.5.3) and only one of $\widetilde{R}_i$ can be $\overline{R_i}$.

Proof. Let's suppose that $\widetilde{R}_1 = \overline{R_1}$, $\widetilde{R}_2 = R_2$ and $\widetilde{R}_3 = R_3$, and the other cases are similar (we will also explain how to prove in the others in the following proof).

Define $S(n, m) := \{(n_1, n_2, n_3) \in \mathbb{Z}^d \times \mathbb{Z}^d \times \mathbb{Z}^d : -n_1 + n_2 + n_3 = n,\ -|n_1|^2 + |n_2|^2 + |n_3|^2 = m,\ n_1 \neq n_2,\ n_3,\ and\ n_i \sim N_i\}$ (For example, if we consider $\mathcal{N}(R_1, \overline{R_2}, R_3)$ case, then the corresponding $S(n, m) := \{(n_1, n_2, n_3) \in \mathbb{Z}^d \times \mathbb{Z}^d \times \mathbb{Z}^d : n_1 - n_2 + n_3 = n,\ |n_1|^2 - |n_2|^2 + |n_3|^2 = m,\ n_2 \neq n_1,\ n_3,\ and\ n_i \sim N_i\}$), where $m \in \mathbb{Z}$ and $n \in \mathbb{Z}^d$. Then we have

$$\begin{aligned} &\mathcal{N}(\overline{R}_1, R_2, R_3) := \mathcal{J}_1 + \mathcal{J}_2 \\ = &\sum_{n \in \mathbb{Z}^d, m \in \mathbb{Z}} e^{in \cdot x + itm} \sum_{S(n,m)} \frac{\overline{g_{n_1}(\omega)}}{\langle n_1 \rangle^{d-1-\alpha}} \frac{g_{n_2}(\omega)}{\langle n_2 \rangle^{d-1-\alpha}} \frac{g_{n_3}(\omega)}{\langle n_3 \rangle^{d-1-\alpha}} \\ + &\sum_{n \in \mathbb{Z}^d, n \sim N_i,\ i=1,2,3} \frac{|g_n(\omega)|^2 g_n(\omega)}{\langle n \rangle^{3d-3-3\alpha}} e^{in \cdot x + it|n|^2} \end{aligned}$$

Step 1 a) First, let's consider $\mathcal{J}_1$ term. By Prop 4.3.3, to estimate $|\int_0^\delta \int_{\mathbb{T}^d} \overline{u}_0 \mathcal{J}_1(\overline{R}_1, R_2, R_3) dx dt|$, we can first consider u_0 as a linear solution $\mathbb{1}_J e^{it\Delta}\phi$ in any small interval $J \subset [0, \delta]$ and get the bound of $\left| \int_{J \times \mathbb{T}^d} \overline{P_{N_0} e^{it\Delta}\phi} \mathcal{J}_1(\overline{R}_1, R_2, R_3)\, dx dt \right|$. Suppose $\phi(x) = \sum_{n \in \mathbb{Z}^d} a_n e^{in \cdot x}$ and $\mathbb{1}_J(t) = \sum_{k \in \mathbb{Z}} b_k e^{ikt}$.

$$
\begin{aligned}
&\left|\int_{J\times\mathbb{T}^d} \overline{P_{N_0}e^{it\Delta}\phi}\,\mathcal{J}_1(\overline{R}_1,R_2,R_3)\,dxdt\right|\\
=&\left|\sum_{\substack{n\in\mathbb{Z}^d,\,|n|\sim N_0\\ k\in\mathbb{Z},\,|k|\lesssim N_1^2}}\ \sum_{S(n,|n|^2+k)} b_k\overline{a_n}\frac{\overline{g_{n_1}(\omega)}}{\langle n_1\rangle^{d-1-\alpha}}\frac{g_{n_2}(\omega)}{\langle n_2\rangle^{d-1-\alpha}}\frac{g_{n_3}(\omega)}{\langle n_3\rangle^{d-1-\alpha}}\right|
\end{aligned}
$$

then by Lemma 4.4.5, we have that $\sum_{|k|\lesssim N_1^2}|b_k|\lesssim \log N_1$. So

$$
\begin{aligned}
&\left|\sum_{\substack{n\in\mathbb{Z}^d,\,|n|\sim N_0\\ k\in\mathbb{Z},\,|k|\lesssim N_1^2}}\ \sum_{S(n,|n|^2+k)} b_k\overline{a_n}\frac{\overline{g_{n_1}(\omega)}}{\langle n_1\rangle^{d-1-\alpha}}\frac{g_{n_2}(\omega)}{\langle n_2\rangle^{d-1-\alpha}}\frac{g_{n_3}(\omega)}{\langle n_3\rangle^{d-1-\alpha}}\right|\\
\leq&\|P_{N_0}\phi\|_{L^2_x}\sum_{|k|\lesssim N_1^2}|b_k|\left(\sum_{n\in\mathbb{Z}^d,|n|\sim N_0}\left|\sum_{S(n,|n|^2+k)}\frac{\overline{g_{n_1}(\omega)}}{\langle n_1\rangle^{d-1-\alpha}}\frac{g_{n_2}(\omega)}{\langle n_2\rangle^{d-1-\alpha}}\frac{g_{n_3}(\omega)}{\langle n_3\rangle^{d-1-\alpha}}\right|^2\right)^{\frac{1}{2}}
\end{aligned}
$$

By Lemma 4.2.5, after choosing a subset Ω^1_δ with $\mathbb{P}(\Omega^1_\delta)\lesssim e^{-\frac{1}{\delta^2}}$, and by Lattice counting lemma (Lemma 4.4.3), we obtain that

$$
\begin{aligned}
&\|P_{N_0}\phi\|_{L^2_x}\sum_{|k|\lesssim N_1^2}|b_k|\left(\sum_{n\in\mathbb{Z}^d,|n|\sim N_0}\left|\sum_{S(n,|n|^2+k)}\frac{\overline{g_{n_1}(\omega)}}{\langle n_1\rangle^{d-1-\alpha}}\frac{g_{n_2}(\omega)}{\langle n_2\rangle^{d-1-\alpha}}\frac{g_{n_3}(\omega)}{\langle n_3\rangle^{d-1-\alpha}}\right|^2\right)^{\frac{1}{2}}\\
\lesssim&\|P_{N_0}\phi\|_{L^2_x}\sum_{|k|\lesssim N_1^2}|b_k|\frac{N_1^\epsilon}{N_1^{d-1-\alpha}N_2^{d-1-\alpha}N_3^{d-1-\alpha}}\\
&\times\left|\{(n,n_1,n_2,n_3)\in\mathbb{Z}^d\times\mathbb{Z}^d\times\mathbb{Z}^d\times\mathbb{Z}^d:(n_1,n_2,n_3)\in S(n,|n|^2+k)\}\right|^{\frac{1}{2}}\\
\leq&\frac{N_1^\epsilon}{N_1^{s_c+\frac{1}{2}-\alpha}N_2^{s_c-\alpha}N_3^{s_c-\alpha}}\|P_{N_0}\phi\|_{L^2_x}.
\end{aligned}
$$

Step 1 b) Second, let's consider $\mathcal{J}_2$, By Lemma 4.2.1, there exists a set Ω^2_δ with $\mathbb{P}(\Omega^{2^c}_\delta)<e^{-1/\delta^\epsilon}$, for all $\omega\in\Omega^2_\delta$, we have $|g_n(\omega)|\lesssim\frac{\log(\langle n\rangle+1)}{\delta^\epsilon}$.

$$
\|\mathcal{J}_2\|^2_{L^2_{x,t}}=\sum_{n\in\mathbb{Z}^4,n\sim N_i,\ i=1,2,3}\left|\frac{|g_n(\omega)|^2g_n(\omega)}{\langle n\rangle^{3d-3-3\alpha}}\right|^2\lesssim\delta^{-6\epsilon}\frac{1}{N_1^{5d-6\alpha-6\epsilon}}.
$$

If we choose $\Omega_\delta = \Omega_\delta^1 \cap \Omega_\delta^2$, then we obtain that

$$\begin{aligned}&\left| \int_{J\times\mathbb{T}^d} \overline{P_{N_0} e^{it\Delta}\phi} \mathcal{N}(\overline{R}_1, R_2, R_3)\, dxdt \right| \\ \lesssim &\delta^{-\epsilon} \frac{N_1^\epsilon}{N_1^{s_c+\frac{1}{2}-\alpha} N_2^{s_c-\alpha} N_3^{s_c-\alpha}} \|P_{N_0}\phi\|_{L_x^2},\end{aligned}$$

(For simplicity, we use ϵ vaguely as a constant which we can choose arbitrary small and $\epsilon' s$ in the different inequalities don't have to be the exactly same.)

Step 2 If we set 1^+ and ∞^- satisfying $\frac{1}{1^+} = 1 - \epsilon$ and $\frac{1}{\infty^-} = \frac{\epsilon}{3}$, by Hölder inequality and Lemma 4.2.4 after excluding a subset of probability $e^{-\frac{1}{\delta^c}}$, we have

$$\begin{aligned}&\left| \int_{J\times\mathbb{T}^d} \overline{P_{N_0} e^{it\Delta}\phi} \mathcal{N}_1(\overline{R}_1, R_2, R_3)\, dxdt \right| \\ =&\left| \int_{J\times\mathbb{T}^d} \overline{P_{N_0} e^{it\Delta}\phi}\ \overline{R}_1 R_2 R_3\, dxdt \right| \\ \leq&\|P_{N_0} e^{it\Delta}\phi\|_{L_t^{1^+} L_x^2(J\times\mathbb{T}^d)} \|R_1\|_{L_t^{\infty^-} L_x^6} \|R_2\|_{L_t^{\infty^-} L_x^6} \|R_3\|_{L_t^{\infty^-} L_x^6} \\ \lesssim&|J|^{1-\epsilon} \frac{N_1^\epsilon}{N_1^{s_c-\alpha} N_2^{s_c-\alpha} N_3^{s_c-\alpha}} \|P_{N_0}\phi\|_{L_x^2}.\end{aligned}$$

And also we have

$$\begin{aligned}&\left| \int_{J\times\mathbb{T}^d} \overline{P_{N_0} e^{it\Delta}\phi} \mathcal{N}_2(\overline{R}_1, R_2, R_3)\, dxdt \right| \\ \lesssim&\left| \int_J \left(\int_{\mathbb{T}^d} \overline{P_{N_0} e^{it\Delta}\phi}\ \overline{R_1}\, dx \right) \left(\int_{\mathbb{T}^d} R_2 R_3\, dx \right) dt \right| \\ \leq&\|P_{N_0} e^{it\Delta}\phi\|_{L_t^{1^+} L_x^2(J\times\mathbb{T}^d)} \|R_1\|_{L_t^{\infty^-} L_x^2} \|R_2\|_{L_t^{\infty^-} L_x^2} \|R_3\|_{L_t^{\infty^-} L_x^2} \\ \lesssim&|J|^{1-\epsilon} \|P_{N_0} e^{it\Delta}\phi\|_{L_t^\infty L_x^2(J\times\mathbb{T}^d)} \|R_1\|_{L_t^{\infty^-} L_x^2} \|R_2\|_{L_t^{\infty^-} L_x^2} \|R_3\|_{L_t^{\infty^-} L_x^2} \\ \lesssim&|J|^{1-\epsilon} \frac{N_1^\epsilon}{N_1^{s_c-\alpha} N_2^{s_c-\alpha} N_3^{s_c-\alpha}} \|P_{N_0}\phi\|_{L_x^2}.\end{aligned}$$

So we obtain that

$$\left| \int_{J\times\mathbb{T}^d} \overline{P_{N_0} e^{it\Delta}\phi} \mathcal{N}(\overline{R}_1, R_2, R_3)\, dxdt \right| \lesssim |J|^{1-\epsilon} \frac{N_1^\epsilon}{N_1^{s_c-\alpha} N_2^{s_c-\alpha} N_3^{s_c-\alpha}} \|P_{N_0}\phi\|_{L_x^2}.$$

Step 3 Average the estimates in Step 1 and Step 2, we obtain that

$$\left| \int_{J\times\mathbb{T}^d} \overline{P_{N_0} e^{it\Delta}\phi} \mathcal{N}(\overline{R}_1, R_2, R_3)\, dxdt \right| \tag{4.5.15}$$

$$\lesssim |J|^{\frac{1}{2}} \delta^{-\epsilon} \frac{N_1^\epsilon}{N_1^{s_c+\frac{1}{4}-\alpha} N_2^{s_c-\alpha} N_3^{s_c-\alpha}} \|P_{N_0}\phi\|_{L_x^2} \tag{4.5.16}$$

By the estimate (4.5.16) in Step 3 and Lemma 4.3.3, we hold that

$$\left|\int_0^\delta \int_{\mathbb{T}^d} \mathcal{N}(\overline{R}_1, R_2, R_3)\overline{u}_0 dxdt\right| \lesssim \delta^{\frac{1}{2}} \frac{N_1^\epsilon}{N_1^{s_c+\frac{1}{4}-\alpha} N_2^{s_c-\alpha} N_3^{s_c-\alpha}} \|u_0\|_{U_\Delta^2}$$

Step 4 Set p_c^+, q as (4.5.13) and $\frac{1}{\infty^-} = \epsilon$. Using Strichartz estimate (4.7.7), we have

$$\begin{aligned}
&\left|\int_{[0,\delta]\times\mathbb{T}^d} \overline{u_0}\mathcal{N}(\overline{R}_1, R_2, R_3)\, dxdt\right| \\
\lesssim&\|u_0\|_{L_{t,x}^{p_c^+}} \|R_1\|_{L_{t,x}^{p_c^+}} \|R_2\|_{L_{t,x}^q} \|R_3\|_{L_{t,x}^q} \\
\lesssim&\delta^{\frac{d+4}{2(d+2)}-\epsilon} \|u_0\|_{L_{t,x}^{p_c^+}} \|R_1\|_{L_{t,x}^{\infty-}} \|R_2\|_{L_{t,x}^{\infty-}} \|R_3\|_{L^{\infty-}} \\
\lesssim&\delta^{\frac{d+4}{2(d+2)}} \frac{1}{N_1^{s_c-\alpha} N_2^{s_c-\alpha} N_3^{s_c-\alpha}} \|u_0\|_{U_\Delta^{p_c^+}}.
\end{aligned}$$

By the interpolation lemma (Lemma 4.3.6) and the embedding property (3.3.5), we obtain that

$$\left|\int_0^\delta \int_{\mathbb{T}^d} \mathcal{N}(\overline{R}_1, R_2, R_3)\overline{u}_0 dxdt\right| \lesssim \delta^{\frac{1}{2}} \frac{N_1^\epsilon}{N_1^{s_c+\frac{1}{4}-\alpha-s} N_2^{s_c-\alpha} N_3^{s_c-\alpha}} \|u_0\|_{Y^{-s}}$$

Since $s < s_c + \frac{1}{4} - \alpha$, we hold the proposition. □

4.5.6 Case B (b)

We consider the all case $(u_1, u_2, u_3) = (R_1, R_2, D_3)$.

Proposition 4.5.8. *Assume N_i, $i = 0, 1, 2, 3$, are dyadic numbers and $N_1 \geq N_2 \geq N_3$, and $0 \leq \delta \leq 1$. For $s_c \leq s < s_c + \frac{1}{4} - \alpha$ and $0 \leq \alpha < \frac{1}{4}$, there exist $c, r > 0$ and subset $\Omega_\delta \subset \Omega$ with $\mathbb{P}(\Omega_\delta^c) \leq e^{-1/\delta^r}$, so that for all $\omega \in \Omega_\delta$ and all $N_1 \geq N_2 \geq N_3$, we can bound the integral:*

$$\left|\int_0^\delta \int_{\mathbb{T}^4} \mathcal{N}(\widetilde{R}_1, \widetilde{R}_2, \widetilde{D}_3)\overline{u}_0 dxdt\right| \lesssim \delta^c (\frac{1}{N_1})^c \|u_0\|_{Y^{-s}} \|D_3\|_{X^s},$$

where $\widetilde{R}_1$, $\widetilde{R}_2$, and $\widetilde{D}_3$ is defined as (4.5.3) and only one of $\{\widetilde{R}_1, \widetilde{R}_2, \widetilde{D}_3\}$ can be the conjuagate.

Proof. Let's suppose that $\widetilde{R}_1 = \overline{R_1}$, $\widetilde{R}_2 = R_2$ and $\widetilde{D}_3 = D_3$, and the other cases are similar.

Define $S_3(n, n_3, m) := \{(n_1, n_2) \in \mathbb{Z}^d \times \mathbb{Z}^d : -n_1 + n_2 + n_3 = n,\ -|n_1|^2 + |n_2|^2 + |n_3|^2 = m,\ n_1 \neq n_2,\ n_3, \text{ and } n_i \sim N_i\}$. Then we have

$$
\begin{aligned}
&\mathcal{N}(\overline{R}_1, R_2, D_3) := \mathcal{J}_1 + \mathcal{J}_2 \\
=& \sum_{n\in\mathbb{Z}^d, m\in\mathbb{Z}} e^{in\cdot x + itm} \sum_{S(n,m)} \frac{\overline{g_{n_1}(\omega)}}{\langle n_1\rangle^{d-1-\alpha}} \frac{g_{n_2}(\omega)}{\langle n_2\rangle^{d-1-\alpha}} \widehat{D_3}(n_3) \\
+& \sum_{n\in\mathbb{Z}^d, n\sim N_i,\ i=1,2,3} \frac{|g_n(\omega)|^2 \widehat{D_3}(n_3)}{\langle n\rangle^{2d-2-2\alpha}} e^{in\cdot x + it|n|^2}
\end{aligned}
$$

Step 1 a) First, let's consider $\mathcal{J}_1$ term. By Prop 4.3.3, to estimate $|\int_0^\delta \int_{\mathbb{T}^d} \overline{u}_0 \mathcal{J}_1(\overline{R}_1, R_2, D_3) dxdt|$, we can first consider u_0 as a linear solution $\mathbb{1}_J e^{it\Delta}\phi$ in any small interval $J \subset [0, \delta]$ and get the bound of $\left|\int_{J\times\mathbb{T}^d} \overline{P_{N_0} e^{it\Delta}\phi} \mathcal{J}_1(\overline{R}_1, R_2, P_{N_3} e^{it\Delta}\phi^{(3)})\, dxdt\right|$. Suppose $\phi(x) = \sum_{n\in\mathbb{Z}^d} a_n e^{in\cdot x}$, $\phi^{(3)}(x) = \sum_{n\in\mathbb{Z}^d} a_n^{(3)} e^{in\cdot x}$ and $\mathbb{1}_J(t) = \sum_{k\in\mathbb{Z}} b_k e^{ikt}$.

$$
\begin{aligned}
&\left|\int_{J\times\mathbb{T}^d} \overline{P_{N_0} e^{it\Delta}\phi} \mathcal{J}_1(\overline{R}_1, R_2, P_{N_3} e^{it\Delta}\phi^{(3)})\, dxdt\right| \\
=& \left| \sum_{\substack{n\in\mathbb{Z}^d,\ |n|\sim N_0 \\ k\in\mathbb{Z},\ |k|\lesssim N_1^2}} \sum_{S(n,|n|^2+k)} b_k \overline{a_n} \frac{\overline{g_{n_1}(\omega)}}{\langle n_1\rangle^{d-1-\alpha}} \frac{g_{n_2}(\omega)}{\langle n_2\rangle^{d-1-\alpha}} a_{n_3}^{(3)} \right|
\end{aligned}
$$

then by Lemma 4.4.5, we have that $\sum_{|k|\lesssim N_1^2} |b_k| \lesssim \log N_1$. So

$$
\begin{aligned}
&\left| \sum_{\substack{n\in\mathbb{Z}^d,\ |n|\sim N_0 \\ k\in\mathbb{Z},\ |k|\lesssim N_1^2}} \sum_{S(n,|n|^2+k)} b_k \overline{a_n} \frac{\overline{g_{n_1}(\omega)}}{\langle n_1\rangle^{d-1-\alpha}} \frac{g_{n_2}(\omega)}{\langle n_2\rangle^{d-1-\alpha}} a_{n_3}^{(3)} \right| \\
\leq& \|P_{N_0}\phi\|_{L_x^2} \|P_{N_3}\phi^{(3)}\|_{L_x^2} \sum_{|k|\lesssim N_1^2} |b_k| \left(\sum_{\substack{n\in\mathbb{Z}^d, |n|\sim N_0 \\ n_3\in\mathbb{Z}^d, |n_3|\sim N_3}} \left| \sum_{S_3(n,n_3,|n|^2+k)} \frac{\overline{g_{n_1}(\omega)}}{\langle n_1\rangle^{d-1-\alpha}} \frac{g_{n_2}(\omega)}{\langle n_2\rangle^{d-1-\alpha}} \right|^2 \right)^{\frac{1}{2}}
\end{aligned}
$$

By Lemma 4.2.5, after choosing a subset Ω_δ^1 with $\mathbb{P}(\Omega_\delta^1) \lesssim e^{-\frac{1}{\delta^2}}$, and by Lattice counting lemma (Lemma 4.4.3), we obtain that

$$
\begin{aligned}
&\|P_{N_0}\phi\|_{L^2_x}\|P_{N_3}\phi^{(3)}\|_{L^2_x}\sum_{|k|\lesssim N_1^2}|b_k|\left(\sum_{\substack{n\in\mathbb{Z}^d,|n|\sim N_0\\ n_3 in\mathbb{Z}^d,|n_3|\sim N_3}}\left|\sum_{S_3(n,n_3,|n|^2+k)}\frac{\overline{g_{n_1}(\omega)}}{\langle n_1\rangle^{d-1-\alpha}}\frac{g_{n_2}(\omega)}{\langle n_2\rangle^{d-1-\alpha}}\right|^2\right)^{\frac{1}{2}}\\
\lesssim&\|P_{N_0}\phi\|_{L^2_x}\|P_{N_3}\phi^{(3)}\|_{L^2_x}\sum_{|k|\lesssim N_1^2}|b_k|\frac{1}{N_1^{d-1-\alpha}N_2^{d-1-\alpha}}\\
&\times\left|\{(n,n_1,n_2,n_3)\in\mathbb{Z}^d\times\mathbb{Z}^d\times\mathbb{Z}^d\times\mathbb{Z}^d:(n_1,n_2,n_3)\in S(n,|n|^2+k)\}\right|^{\frac{1}{2}}\\
\leq&\frac{N_1^{\epsilon}N_3^{\frac{d}{2}}}{N_1^{s_c+\frac{1}{2}-\alpha}N_2^{s_c-\alpha}}\|P_{N_0}\phi\|_{L^2_x}\|P_{N_3}\phi^{(3)}\|_{L^2_x}.
\end{aligned}
$$

Step 1 b) Second, consider $\mathcal{J}_2$.

$$
\left|\int_{J\times\mathbb{T}^d}\overline{P_{N_0}e^{it\Delta}\phi}\mathcal{J}_2(\overline{R}_1,R_2,P_{N_3}e^{it\Delta}\phi^{(3)})\,dxdt\right| \tag{4.5.17}
$$

$$
\leq\left|\sum_{\substack{n\in\mathbb{Z}^d,\,|n|\sim N_0\\ k\in\mathbb{Z},\,|k|\lesssim N_1^2}}b_k\overline{a_n}\frac{\overline{g_n(\omega)}}{\langle n\rangle^{d-1-\alpha}}\frac{g_n(\omega)}{\langle n\rangle^{d-1-\alpha}}a_n^{(3)}\right| \tag{4.5.18}
$$

By Lemma 4.2.1, there exists a set Ω_δ^2 with $\mathbb{P}(\Omega_\delta^{2^c})<e^{-1/\delta^\epsilon}$, for all $\omega\in\Omega_\delta^2$, we have $|g_n(\omega)|\lesssim\frac{\log(\langle n\rangle+1)}{\delta^\epsilon}$.

$$
(4.5.17)\leq\frac{N_1^{2\epsilon}}{N_1^{2d-2-2\alpha}}\left|\sum_{\substack{n\in\mathbb{Z}^d,\,|n|\sim N_0\\ k\in\mathbb{Z},\,|k|\lesssim N_1^2}}b_k\overline{a_n}a_n^{(3)}\right| \tag{4.5.19}
$$

$$
\lesssim\frac{N_1^{2\epsilon}\log(N_1)}{N_1^{2d-2-2\alpha}}\|P_{N_0}\phi\|_{L^2_x}\|P_{N_3}\phi^{(3)}\|_{L^2_x} \tag{4.5.20}
$$

If we choose $\Omega_\delta=\Omega_\delta^1\cap\Omega_\delta^2$, then we obtain that

$$
\begin{aligned}
&\left|\int_{J\times\mathbb{T}^d}\overline{P_{N_0}e^{it\Delta}\phi}\mathcal{N}(\overline{R}_1,R_2,P_{N_3}e^{it\Delta}\phi^{(3)})\,dxdt\right|\\
\lesssim&\frac{N_1^{\epsilon}N_3^{\frac{d}{2}}}{N_1^{s_c+\frac{1}{2}-\alpha}N_2^{s_c-\alpha}}\|P_{N_0}\phi\|_{L^2_x}\|P_{N_3}\phi^{(3)}\|_{L^2_x},
\end{aligned}
$$

Step 2 If we set 2^+, ∞^-, 1^+ and q satisfying $\frac{1}{2^+}=\frac{1}{2}-\epsilon$, $\frac{2}{\infty^-}+\frac{1}{2^+}=\frac{1}{2}$, $\frac{1}{1^+}+\frac{2}{\infty^-}=1$, and $\frac{2}{q}+\frac{d}{2^+}=\frac{d}{2}$. By Hölder inequality and Lemma 4.2.4 after excluding a subset of probability $e^{-\frac{1}{\delta^c}}$, we have

$$
\begin{aligned}
&\left|\int_{J\times\mathbb{T}^d} \overline{P_{N_0}e^{it\Delta}\phi}\mathcal{N}_1(\overline{R}_1, R_2, P_{N_3}e^{it\Delta}\phi^{(3)})\,dxdt\right| \\
=&\left|\int_{J\times\mathbb{T}^d} \overline{P_{N_0}e^{it\Delta}\phi}\ \overline{R}_1 R_2 P_{N_3}e^{it\Delta}\phi^{(3)}\,dxdt\right| \\
\leq&\|P_{N_0}e^{it\Delta}\phi\|_{L_t^\infty L_x^2(J\times\mathbb{T}^d)}\|R_1\|_{L_{t,x}^{\infty-}}\|R_2\|_{L_{t,x}^{\infty-}}\|P_{N_3}e^{it\Delta}\phi^{(3)}\|_{L_t^{1+}L_x^{2+}} \\
\lesssim&|J|^{1-\epsilon}\frac{N_3^\epsilon}{N_1^{s_c-\alpha}N_2^{s_c-\alpha}}\|P_{N_0}\phi\|_{L_x^2}\|P_{N_3}\phi^{(3)}\|_{L_x^2}.
\end{aligned}
$$

And also we have

$$
\begin{aligned}
&\left|\int_{J\times\mathbb{T}^d} \overline{P_{N_0}e^{it\Delta}\phi}\mathcal{N}_2(\overline{R}_1, R_2, P_{N_3}e^{it\Delta}\phi^{(3)})\,dxdt\right| \\
\lesssim&\left|\int_J\left(\int_{\mathbb{T}^d}\overline{P_{N_0}e^{it\Delta}\phi}\ \overline{R_1}\,dx\right)\left(\int_{\mathbb{T}^d}R_2P_{N_3}e^{it\Delta}\phi^{(3)}\,dx\right)dt\right| \\
\leq&\|P_{N_0}e^{it\Delta}\phi\|_{L_t^\infty L_x^2(J\times\mathbb{T}^d)}\|R_1\|_{L_t^{\infty-}L_x^2}\|R_2\|_{L_t^{\infty-}L_x^2}\|P_{N_3}e^{it\Delta}\phi^{(3)}\|_{L_t^{1+}L_x^2} \\
\lesssim&|J|^{1-\epsilon}\frac{1}{N_1^{s_c-\alpha}N_2^{s_c-\alpha}}\|P_{N_0}\phi\|_{L_x^2}\|P_{N_3}\phi^{(3)}\|_{L_x^2}.
\end{aligned}
$$

So we obtain that

$$
\left|\int_{J\times\mathbb{T}^d} \overline{P_{N_0}e^{it\Delta}\phi}\mathcal{N}(\overline{R}_1, R_2, P_{N_3}e^{it\Delta}\phi^{(3)})\,dxdt\right| \lesssim |J|^{1-\epsilon}\frac{N_3^\epsilon}{N_1^{s_c-\alpha}N_2^{s_c-\alpha}}\|P_{N_0}\phi\|_{L_x^2}\|P_{N_3}\phi^{(3)}\|_{L_x^2}.
$$

Step 3 Average the estimates in Step 1 and Step 2, we obtain that

$$
\begin{aligned}
&\left|\int_{J\times\mathbb{T}^d} \overline{P_{N_0}e^{it\Delta}\phi}\mathcal{N}(\overline{R}_1, R_2, P_{N_3}e^{it\Delta}\phi^{(3)})\,dxdt\right| && (4.5.21)\\
\lesssim&|J|^{\frac{1}{2}}\frac{N_1^\epsilon N_3^{\frac{d}{4}}}{N_1^{s_c+\frac{1}{4}-\alpha}N_2^{s_c-\alpha}}\|P_{N_0}\phi\|_{L_x^2}\|P_{N_3}\phi^{(3)}\|_{L_x^2}. && (4.5.22)
\end{aligned}
$$

By the estimate (4.5.22) in Step 3 and Lemma 4.3.3, we hold that

$$
\left|\int_0^\delta\int_{\mathbb{T}^d}\mathcal{N}(\overline{R}_1, R_2, D_3)\overline{u}_0\,dxdt\right| \lesssim \delta^{\frac{1}{2}}\frac{N_1^\epsilon N_3^{\frac{d}{4}}}{N_1^{s_c+\frac{1}{4}-\alpha}N_2^{s_c-\alpha}}\|u_0\|_{U_\Delta^2}\|D_3\|_{U_\Delta^2}
$$

Step 4 Set p_c^+, q as (4.5.13) and $\frac{1}{\infty^-}=\epsilon$ Replacing $\overline{P_{N_0}e^{it\Delta}\phi}$ by $\overline{u_0}$, following the similar idea, and using Strichartz estimate (4.7.7), we have

$$\left|\int_{[0,\delta]\times\mathbb{T}^d} \overline{u_0}\mathcal{N}(\overline{R}_1, R_2, D_3)\,dxdt\right| \lesssim \|u_0\|_{L^{p_c^+}_{t,x}}\|R_1\|_{L^q_{t,x}}\|R_2\|_{L^q_{t,x}}\|D_3\|_{L^{p_c^+}_{t,x}}$$
$$\lesssim \delta^{\frac{4}{2(d+2)}-\epsilon}\|u_0\|_{L^{p_c^+}_{t,x}}\|R_1\|_{L^{\infty-}_{t,x}}\|R_2\|_{L^{\infty-}_{t,x}}\|D_3\|_{L^{p_c^+}}$$
$$\lesssim \delta^{\frac{4}{2(d+2)}-\epsilon}\frac{1}{N_1^{s_c-\alpha-\epsilon}N_2^{s_c-\alpha}}\|u_0\|_{U^{p_c^+}_{\Delta}}\|D_3\|_{U^{p_c^+}_{\Delta}}.$$

By Lemma 4.3.6 and the embedding property (3.3.5), we obtain that

$$\left|\int_0^\delta\int_{\mathbb{T}^d}\mathcal{N}(\overline{R}_1, R_2, D_3)\overline{u}_0dxdt\right| \lesssim \delta^{\frac{1}{2}}\frac{N_1^\epsilon N_3^{\frac{d}{4}}}{N_1^{s_c+\frac{1}{4}-\alpha-s}N_2^{s_c-\alpha}N_3^s}\|u_0\|_{Y^{-s}}\|D_3\|_{X^s}$$

Since $s < s_c + \frac{1}{4} - \alpha$, we hold the proposition. $\square$

4.5.7 Case B (c)

We consider the all case $(u_1, u_2, u_3) = (R_1, D_2, R_3)$. By the similier approach with **Case B (b)**, we can hold following property:

Proposition 4.5.9. *Assume N_i, $i = 0, 1, 2, 3$, are dyadic numbers and $N_1 \geq N_2 \geq N_3$, and $0 \leq \delta \leq 1$. For $s_c \leq s < s_c + \frac{1}{6} - \alpha$ and $0 \leq \alpha < \frac{1}{6}$, there exist $c, r > 0$ and subset $\Omega_\delta \subset \Omega$ with $\mathbb{P}(\Omega_\delta^c) \leq e^{-1/\delta^r}$, so that for all $\omega \in \Omega_\delta$ and all $N_1 \geq N_2 \geq N_3$, we can bound the integral:*

$$\left|\int_0^\delta\int_{\mathbb{T}^4}\mathcal{N}(\widetilde{R}_1, \widetilde{D}_2, \widetilde{R}_3)\overline{u}_0dxdt\right| \lesssim \delta^c(\frac{1}{N_1})^c\|u_0\|_{Y^{-s}}\|D_2\|_{X^s},$$

where $\widetilde{R}_1$, $\widetilde{D}_2$, and $\widetilde{R}_3$ is defined as (4.5.3) and only one of $\{\widetilde{R}_1, \widetilde{D}_2, \widetilde{R}_3\}$ can be the conjuagate.

Proof. For $d \geq 4$.

Following the **Case B (b)**, by choosing a subset $\Omega_\delta \subset \Omega$ with $\mathbb{P}(\Omega_\delta^c) < e^{-1/\delta^r}$, for $\omega \in \Omega_\delta$, we have simliar estimate:

$$\left|\int_0^\delta\int_{\mathbb{T}^d}\mathcal{N}(\widetilde{R}_1, \widetilde{D}_2, \widetilde{R}_3)\overline{u}_0dxdt\right| \lesssim \delta^{\frac{1}{2}}\frac{N_1^\epsilon N_2^{\frac{d}{4}}}{N_1^{s_c+\frac{1}{4}-\alpha-s}N_3^{s_c-\alpha}N_2^s}\|u_0\|_{Y^{-s}}\|D_2\|_{X^s},$$

since $s < s_c + \frac{1}{4} - \alpha$, the proposition holds.

For $d = 3$. Following the **Case B (b)**, in Step 3, we average the estimates in Step 1 and Step 2 with different weights, we have that

$$\left| \int_0^\delta \int_{\mathbb{T}^3} \mathcal{N}(\widetilde{R}_1, \widetilde{D}_2, \widetilde{R}_3) \overline{u}_0 dx dt \right| \lesssim \delta^{\frac{1}{2}} \frac{N_1^\epsilon N_2^{\frac{1}{2}}}{N_1^{s_c + \frac{1}{6} - \alpha - s} N_3^{s_c - \alpha} N_2^s} \|u_0\|_{Y^{-s}} \|D_2\|_{X^s},$$

since $s_c \le s < s_c + \frac{1}{6} - \alpha$, and $s_c = \frac{1}{2}$, the proposition holds.

□

4.5.8 Case B (d)

We consider the case $(u_1, u_2, u_3) = (R_1, D_2, D_3)$.

Proposition 4.5.10. *Assume N_i, $i = 0, 1, 2, 3$, are dyadic numbers and $N_1 \ge N_2 \ge N_3$, and $0 \le \delta \le 1$. For $d \ge 3$,*

$$s_r(d) = \begin{cases} \frac{1}{7} & d = 3 \\ \frac{4}{19} & d = 4 \\ \frac{1}{4} & d \ge 5. \end{cases}$$

For $s_c \le s < s_c + s_r(d) - \alpha$ and $0 \le \alpha < s_r(d)$, there exist $c, r > 0$ and subset $\Omega_\delta \subset \Omega$ with $\mathbb{P}(\Omega_\delta^c) \le e^{-1/\delta^r}$, so that for all $\omega \in \Omega_\delta$ and all $N_1 \ge N_2 \ge N_3$, we can bound the integral:

$$\left| \int_0^\delta \int_{\mathbb{T}^d} \mathcal{N}(\widetilde{R}_1, \widetilde{D}_2, \widetilde{D}_3) \overline{u}_0 dx dt \right| \lesssim \delta^c (\frac{1}{N_1})^c \|u_0\|_{Y^{-s}} \|D_2\|_{X^s} \|D_3\|_{X^s},$$

where $\widetilde{R}_1$, $\widetilde{D}_2$, and $\widetilde{D}_3$ is defined as (4.5.3) and only one of $\{\widetilde{R}_1, \widetilde{D}_2, \widetilde{D}_3\}$ can be the conjuagate.

Proof. Let's suppose that $\widetilde{R}_1 = R_1$, $\widetilde{D}_2 = \overline{D_2}$ and $\widetilde{D}_3 = D_3$, and the other cases are similar.

Define $S_{2,3}(n, n_2, n_3, m) := \{n_1 \in \mathbb{Z}^d : n_1 - n_2 + n_3 = n,\ |n_1|^2 - |n_2|^2 + |n_3|^2 = m,\ n_2 \neq n_1,\ n_3, \text{ and } n_1 \sim N_1\}$. Then we have

$$\begin{aligned}
&\mathcal{N}(R_1, \overline{D}_2, D_3) := \mathcal{J}_1 + \mathcal{J}_2 \\
&= \sum_{n\in\mathbb{Z}^d, m\in\mathbb{Z}} e^{in\cdot x + itm} \sum_{S(n,m)} \frac{g_{n_1}(\omega)}{\langle n_1\rangle^{d-1-\alpha}} \widehat{\overline{D}_2}(n_2)\widehat{D_3}(n_3) \\
&+ \sum_{n\in\mathbb{Z}^d, n\sim N_i,\ i=1,2,3} \frac{g_n(\omega)\widehat{\overline{D}_2}(n_2)\widehat{D_3}(n_3)}{\langle n\rangle^{d-1-\alpha}} e^{in\cdot x + it|n|^2}
\end{aligned}$$

Step 1 a) First, let's consider $\mathcal{J}_1$ term. By Proposition 4.3.3, to estimate $|\int_0^\delta \int_{\mathbb{T}^d} \overline{u}_0 \mathcal{J}_1(R_1, \overline{D}_2, D_3) dxdt|$, we can first consider u_0 as a linear solution $\mathbb{1}_J e^{it\Delta}\phi$ in any small interval $J \subset [0, \delta]$ and get the bound of $\left|\int_{J\times\mathbb{T}^d} \overline{P_{N_0} e^{it\Delta}\phi} \mathcal{J}_1(R_1, \overline{P_{N_2} e^{it\Delta}\phi^{(2)}}, P_{N_3} e^{it\Delta}\phi^{(3)})\, dxdt\right|$. Suppose $\phi(x) = \sum_{n\in\mathbb{Z}^d} a_n e^{in\cdot x}$, $\phi^i(x) = \sum_{n\in\mathbb{Z}^d} a_n^{(i)} e^{in\cdot x}$ for $i = 2, 3$ and $\mathbb{1}_J(t) = \sum_{k\in\mathbb{Z}} b_k e^{ikt}$.

$$\begin{aligned}
&\left|\int_{J\times\mathbb{T}^d} \overline{P_{N_0} e^{it\Delta}\phi} \mathcal{J}_1(R_1, \overline{P_{N_2} e^{it\Delta}\phi^{(2)}}, P_{N_3} e^{it\Delta}\phi^{(3)})\, dxdt\right| \\
=&\left| \sum_{\substack{n\in\mathbb{Z}^d,\ |n|\sim N_0 \\ k\in\mathbb{Z},\ |k|\lesssim N_1^2}} \sum_{S(n,|n|^2+k)} b_k \overline{a_n} \frac{g_{n_1}(\omega)}{\langle n_1\rangle^{d-1-\alpha}} \overline{a_{n_2}^{(2)}} a_{n_3}^{(3)} \right| \\
\leq& \|P_{N_0}\phi\|_{L_x^2} \|P_{N_3}\phi^{(3)}\|_{L_x^2} \sum_{|k|\lesssim N_1^2} |b_k| \left(\sum_{\substack{n\in\mathbb{Z}^d, |n|\sim N_0 \\ n_3\in\mathbb{Z}^d |n_3|\sim N_3}} \left| \sum_{S_3(n,n_3,|n|^2+k)} \frac{g_{n_1}(\omega)}{\langle n_1\rangle^{d-1-\alpha}} \overline{a_{n_2}^{(2)}} \right|^2 \right)^{\frac{1}{2}}
\end{aligned}$$

Next, fix k. Let's focus on

$$\sum_{\substack{n\in\mathbb{Z}^d, |n|\sim N_0 \\ n_3\in\mathbb{Z}^d |n_3|\sim N_3}} \left| \sum_{S_3(n,n_3,|n|^2+k)} \frac{g_{n_1}(\omega)}{\langle n_1\rangle^{d-1-\alpha}} \overline{a_{n_2}^{(2)}} \right|^2. \tag{4.5.23}$$

To bound (4.5.23), we use the matrix $\mathscr{G}^*\mathscr{G}$ argument in Bourgain's paper [13] as follows.

Fix n_3 and $|n_3| \sim N_3$. Define

$$\mathscr{G} = \mathscr{G}_\omega = (\sigma_{n,n_2})_{\substack{|n|<N_0, |n_2|<N_2, \\ n\neq n_3}}$$

where

$$\sigma_{n,n_2} = \begin{cases} \frac{1}{N_1^{d-1-\alpha}} g_{n+n_2-n_3}(\omega) & \text{if } 2\langle n-n_3, n_2-n_3\rangle = k \\ 0 & \text{otherwise} \end{cases}$$

Then (4.5.23) is bounded by $N_3^d \|\mathscr{G}_\omega^* \mathscr{G}_\omega\|^{\frac{1}{2}}$ and by Lemma 4.4.6, we obtain that

$$\|\mathscr{G}^*\mathscr{G}\| \leq \max_n \left(\sum_{n_2} |\sigma_{n,n_2}|^2\right) + \left(\sum_{n\neq n'} \left|\sum_{n_2} \sigma_{n,n_2}\overline{\sigma_{n',n_2}}\right|^2\right)^{\frac{1}{2}}. \tag{4.5.24}$$

Using Lemma 4.2.1, the first term in (4.5.24) is bounded as follows,

$$\sum_{n_2} \left|\frac{1}{N_1^{d-1-\alpha}} g_{n+n_2-n_3}(\omega)\right|^2 \leq \frac{N_2^d}{N_1^{2(d-1-\alpha)-\epsilon}} \leq \frac{N_2}{N_1^{2s_c+1-2\alpha-\epsilon}}. \tag{4.5.25}$$

Then we will show that the second term in (4.5.24) is bounded as follows

$$\left(\sum_{n\neq n'} \left|\sum_{n_2} \sigma_{n,n_2}\overline{\sigma_{n',n_2}}\right|^2\right)^{\frac{1}{2}} \leq N_1^{-2s_c-1+2\alpha+\epsilon} N_2^{\frac{d-1}{2}} \tag{4.5.26}$$

Indeed, write

$$\sum_{n\neq n'} \left|\sum_{n_2} \sigma_{n,n_2}\overline{\sigma_{n',n_2}}\right|^2 = \frac{1}{N_1^{4(d-1-\alpha)}} \sum_{n\neq n'} \left|\sum_{n_2} g_{n+n_2-n_3}(\omega)\overline{g_{n'+n_2-n_3}}(\omega)\right|^2. \tag{4.5.27}$$

Then we use Lemma 4.2.5, there exists a set Ω_δ^1 with $\mathbb{P}(\Omega_\delta^{1^c}) < e^{-1/\delta^r}$, for all $\omega \in \Omega_\delta^1$, (4.5.27) can be bounded by

$$\begin{aligned} &\frac{1}{N_1^{4(d-1-\alpha)}} |\{(n,n',n_2) : n \neq n', 2\langle n-n_3, n_2-n_3\rangle = k \\ &\qquad 2\langle n'-n_3, n_2-n_3\rangle = k\}| \end{aligned} \tag{4.5.28}$$

To bound the number of the elements in $\{(n,n',n_2) : n \neq n', 2\langle n-n_3, n_2-n_3\rangle = k,\ 2\langle n'-n_3, n_2-n_3\rangle = k\}$, first we can count the number of pair (n,n_2) and by Lemma 4.4.4, it is bounded by $N_1^{d-1+\epsilon}N_2^{d-1}$. And then the number of possible n' is bounded N_1^{d-1} by Lemma 4.4.3. So the size of the upper lattice set is bounded by $N_1^{2(d-1)+\epsilon}N_2^{d-1}$, and hence we hold (4.5.26).

By the estimates (4.5.25) and (4.5.26), we obtain that

$$\begin{aligned}
&\left|\int_{J\times\mathbb{T}^d} \overline{P_{N_0}e^{it\Delta}\phi}\mathcal{J}_1(\overline{R}_1, P_{N_2}e^{it\Delta}\phi^{(2)}, P_{N_3}e^{it\Delta}\phi^{(3)})\,dxdt\right| \\
\lesssim&\frac{N_1^{\epsilon}N_2^{\frac{d-1}{4}}N_3^{\frac{d}{2}}}{N_1^{s_c+\frac{1}{2}-\alpha}}\|P_{N_0}\phi\|_{L_x^2}\|P_{N_2}\phi^{(2)}\|_{L_x^2}\|P_{N_3}\phi^{(3)}\|_{L_x^2},
\end{aligned}\tag{4.5.29}$$

Step 1 b) Second, let's consider $\mathcal{J}_2$. By Lemma 4.2.1, there exists a set Ω_δ^2 with $\mathbb{P}(\Omega_\delta^{2^c}) < e^{-1/\delta^{2\epsilon/3}}$, for all $\omega\in\Omega_\delta^2$, we have $|g_n(\omega)|\lesssim\frac{\langle n\rangle^\epsilon}{\delta^\epsilon}$.

$$\begin{aligned}
&\left|\int_{J\times\mathbb{T}^d} \overline{P_{N_0}e^{it\Delta}\phi}\mathcal{J}_2(\overline{R}_1, P_{N_2}e^{it\Delta}\phi^{(2)}, P_{N_3}e^{it\Delta}\phi^{(3)})\,dxdt\right| \\
\le&\left|\sum_{\substack{n\in\mathbb{Z}^d,\,|n|\sim N_0\\ k\in\mathbb{Z},\,|k|\lesssim N_1^2}} b_k\overline{a_n}\frac{\overline{g_n(\omega)}}{\langle n\rangle^{d-1-\alpha}}a_n^{(2)}a_n^{(3)}\right| \le \frac{N_1^{2\epsilon}}{N_1^{d-1-\alpha}}\left|\sum_{\substack{n\in\mathbb{Z}^d,\,|n|\sim N_0\\ k\in\mathbb{Z},\,|k|\lesssim N_1^2}} b_k\overline{a_n}a_n^{(2)}a_n^{(3)}\right| \\
\lesssim&\frac{N_1^{2\epsilon}\log(N_1)}{N_1^{d-1-\alpha}}\|P_{N_0}\phi\|_{L_x^2}\|P_{N_2}\phi^{(2)}\|_{L_x^2}\|P_{N_3}\phi^{(3)}\|_{L_x^1(\mathbb{T}^d)} \\
\lesssim&\frac{N_1^{2\epsilon}\log(N_1)}{N_1^{d-1-\alpha}}\|P_{N_0}\phi\|_{L_x^2}\|P_{N_2}\phi^{(2)}\|_{L_x^2}\|P_{N_3}\phi^{(3)}\|_{L_x^2(\mathbb{T}^d)}.
\end{aligned}$$

If we choose $\Omega_\delta = \Omega_\delta^1\cap\Omega_\delta^2$, then we obtain that

$$\begin{aligned}
&\left|\int_{J\times\mathbb{T}^d} \overline{P_{N_0}e^{it\Delta}\phi}\mathcal{N}(\overline{R}_1, P_{N_2}e^{it\Delta}\phi^{(2)}, P_{N_3}e^{it\Delta}\phi^{(3)})\,dxdt\right| \\
\lesssim&\frac{N_1^{\epsilon}N_2^{\frac{d-1}{4}}N_3^{\frac{d}{2}}}{N_1^{s_c+\frac{1}{2}-\alpha}}\|P_{N_0}\phi\|_{L_x^2}\|P_{N_2}\phi^{(2)}\|_{L_x^2}\|P_{N_3}\phi^{(3)}\|_{L_x^2},
\end{aligned}\tag{4.5.30}$$

Step 2 If we set 2^+, ∞^-, 1^+ and q satisfying $\frac{1}{2^+}=\frac{1}{2}-\epsilon$, $\frac{2}{\infty^-}+\frac{1}{2^+}=\frac{1}{2}$, $\frac{1}{1^+}+\frac{2}{\infty^-}=1$, and $\frac{2}{q}+\frac{d}{2^+}=\frac{d}{2}$. By Hölder inequality and Lemma 4.2.4 after excluding a subset of probability $e^{-\frac{1}{\delta^c}}$, we have

$$\begin{aligned}
&\left|\int_{J\times\mathbb{T}^d} \overline{P_{N_0}e^{it\Delta}\phi}\mathcal{N}_1(\overline{R}_1, P_{N_2}e^{it\Delta}\phi^{(2)}, P_{N_3}e^{it\Delta}\phi^{(3)})\,dxdt\right| \\
=&\left|\int_{J\times\mathbb{T}^d} \overline{P_{N_0}e^{it\Delta}\phi}\;\overline{R}_1 P_{N_2}e^{it\Delta}\phi^{(2)}P_{N_3}e^{it\Delta}\phi^{(3)}\,dxdt\right| \\
\le&\|P_{N_0}e^{it\Delta}\phi\|_{L_t^\infty L_x^2(J\times\mathbb{T}^d)}\|R_1\|_{L_{t,x}^{\infty^-}}\|P_{N_2}e^{it\Delta}\phi^{(2)}\|_{L_t^{1^+}L_x^{2^+}}\|P_{N_3}e^{it\Delta}\phi^{(3)}\|_{L_{t,x}^{\infty^-}} \\
\lesssim&|J|^{1-\epsilon}\frac{N_2^{\epsilon}N_3^{\frac{d}{2}-\epsilon}}{N_1^{s_c-\alpha}}\|P_{N_0}\phi\|_{L_x^2}\|P_{N_2}\phi^{(2)}\|_{L_x^2}\|P_{N_3}\phi^{(3)}\|_{L_x^2}.
\end{aligned}$$

And also we have

$$
\begin{aligned}
&\left|\int_{J\times\mathbb{T}^d}\overline{P_{N_0}e^{it\Delta}\phi}\mathcal{N}_2(\overline{R}_1,P_{N_2}e^{it\Delta}\phi^{(2)},P_{N_3}e^{it\Delta}\phi^{(3)})\,dxdt\right|\\
\lesssim&\left|\int_J\left(\int_{\mathbb{T}^d}\overline{P_{N_0}e^{it\Delta}\phi}\;\overline{R_1}\,dx\right)\left(\int_{\mathbb{T}^d}P_{N_2}e^{it\Delta}\phi^{(2)}P_{N_3}e^{it\Delta}\phi^{(3)}\,dx\right)dt\right|\\
\le&\|P_{N_0}e^{it\Delta}\phi\|_{L_t^\infty L_x^2(J\times\mathbb{T}^d)}\|R_1\|_{L_t^{\infty-}L_x^2}\|P_{N_2}e^{it\Delta}\phi^{(2)}\|_{L_t^{\infty-}L_x^2}\|P_{N_3}e^{it\Delta}\phi^{(3)}\|_{L_t^{1+}L_x^2}\\
\lesssim&|J|^{1-\epsilon}\frac{1}{N_1^{s_c-\alpha}}\|P_{N_0}\phi\|_{L_x^2}\|P_{N_2}\phi^{(2)}\|_{L_x^2}\|P_{N_3}\phi^{(3)}\|_{L_x^2}.
\end{aligned}
$$

So we obtain that

$$
\begin{aligned}
&\left|\int_{J\times\mathbb{T}^d}\overline{P_{N_0}e^{it\Delta}\phi}\mathcal{N}(\overline{R}_1,P_{N_2}e^{it\Delta}\phi^{(2)},P_{N_3}e^{it\Delta}\phi^{(3)})\,dxdt\right|\\
\lesssim&|J|^{1-\epsilon}\frac{N_2^\epsilon N_3^{\frac{d}{2}-\epsilon}}{N_1^{s_c-\alpha}}\|P_{N_0}\phi\|_{L_x^2}\|P_{N_2}\phi^{(2)}\|_{L_x^2}\|P_{N_3}\phi^{(3)}\|_{L_x^2}.
\end{aligned}
\tag{4.5.31}
$$

Step 3 Average the estimates (4.5.30) (4.5.31) in Step 1 and Step 2, we obtain that

$$
\begin{aligned}
&\left|\int_{J\times\mathbb{T}^d}\overline{P_{N_0}e^{it\Delta}\phi}\mathcal{N}(\overline{R}_1,P_{N_2}e^{it\Delta}\phi^{(2)},P_{N_3}e^{it\Delta}\phi^{(3)})\,dxdt\right|\\
\lesssim&|J|^{\frac{1}{2}}\frac{N_2^{\frac{d-1}{8}}N_3^{\frac{d}{2}}}{N_1^{s_c+\frac{1}{4}-\alpha-\epsilon}}\|P_{N_0}\phi\|_{L_x^2}\|P_{N_2}\phi^{(2)}\|_{L_x^2}\|P_{N_3}\phi^{(3)}\|_{L_x^2}.
\end{aligned}
\tag{4.5.32}
$$

By (4.5.32) and Lemma 4.3.3, we hold that

$$
\left|\int_0^\delta\int_{\mathbb{T}^d}\mathcal{N}(\overline{R}_1,D_2,D_3)\overline{u}_0dxdt\right|\lesssim\delta^{\frac{1}{2}}\frac{N_1^\epsilon N_2^{\frac{d-1}{8}}N_3^{\frac{d}{2}}}{N_1^{s_c+\frac{1}{4}-\alpha}}\|u_0\|_{U_\Delta^2}\|D_2\|_{U_\Delta^2}\|D_3\|_{U_\Delta^2}.\tag{4.5.33}
$$

Step 4 (only for $d=3,4$) If we set $\frac{1}{p_c^+}=\frac{d}{2(d+2)}-\epsilon$ and $\frac{1}{q}=1-\frac{3}{p_c^+}$. By Hölder inequality and Lemma 4.2.4 after excluding a subset of probability $e^{-\frac{1}{\delta^c}}$, we have

$$
\begin{aligned}
&\left|\int_{J\times\mathbb{T}^d}\overline{u_0}\mathcal{N}_1(\overline{R}_1,D_2,D_3)\,dxdt\right|=\left|\int_{J\times\mathbb{T}^d}\overline{u_0}\;\overline{R}_1D_2D_3\,dxdt\right|\\
\le&\|u_0\|_{L_{t,x}^{p_c^+}(J\times\mathbb{T}^d)}\|R_1\|_{L_{t,x}^q}\|D_2\|_{L_{t,x}^{p_c^+}}\|D_3\|_{L_{t,x}^{p_c^+}}\\
\lesssim&\frac{N_1^\epsilon}{N_1^{s_c-\alpha}}\|u_0\|_{U_\Delta^{p_c^+}}\|D_2\|_{U_\Delta^{p_c^+}}\|D_3\|_{U_\Delta^{p_c^+}}.
\end{aligned}
$$

And also we have

$$\left|\int_{J\times\mathbb{T}^d}\overline{u_0}\mathcal{N}_2(\overline{R}_1,D_2,D_3)\,dxdt\right|\lesssim\left|\int_J\left(\int_{\mathbb{T}^d}\overline{u_0}\ \overline{R_1}\,dx\right)\left(\int_{\mathbb{T}^d}D_2D_3\,dx\right)dt\right|$$

$$\le\|u_0\|_{L_t^{p_c^+}L_x^2(J\times\mathbb{T}^d)}\|R_1\|_{L_t^qL_x^2}\|D_2\|_{L_t^{p_c^+}L_x^2}\|D_3\|_{L_t^{p_c^+}L_x^2}$$

$$\lesssim\frac{N_1^\epsilon}{N_1^{s_c-\alpha}}\|u_0\|_{U_\Delta^{p_c^+}}\|D_2\|_{U_\Delta^{p_c^+}}\|D_3\|_{U_\Delta^{p_c^+}}.$$

So we obtain that

$$\left|\int_{J\times\mathbb{T}^d}\overline{u_0}\mathcal{N}(\overline{R}_1,D_2,D_3)\,dxdt\right|\lesssim\frac{N_1^\epsilon}{N_1^{s_c-\alpha}}\|u_0\|_{U_\Delta^{p_c^+}}\|D_2\|_{U_\Delta^{p_c^+}}\|D_3\|_{U_\Delta^{p_c^+}}. \tag{4.5.34}$$

Step 5 Finally, when $d=3,4$, by the embedding properties Remark 3.3.3 and (3.3.5), we average (4.5.33) and (4.5.34) with weights: $(\frac{8d-16}{5d-1},\frac{15-3d}{5d-1})$; when $d\ge 5$, we directly use (4.5.33). Summarizing these two cases, we obtain that

$$\begin{aligned}&\left|\int_0^\delta\int_{\mathbb{T}^d}\mathcal{N}(\overline{R}_1,D_2,D_3)\overline{u}_0dxdt\right|\\ \lesssim&\delta^{4s_r(d)-\epsilon}\frac{N_2^{s_c-s}N_3^{s_c-s}}{N_1^{s_c-s+s_r(d)+\frac{1}{4}-\alpha-\epsilon}}\|u_0\|_{Y^{-s}}\|D_2\|_{X^s}\|D_3\|_{X^s}\end{aligned} \tag{4.5.35}$$

where

$$s_r(d)=\begin{cases}\frac{1}{7} & d=3\\ \frac{4}{19} & d=4\\ \frac{1}{4} & d\ge 5.\end{cases}$$

Since $s<s_c+s_r(d)-\alpha$, we have the proposition. $\square$

Remark 4.5.11. The proofs when the conjugate is on the different position are similar, for example, $\mathcal{N}(\overline{R_1},D_2,D_3)$ in the case $\mathbf{B(d)}$. The only difference between $\mathcal{N}(\overline{R_1},D_2,D_3)$ and $\mathcal{N}(R_1,\overline{D_2},D_3)$ in $\mathbf{B(d)}$ is (4.5.28). In the $\mathcal{N}(\overline{R_1},D_2,D_3)$, the set in (4.5.28) should be $\{(n,n',n_2):n\neq n',2\langle n_3-n,n+n_2\rangle=k,\ 2\langle n_3-n',n'+n_2\rangle=k\}$. First, we can count the number of pair (n,n_2) and by Lemma 4.4.3, it is bounded by $N_1^{d-2+\epsilon}N_2^d$. And then the number of possible n' is bounded $N_1^{d-2+\epsilon}$ by Lemma 4.4.4. So the size of the upper lattice set is bounded by $N_1^{2(d-2)+\epsilon}N_2^d$, which is better than the correponding bound in $\mathcal{N}(R_1,\overline{D_2},D_3)$.

Proof of Proposition 4.5.1

Proof. Suppose dyadic coordinates $N_1 \geq N_2 \geq N_3$, consider arbitrary $w_l^{(i)}$ and $u_l^{(0)}$ satisfing $w_l^{(i)}(t) = w^{(i)}(t)$ and $u_l^{(0)}(t) = u^{(0)}(t)$, for $t \in I$ and $i = 1, 2, 3$. Then we have

$$
\begin{aligned}
&\left|\int_0^\delta \int_{\mathbb{T}^4} \mathcal{N}(w^{(1)} + v_0^\omega, \overline{w^{(2)} + v_0^\omega}, w^{(3)} + v_0^\omega)\overline{u^{(0)}} dxdt\right| \\
=&\left|\int_0^\delta \int_{\mathbb{T}^4} \mathcal{N}(w_l^{(1)} + v_0^\omega, \overline{w_l^{(2)} + v_0^\omega}, w_l^{(3)} + v_0^\omega)\overline{u_l^{(0)}} dxdt\right| \\
=&\sum_{N_0 \lesssim N_1 \geq N_2 \geq N_3} \left|\int_0^\delta \int_{\mathbb{T}^4} \mathcal{N}(P_{N_1}(w_l^{(1)} + v_0^\omega), \overline{P_{N_2}(w_l^{(2)} + v_0^\omega)}, P_{N_3}(w_l^{(3)} + v_0^\omega))\overline{P_{N_0} u_l^{(0)}} dxdt\right|
\end{aligned}
$$

There are only two cases: $N_0 \sim N_1 \geq N_2 \geq N_3$ and $N_0 \lesssim N_1 \sim N_2 \geq N_3$.

By Proposition 5.3 - 5.11, we can always sum up for these two cases to obtain the following estimate:

$$
\begin{aligned}
&\left|\int_0^\delta \int_{\mathbb{T}^d} \mathcal{N}(w^{(1)} + v_0^\omega, \overline{w^{(2)} + v_0^\omega}, w^{(3)} + v_0^\omega)\overline{u^{(0)}}\, dxdt\right| \\
\lesssim&\|u_l^{(0)}\|_{Y^{-s}} \left(\delta^{c\min\{1, s-s_c\}} \|w_l^{(1)}\|_{X^s} \|w_l^{(2)}\|_{X^s} \|w_l^{(3)}\|_{X^s} + \delta^c \sum_{\substack{J \subset \{1,2,3\} \\ J \neq \{1,2,3\}}} \prod_{j \in J} \|w_l^{(j)}\|_{X^s}\right),
\end{aligned}
$$

by the definition of $X^s(I)$ and $Y^{-s}(I)$ in Definition 4.3.1, we obtain Proposition 4.5.1. □

4.6 Proof of the Theorem 4.1.4

To prove Theorem 4.1.4 (especially the case $s = s_c$), we should introduce two weaker norms $Z^s(I)$ and $Z'^s(I)$-norm than $X^s(I)$-norm.

Definition 4.6.1.

$$
\|v\|_{Z^s(I)} := \sup_{J \subset I} \left(\sum_{N \in 2^{\mathbb{Z}}} N^{4s+2-d} \|P_N v\|^4_{L^4(\mathbb{T}^d \times J)}\right)^{\frac{1}{4}} \text{ and } \|v\|_{Z'^s(I)} := \|v\|^{\frac{3}{4}}_{Z^s(I)} \|v\|^{\frac{1}{4}}_{X^s(I)}.
$$

The following property show us that $Z^s(I)$ is a weaker norm than $X^s(I)$.

Proposition 4.6.2.

$$\|v\|_{Z^s(I)} \lesssim \|v\|_{X^s(I)}$$

.

Proof. By the definition of $Z^s(I)$ and the following Strichartz type estimates (Proposition 4.4.1), we obtain that

$$\sup_{J\subset I}\left(\sum_{N \text{ dydic number}} N^{4s+2-d}\|P_N v\|^4_{L^4(\mathbb{T}^d\times J)}\right)^{\frac{1}{4}} \lesssim \left(\sum_{N \text{ dydic number}} N^{4s}\|P_N v\|^4_{U^4_\Delta}\right)^{\frac{1}{4}} \lesssim \|v\|_{X^s(I)}.$$

□

Lemma 4.6.3 (Bilinear estimates in [68])**.** *Assuming $|I| \leq 1$ and $N_1 \geq N_2$, for any $v_1 \in Y^0(I)$ and $v_2 \in Y^{s_c}(I)$, where $s_c = \frac{d}{2} - 1$, there holds that*

$$\|P_{N_1}v_1 P_{N_2}v_2\|_{L^2_{x,t}(\mathbb{T}^d\times I)} \lesssim (\frac{N_2}{N_1} + \frac{1}{N_2})^\kappa \|P_{N_1}v_1\|_{Y^0(I)}\|P_{N_2}v_2\|_{Y^{s_c}(I)} \tag{4.6.1}$$

for some $\kappa > 0$.

Remark 4.6.4. This Bilinear estimate is a simple d-dimension generalization of Proposition 2.8 in [68]. The proof of Lemma 4.6.3 is almost the same as the $d = 4$ case in [68] and heavily rely on L^p estimates in Proposition 4.4.1 (for some $p < 4$). In the proof not only we need the decoupling properties for spatial frequency, but also we need further trip partitions to apply the decoupling properties for time frequency.

Let's introduce an refined nonlinear estimate for $s = s_c$ case, which is a d-dimension generalization of Lemma 3.2 in [72].

Proposition 4.6.5 (Refined nonlinear estimate)**.** *For $v_k \in X^{s_c}(I)$, $k = 1, 2, 3$, $|I| \leq 1$, we hold the estimate*

$$\|\mathcal{I}(\mathcal{N}(\widetilde{v_1}, \widetilde{v_2}, \widetilde{v_3}))\|_{X^{s_c}(I)} \lesssim \sum_{\{i,j,k\}=\{1,2,3\}} \|v_i\|_{X^{s_c}(I)}\|v_j\|_{Z'^{s_c}(I)}\|v_k\|_{Z'^{s_c}(I)} \tag{4.6.2}$$

where $\widetilde{v_k} = v_k$ or $\widetilde{v_k} = \overline{v_k}$ for $k = 1, 2, 3$.

Proof. By Proposition 4.3.8, we suppose N_0, N_1, N_2, N_3 are dyadic, and by the symmetry, we assume $N_1 \geq N_2 \geq N_3$. Since it's easy to check that $\mathcal{N}_2$ is simple to bound, we just need to show the case $\mathcal{N}_1$.

$$\begin{aligned}\|\mathcal{I}(\mathcal{N}_1(\widetilde{v_1}, \widetilde{v_2}, \widetilde{v_3}))\|_{X^{s_c}(I)} &\lesssim \sup_{\|u_0\|_{Y^{-s_c}}=1} \left| \int_{\mathbb{T}^d \times I} \overline{u_0} \prod_{k=1}^{3} \widetilde{v_k}\, dxdt \right| \\ &\leq \sup_{\|u_0\|_{Y^{-s_c}}=1} \sum_{N_0, N_1 \geq N_2 \geq N_3} \left| \int_{\mathbb{T}^d \times I} \overline{P_{N_0} u_0} \prod_{k=1}^{3} P_{N_k} \widetilde{v_k}\, dxdt \right|\end{aligned}$$

Then we know $N_1 \sim \max\{N_2, N_0\}$ by the spatial frequency orthogonality. There are two cases:

1. $N_0 \sim N_1 \geq N_2 \geq N_3$;
2. $N_0 \leq N_2 \sim N_1 \geq N_3$.

Case 1: $N_0 \sim N_1 \geq N_2 \geq N_3$

By Cauchy-Schwarz inequality and Lemma 4.6.3, we have that

$$\begin{aligned}&\left| \int \overline{P_{N_0} u_0} P_{N_1} \widetilde{v_1} P_{N_2} \widetilde{v_2} P_{N_3} \widetilde{v_3}\, dxdt \right| \leq \|P_{N_0} u_0 P_{N_2} v_2\|_{L^2_{x,t}} \|P_{N_1} v_1 P_{N_3} v_3\|_{L^2_{x,t}} \\ \lesssim&\left(\frac{N_3}{N_1} + \frac{1}{N_3}\right)^{\kappa} \left(\frac{N_2}{N_0} + \frac{1}{N_2}\right)^{\kappa} \|P_{N_0} u_0\|_{Y^0(I)} \|P_{N_1} v_1\|_{Y^0(I)} \|P_{N_2} v_2\|_{X^{s_c}(I)} \|P_{N_3} v_3\|_{X^{s_c}(I)}\end{aligned} \tag{4.6.3}$$

Assumme $\{C_j\}$ us a cube partition of size N_2, and $\{C_k\}$ is a cube partition of size N_3. By $\{P_{C_j} P_{N_0} u_0 P_{N_2} v_2\}_j$ and $\{P_{C_k} P_{N_1} v_1 P_{N_3} v_3\}_k$ are both almost orthogonality, Proposition 4.4.1 and definition of Z^{s_c} norm, we obtain that

$$\begin{aligned}&\left| \int \overline{P_{N_0} u_0} P_{N_1} \widetilde{v_1} P_{N_2} \widetilde{v_2} P_{N_3} \widetilde{v_3}\, dxdt \right| \leq \|P_{N_0} u_0 P_{N_2} v_2\|_{L^2_{x,t}} \|P_{N_1} v_1 P_{N_3} v_3\|_{L^2_{x,t}} \\ \lesssim&\Big(\sum_{C_j} \|P_{C_j} P_{N_0} u_0 P_{N_2} v_2\|^2_{L^2_{x,t}}\Big)^{\frac{1}{2}} \Big(\sum_{C_k} \|P_{C_k} P_{N_1} v_1 P_{N_3} v_3\|^2\Big)^{\frac{1}{2}} \\ \lesssim&\Big(\sum_{C_j} \|P_{C_j} P_{N_0} u_0\|^2_{L^4_{x,t}} \|P_{N_2} u_2\|^2_{L^4_{x,t}}\Big)^{\frac{1}{2}} \Big(\sum_{C_k} \|P_{C_k} P_{N_1} v_1\|^2_{L^4_{x,t}} \|P_{N_3} v_3\|^2_{L^4_{x,t}}\Big)^{\frac{1}{2}} \\ \lesssim&\Big(\sum_{C_j} \|P_{C_j} P_{N_0} u_0\|^2_{Y^0(I)} (N_2^{\frac{d-2}{4}} \|P_{N_2} v_2\|_{L^4_{x,t}})^2\Big)^{\frac{1}{2}} \Big(\sum_{C_k} \|P_{C_k} P_{N_1} v_1\|^2_{Y^0(I)} (N_3^{\frac{d-2}{4}} \|P_{N_3} v_3\|_{L^4_{x,t}})^2\Big)^{\frac{1}{2}} \\ \lesssim&\|P_{N_0} u_0\|_{Y^0(I)} \|P_{N_1} v_1\|_{Y^0(I)} \|P_{N_2} v_2\|_{Z^{s_c}(I)} \|P_{N_2} v_2\|_{Z^{s_c}(I)}.\end{aligned} \tag{4.6.4}$$

Interpolate (4.6.3) and (4.6.4), and $N_0 \sim N_1$, we have

$$\begin{aligned}&\left|\int \overline{P_{N_0} u_0} P_{N_1} \widetilde{v_1} P_{N_2} \widetilde{v_2} P_{N_3} \widetilde{v_3}\, dxdt\right| \\ \lesssim&(\frac{N_3}{N_1}+\frac{1}{N_3})^{\kappa_1}(\frac{N_2}{N_0}+\frac{1}{N_2})^{\kappa_1}\|P_{N_0}u_0\|_{Y^{-s_c}(I)}\|P_{N_1}v_1\|_{X^{s_c}(I)}\|P_{N_2}v_2\|_{Z'^{s_c}(I)}\|P_{N_2}v_2\|_{Z'^{s_c}(I)}.\end{aligned} \tag{4.6.5}$$

Sum (4.6.5) over all $N_0 \sim N_1 \geq N_2 \geq N_3$,

$$\begin{aligned}&\sum_{N_0\sim N_1\geq N_2\geq N_3}(\frac{N_3}{N_1}+\frac{1}{N_3})^{\kappa_1}(\frac{N_2}{N_0}+\frac{1}{N_2})^{\kappa_1}\|P_{N_0}u_0\|_{Y^{-s_c}(I)}\|P_{N_1}v_1\|_{X^{s_c}(I)}\|P_{N_2}v_2\|_{Z'^{s_c}(I)}\|P_{N_2}v_2\|_{Z'^{s_c}(I)} \\ \lesssim&\|u_0\|_{Y^{-s_c}(I)}\|v_1\|_{X^{s_c}I}\|v_2\|_{Z'^{s_c}(I)}\|v_3\|_{Z'^{s_c}(I)}.\end{aligned}$$

Case 2: $N_0 \leq N_2 \sim N_1 \geq N_3$

Similar, we have

$$\begin{aligned}&\left|\int \overline{P_{N_0} u_0} P_{N_1} \widetilde{v_1} P_{N_2} \widetilde{v_2} P_{N_3} \widetilde{v_3}\, dxdt\right| \\ \lesssim&(\frac{N_3}{N_1}+\frac{1}{N_3})^{\kappa}(\frac{N_0}{N_2}+\frac{1}{N_0})^{\kappa}\|P_{N_0}u_0\|_{Y^0(I)}\|P_{N_1}v_1\|_{Y^0(I)}\|P_{N_2}v_2\|_{X^{s_c}(I)}\|P_{N_3}u_3\|_{X^{s_c}(I)}.\end{aligned} \tag{4.6.6}$$

Similiar with (4.6.4), we obtain that:

$$\begin{aligned}&\left|\int \overline{P_{N_0} u_0} P_{N_1} \widetilde{v_1} P_{N_2} \widetilde{v_2} P_{N_3} \widetilde{v_3}\, dxdt\right| \\ \lesssim&\|P_{N_0}u_0\|_{Y^0(I)}\|P_{N_1}v_1\|_{Y^0(I)}\|P_{N_2}v_2\|_{Z^{s_c}(I)}\|P_{N_3}v_3\|_{Z^{s_c}(I)}.\end{aligned} \tag{4.6.7}$$

Interpolating (4.6.6) and (4.6.7), and suming over $N_0 \leq N_2 \sim N_1 \geq N_3$, we have

$$\begin{aligned}&\sum_{N_0\leq N_2\sim N_1\geq N_3}\left|\int \overline{P_{N_0} u_0} P_{N_1} \widetilde{v_1} P_{N_2} \widetilde{v_2} P_{N_3} \widetilde{v_3}\, dxdt\right| \\ \lesssim&\|P_{N_0}u_0\|_{Y^{-s_c}(I)}\|P_{N_1}v_1\|_{X^{s_c}(I)}\|P_{N_2}v_2\|_{Z'^{s_c}(I)}\|P_{N_3}v_3\|_{Z'^{s_c}(I)}.\end{aligned}$$

Summarize these two cases, and similarly consider $N_1 \geq N_3 \geq N_2$, $N_2 \geq N_1 \geq N_3$, $N_2 \geq N_3 \geq N_1$, $N_3 \geq N_1 \geq N_2$, and $N_3 \geq N_2 \geq N_1$, then we can get the desired estimate (4.6.2).

$\square$

Proof of Theorem 4.1.4. Suppose $d \geq 3$, $s_r(d)$ is defined as (4.1.4) and $0 \leq \alpha < s_r(d)$.

Consider the mapping

$$\Phi(w) = \mathcal{I}(\mathcal{N}(w + v_0^\omega)),$$

and then the fixed point $w = \Phi(w)$ of the mapping Φ is the solution of IVP (4.1.5).

Case 1: $s_c < s < s_c + s_r(d) - \alpha$

Consider the set

$$S = \{w \in X^s(I) : \|w\|_{X^s(I)} \leq 1\}.$$

where $I = [0, \delta]$ and δ is to be determined.

To show Φ is a contraction mapping in S. Using Proposition 4.3.8, Proposition 4.5.1 and choosing δ small enough, we obtain that

$$\|\Phi(w)\|_{X^s(I)} \lesssim \delta^{c \min\{1, s-s_c\}}(1 + \|w\|_{X^s(I)} + \|w\|^2_{X^s(I)} + \|w\|^3_{X^s(I)}) \leq 1.$$

For any $w, v \in S$, using Proposition 4.3.8, Proposition 4.5.1 and choosing δ small enough, there exists $0 < k < 1$ such that

$$\begin{aligned}\|\Phi(w) - \Phi(v)\|_{X^s(I)} &\lesssim \delta^{c \min\{1, s-s_c\}}(1 + \|w\|_{X^s(I)} + \|v\|_{X^s(I)} + \|w\|^2_{X^s(I)} + \|v\|^2_{X^s(I)})\|w - v\|_{X^s(I)} \\ &\leq k\|w - v\|_{X^s(I)}.\end{aligned}$$

So Φ is a contraction mapping.

Case 2: $s = s_c$

Consider the set

$$S = \{w \in X^{s_c}(I) : \|w\|_{X^{s_c}(I)} \leq 1, \quad \|w\|_{Z'^{s_c}(I)} \leq a\}.$$

where $I = [0, \delta]$. a and δ is to be determined.

To show Φ is a contraction mapping in S. By Proposition 4.3.8, Proposition 4.5.1, Proposition 4.6.5, choosing δ small enough, we obtain that

$$\begin{aligned}\|\Phi(w)\|_{X^{s_c}(I)} &\lesssim \delta^c(1 + \|w\|_{X^{s_c}(I)} + \|w\|^2_{X^{s_c}(I)}) + \|w\|^2_{Z'^{s_c}(I)}\|w\|_{X^{s_c}(I)} \\ &\lesssim \delta^c + a^2.\end{aligned}$$

and also

$$\begin{aligned}\|\Phi(w)\|_{Z'^{s_c}(I)} &\lesssim \delta^c(1+\|w\|_{X^{s_c}(I)}+\|w\|^2_{X^{s_c}(I)})+\|w\|^2_{Z'^{s_c}(I)}\|w\|_{X^{s_c}(I)} \\ &\lesssim \delta^c + a^2.\end{aligned}$$

For any $w, v \in S$, by Proposition 4.3.8, Proposition 4.5.1 and Proposition 4.6.5, choosing δ small enough, there exists $0 < k < 1$ such that

$$\begin{aligned}\|\Phi(w)-\Phi(v)\|_{X^{s_c}(I)} &\lesssim (\|w\|_{Z'^{s_c}(I)}+\|v\|_{Z'^{s_c}(I)})(\|w\|_{X^{s_c}(I)(I)}+\|v\|_{X^{s_c}(I)})\|w-v\|_{X^{s_c}(I)} \\ &\quad + \delta^c(\|w\|_{X^{s_c}(I)}+\|v\|_{X^{s_c}(I)}+1)\|w-v\|_{X^{s_c}(I)} \\ &\lesssim (a+\delta^c)\|w-v\|_{X^{s_c}(I)}.\end{aligned}$$

Set $a = \delta$ and let δ small enough, then we obtain that Φ is a contraction mapping.

□

4.7 The analog result in $X^{s,b}$ space

$X^{s,b}$ spaces (also known as *Fourier restriction spaces* or *Bourgain spaces*) were firstly introduced by Bourgain [7][12] in the context of Schrödinger and KdV equations. A nice summary is give by Tao (Section 2.6 in [102]).

Theorem 4.7.1 (Analog of Theorem 4.1.2 in $X^{s,b}$)**.** *Suppose $d \geq 3$ and $s_r(d)$ is defined by (4.1.4) Let $0 \leq \alpha < s_r(d)$, $s \in (s_c, s_c + s_r(d) - \alpha)$. Then there exist some $b > \frac{1}{2}$, $\delta_0 > 0$ and $r = r(s, \alpha) > 0$ such that for any $0 < \delta < \delta_0$, there exists $\Omega_\delta \in A$ with*

$$\mathbb{P}(\Omega_\delta^c) < e^{-\frac{1}{\delta^r}},$$

and for each $\omega \in \Omega_\delta$ there exists a unique solution u of (4.1.1) in the space

$$S(t)\phi^\omega + X^{s,b}([0,\delta])_{dist},$$

where $S(t)\phi^\omega$ is the linear evolution of the initial data ϕ^ω given by (4.1.2).

Remark 4.7.2. The proof of Theorem 7.1 follows the analog nonlinear estimate in $X^{s,b}$ space (Proposition 4.7.9). In the proof of Proposition 4.7.9, we can see the reason why $s = s_c$ case fails in $X^{s,b}$ space.

4.7.1 $X^{s,b}$ space and some properties

Let's recall the definition and some properties of $X^{s,b}$ spaces.

Definition 4.7.3. Suppose $d \in \mathbb{Z}^+$ and $s, b \in \mathbb{R}$, for any $u : \mathbb{R}\times\mathbb{T}^d \to \mathbb{C}$, $u \in X^{s,d}(\mathbb{R}\times\mathbb{T}^d)$ (short for $X^{s,b}$) if

$$\|u\|_{X^{s,b}} := \|\langle n\rangle^s \langle \lambda + |n|^2\rangle^b\, \widehat{u}(n,\lambda)\|_{l^2_n(\mathbb{Z}^d)L^2_\lambda(\mathbb{R})} < +\infty,$$

where $\widehat{u}(n,\lambda)$ is the space-time Fourier transformation of u. Note that $\|u\|_{X^{s,b}}$ can be also defined as $\|e^{-it\Delta}u\|_{H^b_t H^s_x(\mathbb{R}\times\mathbb{T}^d)}$.

Definition 4.7.4 (The corresponding restriction spaces to a time interval I)**.** Suppose $d \in \mathbb{Z}^+$, $s, b \in \mathbb{R}$ and I is a time interval in $\mathbb{R}$, for any $u : I\times\mathbb{T}^d \to \mathbb{C}$, then $u \in X^{s,b}(I\times\mathbb{T}^d)$ (short for $X^{s,b}(I)$) if

$$\|u\|_{X^{s,b}(I)} := \inf_{v\in X^{s,b}} \{\|v\|_{X^{s,b}} : v(t) = u(t) \text{ for all } t \in I\} < +\infty.$$

Remark 4.7.5 (Some embedding properties)*.* For $s \le s'$ and $b \le b'$, we obtain that

$$X^{s,b} \hookrightarrow X^{s',b'}.$$

Furthermore, if $b > \frac{1}{2}$, then

$$X^{s,b} \hookrightarrow L^\infty_t(\mathbb{R}, H^s(\mathbb{T}^d)).$$

Proposition 4.7.6 (Analog of Proposition 4.3.2 in $X^{s,b}$)**.** *Let $T_0 : L^2_x \times \cdots \times L^2_x \to L^1_{x,loc}(\mathbb{T}^d)$ be an m-linear operator. Assume that for some $1 \le p \le \infty$*

$$\|T_0(e^{it\Delta}\phi_1, \cdots, e^{it\Delta}\phi_m)\|_{L^p(\mathbb{R}\times\mathbb{T}^d)} \lesssim \prod_{i=1}^m \|\phi_i\|_{L^2(\mathbb{T}^d)}. \tag{4.7.1}$$

Then, for any $b > \frac{1}{2}$, there exists an extension $T : X^{0,b} \times \cdots \times X^{0,b} \to L^p(\mathbb{R} \times \mathbb{T}^d)$ satisfying

$$\|T(u_1, \cdots, u_m)\|_{L^p(\mathbb{R}\times\mathbb{T}^d)} \lesssim \prod_{i=1}^{m} \|u_i\|_{X^{0,b}}; \tag{4.7.2}$$

and such that $T(u_1, \cdots, u_m)(t, \cdot) = T_0(u_1(t), \cdots, u_m(t))(\cdot)$, a.e.

Proof. Suppose that for all $i = 1, \cdots, m$,

$$u_i(x,t) = \int_{\mathbb{R}} d\lambda_i \sum_{n_i \in \mathbb{Z}^d} \widehat{u_i}(n_i, \lambda_i) e^{in_i \cdot x + \lambda_i t} = \int_{\mathbb{R}} e^{it\Delta} \phi^{(i)}_{\mu_i} d\mu_i, \tag{4.7.3}$$

where $\phi^{(i)}_{\mu_i} := \sum_{n_i \in \mathbb{Z}^d} \widehat{u_i}(n_i, \mu_i - |n_i|^2) e^{i\mu_i t} e^{in_i \cdot x}$ and $\mu_i = \lambda_i + |n_i|^2$.

Then, by (4.7.3), Minkowski integral inequality and (4.7.1), we obtain that

$$\begin{aligned}
&\|T(u_1, \cdots, u_m)\|_{L^p(\mathbb{R}\times\mathbb{T}^d)} = \|T(\int_{\mathbb{R}} e^{it\Delta} \phi^{(1)}_{\mu_1} d\mu_1, \cdots, \int_{\mathbb{R}} e^{it\Delta} \phi^{(m)}_{\mu_m} d\mu_m)\|_{L^p(\mathbb{R}\times\mathbb{T}^d)} \\
&\leq \int_{\mathbb{R}^m} \prod_{i=1}^{m} d\mu_i \|T_0(e^{it\Delta}\phi^{(1)}_{\mu_1}, \cdots, e^{it\Delta}\phi^{(m)}_{\mu_m})\|_{L^p(\mathbb{R}\times\mathbb{T}^d)} \\
&\lesssim \prod_{i=1}^{m} \int_{\mathbb{R}} \|\phi^{(i)}_{\mu_i}\|_{L^2_x(\mathbb{T}^d)} d\mu_i.
\end{aligned} \tag{4.7.4}$$

For a fixed i and any $b > \frac{1}{2}$, by Hölder inequality and $\int_{\mathbb{R}} \frac{1}{\langle \mu_i \rangle^{2b}} d\mu < +\infty$, we have that

$$\begin{aligned}
\int_{\mathbb{R}} \|\phi^{(i)}_{\mu_i}\|_{L^2_x(\mathbb{T}^d)} d\mu_i &= \int_{\mathbb{R}} \frac{1}{\langle \mu_i \rangle^b} \|\langle \mu_i \rangle^b \phi^{(i)}_{\mu_i}\|_{L^2_x(\mathbb{T}^d)} d\mu_i \\
&\leq \left(\int_{\mathbb{R}} \frac{1}{\langle \mu_i \rangle^{2b}} d\mu_i\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} \|\langle \mu_i \rangle^b \phi^{(i)}_{\mu_i}\|^2_{L^2_x(\mathbb{T}^d)} d\mu_i\right)^{\frac{1}{2}} \\
&= \left(\int_{\mathbb{R}} \sum_{n_i} \langle \lambda_i + |n_i|^2 \rangle^{2b} |\widehat{u_i}(n_i, \lambda_i)|^2 d\lambda\right)^{\frac{1}{2}} \\
&= \|u_i\|_{X^{0,b}}.
\end{aligned} \tag{4.7.5}$$

By (4.7.4) and (4.7.5), we obtain the proposition. □

Following Proposition 4.4.1 and Proposition 4.7.6, we obtain the following corollary.

Corollary 4.7.7 (Analog of Proposition 4.4.1)**.** *Let $b > \frac{1}{2}$ and $p > p_c$, where $p_c = \frac{2(d+2)}{d}$. For all $N \geq 1$ we have*

$$\|P_N u\|_{L^p_{x,t}(\mathbb{T}\times\mathbb{T}^d)} \lesssim N^{\frac{d}{2} - \frac{d+2}{p}} \|P_C u\|_{X^{0,b}}, \tag{4.7.6}$$

$$\|P_C u\|_{L^p_{x,t}(\mathbb{T}\times\mathbb{T}^d)} \lesssim N^{\frac{d}{2} - \frac{d+2}{p}} \|P_C u\|_{X^{0,b}}, \tag{4.7.7}$$

where C is a cube in $\mathbb{Z}^d$ with sides parallel to the axis of side length N.

Proposition 4.7.8 ($X^{s,b}$ norm of Duhamel's formula). *For any $s > \frac{1}{2}$, $s \in \mathbb{R}$, and I is a time interval $[0,\delta]$, we obtain that*

$$\|\mathcal{I}(f)\|_{X^{s,b}(I)} \leq \sup_{\|v\|_{\widetilde{X}^{-s,1-b}(I)}=1} \left| \int_0^\delta \int_{\mathbb{T}^d} f(t,x)\overline{v(t,x)}dxdt \right|. \tag{4.7.8}$$

where $\|v\|_{\widetilde{X}^{-s,1-b}(I)} := \|e^{it\Delta}v\|_{H_t^{1-b}H_x^{-s}(I\times\mathbb{T}^d)}$

4.7.2 Nonlinear estimate in $X^{s,b}$

Proposition 4.7.9 (Analog of Proposition 4.5.1 in $X^{s,b}$). *Suppose $d \geq 3$ and $s_r(d)$ is given in (4.1.4). Let $0 \leq \alpha < s_r'(d)$, $s \in (s_c, s_c + s_r'(d) - \alpha)$, $r > 0$, $0 \leq \delta < 1$, and $I = [0,\delta]$. There exist some $b > \frac{1}{2}$, $\Omega_\delta \subset \Omega$ with $\mathbb{P}(\Omega_\delta^c) < e^{-1/\delta^r}$ and $c > 0$, such that we obtain that*

$$\begin{aligned}&\left| \int_0^\delta \int_{\mathbb{T}^d} \mathcal{N}(w^{(1)} + v_0^\omega, \overline{w^{(2)} + v_0^\omega}, w^{(3)} + v_0^\omega)\overline{u^{(0)}}\, dxdt \right| \\ \lesssim&\|u^{(0)}\|_{\widetilde{X}^{-s,1-b}(I)} \left(\delta^{c\min\{1,s-s_c\}}\|w^{(1)}\|_{X^{s,b}(I)}\|w^{(2)}\|_{X^{s,b}(I)}\|w^{(3)}\|_{X^{s,b}(I)} + \delta^c \sum_{\substack{S_J\subset\{1,2,3\}\\ J\neq\{1,2,3\}}} \prod_{j\in S_J} \|w^{(j)}\|_{X^{s,b}(I)} \right),\end{aligned}$$

where v_0^ω is defined (4.1.8), $u^{(0)} \in \widetilde{X}^{-s,1-b}(I)$ and $w^{(i)} \in X^{s,b}(I)$ for $i = 1,2,3$. (when the subset $S_J = \emptyset$, $\prod_{j\in S_J}\|w^{(j)}\|_{X^{s,b}(I)} = 1$.)

Proof. The proof of Proposition 4.7.9 is similar with the proof of Proposition 4.5.1: we first dyadically decompose the terms in each position of the nonlinear term $\mathcal{N}(u_1,u_2,u_3)$, and then we have the same cases: Denote

$$R_i = P_{N_i}v_0^\omega \text{ and } D_i = P_{N_i}w \text{ for } i \in \{1,2,3\}. \tag{4.7.9}$$

The list of all cases of (u_1,u_2,u_3) is below:

A. $u_1 = D_1$:

(a) (D_1, D_2, D_3);

(b) (D_1, D_2, R_3);

(c) (D_1, R_2, D_3);

(d) (D_1, R_2, R_3);

B. $u_1 = R_1$:

(a) (R_1, R_2, R_3);

(b) (R_1, R_2, D_3);

(c) (R_1, D_2, R_3);

(d) (R_1, D_2, D_3).

For the **Case A**, we can control the nonlinear terms as Proposition 5.3 - 5.6 in a similar approach. For simplicity, let me just show the **Case A (a)** as an example.

Choose $b = \frac{1}{2}+$ which is close to $\frac{1}{2}$ enough. We follow the proof of Proposition 5.3 almost identically, but Proposition 4.7.7 take place of Proposition 4.4.1. Then we obtain that

$$\begin{aligned}&\left|\int_0^\delta \int_{\mathbb{T}^d} \mathcal{N}(\widetilde{D}_1, \widetilde{D}_2, \widetilde{D}_3)\overline{u}_0 \, dxdt\right| \\ \lesssim&\delta^{c\min\{1,s-s_c\}}\left(\frac{N_3 \min\{N_0, N_2\}}{N_2^2}\right)^c \frac{1}{N_2^{s-s_c}} \|u_0\|_{X^{-s,b}} \|D_1\|_{X^{s,b}} \|D_2\|_{X^{s,b}} \|D_3\|_{X^{s,b}}.\end{aligned} \tag{4.7.10}$$

To get the $\|u_0\|_{X^{-s,1-b}}$ instead of $\|u_0\|_{X^{-s,b}}$, we need another nonlinear estimate. By the same cube decomposition, Hölder inequality, and Proposition 4.7.7, we obtain that (here we only consider the main part of $\mathcal{N}(\widetilde{D_1}, \widetilde{D}_2, \widetilde{D}_3)$, and the remaining part is easily bounded)

$$
\begin{aligned}
&\left|\int_0^\delta \int_{\mathbb{T}^d} \overline{u}_0 \widetilde{D}_1 \widetilde{D}_2 \widetilde{D}_3 \, dxdt\right| \\
\leq & \sum_{C_j \sim C_k} \int_0^\delta \int_{\mathbb{T}^d} \left|(P_{C_j} \overline{u}_0)(P_{C_k} \widetilde{D}_1) \widetilde{D}_2 \widetilde{D}_3\right| \, dxdt \\
\lesssim & \sum_{C_j \sim C_k} \|P_{C_j} u_0\|_{L^2_{t,x}} \|P_{C_k} D_1\|_{L^4_{t,x}} \|D_2\|_{L^4_{t,x}} \|D_3\|_{L^\infty_{t,x}} \qquad (4.7.11) \\
\lesssim & \sum_{C_j \sim C_k} N_2^{s_c} N_3^{s_c+1} \|P_{C_j} u_0\|_{X^{0,0}} \|P_{C_k} D_1\|_{X^{0,b}} \|D_2\|_{X^{0,b}} \|D_3\|_{X^{0,b}} \\
\lesssim & \frac{N_3^{1-(s-s_c)}}{N_2^{s-s_c}} \|u_0\|_{X^{-s,0}} \|D_1\|_{X^{s,b}} \|D_2\|_{X^{s_c,b}} \|D_3\|_{X^{s,b}}
\end{aligned}
$$

Using complex interpolation method from $X^{s,b}$ and $X^{s,0}$ to $X^{s,1-b}$ and interpolating (4.7.10) and (4.7.11) (actually we don't interpolate (4.7.10) and (4.7.11) directly but interpolate two estimates in the process of (4.7.10) and (4.7.11)), we obtain that

$$
\begin{aligned}
&\left|\int_0^\delta \int_{\mathbb{T}^d} \mathcal{N}(\widetilde{D}_1, \widetilde{D}_2, \widetilde{D}_3) \overline{u}_0 \, dxdt\right| \\
\lesssim & \delta^{c \min\{1, s-s_c\}} \left(\frac{N_3 \min\{N_0, N_2\}}{N_2^2}\right)^{c-\epsilon} \frac{N_3^\epsilon}{N_2^{s-s_c}} \|u_0\|_{X^{-s,1-b}} \|D_1\|_{X^{s,b}} \|D_2\|_{X^{s,b}} \|D_3\|_{X^{s,b}}.
\end{aligned}
\tag{4.7.12}
$$

Observe the bound in (4.7.12), to sum up over N_2 and N_3, we need $s > s_c + \epsilon$, which is the reason why we can't obtain $s = s_c$ case in $X^{s,b}$ space.

For the **Case B**, we could obtain analogs of Proposition 5.7-5.10 by modifying the proofs a little bit. Let me show the **Case B (d)** as an example. Similar with the proof of Proposition 5.10, we only focus $N_0 \sim N_1 \geq N_2 \geq N_3$. Then

$$
\left|\int_0^\delta \int_{\mathbb{T}^d} \mathcal{N}(\widetilde{R}_1, \widetilde{D}_2, \widetilde{D}_3) \overline{u}_0 dxdt\right| \leq \|R_1\|_{L^\infty_{t,x}} \|D_2\|_{L^3_{t,x}} \|D_3\|_{L^3_{t,x}} \|u_0\|_{L^3_{t,x}}. \tag{4.7.13}
$$

By Hausdorff-Young inequality w.r.p.t the time t and Hölder inequality, we obtain that for a general function u and dyadic number N,

$$\begin{aligned}\|P_N u\|_{L^3_{t,x}} &\lesssim \sum_{|n|\sim N} \|e^{ix\cdot n}\int \widehat{u}(n,\lambda)e^{\lambda t}\,d\lambda\|_{L^3_{t,x}} \\ &\lesssim \sum_{|n|\sim N}\left(\int |\widehat{u}(n,\lambda)|^{\frac{3}{2}}\,d\lambda\right)^{\frac{2}{3}} \lesssim \|P_N u\|_{X^{\epsilon,\frac{1}{6}+\epsilon}}.\end{aligned} \tag{4.7.14}$$

By Corollary 4.2.4 and (4.7.14), we obtain that

$$\begin{aligned}\text{LHS of (4.7.13)} &\lesssim \frac{\log N_1}{N_1^{s_c-\alpha}}\|D_2\|_{X^{\epsilon,\frac{1}{6}+\epsilon}}\|D_3\|_{X^{\epsilon,\frac{1}{6}+\epsilon}}\|u_0\|_{X^{\epsilon,\frac{1}{6}+\epsilon}} \\ &\lesssim \frac{N_1^{s-s_c+\alpha+\epsilon}}{N_2^s N_3^s}\|D_2\|_{X^{s,\frac{1}{6}+\epsilon}}\|D_3\|_{X^{s,\frac{1}{6}+\epsilon}}\|u_0\|_{X^{-s,\frac{1}{6}+\epsilon}}\end{aligned} \tag{4.7.15}$$

Thus the estimate (4.7.15) is conclusive provided for some deterministic term, we consider the contribution $\widehat{u_0}|_{\langle\lambda+|n|^2\rangle > N_1^{4(s-s_c+\alpha)}}$. Thus in this case, LHS of (4.7.13) mab be estimated assuming

$$\langle\lambda+|n|^2\rangle \leq N_1^{4(s-s_c+\alpha)}.$$

For $s-s_c+\alpha<\frac{1}{2}$, replacing Proposition 4.3.2 and 4.3.3 by Proposition 4.7.6, we could recover the proof of Proposition 5.10 and obtain an analog of (4.5.35),

$$\begin{aligned}&\left|\int_0^\delta\int_{\mathbb{T}^d}\mathcal{N}(\overline{R}_1,D_2,D_3)\overline{u}_0dxdt\right| \\ \lesssim&\delta^{4s_r(d)-\epsilon}\frac{N_2^{s_c-s}N_3^{s_c-s}}{N_1^{s_c-s+s_r(d)+\frac{1}{4}-\alpha-\epsilon}}\|u_0\|_{\widetilde{X}^{-s,b}}\|D_2\|_{X^{s,b}}\|D_3\|_{X^{s,b}} \\ \lesssim&\delta^{4s_r(d)-\epsilon}\frac{N_1^{4(s-s_c+\alpha)(2b-1)}N_2^{s_c-s}N_3^{s_c-s}}{N_1^{s_c-s+s_r(d)+\frac{1}{4}-\alpha-\epsilon}}\|u_0\|_{\widetilde{X}^{-s,1-b}}\|D_2\|_{X^{s,b}}\|D_3\|_{X^{s,b}}.\end{aligned} \tag{4.7.16}$$

where $s_r(d)$ is defined by (4.1.4), since $s<s_c+s_r(d)-\alpha$, we have the proposition.

In a similar idea, we can also recover the other cases in $X^{s,b}$. □

APPENDIX

PROFILE DECOMPOSITION FOR THE LINEAR SCHRÖDINGER PROPAGATOR ON TORUS

In this chapter, we prove the profile decomposition for the linear Schrödinger propagator on $\mathbb{T}^4$ (Proposition 3.6.3). Considering a non-compact mapping between two function spaces, the profile decomposition is a tool for measuring how far is the mapping from being compact. The profile decomposition was first developed by Gerard [52] in the context of Sobolev embeddings relying on an improved Sobolev inequality by Gerard-Meyer-Oru [53]. Such a linear profile decomposition was proven in the context of Schrödinger equations by Merle-Vega [88] (in $L^2(\mathbb{R}^2)$), by Keraani [78] (in $\dot{H}^1(\mathbb{R}^d),\ d \geq 3$), by Carles-Keraani [25] (in $L^2(\mathbb{R})$) and by Bégout-Vargas [4] (in $L^2(\mathbb{R}^d),\ d \geq 3$). The insight profile decompositions provide was first applied to the well-posedness problem of nonlinear dispersive equations by Kenig-Merle [75].

We illustrate the main idea in the context of this thesis. Given a bounded sequence of functions $\{f_k\}_{k\geq 1} \subset H^1(\mathbb{T}^4)$ we would like to find a subsequence of $\{f_k\}_{k\geq 1}$ that makes $\{e^{it\Delta} f_k\}_{k\geq 1}$ converge in the Z-norm. However, the Strichartz inequality (Lemma 2.3.1) is non-compact. The reason why we would fail to extract a convergent subsequence is the non-compact symmetries. The non-compact symmetries of the Strichartz inequality on torus are space and time translations and partial scaling symmetry.

Let us consider a simple example to show how the non-compact symmetries work against us. Suppose $f_k(x) := f(x + x_k)$ with $f \in H^1(\mathbb{T}^4)$ and $\{x_k\} \subset \mathbb{T}^4$ is a sequence constructed by space translation symmetry. Then the Z-norm of $e^{it\Delta} f_k - e^{it\Delta} f_m$ cannot converge to zero. Furthermore, we know that Schrödinger equations do not preserve the whole scaling symmetry on tori, but we can hold a partial scaling symmetry if we consider data supported near a spatial point (since the Schrödinger solution on a torus should look similar as the one on $\mathbb{R}^d$ when we only focus on a spatial point). This is the reason why we need to cutoff the functions before the scaling transform (see (def:euclideanProfile)). This fact about non-compact symmetries indicates that the suitable way to recapture the compactness of the Strichartz inequality is by removing some bubble components constructed by the non-compact symmetries.

Inverse Strichartz estimates

Let us start with a lemma, which is a refinement of the Strichartz estimates (Lemma 2.3.1).

Lemma 0.1 (Refined Strichartz inequality)**.** *Let $f \in H^1(\mathbb{T}^4)$ and $I \subset [0,1]$. Then*

$$\|e^{it\Delta}f\|_{Z(I)} \lesssim (\|f\|_{H^1_x(\mathbb{T}^4)})^{\frac{5}{6}} \sup_{N\in 2^{\mathbb{Z}}} (N^{-1}\|P_N e^{it\Delta}f\|_{L^\infty_{t,x}(I\times\mathbb{T}^4)})^{\frac{1}{6}}.$$

Proof. By the definition of the Z-norm,

$$\|e^{it\Delta}f\|_{Z(I)} = \left(\sum_N N^2\|P_N e^{it\Delta}f\|^4_{L^4_{t,x}}\right)^{\frac{1}{4}} = \left\|N^{\frac{1}{2}}\|P_N e^{it\Delta}f\|_{L^4_{t,x}}\right\|_{l^4_N}.$$

By Hölder inequality and Proposition 2.3.1, we have that

$$\begin{aligned}
&\left\|N^{\frac{1}{2}}\|P_N e^{it\Delta}f\|_{L^4_{t,x}}\right\|_{l^4_N}\\
\lesssim&\left\|\left(N^{\frac{4}{5}}\|P_N e^{it\Delta}f\|_{L^{\frac{10}{3}}{t,x}}\right)^{\frac{5}{6}}\left(N^{-1}\|P_N e^{it\Delta}f\|_{L^\infty_{t,x}}\right)^{\frac{1}{6}}\right\|_{l^4_N}\\
\lesssim&\sup_N\left(N^{-1}\|P_N e^{it\Delta}f\|_{L^\infty_{t,x}}\right)^{\frac{1}{6}}\left(\sum_N N^{\frac{8}{3}}\|P_N e^{it\Delta}f\|^{\frac{10}{3}}_{L^{\frac{10}{3}}_{t,x}}\right)^{\frac{5}{6}}\\
\lesssim&\sup_N\left(N^{-1}\|P_N e^{it\Delta}f\|_{L^\infty_{t,x}}\right)^{\frac{1}{6}}\left(\|f\|_{H^1_x(\mathbb{T}^4)}\right)^{\frac{5}{6}}.
\end{aligned}$$

$\square$

Remark 0.2. The refined Strichartz estimate actually give us the following fact:

$$\sup_{N\in 2^{\mathbb{Z}}}(N^{-1}\|P_N e^{it\Delta}f\|_{L^\infty_{t,x}}) \gtrsim \frac{\|e^{it\Delta}f\|^6_{Z(I)}}{\|f\|^5_{H^1(\mathbb{T}^4)}},$$

i.e. the linear solutions with non-trivial space-time norm must concentrate on at least one frequency annulus and around some points in space-time space. The next lemma goes one step further and shows that they contain a bubble of concentration around some point in space-time.

Lemma 0.3 (Inverse Strichartz estimates)**.** *Let $\{f_k\} \subset H^1(\mathbb{T}^4)$ and $f_k \rightharpoonup 0$ in $H^1(\mathbb{T}^4)$. Suppose that*

$$\lim_{n\to\infty}\|f_k\|_{H^1(\mathbb{T}^4)} = A < \infty \text{ and } \lim_{n\to\infty}\|e^{it\Delta}f_k\|_{Z(I_k)} = \varepsilon > 0,$$

where $I_k = (-T_k, T^k)$ such that $|I_k| \to 0$ as $k \to \infty$. Then there exists $\phi \in \dot{H}^1(\mathbb{R}^4)$, and a Euclidean frame $\mathcal{O} = (N_k, t_k, x_k)_k$ up to subsequences such that

$$\begin{aligned}\liminf_{k\to\infty}(\|f_k\|^2_{\dot{H}^1(\mathbb{T}^4)} - \|f_k - \widetilde{\phi}_{\mathcal{O}_k}\|^2_{\dot{H}^1(\mathbb{T}^4)}) = \|\phi\|^2_{\dot{H}^1(\mathbb{R}^4)} \gtrsim A^2\left(\frac{\varepsilon}{A}\right)^{12} \\ \liminf_{k\to\infty}(\|f_k\|^2_{L^2(\mathbb{T}^4)} - \|f_k - \widetilde{\phi}_{\mathcal{O}_k}\|^2_{L^2(\mathbb{T}^4)}) = \|\phi\|^2_{L^2(\mathbb{R}^4)} \gtrsim A^2\left(\frac{\varepsilon}{A}\right)^{12}.\end{aligned} \tag{0.1}$$

Proof. By the refined Strichartz inequality (Lemma 0.1), there exist a sequence of triples (N_k, t_k, x_k), where $t_k \to t_\infty = 0$ as $k \to \infty$, satisfying

$$N_k^{-1}\left|[e^{it_k\Delta}P_{N_k}f_k](x_k)\right| \gtrsim A\left(\frac{\varepsilon}{A}\right)^6. \tag{0.2}$$

Case 1: If $\{N_k\}$ remain bounded, then up to subsequence, $N_k \to N_\infty$ as $k \to \infty$.

Since $\mathbb{T}^4$ is a compact manifold, there exists $x_\infty \in \mathbb{T}^4$, up to subsequence, we obtain $x_k \to x_\infty$ as $k \to \infty$. In this case, we define $\psi = (1-\Delta)^{-1}N_\infty^{-1}e^{it_\infty\Delta}P_{N_\infty}\delta_0$, where δ_0 is a Dirac delta function in x.

Let us consider the sequence

$$g_k(x) := f_k(x + x_k) = \pi_{-x_k}f_k.$$

Since $\{g_k\}$ are bounded in $H^1(\mathbb{T}^4)$, which is weak compact, passing to a subsequence, we can choose ϕ so that $g_k \rightharpoonup \phi$ weakly in $H^1(\mathbb{T}^4)$. Then we obtain that

$$\begin{aligned}
|\langle \phi, \psi \rangle_{L^2 \times L^2}| &= \lim_{k \to \infty} |\langle g_k, \psi \rangle_{L^2 \times L^2}| \\
&= \lim_{k \to \infty} |\langle f_k, \psi(\cdot - x_k) \rangle_{L^2 \times L^2}| = \lim_{k \to \infty} |\langle f_k, \psi(\cdot - x_\infty) \rangle_{H^1 \times H^1}| \\
&= N_\infty^{-1} [e^{it_\infty \Delta} P_{N_\infty} f_k](x_\infty) \gtrsim A \left(\frac{\varepsilon}{A} \right)^6 .
\end{aligned}$$

In other way, $f_k \rightharpoonup 0$ in $H^1(\mathbb{T}^4)$ implies $\lim_{k \to \infty} |\langle f_k, \psi(\cdot - x_\infty) \rangle_{H^1 \times H^1}| = 0$. Thus, we have $\varepsilon = 0$ which contradicts $\varepsilon > 0$ in the statement of the lemma.

Case 2: If $\{N_k\}$ are unbounded, then up to subsequence, $N_k \to \infty$ as $k \to \infty$. We denote by $\mathcal{O} = (N_k, t_k, x_k)_k$. We fix a spherically symmetric function $\eta \in C_0^\infty(\mathbb{R}^4)$ supported in the ball of radius 2 and equal to 1 in the ball of radius 1. Let us consider the sequence

$$g_k(x) := N_k^{-1} \eta(y / N_k^{\frac{1}{2}}) (\Pi_{-t_k, -x_k})(\Psi(y/N_k)) \in \dot{H}^1(\mathbb{R}^4),$$

where $\Psi : \{x \in \mathbb{R}^4 : |x| < 1\} \to O_0 \subseteq \mathbb{T}^4,\ \Psi(x) = x$. And similarly we can choose ϕ so that $g_n \rightharpoonup \phi$ weakly in $H^1(\mathbb{T}^4)$.

$$\begin{aligned}
A(\frac{\varepsilon}{A})^6 &\lesssim \lim_{k \to \infty} N_k^{-1} \left| [e^{it_k \Delta} P_{N_k} f_k](x_k) \right| \\
&= \lim_{k \to \infty} \left| \langle N_k^{-1} [e^{it_k \Delta} P_{N_k} f_k](x_k), \delta_0(\cdot - x_k) \rangle_{L^2 \times L^2} \right| \\
&= \lim_{k \to \infty} \left| \langle e^{it_k \Delta} f_k, N_k^{-1} P_{N_k} \delta_0(\cdot - x_k) \rangle_{L^2 \times L^2} \right| .
\end{aligned}$$

Let us define a function ψ satisfying

$$\widehat{\psi}(\xi) = |\xi|^{-2} [\eta(\xi) - \eta(2\xi)].$$

Then by Poisson summation formula, we know $N_k^{-1} P_{N_k} \delta_0(\cdot - x_k) = (1 - \Delta)(T_{N_k} \psi)(\cdot - x_k)$.

Thus, we obtain

$$
\begin{aligned}
A(\frac{\varepsilon}{A})^6 &\lesssim \lim_{k\to\infty} \left|\langle e^{it_k\Delta} f_k, N_k^{-1} P_{N_k}\delta_0(\cdot - x_k)\rangle_{L^2\times L^2}\right| \\
&= \lim_{k\to\infty} \left|\langle e^{it_k\Delta} f_k, (1-\Delta)(T_{N_k}\psi)(\cdot - x_k)\rangle_{L^2\times L^2}\right| \\
&= \lim_{k\to\infty} |\langle \Pi_{-t_k,-x_k} f_k, (1-\Delta)(T_{N_k}\psi)\rangle_{L^2\times L^2}| \\
&= \lim_{k\to\infty} |\langle g_k, \psi\rangle_{L^2\times L^2}| \\
&= |\langle \phi, \psi\rangle_{L^2\times L^2}| \\
&\lesssim_\psi \|\phi\|_{\dot{H}^1(\mathbb{R}^4)}.
\end{aligned}
$$

Similarly we also obtain $\|\phi\|_{L^2(\mathbb{R}^4)} \gtrsim_\psi A(\frac{\varepsilon}{A})^6$.

Then we hold

$$
\begin{aligned}
&\|f_k\|^2_{\dot{H}^1(\mathbb{T}^4)} - \|f_k - \widetilde{\phi}_{\mathcal{O}_k}\|^2_{\dot{H}^1(\mathbb{T}^4)} = 2\langle f_k, \widetilde{\phi}_{\mathcal{O}_k}\rangle_{\dot{H}^1(\mathbb{T}^4)\times\dot{H}^1(\mathbb{T}^4)} - \|\widetilde{\phi}_{\mathcal{O}_k}\|^2_{\dot{H}^1(\mathbb{T}^4)} \\
&= 2\langle g_k, \phi\rangle_{\dot{H}^1(\mathbb{R}^4)\times\dot{H}^1(\mathbb{R}^4)} - \|\widetilde{\phi}_{\mathcal{O}_k}\|^2_{\dot{H}^1(\mathbb{T}^4)}.
\end{aligned}
$$

Since $g_k \rightharpoonup \phi$, we have

$$
\begin{aligned}
\lim_{k\to\infty}\left(\|f_k\|^2_{\dot{H}^1(\mathbb{T}^4)} - \|f_k - \widetilde{\phi}_{\mathcal{O}_k}\|^2_{\dot{H}^1(\mathbb{T}^4)}\right) &= \lim_{k\to\infty}\left(2\langle g_k, \phi\rangle_{\dot{H}^1(\mathbb{R}^4)\times\dot{H}^1(\mathbb{R}^4)} - \|\widetilde{\phi}_{\mathcal{O}_k}\|^2_{\dot{H}^1(\mathbb{T}^4)}\right) \\
&= \|\phi\|^2_{\dot{H}^1(\mathbb{R}^4)} \gtrsim A^2(\frac{\varepsilon}{A})^{12}.
\end{aligned}
$$

Similarly we also have $\lim_{k\to\infty}\left(\|f_k\|^2_{L^2(\mathbb{T}^4)} - \|f_k - \widetilde{\phi}_{\mathcal{O}_k}\|^2_{L^2(\mathbb{T}^4)}\right) \gtrsim A^2(\frac{\varepsilon}{A})^{12}$. □

Proof of Proposition 3.6.3

Before proving Proposition 3.6.3 we prove the next lemma.

Lemma 0.4 (Profile decompositions). *Consider* $\{f_k\}_k$ *a sequence of functions in* $H^1(\mathbb{T}^4)$ *and* $0 < A < \infty$ *satisfying*

$$\limsup_{k\to+\infty} \|f_k\|_{H^1(\mathbb{T}^4)} \leq A$$

and a sequence of intervals $I_k = (-T_k, T^k)$ *such that* $T_k, T^k \to 0$ *as* $k \to \infty$. *Up to passing to a subsequence, assume that* $f_k \rightharpoonup 0 \in H^1(\mathbb{T}^4)$. *For fixed a* $\varepsilon > 0$, *there exists* $J \lesssim \varepsilon^{-2}$, *and a sequence of profile* $\widetilde{\psi}_k^\alpha := \widetilde{\psi}^\alpha_{\mathcal{O}_k^\alpha}$ *associated to pairwise orthogonal Euclidean frames* $\mathcal{O}^\alpha$ *and* $\psi^\alpha \in H^1(\mathbb{R}^4)$ *such that extracting a subsequence, we have*

$$f_k = \sum_{1 \leq \alpha \leq J} \widetilde{\psi}_k^\alpha + R_k \tag{0.3}$$

where R_k *satisfies*

$$\limsup_{k \to \infty} \|e^{it\Delta} R_k\|_{Z(I_k)} \leq \varepsilon. \tag{0.4}$$

Besides, we also have the following orthogonality relations:

$$\begin{aligned} \|f_k\|_{L^2}^2 &= \|g\|_{L^2}^2 + \|R_k\|_{L^2}^2 + o_k(1). \\ \|\nabla f_k\|_{L^2}^2 &= \|\nabla g\|_{L^2}^2 + \sum_{\alpha \leq J} \|\nabla_{\mathbb{R}^4} \psi^\alpha\|_{L^2(\mathbb{R}^4)}^2 + \|\nabla R_k\|_{L^2}^2 + o_k(1). \end{aligned} \tag{0.5}$$

Proof. First, we extract a frame from the sequence $\{f_k\}$ via inverse Strichartz estimates (Lemma 0.3). Suppose $\limsup_{k \to \infty} \|e^{it\Delta} f_k\|_{Z(I_k)} \geq \varepsilon$ (otherwise we just choose $R_k = f_k$ and don't need to remove any frames). By the inverse Strichartz estimates (Lemma 0.3), there exists a frame $\mathcal{O}^1 = (N_k^1, t_k^1, x_k^1)_k$ and a function ϕ^1 satisfying

$$\begin{aligned} &\liminf_{k \to \infty} (\|f_k\|_{\dot{H}^1(\mathbb{T}^4)}^2 - \|f_k - \widetilde{\phi^1}_{\mathcal{O}_k^1}\|_{\dot{H}^1(\mathbb{T}^4)}^2) = \|\phi^1\|_{\dot{H}^1(\mathbb{R}^4)}^2 \gtrsim A^2 \left(\frac{\varepsilon}{A}\right)^{12} \\ &\liminf_{k \to \infty} (\|f_k\|_{L^2(\mathbb{T}^4)}^2 - \|f_k - \widetilde{\phi^1}_{\mathcal{O}_k^1}\|_{L^2(\mathbb{T}^4)}^2) = \|\phi^1\|_{L^2(\mathbb{R}^4)}^2 \gtrsim A^2 \left(\frac{\varepsilon}{A}\right)^{12}. \end{aligned} \tag{0.6}$$

Second, suppose $\limsup_{k \to \infty} \|e^{it\Delta}(f_k - \widetilde{\phi^1}_{\mathcal{O}_k^1})\|_{Z(I_k)} \geq \varepsilon$ (otherwise we stop here and set $R_k = f_k - \widetilde{\phi^1}_{\mathcal{O}_k^1}$), we extract a frame from the sequence $\{f_k - \widetilde{\phi^1}_{\mathcal{O}_k^1}\}$ using a similar idea. By (0.6) we obtain $\liminf_{k \to \infty} \|f_k - \widetilde{\phi^1}_{\mathcal{O}_k^1}\|_{H^1(\mathbb{T}^4)}^2 \leq A^2(1 - (\frac{\varepsilon}{A})^{12}) \leq A^2$.

Since $N_k^1 \to \infty$ as $k \to \infty$, it is easy to check, up to subsequences, $\widetilde{\phi^1}_{\mathcal{O}_k^1} \rightharpoonup 0 \in H^1(\mathbb{T}^4)$. By inverse Strichartz estimates (Lemma 0.3), we can extract the second frame $\mathcal{O}^2 = (N_k^2, t_k^2, x_k^2)_k$ and there exists a corresponding function ϕ^2 satisfying

$$\begin{aligned} &\liminf_{k \to \infty} (\|f_k - \widetilde{\phi^1}_{\mathcal{O}_k^1}\|_{\dot{H}^1(\mathbb{T}^4)}^2 - \|f_k - \widetilde{\phi^1}_{\mathcal{O}_k^1} - \widetilde{\phi^2}_{\mathcal{O}_k^2}\|_{\dot{H}^1(\mathbb{T}^4)}^2) = \|\phi^1\|_{\dot{H}^1(\mathbb{R}^4)}^2 \gtrsim A^2 \left(\frac{\varepsilon}{A}\right)^{12} \\ &\liminf_{k \to \infty} (\|f_k - \widetilde{\phi^1}_{\mathcal{O}_k^1}\|_{L^2(\mathbb{T}^4)}^2 - \|f_k - \widetilde{\phi^1}_{\mathcal{O}_k^1} - \widetilde{\phi^2}_{\mathcal{O}_k^2}\|_{L^2(\mathbb{T}^4)}^2) = \|\phi^1\|_{L^2(\mathbb{R}^4)}^2 \gtrsim A^2 \left(\frac{\varepsilon}{A}\right)^{12}. \end{aligned} \tag{0.7}$$

Third, following the first and second steps, we can keep extracting frames from the sequence $\{f_k\}$ via inverse Strichartz estimates (Lemma 0.3) until $\limsup_{k\to\infty} \|e^{it\Delta}(f_k - \sum_{\alpha\leq J} \widetilde{\phi^\alpha}_{\mathcal{O}_k^\alpha})\|_{Z(I_k)} \leq \varepsilon$. Next we need to show that it stops in finite steps (that is, that $J < \infty$).

Let us set $R_k = f_k - \sum_{\alpha\leq J} \widetilde{\phi^\alpha}_{\mathcal{O}_k^\alpha}$. Then we have $\limsup_{k\to\infty} \|e^{it\Delta} R_k\|_{Z(I_k)} \leq \limsup_{k\to\infty} \|f_k - \sum_{\alpha\leq J} \widetilde{\phi^\alpha}_{\mathcal{O}_k^\alpha}\|_{H^1}$. Then using the estimates (0.6) (0.7) and the almost orthogonality of different frames, we obtain that

$$
\begin{aligned}
&\limsup_{k\to\infty} \|f_k - \sum_{\alpha\leq J} \widetilde{\phi^\alpha}_{\mathcal{O}_k^\alpha}\|_{H^1}^2 \\
\leq &\limsup_{k\to\infty} \|f_k - \sum_{\alpha\leq J-1} \widetilde{\phi^\alpha}_{\mathcal{O}_k^\alpha}\|_{H^1}^2 - (\|f_k - \sum_{\alpha\leq J} \widetilde{\phi^\alpha}_{\mathcal{O}_k^\alpha}\|_{H^1}^2 - \|f_k - \sum_{\alpha\leq J-1} \widetilde{\phi^\alpha}_{\mathcal{O}_k^\alpha}\|_{H^1}^2) \\
\leq &\limsup_{k\to\infty} \|f_k - \sum_{\alpha\leq J-1} \widetilde{\phi^\alpha}_{\mathcal{O}_k^\alpha}\|_{H^1}^2 - A^2 \left(\frac{\varepsilon}{A}\right)^{12} \\
&\dots \\
\leq &A^2 - J \times A^2 \left(\frac{\varepsilon}{A}\right)^{12}.
\end{aligned}
$$

If we choose $J = (\frac{A}{\varepsilon})^{12}$ we then can have $\limsup_{k\to\infty} \|e^{it\Delta} R_k\|_{Z(I_k)} \leq \varepsilon$.

□

Now, to prove Proposition 3.6.3 we use Lemma 0.4. We let $\varepsilon_m = \frac{1}{m}$ and for each m we apply Lemma 0.4 to the finite sequence of frames and profiles. In the end, letting $m \to \infty$ and combining all frames and profiles together, we get the sequence of frames and profiles $\mathcal{O}^\alpha$, $\widetilde{\phi^\alpha}_{\mathcal{O}_k^\alpha}$ satisfying Proposition 3.6.3.

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