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# Global well-posedness and scattering for the defocusing quintic nonlinear Schrödinger equation in two dimensions

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# GLOBAL WELL-POSEDNESS AND SCATTERING FOR THE DEFOCUSING QUINTIC NONLINEAR SCHRÖDINGER EQUATION IN TWO DIMENSIONS

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# DEDICATION

To my family.

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## ABSTRACT

## GLOBAL WELL-POSEDNESS AND SCATTERING FOR THE DEFOCUSING QUINTIC NONLINEAR SCHRÖDINGER EQUATION IN TWO DIMENSIONS

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In this thesis we consider the Cauchy initial value problem for the defocusing quintic nonlinear Schrödinger equation in $\mathbb{R}^2$:

$$\begin{cases} i\partial_t u + \Delta u = |u|^4 u \\ u(0,x) = u_0, \end{cases}$$

where $u : \mathbb{R}_t \times \mathbb{R}_x^2 \to \mathbb{C}$ is a complex-valued function of time and space. We take general data in the critical Sobolev space $\dot{H}^{\frac{1}{2}}(\mathbb{R}^2)$. We show that if a solution remains bounded in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^2)$ in its maximal time interval of existence, then the time interval is infinite and the solution scatters.

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# CHAPTER 1

# INTRODUCTIOIN

In quantum mechanics, the Schrödinger equation is a mathematical equation that describes the changes over time of a physical system in which quantum effects are significant. Before discussing the mathematical results, we first introduce its historical background and development in physics.

## 1.1 Schrödinger equations in quantum mechanics

In classical mechanics, Newton's second law ($F = ma$) is used to describe the motion of a body in response to those forces acting on it. A natural question was then whether there is a universal law in quantum mechanics. The Schrödinger equation thus came into play.

### 1.1.1 Derivation of Schrödinger equations

Heuristically, the derivation of the (time-dependent) Schrödinger equation is based on the following developments in physics:

- In quantum mechanics, the total energy of a particle can be expressed as the sum of the kinetic energy $T$ and the potential energy $V$, just the same as in classical physics.

$$E = T + V = \frac{p^2}{2m} + V \tag{1.1.1}$$

where $p$ is momentum and $m$ is mass.

- The photoelectric effect states that the energy E of a photon is proportional to the frequency $f$ (or angular frequency, $\omega = 2\pi f$) of the corresponding quantum wavepacket of light:

  $$E = hf = \hbar\omega$$

  where $h$ is Planck constant, $\hbar = \frac{h}{2\pi}$.

- De Broglie's hypothesis states that any particle can be associated with a wave, and that the momentum $p$ of the particle is inversely proportional to the wavelength $\lambda$ of such a wave (or proportional to the wavenumber, $k = \frac{2\pi}{\lambda}$), in one dimension, by:

  $$p = \frac{h}{\lambda} = \hbar k.$$

Assuming then that the wave function $\Psi(t, x)$ is a complex plane wave, that is

$$\Psi(t, x) = Ae^{i(kx - \omega t)},$$

we can directly compute the partial derivative with respect to time $t$

$$\frac{\partial}{\partial t}\Psi = -i\omega\Psi.$$

Notice that this partial derivative is related to the total energy in the following way

$$E\Psi = \hbar\omega\Psi = i\hbar\frac{\partial}{\partial t}\Psi.$$

Similarly, we can also compute the second partial derivative with respect to space $x$

$$\frac{\partial^2}{\partial x^2}\Psi = -k^2\Psi,$$

realizing that this second partial derivative is related to the momentum,

$$p^2\Psi = \hbar^2 k^2 \Psi = -\hbar^2 \frac{\partial^2}{\partial x^2}\Psi.$$

Finally, we link all the information above with (1.1.1) leading to the Schrödinger equation in one dimension

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi + V(x)\Psi = i\hbar\frac{\partial}{\partial t}\Psi, \tag{1.1.2}$$

where $V(x)$ is the potential energy.

We can generalize the Schrödinger equation above to any higher dimensions:

$$-\frac{\hbar^2}{2m}\Delta\Psi + V(x)\Psi = i\hbar\frac{\partial}{\partial t}\Psi.$$

### 1.1.2 Dispersion relation and group, phase velocities

Schrödinger equations are classified as dispersive partial differential equations. Dispersion means that waves of different wavelength propagate at different phase velocities. Consider a simple example: free particles. In physics, a free particle is a particle that, in some sense, is not bound by an external force, or equivalently not in a region where its potential energy varies. That is, $V(x) = 0$, then (1.1.2) becomes a linear partial differential equation:

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi = i\hbar\frac{\partial}{\partial t}\Psi.$$

This equation always has a plane wave as its solution

$$\Psi(t,x) = Ae^{i(\omega t - kx)}.$$

By plugging this solution into the linear partial differential equation, we have the so-called dispersion relation connecting the frequency and wavenumber of plane waves:

$$\omega = \frac{\hbar}{2m}k^2.$$

Now we can build a wavepacket out of these waves. To do so, we use linear superposition i.e., for each of these waves with wavenumber $k$, we select an amplitude $A(k)$ and then superpose them. Since features such as boundary conditions have not yet come to the rescue by selecting which wavenumbers are "available" to select, we need to choose all possible wavenumbers, meaning that the "summation" over wavenumbers will not be a summation (as we are used to in superspositions) but rather an integration, i.e.,:

$$u(t,x) = \int_{k_0-\delta k}^{k_0+\delta k} A(k)e^{i(\omega t - kx)}\, dk.$$

Then linearization gives

$$\omega(k) \approx w(k_0) + (k - k_0)\omega'(k_0).$$

After some calculation,

$$u(t,x) = e^{i(\omega t - k_0 x)} \int_{k_0-\delta k}^{k_0+\delta k} A(k)e^{i(k-k_0)(\omega'(k_0)t - x)}\, dk \equiv e^{i(\omega t - k_0 x)}\Psi(t,x).$$

The factor $e^{i(\omega_0 t - k_0 x)}$ is the so-called carrier wave, the principal Fourier component of the spatio-temporal evolution. The rest becomes the so-called envelope wave $\Psi(t,x)$, modulating the carrier.

Furthermore, we see that the phase velocity given by

$$v_p = \frac{\omega}{k} = \frac{\hbar}{2m}k$$

is the velocity of carrier wave. The group velocity given by

$$v_g = \frac{d\omega}{dk} = \frac{\hbar}{m}k$$

is the independent speed with which information is transferred and is dubbed the group velocity of the wave.

The envelope travels with a different speed than the carrier: they do not travel at the same rate so that the shape of the pulse be preserved. As a result of these distinct speeds, the wavepacket spreads, i.e., disperses.

Now we give the formal definition of a dispersive equation:

**Definition 1.1.** We say that an evolution equation (defined on $\mathbb{R}^d$), is dispersive if its dispersion relation $\frac{\omega(k)}{|k|} = g(k)$ is a real function such that

$$|g(k)| \to \infty \quad \text{as} \ \ |k| \to \infty.$$

Informally, dispersion means that different frequencies propagate at different velocities (the higher the frequencies are, the faster they travel), thus dispersing the solution over time.

**Remark 1.2.** The wave equation $\partial_t^2 u = c^2 \Delta u$ is what we would call 'partly dispersive', since the frequency of a wave determines the direction of propagation, but not the speed.

## 1.2 Schrödinger equations in mathematics

We consider the initial value problem of the Schrödinger equation whose linear model is

$$\begin{cases} i\partial_t u + \Delta u = 0 \\ u(0,x) = u_0(x), \end{cases}$$

since we can always scale $u$ such that the coefficient $\frac{\hbar}{2m}$ is absorbed by the scaling. In fact, performing a Fourier transformation on both side of the equation, we are able to explicitly solve this equation:

$$\hat{u}(t,\xi) = e^{-it\,|\xi|^2}\hat{u}_0(\xi),$$

then

$$u(t,x) = e^{it\Delta}u_0 = (4\pi i t)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{\frac{i|x-y|^2}{4t}} u_0(y)\, dy.$$

Here, if we assume that the initial data $u_0$ is integrable in space, we can see that the amplitude of the solution $u$ has a power-type decay in time, that is,

$$\|u(t,x)\|_{L_x^\infty(\mathbb{R}^d)} \lesssim t^{-\frac{d}{2}} \|u_0\|_{L_x^1(\mathbb{R}^d)}.$$

This means that as time goes by, the solution $u$ will disperse in space. See Chapter 2 for details.

The behavior of linear Schrödinger equation is well-known thanks to an important family of inequalities capturing dispersion, namely, the Strichartz estimates. These inequalities establish size and decay of solutions in mixed norm Lebesgue spaces, thus quantifying the dispersive behavior of solutions to the Schrödinger equation.

**Theorem 1.3** (Strichartz estimates for linear Schrödinger)**.** *Fix $d \geq 1$ and call a pair $(q,r)$ of exponents admissible if $2 \leq q, r \leq \infty$, $\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$ and $(q,r,d) \neq (2,\infty,2)$. Then for any admissible exponents $(q,r)$ and $(\tilde{q},\tilde{r})$ we have the homogeneous Strichartz estimate*

$$\left\|e^{it\Delta}u_0\right\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^d)} \lesssim_{d,q,r} \|u_0\|_{L_x^2(\mathbb{R}^d)}$$

We also study the nonlinear Schrödinger equation:

$$i\partial_t u + \Delta u = F(u),$$

where $F$ is a function of $u$. We also have:

**Theorem 1.4** (Strichartz estimates for nonlinear Schrödinger)**.** *Fix $d \geq 1$ and call a pair $(q,r)$ of exponents admissible if $2 \leq q, r \leq \infty$, $\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$ and $(q,r,d) \neq$*

$(2,\infty,2)$. *Then for any admissible exponents* $(q,r)$ *and* $(\tilde{q},\tilde{r})$ *we have the homogeneous Strichartz estimate*

$$\left\| e^{it\Delta} u_0 \right\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^d)} \lesssim_{d,q,r} \|u_0\|_{L_x^2(\mathbb{R}^d)}$$

*the dual homogeneous Strichartz estimate*

$$\left\| \int_{\mathbb{R}} e^{-is\Delta} F(s)\, ds \right\|_{L_x^2(\mathbb{R}^d)} \lesssim_{d,\tilde{q},\tilde{r}} \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}(\mathbb{R}\times\mathbb{R}^d)}$$

*and the inhomogeneous Strichartz estimate*

$$\left\| \int_{t'<t} e^{i(t-t')\Delta} F(t')\, dt' \right\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^d)} \lesssim_{d,q,r,\tilde{q},\tilde{r}} \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}(\mathbb{R}\times\mathbb{R}^d)}.$$

For the proofs, see [19, 67, 26] for details. Also see [58] for a proof.

A typical example of these equations is the power-type nonlinear Schrödinger equations (NLS):

$$\begin{cases} i\partial_t u + \Delta u = \lambda\, |u|^{p-1}\, u \\ u(0,x) = u_0, \end{cases} \tag{1.2.1}$$

where $u : \mathbb{R}_t \times \mathbb{R}_x^d \to \mathbb{C}$ is a complex-valued function of time and space and $p > 1$. By rescaling the values of $\lambda$, it is sufficient to consider the cases $\lambda = -1$ or $\lambda = +1$; these are known as the focusing and defocusing equations, respectively. We center the discussion below on the *defocusing case*.

This equation (1.2.1) conserves the mass,

$$M(u(t)) := \int_{\mathbb{R}^d} |u(t,x)|^2\, dx = M(u_0), \tag{1.2.2}$$

the energy,

$$E(u(t)) := \int_{\mathbb{R}^d} \frac{1}{2}\, |\nabla u(t,x)|^2 + \frac{1}{p+1}\, |u(t,x)|^{p+1}\, dx = E(u_0), \tag{1.2.3}$$

and the momentum,

$$\mathcal{P}(u)(t) := \int_{\mathbb{R}^d} \mathrm{Im}[\bar{u}(t,x)\nabla u(t,x)]\, dx = \mathcal{P}(u)(0). \tag{1.2.4}$$

The equation also enjoys a scaling symmetry, namely

$$u(t,x) \mapsto \lambda^{\frac{2}{p-1}} u(\lambda^2 t, \lambda x),$$

which defines the criticality (relative to scaling) for this equation, that is, the regularity of the only homogeneous $L^2_x$-based Sobolev space $\dot{H}^{s_c}(\mathbb{R}^d)$ of initial data whose norm is left invariant by this rescaling; namely $s_c := \frac{d}{2} - \frac{2}{p-1}$. Data in $H^s(\mathbb{R}^d)$ with $s > s_c$ and $s = s_c$, are called subcritical and critical, respectively. Moreover, in the subcritical regime, the norm of the initial data can be made small while at the same time the interval of time is made longer. In this sense, we can say that the scaling is helping us to obtain well-posedness. However, in the critical regime, the norm is invariant while the interval of time is made longer. This looks like a problematic situation.

## 1.3 Well-posedness problem

A natural question to ask is whether there is existence, uniqueness and continuity of the data to solution map for the Cauchy initial value problem (1.2.1), that is, study the well-posedness problem. In the last half-century, the well-posedness problem for nonlinear Schrödinger and other dispersive equations has drawn a lot of attention. The local (in time) well-posedness for data in Sobolev spaces $H^s(\mathbb{R}^d)$ in both subcritical and critical regimes was obtained by Cazenave and Weissler [6, 7, 8, 5]. The subcritical local well-posedness follows from the Strichartz estimates and a fixed point argument with a time of existence depending solely on the $H^s$-norm of the data $s > s_c$. A similar argument gives also local well-posedness for data in $\dot{H}^{s_c}(\mathbb{R}^d)$ but in this case the time of existence depends also on the profile of the data.

In the energy-subcritical regime ($s_c < s = 1$), the conservation of energy gives global well-posedness in $H^1(\mathbb{R}^d)$ by iteration. That is, start with the initial time $t_0$ and construct a solution all the way up to some later time, then use the function at this point as new initial data and apply the local theory again to move forward and continue iterating. The time of existence in each step depends on the $H^1$ norm of the data only, which is controlled by the conserved energy, therefore it never shrinks. In the energy-critical regime (see [8, 5]), as we mentioned above, the time of existence depends on the profile of the data as well, which is controlled by the energy of initial data due to Strichartz estimates. In this case, if the energy of initial data is given to be small enough, the profile is also sufficiently small, hence the solution is known to exist globally in time, and scatters[1] (see Definition 3.5 for details).

In the energy-critical regime ($s_c = 1, \quad p = 1 + \frac{4}{d-2}$) with large initial data, one cannot, however, iterate the time of existence to obtain a global solution, because the time of existence depends also on the profile of the data (not a conserved quantity), which may lead to a shrinking interval of existence when iterating the local well-posedness theorem.

Similarly, in the mass-subcritical regime ($s_c < s = 0$), the conservation of mass gives global well-posedness in $L^2$ (eg. cubic NLS on $\mathbb{R}$, $s_c = -\frac{1}{2}$) while if $s_c = 0$, $p = 1 + \frac{4}{d}$, one cannot extend the local solution to a global solution by iteration due to the same reason.

Nonetheless, in recent years there has been significant progress in the understanding of the large data long time dynamics in the critical regime. We go over further background below.

[1]There exists a solution $u_{\pm}(t)$ to the free Schrödinger equation $i\partial_t u + \Delta u = 0$ s.t. $\lim_{t\to\pm\infty} \|u(t) - u_{\pm}(t)\|_{\dot{H}^1_x} = 0$

## 1.4 Background

In the energy-critical case, Bourgain [1] first introduced an inductive argument on the size of the energy and a refined Morawetz inequality[2] to prove global existence and scattering in three dimensions for large finite energy data which is assumed to be radial. A different proof of the same result is given by Grillakis in [22]. A key ingredient in the latter proof is an *a priori* estimate of the time average of local energy over a parabolic cylinder, which plays a similar role as Bourgain's refined Morawetz inequality. Then, a breakthrough was made by Colliander-Keel-Staffilani-Takaoka-Tao [13]. They removed the radial assumption and proved global well-posedness and scattering of the energy-critical problem in three dimensions for general large data. They relied on Bourgain's induction on energy technique to find the minimal blow-up solutions that concentrate in both space and frequency and proved new interaction Morawetz-type estimates to rule out this kind of minimal blow-up solutions. Later Ryckman-Visan [53], and Visan [65] extended the result in [13] to higher dimensions.

In [27] Kenig and Merle proposed a new methodology, a deep and broad road map (see Section 1.5), to tackle critical problems. In fact, using a contradiction argument they first proved the existence of a critical element such that the global well-posedness and scattering fail. Then relying on a concentration compactness argument they show that this critical element enjoys a compactness property up to the symmetries of this equation. Finally they reduced to a rigidity theorem to

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[2]The refined Morawetz inequality that was used in [1] is given by

$$\int_I \int_{|x|<|I|^{\frac{1}{2}}} \frac{|u(t,x)|^6}{|x|}\, dxdt \lesssim |I|^{\frac{1}{2}}$$

for any smooth solution $u$ on a time interval $I \subset \mathbb{R}$. The refinement is the spatial restriction of the solution.

preclude the existence of such critical element. In this form they were able to prove in particular the global well-posedness and scattering for the focusing radially symmetric energy critical Schrödinger equation in dimensions three four and five under suitable conditions on the data[3]. Following their road map Kenig and Merle also showed [28] the global well-posedness and scattering for the focusing energy-critical wave equation. It is worth mentioning that the concentration compactness method that they applied was first introduced by Gérard [18] in the context of Sobolev embeddings[4], by Bahouri-Gérard in nonlinear wave equations and by Merle-Vega [45] and by Keraani [30, 31] in Schrödinger equations.

The mass-critical global well-posedness and scattering problem was also first studied in the radial case, by Tao-Visan-Zhang [62] and by Killip-Tao-Visan [35]. Then Dodson proved the global well-posedness of the mass-critical problem in any dimension for nonradial data [14, 15, 16]. A key ingredient in Dodson's work is to prove a long time Strichartz estimate to rule out the minimal blow-up solutions. Such estimate also helps to derive a frequency localized Morawetz -type estimate. The proof of Dodson's long time Strichartz estimate heavily relied on the double endpoint Strichartz estimates and the bilinear Strichartz estimates. Furthermore, in dimension one [15] and in dimensions two [16], Dodson introduced suitable versions of atomic spaces to deal with the failure of the endpoint Strichartz estimates in these cases. It should be noted that the interaction Morawetz estimates proved by Planchon-Vega [52] played a fundamental role in ruling out one type of minimal blow-up solutions. Moreover the bilinear estimates in [16] that gave a logarith-

[3]Namely, that the energy and the $\dot{H}^1$ norm of the data are less than those of the ground state.

[4]The concentration compactness phenomenon (or we also call it profile decompostion) describes and measures how far a non-compact mapping between two function spaces is from being compact. Indeed, in the context of Sobolev embeddings, the homogeneous Sobolev embeddings are not compact, so are the wave map and Schrödinger map.

mic improvement over Bourgain's bilinear estimates also relied on the interaction Morawetz estimates of [52].

Unlike the energy- and mass-critical problems, for any other $s_c \neq 0, 1$, there are no conserved quantities that control the growth in time of the $\dot{H}^{s_c}$ norm of the solutions. In [29], Kenig and Merle showed for the first time that if a solution of the defocusing cubic NLS in three dimensions remains bounded in the critical norm $\dot{H}^{\frac{1}{2}}$ on the maximal time interval of existence, then the time interval of existence is infinite and the solution scatters using concentration compactness and rigidity argument. In [49], Murphy extended the $\dot{H}^{\frac{1}{2}}$ critical result in [29] to dimensions four and higher (some other inter-critical problems in dimentions three and higher were also treated in [48, 50]).

## 1.5 The road map

In the critical regime, it is by now we understood that a uniform *a priori* bound for the spacetime $L_{t,x}^{\frac{2(d+2)}{d-2s_c}}$ norm of solutions to the critical NLS implies the global well-posedness of NLS. Indeed, the main reason is that we can always divide the time interval into small subintervals such that on each small interval the spacetime $L_{t,x}^{\frac{2(d+2)}{d-2s_c}}$ norm of the solution is sufficiently small, hence we are able to run a small data argument on each interval to get a local solution, then attach these pieces of such local solution together to obtain a global solution. Therefore, in order to show global well-posedness, it is sufficient to prove such uniform *a priori* bound for the spacetime $L_{t,x}^{\frac{2(d+2)}{d-2s_c}}$ norm of solutions to the critical NLS. To prove such spacetime bound, following the road map by Kenig and Merle, one proceeds by contradiction as follows:

- **Step 1** First assume that the spacetime norm is unbounded for large data. Then the fact that for sufficiently small initial data the solutions are globally well-posed and the fact that the spacetime norm is bounded, imply the existence of a special class of solutions (called *minimal blow-up solutions*) that are concentrated in both space and frequency.

- **Step 2** One then precludes the existence of minimal blow-up solutions by conservation laws and suitable (frequency localized interaction) Morawetz estimates.

## 1.6 Motivation and main result

As we mentioned above, the $\dot{H}^{\frac{1}{2}}$-critical global well-posedness and scattering results were known in dimensions three and higher under the assumption that the critical norm $\dot{H}^{\frac{1}{2}}$ of solutions remains bounded. However, the analogue of the $\dot{H}^{\frac{1}{2}}$-critical result in dimensions two remained open. This was because:

1. the interaction Morawetz estimates in two dimensions are significantly different from those in dimensions three and above (this is the reason why this problem is interesting!), and

2. the endpoint Strichartz estimates fail.

In this thesis research, we focus on how to close this gap. More precisely, we consider the Cauchy problem for the defocusing $\dot{H}^{\frac{1}{2}}$-critical quintic Schrödinger equation in $\mathbb{R}^{1+2}$:

$$\begin{cases} i\partial_t u + \Delta u = |u|^4 u \\ u(0,x) = u_0(x) \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^2). \end{cases} \tag{1.6.1}$$

with $u : \mathbb{R}_t \times \mathbb{R}^2_x \to \mathbb{C}$.

The main result in this thesis is:

**Theorem 1.5** (Main theorem)**.** *Let $u : I \times \mathbb{R}^2 \to \mathbb{C}$ be a maximal-lifespan solution to* (1.6.1) *such that $u \in L_t^\infty \dot{H}_x^{\frac{1}{2}}(I \times \mathbb{R}^2)$. Then $u$ is global, with the spacetime bounds*

$$\int_{\mathbb{R}} \int_{\mathbb{R}^2} |u(t,x)|^8 \, dxdt \le C \left( \|u\|_{L_t^\infty \dot{H}_x^{\frac{1}{2}}(\mathbb{R}\times\mathbb{R}^2)} \right)$$

*for some function $C : [0,\infty) \to [0,\infty)$. And $u$ also scatters, that is, there exist $u_\pm \in \dot{H}_x^{\frac{1}{2}}(\mathbb{R}^2)$, such that*

$$\lim_{t\to\pm\infty} \left\| u(t) - e^{it\Delta} u_\pm \right\|_{\dot{H}_x^{\frac{1}{2}}(\mathbb{R}^2)} = 0.$$

## 1.7 Outline of the proof

The local well-posedness theory to the Cauchy initial value problem (1.6.1) with data in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^2)$ follows from [6, 7, 8, 5] (see Chapter 3 for details). To prove the global well-posedness for arbitrary large data in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^2)$, it is sufficient to prove a uniform *a priori* bound for the spacetime $L_{t,x}^8$ norm of solutions to (1.6.1). In order to prove such a bound, we argue by contradiction and follow Kenig-Merle's road map.

### 1.7.1 Step 1: Existence of minimal blow-up solutions and reduction to almost periodic solutions.

We argue by contradiction argument, and thus assume that the spacetime $L_{t,x}^8$ norm is unbounded for large data. On the other hand, we know that for sufficiently small initial data, the solutions are globally well-posed with bounded $L_{t,x}^8$ norm and scatter (see Section 3.2). Then it is natural to think that if we increase the size of initial data in the sense of $L_t^\infty \dot{H}_x^{\frac{1}{2}}(\mathbb{R}^2)$ norm, the $L_{t,x}^8$ norm will increase correspondingly, and we expect to have a threshold where the $L_{t,x}^8$ norm of the solution blows up with the minimal $\|u\|_{L_t^\infty \dot{H}_x^{\frac{1}{2}}}$. Such solution which blows up at the threshold level is known as *minimal blow-up solutions*. See Chapter 4 for details.

Moreover, we can show that this type of minimal blow-up solution enjoys a concentration property, that is, it suitably concentrates in both space and frequency. Such property is referred to as *almost periodicity*. In our setting, the minimal blow-up solutions (almost periodic solutions) can be shown to concentrate around some spatial center $x(t)$ and at some frequency scale $N(t)$ at any time $t$ in the interval of existence. See Chapter 4 for details.

### 1.7.2 Step 2: Preclusion of almost periodic solutions.

We first introduce a very important tool, namely the Morawetz estimates. For dimensions three and higher, the Morawetz estimates introduced by Lin-Strauss [41] are given by:

$$\int_I \int_{\mathbb{R}^d} \frac{|u(t,x)|^{p+1}}{|x|} \, dxdt \lesssim \|u\|^2_{L_t^\infty \dot{H}_x^{\frac{1}{2}}(I\times\mathbb{R}^d)} . \tag{1.7.1}$$

Note that the upper bound on the right-hand side depends only on the $\dot{H}^{\frac{1}{2}}$ norm of the solutions, so it is relatively easy to handle in the $\dot{H}^{\frac{1}{2}}$-critical regime. These were the Morawetz estimates used in Kenig-Merle [29] and Murphy [49]. However in dimensions two, (1.7.1) does not hold. We employ instead the interaction Morawetz estimates, which were first introduced by Colliander-Keel-Staffilani-Takaoka-Tao [12] in dimensions three and then extended to dimensions four and higher by Ryckman-Visan [53]:

$$\begin{aligned} &- \int_I \iint_{\mathbb{R}^d\times\mathbb{R}^d} |u(t,x)|^2 \, \Delta \left( \frac{1}{|x-y|} \right) |u(t,y)|^2 \, dxdydt \\ &\lesssim \|u\|^2_{L_t^\infty L_x^2(I\times\mathbb{R}^d)} \, \|u\|^2_{L_t^\infty \dot{H}_x^{\frac{1}{2}}(I\times\mathbb{R}^d)} . \end{aligned}$$

In dimensions two, Planchon-Vega [52] and Colliander-Grillakis-Tzirakis [10, 11] proved the following **interaction Morawetz estimates**:

$$\left\| \, |\nabla|^{\frac{1}{2}} \, |u(t,x)|^2 \right\|^2_{L^2_{t,x}(I\times\mathbb{R}^2)} \lesssim \|u\|^2_{L^\infty_t L^2_x(I\times\mathbb{R}^2)} \, \|u\|^2_{L^\infty_t \dot{H}^{\frac{1}{2}}_x(I\times\mathbb{R}^2)} . \tag{1.7.2}$$

Note that the upper bound above depends on the $\dot{H}^{\frac{1}{2}}$ norm as well as on the $L^2$ norm of the solutions. Hence in our case the right hand side need not be finite since there is no *a priori* bound on $\|u(t)\|_{L^2_x}$. Moreover, (1.7.2) scales like $\int_I N(t)\,dt$, where as we mentioned above $N(t)$ denotes the radius of minimal blow-up solutions concentrated on the frequency side. Intuitively, the interaction Morawetz estimates are expected to help us to rule out the existence of solutions satisfying $\int_0^{T_{max}} N(t)\,dt = \infty$ (i.e. cases III, IV below).

Next we use further classify minimal blow-up solutions according to whether $T_{max}(:= \sup I)$ and $\int_0^{T_{max}} N(t)\,dt$ are finite or infinite, more precisely we have the following scenarios:

| | $T_{max} < \infty$ | $T_{max} = \infty$ |
|---|---|---|
| $\int_0^{T_{max}} N(t)\,dt < \infty$ | I | II |
| $\int_0^{T_{max}} N(t)\,dt = \infty$ | III | IV |

where

- I, III are called finite-time blow-up solutions
- I, II are called frequency cascade solutions
- III, IV are called quasi-soliton solutions

In the $\dot{H}^{\frac{1}{2}}$-critical regime, it happens that $\int_0^{T_{max}} N(t)\,dt < \infty$ implies $T_{max} < \infty$, hence all frequency cascade solutions are also finite-time blow-up solutions, i.e. there is no case II in this setting.

Hence we proceed to rule out the existence of almost periodic solutions case by case.

*Cases III, IV (Quasi-soliton solutions)*

These are the cases that we expect the interaction Morawetz estimates will help us to rule out.

We have already observed that the upper bound in (1.7.2) depends on the $\dot{H}^{\frac{1}{2}}$ norm as well as on the $L^2$ norm of the solutions. Hence in our case the right hand side need not be finite. In contrast, the upper bound of the Morawetz estimates used in [29] and in [49] depends only on $\dot{H}^{\frac{1}{2}}$ norm. So the difference in the Morawetz estimates raises the first problem for us. If we were able to have the following Morawetz estimates simliar to the ones used in [29] and [49],

$$\iint_{I\times\mathbb{R}^2} \frac{|u|^6}{|x|}\, dxdt \lesssim \|u\|^2_{L_t^\infty \dot{H}_x^{\frac{1}{2}}(I\times\mathbb{R}^2)},$$

then we could obtain the upper bound for such spacetime norm of $\frac{|u|^6}{|x|}$. Then following the road map of concentration compactness and rigidity method, we could prove that such spacetime norm of $\frac{|u|^6}{|x|}$ was also bounded below by infinity, which would be a contradiction.

In spite of the lack of the finiteness of $L^2$ norm of the solutions, we can truncate the solutions to high frequencies and make the right-hand side of (1.7.2) finite, just as it was done for the energy-critical problem in dimensions three and above. As a result, we need to have a good estimate for the error produced in this procedure (Note that in [29] and in [49] did not need this control over low frequency terms, since they could obtain the spacetime bound directly from Morawetz without truncating). To obtain such bound for the low frequency component of the solutions, **the long time Strichartz estimates**, first introduced in [14] by Dodson, will be a powerful tool for us.

**Tool 1: Long time Strichartz estimates**

Ideally, if we were able to adapt the long time Strichartz estimates in the mass-critical seting to our $\dot{H}^{\frac{1}{2}}$-critical regime, we would then prove the following estimates:

$$\left\| |\nabla|^{\frac{1}{2}} P_{<N} u \right\|_{L_t^2 L_x^\infty (J\times\mathbb{R}^2)} \lesssim (KN)^{\frac{1}{2}} + 1, \tag{1.7.3}$$

where $J$ is an interval satisfying

$$\int_J N(t)\, dt = K.$$

We see that (1.7.3) would give us a bound for the low frequency component of the solutions. And we would then be able to derive a high frequency-localized Morawetz estimate using this bound, which helps rule out the possibility of quasi-soliton solutions.

However, the proof of long time Strichartz estimates heavily relied on the double endpoint Strichartz estimates, that is, the endpoint Strichartz norm of Duhamel's formula is bounded by the dual of endpoint Strichartz norm of the nonlinear function:

$$\left\| \int_0^t e^{i(t-\tau)\Delta} F(\tau, \cdot)\, d\tau \right\|_{L_t^2 L_x^{\frac{2d}{d-2}}} \leq C \left\| F \right\|_{L_t^2 L_x^{\frac{2d}{d+2}}} .$$

In dimensions three and higher $d \geq 3$ the endpoint estimate $L_t^2 L_x^{\frac{2d}{d-2}}$ holds, but unfortunately in dimensions two, it was shown that the $L_t^2 L_x^\infty$ endpoint fails, and it is even impossible for us to have the the double endpoint Strichartz estimates. If we assumed that we had such double endpoint Strichartz estimates in dimensions two, it would be not too hard to derive (1.7.3). Therefore, the failure of the endpoint Strichartz estimates causes the difficulty defining and proving the long time Strichartz estimates.

Lack of endpoint Strichartz estimates is a common difficulty in dimentions two. In [16], Dodson had the same problem proving global well-posedness in the mass-critical regime in two dimentions. In fact, in [16] Dodson truncated the solutions to low frequency to make the right-hand side of (1.7.2) finite (Note that the cutoff in the mass-critical problem and the cutoff in the $\dot{H}^{\frac{1}{2}}$-critical regime are opposite). In order to estimate the errors produced by truncating to low frequency, a suitable bound over the high frequencies is needed. To conquer this defect, in [16] Dodson constructed a new function space $\tilde{X}_{k_0}$ out of the atomic spaces $U^2_\Delta$ in dimensions two, that is, $\tilde{X}_{k_0}$ norm is a delicate summation over a sequence of $U^2_\Delta$ norms of the frequency localized solutions. This construction captured the essential features of the long time Strichartz estimates that he used in dimensions three and higher.

Back to our case, then it is reasonable for us to follow Dodson's idea in [16] and construct a similar but 'upside-down' version long time Strichartz estimate adapted to the $\dot{H}^{\frac{1}{2}}$-critical setting (as we mentioned before, the cutoff in the mass-critical problem and the cutoff in $\dot{H}^{\frac{1}{2}}$ are opposite). This is because in the mass-critical regime, Dodson [16] lost the *a priori* control in the $\dot{H}^{\frac{1}{2}}$-norm and used the long time Strichartz to quantify how bad the $\dot{H}^{\frac{1}{2}}$-norm is out of control, while we lose the *a priori* control of the $L^2$-norm. Hence we define a long time Strichartz estimate over low frequencies, and expect it to give us a good control of the low frequency components of the solutions, which are exactly the error terms produced by truncating to high frequencies in the interaction Morawetz estimates.

We prove the long time Strichartz estimates by backward induction from high frequency to low frequency, so we should expect it to be harder than the regular forward induction. And in the proof of long time Strichartz estimates, we treat the contributions of free solutions and Duhamel terms to (6.2.5) separately. The hard part is the Duhamel term.

**Remark 1.6** (Main differences with [16])**.** After using Littlewood-Paley to decompose the nonlinearity in the Duhamel term, we should be very careful with the high frequency and high frequency interaction into low frequency terms (the worst case is five high frequencies interaction into low frequency). The reason is that instead of proving in Theorem 7.1 that $\|u\|^2_{\tilde{X}_{k_0}([0,T]\times\mathbb{R}^2)} \lesssim 1$ directly, we do a bootstrap argument, that is, we wish to prove $\|u\|^2_{\tilde{X}_{k_0}([0,T]\times\mathbb{R}^2)} \lesssim 1 + \varepsilon \, \|u\|^2_{\tilde{X}_{k_0}([0,T]\times\mathbb{R}^2)}$. From the construction of the atomic $X$-norm, we can see that the high frequency terms require more summability than the others. Therefore, we need to gain more decay than in the mass-critical case to sum over the high frequency terms, and hence close the bootstrap argument as desired. In contrast, these terms were not problematic in mass-critical [16], because the cutoff in the mass-critical problem and the cutoff in $\dot{H}^{\frac{1}{2}}$ are opposite, hence the worse case was all low frequencies interaction into high frequency. However, this case never happens since the contribution of all low frequencies remains low.

After having a good control of the low frequency component of the solutions, which are exactly the error terms produced by truncating to high frequencies in the interaction Morawetz estimates, we can show that the high frequency component of the solutions can be made as small as possible, which is known as

**Tool 2: Frequency-localized interaction Morawetz estimates**

On the other hand, due to the fact that $u$ is almost periodic and it concentrates in both space and frequency, we can show that the high frequency part of the solutions has a nontrivial component. Then we see the impossibility of the quasi-soliton solutions.

*Case I (Finite-time blow-up solutions)*

Next, we turn to the remaining case I (finite-time blow-up solutions). In $\dot{H}^{s_c}$-regime ($s_c \neq \frac{1}{2}$), the long time Strichartz estimates help to prove either additional decay or additional regularity for the solutions belonging to cases I, II, as a result, the solutions in these cases can be shown to have zero mass or energy, which contradicts with the fact they are blow-up solutions. However, in the $\dot{H}^{\frac{1}{2}}$ critical regime, due to the scaling, the long time Strichartz estimates do not provide any additional decay as we desired in the $s_c = \frac{1}{2}$ setting. This forces us to treat case II as finite-time blow-up solutions, instead of as frequency cascade solutions. Since we do not need the additional information from $\int_I N(t)\, dt$, we are allowed to consider the finite-time blow-up solutions (cases I, III) together.

Usually in dimensions three and higher, to rule out the existence of finite-time blow-up solutions, it is sufficient to use the finiteness of $T_{max}$ and Bernstein's inequality on the dual of the endpoint Strichartz norm ($L_t^2 L_x^{\frac{2d}{d+2}}$) to conclude that it has zero mass which contradicts the fact that $u$ blows up. However, we have no such endpoint $L_t^2 L_x^1$ norm in dimensions two and most of the $L^p$ inequalities fail when $p = 1$. Hence we have to seek another way of ruling out this type of solutions. To that effect we consider the rate of change in time of the mass of solutions that is restricted within a spacial bump:

$$y^2(t, R) := \int_{\mathbb{R}^2} \chi_R(x) \ |u(t,x)|^2 \ dx,$$

where $\chi_R(x) = \chi(\frac{x}{R})$ is a smooth cutoff function, such that

$$\chi(x) = \begin{cases} 1 & \text{if} \ \ |x| \leq 1 \\ 0 & \text{if} \ \ |x| > 2. \end{cases}$$

Then we can compute the derivative of $y^2$ with respect to time $t$. First integration by parts gives us a product of three functions, and then by treating the three factors at different frequencies separately using Littlewood-Paley decomposition, we can derive the following bound for the derivative of $y^2$:

$$\left|\frac{\partial y^2}{\partial t}\right| \lesssim \frac{1}{\sqrt{R}}.$$

In other words, the slope of $y^2$ is quite small. Combing the fact the cutoff mass at the blow-up time is zero, we know that if we trace $y^2(t)$ back to the starting point, the cutoff mass at $t = 0$ is very small. Then by taking $R \to \infty$, we conclude that in fact $u$ has zero mass, which contradicts the fact that $u$ blows up. Thus we see the impossibility of this type of minimal blow-up solutions.

Therefore we preclude the existence of both types of minimal blow-up solutions, thus proving the main result Theorem 1.5.

The rest of this thesis is organized as follows: In Chapter 2, we collect some useful lemmata. In Chapters 3 and 4, we recall the local well-posedness theory, a suitable perturbation lemma and the proof of the reduction to almost periodic solutions. Then it is sufficient to rule out the existence of almost periodic solutions in Theorem 4.15. In Chapter 5, we first show the impossibility of finite-time blow-up solutions. Next, in Chapter 6, we review some basic definitions and properties of the atomic spaces, and then prove a decomposition lemma, which is used in the proof of the long time Strichartz estimate in Chapter 7. In Chapter 7, we derive a long time Strichartz estimate adapted in our setting and in Chapter 8 we recall the interaction Morawetz estimates. Finally, we prove the frequency-localized interaction Morawetz estimates, and rule out the existence of quasi-soliton solutions in Chapter 9, which completes the proof of Theorem 4.15.

# CHAPTER 2

# PRELIMINARIES

In this chapter, we introduce notations that we will use in the rest of this thesis, collect some useful lemmata from harmonic analysis, and present some estimates for the linear Schrödinger equation.

## 2.1 Notation

- For nonnegative quantities $X$ and $Y$, we write $X \lesssim Y$ to denote the inequality $X \leq CY$ for some constant $C > 0$. If $X \lesssim Y \lesssim X$, we write $X \sim Y$. The dependence of implicit constants on parameters will be indicated by subscripts, for example, $X \lesssim_u Y$ denotes $X \leq CY$ for some $C = C(u)$.

- We use the expression $\mathcal{O}(X)$ to denote a finite linear combination of terms that resemble $X$ up to Littlewood-Paley projections and complex conjugation.

- For a time interval $I$, we write $L_t^q L_x^r(I \times \mathbb{R}^2)$ for the Banach space of functions $u : I \times \mathbb{R}^2 \to \mathbb{C}$ equipped with the norm

$$\|u\|_{L_t^q L_x^r(I\times\mathbb{R}^2)} := \left( \int_I \|u(t)\|_{L_x^r(\mathbb{R}^2)}^q \, dt \right)^{\frac{1}{q}}$$

with the usual adjustments when $q$ or $r$ is infinity. We will at times use the abbreviations $\|u\|_{L_t^q L_x^r(I\times\mathbb{R}^2)} = \|u\|_{L_t^q L_x^r}$ and $\|u\|_{L_x^r(\mathbb{R}^2)} = \|u\|_{L_x^r}$. Given $1 \leq r \leq \infty$, we write $r'$ for the solution to $\frac{1}{r} + \frac{1}{r'} = 1$.

## 2.2 Littlewood-Paley theory

We first define the Fourier transform on $\mathbb{R}^2$ by

$$\hat{f}(\xi) := (2\pi)^{-1} \int_{\mathbb{R}^2} e^{-ix\cdot\xi} f(x)\, dx.$$

The fractional differentiation operators $|\nabla|^s$ are defined via

$$\widehat{|\nabla|^s f}(\xi) := |\xi|^s \hat{f}(\xi).$$

**Definition 2.1** (Littlewood-Paley decomposition)**.** Let $\phi \in C_0^\infty(\mathbb{R}^2)$ be a radial and decreasing function,

$$\phi(x) = \begin{cases} 1, & |x| \leq 1 \\ 0, & |x| > 2. \end{cases}$$

Define the partition of unity

$$1 = \phi(x) + \sum_{j=1}^{\infty} \left[\phi(2^{-j}x) - \phi(2^{-j+2}x)\right] = \psi_0(x) + \sum_{j=1}^{\infty} \psi_j(x).$$

For any integer $j \geq 0$, let

$$P_{2^j} f = \mathcal{F}^{-1}(\psi_j(\xi)\hat{f}(\xi)) = \int K_j(x-y) f(y)\, dy,$$

where $K_j$ is an $L^1$ kernel. Let

$$P_{2^{j_1} \leq \cdot \leq 2^{j_2}} f = \sum_{2^{j_1} \leq \cdot \leq 2^{j_2}} P_{2^j} f.$$

We also define the frequency function

$$P_{\leq 2^j} f = \mathcal{F}\left(\phi(\frac{\xi}{2^j})\hat{f}(\xi)\right).$$

The Littlewood-Paley decomposition respects $L^p$ norms for $1 < p < \infty$.

**Lemma 2.2** (Littlewood-Paley theorem in [42, 43, 44, 55]). *For $1 < p < \infty$,*

$$\|f\|_{L^p(\mathbb{R}^2)} \sim_p \left\| \left( \sum_{j=-\infty}^{\infty} |P_{2^j} f|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^2)}.$$

**Remark 2.3.** Note that $\sum_{N \in \mathbb{Z}} P_N f$ gives a decomposition of $f$ as a superposition of frequency localized functions $\{P_N(f)\}$ each of which lives at frequencies $|\xi| \sim N$. Lower values of $N$ represent lower frequencies components of $f$. Higher values of $N$ represent higher frequencies components of $f$.

**Lemma 2.4** (Bernstein inequalities). *For $1 \leq r \leq q \leq \infty$ and $s \geq 0$,*

$$\begin{aligned}
\left\| |\nabla|^{\pm s} P_N f \right\|_{L^r_x(\mathbb{R}^2)} &\lesssim N^{\pm s} \, \|P_N f\|_{L^r_x(\mathbb{R}^2)}, \\
\| |\nabla|^s P_{\leq N} f\|_{L^r_x(\mathbb{R}^2)} &\lesssim N^s \, \|P_{\leq N} f\|_{L^r_x(\mathbb{R}^2)}, \\
\|P_{\geq N} f\|_{L^r_x(\mathbb{R}^2)} &\lesssim N^{-s} \, \| |\nabla|^s P_{\geq N} f\|_{L^r_x(\mathbb{R}^2)}, \\
\|P_{\leq N} f\|_{L^q_x(\mathbb{R}^2)} &\lesssim N^{\frac{2}{r} - \frac{2}{q}} \, \|P_{\leq N} f\|_{L^r_x(\mathbb{R}^2)}.
\end{aligned}$$

## 2.3 Estimates from harmonic analysis

Next we recall some fractional calculus estimates that appear originally in [9]. For a textbook treatment, one can refer to [64].

**Lemma 2.5** (Fractional product rule). *Let $s > 0$ and let $1 < 1, r_1, r_2, q_1, q_2 < \infty$ satisfy $\frac{1}{q} = \frac{1}{r_1} + \frac{1}{r_2}$, $\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2}$. Then*

$$\| |\nabla|^s (fg)\|_{L^q_x(\mathbb{R}^d)} \lesssim \|f\|_{L^{r_1}_x(\mathbb{R}^d)} \, \| |\nabla|^s g\|_{L^{r_2}_x(\mathbb{R}^d)} + \| |\nabla|^s f\|_{L^{p_1}_x(\mathbb{R}^d)} \, \|g\|_{L^{p_2}_x(\mathbb{R}^d)}.$$

**Lemma 2.6** (Chain rule for fractional derivatives)**.** *If $F \in C^2$, with $F(0) = 0$, $F' = 0$, and $|F''(a+b)| \le C\,[|F''(a)| + |F''(b)|]$, and $|F'(a+b)| \le [|F'(a)| + |F'(b)|]$, we have, for* $0 < \alpha < 1$,

$$\left\| |\nabla|^{\alpha} F(u) \right\|_{L_x^q(\mathbb{R}^d)} \le C \left\| F'(u) \right\|_{L_x^{p_1}(\mathbb{R}^d)} \left\| |\nabla|^{\alpha} u \right\|_{L_x^{p_2}(\mathbb{R}^d)},$$

*where* $\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2}$.
*In particular,*

$$\begin{aligned}
&\left\| |\nabla|^{\alpha} [F(u) - F(v)] \right\|_{L_x^q(\mathbb{R}^d)} \\
&\le C \left[ \|F''(u)\|_{L_x^{r_1}(\mathbb{R}^d)} + \|F''(v)\|_{L_x^{r_1}(\mathbb{R}^d)} \right] \left[ \left\| |\nabla|^{\alpha} u \right\|_{L_x^{r_2}(\mathbb{R}^d)} + \left\| |\nabla|^{\alpha} v \right\|_{L_x^{r_2}(\mathbb{R}^d)} \right] \|u - v\|_{L_x^{r_3}(\mathbb{R}^d)} \\
&+ C \left[ \|F'(u)\|_{L_x^{p_1}(\mathbb{R}^d)} + \|F'(v)\|_{L_x^{p_2}(\mathbb{R}^d)} \right] \left\| |\nabla|^{\alpha} (u - v) \right\|_{L_x^{p_2}(\mathbb{R}^d)},
\end{aligned}$$

*where* $\frac{1}{q} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$, $\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2}$.

The inequalities in the rest of this section are from harmonic analysis. For a proof we refer the reader to for example [51, 56].

**Lemma 2.7** (Sobolev embedding)**.** *For $\forall v \in C_0^{\infty}(\mathbb{R}^d)$, $\frac{1}{p} - \frac{1}{q} = \frac{s}{d}$ and $s > 0$, we have*

$$\|v\|_{L_x^q(\mathbb{R}^d)} \le C \left\| |\nabla|^s v \right\|_{L_x^p(\mathbb{R}^d)}, \ \textit{i.e.}\ \dot{W}^{s,p}(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d).$$

**Lemma 2.8** (Gagliardo-Nirenberg interpolation inequality)**.** *Let $1 < p < q \le \infty$ and $s > 0$ be such that $\frac{1}{q} = \frac{1}{p} - \frac{s\theta}{d}$ for some $0 < \theta = \theta(d,p,q,s) < 1$. Then for any $u \in \dot{W}^{s,p}(\mathbb{R}^d)$, we have*

$$\|u\|_{L^q(\mathbb{R}^d)} \lesssim_{d,p,q,s} \|u\|_{L^p(\mathbb{R}^d)}^{1-\theta} \|u\|_{\dot{W}^{s,p}(\mathbb{R}^d)}^{\theta}.$$

**Lemma 2.9** (Hardy's inequality)**.** *For any $0 \le s < d/2$, there exists $c = c(s,d) > 0$, such that*

$$\left\| \frac{f(\cdot)}{|x|^s} \right\|_{L^2(\mathbb{R}^d)} \le c(s,d) \|f\|_{\dot{H}^s(\mathbb{R}^d)}.$$

**Corollary 2.10.** *For any $0 < s < d$ and $1 < r < d/s$, there exists $c = c(s,d) > 0$, such that*

$$\left\| \frac{f(\cdot)}{|x|^s} \right\|_{L^r(\mathbb{R}^d)} \leq c(s,d) \, \|f\|_{\dot{W}^{s,r}(\mathbb{R}^d)} .$$

*Proof.* Using Littlewood-Paley decomposition, it suffices to show that

$$\int_{\mathbb{R}^d} \frac{|f(x)|^r}{|x|^{sr}} \, dx \lesssim \left\| \left( \sum_N N^{2s} \, |P_N f|^2 \right)^{\frac{1}{2}} \right\|_{L^r(\mathbb{R}^d)}^r = \int_{\mathbb{R}^d} \left| \sum_N N^{2s} \, |P_N f|^2 \right|^{\frac{r}{2}} \, dx.$$

We subdivide the left-hand side into dyadic shells and estimate

$$\int_{\mathbb{R}^d} \frac{|f(x)|^r}{|x|^{sr}} \, dx \lesssim \sum_R R^{-sr} \int_{|x| \leq R} |f(x)|^r \, dx = \sum_R R^{-sr} \, \|f\|_{L^r(|x| \leq R)}^r ,$$

where R ranges over dyadic numbers. On the one hand, use Littlewood-Paley decomposition and triangle inequality,

$$\|f\|_{L^r(\mathbb{R}^d)} \sim_r \left\| \left( \sum_N |P_N f|^2 \right)^{\frac{1}{2}} \right\|_{L^r(\mathbb{R}^d)} \leq \left\| \sum_N |P_N f| \right\|_{L^r(\mathbb{R}^d)} \leq \sum_N \|P_N f\|_{L^r(\mathbb{R}^d)} ,$$

$$\|f\|_{L^r(|x| \leq R)} \lesssim \sum_N \|P_N f\|_{L^r(|x| \leq R)} .$$

On the other hand, by Hölder's inequality and Bernstein's inequality, we have

$$\|P_N f\|_{L^r(|x| \leq R)} \lesssim R^{\frac{d}{r}} \, \|P_N f\|_{L^\infty(\mathbb{R}^d)} \lesssim R^{\frac{d}{r}} N^{\frac{d}{r}} \, \|P_N f\|_{L^r(\mathbb{R}^d)} .$$

So put all back together and use Minkowski inequality and Hölder's inequality, and $l^2 \hookrightarrow l^p$ for $p > 2$

$$\begin{aligned}
\int_{\mathbb{R}^d} \frac{|f(x)|^r}{|x|^{sr}} \, dx &\lesssim \sum_R R^{-sr} \left( \sum_N \min \left\{ 1, (RN)^{\frac{d}{r}} \right\} \|P_N f\|_{L^r(\mathbb{R}^d)} \right)^r \\
&= \sum_R \left( \sum_N \min \left\{ (RN)^{-s}, (RN)^{\frac{d}{r} - s} \right\} \left( N^s \, \|P_N f\|_{L^r(\mathbb{R}^d)} \right) \right)^r \\
&\lesssim \left( \sum_N \left( \sum_R \left( \min \left\{ (RN)^{-s}, (RN)^{\frac{d}{r} - s} \right\} \left( N^s \, \|P_N f\|_{L^r(\mathbb{R}^d)} \right) \right)^r \right)^{\frac{1}{r}} \right)^r
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \left(\sum_N \min\left\{N^{-s}, N^{\frac{d}{r}-s}\right\}\left(N^s \left\|P_N f\right\|_{L^r(\mathbb{R}^d)}\right)\right)^r \\
&\lesssim \left(\sum_N \left(N^s \left\|P_N f\right\|_{L^r(\mathbb{R}^d)}\right)^p\right)^{\frac{r}{p}} = \left\| \left\|N^s \left|P_N f\right|\right\|_{L^r_x}\right\|_{l^p_N}^r \\
&\lesssim \left\| \left\|N^s \left|P_N f\right|\right\|_{l^p_N}\right\|_{L^r_x}^r \lesssim \left\| \left\|N^s \left|P_N f\right|\right\|_{l^2_N}\right\|_{L^r_x}^r,
\end{aligned}
$$

where we may choose $p = \frac{d}{s} + 2 > \max\{2, r\}$. Since $0 < s < d/r$, the kernel $\min\left\{M^{-s}, M^{\frac{d}{r}-s}\right\}$ is absolutely convergent over dyadic numbers. □

**Lemma 2.11** (Hardy-Littlewood-Sobolev inequality)**.** *Let* $0 < r < d$*,* $1 < p < q < \infty$*,* $\frac{1}{p} - \frac{1}{q} = 1 - \frac{\gamma}{d}$*. then if* $R_\gamma(x) = \frac{1}{|x|^\gamma}$*,*

$$
\left\| |\cdot|^{-\gamma} * f\right\|_{L^q(\mathbb{R}^d)} = \left\|R_\gamma * f\right\|_{L^q(\mathbb{R}^d)} \lesssim \left\|f\right\|_{L^p(\mathbb{R}^d)}.
$$

## 2.4 Strichartz estimates

In this section, we record the Strichartz estimates for Schrödinger equations. We first take the linear Schrödinger equations

$$
\begin{cases}
i\partial_t u + \Delta u = 0 \\
u(0,x) = u_0(x).
\end{cases}
$$

In fact, performing Fourier transformation on both sides of the equation above gives us an ODE

$$
\begin{cases}
i\partial_t \hat{u}(t,\xi) - |\xi|^2 \hat{u} = 0 \\
\hat{u}(0,\xi) = \hat{u}_0(\xi),
\end{cases}
$$

and the solution is

$$
\hat{u}(t,\xi) = e^{-it|\xi|^2}\hat{u}_0(\xi).
$$

In particular, solutions with Schwartz initial data (for now, just consider the initial data are smooth enough) are Schwartz for all $t \in \mathbb{R}$. Then by taking the inverse Fourier transforms, we get

$$u(t,x) = \left(e^{-it\,|\xi|^2}\hat{u}_0(\xi)\right)^{\vee} = \left(e^{-it\,|\xi|^2}\right)^{\vee} * u_0 = K(t,x) * u_0,$$

where $K(t,x) = (4\pi i t)^{-\frac{d}{2}} e^{i\frac{|x|^2}{4t}}$.

**Definition 2.12** (Propagator $e^{it\Delta}$)**.** We define operator $e^{it\Delta}$ by $\widehat{e^{it\Delta} f} = e^{-it\,|\xi|^2}\hat{f}(\xi)$, that is,

$$e^{it\Delta} f(x) := \left(e^{-it\,|\xi|^2}\hat{f}(\xi)\right)^{\vee} = (4\pi i t)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{\frac{i\,|x-y|^2}{4t}} f(y)\,dy.$$

Since $e^{it\Delta}$ is unitary, we obtain the $L^2_x$ conservation law

$$\left\| e^{it\Delta} u_0 \right\|_{L^2_x(\mathbb{R}^2)} = \left\| u_0 \right\|_{L^2_x(\mathbb{R}^2)},$$

and then

$$\left\| e^{it\Delta} u_0 \right\|_{H^s_x(\mathbb{R}^2)} = \left\| u_0 \right\|_{H^s_x(\mathbb{R}^2)}.$$

From the explicit formula for $e^{it\Delta}$,

$$[e^{it\Delta} f](x) = (4\pi i t)^{-1} \int_{\mathbb{R}^2} e^{i\frac{|x-y|^2}{4t}} f(y)\,dy \quad \text{for } t \neq 0,$$

we have the dispersive inequality

$$\left\| e^{it\Delta} u_0 \right\|_{L^\infty_x(\mathbb{R}^2)} \lesssim t^{-1} \left\| u_0 \right\|_{L^1_x(\mathbb{R}^2)} \quad \text{for } t \neq 0.$$

We can interpolate this with the $L^2_x$ conservation law and obtain

$$\left\| e^{it\Delta} u_0 \right\|_{L^{p'}_x(\mathbb{R}^2)} \lesssim t^{-(\frac{2}{p}-1)} \left\| u_0 \right\|_{L^p_x(\mathbb{R}^2)} \quad \text{for } t \neq 0$$

for all $1 \leq p \leq 2$, where $p'$ is the dual exponent of $p$, defined by the formula $\frac{1}{p} + \frac{1}{p'} = 1$.

By combining the above dispersive estimates with some duality arguments, one can obtain an extremely useful set of estimates, known as Strichartz estimates.

First, we define the following spaces:

**Definition 2.13** (Strichartz spaces)**.** We call a pair of exponents $(q, r)$ admissible if

$$\frac{2}{q} + \frac{2}{r} = 1$$

for any $2 < q \leq \infty$, $2 \leq r < \infty$. Notice that we don't have the endpoint in two dimensions, i.e. $(q, r) \neq (2, \infty)$.

For an interval $I$ and $s \geq 0$, we define the Strichartz norm by

$$\|u\|_{\dot{S}^s(I)} := \sup \left\{ \| |\nabla|^s u\|_{L_t^q L_x^r(I\times\mathbb{R}^2)} : (q, r) \text{ is admissible} \right\}.$$

**Lemma 2.14** (Strichartz estimates)**.** *For any admissible exponents* $(q, r)$ *and* $(\tilde{q}, \tilde{r})$ *we have the homogeneous Strichartz estimate*

$$\left\|e^{it\Delta}u_0\right\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^2)} \lesssim_{q,r} \|u_0\|_{L_x^2(\mathbb{R}^2)}$$

*the dual homogeneous Strichartz estimate*

$$\left\|\int_{\mathbb{R}} e^{-is\Delta}F(s)\,ds\right\|_{L_x^2(\mathbb{R}^2)} \lesssim_{\tilde{q},\tilde{r}} \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}(\mathbb{R}\times\mathbb{R}^2)}$$

*and the inhomogeneous Strichartz estimate*

$$\left\|\int_{t'<t} e^{i(t-t')\Delta}F(t')\,dt'\right\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^2)} \lesssim_{q,r,\tilde{q},\tilde{r}} \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}(\mathbb{R}\times\mathbb{R}^2)}.$$

For the proofs, see [19, 67, 26] for details. Also see [58] for a proof.

Then we also recall the following bilinear Strichartz estimate:

**Lemma 2.15** (Bilinear estimates in [1, 4])**.** *If* $\hat{u}_0$ *is supported on* $|\xi| \sim N$, $\hat{v}_0$ *is supported on* $|\xi| \sim M$, $M \ll N$,

$$\left\|(e^{it\Delta}u_0)(e^{it\Delta}v_0)\right\|_{L_t^2 L_x^2(\mathbb{R}\times\mathbb{R}^2)} \lesssim \left(\frac{M}{N}\right)^{\frac{1}{2}} \|u_0\|_{L_x^2(\mathbb{R}^2)} \|v_0\|_{L_x^2(\mathbb{R}^2)}.$$

**Remark 2.16.** Lemma 2.15 also holds for $(e^{it\Delta}u_0)(\overline{e^{it\Delta}v_0})$. In fact,

$$\begin{aligned}
\left\|(e^{it\Delta}u_0)(e^{it\Delta}v_0)\right\|_{L_t^2L_x^2}^2 &= \left\|(e^{it\Delta}u_0)(e^{it\Delta}v_0)(\overline{e^{it\Delta}u_0})(\overline{e^{it\Delta}v_0})\right\|_{L_t^1L_x^1} \\
&= \left\|(e^{it\Delta}u_0)(\overline{e^{it\Delta}v_0})\right\|_{L_t^2L_x^2}^2 .
\end{aligned}$$

**Remark 2.17.** Note that naively consider the bilinear Strichartz estimate, then we have

$$\begin{aligned}
\left\|(e^{it\Delta}u_0)(e^{it\Delta}v_0)\right\|_{L_t^2L_x^2(\mathbb{R}\times\mathbb{R}^2)} &\lesssim \left\|e^{it\Delta}u_0\right\|_{L_t^4L_x^4(\mathbb{R}\times\mathbb{R}^2)} \left\|e^{it\Delta}v_0\right\|_{L_t^4L_x^4(\mathbb{R}\times\mathbb{R}^2)} \\
&\lesssim \left\|u_0\right\|_{L_x^2(\mathbb{R}^2)} \left\|v_0\right\|_{L_x^2(\mathbb{R}^2)} .
\end{aligned} \tag{2.4.1}$$

Therefore, we see that this bilinear Strichartz estimate Lemma 2.15 is better the product of two Strichartz estimates, since the cancellation caused by the different supports of $u_0$ and $v_0$ will give more decay than (2.4.1), which is the advantage of this bilinear Strichartz estimate.

# CHAPTER 3

# LOCAL THEORY AND STABILITY

In this chapter we will review the local well-posedness theory of Cauchy problem and stability. The local well-posedness theory is standard. We adapt the local theory in the energy-critical setting in [27] to our $\dot{H}^{\frac{1}{2}}$-critical regime. However, we will develop a refined stability result. That is, we do not require errors to be small in spaces with derivatives. This original stability result is first shown in [13], and the refined result is proved in [49] but for data in dimensions four and above. We will extend it to two dimensions.

## 3.1 Scaling and conservation laws

In this section, we review the scaling of the NLS (1.6.1) and three conservation laws.

**Lemma 3.1.** *The NLS* (1.6.1) *is $\dot{H}^{\frac{1}{2}}$-critical, i.e. if $u(t,x)$ solves* (1.6.1) *with initial data $u_0$, so does $u_\lambda(x) = \sqrt{\lambda}u(\lambda^2 t, \lambda x)$, with initial data $u_{0,\lambda}(x) = \sqrt{\lambda}u_0(\lambda x)$ and $\|u_{0,\lambda}\|_{\dot{H}_x^{\frac{1}{2}}} = \|u_0\|_{\dot{H}_x^{\frac{1}{2}}}$.*

*Proof.* Suppose $u$ be a solution of (1.6.1).

(i) Let $u_\lambda = \lambda^\alpha u(\lambda^2 t, \lambda x)$, then

$$\partial_t u_\lambda = \lambda^{\alpha+2} u_t(\lambda^2 t, \lambda x), \quad \Delta u_\lambda = \lambda^{\alpha+2} \Delta u(\lambda^2 t, \lambda x), \quad |u_\lambda|^4 u_\lambda = \lambda^{5\alpha} |u|^4 u.$$

If $u_\lambda$ solves (1.6.1), i.e.

$$i\partial_t u_\lambda + \Delta u_\lambda - |u_\lambda|^4 u_\lambda = 0,$$

then we demand $\lambda^{\alpha+2} = \lambda^{5\alpha}$, which implies $\alpha = \frac{1}{2}$. Therefore $u_\lambda(x) = \sqrt{\lambda} u(\lambda^2 t, \lambda x)$ is also a solution of (1.6.1).

(ii) By the definition of Fourier transformation and a change of variables $\eta = \frac{\xi}{\lambda}$, we have

$$\begin{aligned} \|u_{0,\lambda}\|_{\dot{H}_x^{\frac{1}{2}}} &= \left\| \sqrt{\lambda} u_0(\lambda x) \right\|_{\dot{H}_x^{\frac{1}{2}}} = \left\| |\xi|^{\frac{1}{2}} \sqrt{\lambda} \widehat{u_0(\lambda x)} \right\|_{L_x^2} \\ &= \left\| |\xi|^{\frac{1}{2}} \sqrt{\lambda} \lambda^{-2} \hat{u}_0(\frac{\xi}{\lambda}) \right\|_{L_x^2} = \left\| |\eta|^{\frac{1}{2}} \hat{u}_0(\eta) \right\|_{L_x^2} = \|u_0\|_{\dot{H}_x^{\frac{1}{2}}}. \end{aligned}$$

Therefore, the $\dot{H}^{\frac{1}{2}}$ norm of initial data remains the same under the scaling $u_{0,\lambda}(x) = \sqrt{\lambda} u_0(\lambda x)$.

$\square$

**Theorem 3.2** (Mass, energy and momentum conservation laws). *A solution to* (1.6.1) *conserves the mass,*

$$M(u(t)) := \int_{\mathbb{R}^2} |u(t,x)|^2 \, dx = M(u_0)$$

*the energy,*

$$E(u(t)) := \int_{\mathbb{R}^2} \frac{1}{2} |\nabla u(t,x)|^2 + \frac{1}{6} |u(t,x)|^6 \, dx = E(u_0)$$

*and the momentum*

$$\mathcal{P}(u)(t) := \int_{\mathbb{R}^2} \mathrm{Im}[\bar{u}(t,x) \nabla u(t,x)] \, dx = \mathcal{P}(u)(0).$$

*Proof.* • Mass:

We compute the derivative of the mass with respect to time $t$. Bring the time derivative inside the integral and combining with the product rule and the equation itself, we have

$$\begin{aligned}\frac{dM(u)}{dt} &= \frac{d}{dt}\int_{\mathbb{R}^2} |u(t,x)|^2 \, dx = \int_{\mathbb{R}^2} \frac{d}{dt} |u(t,x)|^2 \, dx = 2\int_{\mathbb{R}^2} \mathrm{Re}[\bar{u}u_t] \, dx \\ &= 2\int_{\mathbb{R}^2} \mathrm{Im}\ \left[\bar{u}(\ |u|^4 u - \Delta u)\right] \, dx = 2\,\mathrm{Im}\int_{\mathbb{R}^2} |\nabla u|^2 \, dx = 0.\end{aligned}$$

Then $M(u(t)) \equiv M(u(0))$ is conserved.

- Energy:

We proceed the same as we did in the proof of the mass conservation law.

$$\begin{aligned}\frac{dE(u)}{dt} &= \frac{d}{dt}\int_{\mathbb{R}^2} \frac{1}{2} |\nabla u(t,x)|^2 + \frac{1}{6} |u(t,x)|^6 \, dx \\ &= \mathrm{Re}\left[\int_{\mathbb{R}^2} 2\nabla u \cdot \nabla \bar{u}_t + |u|^4 u\bar{u}_t \, dx\right] \\ &= \mathrm{Re}\left[-2\int_{\mathbb{R}^2} \Delta u \bar{u}_t \, dx\right] + \mathrm{Re}\left[\int_{\mathbb{R}^d} (iu_t + \Delta u)\bar{u}_t \, dx\right] \\ &= \mathrm{Re}\left[\int_{\mathbb{R}^2} i\ |u_t|^2 \, dx\right] = 0.\end{aligned}$$

Then $E(u(t)) \equiv E(u(0))$ is conserved.

- Momentum: Similarly,

$$\begin{aligned}\frac{d\mathcal{P}(u)(t)}{dt} &= \frac{d}{dt}\int_{\mathbb{R}^2} \mathrm{Im}[\bar{u}(t,x)\nabla u(t,x)] \, dx = \int_{\mathbb{R}^2} \mathrm{Im}[\bar{u}_t \nabla u + \bar{u}\nabla u_t] \, dx \\ &= \int_{\mathbb{R}^2} \mathrm{Im}[\bar{u}_t \nabla u - u_t \nabla \bar{u}] \, dx = \int_{\mathbb{R}^2} 2\,\mathrm{Im}[\bar{u}_t \nabla u] \, dx \\ &= \int_{\mathbb{R}^2} 2\,\mathrm{Im}\ \left[(i\ |u|^4 \bar{u} - i\Delta\bar{u})\nabla u\right] \, dx = \int_{\mathbb{R}^2} 2\,\mathrm{Re}\ \left[\ |u|^4 \bar{u}\nabla u - \Delta\bar{u}\nabla u\right] \, dx \\ &= \int_{\mathbb{R}^2} \left(|u|^4 \bar{u}\nabla u + |u|^4 u\nabla\bar{u}\right) \, dx - 2\int_{R^2} \mathrm{Re}\,\Delta\bar{u}\nabla u \, dx.\end{aligned}$$

The first integral above is 0, because of the identity $|u|^4\,\bar{u}\nabla u + |u|^4\,u\nabla\bar{u} = \frac{1}{3}\nabla(|u|^6)$. For the second integral, integration by parts and properties of complex conjugate give

$$\int_{R^2} \operatorname{Re}\Delta\bar{u}\nabla u\,dx = -\int_{R^2} \operatorname{Re}\Delta u\nabla\bar{u}\,dx = -\int_{R^2} \operatorname{Re}\Delta\bar{u}\nabla u\,dx.$$

Then $\int_{R^2} \operatorname{Re}\Delta\bar{u}\nabla u\,dx = 0$. Hence $\mathcal{P}(u(t)) \equiv \mathcal{P}(u(0))$ is conserved.

$\square$

**Remark 3.3.** In the mass- and energy-critical regimes, we have mass conservation law and energy conservation law, and they control the $L^2$ and $\dot{H}^1$ norms respectively. However, in $\dot{H}^{\frac{1}{2}}$-critical regime, although the conserved quantity, the momentum, scales like $\dot{H}^{\frac{1}{2}}$, it does not control the $\dot{H}^{\frac{1}{2}}$ norm of the solutions.

In fact, the additional assumption $\sup\limits_{t\in I}\|u(t)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^2)} < +\infty$ plays a similar role as the conservation laws in the mass- and energy-critical regimes. In this thesis, we assume that we have this assumption. We are not going to show that for all data $u_0 \in \dot{H}^{\frac{1}{2}}$, we must have $\sup\limits_{t\in I}\|u(t)\|_{\dot{H}_x^{\frac{1}{2}}(\mathbb{R}^2)} < +\infty$, for solutions of the quintic defocusing NLS in two dimensions.

## 3.2 Local theory

In this section, we record the local well-posedness theory of 1.6.1, which is adapted from the local theory in the energy-critical setting in [27].

First, we give the formal definition of the solution to (1.6.1):

**Definition 3.4** (Solution)**.** A function $u : I \times \mathbb{R}^2 \to C$ on a time interval $I(\ni 0)$ is a solution to (1.6.1) if it belongs to $C_t\dot{H}_x^{\frac{1}{2}}(K\times\mathbb{R}^2)\cap L_{t,x}^8(K\times\mathbb{R}^2)$ for every compact

$K \subset I$ and obeys the Duhamel formula

$$u(t) = e^{it\Delta}u_0 - i\int_0^t e^{i(t-t')\Delta}\,\left(|u|^4 u\right)(t', x)\, dt'$$

for all $t \in I$. We call $I$ the lifespan of $u$; we say $u$ is a maximal-lifespan solution if it cannot be extended to any strictly larger interval. If $I = R$, we say $u$ is global.

If a solution $u$ is global, we will care about its long time behavior, so-called scattering, that is,

**Definition 3.5** (Scattering)**.** A solution to (1.6.1) is said to scatter forward in time if there exists $u_+ \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^2)$ such that

$$\lim_{t\to+\infty} \left\| u(t) - e^{it\Delta}u_+ \right\|_{\dot{H}_x^{\frac{1}{2}}(\mathbb{R}^2)} = 0.$$

Similarly, a solution to (1.6.1) is said to scatter backward in time if there exists $u_- \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^2)$ such that

$$\lim_{t\to-\infty} \left\| u(t) - e^{it\Delta}u_- \right\|_{\dot{H}_x^{\frac{1}{2}}(\mathbb{R}^2)} = 0. \tag{3.2.1}$$

**Definition 3.6** (Scattering size and blow up)**.** We define the scattering size of a solution $u$ to (1.6.1) on a time interval $I$ by

$$S_I(u) = \int_I \int_{\mathbb{R}^2} |u(t,x)|^8 \; dxdt.$$

If there exists $t \in I$ such that $S_{[t, \sup I)}(u) = \infty$, then we say $u$ blows up forward in time. Similarly, if there exists $t \in I$ such that $S_{(\inf I, t]}(u) = \infty$, then we say $u$ blows up backward in time.

The local theory for (1.6.1) has been worked out by Cazenave-Weissler [6, 7, 8, 5]. We first present the statement of the local theory.

**Theorem 3.7** (Local well-posedness)**.** *Assume* $u_0 \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^2)$. *Then there exists a unique maximal-lifespan solution* $u : I \times \mathbb{R}^2 \to \mathbb{C}$ *to* (1.6.1) *such that:*

1. *(Local existence) $I$ is an open interval containing* $0$.

2. *(Blowup criterion) If* $\sup I$ *is finite, then the solution $u$ blows up forward in time. If* $\inf I$ *is finite, then the solution $u$ blows up backward in time.*

3. *(Scattering) If $u$ does not blow up forward in time then* $\sup I = \infty$ *and $u$ scatters forward in time. If $u$ does not blow up backward in time then* $\inf I = \infty$ *and $u$ scatters backward in time.*

4. *(Small-data global existence) If $\|u_0\|_{\dot{H}_x^{\frac{1}{2}}}$ is sufficiently small then the solution $u$ is global, scatters and does not blow up either forward or backward in time, with $S_{\mathbb{R}}(u) \lesssim \|u_0\|_{\dot{H}_x^{\frac{1}{2}}}^8$.*

Now we recall the proofs of the local theory above.

**Definition 3.8.** Let us define the $S(I)$, $T(I)$ , $N(I)$, $X(I)$, $Y(I)$, $Z(I)$ norm for a time interval $I$ by

$$\|v\|_{S(I)} := \|v\|_{L_t^8 L_x^8(I\times\mathbb{R}^2)}, \qquad \|v\|_{T(I)} := \|v\|_{L_t^3 L_x^6(I\times\mathbb{R}^2)},$$
$$\|v\|_{N(I)} := \|v\|_{L_t^{\frac{3}{2}} L_x^{\frac{6}{5}}(I\times\mathbb{R}^2)}, \qquad \|v\|_{X(I)} := \|v\|_{L_t^{12} L_x^6(I\times\mathbb{R}^2)},$$
$$\|v\|_{Y(I)} := \|v\|_{L_t^{\frac{12}{5}} L_x^{\frac{6}{5}}(I\times\mathbb{R}^2)}, \qquad \|v\|_{Z(I)} := \|v\|_{L_t^{\frac{4}{3}} L_x^{\frac{4}{3}}(I\times\mathbb{R}^2)}.$$

Recall the definition of the space $S^0(I \times \mathbb{R}^2)$, which is the closure of the Schwartz functions under the norm

$$\|f\|_{S^0(I\times\mathbb{R}^2)} := \sup_{(q,r) \text{ admissible}} \|f\|_{L_t^q L_x^r(I\times\mathbb{R}^2)}.$$

Then the space $\dot{S}^{\frac{1}{2}}(I \times \mathbb{R}^2)$ is the closure is taken with respect to the norm

$$\|f\|_{\dot{S}^{\frac{1}{2}}(I\times\mathbb{R}^2)} := \left\| |\nabla|^{\frac{1}{2}} f \right\|_{S^0(I\times\mathbb{R}^2)}.$$

**Theorem 3.9** (Standard local well-posedness). *Assume* $u_0 \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^2)$, $t_0 \in I$, $\|u_0\|_{\dot{H}_x^{\frac{1}{2}}(\mathbb{R}^2)} \leq A$. *Then there exists* $\delta = \delta(A)$ *such that if* $\left\|e^{i(t-t_0)\Delta}u_0\right\|_{X(I)} < \delta$, *there exists a unique solution* $u$ *to* (1.6.1) *in* $I \times \mathbb{R}^2$, *with* $u \in C(I; \dot{H}^{\frac{1}{2}}(\mathbb{R}^2))$*:*

$$\left\| |\nabla|^{\frac{1}{2}} u \right\|_{T(I)} + \sup_{t\in I} \left\| |\nabla|^{\frac{1}{2}} u \right\|_{L_x^2} \leq CA, \quad \|u\|_{X(I)} \leq 2\delta.$$

*Moreover, if* $u_{0,k} \to u_0$ *in* $\dot{H}^{\frac{1}{2}}(\mathbb{R}^2)$*, the corresponding solutions* $u_k \to u$ *in* $C(I; \dot{H}^{\frac{1}{2}}(\mathbb{R}^2))$*.*

*Proof.* (i) Existence and uniqueness.

The NLS (1.6.1) is equivalent to the following integral equation

$$u(t) = e^{it\Delta}u_0 - \int_0^t e^{i(t-t')\Delta} F(u)\, dt', \text{ where } F(u) = |u|^4 u.$$

To show the existence and the uniqueness of the solution, we rely on the contraction mapping theorem (fixed-point theorem). We first define a set

$$B_{a,b} = \left\{ v \text{ on } I \times \mathbb{R}^2 : \|v\|_{X(I)} \leq a, \left\| |\nabla|^{\frac{1}{2}} v \right\|_{T(I)} \leq b \right\}$$

and a map

$$\Phi_{u_0}(v) = e^{it\Delta}u_0 - \int_0^t e^{i(t-t')\Delta} F(u)\, dt'.$$

We will next choose $\delta, a, b$, so that $\Phi_{u_0}(v) : B_{a,b} \to B_{a,b}$ and is a contraction mapping.

- $\Phi$ is a self map.

  By Strichartz estimates and Hölder's inequality,

$$\begin{aligned}
\|\Phi_{u_0}(v)\|_{X(I)} &\leq \left\| |\nabla|^{\frac{1}{2}} e^{it\Delta}u_0 \right\|_{X(I)} + \left\| \int_0^t e^{i(t-t')\Delta} F(u)\, dt' \right\|_{X(I)} \\
&\leq \delta + C \left\| |\nabla|^{\frac{1}{2}} \; |v|^4 v \right\|_{N(I)} \leq \delta + C_1 \left\| |\nabla|^{\frac{1}{2}} v \right\|_{T(I)} \|v\|_{X(I)}^4 \\
&\leq \delta + C_1 a^4 b
\end{aligned}$$

and

$$\left\| \, |\nabla|^{\frac{1}{2}} \, \Phi_{u_0}(v) \right\|_{T(I)} \leq C \, \|u_0\|_{\dot{H}_x^{\frac{1}{2}}(\mathbb{R}^2)} + \left\| \, |\nabla|^{\frac{1}{2}} \int_0^t e^{i(t-t')\Delta} F(u) \, dt' \right\|_{T(I)}$$

$$\leq C'A + C' \, \left\| \, |\nabla|^{\frac{1}{2}} \, |v|^4 \, v \right\|_{N(I)} \leq C_2 A + C_2 a^4 b.$$

Now we demand $\delta$, $a$, $b$ such that

$$\begin{cases} \|\Phi_{u_0}(v)\|_{X(I)} \leq \delta + C_1 a^4 b \leq a, \\ \left\| \, |\nabla|^{\frac{1}{2}} \, \Phi_{u_0}(v) \right\|_{T(I)} \leq C_2 A + C_2 a^4 b \leq b. \end{cases}$$

It is possible for us to choose $a$ and $b$ satisfying

$$\begin{cases} \delta = \frac{a}{2} & C_1 a^4 b \leq \frac{a}{2} \\ C_2 A = \frac{b}{2} & C_2 a^4 b \leq \frac{b}{2} \end{cases}$$

So $\Phi_{u_0} : B_{a,b} \to B_{a,b}$.

- Next, to prove the contraction, we use the same argument.

$$\|\Phi_{u_0}(v) - \Phi_{u_0}(v')\|_{X(I)} \leq C \, \left\| \, |\nabla|^{\frac{1}{2}} \, (F(v) - F(v')) \right\|_{N(I)}$$

$$\leq C \, \left\| \, |\nabla|^{\frac{1}{2}} \, v \right\|_{T(I)} \, \|v - v'\|_{X(I)}^4 + C \, \left\| \, |\nabla|^{\frac{1}{2}} \, (v - v') \right\|_{T(I)} \, \|v\|_{X(I)}^4$$

$$\leq Cb( \, \|v\|_{X(I)}^3 + \, \|v'\|_{X(I)}^3) \, \|v - v'\|_{X(I)} + C \, \left\| \, |\nabla|^{\frac{1}{2}} \, (v - v') \right\|_{T(I)} a^4$$

$$\leq Ca^3 b \, \|v - v'\|_{X(I)} + Ca^4 \, \left\| \, |\nabla|^{\frac{1}{2}} \, (v - v') \right\|_{T(I)}$$

and

$$\left\| \, |\nabla|^{\frac{1}{2}} \, \Phi_{u_0}(v) - \, |\nabla|^{\frac{1}{2}} \, \Phi_{u_0}(v') \right\|_{T(I)} \leq Ca^3 b \, \|v - v'\|_{X(I)} + Ca^4 \, \left\| \, |\nabla|^{\frac{1}{2}} \, (v - v') \right\|_{T(I)}.$$

Then we have

$$\begin{cases} \|\Phi_{u_0}(v) - \Phi_{u_0}(v')\|_{X(I)} \leq Ca^3 b \, \|v - v'\|_{X(I)} + Ca^4 \, \left\| \, |\nabla|^{\frac{1}{2}} \, (v - v') \right\|_{T(I)} \\ \left\| \, |\nabla|^{\frac{1}{2}} \, [\Phi_{u_0}(v) - \Phi_{u_0}(v')] \right\|_{T(I)} \leq Ca^3 b \, \|v - v'\|_{X(I)} + Ca^4 \, \left\| \, |\nabla|^{\frac{1}{2}} \, (v - v') \right\|_{T(I)}. \end{cases}$$

Thus we establish the contraction property. We then find a unique $u \in B_{a,b}$ solving $\Phi_{u_0}(u) = u$.

(ii) Continuity.

Note that the free solution part is always continuous, that is, $e^{it\Delta}u_0 \in C(I; \dot{H}^{\frac{1}{2}}(\mathbb{R}^2))$ with norm bounded by $A$. Then to show that $u \in C(I; \dot{H}^{\frac{1}{2}}(\mathbb{R}^2))$, we only consider the Duhamel term: $\int_0^t e^{i(t-t')\Delta} F(u)\, dt'$. By Strichartz estimates and Hölder's inequality, we write

$$
\begin{aligned}
&\left\| \int_0^t e^{i(t-t')\Delta} F(u)\, dt' - \int_0^s e^{i(s-t')\Delta} F(u)\, dt' \right\|_{\dot{H}_x^{\frac{1}{2}}} \\
&= \left\| e^{it\Delta} \int_0^t e^{-it'\Delta} F(u)\, dt' - e^{is\Delta} \int_0^s e^{-it'\Delta} F(u)\, dt' \right\|_{\dot{H}_x^{\frac{1}{2}}} \\
&\leq \left\| e^{it\Delta} \int_s^t e^{-it'\Delta} F(u)\, dt' \right\|_{\dot{H}_x^{\frac{1}{2}}} + \left\| \left(e^{it\Delta} - e^{is\Delta}\right) \int_0^s e^{-it'\Delta} F(u)\, dt' \right\|_{\dot{H}_x^{\frac{1}{2}}} \\
&\leq C \left\| \, |\nabla|^{\frac{1}{2}} F(u) \right\|_{N([s,t])} + C \left\| \left(e^{i(t-s)\Delta} - 1\right) \int_0^s e^{i(s-t')\Delta} F(u)\, dt' \right\|_{\dot{H}_x^{\frac{1}{2}}} \\
&\leq C \left\| \, |\nabla|^{\frac{1}{2}} u \right\|_{T([s,t])} \|u\|_{X([s,t])}^4 + C\, |t-s| \left\| \int_0^s e^{-it'\Delta} F(u)\, dt' \right\|_{\dot{H}_x^{\frac{1}{2}}} \\
&\leq C \left\| \, |\nabla|^{\frac{1}{2}} u \right\|_{T([s,t])} \|u\|_{X([s,t])}^4 + C\, |t-s| \left\| \, |\nabla|^{\frac{1}{2}} F(u) \right\|_{N(I)} \\
&\leq C \left\| \, |\nabla|^{\frac{1}{2}} u \right\|_{T([s,t])} \|u\|_{X([s,t])}^4 + C\, |t-s| \left\| \, |\nabla|^{\frac{1}{2}} u \right\|_{T(I)} \|u\|_{X(I)}^4 .
\end{aligned}
$$

Then we can see $u \in C(I; \dot{H}^{\frac{1}{2}}(\mathbb{R}^2))$.

The last continuity statement is an easy consequence of the fixed point argument, see Remark 3.18.

(iii) Boundness of Strichartz norms.

It is easy to see $|\nabla|^{\frac{1}{2}} u \in L_I^q L_x^r$ for any admissible index $(q, r)$, i.e. $u \in \dot{S}^{\frac{1}{2}}$.

In fact, by Strichartz estimates and Hölder's inequality,

$$\begin{aligned}
\|u\|_{\dot{S}^{\frac{1}{2}}(I)} &\lesssim \left\|e^{it\Delta}u_0\right\|_{\dot{S}^{\frac{1}{2}}(I)} + \left\|\int_0^t e^{i(t-t')\Delta}F(u)\,dt'\right\|_{\dot{S}^{\frac{1}{2}}(I)} \\
&\lesssim \|u_0\|_{\dot{H}_x^{\frac{1}{2}}} + \left\||\nabla|^{\frac{1}{2}}F(u)\right\|_{N(I)} \\
&\lesssim \|u_0\|_{\dot{H}_x^{\frac{1}{2}}} + \left\||\nabla|^{\frac{1}{2}}u\right\|_{T(I)} \|u\|_{X(I)}^4 \\
&\lesssim \|u_0\|_{\dot{H}_x^{\frac{1}{2}}} + \delta\,\|u\|_{\dot{S}(I)}^4 .
\end{aligned}$$

This means

$$\|u\|_{\dot{S}^{\frac{1}{2}}(I)} \leq C\,\|u_0\|_{\dot{H}_x^{\frac{1}{2}}} + C\delta\,\|u\|_{\dot{S}^{\frac{1}{2}}(I)}^4 .$$

By choosing $\delta$ small enough, we know that $\|u\|_{\dot{S}^{\frac{1}{2}}}$ is bounded, hence $u \in \dot{S}^{\frac{1}{2}}$.

$\square$

**Remark 3.10** (Small data argument)**.** There exists $\tilde{\delta}$ such that if $\|u_0\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^2)} \leq \tilde{\delta}$, the conclusion of Theorem 3.9 holds, by Strichartz estimates and Sobolev embedding.

**Remark 3.11.** Given $u_0 \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^2)$, there exists $(0 \in)I$ such that the hypothesis of Theorem 3.9 holds on $I$. This is clear from Strichartz estimates and Sobolev embedding.

**Remark 3.12.** The uniqueness of solutions of (1.6.1) allows us to define a maximal interval $I(u_0) = (-T_-(u_0), T_+(u_0))$, where the solution is defined.

**Lemma 3.13** (Standard finite blow-up criterion)**.** *If $T_+(u_0) < +\infty$, then*

$$\|u\|_{S([t_0, t_0+T_+(u_0)))} = +\infty.$$

*A corresponding result holds for $T_-(u_0)$.*

*Proof.* Assume that both $T_+(u_0)$ and $\|u\|_{S([t_0,t_0+T_+(u_0)])}$ are finite. Then interpolating with the uniform bound of $\|u\|_{L_t^\infty \dot{H}_x^{\frac{1}{2}}}$, we know that $\|u\|_{X([t_0,t_0+T_+(u_0))}$ is also finite. Define

$$M := \|u\|_{X([t_0,t_0+T_+(u_0)])}.$$

We can divide $[[t_0, t_0 + T_+(u_0)]$ into $N = N(\varepsilon)$ subintervals $I_j$, such that $\cup_{j=1}^N =$ $[t_0, t_0 + T_+(u_0)]$ and $\|u\|_{X(I_j)} \leq \varepsilon$ ($\varepsilon$ will be chosen later).

- Step 1: Show $\|u\|_{L_t^\infty \dot{H}_x^{\frac{1}{2}}([t_0,t_0+T_+(u_0)]\times\mathbb{R}^2)} + \left\| |\nabla|^{\frac{1}{2}} u \right\|_{T([t_0,t_0+T_+(u_0)])} < \infty$ We write the integral equation on each interval $I_j$, to deduce

$$\begin{aligned}&\sup \|u(t)\|_{\dot{H}_x^{\frac{1}{2}}} + \left\| |\nabla|^{\frac{1}{2}} u \right\|_{T(I_j)} \\ \leq\ & C \|u(t)\|_{\dot{H}_x^{\frac{1}{2}}} + C \left\| |\nabla|^{\frac{1}{2}} u \right\|_{T(I_j)} \|u\|_{X(I_j)}^4 \leq C \|u(t_j)\|_{\dot{H}_x^{\frac{1}{2}}} + C\varepsilon^4 \left\| |\nabla|^{\frac{1}{2}} \right\|_{T(I_j)}\end{aligned}$$

  where $t_j$ is any fixed point in $I_j$.

  Then by choosing $C\varepsilon^4 < \frac{1}{2}$, we have $\sup \|u(t)\|_{\dot{H}_x^{\frac{1}{2}}} + \left\| |\nabla|^{\frac{1}{2}} u \right\|_{T(I_j)} < \infty$ Therefore, by putting all subintervals together, we have

$$\|u\|_{L_t^\infty \dot{H}_x^{\frac{1}{2}}([t_0,t_0+T_+(u_0)]\times\mathbb{R}^2)} + \left\| |\nabla|^{\frac{1}{2}} u \right\|_{T([t_0,t_0+T_+(u_0)])} < \infty.$$

- Step 2: We then choose $t_n \nearrow t_0 + T_+(u_0)$. Then using the integral equation again, we get

$$\begin{aligned}&\left\| e^{i(t-t_n)\Delta} u(t_n) \right\|_{X([t_n,t_0+T_+(u_0)])} \\ \leq\ & \|u\|_{X([t_n,t_0+T_+(u_0)])} + C \|F(u)\|_{Y([t_n,t_0+T_+(u_0)])} \\ \leq\ & \varepsilon + C \|u\|_{X([t_n,t_0+T_+(u_0)])}^5 \leq \varepsilon + C\varepsilon^5 \leq \frac{\delta}{2}\end{aligned}$$

  for $n$ large and $\varepsilon$ small enough. But then, for $n$ large but fixed, and some $\varepsilon_0 > 0$, $\left\| e^{i(t-t_n)\Delta} u(t_n) \right\|_{X([t_n,t_0+T_+(u_0)+\varepsilon_0))} \leq \delta$. Now, we applt Theorem 3.9 and reach a contradiction.

$\square$

**Remark 3.14** (Scattering)**.** If $u$ is a solution of (1.6.1) in $I \times \mathbb{R}^2$, $I = [a, +\infty)$ (or $I = (-\infty, a])$, there exists $u_+ \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^2)$ such that

$$\lim_{t \to +\infty} \left\| u(t) - e^{it\Delta} u_+ \right\|_{\dot{H}_x^{\frac{1}{2}}(\mathbb{R}^2)} = 0.$$

This is a consequence of the fact that $\|u\|_{S(I)} < \infty$.

*Proof.* If $u$ is a solution of (1.6.1) in $I \times \mathbb{R}^2$. $I = [a, +\infty)$ (or $I = (-\infty, a]$) there exists $u_+ \in \dot{H}^{\frac{1}{2}}$ such that

$$\lim_{t \to +\infty} \left\| u(t) - e^{it\Delta} u_+ \right\|_{\dot{H}_x^{\frac{1}{2}}} = \lim_{t \to +\infty} \left\| e^{-it\Delta} u(t) - u_+ \right\|_{\dot{H}_x^{\frac{1}{2}}} = 0.$$

Define

$$u_+ := e^{-ia\Delta} u_0 + \int_a^\infty e^{-it'\Delta} F(u)\, dt'.$$

The Duhamel's formula

$$u(t) = e^{i(t-a)\Delta} u_0 + \int_a^t e^{i(t-t')\Delta} F(u)\, dt',$$

implies

$$e^{-it\Delta} u(t) = e^{-ia\Delta} u_0 + \int_a^t e^{-it'\Delta} F(u)\, dt'.$$

We want to find $u_+$ such that $e^{-it\Delta} u(t) - u_+$ converges to 0 in $\dot{H}^{\frac{1}{2}}$ as $t$ goes to infinity. It is sufficient to show that $\int_a^\infty e^{-it'\Delta} F(u)\, dt'$ converges to 0 in $\dot{H}^{\frac{1}{2}}$ as $t$ goes to infinity. In fact,

$$\left\| \int_a^\infty e^{-it'\Delta} F(u)\, dt' \right\|_{\dot{H}_x^{\frac{1}{2}}} \lesssim \left\| |\nabla|^{\frac{1}{2}} F(u) \right\|_{N(I)} \lesssim \left\| |\nabla|^{\frac{1}{2}} u \right\|_{T(I)} \|u\|_{X(I)}^4 .$$

Then $\int_a^\infty e^{-it'\Delta} F(u)\, dt' \to 0$ in $\dot{H}^{\frac{1}{2}}$ as $t$ goes to infinity. $\square$

## 3.3 Stability

In this section, we present a refined stability result. This original stability result is first shown in [13], and the refined result is proved in [49] but for data in dimensions four and above. The refinement means that we do not require errors to be small in spaces with derivatives. We will extend it to two dimensions.

**Theorem 3.15** (Short-time perturbations)**.** *Let $I \subset \mathbb{R}$ be a compact time interval, and $t_0$ in $I$. Let $\tilde{u} : I \times \mathbb{R}^2 \to \mathbb{C}$ be a solution to $(i\partial_t + \Delta)\tilde{u} = |\tilde{u}|^4 \tilde{u} + e$ with $\tilde{u}(t_0) = \tilde{u}_0 \in \dot{H}^{\frac{1}{2}}$. Suppose that*

$$\|\tilde{u}(t)\|_{\dot{S}^{\frac{1}{2}}(I)} \leq E, \quad \left\| |\nabla|^{\frac{1}{2}} \tilde{u} \right\|_{Z(I)} \leq E$$

*for some $E > 0$. Let $u_0 \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^2)$ be such that $\|u_0 - \tilde{u}_0\|_{\dot{H}^{\frac{1}{2}}_x} \leq E$, and suppose that we have the smallness conditions:*

$$\begin{aligned} &\|\tilde{u}\|_{X(I)} \leq \delta \\ &\left\|e^{i(t-t_0)\Delta}(u_0 - \tilde{u}_0)\right\|_{X(I)} + \|e\|_{Y(I)} \leq \varepsilon \end{aligned}$$

*for some small $0 < \delta = \delta(E)$ and $0 < \varepsilon < \varepsilon_0(E)$.*
*Then there exists $u : I \times \mathbb{R}^2 \to \mathbb{C}$ solving (1.6.1) with $u(t_0) = u_0$ such that:*

$$\begin{aligned} &\|u - \tilde{u}\|_{X(I)} + \|F(u) - F(\tilde{u})\|_{Y(I)} \lesssim \epsilon \\ &\|u - \tilde{u}\|_{\dot{S}^{\frac{1}{2}}(I)} + \left\| |\nabla|^{\frac{1}{2}} (F(u) - F(\tilde{u})) \right\|_{L_t^{q'} L_x^{r'}} \lesssim_E 1, \end{aligned}$$

*where $F(u) = |u|^4 u$ and $(q', r')$ is any duel Strichartz pair.*

*Proof.* By triangle inequality and the integral equation, we know that

$$\left\|e^{i(t-t_0)\Delta}\tilde{u}_0\right\|_{X(I)} \lesssim \|\tilde{u}\|_{X(I)} + \|F(\tilde{u})\|_{Y(I)} + \|e\|_{Y(I)} \lesssim \delta + \delta^5 + \varepsilon.$$

Then

$$\left\|e^{i(t-t_0)\Delta}u_0\right\|_{X(I)} \leq \left\|e^{i(t-t_0)\Delta}\tilde{u}_0\right\|_{X(I)} + \left\|e^{i(t-t_0)\Delta}(u_0-\tilde{u}_0)\right\|_{X(I)} \lesssim \delta+\delta^5+\varepsilon.$$

for $\delta$ and $\varepsilon \lesssim \delta$ sufficiently small. By theorem 3.9, we know that there exists a solution to (1.6.1), then we just need to prove the conclusions as *a priori* estimates.

- Step 1: We first show $\|u\|_{X(I)} \leq \delta$. In fact,

  $$\|u\|_{X(I)} \leq \left\|e^{i(t-t_0)\Delta}u_0\right\|_{X(I)} + \|F(u)\|_{Y(I)} \lesssim \delta + \|u\|_{X(I)}^5$$

  Then by a continuity argument and by taking $\delta$ sufficiently small, we can have $\|u\|_{X(I)} \leq \delta$.

- Step 2: Show $\|u-\tilde{u}\|_{X(I)} + \|F(u)-F(\tilde{u})\|_{Y(I)} \lesssim \varepsilon$.

  Let $w := u-\tilde{u}$, then we know that $u$ solves

  $$(i\partial_t+\Delta)u = |u|^4 u = F(u)$$

  with initial data $u(t_0)=u_0$, and $\tilde{u}$ solves

  $$(i\partial_t+\Delta)\tilde{u} = |\tilde{u}|^4 \tilde{u} + e = F(\tilde{u})+e$$

  with initial data $\tilde{u}(t_0)=\tilde{u}_0$. So $w$ satisfies

  $$(i\partial_t+\Delta)w = F(u)-F(\tilde{u})-e$$

  with initial data $w(t_0)=u_0-\tilde{u}_0$. Then, using Strichartz estimates and the smallness conditions, we have

  $$\begin{aligned}\|w\|_{X(I)} &\lesssim \left\|e^{i(t-t_0)\Delta}(u_0-\tilde{u}_0)\right\|_{X(I)} + \|F(u)-F(\tilde{u})\|_{Y(I)} + \|e\|_{Y(I)} \\ &\lesssim \varepsilon + (\|u\|_{X(I)}^4 + \|\tilde{u}\|_{Y(I)}^4)\, \|w\|_{X(I)} \lesssim \varepsilon+\delta^4\, \|w\|_{X(I)}.\end{aligned}$$

Taking $\delta$ small enough, we can see that $\|u-\tilde{u}\|_{X(I)} \lesssim \varepsilon$ holds.

Therefore,

$$\|F(u)-F(\tilde{u})\|_{Y(I)} \lesssim (\, \|u\|_{X(I)}^4 + \|\tilde{u}\|_{Y(I)}^4)\, \|w\|_{X(I)} \lesssim \delta^4 \varepsilon \lesssim \varepsilon.$$

Then the first estimate follows.

- Step 3: Show the second estimate. First, by Strichartz estimates and Hölder's inequality, we get

$$\begin{aligned}
\|w\|_{\dot{S}^{\frac{1}{2}}(I)} &\lesssim \|u_0-\tilde{u}_0\|_{\dot{H}_x^{\frac{1}{2}}} + \left\| |\nabla|^{\frac{1}{2}} e \right\|_{Z(I)} + \left\| |\nabla|^{\frac{1}{2}} (F(u)-F(\tilde{u})) \right\|_{L_t^{q'} L_x^{r'}} \\
&\lesssim_E 1 + \|\tilde{u}\|_{\dot{S}^{\frac{1}{2}}(I)} \|w\|_{X(I)}^4 + \|w\|_{\dot{S}^{\frac{1}{2}}(I)} \|u\|_{X(I)}^4 \\
&\lesssim_E 1 + \delta^4 \|w\|_{\dot{S}^{\frac{1}{2}}(I)} .
\end{aligned}$$

Taking $\delta = \delta(E)$ small enough, we can conclude $\|w\|_{\dot{S}^{\frac{1}{2}}(I)} \lesssim_E 1$

We also have

$$\left\| |\nabla|^{\frac{1}{2}} (F(u)-F(\tilde{u})) \right\|_{L_t^{q'} L_x^{r'}} \lesssim_E \delta^4 \|w\|_{\dot{S}^{\frac{1}{2}}(I)} \lesssim_E 1.$$

Then the second estimate follows.

□

**Remark 3.16.** Here, the error $e$ is only required to be small in a space without derivatives. This will also be the case in Theorem 3.17. We will see the benefit of this refinement in the proof of Theorem 4.1.

**Theorem 3.17** (Stability). *Let $I$ be a compact time interval, with $t_0 \in I$. Suppose $\tilde{u}$ is a solution to $(i\partial_t + \Delta)u = |u|^4 u + e$, with $\tilde{u}(t_0) = \tilde{u}_0$. Suppose*

$$\|\tilde{u}\|_{\dot{S}^{\frac{1}{2}}(I)} \leq E \text{ and } \left\| |\nabla|^{\frac{1}{2}} e \right\|_{Z(I)} \leq E$$

*for some $E > 0$. Let $u_0 \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^2)$ and suppose we have the smallness conditions*

$$\|u_0 - \tilde{u}_0\|_{\dot{H}_x^{\frac{1}{2}}(\mathbb{R}^2)} + \|e\|_{Y(I)} \leq \varepsilon$$

*for some small $0 < \varepsilon < \varepsilon_1(E)$.*

*Then, there exists $u : I \times \mathbb{R}^2 \to \mathbb{C}$ solving (1.6.1) with $u(t_0) = u_0$, and there exists $0 < c < 1$ such that*

$$\|u - \tilde{u}\|_{S(I)} \lesssim_E \varepsilon^c.$$

*Proof.* The assumption $\|\tilde{u}\|_{\dot{S}^{\frac{1}{2}}(I)} \leq E$ implies that $\|\tilde{u}\|_{X(I)} \lesssim E$. We can then subdivide the interval $I$ into $N$ subintervals $I_j := [T_j, T_{j+1}]$, so that on each $I_j$ we have $\|\tilde{u}\|_{X(I)} \leq \varepsilon$. Now on each subinterval $I_j$, all the assumptions in Theorem 3.15 are verified.

By Theorem 3.15, we have

$$\begin{aligned}
&\|u - \tilde{u}\|_{X(I_j)} + \|F(u) - F(\tilde{u})\|_{Y(I_j)} \lesssim \epsilon, \\
&\|u - \tilde{u}\|_{\dot{S}^{\frac{1}{2}}(I_j)} + \left\| |\nabla|^{\frac{1}{2}} \left(F(u) - F(\tilde{u})\right) \right\|_{L_t^{q'} L_x^{r'}} \lesssim_E 1.
\end{aligned}$$

Using Duhamel's formula, we write

$$\begin{aligned}
u(T_{j+1}) &= e^{i(T_{j+1}-T_j)\Delta} u(T_j) - i \int_{T_j}^{T_{j+1}} e^{i(T_{j+1}-t')\Delta} F(u)\, dt', \\
\tilde{u}(T_{j+1}) &= e^{i(T_{j+1}-T_j)\Delta} \tilde{u}(T_j) - i \int_{T_j}^{T_{j+1}} e^{i(T_{j+1}-t')\Delta} (F(\tilde{u}) + e)\, dt'.
\end{aligned}$$

A simple calculation gives

$$\begin{aligned}
e^{i(t-T_{j+1})\Delta} u(T_{j+1}) &= e^{i(t-T_j)\Delta} u(T_j) - i e^{i(t-T_{j+1})\Delta} \int_{T_j}^{T_{j+1}} e^{i(T_{j+1}-t')\Delta} F(u)\, dt' \\
&= e^{i(t-T_j)\Delta} u(T_j) - i \int_{T_j}^{T_{j+1}} e^{i(t-t')\Delta} F(u)\, dt'.
\end{aligned}$$

Then by Strichartz estimates and Hölder's inequality, we have

$$
\begin{aligned}
&\left\|e^{i(t-T_{j+1})\Delta}u(T_{j+1})-\tilde{u}(T_{j+1})\right\|_{X(I_{j+1})}\\
\leq\ &\left\|e^{i(t-T_j)\Delta}u(T_j)-\tilde{u}(T_j)\right\|_{X(I_{j+1})}+\left\|i\int_{T_j}^{T_{j+1}}e^{i(t-t')\Delta}(F(u)-F(\tilde{u})-e)\,dt'\right\|_{X(I_{j+1})}\\
\leq\ &\left\|e^{i(t-T_j)\Delta}u(T_j)-\tilde{u}(T_j)\right\|_{X(I_{j+1})}+C\left\|F(u)-F(\tilde{u})\right\|_{Y(I_{j+1})}+\left\|e\right\|_{Y(I_{j+1})}\\
\leq\ &\left\|e^{i(t-T_j)\Delta}u(T_j)-\tilde{u}(T_j)\right\|_{X(I_{j+1})}+C(j)\varepsilon,
\end{aligned}
$$

$$
\begin{aligned}
&\left\|u(T_{j+1})-\tilde{u}(T_{j+1})\right\|_{\dot{H}_x^{\frac{1}{2}}}\\
\leq\ &\left\|e^{i(T_{j+1}-T_j)\Delta}(u(T_j)-\tilde{u}(T_j))\right\|_{\dot{H}_x^{\frac{1}{2}}}+\left\|i\int_{T_j}^{T_{j+1}}e^{i(T_{j+1}-t')\Delta}(F(u)-F(\tilde{u})-e)\,dt'\right\|_{\dot{H}_x^{\frac{1}{2}}}\\
\leq\ &\left\|e^{i(T_{j+1}-T_j)\Delta}(u(T_j)-\tilde{u}(T_j))\right\|_{\dot{H}_x^{\frac{1}{2}}}+C\left\|F(u)-F(\tilde{u})\right\|_{Y(I_j)}+\left\|e\right\|_{Y(I_j)}\\
\leq\ &\left\|e^{i(T_{j+1}-T_j)\Delta}(u(T_j)-\tilde{u}(T_j))\right\|_{\dot{H}_x^{\frac{1}{2}}}+C(j)\varepsilon.
\end{aligned}
$$

These calculations allow us to continue the induction (if $\varepsilon_1$ is sufficiently small depending on $E$, $N$), then by iterating the argument above, we obtain the bounds

$$\left\|u-\tilde{u}\right\|_{X(I)}\lesssim\varepsilon \text{ and } \left\|u-\tilde{u}\right\|_{\dot{S}^{\frac{1}{2}}(I)}\lesssim_E 1.$$

Then by Gagliardo-Nirenberg inequality (Lemma 2.8),

$$\left\|u\right\|_{L_t^8L_x^8(I\times\mathbb{R}^2)}\lesssim\left\|u\right\|_{X(I)}^{\frac{5}{6}}\left\|u\right\|_{L_t^3\dot{W}_x^{\frac{1}{2},6}(I\times\mathbb{R}^2)}^{\frac{1}{6}}\leq\left\|u\right\|_{X(I)}^{\frac{5}{6}}\left\|u\right\|_{\dot{S}^{\frac{1}{2}}(I)}^{\frac{1}{6}}.$$

Now we conclude $\left\|u-\tilde{u}\right\|_{S(I)}\lesssim_E\varepsilon^{\frac{5}{6}}$. □

**Remark 3.18.** Theorem 3.15 yields the following continuity fact:
Let $\tilde{u}_0\in\dot{H}^{\frac{1}{2}}(\mathbb{R}^2)$, $\left\|\tilde{u}_0\right\|_{\dot{H}_x^{\frac{1}{2}}(\mathbb{R}^2)}\leq A$, and $\tilde{u}$ be a solution of (1.6.1), $t_0=0$, with maximal interval of existence $(-T_-(u_0),T_+(u_0))$. Let $u_{0,n}\to\tilde{u}_0$ in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^2)$, and let $u_n$ be the corresponding solution of (1.6.1), with the maximal interval of existence

$(-T_-(u_{0,n}), T_+(u_{0,n}))$. Then $-T_-(u_0) \geq -T_-(u_{0,n})$, for all $n$ large. Moreover, for each $t \in (-T_-(u_0), T_+(u_0))$, $u_n(t) \to \tilde{u}(t)$ in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^2)$.

**Remark 3.19.** Theorem 3.15 also yields the following:

Let $K \subset\subset \dot{H}^{\frac{1}{2}}(\mathbb{R}^2)$ be such that $\bar{K}$ is compact. Then there exist $T_K^+, T_K^-$ such that for all $u_0 \in K$ we have $T_+(u_0) > T_K^+$, $T_-(u_0) > T_K^-$. Moreover, the family $\{u(t) : t \in [-T_K^-, T_K^+], u_0 \in K\}$ has compact closure in $C([-T_K^-, T_K^+]; \dot{H}^{\frac{1}{2}}(\mathbb{R}^2))$ and hence is equicontinuous and bounded.

**Definition 3.20** (Nonlinear profile)**.** Let $v_0 \in \dot{H}^{\frac{1}{2}}$, $v(t) = e^{it\Delta}v_0$ and let $\{t_n\}$ be a sequence with $\lim\limits_{n\to\infty} = \bar{t} \in [-\infty, +\infty]$. We say that $u(x,t)$ is a nonlinear profile associated with $(v_0, \{t_n\})$ if there exists an interval $I$ , with $\bar{t} \in I$ (if $t = \pm\infty$, $I = [a, +\infty)$ or $(-\infty, a]$) such that $u$ is a solution of (1.6.1) in $I$ and

$$\lim_{n\to\infty} \|u(\cdot, t_n) - v(\cdot, t_n)\|_{\dot{H}_x^{\frac{1}{2}}} = 0.$$

**Remark 3.21.** There always exists a non-linear profile associated to $(v_0, \{t_n\})$. We can hence define a maximal interval $I$ of existence for the non-linear profile associated to $(v_0, \{t_n\})$.

*Proof.* (i) Existence.

If $\bar{t} \in (-\infty, +\infty)$, let $u_0 = v(x, \bar{t}) = e^{i\bar{t}\Delta}v_0$, then $v(t_n) = e^{it_n\Delta}v_0$. Hence we write

$$\begin{aligned} u(x,t) &= e^{i(t-\bar{t})\Delta}u_0 + \int_{\bar{t}}^{t} e^{i(t-t')\Delta}F(u)\,dt' \\ u(x,t_n) &= e^{i(t_n-\bar{t})\Delta}e^{i\bar{t}\Delta}v_0 + \int_{\bar{t}}^{t_n} e^{i(t_n-t')\Delta}F(u)\,dt' \\ &= e^{it_n\Delta}v_0 + \int_{\bar{t}}^{t_n} e^{i(t_n-t')\Delta}F(u)\,dt' \end{aligned}$$

Therefore,

$$\|u(t_n) - v(t_n)\|_{\dot{H}_x^{\frac{1}{2}}} = \left\| \int_{\bar{t}}^{t_n} e^{i(t_n - t')\Delta} F(u) dt' \right\|_{\dot{H}_x^{\frac{1}{2}}}$$

$$\lesssim \left\| |\nabla|^{\frac{1}{2}} F(u) \right\|_{N([\bar{t}, t_n])} \lesssim \|u\|_{X([\bar{t}, t_n])}^4 \left\| |\nabla|^{\frac{1}{2}} u \right\|_{T([\bar{t}, t_n])} \to 0$$

as $t_n \to \bar{t}$.

If $t = +\infty$, we solve the integral equation

$$u(t) = e^{it\Delta} u_0 + \int_t^{\infty} e^{i(t-t')\Delta} F(u)\, dt'$$

on $(t_{n_0}, +\infty) \times \mathbb{R}^2$, for $n_0$ large enough, such that $\left\| e^{it\Delta} v_0 \right\|_{X([t_{n_0}, +\infty)} \leq \delta$, where $\delta$ is as in Theorem 3.9. Then, if $n$ is large,

$$u(t_n) - v(t_n) = \int_{t_n}^{+\infty} e^{i(t_n - t')\Delta} F(u)\, dt',$$

then

$$\|u(t_n) - v(t_n)\|_{\dot{H}_x^{\frac{1}{2}}} = \left\| \int_{t_n}^{+\infty} e^{i(t_n - t')\Delta} F(u)\, dt' \right\|_{\dot{H}_x^{\frac{1}{2}}}$$

$$\lesssim \left\| |\nabla|^{\frac{1}{2}} F(u) \right\|_{N((t_n, +\infty))} \lesssim \|u\|_{X((t_n, +\infty))}^4 \left\| |\nabla|^{\frac{1}{2}} u \right\|_{T((t_n, +\infty))} \to 0$$

A similar argument applies when $t = -\infty$.

(ii) Uniqueness:

If $u^{(1)}, u^{(2)}$ are both non-linear profiles associated to $(v_0, \{t_n\})$ in an interval $I$ with $\bar{t} \in I$, then $u^{(1)} \equiv u^{(2)}$.

If $\bar{t} \in (-\infty, +\infty)$, this is clear from Definition 2.16 and the uniqueness of Definition 2.9, since $\left\| u^{(1)}(t_n), -u^{(2)}(t_n) \right\|_{\dot{H}_x^{\frac{1}{2}}} \to 0$.

If $t = +\infty$, by $\left\| |\nabla|^{\frac{1}{2}} u^{(i)} \right\|_{T(I)} < \infty$, we have $\left\| |\nabla|^{\frac{1}{2}} u^{(i)} \right\|_{T((t_n, +\infty))} \leq \tilde{\delta}$, where $\tilde{\delta}$ is as small as we like. By the proof of Theorem 3.9, we have (with a constant independent of $u$) that for $n \gg n_0$

$$\sup_{t\in(t_0,t_n)} \left\| |\nabla|^{\frac{1}{2}} u^{(1)}(t) - |\nabla|^{\frac{1}{2}} u^{(2)}(t) \right\|_{L_x^2}$$
$$= \sup_{t\in(t_0,t_n)} \left\| u^{(1)}(t) - u^{(2)}(t) \right\|_{\dot{H}_x^{\frac{1}{2}}} \leq \left\| u^{(1)}(t_n) - u^{(2)}(t_n) \right\|_{\dot{H}_x^{\frac{1}{2}}}$$

This easily shows that $u^{(1)} \equiv u^{(2)}$ on $(t_{n_0}, +\infty)$ and hence on $I$.

A similar argument applies when $t = -\infty$.

$\square$

# CHAPTER 4

## ALMOST PERIODIC SOLUTIONS

In this chapter, we first recall the concentration compactness results and define almost periodic solutions, then we discuss the proof of the reduction to almost periodic solutions, which is the first step in the road map. This step is by now standard. In fact, we adapt the result in the energy-crtical regime in [40] and the one in the $\dot{H}^{\frac{1}{2}}$-critical dimensions four and higher in [49] to our $\dot{H}^{\frac{1}{2}}$-critical regime in dimensions two.

Following the road map in [27], argue by contradiction, first, we define $L : [0, \infty) \to [0, \infty]$ by

$$L(E) := \sup \left\{ S_I(u) \middle| u : I \times \mathbb{R}^2 \to \mathbb{C} \text{ solving (1.6.1) with } \|u\|^2_{L_t^\infty \dot{H}_x^{\frac{1}{2}}(I\times\mathbb{R}^2)} \leq E \right\}$$

Note that:

1. $L(E) \geq 0$, since for every $E$, the trivial solution is always in this set, hence
$$\left\{ S_I(u) \middle| u : I \times \mathbb{R}^2 \to \mathbb{C} \text{ solving (1.6.1) with } \|u\|^2_{L_t^\infty \dot{H}_x^{\frac{1}{2}}(I\times\mathbb{R}^2)} \leq E \right\} \neq \varnothing.$$

2. $L$ is non-decreasing. In fact, for any $E_1 < E_2$,
$$\begin{aligned} &\left\{ S_I(u) \middle| u : I \times \mathbb{R}^2 \to \mathbb{C} \text{ solving (1.6.1) with } \|u\|^2_{L_t^\infty \dot{H}_x^{\frac{1}{2}}(I\times\mathbb{R}^2)} \leq E_1 \right\} \\ \subseteq &\left\{ S_I(u) \middle| u : I \times \mathbb{R}^2 \to \mathbb{C} \text{ solving (1.6.1) with } \|u\|^2_{L_t^\infty \dot{H}_x^{\frac{1}{2}}(I\times\mathbb{R}^2)} \leq E_2 \right\}, \end{aligned}$$
hence $L(E_1) \leq L(E_2)$.

3. $L(E) \leq E^4$ for $E$ sufficiently small.

4. $L(E)$ is a continuous function in $E$, by perturbation lemma (see Theorem 3.15).

Thus by the definition of $L(E)$, there must exist a unique critical threshold $E_c \in (0, \infty]$ such that

$$L(E) \begin{cases} < \infty & \text{if } E < E_c \\ = \infty & \text{if } E \geq E_c. \end{cases}$$

Then the failure of main theorem implies that $0 < E_c < \infty$.

Next, we show that at the critical level $E_c$, we can find a solution $u$ to (1.6.1) such that:

1. $\|u\|^2_{L_t^\infty \dot{H}_x^{\frac{1}{2}}(I \times \mathbb{R}^2)} = E_c$,

2. $u$ blows up forward and backward in time,

3. $u$ is almost periodic (see Definition 4.5).

More precisely,

**Theorem 4.1** (Existence of minimal counterexamples)**.** *Suppose main theorem 1.5 fails to be true. Then there exists a critical threshold* $0 < E_c < \infty$ *and a maximal-lifespan solution* $u : I \times \mathbb{R}^2 \to \mathbb{C}$ *to the defocusing* $\dot{H}^{\frac{1}{2}}$*-critical NLS with* $\|u\|^2_{L_t^\infty \dot{H}_x^{\frac{1}{2}}(I \times \mathbb{R}^2)} = E_c$*, which blows up in both time directions in the sense that*

$$S_{\geq 0}(u) = S_{\leq 0}(u) = \infty,$$

*and whose orbit* $\{u(t) : t \in \mathbb{R}\}$ *is precompact in* $\dot{H}^{\frac{1}{2}}$ *modulo scaling and spatial translations.*

Before proving Theorem 4.1, we first introduce the concentration compactness techniques.

## 4.1 Concentration compactness

In this section, we record a linear profile decomposition for $e^{it\Delta}$.

**Definition 4.2** (Symmetry group)**.** For any position $x_0 \in \mathbb{R}^2$ and scaling parameter $\lambda > 0$, we define a unitary transformation $g_{x_0,\lambda} : \dot{H}_x^{\frac{1}{2}} \to \dot{H}_x^{\frac{1}{2}}$ by

$$[g_{x_0,\lambda} f](x) := \lambda^{-\frac{1}{2}} f\left(\lambda^{-1}(x - x_0)\right).$$

We let $G$ denote the collection of such transformations. For a function $u : I \times \mathbb{R}^2 \to \mathbb{C}$ we define $T_{g_{x_0,\lambda}} u : \lambda^2 I \times \mathbb{R}^2 \to \mathbb{C}$ by the formula

$$[T_{g_{x_0,\lambda}} u](t, x) := \lambda^{-\frac{1}{2}} u\left(\lambda^{-2} t, \lambda^{-1}(x - x_0)\right),$$

where $\lambda^2 I := \{\lambda^2 t : t \in I\}$. Note that if $u$ is a solution to (1.6.1), then $T_g u$ is a solution to (1.6.1) with initial data $g u_0$.

**Remark 4.3.** We remark here that $G$ forms a group under composition. The map $u \mapsto T_g u$ takes solutions to (1.6.1) to solutions with the same scattering size. Furthermore, $u$ is a maximal-lifespan solution if and only if $T_g u$ is a maximal-lifespan solution.

**Lemma 4.4** (Linear profile decomposition)**.** *Let $\{u_n\}$ be a bounded sequence in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^2)$. After passing to a subsequence if necessary, there exist functions $\{\phi\} \subset \dot{H}^{\frac{1}{2}}(\mathbb{R}^2)$, group elements $g_n^j \in G$ (with parameters $x_n^j$ and $\lambda_n^j$), and times $t_n^j \in \mathbb{R}$ such that for all $J \geq 1$, we have the decomposition*

$$u_n = \sum_{j=1}^{J} g_n^j e^{it_n^j \Delta} \phi^j + w_n^J$$

*with the following properties:*

*(i) For each $j$, either $t_n^j \equiv 0$ or $t_n^j \to \pm\infty$ as $n \to \infty$,*

*(ii) For all $n$ and all $J \geq 1$, we have $w_n^j \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^2)$, with*

$$\lim_{J\to\infty} \limsup_{n\to\infty} \left\| e^{it\Delta} w_n^j \right\|_{L_t^8 L_x^8(\mathbb{R}\times\mathbb{R}^2)} = 0,$$

*(iii) For any $j \neq k$, we have the following asymptotic orthogonality of parameters:*

$$\frac{\lambda_n^j}{\lambda_n^k} + \frac{\lambda_n^k}{\lambda_n^j} + \frac{\left|t_n^j(\lambda_n^j)^2 - t_n^k(\lambda_n^k)^2\right|}{\lambda_n^j \lambda_n^k} \to \infty \qquad \text{as } n \to \infty,$$

*(iv) We have the decoupling properties: for any $J \geq 1$,*

$$\lim_{n\to\infty} \left[ \left\| |\nabla|^{\frac{1}{2}} u_n \right\|_{L_x^2}^2 - \sum_{j=1}^{J} \left\| |\nabla|^{\frac{1}{2}} \phi^j \right\|_{L_x^2}^2 - \left\| |\nabla|^{\frac{1}{2}} w_n^J \right\|_{L_x^2}^2 \right] = 0$$

*and for any $1 \leq j \leq J$,*

$$e^{-it_n^j\Delta}[(g_n^j)^{-1} w_n^J] \rightharpoonup 0 \qquad \text{weakly in } \dot{H}_x^{\frac{1}{2}} \qquad \text{as } n \to \infty.$$

## 4.2 Definition of almost periodic solutions and properties

In this section, we are ready to give the definition of almost periodic solutions.

**Definition 4.5** (Almost periodicity)**.** A solution $u$ to (1.6.1) with lifespan $I$ is said to be almost periodic (modulo symmetries) if $u \in L_t^\infty \dot{H}_x^{\frac{1}{2}}(I \times \mathbb{R}^2)$ and there exist (possibly discontinuous) functions: $N : I \to \mathbb{R}^+$, $x : I \to \mathbb{R}^2$ and $C : \mathbb{R}^+ \to \mathbb{R}^+$ such that:

$$\int_{|x-x(t)|\geq \frac{C(\eta)}{N(t)}} \left| |\nabla|^{\frac{1}{2}} u(t,x) \right|^2 dx + \int_{|\xi|\geq C(\eta)N(t)} |\xi| \, |\hat{u}(t,\xi)|^2 \, d\xi \leq \eta \tag{4.2.1}$$

for all $t \in I$ and $\eta > 0$. We refer to the function $N$ as the frequency scale function, $x$ is the spatial center function, and $C$ is the compactness modulus function.

Notice that the Galilean transformation only preserves the $L^2$ norm of $u$ and not the $\dot{H}^{s_c}$ norm where $s_c > 0$, hence we have no Galilean transformation in our case and frequency center $\xi(t)$ is the origin. We will use this later in Definition 6.18.

Another consequence of the precompactness in $\dot{H}^{\frac{1}{2}}$ modulo symmetries of the orbit of the solution found in Theorem 4.1 is that for every $\eta > 0$ there exists $c(\eta) > 0$ such that

$$\int_{|x-x(t)|\leq \frac{c(\eta)}{N(t)}} \left| |\nabla|^{\frac{1}{2}} u(t,x) \right|^2 dx + \int_{|\xi|\leq c(\eta)N(t)} |\xi| \, |\hat{u}(t,\xi)|^2 \, d\xi \leq \eta \tag{4.2.2}$$

uniformly for all $t \in I$.

For these almost periodic solutions, $N(t)$ enjoys the following properties: Lemma 4.6, Corollary 4.8, Lemma 4.9 and Lemma 4.10 (see [32] for details):

**Lemma 4.6** (Local constancy of $N(t)$ and $x(t)$)**.** *Let $u : I \times \mathbb{R}^2 \to \mathbb{C}$ be a non-zero almost periodic modulo symmetries solution to* (1.6.1) *with parameters $N(t)$ and $x(t)$. Then there exists a small number $\delta$, depending on $u$, such that for every $t_0 \in I$ we have*

$$[t_0 - \delta N(t_0)^{-2}, t_0 + \delta N(t_0)^{-2}] \subset I$$

*and*

$$N(t) \sim N(t_0) \text{ and } |x(t) - x(t_0)| \lesssim N(t_0)^{-1}$$

*whenever $|t - t_0| \leq \delta N(t_0)^{-2}$.*

**Remark 4.7.** If $J$ is an interval with

$$\|u\|_{L_t^8 L_x^8(J\times\mathbb{R}^2)} = 1,$$

then for $t_1, t_2 \in J$,

$$N(t_1) \sim N(t_2).$$

Then combine with Lemma 4.6, we can choose $N(t)$ such that

$$\left|\frac{d}{dt}N(t)\right| \lesssim N(t)^3 \left(\implies \left|\frac{d}{dt}\left(\frac{1}{N(t)}\right)\right| \lesssim N(t)\right).$$

Define

$$N(J) = \inf_{t\in J} N(t),$$

then

$$\frac{1}{N(J)} \sim \int_J N(t)\,dt.$$

**Corollary 4.8** ($N(t)$ at blow-up). *Let $u : I \times \mathbb{R}^2 \to \mathbb{C}$ be a non-zero maximal-lifespan solution to* (1.6.1) *that is almost periodic modulo symmetries with frequency scale function $N : I \to \mathbb{R}^+$. If $T$ is any finite endpoint of the lifespan $I$, then $N(t) \gtrsim |T-t|^{-\frac{1}{2}}$; in particular, $\lim_{t\to T} N(t) = \infty$. If $I$ is infinite or semi-infinite, then for any $t_0 \in I$ we have $N(t) \gtrsim \min\{N(t_0), |t-t_0|^{-\frac{1}{2}}\}$.*

**Lemma 4.9** (Local quasi-boundedness of $N(t)$). *Let $u$ be a non-zero solution to* (1.6.1) *with lifespan $I$ that is almost periodic modulo symmetries with frequency scale function $N : I \to \mathbb{R}^+$. If $K$ is any compact subset of $I$, then*

$$0 < \inf_{t\in K} N(t) \le \sup_{t\in K} N(t) < \infty.$$

**Lemma 4.10** (Strichartz norms via $N(t)$). *Let $u : I \times \mathbb{R}^2 \to \mathbb{C}$ be a non-zero almost periodic modulo symmetries solution to* (1.6.1) *with frequency scale function $N : I \to \mathbb{R}^+$. Then*

$$\int_I N(t)^2\,dt \lesssim \int_I \int_{\mathbb{R}^2} |u(t,x)|^8\,dxdt \lesssim 1 + \int_I N(t)^2\,dt.$$

## 4.3 Reduction to almost periodic solutions

In this section, we show that at this critical level $E_c$, we can find minimal blow-up solutions, and reduce them to almost periodic solutions. We adapt the result in the energy-crtical regime in [40] and the one in the $\dot{H}^{\frac{1}{2}}$-critical dimensions four and higher in [49] to our $\dot{H}^{\frac{1}{2}}$-critical regime in dimensions two.

**Proposition 4.11** (Palais-Smale condition)**.** *Let* $u_n : I_n \times \mathbb{R}^2 \to \mathbb{C}$ *be a sequence of solutions to* (1.6.1) *such that* $\limsup_{n\to\infty} \|u_n\|^2_{L_t^\infty \dot{H}_x^{\frac{1}{2}}(I_n\times\mathbb{R}^2)} = E_c$*. Suppose that* $t_n \in I_n$ *are such that* $\lim_{n\to\infty} S_{[t_n,\sup I_n)}(u_n) = \lim_{n\to\infty} S_{(\inf I_n, t_n]}(u_n) = \infty$*. Then,* $\{u_n(t_n)\}$ *converges along a subsequence in* $\dot{H}_x^{\frac{1}{2}}(\mathbb{R}^2)/G$*, where* $G$ *is the semi-group defined before.*

*Proof.* We first translate so that each $t_n = 0$ for any $n$ by time translation symmetry, thus

$$\lim_{n\to\infty} S_{\geq 0}(u_n) = \lim_{n\to\infty} S_{\leq 0}(u_n) = +\infty. \tag{4.3.1}$$

By profile decomposition, $\{u_n(0)\}$ is a bounded subsequence in $\dot{H}_x^{\frac{1}{2}}$. By passing to a subsequence if necessary, we decompose

$$u_n(0) = \sum_{j=1}^{J} g_n^j e^{it_n\Delta}\phi^j + w_n^j.$$

In particular, for any finite $0 \leq J \leq J^*$, we have the $\dot{H}_x^{\frac{1}{2}}$ norm decoupling condition:

$$\lim_{n\to\infty} \left[ \|u_n\|^2_{\dot{H}_x^{\frac{1}{2}}} - \sum_{j=1}^{J} \left\|\phi^j\right\|^2_{\dot{H}_x^{\frac{1}{2}}} - \left\|w_n^J\right\|^2_{\dot{H}_x^{\frac{1}{2}}} \right] = 0. \tag{4.3.2}$$

To prove Proposition 4.11, we need to show that we have only one profile with error small enough, that is,

1. $J^* = 1$

2. $w_n^J \to 0$ in $\dot{H}_x^{\frac{1}{2}}$

3. $t_n^1 \equiv 0$

To this end, we consider the following two cases:

- Case 1: $\sup_j \limsup_{n\to\infty} \left\| e^{it_n^j \Delta}\phi^j \right\|_{\dot H_x^{\frac12}} = E_c$.

  From the non-triviality of the profiles, we have

$$\liminf_{n\to\infty} \left\| e^{it_n^j \Delta}\phi^j \right\|_{\dot H_x^{\frac12}} > 0$$

  for any finite $0 \le J \le J^*$.

  Then by decoupling of $\dot H_x^{\frac12}$ norm (4.3.2) and the hypothesis $E(u_n) \to E_c$ (and passing to a subsequence if necessary), we deduce that there is a single profile in the decomposition, i.e. $J^* = 1$. Therefore, we write

$$u_n(0) = g_n^j e^{it_n\Delta}\phi + w_n$$

  with $\lim_{n\to\infty} \|w_n\|_{\dot H_x^{\frac12}}$ and $t_n \equiv 0$ or $t_n \to \pm\infty$.

  If $t_n \equiv 0$, then we are done. Now we only need to preclude the case when $t_n \to \pm\infty$. Without loss of generality, we may assume $t_n \to +\infty$. Then Strichartz estimates and the monotone convergence theorem yield,

$$\begin{aligned} S_{\ge 0}(e^{it\Delta}u_n(0)) &\lesssim S_{\ge 0}(g_n e^{i(t+t_n)\Delta}\phi) + S_{\ge 0}(e^{it\Delta}w_n) \\ &= S_{\ge t_n}(g_n e^{it\Delta}\phi) + S_{\ge 0}(e^{it\Delta}w_n) \\ &= S_{\ge t_n}(e^{it\Delta}\phi) + S_{\ge 0}(e^{it\Delta}w_n) \to 0 \end{aligned}$$

  as $n \to 0$.

  This implies that $S_{\ge t_n}(u_n) \to 0$ by Theorem 3.17, which contradicts (4.3.1).

- Case 2: $\sup_j \limsup_{n\to\infty} \left\| e^{it_n^j \Delta}\phi^j \right\|_{\dot H_x^{\frac12}} \le E_c - 2\delta$ for some $\delta > 0$.

  We first observe that in this case, for each finite $J \le J^*$, we have

$$\left\| e^{it_n^j \Delta}\phi^j \right\|_{\dot H_x^{\frac12}} \le E_c - 2\delta$$

  for all $1 \le j \le J$ and $n$ sufficiently large.

Next, we define nonlinear profiles corresponding to each bubble in the decomposition of $u_n(0)$:

1. If $t_n^j \equiv 0$, define $v^j : I^j \times \mathbb{R}^2 \to \mathbb{C}$ to be the maximal-lifespan solution to (1.6.1) with initial data $v^j(0) = \phi^j$.
2. If $t_n^j \to \pm\infty$, define $v^j : I^j \times \mathbb{R}^2 \to \mathbb{C}$ to be the maximal-lifespan solution to (1.6.1) which scatters to $\phi^j$ as $t \to \pm\infty$.

Now define $v_n^j := T_n^j v^j$. Then $v_n^j$ is a solution of (1.6.1) on the time interval $I_n^j := \{t : (\lambda_n^j)^{-2} t + t_n^j \in I^j\}$. In particular, for $n$ sufficiently large, we have $0 \in I_n^j$ and

$$\lim_{n\to\infty} \left\| v_n^j(0) - g_n^j e^{it_n^j \Delta} \phi^j \right\|_{\dot{H}_x^{\frac{1}{2}}} = 0.$$

Combining this with $\left\| e^{it_n^j \Delta} \phi^j \right\|_{\dot{H}_x^{\frac{1}{2}}} \leq E_c - \delta < E$, and the definition of $E_c$, we deduce that for $n$ sufficiently large, $v_n^j$ (and so also $v^j$) are global solutions that satisfy

$$S_{\mathbb{R}}(v^j) = S_{\mathbb{R}}(v_n^j) \leq L(E_c - \delta) < \infty.$$

This means that all Strichartz norms of $v^j$ and $v_n^j$ are finite, that is,

$$\left\| v^j \right\|_{\dot{S}^{\frac{1}{2}}(\mathbb{R})} < +\infty, \quad \left\| v_n^j \right\|_{\dot{S}^{\frac{1}{2}}(\mathbb{R})} < +\infty.$$

In particular, if we define the space $\dot{X}^{\frac{1}{2}} := L_t^8 L_x^8 \cap L_t^4 \dot{W}_x^{\frac{1}{2},4}$, then

$$\left\| v^j \right\|_{\dot{X}^{\frac{1}{2}}(\mathbb{R}\times\mathbb{R}^2)} = \left\| v_n^j \right\|_{\dot{X}^{\frac{1}{2}}(\mathbb{R}\times\mathbb{R}^2)} \lesssim_{E_c,\delta} 1.$$

Note that $C_c^\infty \subseteq \dot{X}^{\frac{1}{2}} = L_t^8 L_x^8 \cap L_t^4 \dot{W}_x^{\frac{1}{2},4}$. This allows us to approximate $v_n^j$ in $\dot{X}^{\frac{1}{2}}$ by $C_c^\infty(\mathbb{R} \times \mathbb{R}^2)$ functions. More precisely, for any $\varepsilon > 0$, there exists $\psi_\varepsilon^j \in C_c^\infty(\mathbb{R} \times \mathbb{R}^2)$ so that

$$\left\| v_n^j - T_n^j \psi_\varepsilon^j \right\|_{\dot{X}^{\frac{1}{2}}(\mathbb{R}\times\mathbb{R}^2)} < \varepsilon.$$

In the remainder of the proof all spacetime norms are over $\mathbb{R} \times \mathbb{R}^2$, unless indicated otherwise.

After defining the space $\dot{X}^{\frac{1}{2}}$, it is easy to see that for all data with $\|u_0\|_{\dot{H}_x^{\frac{1}{2}}} \leq \eta_0$ (where $\eta_0$ denotes the small data threshold), we have

$$\|u\|_{\dot{X}^{\frac{1}{2}}(\mathbb{R}\times\mathbb{R}^2)} \lesssim E(u_0)^{\frac{1}{2}} = \|u_0\|_{\dot{H}_x^{\frac{1}{2}}} .$$

In fact, by Strichartz estimates and Hölder's inequality,

$$\begin{aligned} \|u\|_{\dot{X}^{\frac{1}{2}}} = \|u\|_{L_t^8 L_x^8} + \left\| |\nabla|^{\frac{1}{2}} u \right\|_{L_t^4 L_x^4} &\lesssim \left\| |\nabla|^{\frac{1}{2}} u \right\|_{L_t^8 L_x^{\frac{8}{3}}} + \left\| |\nabla|^{\frac{1}{2}} u \right\|_{L_t^4 L_x^4} \\ &\lesssim \|u_0\|_{\dot{H}_x^{\frac{1}{2}}} + \left\| |\nabla|^{\frac{1}{2}} ( |u|^4 u) \right\|_{L_t^{\frac{4}{3}} L_x^{\frac{4}{3}}} \lesssim \|u_0\|_{\dot{H}_x^{\frac{1}{2}}} + \|u\|_{\dot{X}^{\frac{1}{2}}}^5 . \end{aligned}$$

Therefore, $\|u\|_{\dot{X}^{\frac{1}{2}}} \lesssim \|u_0\|_{\dot{H}_x^{\frac{1}{2}}} \leq \eta$ by a continuity argument.

Then together with our bounds on the space-time norms of $v_n^j$ and the finiteness of $E_c$ to deduce that

$$\left\|v_n^j\right\|_{\dot{X}^{\frac{1}{2}}(\mathbb{R}\times\mathbb{R}^2)} \lesssim_{E_c,\delta} \left\|e^{it_n^j\Delta}\phi^j\right\|_{\dot{H}_x^{\frac{1}{2}}} \lesssim_{E_c,\delta} 1. \tag{4.3.3}$$

Combining this with the decoupling of $\dot{H}^{\frac{1}{2}}$ norm, we deduce that

$$\limsup_{n\to\infty} \sum_{j=1}^{J} \left\|v_n^j\right\|_{\dot{X}^{\frac{1}{2}}(\mathbb{R}\times\mathbb{R}^2)}^2 \lesssim_{E_c,\delta} \limsup_{n\to\infty} \sum_{j=1}^{J} \left\|e^{it_n^j\Delta}\phi^j\right\|_{\dot{H}_x^{\frac{1}{2}}}^2 \lesssim_{E_c,\delta} 1. \tag{4.3.4}$$

uniformly for finite $J \leq J^*$.

Before we continue our proof, we first show two decoupling lemmata. Lemma 4.12 shows the asymptotic decoupling behavior of functions in $C_c^\infty$ while Lemma 4.13 describes the decoupling of nonlinear profiles.

**Lemma 4.12** (Asymptotic decoupling)**.** *Define*

$$(T_n^j u)(t,x) := (\lambda_n^j)^{-\frac{1}{2}} u \left( \frac{t}{(\lambda_n^j)^2} + t_n^j, \frac{x - x_n^j}{\lambda_n^j} \right).$$

*(These act on linear solutions in a manner corresponding to the action of $g_n e^{it\Delta}$ on initial data.) $(\lambda_n^j, t_n^j, x_n^j)$ are defined in linear profile decomposition. Suppose that the parameters associated to $j, k$ are orthogonal in the sense of*

$$\frac{\lambda_n^j}{\lambda_n^k} + \frac{\lambda_n^k}{\lambda_n^j} + \frac{\left|x_n^j - x_n^k\right|^2}{\lambda_n^j \lambda_n^k} + \frac{\left|t_n^j (\lambda_n^j)^2 - t_n^k (\lambda_n^k)^2\right|}{\lambda_n^j \lambda_n^k} \to +\infty$$

*as $n \to \infty$.*

*Then for any $\psi^j, \psi^k \in C_c^\infty(\mathbb{R} \times \mathbb{R}^2)$*

$$\begin{aligned} &\left\|T_n^j \psi^j T_n^k \psi^k\right\|_{L_t^4 L_x^4} + \left\| \left( |\nabla|^{\frac{1}{2}} T_n^j \psi^j \right) \left( |\nabla|^{\frac{1}{2}} T_n^k \psi^k \right) \right\|_{L_t^2 L_x^2} \\ &\quad + \left\| \left| |\nabla|^{\frac{1}{2}} F(T_n^j \psi^j) \right| \left| |\nabla|^{\frac{1}{2}} F(T_n^k \psi^k) \right|^{\frac{1}{3}} \right\|_{L_t^1 L_x^1} \to 0 \end{aligned}$$

*as $n \to \infty$.*

*Proof.* For the first term, by changing of variables,

$$\left\|T_n^j \psi^j T_n^k \psi^k\right\|_{L_t^4 L_x^4} = \left\|\psi^j (T_n^j)^{-1} T_n^k \psi^k\right\|_{L_t^4 L_x^4}$$

Note that

$$\begin{aligned} &\left[(T_n^j)^{-1} T_n^k \psi^k\right](t,x) \\ = &\left(\frac{\lambda_n^j}{\lambda_n^k}\right)^{\frac{1}{2}} \psi^k \left[ \left(\frac{\lambda_n^j}{\lambda_n^k}\right)^2 \left(t - \frac{t_n^j (\lambda_n^j)^2 - t_n^k (\lambda_n^k)^2}{\lambda_n^j \lambda_n^k}\right), \frac{\lambda_n^j}{\lambda_n^k} \left(x - \frac{x_n^j - x_n^k}{\lambda_n^j \lambda_n^k}\right) \right]. \end{aligned}$$

We first assume $\frac{\lambda_n^j}{\lambda_n^k} + \frac{\lambda_n^k}{\lambda_n^j} \to \infty$. Then

$$\begin{aligned} &\left\|\psi^j (T_n^j)^{-1} T_n^k \psi^k\right\|_{L_t^4 L_x^4} \\ \le &\min\left\{ \left\|\psi^j\right\|_{L_t^\infty L_x^\infty} \left\|(T_n^j)^{-1} T_n^k \psi^k\right\|_{L_t^4 L_x^4}, \left\|\psi^j\right\|_{L_t^4 L_x^4} \left\|(T_n^j)^{-1} T_n^k \psi^k\right\|_{L_t^\infty L_x^\infty} \right\} \\ \lesssim &\min\left\{ \left(\frac{\lambda_n^j}{\lambda_n^k}\right)^{-\frac{1}{2}}, \left(\frac{\lambda_n^j}{\lambda_n^k}\right)^{\frac{1}{2}} \right\} \to 0. \end{aligned}$$

So we may assume $\frac{\lambda_n^j}{\lambda_n^k} \to \lambda_0 \in (0, \infty)$.

If $\frac{\left|t_n^j(\lambda_n^j)^2 - t_n^k(\lambda_n^k)^2\right|}{\lambda_n^j\lambda_n^k} \to \infty$, then the temporal supports of $\psi^j$ and $(T_n^j)^{-1}T_n^k\psi^k$ become disjoint for $n$ sufficiently large. Therefore

$$\lim_{n\to\infty} \left\|\psi^j(T_n^j)^{-1}T_n^k\psi^k\right\|_{L_t^4L_x^4} = 0.$$

If $\frac{\lambda_n^j}{\lambda_n^k} \to \lambda_0 \in (0,\infty)$, $\frac{\left|t_n^j(\lambda_n^j)^2 - t_n^k(\lambda_n^k)^2\right|}{\lambda_n^j\lambda_n^k} \to t_0 \in (0,\infty)$, $\frac{\left|x_n^j - x_n^k\right|^2}{\lambda_n^j\lambda_n^k} \to \infty$, then the spatial supports of $\psi^j$ and $(T_n^j)^{-1}T_n^k\psi^k$ become disjoint for $n$ sufficiently large. Then we have

$$\frac{\left|x_n^j - x_n^k\right|}{\lambda_n^j} = \frac{\left|x_n^j - x_n^k\right|}{\sqrt{\lambda_n^j\lambda_n^k}}\sqrt{\frac{\lambda_n^k}{\lambda_n^j}} \to \infty.$$

Then $\left\|T_n^j\psi^jT_n^k\psi^k\right\|_{L_t^4L_x^4} \to 0$ as $n\to\infty$.

Then the other two terms can be proved similarly. □

**Lemma 4.13** (Decoupling of nonlinear profiles)**.** *For $j \neq k$ we have*

$$\lim_{n\to\infty} \left\|v_n^jv_n^k\right\|_{L_t^4L_x^4} + \left\|\left(|\nabla|^{\frac{1}{2}} v_n^j\right)\left(|\nabla|^{\frac{1}{2}} v_n^k\right)\right\|_{L_t^2L_x^2}$$
$$+ \left\|\left||\nabla|^{\frac{1}{2}} F(v_n^j)\right|\left||\nabla|^{\frac{1}{2}} F(v_n^k)\right|^{\frac{1}{3}}\right\|_{L_t^1L_x^1} = 0.$$

*Proof.* For the first term, recall that for any $\varepsilon > 0$, there exists $\psi_\varepsilon^j \in C_c^\infty(\mathbb{R}\times\mathbb{R}^2)$, so that

$$\left\|v_n^j - T_n^j\psi_\varepsilon^j\right\|_{\dot{X}^{\frac{1}{2}}(\mathbb{R}\times\mathbb{R}^2)} + \left\|v_n^k - T_n^k\psi_\varepsilon^k\right\|_{\dot{X}^{\frac{1}{2}}(\mathbb{R}\times\mathbb{R}^2)} < \varepsilon.$$

Then by (4.3.3) and Lemma 4.12, we write

$$\begin{aligned}
&\left\|v_n^jv_n^k\right\|_{L_t^4L_x^4}\\
\leq{}& \left\|v_n^j(v_n^k - T_n^k\psi_\varepsilon^k)\right\|_{L_t^4L_x^4} + \left\|(v_n^j - T_n^j\psi_\varepsilon^j)T - n^j\psi_\varepsilon^k\right\|_{L_t^4L_x^4} + \left\|T_n^j\psi_\varepsilon^jT_n^j\psi_\varepsilon^k\right\|_{L_t^4L_x^4}\\
\lesssim{}& \left\|v_n^j\right\|_{L_t^8L_x^8}\left\|v_n^k - T_n^k\psi_\varepsilon^k\right\|_{L_t^8L_x^8} + \left\|v_n^j - T_n^j\psi_\varepsilon^j\right\|_{L_t^8L_x^8}\left\|T_n^j\psi_\varepsilon^k\right\|_{L_t^8L_x^8} + \left\|T_n^j\psi_\varepsilon^jT_n^j\psi_\varepsilon^k\right\|_{L_t^4L_x^4}\\
\lesssim{}& \left\|v_n^j\right\|_{\dot{X}^{\frac{1}{2}}}\left\|v_n^k - T_n^k\psi_\varepsilon^k\right\|_{\dot{X}^{\frac{1}{2}}} + \left\|v_n^j - T_n^j\psi_\varepsilon^j\right\|_{\dot{X}^{\frac{1}{2}}}\left\|T_n^j\psi_\varepsilon^k\right\|_{\dot{X}^{\frac{1}{2}}} + \left\|T_n^j\psi_\varepsilon^jT_n^j\psi_\varepsilon^k\right\|_{L_t^4L_x^4}\\
\lesssim_{E_c,\delta}{}& \varepsilon + o(1) \quad \text{ as } n\to\infty.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, this proves that $\left\|v_n^j v_n^k\right\|_{L_t^4 L_x^4} \to 0$, as $n \to \infty$ The other two terms can be proved similarly. $\square$

As a consequence of this decoupling we can bound the sum of the nonlinear profiles in $\dot{X}^{\frac{1}{2}}$ as follows:

$$\limsup_{n\to\infty} \left\|\sum_{j=1}^{J} v_n^j\right\|_{\dot{X}^{\frac{1}{2}}(\mathbb{R}\times\mathbb{R}^2)} \lesssim_{E_c,\delta} 1 \tag{4.3.5}$$

uniformly for finite $T \le J^*$.

In fact, by Young's inequality, (4.3.3), (4.3.4) and Lemma 4.13, we have

$$S_{\mathbb{R}}(\sum_{j=1}^{J} v_n^j) \lesssim \sum_{j=1}^{J} S_{\mathbb{R}}(v_n^j) + C_j \sum_{j\neq k} \left\|v_n^j v_n^k\right\|_{L_t^4 L_x^4}^4 \lesssim_{E_c,\delta} 1 + C_j o(1)$$

as $n \to \infty$. Similarly,

$$\begin{aligned} \left\|\sum_{j=1}^{J} |\nabla|^{\frac{1}{2}} v_n^j\right\|_{L_t^4 L_x^4}^2 &= \left\|\left(\sum_{j=1}^{J} |\nabla|^{\frac{1}{2}} v_n^j\right)^2\right\|_{L_t^2 L_x^2} \\ \lesssim \sum_{j=1}^{J} \left\||\nabla|^{\frac{1}{2}} v_n^j\right\|_{L_t^4 L_x^4}^2 &+ \sum_{j\neq k} \left\|\left(|\nabla|^{\frac{1}{2}} v_n^j\right)\left(|\nabla|^{\frac{1}{2}} v_n^k\right)\right\|_{L_t^2 L_x^2} \\ \lesssim_{E_c,\delta} 1 + o(1) \end{aligned}$$

as $n \to \infty$. Therefore, (4.3.5) holds.

The same argument combined with the decoupling of $\dot{H}^{\frac{1}{2}}$ norm (4.3.2) shows that given $\eta > 0$, there exists $J' = J'(\eta)$ such that

$$\limsup_{n\to\infty} \left\|\sum_{j=J'}^{J} v_n^j\right\|_{\dot{X}^{\frac{1}{2}}(\mathbb{R}\times\mathbb{R}^2)} \le \eta$$

uniformly in $J \ge J'$.

Now we are ready to construct an approximation solution to (1.6.1). For each $n$ and $J$, define

$$u_n^J := \sum_{j=1}^{J} v_n^j + e^{it\Delta} w_n^J.$$

Define

$$e_n^J := (i\partial_t + \Delta)u_n^J - \left|u_n^J\right|^4 u_n^J = \sum_{j=1}^{J} \left|v_n^J\right|^4 v_n^J - \left|u_n^J\right|^4 u_n^J.$$

We know that $u_n^J$ is defined globally in time. In order to apply Theorem 3.17, we need to the following four assumptions in the theorem:

Claim:

1. $\limsup_{J \to J^*} \limsup_{n \to \infty} \left\|u_n^J(0) - u_n(0)\right\|_{\dot{H}_x^{\frac{1}{2}}} = 0$ ,
2. $\limsup_{J \to J^*} \limsup_{n \to \infty} \left\|u_n^J\right\|_{\dot{S}^{\frac{1}{2}}(\mathbb{R})} + \left\|u_n^J\right\|_{S(\mathbb{R})} \lesssim_{E_c,\delta} 1$,
3. $\limsup_{J \to J^*} \limsup_{n \to \infty} \left\| |\nabla|^{\frac{1}{2}} e_n^J\right\|_{Z([0,\infty))} \lesssim_{E_c} 1$,
4. $\limsup_{J \to J^*} \limsup_{n \to \infty} \left\|e_n^J\right\|_{Y([0,\infty))} \lesssim_{E_c} = 0$.

The claim above implies that for sufficiently large $n$ and $J$, $u_n^J$ is an approximate solution to (1.6.1) with finite scattering size, which asymptotically approaches $u_n(0)$ at time $t = 0$. Using Theorem 3.17, we see that for $n$, $J$ sufficiently large, in Case 2, $S_{\mathbb{R}}(u_n) \lesssim_{E_c} 1$, which contradicts (4.3.1). Then we have Proposition 4.11.

Now we are left to proof the claim above.

*Proof of claim.* 1. This is automatically true by the definition of $u_n^J$.

2. For $S(\mathbb{R})$ norm:

$$\lim_{n \to \infty} \left\|u_n^J\right\|_{S(\mathbb{R})} = S_{\mathbb{R}}(u_n^J) \leq \sum_{j=1}^{J} S_{\mathbb{R}}(v_n^J) + S_{\mathbb{R}}(e^{it\Delta} w_n^J) \lesssim 1.$$

For $\dot{S}^{\frac{1}{2}}$ norm, we prove this later.

3. To begin, we will derive the bound $\limsup_{J\to J^*}\limsup_{n\to\infty}\left\| |\nabla|^{\frac12} u_n^J\right\|_{L_t^4L_x^4}^4 \lesssim_{E_c} 1.$

   In fact, by Lemma 4.13,

$$\begin{aligned}
&\limsup_{J\to J^*}\limsup_{n\to\infty}\left\| |\nabla|^{\frac12} u_n^J\right\|_{L_t^4L_x^4}^4\\
&\lesssim \limsup_{J\to J^*}\limsup_{n\to\infty}\left\|\sum_{j=1}^J |\nabla|^{\frac12} v_n^J\right\|_{L_t^4L_x^4}^4 + \limsup_{J\to J^*}\limsup_{n\to\infty}\left\| |\nabla|^{\frac12} e^{it\Delta}w_n^J\right\|_{L_t^4L_x^4}^4\\
&\lesssim \limsup_{J\to J^*}\limsup_{n\to\infty}\sum_{j=1}^J \left\|\left(|\nabla|^{\frac12} v_n^J\right)^2\right\|_{L_t^2L_x^2}^2 + \limsup_{J\to J^*}\limsup_{n\to\infty}\left\|e^{it\Delta}w_n^J\right\|_{\dot H_x^{\frac12}}^4\\
&\lesssim 1.
\end{aligned}$$

   To complete the proof of (3), it remains to show that

$$\limsup_{J\to J^*}\limsup_{n\to\infty}\left\|\sum_{j=1}^J |\nabla|^{\frac12}\left(\left|v_n^j\right|^4 v_n^j\right)\right\|_{Z(\mathbb{R})}^{\frac43}\lesssim 1.$$

   We claim that it will suffice to show that

$$\limsup_{J\to J^*}\limsup_{n\to\infty}\sum_{j=1}^J\left\| |\nabla|^{\frac12}\left(\left|v_n^j\right|^4 v_n^j\right)\right\|_{Z(\mathbb{R})}^{\frac43}\lesssim_{E_c} 1.$$

   In fact, by Lemma 4.13, the mixed terms vanish

$$\begin{aligned}
&\left|\left\|\sum_{j=1}^J |\nabla|^{\frac12} F(v_n^j)\right\|_{Z(\mathbb{R})}^{\frac43} - \sum_{j=1}^J\left\| |\nabla|^{\frac12} F(v_n^j)\right\|_{Z(\mathbb{R})}^{\frac43}\right|\\
&\lesssim_J \sum_{j\neq k}\left\|\left| |\nabla|^{\frac12} F(v_n^j)\right|\left| |\nabla|^{\frac12} F(v_n^k)\right|\right\|_{L_t^1L_x^1}\to 0
\end{aligned}$$

   as $n\to\infty$, where $F(u) = |u|^4 u$.

   Then we use the fractional chain rule and Sobolev embedding to see

$$\begin{aligned}
&\limsup_{J\to J^*}\limsup_{n\to\infty}\sum_{j=1}^J\left\| |\nabla|^{\frac12}\left(\left|v_n^j\right|^4 v_n^j\right)\right\|_{Z(\mathbb{R})}\\
&\lesssim \limsup_{J\to J^*}\limsup_{n\to\infty}\sum_{j=1}^J\left(\left\| |\nabla|^{\frac12} v_n^j\right\|_{L_t^4L_x^4}\left\|v_n^j\right\|_{S(\mathbb{R})}^4\right)^{\frac43}\\
&\lesssim \limsup_{J\to J^*}\limsup_{n\to\infty}\sum_{j=1}^J\left\|v_n^j\right\|_{\dot S^{\frac12}}^{\frac{20}{3}}\lesssim 1.
\end{aligned}$$

Now the bound for $\dot{S}^{\frac{1}{2}}$ follows,

$$\left\|u_n^J\right\|_{\dot{S}^{\frac{1}{2}}} \lesssim \left\|u_n^J(0)\right\|_{\dot{H}_x^{\frac{1}{2}}} + \left\| |\nabla|^{\frac{1}{2}} F(v_n^j) \right\|_{Z(\mathbb{R})} + \left\|e^{it\Delta} w_n^J\right\|_{\dot{S}^{\frac{1}{2}}} \lesssim 1.$$

4. Rewrite the error term into:

$$\begin{aligned} e_n^J &= (\sum_{j=1}^{J} F(v_n^j) - F(\sum_{j=1}^{J} v_n^j)) + (F(u_n^j - e^{it\Delta} w_n^J) - F(u_n^J)) \\ &:= (e_n^J)_1 + (e_n^J)_2. \end{aligned}$$

Then using Höder's inequality, Strichartz estimates, Sobolev embedding and Lemma 4.13, we estimate these two terms above:

$$\begin{aligned} &\left\|(e_n^J)_1\right\|_{Y(\mathbb{R})} \\ \lesssim_J &\sum_{j\neq k} \left\| \left|v_n^k\right| \left|v_n^j\right|^4 \right\|_{Y(\mathbb{R})} \lesssim_J \sum_{j\neq k} \left\|v_n^j\right\|^3_{L_t^{18} L_x^{\frac{36}{7}}} \left\|v_n^j v_n^k\right\|_{L_t^4 L_x^4} \\ \lesssim_J &\sum_{j\neq k} \left\|v_n^j\right\|^3_{L_t^{18} \dot{W}_x^{\frac{1}{2},\frac{9}{4}}} \left\|v_n^j v_n^k\right\|_{L_t^4 L_x^4} \to 0 \quad \text{as } n\to\infty, \\ &\left\|(e_n^J)_2\right\|_{Y(\mathbb{R})} \\ \lesssim &\left\|e^{it\Delta} w_n^J\right\|_{X(\mathbb{R})} ( \left\|u_n^J\right\|^4_{X(\mathbb{R})} + \left\|u_n^J - e^{it\Delta} w_n^J\right\|^4_{X(\mathbb{R})}) \\ \lesssim &\left\|e^{it\Delta} w_n^J\right\|^{\frac{2}{3}}_{S(\mathbb{R})} \left\|e^{it\Delta} w_n^J\right\|^{\frac{1}{3}}_{L_t^\infty L_x^4} \left( \left\| |\nabla|^{\frac{1}{2}} u_n^J \right\|^4_{L_t^{12} L_x^{\frac{12}{5}}} + \left\| |\nabla|^{\frac{1}{2}} e^{it\Delta} w_n^J \right\|^4_{L_t^{12} L_x^{\frac{12}{5}}} \right) \\ \lesssim &\left\|e^{it\Delta} w_n^J\right\|^{\frac{2}{3}}_{S(\mathbb{R})} \left\|e^{it\Delta} w_n^J\right\|^{\frac{1}{3}}_{L_t^\infty \dot{H}_x^{\frac{1}{2}}} \left( \left\| |\nabla|^{\frac{1}{2}} u_n^J \right\|^4_{\dot{S}^{\frac{1}{2}}} + \left\|w_n^J\right\|^4_{\dot{H}_x^{\frac{1}{2}}} \right) \\ \lesssim &\left\|e^{it\Delta} w_n^J\right\|^{\frac{2}{3}}_{S(\mathbb{R})} \left\|w_n^J\right\|^{\frac{1}{3}}_{\dot{H}_x^{\frac{1}{2}}} \to 0 \quad \text{as } n\to\infty. \end{aligned}$$

Now we complete the proof of the claim. □

Therefore, we complete the proof of Proposition 4.11.

□

**Theorem 4.14** (Reduction to almost periodic solution)**.** *If Theorem 1.5 failed, then there exists a maximal-lifespan solution $u : I \times \mathbb{R}^2 \to \mathbb{C}$ such that: $u$ is almost periodic, and blows up in both time directions and* $\sup_{t\in I} \left\| |\nabla|^{\frac{1}{2}} u(t) \right\|^2_{\dot{H}_x^{\frac{1}{2}}(\mathbb{R}^2)} = E_c$.

*Proof.* 

- By the definition of $E_c$ and the continuity of $L$, we can find a sequence of $u_n : I_n \times \mathbb{R}^2 \to \mathbb{C}$ of maximal-lifespan solutions with

$$\sup_{t\in I} \|u_n(t)\|^2_{\dot{H}_x^{\frac{1}{2}}(\mathbb{R}^2)} \leq E_c \text{ and } \lim_{n\to\infty} S(u_n) = +\infty.$$

  We choose $t_n \in I$ to be the median time of the $L_t^8 L_x^8$ norm of $u_n$, i.e.

$$S_{\leq t_n}(u_n) = S_{\geq t_n}(u_n) = \frac{1}{2} S_I(u_n).$$

  Then by time-translation invariance, we may take $t_n = 0$.

  Using Palais-Smale condition (Proposition 4.11) and passing to a subsequence, we find group element $g_n \in G$ such that $g_n u_n(0)$ converges strongly in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^2)$ to some $u_0 \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^2)$.

  Applying the group action $T_{g_n}$ to the solution $u_n$, we may take $g_n$ to be the identity, thus, $u_n(0)$ converge strongly $\dot{H}^{\frac{1}{2}}(\mathbb{R}^2)$ to $u_0$. In particular, this implies

$$\sup_{t\in I} \|u(t)\|^2_{\dot{H}_x^{\frac{1}{2}}(\mathbb{R}^2)} \leq E_c.$$

  Let $u : I \times \mathbb{R}^2 \to \mathbb{C}$ be the maximal-lifespan solution to NLS with initial data $u(0) = u_0$.

  We claim that $u$ blows up both forward and backward in time. Indeed, if $u$ does not blow up forward in time, then $[0, +\infty) \subseteq I$, and $S_{\geqslant 0}(u) < \infty$. By perturbation lemma (Theorem 3.17), this implies that for sufficiently large $n$, we have $[0, +\infty) \subseteq I_n$, and $\limsup_{n\to\infty} S_{\geq 0}(u_n) < \infty$. Contradiction!

  The definition of $E_c$ forces $\sup_{t\in I} \|u\|^2_{\dot{H}_x^{\frac{1}{2}}} \geq E_c$, hence $\sup_{t\in I} \|u\|^2_{\dot{H}_x^{\frac{1}{2}}} = E_c$.

- It remains to show that $u$ is almost periodic modulo symmetries. Consider an arbitrary sequence $\tau_n \in I$. Then $S_{\leq \tau_n}(u) = S_{\geq \tau_n}(u) = +\infty$, since $u$ blows up in both time directions. Using Palais-Smale condition (Proposition 4.11), we conclude that $u(\tau_n)$ admits a convergent subsequence in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^2)$ modulo symmetries. Thus the orbit $\{Gu(t) : t \in I\}$ is precompact in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^2)/G$. Then the theorem holds.

$\square$

With the above setup and properties in hand, we arrive at the following theorem:

**Theorem 4.15** (two special scenarios for blow-up)**.** *Suppose Theorem 1.5 failed. Then there exists an almost periodic solution $u : [0, T_{max}) \times \mathbb{R}^2 \to \mathbb{C}$, such that (4.2.1), (4.2.2),*

$$\|u\|_{L^8_{t,x}([0,T_{max})\times\mathbb{R}^2)} = +\infty,$$

*$N(0) = 1$, and $N(t) \geq 1$ on $[0, \infty)$, $\left|\frac{d}{dt}N(t)\right| \lesssim N(t)^3$.*

*Furthermore, one of the following holds:*

1. *The finite-time blow-up solutions,*

$$T_{max} < \infty,$$

2. *The quasi-soliton,*

$$\int_0^\infty N(t)\,dt = \infty.$$

The rest of this thesis is organized as follows:

- In Chapter 5, we will rule out the existence of finite-time blow-up solutions.

- To preclude the quasi-soliton solutions, we follow the following steps:

Step 1: Prove a suitable long time Strichartz estimate in Chapters 6 and 7;

Step 2: Derive a frequency localized Morawetz estimate with error terms estimated by the long time Strichartz estimate in Chapters 8 and 9;

Step 3: Using the frequency localized Morawetz estimate, rule out the quasi-soliton solutions (recall that Morawetz inequality scales like $\int_I N(t)\,dt$) in Chapter 9.

# CHAPTER 5

# IMPOSSIBILITY OF FINITE-TIME BLOW-UP SOLUTIONS

In this chapter, we rule out the existence of finite-time blow-up solutions in Theorem 5.1.

As we mentioned before, in $\dot{H}^{\frac{1}{2}}$ critical regime, due to the scaling, the long time Strichartz estimates do not provide any additional decay as we desired in the other $s_c \neq \frac{1}{2}$ settings. So in order to preclude the existence of finite-time blow-up solutions, we consider the mass of solutions restricted within a spatial bump instead, then show that the mass is indeed zero, which contradicts the fact of minimal blow-up.

**Theorem 5.1** (Impossibility of finite-time blow-up solutions)**.** *If $u$ is an almost periodic solution to* (1.6.1) *in the form of Theorem 4.15 and $T_{max} < \infty$, then $u \equiv 0$.*

*Proof.* We consider the following quantity $y^2(t,R)$:

$$y^2(t,R) := \int_{\mathbb{R}^2} \chi_R(x) \, |u(t,x)|^2 \, dx, \tag{5.0.1}$$

where $\chi_R(x) = \chi(\frac{x}{R})$ is a smooth cutoff function, such that

$$\chi(x) = \begin{cases} 1, & \text{if } |x| \leq 1 \\ 0, & \text{if } |x| > 2, \end{cases}$$

and

$$\left\| |\nabla|^{k} \chi(x) \right\|_{L_x^\infty} \leq 1 \quad \text{ for } k = \frac{3}{2}, 3. \tag{5.0.2}$$

In fact, the quantity defined in (5.0.1) can be thought of the mass of the solution $u$ restricted within the spatial bump with radius $\sim R$.

Recall the NLS equation (1.6.1) and the properties of complex numbers, then we can write

$$\begin{aligned}\frac{\partial}{\partial t} |u|^2 &= 2\,\mathrm{Re}[u_t \bar{u}] = 2\,\mathrm{Re}\,\left[(-i\,|u|^4\,u + i\Delta u)\bar{u}\right] \\ &= 2\,\mathrm{Re}\,\left[-i\,|u|^6 + i\Delta u\bar{u}\right] = -2\,\mathrm{Im}\,[\Delta u \bar{u}]\,.\end{aligned}$$

Now we compute the rate of change in time of $y^2(t,R)$, that is, the derivative of $y^2(t,R)$ with respect to time $t$. Passing the time derivative inside the integral, we have

$$\frac{\partial y^2}{\partial t} = \frac{\partial}{\partial t}\int_{\mathbb{R}^2} \chi_R(x)\,|u|^2\,dx = \int_{\mathbb{R}^2}\chi_R(x)\left[\frac{\partial}{\partial t}\,|u|^2\right]dx = -2\,\mathrm{Im}\int_{\mathbb{R}^2}\chi_R(x)\Delta u\bar{u}\,dx.$$

Then preforming integration by parts and the product rule in calculus, we obtain

$$\begin{aligned}-\,\mathrm{Im}\int_{\mathbb{R}^2}\chi_R(x)\Delta u\bar{u}\,dx &= \mathrm{Im}\int_{\mathbb{R}^2}\nabla\,\left(\chi_R(x)\bar{u}\right)\cdot\nabla u\,dx \\ = \mathrm{Im}\int_{\mathbb{R}^2}\nabla\chi_R(x)\cdot\nabla u\bar{u}\,dx &+ \mathrm{Im}\int_{\mathbb{R}^2}\nabla\bar{u}\cdot\nabla u\chi_R(x)\,dx \\ = \mathrm{Im}\int_{\mathbb{R}^2}\nabla\chi_R(x)\cdot\nabla u\bar{u}\,dx.&\end{aligned} \tag{5.0.3}$$

The second term above is zero, since $\nabla\bar{u}\cdot\nabla u = |u|^2$ is a real number.

Now to estimate the time derivative of $y^2(t,R)$ is equivalent to estimating (5.0.3). Notice that this is a product of three functions, hence it is natural to employ Littlewood-Paley decomposition and treat these three functions at different frequency scales separately, i.e. consider

$$\sum_{N_1,N_2,N_3}\int_{\mathbb{R}^2} P_{N_1}(\nabla\chi_R(x))\,P_{N_2}(\nabla u)\,P_{N_3}\bar{u}\,dx. \tag{5.0.4}$$

In fact, by Littlewood-Paley decomposition, it is sufficient for us to consider the following three cases:

$$\begin{cases} \text{case 1: } N_1 \sim N_2 \geq N_3 \\ \text{case 2: } N_2 \sim N_3 \geq N_1 \\ \text{case 3: } N_3 \sim N_1 \geq N_2 \end{cases}$$

Next, we will deal with the three cases above one by one:

- Case 1: $N_1 \sim N_2 \geq N_3$.

  First, by Hölder's inequality, Bernstein's inequality, $N_1 \sim N_2$ and Sobolev embedding, we get

$$\begin{aligned} &\int_{\mathbb{R}^2} P_{N_1}(\nabla\chi_R)\, P_{N_2}(\nabla u)\, P_{N_3}\bar{u}\, dx \\ \lesssim\ & \|P_{N_1}\nabla\chi_R\|_{L_x^{\frac{82}{21}}}\ \|P_{N_2}\nabla u\|_{L_x^2}\ \|P_{N_3}u\|_{L_x^{\frac{41}{10}}} \\ \simeq\ & N_1^{\frac{20}{41}}\ \|P_{N_1}\nabla\chi_R\|_{L_x^2}\, N_2^{\frac{1}{2}}\ \left\|P_{N_2}\ |\nabla|^{\frac{1}{2}}\, u\right\|_{L_x^2} N_3^{\frac{1}{82}}\ \|P_{N_3}u\|_{L_x^4} \\ \simeq\ & N_1\ \|P_{N_1}\nabla\chi_R\|_{L_x^2} \left\|P_{N_2}\ |\nabla|^{\frac{1}{2}}\, u\right\|_{L_x^2} \left\|P_{N_3}\ |\nabla|^{\frac{1}{2}}\, u\right\|_{L_x^2} \left(\frac{N_3}{N_2}\right)^{\frac{1}{82}}. \end{aligned} \tag{5.0.5}$$

  Then we are left to sum $N_1$, $N_2$ and $N_3$ over (5.0.5). For the sums over $N_2$ and $N_3$, taking the components only depending on $N_2$ and $N_3$ in (5.0.5) and applying Cauchy-Schwarz inequality, we write

$$\begin{aligned} &\sum_{N_3\leq N_2} \left\|P_{N_2}\ |\nabla|^{\frac{1}{2}}\, u\right\|_{L_x^2} \left\|P_{N_3}\ |\nabla|^{\frac{1}{2}}\, u\right\|_{L_x^2} \left(\frac{N_3}{N_2}\right)^{\frac{1}{82}} \qquad (5.0.6) \\ \lesssim &\left(\sum_{N_3\leq N_2} \left(\frac{N_3}{N_2}\right)^{\frac{1}{82}} \left\|P_{N_2}\ |\nabla|^{\frac{1}{2}}\, u\right\|_{L_x^2}^2\right)^{\frac{1}{2}} \left(\sum_{N_3\leq N_2} \left(\frac{N_3}{N_2}\right)^{\frac{1}{82}} \left\|P_{N_3}\ |\nabla|^{\frac{1}{2}}\, u\right\|_{L_x^2}^2\right)^{\frac{1}{2}}. \end{aligned}$$

  **Remark 5.2.** Note that the power of the fraction $\frac{N_3}{N_2}$ here is not necessary to be $\frac{1}{82}$. In fact, any positive power $\varepsilon$ would help us to sum $N_2$ and $N_3$ above (we will see this in the following calculation), while without the factor $(\frac{N_3}{N_2})^{\varepsilon}$, this sum would be not summable. The reason why we choose this number $\frac{1}{82}$ is only to avoid the confusion caused by the notation $\varepsilon$.

A direct computation gives us

$$\sum_{N_3 \leq N_2} \left(\frac{N_3}{N_2}\right)^{\frac{1}{82}} \left\| P_{N_2} \, |\nabla|^{\frac{1}{2}} u \right\|_{L_x^2}^2 = \sum_{N_2} \sum_{N_3 : N_3 \leq N_2} \left(\frac{N_3}{N_2}\right)^{\frac{1}{82}} \left\| P_{N_2} \, |\nabla|^{\frac{1}{2}} u \right\|_{L_x^2}^2$$
$$\lesssim \sum_{N_2} \left\| P_{N_2} \, |\nabla|^{\frac{1}{2}} u \right\|_{L_x^2}^2 \sim \left\| \, |\nabla|^{\frac{1}{2}} u \right\|_{L_x^2}^2 ,$$

and

$$\sum_{N_3 \leq N_2} \left(\frac{N_3}{N_2}\right)^{\frac{1}{82}} \left\| P_{N_3} \, |\nabla|^{\frac{1}{2}} u \right\|_{L_x^2}^2 = \sum_{N_3} \sum_{N_2 : N_2 \geq N_3} \left(\frac{N_3}{N_2}\right)^{\frac{1}{82}} \left\| P_{N_3} \, |\nabla|^{\frac{1}{2}} u \right\|_{L_x^2}^2$$
$$\lesssim \sum_{N_3} \left\| P_{N_3} \, |\nabla|^{\frac{1}{2}} u \right\|_{L_x^2}^2 \sim \left\| \, |\nabla|^{\frac{1}{2}} u \right\|_{L_x^2}^2 ,$$

hence

$$(5.0.6) \lesssim \left\| \, |\nabla|^{\frac{1}{2}} u \right\|_{L_x^2}^2 . \tag{5.0.7}$$

Now we are left to sum $N_1$ over (5.0.5). Splitting the sum over $N_1$ into $\sum_{N_1 < 1}$ and $\sum_{N_1 \geq 1}$, taking the component in (5.0.5) only depending on $N_1$ and applying Bernstein's inequality, we obtain

$$\begin{aligned} &\sum_{N_1} N_1 \left\| P_{N_1} \nabla \chi_R \right\|_{L_x^2} \\ &= \sum_{N_1 < 1} N_1 \left\| P_{N_1} \nabla \chi_R \right\|_{L_x^2} + \sum_{N_1 \geq 1} N_1 \left\| P_{N_1} \nabla \chi_R \right\|_{L_x^2} \\ &\lesssim \sum_{N_1 < 1} N_1^{\frac{1}{2}} \left\| P_{N_1} \, |\nabla|^{\frac{3}{2}} \chi_R \right\|_{L_x^2} + \sum_{N_1 \geq 1} \frac{1}{N_1} \left\| P_{N_1} \nabla^3 \chi_R \right\|_{L_x^2} . \end{aligned} \tag{5.0.8}$$

Then by Cauchy-Schwarz inequality, Littlewood-Paley theory, Hölder's inequality and the compactness of the smooth cutoff function $\chi_R(x)$,

$$
\begin{aligned}
(5.0.8) \lesssim & \left(\sum_{N_1<1} N_1\right)^{\frac{1}{2}} \left(\sum_{N_1<1} \left\| P_{N_1} \, |\nabla|^{\frac{3}{2}} \, \chi_R \right\|_{L_x^2}^2 \right)^{\frac{1}{2}} \\
& + \left(\sum_{N_1\geq 1} \frac{1}{N_1^2}\right)^{\frac{1}{2}} \left(\sum_{N_1\geq 1} \left\| P_{N_1} \nabla^3 \chi_R \right\|_{L_x^2}^2 \right)^{\frac{1}{2}} \\
\lesssim & \left\| \, |\nabla|^{\frac{3}{2}} \, \chi_R \right\|_{L_x^2} + \left\| \nabla^3 \chi_R \right\|_{L_x^2} \\
\lesssim & \left\| \, |\nabla|^{\frac{3}{2}} \, \chi_R \right\|_{L_x^\infty} \left\| \mathbf{1}_{2R} \right\|_{L_x^2} + \left\| \nabla^3 \chi_R \right\|_{L_x^\infty} \left\| \mathbf{1}_{2R} \right\|_{L_x^2} \\
\lesssim & \left\| \, |\nabla|^{\frac{3}{2}} \, \chi_R \right\|_{L_x^\infty} R + \left\| \nabla^3 \chi_R \right\|_{L_x^\infty} R.
\end{aligned}
$$

Realizing that (5.0.2) implies $\left\| \, |\nabla|^k \, \chi_R(x) \right\|_{L_x^\infty} \leq \frac{1}{R^k}$, hence we have

$$
\sum_{N_1} N_1 \left\| P_{N_1} \nabla \chi_R \right\|_{L_x^2} \lesssim \left\| \, |\nabla|^{\frac{3}{2}} \, \chi_R \right\|_{L_x^\infty} R + \left\| \nabla^3 \chi_R \right\|_{L_x^\infty} R \lesssim \frac{1}{\sqrt{R}}. \tag{5.0.9}
$$

Finally by putting the information above together, we obtain

$$
\begin{aligned}
& \sum_{N_1 \sim N_2 \geq N_3} \int_{\mathbb{R}^2} P_{N_1}(\nabla \chi_R) P_{N_2}(\nabla u) P_{N_3} \bar{u} \, dx \\
\lesssim & \sum_{N_1} \sum_{N_3 \leq N_2} N_1 \left\| P_{N_1} \nabla \chi_R \right\|_{L_x^2} \left\| P_{N_2} \, |\nabla|^{\frac{1}{2}} \, u \right\|_{L_x^2} \left\| P_{N_3} \, |\nabla|^{\frac{1}{2}} \, u \right\|_{L_x^2} \left(\frac{N_3}{N_2}\right)^{\frac{1}{82}} \\
\lesssim & \sum_{N_1} N_1 \left\| P_{N_1} \nabla \chi_R \right\|_{L_x^2} \left\| \, |\nabla|^{\frac{1}{2}} \, u \right\|_{L_x^2}^2 \\
\lesssim & \frac{1}{\sqrt{R}} \left\| \, |\nabla|^{\frac{1}{2}} \, u \right\|_{L_x^2}^2 \lesssim \frac{1}{\sqrt{R}}.
\end{aligned}
$$

- Case 2: $N_2 \sim N_3 \geq N_1$.

  In this case, $N_2 \sim N_3$, we can pass half derivative from $\nabla u$ to $\bar{u}$, then bound these two terms by $\dot{H}^{\frac{1}{2}}$ norm of the solution. By Hölder's inequality and

Bernstein's inequality, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^2} P_{N_1}(\nabla\chi_R)P_{N_2}(\nabla u)P_{N_3}\bar{u}\,dx \\
&\lesssim \|P_{N_1}\nabla\chi_R\|_{L_x^\infty}\ \|P_{N_2}\nabla u\|_{L_x^2}\ \|P_{N_3}u\|_{L_x^2} \\
&\simeq N_1\ \|P_{N_1}\nabla\chi_R\|_{L_x^2}\ \left\|P_{N_2}\ |\nabla|^{\frac{1}{2}}u\right\|_{L_x^2}\ \left\|P_{N_3}\ |\nabla|^{\frac{1}{2}}u\right\|_{L_x^2}\left(\frac{N_2}{N_3}\right)^{\frac{1}{2}}.
\end{aligned}
$$

Again Cauchy-Schwarz inequality gives

$$
\sum_{N_3\sim N_2}\left\|P_{N_2}\ |\nabla|^{\frac{1}{2}}u\right\|_{L_x^2}\ \left\|P_{N_3}\ |\nabla|^{\frac{1}{2}}u\right\|_{L_x^2}\left(\frac{N_2}{N_3}\right)^{\frac{1}{2}}\lesssim\left\|\,|\nabla|^{\frac{1}{2}}u\right\|_{L_x^2}^2.
$$

Therefore, by (5.0.9)

$$
\begin{aligned}
&\sum_{N_2\sim N_3\geq N_1}\int_{\mathbb{R}^2} P_{N_1}(\nabla\chi_R)P_{N_2}(\nabla u)P_{N_3}\bar{u}\,dx \\
&\lesssim \sum_{N_1} N_1\ \|P_{N_1}\nabla\chi_R\|_{L_x^2}\sum_{N_2\sim N_3}\left\|P_{N_2}\ |\nabla|^{\frac{1}{2}}u\right\|_{L_x^2}\ \left\|P_{N_3}\ |\nabla|^{\frac{1}{2}}u\right\|_{L_x^2}\left(\frac{N_2}{N_3}\right)^{\frac{1}{2}} \\
&\lesssim \frac{1}{\sqrt{R}}\left\|\,|\nabla|^{\frac{1}{2}}u\right\|_{L_x^2}^2\lesssim\frac{1}{\sqrt{R}}.
\end{aligned}
$$

- Case 3: $N_3\sim N_1\geq N_2$.

  In this case, $N_2\leq N_3$, we use the same idea in Case 2, that is, we pass half derivative from $\nabla u$ to $\bar{u}$. By Hölder's inequality and Bernstein's inequality, we have that

$$
\begin{aligned}
&\int_{\mathbb{R}^2} P_{N_1}(\nabla\chi_R)P_{N_2}(\nabla u)P_{N_3}\bar{u}\,dx \\
&\lesssim \|P_{N_1}\nabla\chi_R\|_{L_x^\infty}\ \|P_{N_2}\nabla u\|_{L_x^2}\ \|P_{N_3}u\|_{L_x^2} \\
&\simeq N_1\ \|P_{N_1}\nabla\chi_R\|_{L_x^2}\ \left\|P_{N_2}\ |\nabla|^{\frac{1}{2}}u\right\|_{L_x^2}\ \left\|P_{N_3}\ |\nabla|^{\frac{1}{2}}u\right\|_{L_x^2}\left(\frac{N_2}{N_3}\right)^{\frac{1}{2}}.
\end{aligned}
$$

  Using Cauchy-Schwarz inequality again,

$$
\sum_{N_2\lesssim N_3}\left\|P_{N_2}\ |\nabla|^{\frac{1}{2}}u\right\|_{L_x^2}\ \left\|P_{N_3}\ |\nabla|^{\frac{1}{2}}u\right\|_{L_x^2}\left(\frac{N_2}{N_3}\right)^{\frac{1}{2}}\lesssim\left\|\,|\nabla|^{\frac{1}{2}}u\right\|_{L_x^2}^2.
$$

Therefore, by (5.0.9)

$$\sum_{N_3 \sim N_1 \geq N_2} \int_{\mathbb{R}^2} P_{N_1}(\nabla \chi_R) P_{N_2}(\nabla u) P_{N_3} \bar{u}\, dx$$

$$\lesssim \sum_{N_1} N_1 \, \| P_{N_1} \nabla \chi_R \|_{L_x^2} \sum_{N_2 \leq N_3} \left\| P_{N_2} \, |\nabla|^{\frac{1}{2}} u \right\|_{L_x^2} \left\| P_{N_3} \, |\nabla|^{\frac{1}{2}} u \right\|_{L_x^2} \left( \frac{N_2}{N_3} \right)^{\frac{1}{2}}$$

$$\lesssim \frac{1}{\sqrt{R}} \left\| \, |\nabla|^{\frac{1}{2}} u \right\|_{L_x^2}^2 \lesssim \frac{1}{\sqrt{R}}.$$

Hence, all these three cases give us the same estimate for $\frac{\partial y^2}{\partial t}$:

$$\left| \frac{\partial y^2}{\partial t} \right| \lesssim \frac{1}{\sqrt{R}}. \tag{5.0.10}$$

At this point, we claim that $y^2(T_{max}, R) = 0$, that is, for any $R$ fixed,

$$\lim_{t \to T_{max}} y^2(t,R) = \lim_{t \to T_{max}} \int_{\mathbb{R}^2} \chi_R(x) \, |u(t,x)|^2 \, dx = 0. \tag{5.0.11}$$

Assuming that (5.0.11) is true, we are able to complete the proof. In fact, by the definition of limit, we fix an arbitrary small number $\varepsilon$, then there exists $t_0$, such that $y^2(t_0, R) < \varepsilon$. We can think of $y^2(t,R)$ as a function of $t$, whose slope is bounded by $\frac{1}{\sqrt{R}}$. Then by the fundamental theorem of calculus, $y^2$ itself should be bounded by

$$y^2(t,R) \lesssim \frac{|t_0 - t|}{\sqrt{R}} + y^2(t_0,R) \leq \frac{T_{max}}{\sqrt{R}} + \varepsilon \quad \text{ for any } t < T_{max},$$

especially at time $t = 0$,

$$y^2(0,R) \lesssim \frac{T_{max}}{\sqrt{R}} + \varepsilon.$$

Next we let $R$ go to infinity. Due to the finiteness of $T_{max}$, it is easy to see that

$$\lim_{R \to \infty} y^2(0,R) \lesssim \varepsilon \quad \text{ for any arbitrary } \varepsilon,$$

which implies

$$\lim_{R \to \infty} y^2(0,R) = 0.$$

Therefore, by passing the limit inside, we have

$$0 = \lim_{R\to\infty} y^2(0,R) = \lim_{R\to\infty} \int_{\mathbb{R}^2} \chi_R(x)\ |u(0,x)|^2\ dx = \int_{\mathbb{R}^2} |u(0,x)|^2\ dx = \ \|u_0\|_{L_x^2}^2 .$$

This implies $u_0 \equiv 0$, which contradicts with the fact that $u$ is a minimal blow-up solution, so we prove the impossibility of finite-time blow-up solutions.

However, it still remains to prove the claim (5.0.11) above:

*Proof of* (5.0.11). By the definition of limit, it is equivalent to proving that: for any $\varepsilon > 0$, there exists $t_0$, such that for any $t > t_0$,

$$\int_{\mathbb{R}^2} \chi_R(x)\ |u(t,x)|^2\, dx < \varepsilon.$$

By the almost periodicity of the solution (Definition 4.5), we know that for $\eta = \frac{\varepsilon}{2R}$ fixed, there exist $c(\eta)$, $N(t)$, and $x(t)$ such that

$$\left(\int_{|x-x(t)|>\frac{c(\eta)}{N(t)}} |u(t,x)|^4\ dx\right)^{\frac{1}{2}} < \eta = \frac{\varepsilon}{2R}.$$

Hence, if we consider the cutoff mass $y^2(t,R)$ inside the bump $\mathbf{1}_{|x|\le\frac{c(\eta)}{N(t)}}$ and outside the bump $\chi_R\mathbf{1}_{|x|>\frac{c(\eta)}{N(t)}}$ separately, the mass inside will be small because the measure of the bump is small and the mass outside will be also small due to the almost periodicity of the solution. By Hölder's inequality, the almost periodicity of the solution, Sobolev embedding and the uniform $\dot{H}^{\frac{1}{2}}$ norm bound for the solution, we have

$$\begin{aligned}
y^2(t,R) &= \int_{\mathbb{R}^2} \chi_R(x)\ |u(t,x)|^2\ dx \\
&\le \int \mathbf{1}_{|x-x(t)|\le\frac{c(\eta)}{N(t)}}\ |u|^2\ dx + \int \chi_R \mathbf{1}_{|x-x(t)|>\frac{c(\eta)}{N(t)}}\ |u|^2\ dx \\
&\lesssim \left\| \mathbf{1}_{|x-x(t)|\le\frac{c(\eta)}{N(t)}} \right\|_{L_x^2} \|u\|_{L_x^4}^2 + \ \|\chi_R\|_{L_x^2} \left\| u\mathbf{1}_{|x-x(t)|>\frac{c(\eta)}{N(t)}} \right\|_{L_x^4}^2 \\
&\lesssim \frac{c(\eta)}{N(t)}\ \|u\|_{L_t^\infty \dot{H}_x^{\frac{1}{2}}}^2 + R\frac{\varepsilon}{2R} \\
&= \frac{c(\eta)}{N(t)}\ \|u\|_{L_t^\infty \dot{H}_x^{\frac{1}{2}}}^2 + \frac{\varepsilon}{2}.
\end{aligned}$$

The first term $\frac{c(\eta)}{N(t)} \|u\|^2_{L_t^\infty \dot{H}_x^{\frac{1}{2}}}$ can be made less than $\frac{\varepsilon}{2}$. In fact $N(t)$ goes to infinity as $t$ approaches to $T_{max}$ by Corollary 4.8, so it is always possible for us to choose some $t$ close enough to $T_{max}$ such that $N(t)$ is large enough. Then the claim follows. $\square$

The proof of Theorem 5.1 is complete. $\square$

# CHAPTER 6

# ATOMIC SPACES AND THE $X$-NORM

In this chapter, we recall some basic definitions and properties of atomic spaces, then prove a decomposition lemma, which will be used in the proof of the long time Strichartz estimates in Chapter 7. At the end of this chapter, we give the definition of $\tilde{X}_{k_0}$ norm, which will be also used in Chapter 7 in defining the long time Strichartz estimates.

## 6.1 Basic definitions and properties of the atomic spaces

The atomic spaces were first introduced in partial differential equations in [37], and then applied to KP-II equations in [23] and nonlinear Schrödinger equations in [38, 39, 24, 25]. They are useful in many different settings, and it is worth mentioning that they are quite helpful in fixing the lack of Strichartz endpoint problems.

First, we recall some basic definitions and properties of the atomic spaces.
Let $\mathcal{Z}$ denote the set of finite partitions

$$-\infty < t_0 < t_1 < \cdots < t_K \leq \infty$$

of the real line. If $t_K = \infty$, we use the convention that $v(t_K) := 0$ for all functions

$v : \mathbb{R} \to L^2$. Let $\chi_I : \mathbb{R} \to \mathbb{R}$ denote the sharp characteristic function of a set $I \subset \mathbb{R}$.

**Definition 6.1** ($U^p$ spaces, Definition 2.1 in [23]). Let $1 \leq p < \infty$. For $\{t_k\}_{k=0}^K \in \mathcal{Z}$ and $\{\phi_k\}_{k=0}^{K-1} \subset L^2$ with $\sum\limits_{k=0}^{K-1} \|\phi_k\|_{L^2}^p = 1$, we call the piecewise defined function $a : \mathbb{R} \to L^2$:

$$a = \sum_{k=1}^{K} \chi_{[t_{k-1}, t_k)} \phi_{k-1}$$

a $U^p$-atom, and we define the atomic space $U^p(\mathbb{R}, L^2)$ of all functions $u : \mathbb{R} \to L^2$ such that

$$u = \sum_{j=1}^{\infty} \lambda_j a_j \quad \text{for } U^p\text{-atoms } a_j,\ \{\lambda_j\} \in l^1,$$

with norm

$$\|u\|_{U^p} := \inf \left\{ \sum_{j=1}^{\infty} |\lambda_j| : u = \sum_{j=1}^{\infty} \lambda_j a_j, \lambda_j \in \mathbb{C} \text{ and } a_j \text{ are } U^p\text{-atoms} \right\}.$$

If $J \subset \mathbb{R}$ is an interval, then we say that $u_\lambda$ is a $U^p(J)$-atom if $t_k \in J$ for all $1 \leq k \leq K$. Then for any $1 \leq p < \infty$, let

$$\|u\|_{U^p(J \times \mathbb{R}^2)} = \inf \left\{ \sum_{j=1}^{\infty} |\lambda_j| : u = \sum_{j=1}^{\infty} \lambda_j a_j, \lambda \in \mathbb{C}, a_j \text{ are } U^p(J)\text{-atoms} \right\}.$$

For any $1 \leq p < \infty$, $U^p(J \times \mathbb{R}^2) \hookrightarrow L^\infty L^2(J \times \mathbb{R}^2)$. Additionally, $U^p$-functions are continuous except at countably many points and right-continuous everywhere.

**Definition 6.2** ($V^p$ spaces, Definition 2.3 in [23]). Let $1 \leq p < \infty$.

1. We define $V^p(\mathbb{R}, L^2)$ as the space of all functions $v : \mathbb{R} \to L^2$ such that

$$\|v\|_{V^p} := \sup_{\{t_k\}_{k=0}^K \in \mathcal{Z}} \left( \sum_{k=1}^{K} \|v(t_k) - v(t_{k-1})\|_{L^2}^p \right)^{\frac{1}{p}} \tag{6.1.1}$$

   is finite. Notice that here we use the convention $v(\infty) = 0$.

2. Likewise, let $V^p_{rc}(\mathbb{R}, L^2)$ denote the closed subspace of all right-continuous functions: $v : \mathbb{R} \to L^2$ such that $\lim_{t\to-\infty} v(t) = 0$, endowed with the same norm (6.1.1).

3. If $J \subset \mathbb{R}$, then

$$\|v\|_{V^p(J\times\mathbb{R}^2)} := \sup_{\{t_k\}_{k=0}^K \in \mathcal{Z}} \left( \|v(t_0)\|^p_{L^2} + \sum_{k=1}^{K} \|v(t_k) - v(t_{k-1})\|^p_{L^2} + \|v(t_K)\|^p_{L^2} \right)^{\frac{1}{p}},$$

where each $t_k$ lies in $J$. Note that $\{t_k\}$ may be a finite or infinite sequence.

**Proposition 6.3** (Embedding, Proposition 2.2, Proposition 2.4 and Corollary 2.6 in [23])**.** *For $1 \leq p \leq q < \infty$,*

$$U^p(\mathbb{R}, L^2) \hookrightarrow U^q(\mathbb{R}, L^2) \hookrightarrow L^\infty(\mathbb{R}, L^2),$$

*and functions in $U^p(\mathbb{R}, L^2)$ are right continuous and $\lim_{t\to-\infty} u(t) = 0$ for each $u \in U^p(\mathbb{R}, L^2)$.*

*The Banach subspace of all right continuous functions endowed with $\|\cdot\|_{V^p}$ is denoted by $V^p_{rc}(\mathbb{R}, L^2)$. Note that,*

$$U^p(\mathbb{R}, L^2) \hookrightarrow V^p_{rc}(\mathbb{R}, L^2) \hookrightarrow L^\infty(\mathbb{R}, L^2).$$

*Moreover, for $1 \leq p \leq q < \infty$,*

$$U^p(\mathbb{R}, L^2) \hookrightarrow V^p_{rc}(\mathbb{R}, L^2) \hookrightarrow U^q(\mathbb{R}, L^2) \hookrightarrow L^\infty(\mathbb{R}, L^2).$$

**Definition 6.4** ($U^p_\Delta$ and $V^p_\Delta$ spaces, Definition 2.15 in [23])**.** For $s \in \mathbb{R}$, we let $U^p_\Delta L^2_x$ (respectively $V^p_\Delta L^2_x$) be the space of all functions $u : \mathbb{R} \to L^2_x(\mathbb{R}^d)$ such that $t \mapsto e^{-it\Delta}u(t)$ is in $U^p(\mathbb{R}, L^2_x)$ (respectively in $V^p(\mathbb{R}, L^2_x)$), with norms

$$\|u\|_{U^p_\Delta L^2} := \left\|e^{-it\Delta}u(t)\right\|_{U^p(\mathbb{R},L^2_x)}, \quad \|u\|_{V^p_\Delta L^2} := \left\|e^{-it\Delta}u(t)\right\|_{V^p(\mathbb{R},L^2_x)}.$$

**Proposition 6.5.** *For $1 \le p \le q < \infty$,*

$$U^p_\Delta(\mathbb{R}, L^2) \hookrightarrow V^p_\Delta(\mathbb{R}, L^2) \hookrightarrow U^q_\Delta(\mathbb{R}, L^2) \hookrightarrow L^\infty(\mathbb{R}, L^2).$$

**Lemma 6.6** ((29) in [39], Lemma 3.3 in [16])**.** *Suppose $J = I_1 \cup I_2$, $I_1 = [a, b]$, $I_2 = [b, c]$, $a \le b \le c$. Then*

$$\|u\|^p_{U^p_\Delta(J\times\mathbb{R}^2)} \le \|u\|^p_{U^p_\Delta(I_1\times\mathbb{R}^2)} + \|u\|^p_{U^p_\Delta(I_2\times\mathbb{R}^2)}.$$

**Proposition 6.7** (Duality, Theorem 2.8 in [23])**.** *Let $DU^p_\Delta$ be the space of functions*

$$DU^p_\Delta = \{(i\partial_t + \Delta)u : u \in U^p_\Delta\},$$

*and the $DU^p_\Delta = (V^{p'}_\Delta)^*$, with $\frac{1}{p} + \frac{1}{p'} = 1$. Then $(0 \in J)$*

$$\left\| \int_0^t e^{i(t-t')\Delta} F(u)(t')\, dt' \right\|_{U^p_\Delta(J\times\mathbb{R}^d)} \lesssim \sup \left\{ \int_J \langle v, F\rangle \, dt : \|v\|_{V^{p'}_\Delta} = 1 \right\}.$$

**Lemma 6.8** (Decomposition lemma)**.** *Suppose $[a, b] = J = \cup_{k=1}^K P_k$, where $P_k = [a_k, b_k]$ $(b_k = a_{k+1})$ are consecutive intervals. Also suppose that $|\nabla|^{\frac{1}{2}} F \in L^1_t L^2_x(J \times \mathbb{R}^2)$, then for any $t_0 \in J$,*

$$\left\| |\nabla|^{\frac{1}{2}} \int_{t_0}^t e^{i(t-t')\Delta} F(t')\, dt' \right\|_{U^2_\Delta(J\times\mathbb{R}^2)} \lesssim \sum_{k=1}^K \sup_{\|v\|_{V^2_\Delta(P_k\times\mathbb{R}^2)}=1} \int_{P_k} \left\langle v, |\nabla|^{\frac{1}{2}} F \right\rangle dt.$$

*Note that the implicit constant will not depend on $\left\| |\nabla|^{\frac{1}{2}} F \right\|_{L^1_t L^2_x}$.*

*Proof.* We first consider $t > t_0$, and $t_0 \in P_{k^*} = [a_{k^*}, b_{k^*}]$. Then by duality (Proposition 6.7) and the partition of the interval $J$, we write

$$\left\| |\nabla|^{\frac{1}{2}} \int_{t_0}^t e^{i(t-t')\Delta} F(t')\, dt' \right\|_{U^2_\Delta([t_0,b]\times\mathbb{R}^2)} \lesssim \sup_{\|v\|_{V^2_\Delta([t_0,b]\times\mathbb{R}^2)=1}} \int_{[t_0,b]} \left\langle v, |\nabla|^{\frac{1}{2}} F \right\rangle dt$$

$$= \sup_{\|v\|_{V^2_\Delta([t_0,b]\times\mathbb{R}^2)}=1} \left\{ \int_{\cup_{k>k^*} P_k} \left\langle \sum_{P_k} v\mathbf{1}_{P_k}, |\nabla|^{\frac{1}{2}} F \right\rangle dt + \int_{[t_0,b_{k^*}]} \left\langle v\mathbf{1}_{[t_0,b_{k^*}]}, |\nabla|^{\frac{1}{2}} F \right\rangle dt \right\}$$

$$\le \sum_{k>k^*} \sup_{\|v\|_{V^2_\Delta(P_k\times\mathbb{R}^2)}=1} \int_{P_k} \left\langle v, |\nabla|^{\frac{1}{2}} F \right\rangle dt + \sup_{\|v\|_{V^2_\Delta([a_{k^*},b_{k^*}]\times\mathbb{R}^2)}=1} \int_{[a_{k^*},b_{k^*}]} \left\langle v, |\nabla|^{\frac{1}{2}} F \right\rangle dt$$

$$= \sum_{k\ge k^*} \sup_{\|v\|_{V^2_\Delta(P_k\times\mathbb{R}^2)}=1} \int_{P_k} \left\langle v, |\nabla|^{\frac{1}{2}} F \right\rangle dt.$$

If we consider $t < t_0$, similar argument as above gives

$$\left\| |\nabla|^{\frac{1}{2}} \int_{t_0}^{t} e^{i(t-t')\Delta} F(t')\, dt' \right\|_{U^2_\Delta([a,t_0]\times\mathbb{R}^2)} \lesssim \sum_{k\leq k^*} \sup_{\|v\|_{V^2_\Delta(P_k\times\mathbb{R}^2)}=1} \int_{P_k} \left\langle v,\, |\nabla|^{\frac{1}{2}} F \right\rangle dt.$$

Therefore,

$$\left\| |\nabla|^{\frac{1}{2}} \int_{t_0}^{t} e^{i(t-t')\Delta} F(t')\, dt' \right\|_{U^2_\Delta(J\times\mathbb{R}^2)} \lesssim \sum_{k=1}^{K} \sup_{\|v\|_{V^2_\Delta(P_k\times\mathbb{R}^2)}=1} \int_{P_k} \left\langle v,\, |\nabla|^{\frac{1}{2}} F \right\rangle dt.$$

Then the lemma follows. □

**Proposition 6.9** (Transfer Principle, Proposition 2.19 in [23]). *Let*

$$T_0 : L^2 \times \cdots \times L^2 \to L^1_{loc}$$

*be an m-linear operator. Assume that for some* $1 \leq p,\ q \leq \infty$

$$\left\| T_0(e^{it\Delta}\phi_1, \ldots, e^{it\Delta}\phi_m) \right\|_{L^p(\mathbb{R}, L^q_x)} \lesssim \prod_{i=1}^{m} \|\phi_i\|_{L^2}.$$

*Then, there exists an extension* $T : U^p_\Delta \times \cdots \times U^p_\Delta \to L^p(\mathbb{R}, L^q_x)$ *satisfying*

$$\|T(u_1, \ldots, u_m)\|_{L^p(\mathbb{R}, L^q_x)} \lesssim \prod_{i=1}^{m} \|u_i\|_{U^p_\Delta},$$

*and such that* $T(u_1, \ldots, u_m)(t, \cdot) = T_0(u_1(t), \ldots, u_m(t))(\cdot)$, *a.e.*

**Corollary 6.10.** *Suppose that under the same condition on the supports of* $\hat{u}_0$ *and* $\hat{v}_0$ *as in Lemma 2.15, i.e.* $\hat{u}_0$ *is supported on* $|\xi| \sim N$, $\hat{v}_0$ *is supported on* $|\xi| \sim M$, $M \ll N$, *for* $\frac{1}{q} + \frac{1}{q} = 1$, $2 \leq p \leq \infty$

$$\left\| (e^{it\Delta}u_0)(e^{it\Delta}v_0) \right\|_{L^p_t L^q_x(\mathbb{R}\times\mathbb{R}^2)} \lesssim \left(\frac{M}{N}\right)^{\frac{1}{p}} \|u_0\|_{L^2_x(\mathbb{R}^2)} \|v_0\|_{L^2_x(\mathbb{R}^2)}.$$

*Then if* $\hat{u}(t,\xi)$ *and* $\hat{v}(t,\xi)$ *are under the same conditions,*

$$\|uv\|_{L^p_t L^q_x(I\times\mathbb{R}^2)} \lesssim \left(\frac{M}{N}\right)^{\frac{1}{p}} \|u\|_{U^p_\Delta(I\times\mathbb{R}^2)} \|v\|_{U^p_\Delta(I\times\mathbb{R}^2)}.$$

Now we are ready to define the suitable atomic space where we are able to drive a long time Strichartz estimate.

## 6.2 $\tilde{X}_{k_0}$-norm

In this section, we are ready to define the long time Strichartz norm $\tilde{X}_{k_0}$-norm. We will fix some parameters and define some suitable small intervals, then give the construction of this $\tilde{X}_{k_0}$-norm.

### 6.2.1 Choice of $\varepsilon_1$, $\varepsilon_2$ and $\varepsilon_3$

Fix three constants $\varepsilon_1$, $\varepsilon_2$ and $\varepsilon_3$ satisfying

$$0 < \varepsilon_3 \ll \varepsilon_2 \ll \varepsilon_1 < 1. \tag{6.2.1}$$

We will add restrictions to these constants later.

Fix a non-negative integer $k_0$ and suppose that $M = 2^{k_0}$. Let $[a, b]$ be an interval such that

$$\int_a^b \int |u(t, x)|^8 \, dxdt = M, \tag{6.2.2}$$

and

$$\int_a^b N(t)\, dt = \varepsilon_3 M. \tag{6.2.3}$$

Note that we are always able to choose such interval $[a, b]$, since the scalings of (6.2.2) and (6.2.3) are different. We will choose $\varepsilon_1$ and $\varepsilon_2$ later in (6.2.4).

### 6.2.2 Definitions of small intervals: $J_l$ and $J^\alpha$

Next, we consider the following two types of partitions of $[a, b]$ ($J_l$ small intervals and $J^\alpha$ small intervals):

**Definition 6.11** ($J_l$ small intervals)**.** Let $\{J_l\}_{l=0}^{M-1}$ be a set of disjoint intervals, such that $[a, b] = \cup_{l=0}^{M-1} J_l$ and

$$\|u\|_{L^8_{t,x}(J_l \times \mathbb{R}^2)}^8 = 1.$$

**Definition 6.12** ($J^\alpha$ small intervals)**.** Let $\{J^\alpha\}_{\alpha=0}^{M-1}$ be a set of intervals such that $[a,b] = \cup_{\alpha=0}^{M-1} J^\alpha$ and

$$\int_{J^\alpha} (N(t) + \varepsilon_3 \, \|u(t)\|_{L_x^8(\mathbb{R}^2)}^8) \, dt = 2\varepsilon_3.$$

**Remark 6.13** (Differences between $J_l$ and $J^\alpha$ small intervals)**.** The total number of these two types of intervals are the same, since we have fixed $M$ at first. In fact,

$$\|u\|_{L_{t,x}^8(J_l \cup J_{l+1} \times \mathbb{R}^2)}^8 = 2.$$

So every $J^\alpha$ small interval will be covered within at most three $J_l$ small intervals. A $J_l$ interval may intersect $J_\alpha$ intervals and a $J_\alpha$ interval may intersect $J_l$ intervals. The partitions $J_\alpha$ and $J_l$ can be quite different.

**Remark 6.14.** For any Strichartz pair $(q,r)$,

$$\left\| |\nabla|^{\frac{1}{2}} u \right\|_{L_t^q L_x^r (J_l \times \mathbb{R}^2)} \lesssim_q 1 \text{ and } \left\| |\nabla|^{\frac{1}{2}} u \right\|_{L_t^q L_x^r (J^\alpha \times \mathbb{R}^2)} \lesssim_q 1.$$

*Proof of Remark 6.14.* We only prove the second inequality above, and the first one will follow similarly.

By Definition 6.12, we know

$$\|u\|_{L_{t,x}^8(J^\alpha \times \mathbb{R}^2)}^8 = \int_{J^\alpha} \|u(t)\|_{L_x^8}^8 \, dt \leq 2.$$

Then subdivide the interval $J^\alpha$ into $n$ subintervals $\{I_i\}_{i=1}^n$, such that $J^\alpha = \cup_{i=1}^n I_i$ disjoint and

$$\|u\|_{L_{t,x}^8(I_i \times \mathbb{R}^2)}^8 \leq \eta.$$

Note that $n = n(\eta)$, $n$ is independent of $J^\alpha$.

Then, by the integral equation, Hölder's inequality and Strichartz estimates, we have

$$\begin{aligned}\left\| |\nabla|^{\frac{1}{2}} u \right\|_{L_t^q L_x^r (I_i \times \mathbb{R}^2)} &\leq \|u\|_{L_t^\infty \dot{H}_x^{\frac{1}{2}} (I_i \times \mathbb{R}^2)} + C_1 \left\| |\nabla|^{\frac{1}{2}} F(u) \right\|_{L_t^{q'} L_x^{r'} (I_i \times \mathbb{R}^2)} \\ &\leq \|u\|_{L_t^\infty \dot{H}_x^{\frac{1}{2}} (I_i \times \mathbb{R}^2)} + C_1 C_2 \left\| |\nabla|^{\frac{1}{2}} u \right\|_{L_t^q L_x^r (I_i \times \mathbb{R}^2)} \|u\|^4_{L^8_{t,x} (I_i \times \mathbb{R}^2)} \\ &\leq \|u\|_{L_t^\infty \dot{H}_x^{\frac{1}{2}} (I_i \times \mathbb{R}^2)} + C_1 C_2 \left\| |\nabla|^{\frac{1}{2}} u \right\|_{L_t^q L_x^r (I_i \times \mathbb{R}^2)} \sqrt{\eta},\end{aligned}$$

where $(q', r')$ is a suitable dual Strichartz pair that makes Hölder's inequality valid, $C_1$ is the constant in Strichartz estimates and $C_2$ is the constant in the chain rule for fractional derivatives.

Choose $\eta$ small enough such that $C_1 C_2 \sqrt{\eta} < 1$, and then by a continuity argument, we have

$$\left\| |\nabla|^{\frac{1}{2}} u \right\|_{L_t^q L_x^r (I_i \times \mathbb{R}^2)} \lesssim_\eta \|u\|_{L_t^\infty \dot{H}_x^{\frac{1}{2}} (I_i \times \mathbb{R}^2)} \leq \|u\|_{L_t^\infty \dot{H}_x^{\frac{1}{2}} ([0,T] \times \mathbb{R}^2)}.$$

Next we put all $I_i$ intervals together, hence

$$\begin{aligned}\left\| |\nabla|^{\frac{1}{2}} u \right\|^q_{L_t^q L_x^r (J^\alpha \times \mathbb{R}^2)} &= \sum_{i=1}^n \left\| |\nabla|^{\frac{1}{2}} u \right\|^q_{L_t^q L_x^r (I_i \times \mathbb{R}^2)} \\ \lesssim_\eta \sum_{i=1}^n \|u\|^q_{L_t^\infty \dot{H}_x^{\frac{1}{2}} ([0,T] \times \mathbb{R}^2)} &= n \|u\|^q_{L_t^\infty \dot{H}_x^{\frac{1}{2}} ([0,T] \times \mathbb{R}^2)}.\end{aligned}$$

Therefore

$$\left\| |\nabla|^{\frac{1}{2}} u \right\|_{L_t^q L_x^r (J^\alpha \times \mathbb{R}^2)} \lesssim_\eta \sqrt[q]{n(\eta)} \|u\|_{L_t^\infty \dot{H}_x^{\frac{1}{2}} ([0,T] \times \mathbb{R}^2)} \lesssim 1.$$

□

### 6.2.3 Construction of $G_k^j$ intervals and restrictions of $\varepsilon_1$, $\varepsilon_2$ and $\varepsilon_3$

**Definition 6.15.** For an integer $0 \leq j < k_0$, $0 \leq k < 2^{k_0 - j}$, let

$$G_k^j = \cup_{\alpha = k2^j}^{(k+1)2^j - 1} J^\alpha.$$

For $j \geq k_0$ let $G_k^j = [a, b]$.

**Remark 6.16.** After defining these two types of small intervals, we recall Remark 4.7, then we have

$$\frac{1}{N(J_l)} = N(J_l) \cdot N^{-2}(J_l) \sim N(J_l)\ |J_l|\,,$$
$$\sum_{J_l \subset G_k^j} \frac{1}{N(J_l)} \sim \sum_{J_l \subset G_k^j} N(J_l)\ |J_l| \sim \int_{G_k^j} N(t)\,dt \sim 2^j \varepsilon_3.$$

Recall (4.2.2), Lemma 4.6 and Remark 4.7. It is possible for us to choose $\varepsilon_1$, $\varepsilon_2$ and $\varepsilon_3$ which satisfy (6.2.1) and also the following conditions:

$$\begin{cases} \left|\frac{d}{dt}N(t)\right| \leq \frac{N^3(t)}{\varepsilon_1^{1/2}} \qquad \Longrightarrow \left|\frac{d}{dt}\left(\frac{1}{N(t)}\right)\right| \leq \frac{N(t)}{\varepsilon_1^{1/2}} \\ \int_{|x-x(t)|\leq \frac{\varepsilon_3^{1/4}}{N(t)}} \left||\nabla|^{\frac{1}{2}} u(t,x)\right|^2 dx + \int_{|\xi|\leq \varepsilon_3^{1/4} N(t)} |\xi|\ |\hat{u}(t,\xi)|^2\ d\xi \leq \varepsilon_2^2 \\ \varepsilon_3 < \varepsilon_2^{24} \end{cases} \tag{6.2.4}$$

The smallness conditions above will be used in Chapter 7.

**Remark 6.17.** Combine Definition 6.12 and (6.2.4), then the difference of $\frac{1}{N(t)}$ on $G_\alpha^i$ is at most,

$$\int_{G_\alpha^i} \left|\frac{d}{dt}\left(\frac{1}{N(t)}\right)\right| dt \leq \int_{G_\alpha^i} \frac{N(t)}{\varepsilon_1^{1/2}} dt \leq \varepsilon_1^{-1/2} \varepsilon_3 2^i.$$

### 6.2.4 Definition of $\tilde{X}_{k_0}$ spaces and properties

For the rest of the paper, the Littlewood-Paley projection $P_{2^{-i-2}\leq \cdot \leq 2^{2^{-i+2}}} u$ will be abbreviated to $P_{2^{-i}} u$.

**Definition 6.18** ($\tilde{X}_{k_0}$ spaces)**.** For any $G_k^j \subset [a,b]$, let

$$\begin{aligned} \|u\|_{X(G_k^j\times\mathbb{R}^2)}^2 := & \sum_{i:0\leq i<j} 2^{i-j} \sum_{G_\alpha^i \subset G_k^j} \left\| |\nabla|^{\frac{1}{2}} P_{2^{-i}} u \right\|_{U_\Delta^2(G_\alpha^i \times\mathbb{R}^2)}^2 \\ & + \sum_{i:i\geq j} \left\| |\nabla|^{\frac{1}{2}} P_{2^{-i}} u \right\|_{U_\Delta^2(G_k^j \times\mathbb{R}^2)}^2 . \end{aligned} \tag{6.2.5}$$

Then define $\tilde{X}_{k_0}$ to be the supremum of (6.2.5) over all intervals $G_k^j \subset [a,b]$ with $k \leq k_0$.

$$\|u\|^2_{\tilde{X}_{k_0}([a,b]\times\mathbb{R}^2)} := \sup_{j:0\leq j\leq k_0} \sup_{G_k^j \subset [a,b]} \|u\|^2_{X(G_k^j\times\mathbb{R}^2)}.$$

Also for $0 \leq k_* \leq k_0$, let

$$\|u\|^2_{\tilde{X}_{k_*}([a,b]\times\mathbb{R}^2)} := \sup_{j:0\leq j\leq k_*} \sup_{G_k^j \subset [a,b]} \|u\|^2_{X(G_k^j\times\mathbb{R}^2)}.$$

$\|u\|_{\tilde{X}_{k_*}(G_k^j\times\mathbb{R}^2)}$, $k_* \leq j$ is defined in a similar manner:

$$\|u\|^2_{\tilde{X}_{k_*}(G_k^j\times\mathbb{R}^2)} := \sup_{i:0\leq i\leq k_*} \sup_{G_\alpha^i \subset G_k^j} \|u\|^2_{X(G_\alpha^i\times\mathbb{R}^2)}.$$

Recall that we have no Galilean transformation in our case, therefore the frequency center $\xi(t)$ is the origin.

**Remark 6.19** (Construction of $G_k^j$ intervals)**.** For example, in Figure 1, we take $k_0 = 4$, $M = 2^{k_0} = 16$, hence there are 16 small intervals: $J^0, J^1, \ldots, J^{15}$ at frequency level $2^0$. We may treat them as the building blocks in the process of constructing $X$-norm. Then in order to build a lower level $2^{-1}$, we combine every two consecutive small intervals into a larger interval, that is, at a lower level $2^{-1}$, the unions of two consecutive small intervals give us $G_0^1 = J^0 \cup J^1, \ldots, G_7^1 = J^{14} \cup J^{15}$. Note that upper indices in $G_k^j$ indicate the length of the time interval (more precisely, $2^j$ is the number of small intervals inside $G_k^j$) and the lower indices in $G_k^j$ are the locations of the time intervals. In particular, for an interval with its upper indices 0, it means that it is a $J^\alpha$-type small interval, i.e. $G_k^0 = J^k$. Then we continue moving onto the next level $2^{-2}$ to get $G_0^2, G_1^2, G_2^2$ and $G_3^2$, and even lower levels to get $G_0^3$, $G_1^3$ and $G_0^4$. We can see that $G_0^4$ is the whole interval $[a,b]$, so we stop building intervals here. For any lever lower than $2^{-4} = 2^{k_0}$, we just take the whole interval $[a,b]$.

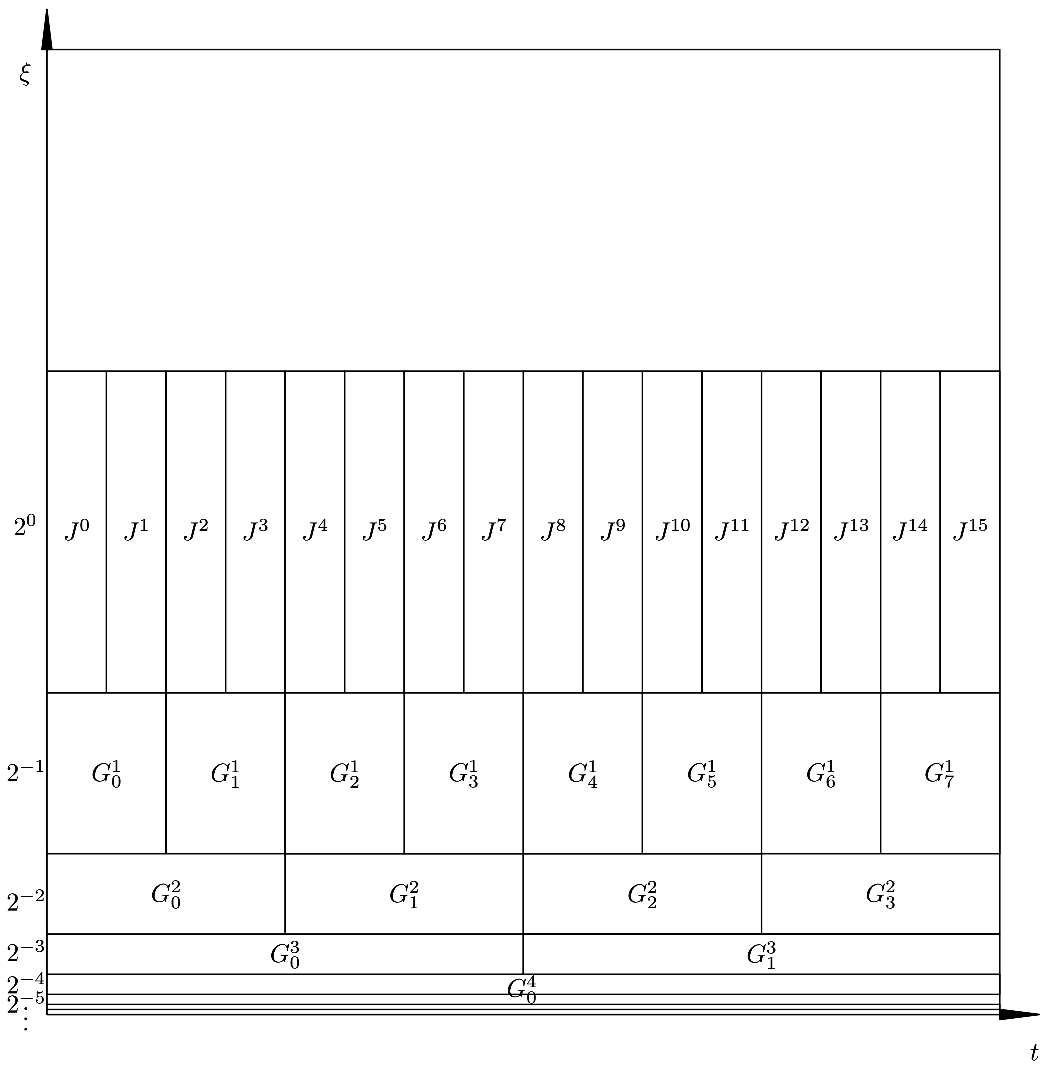


**Figure 1: $\tilde{X}_{k_0}$ norm**

**Remark 6.20** (Constriction of the $X$ norm)**.** We can see the structure of $X$ norm from the figure above. First, we localize the solution $u$ at different frequencies. Then the first term in $X$ norm is:

$$\sum_{i:0\leq i<j} 2^{i-j} \sum_{G_\alpha^i \subset G_k^j} \left\| |\nabla|^{\frac{1}{2}} P_{2^{-i}} u \right\|^2_{U^2_\Delta(G_\alpha^i \times \mathbb{R}^2)}.$$

In fact, for a fixed frequency level $2^{-i}$ (higher than $2^{-j}$), compute the average of frequency localized $U^2_\Delta$ norms on all the corresponding time intervals $G_\alpha^i$'s, then sum over all the frequencies higher than $2^{-j}$.

The second term

$$\sum_{i:i\geq j} \left\| |\nabla|^{\frac{1}{2}} P_{2^{-i}} u \right\|^2_{U^2_\Delta(G_k^j \times \mathbb{R}^2)}.$$

is the summation of frequency localized $U^2_\Delta$ norms over all the frequencies lower than $2^{-j}$ on the time interval $G_k^j$.

**Remark 6.21** ($\tilde{X}_{k_0}$ norm)**.** After we compute every $X$ norm over interval $G_k^j$, we are only two supremums away from the $\tilde{X}_{k_0}$ norm on the interval $[a,b]$. First, we fix a frequency level $2^{-j}$ and take the supremum over all the intervals at this level. This step picks out the largest candidates from each level horizontally. Then we choose the largest one from these candidates (vertically). This is the second supremum.

Now let us link the Strichartz norms of the solution to the $\tilde{X}$ norms that we just defined. We will use the following estimates in Chapter 9.

**Proposition 6.22** (Some properties of $\tilde{X}_j(G_k^j \times \mathbb{R}^2)$ norm)**.** *For $i \leq j$, let $(q,r)$ be any admissible pair, then we have:*

1. $\left\| |\nabla|^{\frac{1}{2}} P_{2^{-i}} u \right\|_{L_t^q L_x^r(G_k^j \times \mathbb{R}^2)} \lesssim_{q,r} 2^{\frac{j-i}{q}} \|u\|_{\tilde{X}_j(G_k^j \times \mathbb{R}^2)}$,
2. $\|P_{>2^{-i}} u\|_{L_t^q L_x^r(G_k^j \times \mathbb{R}^2)} \lesssim 2^{\frac{j}{2}} \|u\|_{\tilde{X}_j(G_k^j \times \mathbb{R}^2)}$,

*3.* $\left\| |\nabla|^{\frac{1}{2}} P_{\leq 2^{-j}} u \right\|_{L_t^q L_x^r(G_k^j \times \mathbb{R}^2)} \lesssim \|u\|_{\tilde{X}_j(G_k^j \times \mathbb{R}^2)}$.

*Proof.* 1. For $i \leq j$, $(q, r)$ an admissible pair, by Definition 6.2.5, we write

$$\left\| |\nabla|^{\frac{1}{2}} P_{2^{-i}} u \right\|^q_{L_t^q L_x^r(G_k^j \times \mathbb{R}^2)} = \sum_{G_\alpha^i \subset G_k^j} \left\| |\nabla|^{\frac{1}{2}} P_{2^{-i}} u \right\|^q_{L_t^q L_x^r(G_\alpha^i \times \mathbb{R}^2)}$$
$$\leq \sum_{G_\alpha^i \subset G_k^j} \left\| |\nabla|^{\frac{1}{2}} P_{2^{-i}} u \right\|^2_{L_t^q L_x^r(G_\alpha^i \times \mathbb{R}^2)} \sup_{G_\alpha^i \subset G_k^j} \left\| |\nabla|^{\frac{1}{2}} P_{2^{-i}} u \right\|^{q-2}_{L_t^q L_x^r(G_\alpha^i \times \mathbb{R}^2)}$$
$$\lesssim_{q,r} \sum_{G_\alpha^i \subset G_k^j} \left\| |\nabla|^{\frac{1}{2}} P_{2^{-i}} u \right\|^2_{U^2_\Delta(G_\alpha^i \times \mathbb{R}^2)} \sup_{G_\alpha^i \subset G_k^j} \left\| |\nabla|^{\frac{1}{2}} P_{2^{-i}} u \right\|^{q-2}_{U^2_\Delta(G_\alpha^i \times \mathbb{R}^2)}$$
$$\lesssim_{q,r} 2^{j-i} \|u\|^2_{X(G_k^j \times \mathbb{R}^2)} \|u\|^{q-2}_{\tilde{X}_j(G_k^j \times \mathbb{R}^2)} \lesssim_{q,r} 2^{j-i} \|u\|^q_{\tilde{X}_j(G_k^j \times \mathbb{R}^2)}.$$

2. For any $i \leq j$, by Littlewood-Paley theorem and the estimate in the first part, we have

$$\|P_{>2^{-i}} u\|_{L_t^q L_x^r(G_k^j \times \mathbb{R}^2)} \leq \sum_{l:l<i} 2^{\frac{l}{2}} \left\| |\nabla|^{\frac{1}{2}} P_{2^{-l}} u \right\|_{L_t^q L_x^r(G_k^j \times \mathbb{R}^2)}$$
$$\lesssim \sum_{l:l<i} 2^{\frac{l}{2}} 2^{\frac{j-l}{q}} \|u\|_{\tilde{X}_j(G_k^j \times \mathbb{R}^2)} \lesssim 2^{\frac{j}{2}} \|u\|_{\tilde{X}_j(G_k^j \times \mathbb{R}^2)}.$$

3. Then by Littlewood-Paley theorem and Minkowski inequality, we get

$$\left\| |\nabla|^{\frac{1}{2}} P_{\leq 2^{-j}} u \right\|_{L_t^q L_x^r(G_k^j \times \mathbb{R}^2)} \sim \left\| \left( \sum_{l:l \geq j} \left| |\nabla|^{\frac{1}{2}} P_{2^{-l}} u \right|^2 \right)^{\frac{1}{2}} \right\|_{L_t^q L_x^r(G_k^j \times \mathbb{R}^2)}$$
$$\lesssim \left( \sum_{l:l \geq j} \left\| |\nabla|^{\frac{1}{2}} P_{2^{-l}} u \right\|^2_{L_t^q L_x^r(G_k^j \times \mathbb{R}^2)} \right)^{\frac{1}{2}}$$
$$\lesssim_{q,r} \|u\|_{X(G_k^j \times \mathbb{R}^2)} \lesssim \|u\|_{\tilde{X}_j(G_k^j \times \mathbb{R}^2)}.$$

$\square$

# CHAPTER 7

# LONG TIME STRICHARTZ ESTIMATE

In this chapter, we recall the long time Strichartz estimate introduced in [16] and prove a long time Strichartz estimate adapted in our $\dot{H}^{\frac{1}{2}}$ setting based on the $\tilde{X}_{k_0}$ norm defined in Definition 6.18. This long time Strichartz estimate, giving us a good control of the low frequency component of the solutions, will be used to in the proof of frequency-localized Morawetz estimate in Chapter 9.

## 7.1 Long time Strichartz estimate in the mass-critical regime in two dimensions in [16]

In dimensions two, the endpoint of Strichartz estimates is false, more precisely: Let $P$ be a Fourier multiplier with symbol in $C_0^\infty(\mathbb{R}^2)$ (thus $\widehat{Pf} = \phi \hat{f}$ for some $\phi \in C_0^\infty(\mathbb{R}^2)$) which is not identically zero. Then there does not exist a constant $C > 0$ for which one has the estimate

$$\left\| e^{it\Delta} P f \right\|_{L_t^2 L_x^\infty(\mathbb{R}\times\mathbb{R}^2)} \leq C \left\| f \right\|_{L_x^2(\mathbb{R}^2)}$$

for all $f \in L_x^2(\mathbb{R}^2)$. This makes us unable to choose the regular Strichartz space. However the long time Strichartz estimate highly relies on the double endpoint

Strichartz. Therefore, we have to prove new long time Strichartz estimate adapted to two dimensions.

In two dimensional mass-critical regime, Dodson [16] defined a new space on which to compute the long time Strichartz: For any $G_k^j \subset [a,b]$,

$$\begin{aligned} \|u\|_{X(G_k^j\times\mathbb{R}^2)}^2 := \sum_{i:0\le i<j} 2^{i-j} \sum_{G_\alpha^i\subset G_k^j} \left\|P_{\xi(G_\alpha^i),2^i} u\right\|_{U_\Delta^2(G_\alpha^i\times\mathbb{R}^2)}^2 \\ + \sum_{i:i\ge j} \left\|P_{\xi(G_k^j),2^i} u\right\|_{U_\Delta^2(G_k^j\times\mathbb{R}^2)}^2 . \end{aligned} \tag{7.1.1}$$

Then define

$$\|u\|_{\tilde{X}_{k_0}([a,b]\times\mathbb{R}^2)}^2 := \sup_{j:0\le j\le k_0} \sup_{G_k^j\subset[a,b]} \|u\|_{X(G_k^j\times\mathbb{R}^2)}^2 .$$

Dodson showed that $\|u\|_{\tilde{X}_{k_0}([0,T]\times\mathbb{R}^2)} \lesssim 1$ as the new long time Strichartz estimate in dimensions two, which played a similar role as the long time Strichartz estimate in dimensions three and higher:

$$\left\|P_{|\xi-\xi(t)|>N} u\right\|_{L_t^2 L_x^{\frac{2d}{d-2}}(J\times\mathbb{R}^d)} \lesssim \left(\frac{K}{N}\right)^{\frac{1}{2}} + 1,$$

where $J$ is an interval satisfying

$$\int_J N(t)^3\, dt = K.$$

## 7.2 Long time Strichartz estimate

In contrast, we focus on the low frequency instead of high frequency, and define that for any $G_k^j \subset [a,b]$,

$$\|u\|_{X(G_k^j\times\mathbb{R}^2)}^2 := \sum_{i:0\le i<j} 2^{i-j} \sum_{G_\alpha^i\subset G_k^j} \left\| |\nabla|^{\frac{1}{2}} P_{2^{-i}} u\right\|_{U_\Delta^2(G_\alpha^i\times\mathbb{R}^2)}^2 + \sum_{i:i\ge j} \left\| |\nabla|^{\frac{1}{2}} P_{2^{-i}} u\right\|_{U_\Delta^2(G_k^j\times\mathbb{R}^2)}^2 .$$

Also define

$$\|u\|_{\tilde{X}_{k_0}([a,b]\times\mathbb{R}^2)}^2 := \sup_{j:0\le j\le k_0} \sup_{G_k^j\subset[a,b]} \|u\|_{X(G_k^j\times\mathbb{R}^2)}^2 .$$

Similarly, we want to show $\|u\|_{\tilde{X}_{k_0}([0,T]\times\mathbb{R}^2)} \lesssim 1$, which captures the essential feature in the case that if we assume the double endpoint were true:

$$\left\| |\nabla|^{\frac{1}{2}} P_{<N} u \right\|_{L_t^2 L_x^\infty(J\times\mathbb{R}^2)} \lesssim (KN)^{\frac{1}{2}} + 1,$$

where $J$ is an interval satisfying

$$\int_J N(t)\,dt = K.$$

More precisely, we want to show that

**Theorem 7.1** (Long time Strichartz estimate). *If $u$ is an almost periodic solution to* (1.6.1) *then for any $M = 2^{k_0}$, $\varepsilon_1$, $\varepsilon_2$, $\varepsilon_3$ satisfying* (6.2.4), *$\int_0^T N(t)\,dt = \varepsilon_3 M$, and $\int_0^T |u(t,x)|^8\,dxdt = M$, we have*

$$\|u\|^2_{\tilde{X}_{k_0}([0,T]\times\mathbb{R}^2)} \lesssim 1. \tag{7.2.1}$$

**Remark 7.2.** Throughout this section the implicit constant depends only on $\|u\|_{L_t^\infty \dot{H}_x^{\frac{1}{2}}}$, and not on $M$, or $\varepsilon_1$, $\varepsilon_2$, $\varepsilon_3$.

## 7.3 Proof of Theorem 7.1

We want to show that for any $0 \le j \le k$ and $G_k^j \subset [0,T]$ by induction on $k_*$,

$$\sum_{i:0\le i<j} 2^{i-j} \sum_{G_\alpha^i \subset G_k^j} \left\| |\nabla|^{\frac{1}{2}} P_{2^{-i}} u \right\|^2_{U_\Delta^2(G_\alpha^i\times\mathbb{R}^2)} + \sum_{i:i\ge j} \left\| |\nabla|^{\frac{1}{2}} P_{2^{-i}} u \right\|^2_{U_\Delta^2(G_k^j\times\mathbb{R}^2)} \lesssim 1.$$

### 7.3.1 Base case

First, we start with the base case ($k_* = 0$), that is, $\|u\|_{\tilde{X}_0([0,T]\times\mathbb{R}^2)} \lesssim 1$.

Let $J^\alpha = [a_\alpha, b_\alpha]$. By the integral equation, Strichartz estimates, duality (Proposition 6.7), $V^2_\Delta \hookrightarrow U^4_\Delta \hookrightarrow L^4_t L^4_x$ (Theorem 6.5), Definition 6.12, and Remark 6.14, we write

$$\begin{aligned}
\left\| |\nabla|^{\frac{1}{2}} u \right\|_{U^2_\Delta(J^\alpha \times \mathbb{R}^2)} &\lesssim \left\| |\nabla|^{\frac{1}{2}} u(a_\alpha) \right\|_{L^2_x} + \left\| |\nabla|^{\frac{1}{2}} \int_{a^\alpha}^t e^{i(t-t')\Delta} F(u)\, dt' \right\|_{U^2_\Delta(J^\alpha \times \mathbb{R}^2)} \\
&\lesssim \|u\|_{L^\infty_t \dot{H}^{\frac{1}{2}}_x (J^\alpha \times \mathbb{R}^2)} + \sup_{\|v\|_{V^2_\Delta}=1} \int_{J^\alpha} \left\langle v, |\nabla|^{\frac{1}{2}} F(u) \right\rangle dt \\
&\lesssim 1 + \sup_{\|v\|_{V^2_\Delta}=1} \|v\|_{L^4_t L^4_x(J^\alpha \times \mathbb{R}^2)} \left\| |\nabla|^{\frac{1}{2}} u \right\|_{L^4_t L^4_x(J^\alpha \times \mathbb{R}^2)} \|u\|^4_{L^8_t L^8_x(J^\alpha \times \mathbb{R}^2)} \\
&\lesssim 1.
\end{aligned}$$

To compute $\|u\|_{X(J^\alpha \times \mathbb{R}^2)}$, we know that at the base case level, the only small interval inside a small interval $J^\alpha$ is itself, hence we have no first term in (6.2.5). Then by Littlewood-Paley theorem, Minkowski inequality and Remark 6.14, we obtain

$$\begin{aligned}
\|u\|^2_{X(J^\alpha \times \mathbb{R}^2)} &= \sum_{i: i \geq 0} \left\| |\nabla|^{\frac{1}{2}} P_{2^{-i}} u \right\|^2_{U^2_\Delta(J^\alpha \times \mathbb{R}^2)} \\
&\lesssim \sum_{i: i \geq 0} \left( \left\| |\nabla|^{\frac{1}{2}} P_{2^{-i}} u(a_\alpha) \right\|^2_{L^2_x} + \left\| |\nabla|^{\frac{1}{2}} P_{2^{-i}} F(u) \right\|^2_{L^1_t L^2_x(J^\alpha \times \mathbb{R}^2)} \right) \\
&\lesssim \left\| |\nabla|^{\frac{1}{2}} P_{\leq 2^2} u \right\|^2_{L^\infty_t L^2_x(J^\alpha \times \mathbb{R}^2)} + \left\| |\nabla|^{\frac{1}{2}} P_{\leq 2^2} F(u) \right\|^2_{L^1_t L^2_x(J^\alpha \times \mathbb{R}^2)} \\
&\lesssim \left\| |\nabla|^{\frac{1}{2}} u \right\|^2_{L^\infty_t L^2_x(J^\alpha \times \mathbb{R}^2)} + \left\| |\nabla|^{\frac{1}{2}} u \right\|^2_{L^4_t L^4_x(J^\alpha \times \mathbb{R}^2)} \|u\|^8_{L^{\frac{16}{3}}_t L^{16}_x(J^\alpha \times \mathbb{R}^2)} \\
&\lesssim 1.
\end{aligned}$$

Note that when $k_* = 0$, the supremum of (6.2.5) over all intervals $G^j_k \subset [0,T]$ becomes the the supremum of over all small intervals $J^\alpha \subset [0,T]$. Therefore, by Definition 6.18,

$$\|u\|^2_{\tilde{X}_0([0,T] \times \mathbb{R}^2)} = \sup_{j=0} \sup_{J^\alpha \subset [0,T]} \|u\|^2_{X(J^\alpha \times \mathbb{R}^2)} \leq C(u). \tag{7.3.1}$$

Notice that $C(u)$ only depends on $\|u\|_{L^\infty_t \dot{H}^{\frac{1}{2}}_x [0,T] \times \mathbb{R}^2}$.

### 7.3.2 Induction

By Definition 6.18 and Lemma 6.6, we have

$$
\begin{aligned}
&\|u\|^2_{X(G_k^{j+1}\times\mathbb{R}^2)} \\
&= \sum_{i:0\le i<j+1} 2^{i-j-1} \sum_{G_\alpha^i\subset G_k^{j+1}} \left\| |\nabla|^{\frac{1}{2}} P_{2^{-i}} u\right\|^2_{U^2_\Delta(G_\alpha^i\times\mathbb{R}^2)} + \sum_{i:i\ge j+1} \left\| |\nabla|^{\frac{1}{2}} P_{2^{-i}} u\right\|^2_{U^2_\Delta(G_k^{j+1}\times\mathbb{R}^2)} \\
&\le \frac{1}{2} \sum_{i:0\le i<j+1} 2^{i-j} \sum_{G_\alpha^i\subset G_{2k}^{j}} \left\| |\nabla|^{\frac{1}{2}} P_{2^{-i}} u\right\|^2_{U^2_\Delta(G_\alpha^i\times\mathbb{R}^2)} + \sum_{i:i\ge j+1} \left\| |\nabla|^{\frac{1}{2}} P_{2^{-i}} u\right\|^2_{U^2_\Delta(G_{2k}^{j}\times\mathbb{R}^2)} \\
&+ \frac{1}{2} \sum_{i:0\le i<j+1} 2^{i-j} \sum_{G_\alpha^i\subset G_{2k+1}^{j}} \left\| |\nabla|^{\frac{1}{2}} P_{2^{-i}} u\right\|^2_{U^2_\Delta(G_\alpha^i\times\mathbb{R}^2)} + \sum_{i:i\ge j+1} \left\| |\nabla|^{\frac{1}{2}} P_{2^{-i}} u\right\|^2_{U^2_\Delta(G_{2k+1}^{j}\times\mathbb{R}^2)} \\
&\le 2\, \|u\|^2_{\tilde{X}_{k_*}([0,T]\times\mathbb{R}^2)}.
\end{aligned}
$$

Then for any $0\le k_*\le k_0$,

$$\|u\|^2_{\tilde{X}_{k_*+1}([0,T]\times\mathbb{R}^2)} \le 2\, \|u\|^2_{\tilde{X}_{k_*}([0,T]\times\mathbb{R}^2)}. \tag{7.3.2}$$

Therefore, by (7.3.1) and (7.3.2)

$$\|u\|^2_{\tilde{X}_{11}([0,T]\times\mathbb{R}^2)} \le 2^{11} C(u). \tag{7.3.3}$$

### 7.3.3 Bootstrap

For $j>11$ and $G_k^j\subset[0,T]$, we want to prove $\|u\|_{\tilde{X}_j(G_k^j\times\mathbb{R}^2)} \le 2^{11}C(u)$ by bootstrap argument.

First consider the terms in the $X$ norm in (6.2.5) with frequencies localized higher than $2^{-11}$, and they are bounded due to (7.3.3) (see Step 1). For the terms in the $X$ norm with frequencies localized lower than $2^{-11}$, we can use the integral equation to rewrite $u$ into the free solution and the Duhamel term, then compute the contributions of these two terms to the first term (A) and the second term (B) of the $X$ norm respectively. As a result, we have a free solution term (AF) and a Duhamel term (AD) contributing to A, and a free solution term (BF) and a Duhamel term (BD) contributing to B.

$$X(G_k^j)\text{ norm }\begin{cases}\text{frequencies higher than } 2^{-11} \text{ (Step 1)} \\ \text{frequencies lower than } 2^{-11} \begin{cases} \text{1st term } A \begin{cases} \text{free solution } AF \text{ (Step 2)} \\ \text{Duhamel term } AD \text{ (Step 4)} \end{cases} \\ \text{2nd term } B \begin{cases} \text{free solution } BF \text{ (Step 3)} \\ \text{Duhamel term } BD \text{ (Step 4)} \end{cases} \end{cases} \end{cases}$$

We will consider the terms with frequencies higher than $2^{-11}$ in Step 1. And estimates the free solution terms in A and B in Step 2 and Step 3 respectively. In this proof, the hardest part is to estimate AD and BD. We will bound them in Step 4.

It is worth mentioning that in Step 4, we treat two different types of $G_\alpha^i$ intervals in two cases (see the classification of these two cases in Step 4). For case 1, we compute directly, while for case 2, we will prove a bootstrap argument Proposition 7.3. In the proof we decompose the nonlinear term $|u|^4 u$ into different frequencies and consider them in Lemma 7.4 and Lemma 7.5.

$$\text{Step 4 } (AD \,\&\, BD) \to \begin{cases} \text{case 1} \\ \text{case 2} \to \text{Proposition 7.3} \begin{cases} \text{Lemma 7.4} \\ \text{Lemma 7.5} \end{cases} \end{cases}$$

With the structure of the proof in hand, we are now ready to start the proof.

**Step 1: Frequency higher than $2^{-11}$**

Note that (7.3.3) implies that for any $j > 11$ and $G_k^j \subset [0, T]$,

$$\sum_{i:0\leq i\leq 11} 2^{i-j} \sum_{G_\alpha^i \subset G_k^j} \left\| \, |\nabla|^{\frac{1}{2}} P_{2^{-i}} u \right\|^2_{U^2_\Delta(G_\alpha^i \times \mathbb{R}^2)} \leq 2^{11} C(u).$$

To see this, note that $G_j^k$ overlaps $2^{j-11}$ intervals $G_\beta^{11}$ and $G_\beta^{11}$ overlaps $2^{11-i}$ intervals $G_\alpha^i$. So by Fubini-Tonelli theorem, we have

$$
\begin{aligned}
&\sum_{i:0\leq i\leq 11} 2^{i-j} \sum_{G_\alpha^i\subset G_k^j} \left\| |\nabla|^{\frac{1}{2}} P_{2^{-i}} u\right\|^2_{U^2_\Delta(G_\alpha^i\times\mathbb{R}^2)} \\
&= 2^{11-j} \sum_{i:0\leq i\leq 11} 2^{i-11} \sum_{G_\alpha^i\subset G_k^j} \left\| |\nabla|^{\frac{1}{2}} P_{2^{-i}} u\right\|^2_{U^2_\Delta(G_\alpha^i\times\mathbb{R}^2)} \\
&= 2^{11-j} \sum_{i:0\leq i\leq 11} 2^{i-11} \sum_{G_\beta^{11}\subset G_k^j} \sum_{G_\alpha^i\subset G_\beta^{11}} \left\| |\nabla|^{\frac{1}{2}} P_{2^{-i}} u\right\|^2_{U^2_\Delta(G_\alpha^i\times\mathbb{R}^2)} \\
&= 2^{11-j} \sum_{G_\beta^{11}\subset G_k^j} \sum_{i:0\leq i\leq 11} 2^{i-11} \sum_{G_\alpha^i\subset G_\beta^{11}} \left\| |\nabla|^{\frac{1}{2}} P_{2^{-i}} u\right\|^2_{U^2_\Delta(G_\alpha^i\times\mathbb{R}^2)} \\
&\leq 2^{11-j} \sum_{G_\beta^{11}\subset G_k^j} \|u\|^2_{\tilde{X}_{11}([0,T]\times\mathbb{R}^2)} = \|u\|^2_{\tilde{X}_{11}([0,T]\times\mathbb{R}^2)} \leq 2^{11} C(u).
\end{aligned}
$$

**Step 2: Free solution term in A**

Fix $k_0$, $12 \leq j \leq k_0$, and $G_k^j \subset [0,T]$. For $11 \leq i < j$, Duhamel's principle implies

$$
\begin{aligned}
&\left\| |\nabla|^{\frac{1}{2}} P_{2^{-i}} u\right\|_{U^2_\Delta(G_\alpha^i\times\mathbb{R}^2)} \\
&\lesssim \left\| |\nabla|^{\frac{1}{2}} P_{2^{-i}} u(t_\alpha^i)\right\|_{L_x^2(\mathbb{R}^2)} + \left\| |\nabla|^{\frac{1}{2}} \int_{t_\alpha^i}^t e^{i(t-t')\Delta} P_{2^{-i}} F(u)\,dt'\right\|_{U^2_\Delta(G_\alpha^i\times\mathbb{R}^2)}.
\end{aligned}
$$

Choose $t_\alpha^i$ satisfying

$$
\left\| |\nabla|^{\frac{1}{2}} P_{2^{-i}} u(t_\alpha^i)\right\|_{L_x^2(\mathbb{R}^2)} = \inf_{t\in G_\alpha^i} \left\| |\nabla|^{\frac{1}{2}} P_{2^{-i}} u(t)\right\|_{L_x^2(\mathbb{R}^2)}. \tag{7.3.4}
$$

Then by Definition 6.12, Fubini-Tonelli theorem and (7.3.4),

$$
\begin{aligned}
&\sum_{i:11\leq i<j} 2^{i-j} \sum_{G_\alpha^i\subset G_k^j} \left\| |\nabla|^{\frac{1}{2}} P_{2^{-i-2}\leq\cdot\leq 2^{-i+2}} u(t_\alpha^i)\right\|^2_{L_x^2(\mathbb{R}^2)} \\
&= \frac{1}{2\varepsilon_3} 2^{-j} \sum_{i:11\leq i<j} \sum_{G_\alpha^i\subset G_k^j} \left\| |\nabla|^{\frac{1}{2}} P_{2^{-i}} u(t_\alpha^i)\right\|^2_{L_x^2(\mathbb{R}^2)} \int_{G_\alpha^i} \left(\varepsilon_3 \|u(t)\|^8_{L_x^8(\mathbb{R}^2)} + N(t)\right) dt \\
&\leq \frac{1}{2\varepsilon_3} 2^{-j} \sum_{G_\alpha^i\subset G_k^j} \int_{G_\alpha^i} \sum_{i:11\leq i<j} \left\| |\nabla|^{\frac{1}{2}} P_{2^{-i}} u(t)\right\|^2_{L_x^2(\mathbb{R}^2)} \left(\varepsilon_3 \|u(t)\|^8_{L_x^8(\mathbb{R}^2)} + N(t)\right) dt
\end{aligned}
$$

$$\lesssim \frac{1}{2\varepsilon_3} 2^{-j} \left\| |\nabla|^{\frac{1}{2}} u \right\|^2_{L^\infty_t L^2_x([0,T]\times\mathbb{R}^2)} \int_{G^j_k} \left( \varepsilon_3 \, \|u(t)\|^8_{L^8_x(\mathbb{R}^2)} + N(t) \right) dt$$

$$\lesssim \frac{1}{2\varepsilon_3} 2^{-j} \left\| |\nabla|^{\frac{1}{2}} u \right\|^2_{L^\infty_t L^2_x([0,T]\times\mathbb{R}^2)} \varepsilon_3 2^j \lesssim 1.$$

<u>**Step 3: Free solution term in B**</u>

For $i \geq j$ simply take $t_0$, where $t_0$ is a fixed element of $G^j_k$, say the left endpoint. Then

$$\sum_{i: i\geq j} \left\| |\nabla|^{\frac{1}{2}} P_{2^{-i}} e^{it\Delta} u(t_0) \right\|^2_{U^2_\Delta(G^j_k \times \mathbb{R}^2)}$$
$$\lesssim \sum_{i: i\geq j} \left\| |\nabla|^{\frac{1}{2}} P_{2^{-i}} u(t_0) \right\|^2_{L^2_x(\mathbb{R}^2)} \lesssim \left\| |\nabla|^{\frac{1}{2}} u(t_0) \right\|^2_{L^2_x(\mathbb{R}^2)} \lesssim 1.$$

Therefore, from Step 2 and Step 3, we have the following bound for the free solution terms AF and BF:

$$\sum_{i: 0 \leq i < j} 2^{i-j} \sum_{G^i_\alpha \subset G^j_k} \left\| |\nabla|^{\frac{1}{2}} P_{2^{-i}} u \right\|^2_{L^2_x(\mathbb{R}^2)} + \sum_{i: i \geq j} \left\| |\nabla|^{\frac{1}{2}} P_{2^{-i}} u \right\|^2_{L^2_x(\mathbb{R}^2)} \lesssim 1.$$

Thanks to the calculation above, we have

$$\|u\|^2_{X(G^j_k \times \mathbb{R}^2)} \lesssim 1 + \sum_{i: 11 \leq i < j} 2^{i-j} \sum_{G^i_\alpha \subset G^j_k} \left\| |\nabla|^{\frac{1}{2}} \int_{t^i_\alpha}^t e^{i(t-t')\Delta} P_{2^{-i}} F(u)\, dt' \right\|^2_{U^2_\Delta(G^i_\alpha \times \mathbb{R}^2)}$$
$$+ \sum_{i: i \geq j} \left\| |\nabla|^{\frac{1}{2}} \int_{t_0}^t e^{i(t-t')\Delta} P_{2^{-i}} F(u)\, dt' \right\|^2_{U^2_\Delta(G^j_k \times \mathbb{R}^2)}.$$

<u>**Step 4: Duhamel terms in A and B**</u>

For these two terms, we consider the following two cases

$$\begin{cases} \text{Case } 1: \ G^i_\alpha : G^i_\alpha \subset G^j_k \text{ and } N(G^i_\alpha) \leq \varepsilon_3^{-1/2} 2^{-i}; \text{ and } G^j_k : N(G^j_k) \leq \varepsilon_3^{-1/2} 2^{-j} \\ \text{Case } 2: \ G^i_\alpha : G^i_\alpha \subset G^j_k \text{ and } N(G^i_\alpha) \geq \varepsilon_3^{-1/2} 2^{-i}; \text{ and } G^j_k : N(G^j_k) \geq \varepsilon_3^{-1/2} 2^{-j} \end{cases}$$

- **Case 1 in Step 4**

There are at most two small intervals, call them $J_1$ and $J_2$, that intersect $G_k^j$ but are not contained in $G_k^j$. Therefore, by Minkowski inequality and Remark 6.14

$$
\begin{aligned}
&\sum_{i:11\leq i<j} 2^{i-j} \sum_{G_\alpha^i\subset G_k^j} \left\| |\nabla|^{\frac{1}{2}} F(u)\right\|^2_{L_t^1 L_x^2(G_\alpha^i\cap(J_1\cup J_2)\times\mathbb{R}^2)} \\
&\lesssim \sum_{i:11\leq i<j} 2^{i-j} \left\| |\nabla|^{\frac{1}{2}} F(u)\right\|^2_{L_t^1 L_x^2(J_1\cup J_2\times\mathbb{R}^2)} \\
&\lesssim \sum_{i:11\leq i<j} 2^{i-j} \left\| |\nabla|^{\frac{1}{2}} u\right\|^2_{L_t^4 L_x^4(J_1\cup J_2\times\mathbb{R}^2)} \|u\|^8_{L_t^{\frac{16}{3}} L_x^{16}(J_1\cup J_2\times\mathbb{R}^2)} \lesssim 1.
\end{aligned}
\tag{7.3.5}
$$

Next observe that $N(G_\alpha^i) \leq \varepsilon_3^{-1/2} 2^{-i}$ implies that $N(t) \leq \varepsilon_3^{-1/2} 2^{-i+1}$ for all $t \in G_\alpha^i$. In fact, by Remark 6.17 the difference of $\frac{1}{N(t)}$ on $G_\alpha^i$ is at most $\varepsilon_1^{-1/2}\varepsilon_3 2^i$, hence

$$
N(t) \leq N(G_\alpha^i) + \varepsilon_1^{-1/2}\varepsilon_3 2^i \leq \varepsilon_3^{-1/2} 2^{-i+1} \quad \text{ for all } t \in G_\alpha^i.
$$

Now by Definition 6.11 and Definition 6.12, the number of $J_l$ intervals inside $G_\alpha^i$ is

$$
\#\{J_l : J_l \subset G_\alpha^i\} \sim \int_{\cup J_l} N(t)^2\, dt \lesssim \int_{G_\alpha^i} N(t)^2\, dt \leq \varepsilon_3^{-1/2} 2^{-i+1} \int_{G_\alpha^i} N(t)\, dt \leq \varepsilon_3^{1/2} 2^2.
$$

This implies that the number of intervals $J_l$ such that $N(t) \leq \varepsilon_3^{-1/2} 2^{-i+1}$ for all $t \in G_\alpha^i$ is finite and does not depend on $G_\alpha^i$. By (7.3.5), Fubini-Tonelli theorem and Remark 6.16, we have

$$
\begin{aligned}
&\sum_{i:11\leq i<j} 2^{i-j} \sum_{\substack{G_\alpha^i\subset G_k^j \\ N(G_\alpha^i)\leq \varepsilon_3^{1/2} 2^{-i}}} \left\| |\nabla|^{\frac{1}{2}} \int_{t_\alpha^i}^t e^{i(t-t')\Delta} P_{2^{-i}} F(u)\, dt' \right\|^2_{U_\Delta^2(G_\alpha^i\times\mathbb{R}^2)} \\
&\lesssim \sum_{i:11\leq i<j} 2^{i-j} \sum_{G_\alpha^i\subset G_k^j} \left\| |\nabla|^{\frac{1}{2}} F(u)\right\|^2_{L_t^1 L_x^2(G_\alpha^i\cap(J_1\cup J_2)\times\mathbb{R}^2)} \\
&+ \sum_{i:11\leq i<j} 2^{i-j} \sum_{\substack{J_l\subset G_k^j \\ N(J_l)\leq \varepsilon_3^{-1/2} 2^{-i+1}}} \left\| |\nabla|^{\frac{1}{2}} F(u)\right\|^2_{L_t^1 L_x^2(J_l\times\mathbb{R}^2)} \\
&\lesssim 1 + \sum_{J_l\subset G_k^j} \sum_{\substack{i:11\leq i<j \\ 2^i\leq 2\varepsilon_3^{-1/2} N^{-1}(J_l)}} 2^{i-j} \lesssim 1 + 2^{-j} \sum_{J_l\subset G_k^j} \varepsilon_3^{-1/2} \frac{1}{N(J_l)} \lesssim 1.
\end{aligned}
$$

Similarly, if $N(G_k^j) \leq \varepsilon_3^{-1/2} 2^{-j}$, then $N(t) \leq \varepsilon_3^{-1/2} 2^{-j+1}$ for all $t \in G_k^j$. This implies that

$$\|u\|_{L_t^8 L_x^8(G_k^j \times \mathbb{R}^2)}^8 \sim \int_{G_k^j} N(t)^2 \, dt \leq \varepsilon_3^{-1/2} 2^{-j+1} \int_{G_k^j} N(t) \, dt \leq \varepsilon_3^{-1/2} 2^{-j+1} \varepsilon_3 2^j \lesssim 1.$$

Hence, for any admissible pair $(q, r)$,

$$\left\| \, |\nabla|^{\frac{1}{2}} \, u \right\|_{L_t^q L_x^r(G_k^j \times \mathbb{R}^2)} \lesssim 1. \tag{7.3.6}$$

Therefore, by Minkowski inequality, Hölder's inequality and (7.3.6)

$$\begin{aligned}
&\sum_{\substack{i: i \geq j \\ N(G_k^j) \leq \varepsilon_3^{-1/2} 2^{-j}}} \left\| \, |\nabla|^{\frac{1}{2}} \, P_{2^{-i}} F(u) \right\|_{L_t^1 L_x^2(G_k^j \times \mathbb{R}^2)}^2 \\
\lesssim & \left\| \, |\nabla|^{\frac{1}{2}} \, P_{\leq 2^{-j+2}} F(u) \right\|_{L_t^1 L_x^2(G_k^j \times \mathbb{R}^2)}^2 \lesssim \left\| \, |\nabla|^{\frac{1}{2}} \, u \right\|_{L_t^4 L_x^4(G_k^j \times \mathbb{R}^2)}^2 \|u\|_{L_t^{\frac{16}{3}} L_x^{16}(G_k^j \times \mathbb{R}^2)}^8 \lesssim 1.
\end{aligned}$$

Then all the computations above yield

$$\begin{aligned}
&\|u\|_{X(G_k^j \times \mathbb{R}^2)}^2 \\
\lesssim & \, 1 + \sum_{i; 11 \leq i < j} 2^{i-j} \sum_{\substack{G_\alpha^i \subset G_k^j \\ N(G_\alpha^i) \geq \varepsilon_3^{-1/2} 2^{-i}}} \left\| \, |\nabla|^{\frac{1}{2}} \int_{t_\alpha^i}^t e^{i(t-t')\Delta} P_{2^{-i}} F(u) \, dt' \right\|_{U_\Delta^2(G_\alpha^i \times \mathbb{R}^2)}^2 \\
+ & \sum_{\substack{i: i \geq j \\ N(G_k^j) \geq \varepsilon_3^{-1/2} 2^{-j}}} \left\| \, |\nabla|^{\frac{1}{2}} \int_{t_0}^t e^{i(t-t')\Delta} P_{2^{-i}} F(u) \, dt' \right\|_{U_\Delta^2(G_k^j \times \mathbb{R}^2)}^2 .
\end{aligned} \tag{7.3.7}$$

• **Case 2 in Step 4**

From now on, we take the intervals $G_\alpha^i$ with $N(G_\alpha^i) \geq \varepsilon_3^{-1/2} 2^{-i}$ and intervals $G_k^j$ with $N(G_k^j) \geq \varepsilon_3^{-1/2} 2^{-j}$.

Continue from (7.3.7). To close the proof of the long time Strichartz, it suffices to prove the following proposition:

**Proposition 7.3.**

$$\begin{aligned}
&\sum_{i:11\leq i<j} 2^{i-j} \sum_{\substack{G_\alpha^i\subset G_k^j\\ N(G_\alpha^i)\geq \varepsilon_3^{-1/2}2^{-i}}} \left\| |\nabla|^{\frac{1}{2}} \int_{t_\alpha^i}^t e^{i(t-t')\Delta} P_{2^{-i}} F(u)\, dt' \right\|_{U_\Delta^2(G_\alpha^i\times\mathbb{R}^2)}^2 \\
&+ \sum_{\substack{i:i\geq j\\ N(G_k^j)\geq \varepsilon_3^{-1/2}2^{-j}}} \left\| |\nabla|^{\frac{1}{2}} \int_{t_0}^t e^{i(t-t')\Delta} P_{2^{-i}} F(u)\, dt' \right\|_{U_\Delta^2(G_k^j\times\mathbb{R}^2)}^2 \\
&\lesssim \varepsilon_2^4 \, \|u\|_{\tilde{X}_j([0,T]\times\mathbb{R}^2)}^6 + \varepsilon_2^2 \, \|u\|_{\tilde{X}_j([0,T]\times\mathbb{R}^2)}^8 .
\end{aligned} \tag{7.3.8}$$

Indeed, assuming that Proposition 7.3 is true, we can run a bootstrap argument. Suppose

$$\|u\|_{\tilde{X}_{k_*}([0,T]\times\mathbb{R}^2)}^2 \leq C_0. \tag{7.3.9}$$

Then by (7.3.2)

$$\|u\|_{\tilde{X}_{k_*+1}([0,T]\times\mathbb{R}^2)}^2 \leq 2C_0.$$

Then by (7.3.7) and Proposition 7.3, we have

$$\|u\|_{\tilde{X}_{k_*+1}([0,T]\times\mathbb{R}^2)}^2 \leq C(u) \, \left(1+\varepsilon_2^4(2C_0)^3+\varepsilon_2^2(2C_0)^4\right).$$

Taking $C_0 = 2^{11}C(u)$ and $\varepsilon_2 > 0$ sufficiently small, it implies that (7.3.9) holds for $k_* = 11$. The bootstrap argument is closed, since we obtain

$$\|u\|_{\tilde{X}_{k_*+1}([0,T]\times\mathbb{R}^2)}^2 \leq C_0.$$

Hence the long time Strichartz estimates follow by (7.3.1), (7.3.2) and induction on $k_*$.

Now we are left to prove Proposition 7.3.

*Proof of Proposition 7.3.* We first write $F(u)$ into the Littlewood-Paley decomposition. Without loss of generality, we assume that $2^{-n_1} \geq 2^{-n_2} \geq 2^{-n_3} \geq 2^{-n_4} \geq 2^{-n_5}$, i.e. $n_1 \leq n_2 \leq n_3 \leq n_4 \leq n_5$. In the proof, since we utilize Lebesgue norms $L_x^q L_x^r$ and Lebesgue norms do not see the difference between $u$ and $\bar{u}$, i.e, $\|P_N u\|_{L_x^q L_x^r} = \|P_N \bar{u}\|_{L_x^q L_x^r}$, it is safe for us to write

$$P_{2^{-i}} F(u) \simeq P_{2^{-i}} \left( \sum_{n_1 \leq n_2 \leq n_3 \leq n_4 \leq n_5} f \right),$$

where $f = (P_{2^{-n_1}} u)(P_{2^{-n_2}} u)(P_{2^{-n_3}} u)(P_{2^{-n_4}} u)(P_{2^{-n_5}} u)$.

Then we compare the largest frequency $2^{-n_1}$ with $2^{-i}$:

- If $2^{-i} \gg 2^{-n_1}$, it is impossible since $P_{2^{-i}}((P_{\leq 2^{-i-7}} u)^5) = 0$;
- If $2^{-i} \sim 2^{-n_1}$ (i.e. $2^{-n_1-7} \leq 2^{-i} \leq 2^{-n_1+7}$), then this is our first case:

$$2^{-i} \sim 2^{-n_1} \geq 2^{-n_2} \geq 2^{-n_3} \geq 2^{-n_4} \geq 2^{-n_5};$$

- If $2^{-i} \ll 2^{-n_1}$, then $2^{-n_2}$ must have a similar frequency to $2^{-n_1}$, otherwise the projection to $2^{-i}$ frequency of this term will be zero. Hence the second case becomes

$$2^{-i} \leq 2^{-n_1} \sim 2^{-n_2}; 2^{-n_1} \geq 2^{-n_2} \geq 2^{-n_3} \geq 2^{-n_4} \geq 2^{-n_5}.$$

Therefore, it is sufficient to consider the following two terms:

$$P_{2^{-i}} F(u) \simeq P_{2^{-i}} \left( \sum_{\substack{i \sim n_1; \\ n_1 \leq n_2 \leq n_3 \leq n_4 \leq n_5}} f \right) + P_{2^{-i}} \left( \sum_{\substack{i > n_1; n_1 \sim n_2; \\ n_1 \leq n_2 \leq n_3 \leq n_4 \leq n_5}} f \right) := \tilde{F}_1 + \tilde{F}_2.$$

Next, we compute the contributions of $\tilde{F}_1$ and $\tilde{F}_2$ to (7.3.8) in Lemma 7.4 and Lemma 7.5 respectively.

**Lemma 7.4** (Contribution of $\tilde{F}_1$)**.** *For a fixed $G_k^j \subset [0,T]$, $j > 11$,*

$$\sum_{i:11\le i<j} 2^{i-j} \sum_{\substack{G_\alpha^i \subset G_k^j \\ N(G_\alpha^i) \ge \varepsilon_3^{-1/2} 2^{-i}}} \left\| |\nabla|^{\frac{1}{2}} \int_{t_\alpha^i}^t e^{i(t-t')\Delta} \tilde{F}_1 \, dt' \right\|_{U_\Delta^2(G_\alpha^i \times \mathbb{R}^2)}^2 \tag{7.3.10}$$

$$+ \sum_{\substack{i:i\ge j \\ N(G_k^j)\ge \varepsilon_3^{-1/2} 2^{-j}}} \left\| |\nabla|^{\frac{1}{2}} \int_{t_0}^t e^{i(t-t')\Delta} \tilde{F}_1 \, dt' \right\|_{U_\Delta^2(G_k^j \times \mathbb{R}^2)}^2 \tag{7.3.11}$$

$$\lesssim \varepsilon_2^4 \, \|u\|_{\tilde{X}_j([0,T]\times\mathbb{R}^2)}^6 . \tag{7.3.12}$$

*Proof.* By Proposition 6.7,

$$\left\| |\nabla|^{\frac{1}{2}} \int_{t_\alpha^i}^t e^{i(t-t')\Delta} \tilde{F}_1 \, dt' \right\|_{U_\Delta^2(G_\alpha^i \times \mathbb{R}^2)} \lesssim \sup_{\|v\|_{V_\Delta^2(G_\alpha^i\times\mathbb{R}^2)}=1} \int_{G_\alpha^i} \left\langle v, \, |\nabla|^{\frac{1}{2}} \, \tilde{F}_1 \right\rangle dt$$

$$\simeq \sup_{\|v\|_{V_\Delta^2(G_\alpha^i\times\mathbb{R}^2)}=1} \int_{G_\alpha^i} \left\langle |\nabla|^{\frac{1}{2}} P_{2^{-i}} v, \sum_{\substack{i\sim n_1; \\ n_1\le n_2\le n_3\le n_4\le n_5}} f \right\rangle dt$$

$$\lesssim \sum_{\substack{i\sim n_1; \\ n_1\le n_2\le n_3\le n_4\le n_5}} \sup_{\|v\|_{V_\Delta^2(G_\alpha^i\times\mathbb{R}^2)}=1} \int_{G_\alpha^i} \left\langle |\nabla|^{\frac{1}{2}} P_{2^{-i}} v, f \right\rangle dt. \tag{7.3.13}$$

Lemma 4.9 implies that $N(t)$ is bounded on $G_\alpha^i$. Recall $N(G_\alpha^i) = \inf_{t\in G_\alpha^i} N(t)$, $2^{-n_5} \le 2^{-n_4} \le \varepsilon_3^{1/2} N(t)$ and (6.2.4). Then we know

$$\begin{aligned} \|P_{2^{-n_4}} u\|_{L_t^\infty L_x^4(G_\alpha^i\times\mathbb{R}^2)} &\lesssim \varepsilon_2, \\ \|P_{2^{-n_5}} u\|_{L_t^\infty L_x^4(G_\alpha^i\times\mathbb{R}^2)} &\lesssim \varepsilon_2. \end{aligned} \tag{7.3.14}$$

Then using Hölder's inequality, Bernstein's inequality, we write

$$\int_{G_\alpha^i} \left\langle |\nabla|^{\frac{1}{2}} P_{2^{-i}} v, f \right\rangle dt$$

$$= \int_{G_\alpha^i} \left\langle |\nabla|^{\frac{1}{2}} P_{2^{-i}} v, (P_{2^{-n_1}} u)(P_{2^{-n_2}} u)(P_{2^{-n_3}} u)(P_{2^{-n_4}} u)(P_{2^{-n_5}} u) \right\rangle dt$$

$$\lesssim 2^{-\frac{i}{2}} \, \|v\|_{L_t^4 L_x^4(G_\alpha^i\times\mathbb{R}^2)} \, \|P_{2^{-n_1}} u\|_{L_t^4 L_x^4(G_\alpha^i\times\mathbb{R}^2)} \, \|P_{2^{-n_2}} u\|_{L_t^4 L_x^4(G_\alpha^i\times\mathbb{R}^2)} \tag{7.3.15}$$

$$\times \, \|P_{2^{-n_3}} u\|_{L_t^4 L_x^4(G_\alpha^i\times\mathbb{R}^2)} \, \|P_{2^{-n_4}} u\|_{L_t^\infty L_x^\infty(G_\alpha^i\times\mathbb{R}^2)} \, \|P_{2^{-n_5}} u\|_{L_t^\infty L_x^\infty(G_\alpha^i\times\mathbb{R}^2)} .$$

By $V^2_\Delta \hookrightarrow U^4_\Delta$ (Theorem 6.5), Bernstein's inequality and (7.3.14)

$$\begin{aligned}
(7.3.15) \lesssim\ & 2^{-\frac{i}{2}} \left\| v \right\|_{V^2_\Delta(G^i_\alpha \times \mathbb{R}^2)} 2^{\frac{n_1}{2}} \left\| |\nabla|^{\frac{1}{2}} P_{2^{-n_1}} u \right\|_{L^4_t L^4_x(G^i_\alpha \times \mathbb{R}^2)} \\
& \times 2^{\frac{n_2}{2}} \left\| |\nabla|^{\frac{1}{2}} P_{2^{-n_2}} u \right\|_{L^4_t L^4_x(G^i_\alpha \times \mathbb{R}^2)} 2^{\frac{n_3}{2}} \left\| |\nabla|^{\frac{1}{2}} P_{2^{-n_3}} u \right\|_{L^4_t L^4_x(G^i_\alpha \times \mathbb{R}^2)} \\
& \times 2^{-\frac{n_4}{2}} \left\| P_{2^{-n_4}} u \right\|_{L^\infty_t L^4_x(G^i_\alpha \times \mathbb{R}^2)} 2^{-\frac{n_5}{2}} \left\| P_{2^{-n_5}} u \right\|_{L^\infty_t L^4_x(G^i_\alpha \times \mathbb{R}^2)} \\
\lesssim\ & 2^{\frac{n_1}{2} - \frac{i}{2} + \frac{n_2}{2} + \frac{n_3}{2} - \frac{n_4}{2} - \frac{n_5}{2}} \left\| |\nabla|^{\frac{1}{2}} P_{2^{-n_1}} u \right\|_{U^2_\Delta(G^i_\alpha \times \mathbb{R}^2)} \left\| |\nabla|^{\frac{1}{2}} P_{2^{-n_2}} u \right\|_{U^2_\Delta(G^i_\alpha \times \mathbb{R}^2)} \\
& \times \left\| |\nabla|^{\frac{1}{2}} P_{2^{-n_3}} u \right\|_{U^2_\Delta(G^i_\alpha \times \mathbb{R}^2)} \varepsilon_2^2.
\end{aligned}$$

Therefore (7.3.13) becomes

$$\begin{aligned}
(7.3.13) \lesssim & \sum_{\substack{n_1 \sim i; \\ n_1 \leq n_2 \leq n_3 \leq n_4 \leq n_5}} 2^{\frac{n_1}{2} - \frac{i}{2} + \frac{n_2}{2} + \frac{n_3}{2} - \frac{n_4}{2} - \frac{n_5}{2}} \left\| |\nabla|^{\frac{1}{2}} P_{2^{-n_1}} u \right\|_{U^2_\Delta(G^i_\alpha \times \mathbb{R}^2)} \\
& \times \left\| |\nabla|^{\frac{1}{2}} P_{2^{-n_2}} u \right\|_{U^2_\Delta(G^i_\alpha \times \mathbb{R}^2)} \left\| |\nabla|^{\frac{1}{2}} P_{2^{-n_3}} u \right\|_{U^2_\Delta(G^i_\alpha \times \mathbb{R}^2)} \varepsilon_2^2. \qquad (7.3.16)
\end{aligned}$$

We first take the sum over $n_2$, $n_3$, $n_4$ and $n_5$ and the component in (7.3.16) depending on these frequencies. By Cauchy-Schwarz inequality and Definition 6.18, we obtain

$$\begin{aligned}
& \sum_{\substack{n_2, n_3, n_4, n_5: \\ i \leq n_2 \leq n_3 \leq n_4 \leq n_5}} 2^{\frac{n_2}{2} + \frac{n_3}{2} - \frac{n_4}{2} - \frac{n_5}{2}} \left\| |\nabla|^{\frac{1}{2}} P_{2^{-n_2}} u \right\|_{U^2_\Delta(G^i_\alpha \times \mathbb{R}^2)} \left\| |\nabla|^{\frac{1}{2}} P_{2^{-n_3}} u \right\|_{U^2_\Delta(G^i_\alpha \times \mathbb{R}^2)} \\
\lesssim & \sum_{\substack{n_2, n_3: \\ i \leq n_2 \leq n_3}} 2^{\frac{n_2}{2} - \frac{n_3}{2}} \left\| |\nabla|^{\frac{1}{2}} P_{2^{-n_2}} u \right\|_{U^2_\Delta(G^i_\alpha \times \mathbb{R}^2)} \left\| |\nabla|^{\frac{1}{2}} P_{2^{-n_3}} u \right\|_{U^2_\Delta(G^i_\alpha \times \mathbb{R}^2)} \\
\lesssim & \left( \sum_{n_2 : n_2 \geq i} \left\| |\nabla|^{\frac{1}{2}} P_{2^{-n_2}} u \right\|^2_{U^2_\Delta(G^i_\alpha \times \mathbb{R}^2)} \right)^{\frac{1}{2}} \left( \sum_{n_3 : n_3 \geq i} \left\| |\nabla|^{\frac{1}{2}} P_{2^{-n_3}} u \right\|^2_{U^2_\Delta(G^i_\alpha \times \mathbb{R}^2)} \right)^{\frac{1}{2}} \\
\lesssim & \left\| u \right\|^2_{\tilde{X}_i([0,T] \times \mathbb{R}^2)} \leq \left\| u \right\|^2_{\tilde{X}_j([0,T] \times \mathbb{R}^2)}.
\end{aligned}$$

Then we are left with the sum over $n_1$ in (7.3.16). Due to the fact that the sum over $n_1$ is a finite sum, we can write

$$\begin{aligned}
&\sum_{n_1 : n_1 \sim i} 2^{\frac{n_1}{2} - \frac{i}{2}} \left\| |\nabla|^{\frac{1}{2}} P_{2^{-n_1}} u \right\|_{U^2_\Delta (G^i_\alpha \times \mathbb{R}^2)} \\
&\lesssim \left( \sum_{n_1 : n_1 \sim i} 2^{n_1 - i} \left\| |\nabla|^{\frac{1}{2}} P_{2^{-n_1}} u \right\|^2_{U^2_\Delta (G^i_\alpha \times \mathbb{R}^2)} \right)^{\frac{1}{2}} . \qquad (7.3.17)
\end{aligned}$$

For $n_1$ satisfying $n_1 \sim i$ and $n_1 \le i$, we can decompose the $U^2_\Delta$ norm on $G^i_\alpha$ into the $U^2_\Delta$ norms on subintervals $G^{n_1}_\beta$ using Lemma 6.6, therefore

$$\begin{aligned}
(7.3.17)^2 \lesssim & \sum_{\substack{n_1 : n_1 \sim i; \\ n_1 \le i}} 2^{n_1 - i} \sum_{G^{n_1}_\beta \subset G^i_\alpha} \left\| |\nabla|^{\frac{1}{2}} P_{2^{-n_1}} u \right\|^2_{U^2_\Delta (G^{n_1}_\beta \times \mathbb{R}^2)} \\
& + \sum_{\substack{n_1 : n_1 \sim i; \\ i < n_1 < j}} 2^{n_1 - i} \left\| |\nabla|^{\frac{1}{2}} P_{2^{-n_1}} u \right\|^2_{U^2_\Delta (G^i_\alpha \times \mathbb{R}^2)} \qquad (7.3.18) \\
& + \sum_{\substack{n_1 : n_1 \sim i; \\ n_1 \ge j}} 2^{n_1 - i} \left\| |\nabla|^{\frac{1}{2}} P_{2^{-n_1}} u \right\|^2_{U^2_\Delta (G^i_\alpha \times \mathbb{R}^2)} .
\end{aligned}$$

In fact, the calculations above give us

$$\begin{aligned}
(7.3.10) \lesssim & \sum_{i : 11 \le i < j} 2^{i-j} \sum_{\substack{G^i_\alpha \subset G^j_k \\ N(G^i_\alpha) \ge \varepsilon_3^{-1/2} 2^{-i}}} (7.3.13)^2 \\
\lesssim & \sum_{i : 11 \le i < j} 2^{i-j} \sum_{\substack{G^i_\alpha \subset G^j_k \\ N(G^i_\alpha) \ge \varepsilon_3^{-1/2} 2^{-i}}} (7.3.16)^2 \\
\lesssim & \sum_{i : 11 \le i < j} 2^{i-j} \sum_{\substack{G^i_\alpha \subset G^j_k \\ N(G^i_\alpha) \ge \varepsilon_3^{-1/2} 2^{-i}}} (7.3.17)^2 \, \|u\|^4_{\tilde{X}_j([0,T] \times \mathbb{R}^2)} \, \varepsilon_2^4 \\
\lesssim & \sum_{i : 11 \le i < j} 2^{i-j} \sum_{\substack{G^i_\alpha \subset G^j_k \\ N(G^i_\alpha) \ge \varepsilon_3^{-1/2} 2^{-i}}} (7.3.18) \, \|u\|^4_{\tilde{X}_j([0,T] \times \mathbb{R}^2)} \, \varepsilon_2^4 .
\end{aligned}$$

Now to close the argument, we only need to show

$$\sum_{i : 11 \le i < j} 2^{i-j} \sum_{\substack{G^i_\alpha \subset G^j_k \\ N(G^i_\alpha) \ge \varepsilon_3^{-1/2} 2^{-i}}} (7.3.18) \lesssim \|u\|^2_{\tilde{X}_j([0,T] \times \mathbb{R}^2)} .$$

For $n_1$ satisfying $n_1 \sim i$ and $n_1 \geq i$, we know that the number of $G_\alpha^i$ such that $G_\alpha^i \subset G_\beta^{n_1}$ is at most $2^7$. For $n_1$ satisfying $n_1 \sim i$ and $n_1 \geq j$ (implies $i \sim j$), we know that the number of $G_\alpha^i$ such that $G_\alpha^i \subset G_k^j$ is at most $2^7$ as well. By Fubini-Tonelli theorem and Definition 6.18,

$$\sum_{i:11\leq i<j} 2^{i-j} \sum_{\substack{G_\alpha^i \subset G_k^j \\ N(G_\alpha^i)\geq \varepsilon_3^{-1/2}2^{-i}}} \quad (7.3.18)$$

$$= \sum_{i:11\leq i<j} 2^{i-j} \sum_{\substack{G_\alpha^i \subset G_k^j \\ N(G_\alpha^i)\geq \varepsilon_3^{-1/2}2^{-i}}} \sum_{\substack{n_1:n_1\sim i; \\ n_1\leq i}} 2^{n_1-i} \sum_{G_\beta^{n_1}\subset G_\alpha^i} \left\| |\nabla|^{\frac{1}{2}} P_{2^{-n_1}} u \right\|_{U_\Delta^2(G_\beta^{n_1}\times\mathbb{R}^2)}^2$$

$$+ \sum_{i:11\leq i<j} 2^{i-j} \sum_{\substack{G_\alpha^i \subset G_k^j \\ N(G_\alpha^i)\geq \varepsilon_3^{-1/2}2^{-i}}} \sum_{\substack{n_1:n_1\sim i; \\ i<n_1<j}} 2^{n_1-i} \left\| |\nabla|^{\frac{1}{2}} P_{2^{-n_1}} u \right\|_{U_\Delta^2(G_\alpha^i\times\mathbb{R}^2)}^2$$

$$+ \sum_{i:11\leq i<j} 2^{i-j} \sum_{\substack{G_\alpha^i \subset G_k^j \\ N(G_\alpha^i)\geq \varepsilon_3^{-1/2}2^{-i}}} \sum_{\substack{n_1:n_1\sim i; \\ n_1\geq j}} 2^{n_1-i} \left\| |\nabla|^{\frac{1}{2}} P_{2^{-n_1}} u \right\|_{U_\Delta^2(G_\alpha^i\times\mathbb{R}^2)}^2$$

$$\lesssim \sum_{n_1:0\leq n_1<j} 2^{n_1-j} \sum_{G_\beta^{n_1}\subset G_k^j} \left\| |\nabla|^{\frac{1}{2}} P_{2^{-n_1}} u \right\|_{U_\Delta^2(G_\beta^{n_1}\times\mathbb{R}^2)}^2 + \sum_{n_1:n_1\geq j} \left\| |\nabla|^{\frac{1}{2}} P_{2^{-n_1}} u \right\|_{U_\Delta^2(G_k^j\times\mathbb{R}^2)}^2$$

$$\leq \|u\|_{\tilde{X}_j([0,T]\times\mathbb{R}^2)}^2 .$$

Then first term (7.3.10) in Lemma 7.4 has the following bound:

$$(7.3.10) \lesssim \varepsilon_2^4 \, \|u\|_{\tilde{X}_j([0,T]\times\mathbb{R}^2)}^6 .$$

Similarly, the second term (7.3.11) in Lemma 7.4 becomes,

$$(7.3.11) \lesssim \sum_{\substack{i:i\geq j \\ N(G_k^j)\geq \varepsilon_3^{-1/2}2^{-j}}} \left\| |\nabla|^{\frac{1}{2}} P_{2^{-i}} u \right\|_{U_\Delta^2(G_k^j\times\mathbb{R}^2)}^2 \varepsilon_3^4 \, \|u\|_{\tilde{X}_j([0,T]\times\mathbb{R}^2)}^4 \lesssim \varepsilon_2^4 \, \|u\|_{\tilde{X}_j([0,T]\times\mathbb{R}^2)}^6 .$$

Therefore, the proof of Lemma 7.4 is complete. $\square$

Next, we estimate the contribution of $\tilde{F}_2$ in the decomposition in Proposition 7.3.

**Lemma 7.5** (Contribution of $\tilde{F}_2$)**.** *For a fixed $G_k^j \subset [0,T]$, $j > 11$,*

$$\sum_{i:11\leq i<j} 2^{i-j} \sum_{\substack{G_\alpha^i \subset G_k^j \\ N(G_\alpha^i)\geq \varepsilon_3^{-1/2}2^{-i}}} \left\| |\nabla|^{\frac{1}{2}} \int_{t_\alpha^i}^t e^{i(t-t')\Delta} \tilde{F}_2 \, dt' \right\|_{U_\Delta^2(G_\alpha^i \times \mathbb{R}^2)}^2 \tag{7.3.19}$$

$$+ \sum_{\substack{i:i\geq j \\ N(G_k^j)\geq \varepsilon_3^{-1/2}2^{-j}}} \left\| |\nabla|^{\frac{1}{2}} \int_{t_0}^t e^{i(t-t')\Delta} \tilde{F}_2 \, dt' \right\|_{U_\Delta^2(G_k^j \times \mathbb{R}^2)}^2 \tag{7.3.20}$$

$$\lesssim \varepsilon_2^2 \, \|u\|_{\tilde{X}_j([0,T]\times\mathbb{R}^2)}^8$$

*Proof.* First, we consider (7.3.19). By Lemma 6.8, we decompose

$$\left\| |\nabla|^{\frac{1}{2}} \int_{t_\alpha^i}^t e^{i(t-t')\Delta} \tilde{F}_2 \, dt' \right\|_{U_\Delta^2(G_\alpha^i \times \mathbb{R}^2)}$$

$$\leq \sum_{n_1:0\leq n_1\leq i} \sum_{\substack{n_2,n_3,n_4,n_5: \\ n_1\sim n_2\leq n_3\leq n_4\leq n_5}} \sum_{G_\beta^{n_1}\subset G_\alpha^i} \sup_{\|v\|_{V_\Delta^2(G_\beta^{n_1}\times\mathbb{R}^2)}=1} \int_{G_\beta^{n_1}} \left\langle |\nabla|^{\frac{1}{2}} P_{2^{-i}} v, f \right\rangle dt. \tag{7.3.21}$$

By Hölder's inequality,

$$\int_{G_\beta^{n_1}} \left\langle |\nabla|^{\frac{1}{2}} P_{2^{-i}} v, F_2 \right\rangle dt$$

$$\lesssim \left\| |\nabla|^{\frac{1}{2}} P_{2^{-i}} v \right\|_{L_t^\infty L_x^\infty(G_\beta^{n_1}\times\mathbb{R}^2)} \| (P_{2^{-n_1}} u) \ (P_{2^{-n_3}} u)\|_{L_t^2 L_x^2(G_\beta^{n_1}\times\mathbb{R}^2)} \tag{7.3.22}$$

$$\times \| (P_{2^{-n_2}} u) \ (P_{2^{-n_4}} u)\|_{L_t^2 L_x^2(G_\beta^{n_1}\times\mathbb{R}^2)} \ \|P_{2^{-n_5}} u\|_{L_t^\infty L_x^\infty(G_\beta^{n_1}\times\mathbb{R}^2)}.$$

By Bernstein's inequality and $\|v\|_{V_\Delta^2(G_\beta^{n_1}\times\mathbb{R}^2)} = 1$,

$$\left\| |\nabla|^{\frac{1}{2}} P_{2^{-i}} v \right\|_{L_t^\infty L_x^\infty(G_\beta^{n_1}\times\mathbb{R}^2)} \lesssim 2^{-\frac{3i}{2}} \ \|P_{2^{-i}} v\|_{L_t^\infty L_x^2(G_\beta^{n_1}\times\mathbb{R}^2)} \lesssim 2^{-\frac{3i}{2}} \ \|v\|_{V_\Delta^2(G_\beta^{n_1}\times\mathbb{R}^2)} \leq 2^{-\frac{3i}{2}}.$$

Next, we employ Bourgain's bilinear estimates (Lemma 2.15) and Bernstein's inequality to obtain the following bound:

$$\| (P_{2^{-n_1}} u) \ (P_{2^{-n_3}} u)\|_{L_t^2 L_x^2(G_\beta^{n_1}\times\mathbb{R}^2)}$$

$$\lesssim \left(\frac{2^{-n_3}}{2^{-n_1}}\right)^{\frac{1}{2}} \ \|P_{2^{-n_1}} u\|_{U_\Delta^2(G_\beta^{n_1}\times\mathbb{R}^2)} \ \|P_{2^{-n_3}} u\|_{U_\Delta^2(G_\beta^{n_1}\times\mathbb{R}^2)}$$

$$\lesssim 2^{n_1} \ \left\| |\nabla|^{\frac{1}{2}} P_{2^{-n_1}} u \right\|_{U_\Delta^2(G_\beta^{n_1}\times\mathbb{R}^2)} \ \left\| |\nabla|^{\frac{1}{2}} P_{2^{-n_3}} u \right\|_{U_\Delta^2(G_\beta^{n_1}\times\mathbb{R}^2)}.$$

Similarly,

$$
\begin{aligned}
&\| (P_{2^{-n_2}} u) \ (P_{2^{-n_4}} u) \|_{L_t^2 L_x^2 (G_\beta^{n_1} \times \mathbb{R}^2)} \\
\lesssim \ & 2^{n_2} \left\| |\nabla|^{\frac{1}{2}} P_{2^{-n_2}} u \right\|_{U_\Delta^2 (G_\beta^{n_1} \times \mathbb{R}^2)} \left\| |\nabla|^{\frac{1}{2}} P_{2^{-n_4}} u \right\|_{U_\Delta^2 (G_\beta^{n_1} \times \mathbb{R}^2)} .
\end{aligned}
$$

For $P_{2^{-n_5}} u$ term, we treat $2^{-n_5} \leq \varepsilon_3^{1/4} N(t)$ and $2^{-n_5} \geq \varepsilon_3^{1/4} N(t)$ separately:

- If $2^{-n_5} \leq \varepsilon_3^{1/4} N(t)$, then by Bernstein's inequality and (6.2.4),

$$
\| P_{2^{-n_5}} u \|_{L_t^\infty L_x^\infty (G_\beta^{n_1} \times \mathbb{R}^2)} \lesssim 2^{-\frac{n_5}{2}} \ \| P_{2^{-n_5}} u \|_{L_t^\infty L_x^4 (G_\beta^{n_1} \times \mathbb{R}^2)} \varepsilon_2 \lesssim 2^{-\frac{n_5}{2}} \varepsilon_2 .
$$

- If $2^{-n_5} \geq \varepsilon_3^{1/4} N(t)$, then the assumption $N(t) \geq \varepsilon_3^{-1/2} 2^{-i}$ in **Case 2** implies $2^{n_5} \leq \varepsilon_3^{1/4} 2^i$. Then by Bernstein's inequality, we have

$$
\begin{aligned}
&\| P_{2^{-n_5}} u \|_{L_t^\infty L_x^\infty (G_\beta^{n_1} \times \mathbb{R}^2)} \lesssim 2^{-\frac{n_5}{2}} \ \| P_{2^{-n_5}} u \|_{L_t^\infty L_x^4 (G_\beta^{n_1} \times \mathbb{R}^2)} \\
\lesssim \ & 2^{-\frac{n_5}{2}} = 2^{\frac{n_5}{6}} 2^{-\frac{2n_5}{3}} \leq \varepsilon_3^{1/24} 2^{\frac{i}{6} - \frac{2n_5}{3}} .
\end{aligned}
$$

Using (6.2.4) again, we have the following bound for $P_{2^{-n_5}} u$ term,

$$
\| P_{2^{-n_5}} u \|_{L_t^\infty L_x^\infty (G_\beta^{n_1} \times \mathbb{R}^2)} \lesssim \varepsilon_2 2^{-\frac{n_5}{2}} \left( 1 + 2^{\frac{1}{6}(i - n_5)} \right) .
$$

Putting the computations above together, combing with $\|v\|_{V_\Delta^2 (G_\beta^{n_1} \times \mathbb{R}^2)} = 1$ and (6.2.4), we obtain

$$
\begin{aligned}
(7.3.22) \lesssim \ & 2^{n_1 + n_2 - \frac{n_5}{2} - \frac{3i}{2}} \varepsilon_2 \left\| |\nabla|^{\frac{1}{2}} P_{2^{-n_1}} u \right\|_{U_\Delta^2 (G_\beta^{n_1} \times \mathbb{R}^2)} \left\| |\nabla|^{\frac{1}{2}} P_{2^{-n_2}} u \right\|_{U_\Delta^2 (G_\beta^{n_1} \times \mathbb{R}^2)} \\
& \times \left\| |\nabla|^{\frac{1}{2}} P_{2^{-n_3}} u \right\|_{U_\Delta^2 (G_\beta^{n_1} \times \mathbb{R}^2)} \left\| |\nabla|^{\frac{1}{2}} P_{2^{-n_4}} u \right\|_{U_\Delta^2 (G_\beta^{n_1} \times \mathbb{R}^2)} \qquad (7.3.23) \\
& + 2^{n_1 + n_2 - \frac{n_5}{2} - \frac{3i}{2}} 2^{\frac{1}{6}(i - n_5)} \varepsilon_2 \left\| |\nabla|^{\frac{1}{2}} P_{2^{-n_1}} u \right\|_{U_\Delta^2 (G_\beta^{n_1} \times \mathbb{R}^2)} \left\| |\nabla|^{\frac{1}{2}} P_{2^{-n_2}} u \right\|_{U_\Delta^2 (G_\beta^{n_1} \times \mathbb{R}^2)} \\
& \times \left\| |\nabla|^{\frac{1}{2}} P_{2^{-n_3}} u \right\|_{U_\Delta^2 (G_\beta^{n_1} \times \mathbb{R}^2)} \left\| |\nabla|^{\frac{1}{2}} P_{2^{-n_4}} u \right\|_{U_\Delta^2 (G_\beta^{n_1} \times \mathbb{R}^2)} . \qquad (7.3.24)
\end{aligned}
$$

Next, we consider the summations in (7.3.21).

Take the sum over $n_3$, $n_4$ and $n_5$ acting on the first term (7.3.23) and the component in (7.3.23) depending on these frequencies. Applying Fubini-Tonelli theorem, Cauchy-Schwarz inequality and Definition 6.18, we obtain

$$\begin{aligned}
&\sum_{\substack{n_3,n_4,n_5:\\ n_1\le n_3\le n_4\le n_5}} 2^{n_1+n_2-\frac{n_5}{2}-\frac{3i}{2}} \left\| |\nabla|^{\frac12} P_{2^{-n_3}}u\right\|_{U^2_\Delta(G^{n_1}_\beta\times\mathbb{R}^2)} \left\| |\nabla|^{\frac12} P_{2^{-n_4}}u\right\|_{U^2_\Delta(G^{n_1}_\beta\times\mathbb{R}^2)}\\
&\lesssim \sum_{\substack{n_3,n_4:\\ n_1\le n_3\le n_4}} 2^{n_1+n_2-\frac{n_4}{2}-\frac{3i}{2}} \left\| |\nabla|^{\frac12} P_{2^{-n_3}}u\right\|_{U^2_\Delta(G^{n_1}_\beta\times\mathbb{R}^2)} \left\| |\nabla|^{\frac12} P_{2^{-n_4}}u\right\|_{U^2_\Delta(G^{n_1}_\beta\times\mathbb{R}^2)}\\
&\lesssim \sum_{n_3:n_3\ge n_1} 2^{n_1+n_2-\frac{3i}{2}} \left\| |\nabla|^{\frac12} P_{2^{-n_3}}u\right\|_{U^2_\Delta(G^{n_1}_\beta\times\mathbb{R}^2)} \left(\sum_{n_4:n_4\ge n_3} 2^{-n_4}\right)^{\frac12}\\
&\qquad\times\left(\sum_{n_4:n_4\ge n_1} \left\| |\nabla|^{\frac12} P_{2^{-n_4}}u\right\|^2_{U^2_\Delta(G^{n_1}_\beta\times\mathbb{R}^2)}\right)^{\frac12}\\
&\lesssim 2^{n_1+n_2-\frac{3i}{2}} \left(\sum_{n_3:n_3\ge n_1} 2^{-n_3}\right)^{\frac12} \left(\sum_{n_3:n_3\ge n_1} \left\| |\nabla|^{\frac12} P_{2^{-n_3}}u\right\|^2_{U^2_\Delta(G^{n_1}_\beta\times\mathbb{R}^2)}\right)^{\frac12} \|u\|_{X(G^{n_1}_\beta\times\mathbb{R}^2)}\\
&\lesssim 2^{\frac{n_1}{2}+n_2-\frac{3i}{2}} \|u\|^2_{\tilde{X}_j([0,T]\times\mathbb{R}^2)} \sim 2^{\frac{3n_1}{2}-\frac{3i}{2}} \|u\|^2_{\tilde{X}_j([0,T]\times\mathbb{R}^2)}.
\end{aligned}$$

Then we take the sum over $n_2$ in (7.3.21) and corresponding component. Note that this is a finite sum, hence by Cauchy-Schwarz inequality and Definition 6.2.5, we have

$$\begin{aligned}
\sum_{\substack{n_2:n_2\ge n_1;\\ n_2\sim n_1}} \left\| |\nabla|^{\frac12} P_{2^{-n_2}}u\right\|_{U^2_\Delta(G^{n_1}_\beta\times\mathbb{R}^2)} &\lesssim \left(\sum_{\substack{n_2:n_2\ge n_1;\\ n_2\sim n_1}} \left\| |\nabla|^{\frac12} P_{2^{-n_2}}u\right\|^2_{U^2_\Delta(G^{n_1}_\beta\times\mathbb{R}^2)}\right)^{\frac12}\\
&\le \|u\|_{\tilde{X}_j([0,T]\times\mathbb{R}^2)}.
\end{aligned}$$

Therefore, putting the sum over $n_2$, $n_3$, $n_4$ and $n_5$ together, we have

$$\sum_{\substack{n_2:n_2\ge n_1;\\ n_2\sim n_1}} \sum_{\substack{n_3,n_4,n_5:\\ n_1\le n_3\le n_4\le n_5}} (7.3.23) \lesssim 2^{\frac32(n_1-i)} \|u\|^3_{\tilde{X}_j([0,T]\times\mathbb{R}^2)} \varepsilon_2 \left\| |\nabla|^{\frac12} P_{2^{-n_1}}u\right\|_{U^2_\Delta(G^{n_1}_\beta\times\mathbb{R}^2)}.$$

Similarly, the sum over $n_2$, $n_3$, $n_4$ and $n_5$ acting on the second term (7.3.24) of

(7.3.22) yields

$$\sum_{\substack{n_2:n_2\geq n_1;\\ n_2\sim n_1}} \sum_{\substack{n_3,n_4,n_5:\\ n_1\leq n_3\leq n_4\leq n_5}} (7.3.24) \lesssim 2^{\frac{4}{3}(n_1-i)} \, \|u\|^3_{\tilde{X}_j([0,T]\times\mathbb{R}^2)} \, \varepsilon_2 \, \left\| |\nabla|^{\frac{1}{2}} P_{2^{-n_1}} u \right\|_{U^2_\Delta(G_\beta^{n_1}\times\mathbb{R}^2)}.$$

Note that $G_\beta^{n_1}$ overlaps $2^{i-n_1}$ intervals $G_\alpha^i$. By Cauchy-Schwarz inequality and Definition 6.18, we have

$$\begin{aligned}
&(7.3.21)\\
&\lesssim \sum_{n_1:0\leq n_1\leq i} \sum_{G_\beta^{n_1}\subset G_\alpha^i} \sum_{\substack{n_2:n_2\geq n_1;\\ n_2\sim n_1}} \sum_{\substack{n_3,n_4,n_5:\\ n_1\leq n_3\leq n_4\leq n_5}} (7.3.22)\\
&\lesssim \sum_{n_1:0\leq n_1\leq i} \sum_{G_\beta^{n_1}\subset G_\alpha^i} \sum_{\substack{n_2:n_2\geq n_1;\\ n_2\sim n_1}} \sum_{\substack{n_3,n_4,n_5:\\ n_1\leq n_3\leq n_4\leq n_5}} (7.3.23)+(7.3.24)\\
&\lesssim \sum_{n_1:0\leq n_1\leq i} \sum_{G_\beta^{n_1}\subset G_\alpha^i} 2^{\frac{4}{3}(n_1-i)} \, \|u\|^3_{\tilde{X}_j([0,T]\times\mathbb{R}^2)} \, \varepsilon_2 \, \left\| |\nabla|^{\frac{1}{2}} P_{2^{-n_1}} u \right\|_{U^2_\Delta(G_\beta^{n_1}\times\mathbb{R}^2)}\\
&\lesssim \varepsilon_2 \, \|u\|^3_{\tilde{X}_j([0,T]\times\mathbb{R}^2)} \left( \sum_{n_1:0\leq n_1\leq i} \sum_{G_\beta^{n_1}\subset G_\alpha^i} 2^{(n_1-i)1+} \right)^{\frac{1}{2}}\\
&\qquad \times \left( \sum_{n_1:0\leq n_1\leq i} \sum_{G_\beta^{n_1}\subset G_\alpha^i} 2^{(n_1-i)\frac{5}{3}-} \left\| |\nabla|^{\frac{1}{2}} P_{2^{-n_1}} u \right\|^2_{U^2_\Delta(G_\beta^{n_1}\times\mathbb{R}^2)} \right)^{\frac{1}{2}}\\
&\lesssim \varepsilon_2 \, \|u\|^3_{\tilde{X}_j([0,T]\times\mathbb{R}^2)} \left( \sum_{n_1:0\leq n_1\leq i} \sum_{G_\beta^{n_1}\subset G_\alpha^i} 2^{(n_1-i)\frac{5}{3}-} \left\| |\nabla|^{\frac{1}{2}} P_{2^{-n_1}} u \right\|^2_{U^2_\Delta(G_\beta^{n_1}\times\mathbb{R}^2)} \right)^{\frac{1}{2}}.
\end{aligned} \tag{7.3.25}$$

Therefore, by Cauchy-Schwarz inequality, Fubini-Tonelli theorem and Definition 6.18, we obtain

$$(7.3.19) \lesssim \sum_{i:11\leq i<j} 2^{i-j} \sum_{\substack{G_\alpha^i\subset G_k^j\\ N(G_\alpha^i)\geq \varepsilon_3^{-1/2}2^{-i}}} (7.3.21)^2 \lesssim \sum_{i:11\leq i<j} 2^{i-j} \sum_{\substack{G_\alpha^i\subset G_k^j\\ N(G_\alpha^i)\geq \varepsilon_3^{-1/2}2^{-i}}} (7.3.25)^2$$

$$\lesssim \varepsilon_2^2 \, \|u\|^6_{\tilde{X}_j([0,T]\times\mathbb{R}^2)}$$

$$\times \left( \sum_{i:11\leq i<j} 2^{i-j} \sum_{\substack{G_\alpha^i \subset G_k^j \\ N(G_\alpha^i)\geq \varepsilon_3^{-1/2} 2^{-i}}} \sum_{n_1:0\leq n_1\leq i} \sum_{G_\beta^{n_1}\subset G_\alpha^i} 2^{(n_1-i)\frac{5}{3}-} \left\| |\nabla|^{\frac{1}{2}} P_{2^{-n_1}} u \right\|^2_{U^2_\Delta(G_\beta^{n_1}\times\mathbb{R}^2)} \right)$$

$$\lesssim \varepsilon_2^2 \, \|u\|^6_{\tilde{X}_j([0,T]\times\mathbb{R}^2)} \sum_{n_1:0\leq n_1<j} 2^{n_1-j} \sum_{i:n_1\leq i<j} 2^{(n_1-i)\frac{2}{3}-} \sum_{G_\beta^{n_1}\subset G_\alpha^i} \left\| |\nabla|^{\frac{1}{2}} P_{2^{-n_1}} u \right\|^2_{U^2_\Delta(G_\beta^{n_1}\times\mathbb{R}^2)}$$

$$\lesssim \varepsilon_2^2 \, \|u\|^6_{\tilde{X}_j([0,T]\times\mathbb{R}^2)} \sum_{n_1:0\leq n_1<j} 2^{n_1-j} \sum_{G_\beta^{n_1}\subset G_\alpha^i} \left\| |\nabla|^{\frac{1}{2}} P_{2^{-n_1}} u \right\|^2_{U^2_\Delta(G_\beta^{n_1}\times\mathbb{R}^2)}$$

$$\lesssim \varepsilon_2^2 \, \|u\|^8_{\tilde{X}_j([0,T]\times\mathbb{R}^2)}. \tag{7.3.26}$$

Then take (7.3.20). Again by Lemma 6.8, we decompose

$$\left\| |\nabla|^{\frac{1}{2}} \int_{t_\alpha^i}^t e^{i(t-t')\Delta} \tilde{F}_2 \, dt' \right\|_{U^2_\Delta(G_k^j\times\mathbb{R}^2)}$$

$$\lesssim \sum_{n_1:0\leq n_1\leq j} \sum_{\substack{n_2,n_3,n_4,n_5: \\ n_1\sim n_2\leq n_3\leq n_4\leq n_5}} \sum_{G_\beta^{n_1}\subset G_k^j} \sup_{\|v\|_{V^2_\Delta(G_\beta^{n_1}\times\mathbb{R}^2)}=1} \int_{G_\beta^{n_1}} \left\langle |\nabla|^{\frac{1}{2}} P_{2^{-i}} v, f \right\rangle dt \tag{7.3.27}$$

$$+ \sum_{n_1:j\leq n_1\leq i} \sum_{\substack{n_2,n_3,n_4,n_5: \\ n_1\sim n_2\leq n_3\leq n_4\leq n_5}} \sup_{\|v\|_{V^2_\Delta(G_k^j\times\mathbb{R}^2)}=1} \int_{G_k^j} \left\langle |\nabla|^{\frac{1}{2}} P_{2^{-i}} v, f \right\rangle dt. \tag{7.3.28}$$

Next we estimate these two terms separately.

By the same calculation from (7.3.22) to (7.3.25), we have

$$(7.3.27) \lesssim \varepsilon_2 \, \|u\|^3_{\tilde{X}_j([0,T]\times\mathbb{R}^2)} \left( \sum_{n_1:0\leq n_1\leq j} \sum_{G_\beta^{n_1}\subset G_k^j} 2^{(n_1-i)\frac{5}{3}-} \left\| |\nabla|^{\frac{1}{2}} P_{2^{-n_1}} u \right\|^2_{U^2_\Delta(G_\beta^{n_1}\times\mathbb{R}^2)} \right)^{\frac{1}{2}},$$

$$(7.3.28) \lesssim \varepsilon_2 \, \|u\|^3_{\tilde{X}_j([0,T]\times\mathbb{R}^2)} \left( \sum_{n_1:j\leq n_1\leq i} 2^{(n_1-i)\frac{5}{3}-} \left\| |\nabla|^{\frac{1}{2}} P_{2^{-n_1}} u \right\|^2_{U^2_\Delta(G_k^j\times\mathbb{R}^2)} \right)^{\frac{1}{2}}.$$

Then by Fubini-Tonelli theorem and Definition 6.18 and the similar calculation as in (7.3.26), we have

$$(7.3.20)$$

$$\lesssim \varepsilon_2^2 \, \|u\|^6_{\tilde{X}_j([0,T]\times\mathbb{R}^2)} \sum_{\substack{i:i\geq j \\ N(G_k^j)\geq \varepsilon_3^{-1/2} 2^{-j}}} \sum_{n_1:0\leq n_1\leq j} \sum_{G_\beta^{n_1}\subset G_k^j} 2^{(n_1-i)\frac{5}{3}-} \left\| |\nabla|^{\frac{1}{2}} P_{2^{-n_1}} u \right\|^2_{U^2_\Delta(G_\beta^{n_1}\times\mathbb{R}^2)}$$

$$+ \varepsilon_2^2 \|u\|^6_{\tilde{X}_j([0,T]\times\mathbb{R}^2)} \sum_{\substack{i:i\geq j \\ N(G_k^j)\geq \varepsilon_3^{-1/2}2^{-j}}} \sum_{n_1:j\leq n_1\leq i} 2^{(n_1-i)\frac{5}{3}-} \left\| |\nabla|^{\frac{1}{2}} P_{2^{-n_1}} u \right\|^2_{U^2_\Delta(G_k^j\times\mathbb{R}^2)}$$

$$\lesssim \varepsilon_2^2 \|u\|^8_{\tilde{X}_j([0,T]\times\mathbb{R}^2)}.$$

Therefore, we complete the proof of Lemma 7.5. □

Then Proposition 7.3 follows from Lemma 7.4 and Lemma 7.5. □

Now the proof of Theorem 7.1 is complete.

**Remark 7.6** (Simplification for radial data)**.** In the proof of Theorem 7.1, the reason why we consider general data instead of radial data is that even if we have the radial assumption, there is no simplified Morawetz estimates as the ones used in [29, 49] for us to employ. But if we restrict our initial data with radial symmetry, we do have some simplification in the proof. In fact, we can apply the bilinear estimates in Corollary 6.5 in [54], that is,

**Lemma 7.7** (Radial bilinear Strichartz estimates)**.** *If $\hat{u}_0$ is supported on $|\xi| \sim M_1$, $\hat{v}_0$ is supported on $|\xi| \sim M_2$, $M_2 \ll M_1$, then*

1. *for* $\frac{d+1}{d} < q \leq 2$,

$$\left\|e^{it\Delta}u_0 e^{it\Delta}v_0\right\|_{L^q_{t,x}(\mathbb{R}\times\mathbb{R}^d)} \lesssim M_1^{-\frac{1}{2}} M_2^{\frac{2d+1}{2}-\frac{d+2}{q}} \|u_0\|_{L^2_x(\mathbb{R}^d)} \|v_0\|_{L^2_x(\mathbb{R}^d)},$$

2. *for* $2 \leq q \leq \frac{2(2d+1)}{2d-1}$,

$$\left\|e^{it\Delta}u_0 e^{it\Delta}v_0\right\|_{L^q_{t,x}(\mathbb{R}\times\mathbb{R}^d)} \lesssim M_1^{-\frac{3}{2q}+\frac{1}{4}} M_2^{\frac{4d-1}{4}-\frac{2d+1}{2q}} \|u_0\|_{L^2_x(\mathbb{R}^d)} \|v_0\|_{L^2_x(\mathbb{R}^d)},$$

3. *for* $q \geq \frac{2(2d+1)}{2d-1}$,

$$\left\|e^{it\Delta}u_0 e^{it\Delta}v_0\right\|_{L^q_{t,x}(\mathbb{R}\times\mathbb{R}^d)} \lesssim M_1^{\frac{d}{2}-\frac{d+2}{q}} M_2^{\frac{d}{2}} \|u_0\|_{L^2_x(\mathbb{R}^d)} \|v_0\|_{L^2_x(\mathbb{R}^d)}.$$

By applying the bilinear estimates above, we are able to have more decay in, for example, (7.3.15) and (7.3.22). Then the extra decay obtained here will help us to sum over $n_1$, $n_2$, $n_3$, $n_4$ and $n_5$. However, these radial bilinear estimates will not shorten the proof itself, hence we did not present the proof with radial data separately.

## 7.4 Main differences with [16]

After using Littlewood-Paley to decompose the nonlinearity in the Duhamel term, we should be very careful with the high frequency and high frequency interaction into low frequency terms (the worst case is five high frequencies interaction into a low frequency). The reason here is that instead of proving Theorem 7.1 directly, we are doing a bootstrap argument, that is, we wish to prove

$$\|u\|^2_{\tilde{X}_{k_0}([0,T]\times\mathbb{R}^2)} \lesssim 1 + \varepsilon \, \|u\|^2_{\tilde{X}_{k_0}([0,T]\times\mathbb{R}^2)} \, . \tag{7.4.1}$$

From the construction of the atomic $X$-norm, we can see that the high frequency terms require more summability than the others. Therefore, in order to close the bootstrap argument as desired, we should gain more decay than the mass-critical case to sum over the high frequency terms. In contrast, these terms were not problematic in mass-critical [16], because the cutoff in the mass-critical problem and the cutoff in $\dot{H}^{\frac{1}{2}}$ are opposite, hence the worse case was all low frequencies interaction into high frequency. However, this case never happens since the contribution of all low frequencies remains low.

# CHAPTER 8

## INTERACTION MORAWETZ ESTIMATE IN 2D

In this chapter, we go over the proof of the interaction Morawetz estimate in dimensions two, with modified nonlinear terms, that is, we consider equations

$$i\partial_t v + \Delta v = |v|^4 v + \mathcal{N}_1, \quad i\partial_t w + \Delta w = |w|^4 w + \mathcal{N}_2,$$

instead of

$$i\partial_t v + \Delta v = |v|^4 v, \quad i\partial_t w + \Delta w = |w|^4 w,$$

in [52]. We will use this interaction Morawetz estimate in Chapter 9 to derive a frequency-localized interaction Morawetz estimate.

**Theorem 8.1** (Interaction Morawetz estimate)**.** *Let $v$ and $u$ solve the following equations respectively,*

$$i\partial_t v(t,x) + \Delta v(t,x) = F(v)(t,x) = |v|^4 v(t,x) + \mathcal{N}_1(t,x), \tag{8.0.1}$$

$$i\partial_t w(t,y) + \Delta w(t,y) = G(w)(t,y) = |w|^4 w(t,y) + \mathcal{N}_2(t,y), \tag{8.0.2}$$

*and define*

$$\begin{aligned} M_\omega(t) := &\iint |w(t,y)|^2 \frac{(x-y)_\omega}{|(x-y)_\omega|} \operatorname{Im}[\bar{v}\partial_\omega v](t,x)\, dxdy \\ &+ \iint |v(t,x)|^2 \frac{(y-x)_\omega}{|(y-x)_\omega|} \operatorname{Im}[\bar{w}\partial_\omega w](t,y)\, dxdy \end{aligned} \tag{8.0.3}$$

*where*

$$\frac{(x-y)_\omega}{|(x-y)_\omega|} = \frac{(x-y)\cdot\omega}{|(x-y)\cdot\omega|}, \qquad \textit{for any } \omega \textit{ on the unit circle } S^1.$$

*Then,*

$$\begin{aligned}
\frac{d}{dt}M_\omega(t) &= 4\iint_{x_\omega=y_\omega} \left|\partial_\omega(\overline{w(t,y)}v(t,x))\right|^2 \, dxdy \\
&\quad + 2\iint |w(t,y)|^2 \, \frac{(x-y)_\omega}{|(x-y)_\omega|} \operatorname{Re}[\mathcal{N}_1\partial_\omega\bar{v} - v\partial_\omega\bar{\mathcal{N}_1}](t,x)\, dxdy \\
&\quad + 2\iint |v(t,x)|^2 \, \frac{(y-x)_\omega}{|(y-x)_\omega|} \operatorname{Re}[\mathcal{N}_2\partial_\omega\bar{w} - w\partial_\omega\bar{\mathcal{N}_2}](t,y)\, dxdy \\
&\quad + 2\iint \operatorname{Im}[\bar{w}\partial_\omega w](t,y) \frac{(y-x)_\omega}{|(y-x)_\omega|} \operatorname{Im}[\bar{v}\mathcal{N}_1](t,x)\, dxdy \\
&\quad + 2\iint \operatorname{Im}[\bar{v}\partial_\omega v](t,x) \frac{(x-y)_\omega}{|(x-y)_\omega|} \operatorname{Im}[\bar{w}\mathcal{N}_2](t,y)\, dxdy \\
&\quad + \frac{2}{3}\iint_{x_\omega=y_\omega} |w(t,y)|^2 \, |v(t,x)|^6 \, dxdy \\
&\quad + \frac{2}{3}\iint_{x_\omega=y_\omega} |v(t,x)|^2 \, |w(t,y)|^6 \, dxdy.
\end{aligned}$$

*Proof.* We compute the derivative of (8.0.3) directly,

$$\begin{aligned}
\frac{d}{dt}M_\omega(t) &= \iint \left(\frac{\partial}{\partial_t}\, |w(t,y)|^2\right) \frac{(x-y)_\omega}{|(x-y)_\omega|} \operatorname{Im}[\bar{v}\partial_\omega v](t,x)\, dxdy \\
&\quad + \iint |w(t,y)|^2 \, \frac{(x-y)_\omega}{|(x-y)_\omega|} \left(\frac{\partial}{\partial_t}\operatorname{Im}[\bar{v}\partial_\omega v](t,x)\right) dxdy \\
&\quad - \iint \left(\frac{\partial}{\partial_t}\, |v(t,x)|^2\right) \frac{(x-y)_\omega}{|(x-y)_\omega|} \operatorname{Im}[\bar{w}\partial_\omega w](t,y)\, dxdy \\
&\quad - \iint |v(t,x)|^2 \, \frac{(x-y)_\omega}{|(x-y)_\omega|} \left(\frac{\partial}{\partial_t}\operatorname{Im}[\bar{w}\partial_\omega w](t,y)\right) dxdy \\
&:= M_1 + M_2 + M_3 + M_4.
\end{aligned}$$

Then by the symmetry between $x$ and $y$, we only need to compute $M_1$ and $M_2$.

First, take $M_1$. Using the equation (8.0.2) and the product rule, we write

$$\begin{aligned}
\frac{\partial}{\partial_t}\, |w|^2 &= 2\operatorname{Re}[w_t\bar{w}] = 2\operatorname{Re}[i\Delta w\bar{w} - iG\bar{w}] = -2\operatorname{Im}[\Delta w\bar{w} - \bar{w}G] \\
&= -2\nabla\cdot\operatorname{Im}[\bar{w}\nabla w] + 2\operatorname{Im}[\bar{w}G] = -2\nabla\cdot\operatorname{Im}[\bar{w}\nabla w] + 2\operatorname{Im}[\bar{w}\mathcal{N}_2].
\end{aligned}$$

Then integration by parts yields,

$$
\begin{aligned}
&\iint \nabla \cdot \operatorname{Im}[\bar{w}\nabla w](t,y) \frac{(x-y)_\omega}{|(x-y)_\omega|} \operatorname{Im}[\bar{v}\partial_\omega v](t,x)\, dxdy \\
&= -\iint \operatorname{Im}[\bar{w}\nabla w](t,y) \cdot \nabla_y \left( \frac{(x-y)_\omega}{|(x-y)_\omega|} \right) \operatorname{Im}[\bar{v}\partial_\omega v](t,x)\, dxdy \\
&= \iint \operatorname{Im}[\bar{w}\nabla w](t,y) \cdot\, (2\delta_{x_\omega = y_\omega}\omega) \operatorname{Im}[\bar{v}\partial_\omega v](t,x)\, dxdy \\
&= 2 \iint_{x_\omega = y_\omega} \operatorname{Im}[\bar{w}\partial_\omega w](t,y) \operatorname{Im}[\bar{v}\partial_\omega v](t,x)\, dxdy.
\end{aligned}
$$

Hence

$$
\begin{aligned}
M_1 = -4 &\iint_{x_\omega = y_\omega} \operatorname{Im}[\bar{w}\partial_\omega w](t,y) \operatorname{Im}[\bar{v}\partial_\omega v](t,x)\, dxdy \\
&+ 2 \iint \operatorname{Im}[\bar{w}\mathcal{N}_2](t,y) \frac{(x-y)_\omega}{|(x-y)_\omega|} \operatorname{Im}[\bar{v}\partial_\omega v](t,x)\, dxdy.
\end{aligned}
$$

Next take $M_2$. We rewrite the equation $iv_t + \Delta v = F$ into

$$
v_t = i(\Delta v - F) \text{ and } \bar{v}_t = i(-\Delta \bar{v} + \bar{F}),
$$

and obtain

$$
\begin{aligned}
&\frac{\partial}{\partial_t} \operatorname{Im}[\bar{v}\partial_\omega v] = \frac{\partial}{\partial_t} \operatorname{Im}[\bar{v}\nabla v \cdot \omega] = \frac{\partial}{\partial_t} \left[ \frac{(\bar{v}\nabla v - v\nabla \bar{v}) \cdot \omega}{2i} \right] \\
&= \frac{(\bar{v}_t \nabla v + \bar{v}\nabla v_t - v_t \nabla \bar{v} - v\nabla \bar{v}_t) \cdot \omega}{2i} \qquad (8.0.4) \\
&= \frac{1}{2} \left[ (-\Delta \bar{v} + \bar{F})\nabla v + \bar{v}\nabla(\Delta v - F) - (\Delta v - F)\nabla \bar{v} - v\nabla(-\Delta \bar{v} + \bar{F}) \right] \cdot \omega.
\end{aligned}
$$

Recall $F = |v|^4 v + \mathcal{N}_1$. Combining the following identity

$$
|v|^4 \bar{v}\nabla v + |v|^4 v\nabla \bar{v} - \bar{v}\nabla(|v|^4 v) - v\nabla(|v|^4 \bar{v}) = -2\, |v|^2 \nabla(|v|^4),
$$

we have

$$
\begin{aligned}
&(8.0.4) \\
&= \frac{1}{2} \left[ -\Delta \bar{v}\nabla v - \Delta v \nabla \bar{v} + \bar{v}(\nabla \Delta v) + v(\nabla \Delta \bar{v}) + \bar{F}\nabla v + F\nabla \bar{v} - \bar{v}\nabla F - v\nabla \bar{F} \right] \cdot \omega
\end{aligned}
$$

$$= \frac{1}{2}\left[-\Delta\bar{v}\nabla v - \Delta v\nabla\bar{v} + \bar{v}(\nabla\Delta v) + v(\nabla\Delta\bar{v}) + \bar{\mathcal{N}_1}\nabla v + \mathcal{N}_1\nabla\bar{v} - \bar{v}\nabla\mathcal{N}_1 - v\nabla\bar{\mathcal{N}_1}\right]\cdot\omega$$
$$+ \frac{1}{2}\left[\,|v|^4\,\bar{v}\nabla v + |v|^4\,v\nabla\bar{v} - \bar{v}\nabla(\,|v|^4\,v) - v\nabla(\,|v|^4\,\bar{v})\right]\cdot\omega$$
$$= \frac{1}{2}\left[-\Delta\bar{v}\nabla v - \Delta v\nabla\bar{v} + \bar{v}(\nabla\Delta v) + v(\nabla\Delta\bar{v}) + \bar{\mathcal{N}_1}\nabla v + \mathcal{N}_1\nabla\bar{v} - \bar{v}\nabla\mathcal{N}_1 - v\nabla\bar{\mathcal{N}_1}\right]\cdot\omega$$
$$- \,|v|^2\,\nabla(\,|v|^4)\cdot\omega$$
$$:= \frac{1}{2}\left[M_{21} + M_{22} + M_{23} + M_{24} + M_{25} + M_{26} + M_{27} + M_{28}\right] + M_{29}.$$

Now we compute these nine terms one by one.

For $M_{21}$, integration by parts and the product rule yield,

$$\begin{aligned}
&\iint |w(t,y)|^2\,\frac{(x-y)_\omega}{|(x-y)_\omega|}\,\left[-\Delta\bar{v}\nabla v\cdot\omega\right](t,x)\,dxdy\\
&= \iint |w(t,y)|^2\,\nabla\bar{v}\cdot\nabla\left(\frac{(x-y)_\omega}{|(x-y)_\omega|}\nabla v\cdot\omega\right)dxdy\\
&= \iint |w(t,y)|^2\,\nabla\bar{v}\cdot\nabla_x\left(\frac{(x-y)_\omega}{|(x-y)_\omega|}\right)\nabla v\cdot\omega\,dxdy\\
&\quad + \iint |w(t,y)|^2\,\nabla\bar{v}\cdot\frac{(x-y)_\omega}{|(x-y)_\omega|}\nabla\cdot\nabla v\cdot\omega\,dxdy\\
&= 2\iint_{x_\omega=y_\omega} |w(t,y)|^2\,|\partial_\omega v(t,x)|^2\,dxdy\\
&\quad + \iint |w(t,y)|^2\,\frac{(x-y)_\omega}{|(x-y)_\omega|}\partial_\omega\bar{v}\Delta v(t,x)\,dxdy.
\end{aligned}$$

Similarly, for $M_{22}$,

$$\begin{aligned}
&\iint |w(t,y)|^2\,\frac{(x-y)_\omega}{|(x-y)_\omega|}\,\left[-\Delta v\nabla\bar{v}\cdot\omega\right](t,x)\,dxdy\\
&= 2\iint_{x_\omega=y_\omega} |w(t,y)|^2\,|\partial_\omega v(t,x)|^2\,dxdy\\
&\quad + \iint |w(t,y)|^2\,\frac{(x-y)_\omega}{|(x-y)_\omega|}\partial_\omega v\Delta\bar{v}(t,x)\,dxdy.
\end{aligned}$$

For $M_{23}$, integration by parts and the product rule again yield,

$$\begin{aligned}
&\iint |w(t,y)|^2\,\frac{(x-y)_\omega}{|(x-y)_\omega|}\,\left[\bar{v}(\nabla\Delta v)\cdot\omega\right](t,x)\,dxdy\\
&= -\iint |w(t,y)|^2\,\Delta v\omega\cdot\nabla\left(\frac{(x-y)_\omega}{|(x-y)_\omega|}\bar{v}\right)dxdy
\end{aligned}$$

$$
\begin{aligned}
&= -\iint |w(t,y)|^2 \, \Delta v \omega \cdot \nabla_x \left( \frac{(x-y)_\omega}{|(x-y)_\omega|} \right) \bar{v} \, dxdy \\
&\quad - \iint |w(t,y)|^2 \, \Delta v \omega \cdot \frac{(x-y)_\omega}{|(x-y)_\omega|} \nabla \bar{v} \, dxdy \\
&= -2 \iint_{x_\omega = y_\omega} |w(t,y)|^2 \, \Delta v \bar{v}(t,x) \, dxdy \\
&\quad - \iint |w(t,y)|^2 \, \frac{(x-y)_\omega}{|(x-y)_\omega|} \Delta v \partial_\omega \bar{v}(t,x) \, dxdy.
\end{aligned}
$$

Similarly, for $M_{24}$,

$$
\begin{aligned}
&\iint |w(t,y)|^2 \, \frac{(x-y)_\omega}{|(x-y)_\omega|} \, [v(\nabla \Delta \bar{v}) \cdot \omega] \, (t,x) \, dxdy \\
&= -2 \iint_{x_\omega = y_\omega} |w(t,y)|^2 \, \Delta \bar{v} v(t,x) \, dxdy \\
&\quad - \iint |w(t,y)|^2 \, \frac{(x-y)_\omega}{|(x-y)_\omega|} \Delta \bar{v} \partial_\omega v(t,x) \, dxdy.
\end{aligned}
$$

For $M_{25} + M_{26} + M_{27} + M_{28}$

$$
\begin{aligned}
&\iint |w(t,y)|^2 \, \frac{(x-y)_\omega}{|(x-y)_\omega|} \, \left[ \bar{\mathcal{N}_1} \nabla v + \mathcal{N}_1 \nabla \bar{v} - \bar{v} \nabla \mathcal{N}_1 - v \nabla \bar{\mathcal{N}_1} \right] (t,x) \, dxdy \\
&= 2 \iint |w(t,y)|^2 \, \frac{(x-y)_\omega}{|(x-y)_\omega|} \operatorname{Re} \left[ \mathcal{N}_1 \nabla \bar{v} - v \nabla \bar{\mathcal{N}_1} \right] (t,x) \, dxdy.
\end{aligned}
$$

For $M_{29}$, integrate by parts again,

$$
\begin{aligned}
&\iint |w(t,y)|^2 \, \frac{(x-y)_\omega}{|(x-y)_\omega|} \, \left[ - |v|^2 \, \nabla( |v|^4) \cdot \omega \right] (t,x) \, dxdy \\
&= \iint |w(t,y)|^2 \, \frac{(x-y)_\omega}{|(x-y)_\omega|} \, \left[ -( |v|^4)^{\frac{1}{2}} \nabla( |v|^4) \cdot \omega \right] (t,x) \, dxdy \\
&= \iint |w(t,y)|^2 \, \frac{(x-y)_\omega}{|(x-y)_\omega|} \, \left[ -\frac{2}{3} \nabla( |v|^6) \cdot \omega \right] (t,x) \, dxdy \\
&= \frac{2}{3} \iint |w(t,y)|^2 \, \omega \cdot \nabla_x \left( \frac{(x-y)_\omega}{|(x-y)_\omega|} \right) \, |v(t,x)|^6 \, dxdy \\
&= \frac{4}{3} \iint_{x_\omega = y_\omega} |w(t,y)|^2 \, |v(t,x)|^6 \, dxdy.
\end{aligned}
$$

Then we put all the calculation together and a further simplification yields

$$
\begin{aligned}
M_2 &= \iint |w(t,y)|^2 \frac{(x-y)_\omega}{|(x-y)_\omega|} \left(\frac{1}{2}\left[M_{21}+M_{22}+M_{23}+M_{24}\right]\right) dxdy \\
&\quad + \iint |w(t,y)|^2 \frac{(x-y)_\omega}{|(x-y)_\omega|} \left(\frac{1}{2}\left[M_{25}+M_{26}+M_{27}+M_{28}\right]+M_{29}\right) dxdy \\
&= 2\iint_{x_\omega=y_\omega} |w(t,y)|^2 \, |\partial_\omega v(t,x)|^2 \, dxdy \\
&\quad - \iint_{x_\omega=y_\omega} |w(t,y)|^2 (\Delta v\bar{v}+\Delta\bar{v}v)(t,x)\, dxdy \\
&\quad + \iint |w(t,y)|^2 \frac{(x-y)_\omega}{|(x-y)_\omega|} \operatorname{Re}\left[\mathcal{N}_1\nabla\bar{v} - v\nabla\bar{\mathcal{N}}_1\right](t,x)\, dxdy \\
&\quad + \frac{4}{3}\iint_{x_\omega=y_\omega} |w(t,y)|^2 \, |v(t,x)|^6 \, dxdy.
\end{aligned}
$$

Notice that integration by parts and by symmetry

$$
\nabla_x\left(\frac{(x-y)_\omega}{|(x-y)_\omega|}\right) = -\nabla_y\left(\frac{(x-y)_\omega}{|(x-y)_\omega|}\right),
$$

give us

$$
\begin{aligned}
&- \iint_{x_\omega=y_\omega} |w(t,y)|^2 \, \Delta v\bar{v}(t,x)\, dxdy \\
&= -\frac{1}{2}\iint |w(t,y)|^2 \, \Delta_x v\omega\cdot\nabla_x\left(\frac{(x-y)_\omega}{|(x-y)_\omega|}\right)\bar{v}\, dxdy \\
&= \frac{1}{2}\iint |w(t,y)|^2 \, \Delta_x v\omega\cdot\nabla_y\left(\frac{(x-y)_\omega}{|(x-y)_\omega|}\right)\bar{v}\, dxdy \\
&= -\frac{1}{2}\iint \nabla_y \, |w(t,y)|^2 \, \frac{(x-y)_\omega}{|(x-y)_\omega|}\cdot\omega\Delta_x v\bar{v}\, dxdy \\
&= \frac{1}{2}\iint \partial_\omega \, |w(t,y)|^2 \, \nabla_x v\cdot\nabla_x\left(\frac{(x-y)_\omega}{|(x-y)_\omega|}\bar{v}\right) dxdy \\
&= \frac{1}{2}\iint \partial_\omega \, |w(t,y)|^2 \, \nabla_x v\cdot\nabla_x\left(\frac{(x-y)_\omega}{|(x-y)_\omega|}\right)\bar{v}\, dxdy \\
&\quad + \frac{1}{2}\iint \partial_\omega \, |w(t,y)|^2 \, \nabla_x v\cdot\frac{(x-y)_\omega}{|(x-y)_\omega|}\nabla\bar{v}\, dxdy \\
&= \iint_{x_\omega=y_\omega} \left[\partial_\omega \, |w(t,y)|^2\right]\partial_\omega v\bar{v}\, dxdy \\
&\quad - \frac{1}{2}\iint |w(t,y)|^2 \, \nabla_y\left(\frac{(x-y)_\omega}{|(x-y)_\omega|}\right)\cdot\omega \, |\nabla v|^2 \, dxdy
\end{aligned}
$$

$$= \iint_{x_\omega = y_\omega} \left[\partial_\omega\ |w(t,y)|^2\right] \partial_\omega v \bar{v}\, dxdy + \iint_{x_\omega = y_\omega} |w(t,y)|^2\ |\partial_\omega v(t,x)|^2\ dxdy.$$

similarly,

$$\begin{aligned}
&- \iint_{x_\omega = y_\omega} |w(t,y)|^2\, \Delta \bar{v} v(t,x)\, dxdy \\
&= \iint_{x_\omega = y_\omega} \left[\partial_\omega\ |w(t,y)|^2\right] \partial_\omega \bar{v} v\, dxdy + \iint_{x_\omega = y_\omega} |w(t,y)|^2\ |\partial_\omega v(t,x)|^2\ dxdy.
\end{aligned}$$

Then

$$\begin{aligned}
&- \iint_{x_\omega = y_\omega} |w(t,y)|^2\, (\Delta v \bar{v} + \Delta \bar{v} v)(t,x)\, dxdy \\
&= \iint_{x_\omega = y_\omega} \left[\partial_\omega\ |w(t,y)|^2\right]\ \left[\partial_\omega\ |v(t,x)|^2\right]\ dxdy \\
&\quad + 2 \iint_{x_\omega = y_\omega} |w(t,y)|^2\ |\partial_\omega v(t,x)|^2\ dxdy.
\end{aligned}$$

Therefore,

$$\begin{aligned}
M_2 = 4 &\iint_{x_\omega = y_\omega} |w(t,y)|^2\ |\partial_\omega v(t,x)|^2\ dxdy \\
&+ \iint_{x_\omega = y_\omega} \left[\partial_\omega\ |w(t,y)|^2\right]\ \left[\partial_\omega\ |v(t,x)|^2\right]\ dxdy \\
&+ \iint |w(t,y)|^2\, \frac{(x-y)_\omega}{|(x-y)_\omega|}\, \mathrm{Re}\ \left[\mathcal{N}_1 \nabla \bar{v} - v \nabla \bar{\mathcal{N}}_1\right](t,x)\, dxdy \\
&+ \frac{4}{3} \iint_{x_\omega = y_\omega} |w(t,y)|^2\ |v(t,x)|^6\ dxdy.
\end{aligned}$$

Finally, using the symmetry and the following identity:

$$\begin{aligned}
&4 \iint_{x_\omega = y_\omega} \left|\partial_\omega (\overline{w(t,y)} v(t,x))\right|^2\ dxdy \\
&= -8 \iint_{x_\omega = y_\omega} \mathrm{Im}[\bar{w} \partial_\omega w](t,y)\, \mathrm{Im}[\bar{v} \partial_\omega v](t,x)\, dxdy \\
&\quad + 2 \iint_{x_\omega = y_\omega} \left[\partial_\omega\ |w(t,y)|^2\right]\ \left[\partial_\omega\ |v(t,x)|^2\right]\ dxdy \\
&\quad + 4 \iint_{x_\omega = y_\omega} |w(t,y)|^2\ |\partial_\omega v(t,x)|^2\ dxdy \\
&\quad + 4 \iint_{x_\omega = y_\omega} |v(t,x)|^2\ |\partial_\omega w(t,y)|^2\ dxdy.
\end{aligned}$$

we compute the derivative

$$\begin{aligned}
\frac{d}{dt}M_\omega(t) &= M_1 + M_2 + M_3 + M_4 \\
&= 4 \iint_{x_\omega = y_\omega} \left|\partial_\omega(\overline{w(t,y)}v(t,x))\right|^2 \, dxdy \\
&\quad + 2 \iint |w(t,y)|^2 \frac{(x-y)_\omega}{|(x-y)_\omega|} \operatorname{Re}[\mathcal{N}_1 \partial_\omega \bar{v} - v \partial_\omega \bar{\mathcal{N}_1}](t,x) \, dxdy \\
&\quad + 2 \iint |v(t,x)|^2 \frac{(y-x)_\omega}{|(y-x)_\omega|} \operatorname{Re}[\mathcal{N}_2 \partial_\omega \bar{w} - w \partial_\omega \bar{\mathcal{N}_2}](t,y) \, dxdy \\
&\quad + 2 \iint \operatorname{Im}[\bar{w}\partial_\omega w](t,y) \frac{(y-x)_\omega}{|(y-x)_\omega|} \operatorname{Im}[\bar{v}\mathcal{N}_1](t,x) \, dxdy \\
&\quad + 2 \iint \operatorname{Im}[\bar{v}\partial_\omega v](t,x) \frac{(x-y)_\omega}{|(x-y)_\omega|} \operatorname{Im}[\bar{w}\mathcal{N}_2](t,y) \, dxdy \\
&\quad + \frac{4}{3} \iint_{x_\omega = y_\omega} |w(t,y)|^2 \, |v(t,x)|^6 \, dxdy \\
&\quad + \frac{4}{3} \iint_{x_\omega = y_\omega} |v(t,x)|^2 \, |w(t,y)|^6 \, dxdy.
\end{aligned}$$

□

**Corollary 8.2.** *If $v$ and $w$ solve the same equation*

$$i\partial_t w + \Delta w = F(w) + \mathcal{N} = |w|^4 w + \mathcal{N},$$

*then*

$$\begin{aligned}
\frac{d}{dt}M_\omega(t) &= 4 \iint_{x_\omega = y_\omega} \left|\partial_\omega(\overline{w(t,y)}w(t,x))\right|^2 \, dxdy \\
&\quad + \frac{8}{3} \iint_{x_\omega = y_\omega} |w(t,x)|^2 \, |w(t,y)|^6 \, dxdy \\
&\quad + 4 \iint |w(t,x)|^2 \frac{(x-y)_\omega}{|(x-y)_\omega|} \operatorname{Re}[\mathcal{N} \partial_\omega \bar{w} - w \partial_\omega \bar{\mathcal{N}}](t,y) \, dxdy \\
&\quad + 4 \iint \operatorname{Im}[\bar{w}\partial_\omega w](t,x) \frac{(x-y)_\omega}{|(x-y)_\omega|} \operatorname{Im}[\bar{w}\mathcal{N}](t,y) \, dxdy.
\end{aligned}$$

*We will use this result in deriving the frequency-localized interaction Morawetz estimates in Chapter 9.*

**Lemma 8.3.** *Define*

$$M_y[w](t) = \int_{\mathbb{R}^2} \frac{x-y}{|x-y|} \cdot \mathrm{Im}[\bar{w}\nabla w](t,x)\,dx.$$

*Then*

$$|M_y[w](t)| \lesssim \|w(t)\|^2_{\dot{H}^{\frac{1}{2}}_x(\mathbb{R}^2)}.$$

*Proof.* Without loss of generality, take $y = 0$, then we want to show that

$$\left| \int \frac{x}{|x|} \cdot \mathrm{Im}[\bar{w}\nabla w](t,x)\,dx \right| \lesssim \|w(t)\|^2_{\dot{H}^{\frac{1}{2}}_x(\mathbb{R}^2)}.$$

By duality, we write

$$\left| \mathrm{Im}\left[ \int_{\mathbb{R}^2} \overline{w(t,x)} \partial_r w(t,x)\,dx \right] \right| \leq \|w\|_{\dot{H}^{\frac{1}{2}}_x(\mathbb{R}^2)} \|\partial_r w\|_{\dot{H}^{-\frac{1}{2}}_x(\mathbb{R}^2)},$$

where $\partial_r = \frac{x}{|x|} \cdot \nabla$. Then it is sufficient to show

$$\|\partial_r w\|_{\dot{H}^{-\frac{1}{2}}_x(\mathbb{R}^2)} \leq \|w\|_{\dot{H}^{\frac{1}{2}}_x(\mathbb{R}^2)}.$$

which is equivalent to showing

$$\left\| \frac{x}{|x|} f \right\|_{\dot{H}^{\frac{1}{2}}_x(\mathbb{R}^2)} \leq \|f\|_{\dot{H}^{\frac{1}{2}}_x(\mathbb{R}^2)}.$$

for any $f$ for which the right hand side is finite.

In fact, the inequality above follows from interpolating between the following two bounds,

$$\left\| \frac{x}{|x|} f \right\|_{L^3_x(\mathbb{R}^2)} \leq \|f\|_{L^3_x(\mathbb{R}^2)},$$

$$\left\| \frac{x}{|x|} f \right\|_{\dot{W}^{1,\frac{3}{2}}_x(\mathbb{R}^2)} \leq \|f\|_{\dot{W}^{1,\frac{3}{2}}_x(\mathbb{R}^2)}.$$

The first estimate is trivial. For the second one, we recall Hardy's inequality, that is, for $0 < s < d$ and $1 < r < \frac{d}{s}$,

$$\left\| \frac{1}{|x|^s} f \right\|_{L^r_x(\mathbb{R}^d)} \lesssim \|\nabla^s f\|_{L^r_x(\mathbb{R}^d)}.$$

Hence, taking $s=1$, $r=\frac{3}{2}$ and $d=2$, we obtain

$$\begin{aligned}
&\left\|\frac{x}{|x|}f\right\|_{\dot{W}_x^{1,\frac{3}{2}}(\mathbb{R}^2)} = \left\|\nabla\left(\frac{x}{|x|}f\right)\right\|_{\dot{W}_x^{0,\frac{3}{2}}(\mathbb{R}^2)} \\
\leq &\left\|\nabla\left(\frac{x}{|x|}\right)f\right\|_{L_x^{\frac{3}{2}}(\mathbb{R}^2)} + \left\|\frac{x}{|x|}\cdot\nabla f\right\|_{L^{\frac{3}{2}}(\mathbb{R}^2)} \\
\lesssim &\left\|\frac{1}{|x|}f\right\|_{L_x^{\frac{3}{2}}(\mathbb{R}^2)} + \|f\|_{\dot{W}^{1,\frac{3}{2}}(\mathbb{R}^2)} \\
\lesssim &\|\nabla f\|_{L_x^{\frac{3}{2}}(\mathbb{R}^2)} + \|f\|_{\dot{W}_x^{1,\frac{3}{2}}(\mathbb{R}^2)} \simeq \|f\|_{\dot{W}_x^{1,\frac{3}{2}}(\mathbb{R}^2)}.
\end{aligned}$$

Now we complete the proof of Lemma 8.3. $\square$

# CHAPTER 9

# IMPOSSIBILITY OF QUASI-SOLITON SOLUTIONS

In this chapter, we prove a frequency-localized interaction Morawetz estimate and use it to preclude the existence of quasi-soliton solutions.

## 9.1 Frequency-localized interaction Morawetz estimate

**Theorem 9.1** (Frequency-localized interaction Morawetz estimate)**.** *If $u$ is an almost periodic solution to* (1.6.1) *on* $[0,T]$ *with* $\int_0^T N(t)\,dt = K$*, then*

$$\left\| |\nabla|^{\frac{1}{2}} \left|P_{\geq \varepsilon_3 K^{-1}} u(t,x)\right|^2 \right\|_{L_t^2 L_x^2([0,T]\times\mathbb{R}^2)} \lesssim o(K), \tag{9.1.1}$$

*where $o(K)$ is a quantity, such that $\frac{o(K)}{K} \to 0$ as $K \nearrow \infty$.*

*Proof.* Suppose $[0,T]$ is an interval such that for some integer $k_0$,

$$\int_0^T \int_{\mathbb{R}^2} |u(t,x)|^8 \, dxdt = 2^{k_0}.$$

Note that $\int_0^T N(t)dt = K$. Hence in order to apply Theorem 7.1, we need to do the scaling $u_\lambda(x) = \sqrt{\lambda} u(\lambda^2 t, \lambda x)$, where $\lambda = \frac{K}{\varepsilon_3 2^{k_0}}$. Since $\int N(t)^2\,dt$ scales like $\|u\|_{L_t^8 L_x^8}^8$, $N(t)$ under the scaling should be $N_\lambda(t) = \lambda N(\lambda^2 t)$, therefore

$$\int_0^{\frac{T}{\lambda^2}} N_\lambda(t)\,dt = \varepsilon_3 2^{k_0}.$$

Now we can apply Theorem 7.1, and have

$$\|u_\lambda\|_{\tilde{X}_{k_0}([0,\frac{T}{\lambda^2}]\times\mathbb{R}^2)} \lesssim 1.$$

Note that in Theorem 7.1 we only care about the low frequency component of the solution $u$, and already had a good upper bound for it. From now on, we will focus on the high frequency component of $u$.

Let $w = P_{\geq 2^{-k_0}} u_\lambda$, hence $w$ satisfies the following equation:

$$i\partial_t w + \Delta w = P_{\geq 2^{-k_0}} F(u_\lambda) := F(w) + \mathcal{N},$$

where

$$\mathcal{N} = P_{\geq 2^{-k_0}} F(u_\lambda) - F(w).$$

Let

$$M_\omega(t) = \iint_{\mathbb{R}^2\times\mathbb{R}^2} |w(t,y)|^2 \frac{(x-y)_\omega}{|(x-y)_\omega|} \cdot \mathrm{Im}[\bar{w}\partial_\omega w](t,x)\, dxdy.$$

By Corollary 8.2, we get

$$\begin{aligned}
\frac{d}{dt}M_\omega(t) = {} & 4\iint_{x_\omega=y_\omega} \left|\partial_\omega(\overline{w(t,y)}w(t,x))\right|^2 dxdy + \frac{8}{3}\iint_{x_\omega=y_\omega} |w(t,x)|^2\, |w(t,y)|^6\, dxdy \\
& + 4\iint |w(t,x)|^2 \frac{(x-y)_\omega}{|(x-y)_\omega|} \mathrm{Re}[\mathcal{N}\partial_\omega \bar{w} - w\partial_\omega \bar{\mathcal{N}}](t,y)\, dxdy \qquad (9.1.2) \\
& + 4\iint \mathrm{Im}[\bar{w}\partial_\omega w](t,x) \frac{(x-y)_\omega}{|(x-y)_\omega|} \mathrm{Im}[\bar{w}\mathcal{N}](t,y)\, dxdy.
\end{aligned}$$

Recall $\omega \in S^1$, then we can write $\partial_\omega = \nabla\cdot\omega = \cos(\omega)\partial_1 + \sin(\omega)\partial_2$. Hence there exists $C$ such that

$$\int_{\omega\in S^1} \frac{x_\omega}{|x_\omega|}(\nabla\cdot\omega)\, d\omega = C\frac{x}{|x|}\cdot\nabla. \qquad (9.1.3)$$

Therefore,

$$M(t) = \int_{\omega\in S^1} M_\omega(t)\, d\omega = C\iint |w(t,y)|^2 \frac{x-y}{|x-y|} \mathrm{Im}[\bar{w}\nabla w](t,x)\, dxdy. \qquad (9.1.4)$$

Now we move the last two terms in (9.1.2) to the left hand side and integrate on both sides over $\omega$. The properties of the Radon transform in [52] imply:

$$\iiint_{x_\omega=y_\omega} \left|\partial_\omega(\overline{w(t,y)}w(t,x))\right|^2 dxdyd\omega + \iiint_{x_\omega=y_\omega} |w(t,y)|^2 \, |w(t,x)|^6 \, dxdyd\omega$$
$$\gtrsim \left\| |\nabla|^{\frac{1}{2}} \, |w(t,x)|^2 \right\|^2_{L^2_x(\mathbb{R}^2)}, \tag{9.1.5}$$

combining (9.1.3), then we obtain

$$\begin{aligned}\left\| |\nabla|^{\frac{1}{2}} \, |w(t,x)|^2 \right\|^2_{L^2_x(\mathbb{R}^2)} \lesssim{}& \frac{d}{dt}M(t) \\ &- \iint |w(t,y)|^2 \frac{x-y}{|x-y|} \operatorname{Re}[\mathcal{N}\nabla\bar{w} - w\nabla\bar{\mathcal{N}}](t,x)\,dxdy \\ &- \iint \operatorname{Im}[\bar{w}\nabla w](t,y)\frac{x-y}{|x-y|}\operatorname{Im}[\bar{w}\mathcal{N}](t,x)\,dxdy,\end{aligned} \tag{9.1.6}$$

where $M(t)$ is calculated in (9.1.4).

Then we integrate on both sides of (9.1.6) over time $t$, and the fundamental theorem of calculus in time yields,

$$\left\| |\nabla|^{\frac{1}{2}} \, |w(t,x)|^2 \right\|^2_{L^2_tL^2_x([0,\frac{T}{\lambda^2}]\times\mathbb{R}^2)}$$
$$\lesssim \sup_{t\in[0,\frac{T}{\lambda^2}]} \left| \iint |w(t,y)|^2 \frac{x-y}{|x-y|} \operatorname{Im}[\bar{w}\nabla w](t,x)\,dxdy \right| \tag{9.1.7}$$
$$+ \left| \int_0^{\frac{T}{\lambda^2}} \iint |w(t,y)|^2 \frac{x-y}{|x-y|} \operatorname{Re}[\mathcal{N}\nabla\bar{w} - w\nabla\bar{\mathcal{N}}](t,x)\,dxdydt \right| \tag{9.1.8}$$
$$+ \left| \int_0^{\frac{T}{\lambda^2}} \iint \operatorname{Im}[\bar{w}\nabla w](t,y)\frac{x-y}{|x-y|}\operatorname{Im}[\bar{w}\mathcal{N}](t,x)\,dxdydt \right|. \tag{9.1.9}$$

Next, we will estimate the terms (9.1.7) in Lemma 9.2, (9.1.8) in Lemma 9.4 and (9.1.9) in Lemma 9.5. In the remainder of the proof all spacetime norms are over $[0, \frac{T}{\lambda^2}] \times \mathbb{R}^2$, unless indicated otherwise.

**Lemma 9.2.** *There exists $\eta = \eta(K) > 0$ satisfying*

$$(9.1.7) \lesssim \eta 2^{k_0}.$$

*Proof.* By Lemma 8.3, Bernstein's inequality and (6.2.4), we obtain

$$
\begin{aligned}
(9.1.7) &= \sup_{t\in[0,\frac{T}{\lambda^2}]} \left| \int |w(t,y)|^2 \left( \int \frac{x-y}{|x-y|} \operatorname{Im}[\bar{w}\nabla w](t,x)\,dx \right) dy \right| \\
&\lesssim \|w\|_{L_t^\infty L_x^2}^2 \, \|w\|_{L_t^\infty \dot{H}_x^{\frac{1}{2}}}^2 .
\end{aligned}
$$

Now we claim

$$
\|w\|_{L_t^\infty L_x^2}^2 \lesssim \eta 2^{k_0}. \tag{9.1.10}
$$

Assuming the claim is true, it is easy to see Lemma 9.2 holds.

Then we are left to show the claim (9.1.10).

*Proof of* (9.1.10). By Definition 4.5, we know that for any $\eta$, there exists $c(\eta)$ such that

$$
\left\| \, |\nabla|^{\frac{1}{2}} P_{\leq c(\eta)N(t)} u \right\|_{L_t^\infty L_x^2}^2 \leq \eta,
$$

combining with the fact that $N(t) \geq 1$, then we obtain

$$
\left\| \, |\nabla|^{\frac{1}{2}} P_{\leq c(\eta)} u \right\|_{L_t^\infty L_x^2}^2 \leq \eta.
$$

Therefore under the scaling $u_\lambda(x) = \sqrt{\lambda} u(\lambda^2 t, \lambda x)$, $N_\lambda(t) = \lambda N(\lambda^2 t)$, we have $N_\lambda(t) \geq \lambda$, and

$$
\left\| \, |\nabla|^{\frac{1}{2}} P_{\leq c(\eta)\lambda} u_\lambda \right\|_{L_t^\infty L_x^2}^2 \leq \eta.
$$

Now using Bernstein's inequality, Definition 4.5 and $\lambda = \frac{K}{\varepsilon_3 2^{k_0}}$, we can write

$$
\begin{aligned}
\|w\|_{L_t^\infty L_x^2} = \|P_{>2^{-k_0}} u_\lambda\|_{L_t^\infty L_x^2} &\leq \left\|P_{2^{-k_0} < \cdot < c(\eta)\lambda} u_\lambda\right\|_{L_t^\infty L_x^2} + \left\|P_{>c(\eta)\lambda} u_\lambda\right\|_{L_t^\infty L_x^2} \\
&\lesssim 2^{\frac{k_0}{2}} \left\| \, |\nabla|^{\frac{1}{2}} P_{2^{-k_0} < \cdot < c(\eta)} u_\lambda \right\|_{L_t^\infty L_x^2} + c(\eta)^{-\frac{1}{2}} \lambda^{-\frac{1}{2}} \left\| \, |\nabla|^{\frac{1}{2}} P_{>c(\eta)\lambda} u_\lambda \right\|_{L_t^\infty L_x^2} \\
&\lesssim 2^{\frac{k_0}{2}} \eta^{\frac{1}{2}} + c(\eta)^{-\frac{1}{2}} \lambda^{-\frac{1}{2}} = 2^{\frac{k_0}{2}} \eta^{\frac{1}{2}} + \left( \frac{\varepsilon_3 2^{k_0}}{c(\eta) K} \right)^{\frac{1}{2}}.
\end{aligned}
$$

To prove (9.1.10), we only need to demand $\frac{\varepsilon_3}{c(\eta)K} \leq \eta$. It is possible for us to choose $\eta = \eta(K)$ such that $Kc(\eta)\eta = 1$, therefore we complete the proof of (9.1.10).

□

**Remark 9.3.** For any admissible pair $(q, r)$, by Hölder's inequality, (9.1.10) and Proposition 6.22, we can write

$$\|w\|_{L_t^q L_x^r} \lesssim \|w\|_{L_t^\infty L_x^2}^{\frac{1}{r}} \|w\|_{L_t^{q(1-\frac{1}{r})} L_x^{2r-2}}^{1-\frac{1}{r}} \lesssim \left(\eta^{\frac{1}{2}} 2^{\frac{k_0}{2}}\right)^{\frac{1}{r}} \left(2^{\frac{k_0}{2}} \|u_\lambda\|_{\tilde{X}_{k_0}([0,\frac{T}{\lambda^2}]\times\mathbb{R}^2)}\right)^{1-\frac{1}{r}}$$
$$= \eta^{\frac{1}{2r}} 2^{\frac{k_0}{2}} \|u_\lambda\|_{\tilde{X}_{k_0}([0,\frac{T}{\lambda^2}]\times\mathbb{R}^2)}^{1-\frac{1}{r}}. \tag{9.1.11}$$

□

**Lemma 9.4.** *There exists $\eta = \eta(K) > 0$ satisfying*

$$(9.1.8) \lesssim \eta 2^{k_0} \left(1 + \|u_\lambda\|_{\tilde{X}_{k_0}([0,\frac{T}{\lambda^2}]\times\mathbb{R}^2)}\right)^6.$$

*Proof.* We first define the momentum bracket:

$$\{a, b\}_p = \mathrm{Re}[a\nabla\bar{b} - b\nabla\bar{a}].$$

Realizing that

$$\{F(u), u\}_p = |u|^4 u\nabla\bar{u} - u\nabla(|u|^4 \bar{u}) = -|u|^2 \nabla(|u|^4) = -\frac{2}{3}\nabla(|u|^6),$$

we can rewrite the factor $\mathrm{Re}[\mathcal{N}\nabla\bar{w} - w\nabla\bar{\mathcal{N}}]$ in (9.1.8) into

$$\begin{aligned}
\{\mathcal{N}, w\}_p &= \{F(u_\lambda), u_\lambda\}_p - \{F(u_{lo}), u_{lo}\}_p - \{F(u_{hi}), u_{hi}\}_p \\
&\quad - \{F(u_\lambda) - F(u_{lo}), u_{lo}\}_p - \{P_{lo}F(u_\lambda), u_{hi}\}_p \\
&= -\frac{2}{3}\left(\nabla(|u_\lambda|^6) - \nabla(|u_{lo}|^6) - \nabla(|u_{hi}|^6)\right) - \{F(u_\lambda) - F(u_{lo}), u_{lo}\}_p \\
&\quad - \{P_{lo}F(u_\lambda), u_{hi}\}_p \\
&:= \mathrm{I} + \mathrm{II} + \mathrm{III},
\end{aligned}$$

where $P_{hi} := P_{\geq 2^{-k_0}}$, $P_{lo} := P_{<2^{-k_0}}$, $u_{hi} := P_{\geq 2^{-k_0}} u_\lambda = w$ and $u_{lo} := P_{<2^{-k_0}} u_\lambda$.

After an integration by parts, I contributes to (9.1.8) a multiple of

$$\sum_{i=1}^{5} \int_0^{\frac{T}{\lambda^2}} \iint |w(t,y)|^2 \frac{1}{|x-y|} |u_{hi}(t,x)|^i |u_{lo}(t,x)|^{6-i} \, dxdydt.$$

We know that $\{a,b\}_p = \nabla\mathcal{O}(ab) + \mathcal{O}(a\nabla b)$. We use the expression $\mathcal{O}(X)$ to denote a finite linear combination of terms that resemble $X$ up to Littlewood-Paley projections and complex conjugation. Then

$$\mathrm{II} = -\ \{F(u_\lambda) - F(u_{lo}), u_{lo}\}_p = \sum_{i=1}^{5} \nabla\mathcal{O}(u_{hi}^i u_{lo}^{6-i}) + \sum_{i=1}^{5} \mathcal{O}(u_{hi}^i u_{lo}^{5-i}\nabla u_{lo}).$$

Integrating by parts for the first term and bringing absolute values inside the integrals for the second term, we find that II contributes to (9.1.8) a multiple of

$$\begin{aligned}
&\sum_{i=1}^{5}\int_0^{\frac{T}{\lambda^2}} \iint |w(t,y)|^2 \frac{1}{|x-y|} \ |u_{hi}(t,x)|^i \ |u_{lo}(t,x)|^{6-i} \ dxdydt \\
+&\sum_{i=1}^{5}\int_0^{\frac{T}{\lambda^2}} \iint |w(t,y)|^2 \ |u_{hi}(t,x)|^i \ |u_{lo}(t,x)|^{5-i} \ |\nabla u_{lo}(t,x)| \ dxdydt.
\end{aligned}$$

Finally, integrating by parts when the derivative (from the definition of the momentum bracket) falls on $u_{hi}$, we estimate the contribution of III to (9.1.8) by a multiple of

$$\begin{aligned}
&\int_0^{\frac{T}{\lambda^2}} \iint |w(t,y)|^2 \frac{1}{|x-y|} \ |u_{hi}(t,x)| \ |P_{lo}F(u_\lambda(t,x))| \ dxdydt \\
+&\int_0^{\frac{T}{\lambda^2}} \iint |w(t,y)|^2 \ |u_{hi}(t,x)| \ |\nabla P_{lo}F(u_\lambda(t,x))| \ dxdydt.
\end{aligned}$$

Now the contributions of all these three terms to (9.1.8) are at most a multiple of

$$\begin{aligned}
&\sum_{i=1}^{5}\int_0^{\frac{T}{\lambda^2}} \iint |w(t,y)|^2 \frac{1}{|x-y|} \ |u_{hi}(t,x)|^i \ |u_{lo}(t,x)|^{6-i} \ dxdydt \\
+&\int_0^{\frac{T}{\lambda^2}} \iint |w(t,y)|^2 \frac{1}{|x-y|} \ |u_{hi}(t,x)| \ |P_{lo}F(u_\lambda(t,x))| \ dxdydt \\
+&\sum_{i=1}^{5}\int_0^{\frac{T}{\lambda^2}} \iint |w(t,y)|^2 \ |u_{hi}(t,x)|^i \ |u_{lo}(t,x)|^{5-i} \ |\nabla u_{lo}(t,x)| \ dxdydt \\
+&\int_0^{\frac{T}{\lambda^2}} \iint |w(t,y)|^2 \ |u_{hi}(t,x)| \ |\nabla P_{lo}F(u_\lambda(t,x))| \ dxdydt.
\end{aligned} \tag{9.1.12}$$

Then using Hölder's inequality, Hardy-Littlewood-Sobolev inequality and Bernstein's inequality, we have

$$
\begin{aligned}
(9.1.12) \lesssim & \sum_{i=1}^{4} \|w\|^2_{L_t^6 L_x^3} \left\| u_{hi}^i u_{lo}^{6-i} \right\|_{L_t^{\frac{3}{2}} L_x^{\frac{6}{5}}} && (9.1.13)\\
& + \iint |w(t,y)|^2 \frac{1}{|x-y|} |u_{hi}(t,x)|^5 \, |u_{lo}(t,x)| \, dxdydt && (9.1.14)\\
& + \|w\|^2_{L_t^6 L_x^3} \left\| u_{hi} P_{lo}(u_{hi}^5) \right\|_{L_t^{\frac{3}{2}} L_x^{\frac{6}{5}}} && (9.1.15)\\
& + \sum_{i=1}^{5} \|w\|^2_{L_t^\infty L_x^2} \left\| u_{hi}^i u_{lo}^{5-i} \nabla u_{lo} \right\|_{L_t^1 L_x^1} && (9.1.16)\\
& + \sum_{i=0}^{5} \|w\|^2_{L_t^\infty L_x^2} \left\| u_{hi} \nabla P_{lo} \mathcal{O}(u_{hi}^i u_{lo}^{5-i}) \right\|_{L_t^1 L_x^1}. && (9.1.17)
\end{aligned}
$$

(i) First, consider (9.1.13). By Hölder's inequality, Sobolev embedding, Bernstein's inequality and Proposition 6.22, we can estimate the following four terms in (9.1.13)

$$
\begin{aligned}
\left\| \mathcal{O}(u_{hi} u_{lo}^5) \right\|_{L_t^{\frac{3}{2}} L_x^{\frac{6}{5}}} & \lesssim \|u_{hi}\|_{L_t^3 L_x^6} \|u_{lo}\|^5_{L_t^{15} L_x^{\frac{15}{2}}} \\
& \lesssim \|u_{hi}\|_{L_t^3 L_x^6} 2^{-\frac{k_0}{2}} \left\| |\nabla|^{\frac{1}{2}} u_{lo} \right\|^5_{L_t^{15} L_x^{\frac{30}{13}}} \\
& \lesssim 2^{\frac{k_0}{2}} \|u_\lambda\|_{\tilde{X}_{k_0}} 2^{-\frac{k_0}{2}} \left\| |\nabla|^{\frac{1}{2}} u_{lo} \right\|^5_{L_t^{15} L_x^{\frac{30}{13}}} \\
& \lesssim \|u_\lambda\|^6_{\tilde{X}_{k_0}([0,\frac{T}{\lambda^2}]\times\mathbb{R}^2)},
\end{aligned}
$$

$$
\begin{aligned}
\left\| \mathcal{O}(u_{hi}^2 u_{lo}^4) \right\|_{L_t^{\frac{3}{2}} L_x^{\frac{6}{5}}} & \lesssim \|u_{hi}\|^2_{L_t^3 L_x^6} \|u_{lo}\|^4_{L_t^\infty L_x^8} \\
& \lesssim 2^{k_0} \|u_\lambda\|^2_{\tilde{X}_{k_0}} 2^{-k_0} \|u_{lo}\|^4_{L_t^\infty L_x^4} \\
& \lesssim \|u_\lambda\|^2_{\tilde{X}_{k_0}([0,\frac{T}{\lambda^2}]\times\mathbb{R}^2)},
\end{aligned}
$$

$$
\begin{aligned}
\left\| \mathcal{O}(u_{hi}^3 u_{lo}^3) \right\|_{L_t^{\frac{3}{2}} L_x^{\frac{6}{5}}} & \lesssim \|u_{hi}\|^3_{L_t^{\frac{9}{2}} L_x^{\frac{18}{5}}} \|u_{lo}\|^3_{L_t^\infty L_x^\infty} \\
& \lesssim 2^{\frac{3k_0}{2}} \|u_\lambda\|^3_{\tilde{X}_{k_0}} 2^{-\frac{3k_0}{2}} \|u_{lo}\|^3_{L_t^\infty L_x^4}
\end{aligned}
$$

$$\lesssim \|u_\lambda\|^3_{\tilde{X}_{k_0}([0,\frac{T}{\lambda^2}]\times\mathbb{R}^2)},$$

$$\begin{aligned}
\left\|\mathcal{O}(u_{hi}^4 u_{lo}^2)\right\|_{L_t^{\frac{3}{2}} L_x^{\frac{6}{5}}} &\lesssim \|u_{hi}\|^2_{L_t^\infty L_x^4} \|u_{hi}\|^2_{L_t^3 L_x^6} \|u_{lo}\|^2_{L_t^\infty L_x^\infty} \\
&\lesssim 2^{k_0} \|u_\lambda\|^4_{\tilde{X}_{k_0}} 2^{-k_0} \|u_{lo}\|^2_{L_t^\infty L_x^4} \\
&\lesssim \|u_\lambda\|^4_{\tilde{X}_{k_0}([0,\frac{T}{\lambda^2}]\times\mathbb{R}^2)}.
\end{aligned}$$

Then we have

$$(9.1.13) = \sum_{i=1}^{4} \|w\|^2_{L_t^6 L_x^3} \left\|u_{hi}^i u_{lo}^{6-i}\right\|_{L_t^{\frac{3}{2}} L_x^{\frac{6}{5}}} \lesssim \eta^{\frac{1}{3}} 2^{k_0} \left(1 + \|u_\lambda\|_{\tilde{X}_{k_0}([0,\frac{T}{\lambda^2}]\times\mathbb{R}^2)}\right)^{\frac{22}{3}}.$$

(ii) Next, take (9.1.14). In fact, we consider the following two scenarios:

- If $|u_{lo}| \le \delta\, |u_{hi}|$ for some small $\delta > 0$, this contribution will be absorbed into the following term

$$\begin{aligned}
&\iiint_{x_\omega = y_\omega} |w(t,x)|^2\, |w(t,y)|^6\, dxdyd\omega \\
\simeq &\iint_{x_\omega = y_\omega} \frac{1}{|x-y|}\, |w(t,x)|^2\, |w(t,y)|^6\, dxdy.
\end{aligned}$$

- If $|u_{hi}| \le \delta^{-1}\, |u_{lo}|$, we can estimate the contribution of this term by

$$\int_0^{\frac{T}{\lambda^2}} \iint |w(t,y)|^2 \frac{1}{|x-y|}\, |u_{hi}(t,x)|^4\, |u_{lo}(t,x)|^2\, dxdydt.$$

(iii) Take (9.1.15) and apply Hölder's inequality, (9.1.11), Bernstein's inequality and Proposition 6.22,

$$\begin{aligned}
(9.1.15) = &\|w\|^2_{L_t^6 L_x^3} \left\|u_{hi} P_{lo}\mathcal{O}(u_{hi}^5)\right\|_{L_t^{\frac{3}{2}} L_x^{\frac{6}{5}}} \\
\lesssim &\eta^{\frac{1}{3}} 2^{k_0} \|u_\lambda\|^{\frac{4}{3}}_{\tilde{X}_{k_0}([0,\frac{T}{\lambda^2}]\times\mathbb{R}^2)} \|u_{hi}\|_{L_t^\infty L_x^2} \left\|P_{lo}(u_{hi}^5)\right\|_{L_t^{\frac{3}{2}} L_x^3} \\
\lesssim &\eta^{\frac{1}{3}} 2^{k_0} \|u_\lambda\|^{\frac{4}{3}}_{\tilde{X}_{k_0}([0,\frac{T}{\lambda^2}]\times\mathbb{R}^2)} 2^{\frac{k_0}{2}} \|u_{hi}\|_{L_t^\infty \dot{H}_x^{\frac{1}{2}}} 2^{-\frac{4k_0}{3}} \left\|u_{hi}^5\right\|_{L_t^{\frac{3}{2}} L_x^1}
\end{aligned}$$

$$\lesssim \eta^{\frac{1}{3}} 2^{k_0} \ \|u_\lambda\|^{\frac{4}{3}}_{\tilde{X}_{k_0}([0,\frac{T}{\lambda^2}]\times\mathbb{R}^2)} 2^{-\frac{5k_0}{6}} \ \|u_{hi}\|^{\frac{5}{3}}_{L_t^{\frac{5}{2}} L_x^{10}} \ \|u_{hi}\|^{\frac{10}{3}}_{L_t^\infty L_x^4}$$

$$\lesssim \eta^{\frac{1}{3}} 2^{k_0} \ \|u_\lambda\|^{3}_{\tilde{X}_{k_0}([0,\frac{T}{\lambda^2}]\times\mathbb{R}^2)} .$$

(iv) Take (9.1.16) and use Hölder's inequality, Sobolev embedding, Bernstein's inequality and Proposition 6.22

$$\begin{aligned}\left\|\mathcal{O}(u_{hi}u_{lo}^4)\nabla u_{lo}\right\|_{L_t^1 L_x^1} &\lesssim \|u_{hi}\|_{L_t^4 L_x^4} \ \|u_{lo}\|^4_{L_t^8 L_x^8} \, 2^{-\frac{k_0}{2}} \ \left\| |\nabla|^{\frac{1}{2}} u_{lo} \right\|_{L_t^4 L_x^4} \\ &\lesssim \|u_\lambda\|^6_{\tilde{X}_{k_0}([0,\frac{T}{\lambda^2}]\times\mathbb{R}^2)} ,\end{aligned}$$

$$\begin{aligned}\left\|\mathcal{O}(u_{hi}^2 u_{lo}^3)\nabla u_{lo}\right\|_{L_t^1 L_x^1} &\lesssim \|u_{hi}\|^2_{L_t^4 L_x^4} \ \|u_{lo}\|^3_{L_t^8 L_x^8} \ \|\nabla u_{lo}\|_{L_t^8 L_x^8} \\ &\lesssim \|u_{hi}\|^2_{L_t^4 L_x^4} \ \|u_{lo}\|^3_{L_t^8 L_x^8} \, 2^{-k_0} \ \left\| |\nabla|^{\frac{1}{2}} u_{lo} \right\|_{L_t^8 L_x^{\frac{8}{3}}} \\ &\lesssim \|u_\lambda\|^6_{\tilde{X}_{k_0}([0,\frac{T}{\lambda^2}]\times\mathbb{R}^2)} ,\end{aligned}$$

$$\begin{aligned}\left\|\mathcal{O}(u_{hi}^3 u_{lo}^2)\nabla u_{lo}\right\|_{L_t^1 L_x^1} &\lesssim \|u_{hi}\|^3_{L_t^4 L_x^4} \ \|u_{lo}\|^2_{L_t^8 L_x^8} \, 2^{-\frac{k_0}{2}} \ \left\| |\nabla|^{\frac{1}{2}} u_{lo} \right\|_{L_t^\infty L_x^\infty} \\ &\lesssim \|u_{hi}\|^3_{L_t^4 L_x^4} \ \|u_{lo}\|^2_{L_t^8 L_x^8} \, 2^{-\frac{3k_0}{2}} \ \left\| |\nabla|^{\frac{1}{2}} u_{lo} \right\|_{L_t^\infty L_x^2} \\ &\lesssim \|u_\lambda\|^5_{\tilde{X}_{k_0}([0,\frac{T}{\lambda^2}]\times\mathbb{R}^2)} ,\end{aligned}$$

$$\begin{aligned}\left\|\mathcal{O}(u_{hi}^4 u_{lo})\nabla u_{lo}\right\|_{L_t^1 L_x^1} &\lesssim \|u_{hi}\|^4_{L_t^4 L_x^4} \ \|u_{lo}\|_{L_t^\infty L_x^\infty} \, 2^{-\frac{k_0}{2}} \ \left\| |\nabla|^{\frac{1}{2}} u_{lo} \right\|_{L_t^\infty L_x^\infty} \\ &\lesssim \|u_{hi}\|^4_{L_t^4 L_x^4} \, 2^{-\frac{k_0}{2}} \ \|u_{lo}\|_{L_t^\infty L_x^2} \, 2^{-\frac{3k_0}{2}} \ \left\| |\nabla|^{\frac{1}{2}} u_{lo} \right\|_{L_t^\infty L_x^2} \\ &\lesssim \|u_\lambda\|^4_{\tilde{X}_{k_0}([0,\frac{T}{\lambda^2}]\times\mathbb{R}^2)} ,\end{aligned}$$

$$\begin{aligned}\left\|\mathcal{O}(u_{hi}^5)\nabla u_{lo}\right\|_{L_t^1 L_x^1} &\lesssim \|u_{hi}\|^3_{L_t^3 L_x^6} \ \|u_{hi}\|^2_{L_t^\infty L_x^4} \, 2^{-\frac{k_0}{2}} \ \left\| |\nabla|^{\frac{1}{2}} u_{lo} \right\|_{L_t^\infty L_x^\infty} \\ &\lesssim \|u_{hi}\|^3_{L_t^3 L_x^6} \ \|u_{hi}\|^2_{L_t^\infty L_x^4} \, 2^{-\frac{3k_0}{2}} \ \left\| |\nabla|^{\frac{1}{2}} u_{lo} \right\|_{L_t^\infty L_x^2}\end{aligned}$$

$$\lesssim \|u_\lambda\|^3_{\tilde{X}_{k_0}([0,\frac{T}{\lambda^2}]\times\mathbb{R}^2)} .$$

Therefore, put the calculations above together

$$(9.1.16) = \sum_{i=1}^{5} \|w\|^2_{L_t^\infty L_x^2} \left\|u_{hi}^i u_{lo}^{5-i} \nabla u_{lo}\right\|_{L_t^1 L_x^1} \lesssim \eta 2^{k_0} \left(1 + \|u_\lambda\|_{\tilde{X}_{k_0}([0,\frac{T}{\lambda^2}]\times\mathbb{R}^2)}\right)^6 .$$

(v) Finally, take (9.1.17). Apply Hölder's inequality, Sobolev embedding, Bernstein's inequality and Proposition 6.22

$$\begin{aligned}
\left\|u_{hi} \nabla P_{lo} \mathcal{O}(u_{lo}^5)\right\|_{L_t^1 L_x^1} &\lesssim \|u_{hi}\|_{L_t^\infty L_x^2} \left\|\nabla P_{lo} \mathcal{O}(u_{lo}^5)\right\|_{L_t^1 L_x^2} \\
&\lesssim 2^{\frac{k_0}{2}} \left\| |\nabla|^{\frac{1}{2}} u_{hi}\right\|_{L_t^\infty L_x^2} \|\nabla u_{lo}\|_{L_t^4 L_x^4} \|u_{lo}\|^4_{L_t^{\frac{16}{3}} L_x^{16}} \\
&\lesssim \left\| |\nabla|^{\frac{1}{2}} u_{lo}\right\|_{L_t^4 L_x^4} \|u_{lo}\|^4_{L_t^{\frac{16}{3}} L_x^{16}} \\
&\lesssim \|u_\lambda\|^5_{\tilde{X}_{k_0}([0,\frac{T}{\lambda^2}]\times\mathbb{R}^2)} ,
\end{aligned}$$

$$\begin{aligned}
\left\|u_{hi} \nabla P_{lo} \mathcal{O}(u_{hi} u_{lo}^4)\right\|_{L_t^1 L_x^1} &\lesssim \|u_{hi}\|_{L_t^4 L_x^4} \left\|\nabla P_{lo} \mathcal{O}(u_{hi} u_{lo}^4)\right\|_{L_t^{\frac{4}{3}} L_x^{\frac{4}{3}}} \\
&\lesssim \|u_{hi}\|_{L_t^4 L_x^4} 2^{-k_0} \left\|u_{hi} u_{lo}^4\right\|_{L_t^{\frac{4}{3}} L_x^{\frac{4}{3}}} \\
&\lesssim \|u_{hi}\|_{L_t^4 L_x^4} 2^{-k_0} \|u_{hi}\|_{L_t^\infty L_x^2} \|u_{lo}\|^4_{L_t^{\frac{16}{3}} L_x^{16}} \\
&\lesssim \|u_\lambda\|^5_{\tilde{X}_{k_0}([0,\frac{T}{\lambda^2}]\times\mathbb{R}^2)} ,
\end{aligned}$$

$$\begin{aligned}
\left\|u_{hi} \nabla P_{lo} \mathcal{O}(u_{hi}^2 u_{lo}^3)\right\|_{L_t^1 L_x^1} &\lesssim \|u_{hi}\|_{L_t^4 L_x^4} \left\|\nabla P_{lo} \mathcal{O}(u_{hi}^2 u_{lo}^3)\right\|_{L_t^{\frac{4}{3}} L_x^{\frac{4}{3}}} \\
&\lesssim \|u_{hi}\|_{L_t^4 L_x^4} 2^{-k_0} \|u_{hi}\|^2_{L_t^4 L_x^4} \|u_{lo}\|^3_{L_t^{12} L_x^{12}} \\
&\lesssim \|u_{hi}\|_{L_t^4 L_x^4} 2^{-k_0} \|u_{hi}\|^2_{L_t^4 L_x^4} 2^{-\frac{k_0}{2}} \|u_{lo}\|^3_{L_t^{12} L_x^6} \\
&\lesssim \|u_\lambda\|^3_{\tilde{X}_{k_0}([0,\frac{T}{\lambda^2}]\times\mathbb{R}^2)} ,
\end{aligned}$$

$$\left\|u_{hi} \nabla P_{lo} \mathcal{O}(u_{hi}^3 u_{lo}^2)\right\|_{L_t^1 L_x^1} \lesssim \|u_{hi}\|_{L_t^4 L_x^4} \left\|\nabla P_{lo} \mathcal{O}(u_{hi}^3 u_{lo}^2)\right\|_{L_t^{\frac{4}{3}} L_x^{\frac{4}{3}}}$$

$$
\begin{aligned}
&\lesssim \|u_{hi}\|_{L_t^4 L_x^4} \, 2^{-k_0} \, \|u_{hi}\|_{L_t^4 L_x^4}^3 \, \|u_{lo}\|_{L_t^\infty L_x^\infty}^2 \\
&\lesssim \|u_{hi}\|_{L_t^4 L_x^4} \, 2^{-k_0} \, \|u_{hi}\|_{L_t^4 L_x^4}^3 \, 2^{-k_0} \, \|u_{lo}\|_{L_t^\infty L_x^4}^2 \\
&\lesssim \|u_\lambda\|_{\tilde{X}_{k_0}([0,\frac{T}{\lambda^2}]\times\mathbb{R}^2)}^4 ,
\end{aligned}
$$

$$
\begin{aligned}
\left\|u_{hi} \nabla P_{lo} \mathcal{O}(u_{hi}^4 u_{lo})\right\|_{L_t^1 L_x^1} &\lesssim \|u_{hi}\|_{L_t^4 L_x^4} \, \left\|\nabla P_{lo} \mathcal{O}(u_{hi}^4 u_{lo})\right\|_{L_t^{\frac{4}{3}} L_x^{\frac{4}{3}}} \\
&\lesssim \|u_{hi}\|_{L_t^4 L_x^4} \, 2^{-k_0} \, \|u_{hi}\|_{L_t^\infty L_x^4}^2 \, \|u_{hi}\|_{L_t^{\frac{8}{3}} L_x^8}^2 \, \|u_{lo}\|_{L_t^\infty L_x^\infty} \\
&\lesssim \|u_{hi}\|_{L_t^4 L_x^4} \, 2^{-k_0} \, \|u_{hi}\|_{L_t^{\frac{8}{3}} L_x^8}^2 \, 2^{-\frac{k_0}{2}} \, \|u_{lo}\|_{L_t^\infty L_x^4} \\
&\lesssim \|u_\lambda\|_{\tilde{X}_{k_0}([0,\frac{T}{\lambda^2}]\times\mathbb{R}^2)}^3 ,
\end{aligned}
$$

$$
\begin{aligned}
\left\|u_{hi} \nabla P_{lo} \mathcal{O}(u_{hi}^5)\right\|_{L_t^1 L_x^1} &\lesssim \|u_{hi}\|_{L_t^4 L_x^4} \, \left\|\nabla P_{lo} \mathcal{O}(u_{hi}^5)\right\|_{L_t^{\frac{4}{3}} L_x^{\frac{4}{3}}} \\
&\lesssim \|u_{hi}\|_{L_t^4 L_x^4} \, 2^{-\frac{3k_0}{2}} \, \left\|P_{lo}(u_{hi}^5)\right\|_{L_t^{\frac{4}{3}} L_x^1} \\
&\lesssim \|u_{hi}\|_{L_t^4 L_x^4} \, 2^{-\frac{3k_0}{2}} \, \|u_{hi}\|_{L_t^\infty L_x^4}^3 \, \|u_{hi}\|_{L_t^{\frac{8}{3}} L_x^8}^2 \\
&\lesssim \|u_\lambda\|_{\tilde{X}_{k_0}([0,\frac{T}{\lambda^2}]\times\mathbb{R}^2)}^3 .
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
(9.1.17) &= \sum_{i=0}^{5} \|w\|_{L_t^\infty L_x^2}^2 \, \left\|u_{hi} \nabla P_{lo} \mathcal{O}(u_{hi}^i u_{lo}^{5-i})\right\|_{L_t^1 L_x^1} \\
&\lesssim \eta 2^{k_0} \, \left(1 + \|u_\lambda\|_{\tilde{X}_{k_0}([0,\frac{T}{\lambda^2}]\times\mathbb{R}^2)}\right)^5 .
\end{aligned}
$$

Hence, collect all the estimates, we have

$$
\begin{aligned}
(9.1.8) &= \left| \int_0^{\frac{T}{\lambda^2}} \iint |w(t,y)|^2 \, \frac{x-y}{|x-y|} \, \mathrm{Re}[\mathcal{N} \nabla \bar{w} - w \nabla \bar{\mathcal{N}}](t,x) \, dx dy dt \right| \\
&\lesssim \eta 2^{k_0} \, \left(1 + \|u_\lambda\|_{\tilde{X}_{k_0}([0,\frac{T}{\lambda^2}]\times\mathbb{R}^2)}\right)^7 .
\end{aligned}
$$

$\square$

**Lemma 9.5.** *There exists $\eta = \eta(K) > 0$ satisfying*

$$(9.1.9) \lesssim \eta^{\frac{1}{4}} 2^{k_0} \left(1 + \|u_\lambda\|_{\tilde{X}_{k_0}([0,\frac{T}{\lambda^2}]\times\mathbb{R}^2)}\right)^6$$

*Proof.* By Hölder's inequality and Lemma 8.3,

$$\begin{aligned}
(9.1.9) &= \left|\int_0^{\frac{T}{\lambda^2}} \iint \operatorname{Im}[\bar{w}\nabla w](t,x)\frac{x-y}{|x-y|}\operatorname{Im}[\bar{w}\mathcal{N}](t,y)\,dxdydt\right| \\
&\lesssim \left\| |\nabla|^{\frac{1}{2}} w\right\|^2_{L_t^\infty L_x^2} \|\operatorname{Im}[\bar{w}\mathcal{N}]\|_{L_t^1 L_x^1} \\
&\lesssim \|\operatorname{Im}[\bar{w}\mathcal{N}]\|_{L_t^1 L_x^1}.
\end{aligned}$$

Then we reduce to estimating $\operatorname{Im}[\bar{w}\mathcal{N}]$, that is,

$$\operatorname{Im}[\bar{w}\mathcal{N}] \lesssim \eta^{\frac{1}{4}} 2^{k_0} \left(1 + \|u_\lambda\|_{\tilde{X}_{k_0}([0,\frac{T}{\lambda^2}]\times\mathbb{R}^2)}\right)^6.$$

We first write

$$\operatorname{Im}[\bar{w}\mathcal{N}] = \mathcal{O}(u_{hi}^5 u_{lo}) + \mathcal{O}(u_{hi}^4 u_{lo}^2) + \mathcal{O}(u_{hi}^3 u_{lo}^3) + \mathcal{O}(u_{hi}^2 u_{lo}^4) + \mathcal{O}(u_{hi} P_{hi}(u_{lo}^5)).$$

By Hölder's inequality, Sobolev embedding, Bernstein's inequality, (9.1.11) and Proposition 6.22

$$\begin{aligned}
\left\|\mathcal{O}(u_{hi}^5 u_{lo})\right\|_{L_t^1 L_x^1} &\lesssim \|u_{hi}\|^3_{L_t^\infty L_x^4} \|u_{hi}\|^2_{L_t^{\frac{16}{7}} L_x^{16}} \|u_{lo}\|_{L_t^8 L_x^8} \\
&\lesssim \eta^{\frac{1}{16}} 2^{k_0} \|u_\lambda\|^{\frac{23}{8}}_{\tilde{X}_{k_0}([0,\frac{T}{\lambda^2}]\times\mathbb{R}^2)},
\end{aligned}$$

$$\begin{aligned}
\left\|\mathcal{O}(u_{hi}^4 u_{lo}^2)\right\|_{L_t^1 L_x^1} &\lesssim \|u_{hi}\|^2_{L_t^\infty L_x^4} \|u_{hi}\|^2_{L_t^{\frac{8}{3}} L_x^8} \|u_{lo}\|^2_{L_t^8 L_x^8} \\
&\lesssim \eta^{\frac{1}{8}} 2^{k_0} \|u_\lambda\|^{\frac{15}{4}}_{\tilde{X}_{k_0}([0,\frac{T}{\lambda^2}]\times\mathbb{R}^2)},
\end{aligned}$$

$$\begin{aligned}
\left\|\mathcal{O}(u_{hi}^3 u_{lo}^3)\right\|_{L_t^1 L_x^1} &\lesssim \|u_{hi}\|_{L_t^\infty L_x^4} \|u_{hi}\|^2_{L_t^{\frac{16}{5}} L_x^{\frac{16}{3}}} \|u_{lo}\|^3_{L_t^8 L_x^8} \\
&\lesssim \eta^{\frac{3}{16}} 2^{k_0} \|u_\lambda\|^{\frac{37}{8}}_{\tilde{X}_{k_0}([0,\frac{T}{\lambda^2}]\times\mathbb{R}^2)},
\end{aligned}$$

$$\left\|\mathcal{O}(u_{hi}^2 u_{lo}^4)\right\|_{L_t^1 L_x^1} \lesssim \|u_{hi}\|_{L_t^4 L_x^4}^2 \ \|u_{lo}\|_{L_t^8 L_x^8}^4$$
$$\lesssim \eta^{\frac{1}{4}} 2^{k_0} \ \|u_\lambda\|_{\tilde{X}_{k_0}([0,\frac{T}{\lambda^2}]\times\mathbb{R}^2)}^{\frac{11}{2}},$$

$$\begin{aligned}\left\|\mathcal{O}(u_{hi} P_{hi}(u_{lo})^5)\right\|_{L_t^1 L_x^1} &\lesssim \|u_{hi}\|_{L_t^4 L_x^4} \ \left\|P_{hi}(u_{lo})^5\right\|_{L_t^{\frac{4}{3}} L_x^{\frac{4}{3}}} \\ &\lesssim \|u_{hi}\|_{L_t^4 L_x^4} 2^{\frac{k_0}{2}} \ \left\| |\nabla|^{\frac{1}{2}} (u_{lo})^5 \right\|_{L_t^{\frac{4}{3}} L_x^{\frac{4}{3}}} \\ &\lesssim 2^{k_0} \ \|u_{hi}\|_{L_t^4 L_x^4} \ \left\| |\nabla|^{\frac{1}{2}} u_{lo} \right\|_{L_t^4 L_x^4} \ \|u_{lo}\|_{L_t^8 L_x^8}^4 \\ &\lesssim \eta^{\frac{1}{8}} 2^{k_0} \ \|u_\lambda\|_{\tilde{X}_{k_0}([0,\frac{T}{\lambda^2}]\times\mathbb{R}^2)}^{\frac{23}{4}}.\end{aligned}$$

Therefore,

$$\|\mathrm{Im}[\bar{w}\mathcal{N}]\|_{L_t^1 L_x^1} \lesssim \eta^{\frac{1}{4}} 2^{k_0} \ \left(1 + \ \|u_\lambda\|_{\tilde{X}_{k_0}([0,\frac{T}{\lambda^2}]\times\mathbb{R}^2)}\right)^6.$$

□

Then, combining Lemmata 9.2, 9.4 and 9.5 together, and by long time Strichartz estimates, we have

$$\left\| |\nabla|^{\frac{1}{2}} \ |w(t,x)|^2 \right\|_{L_t^2 L_x^2([0,\frac{T}{\lambda^2}]\times\mathbb{R}^2)}^2 \lesssim \eta^{\frac{1}{4}} 2^{k_0} \ \left(1 + \ \|u_\lambda\|_{\tilde{X}_{k_0}([0,\frac{T}{\lambda^2}]\times\mathbb{R}^2)}\right)^6 \lesssim \eta^{\frac{1}{4}} 2^{k_0}.$$

Undoing the scaling $u(t,x) \mapsto \sqrt{\lambda} u(\lambda^2 t, \lambda x)$, $\lambda = \frac{K}{\varepsilon_3 2^{k_0}}$, we have

$$\left\| |\nabla|^{\frac{1}{2}} \ |P_{\geq \varepsilon_3 K^{-1}} u(t,x)|^2 \right\|_{L_t^2 L_x^2([0,T]\times\mathbb{R}^2)}^2 \lesssim \varepsilon_3^{-1} \eta(K)^{\frac{1}{2}} K.$$

Now we complete the proof of Theorem 9.1. □

**Remark 9.6.** Realizing that Sobolev embedding gives us

$$\|u\|_{L_t^4 L_x^8}^2 = \left\| |u|^2 \right\|_{L_t^2 L_x^4}^2 \lesssim \left\| |\nabla|^{\frac{1}{2}} \ |u|^2 \right\|_{L_t^2 L_x^2}^2,$$

hence Theorem 9.1 implies

$$\|P_{\geq \varepsilon_3 K^{-1}} u(t,x)\|_{L_t^4 L_x^8([0,T]\times\mathbb{R}^2)}^4 \lesssim \eta(K)^{\frac{1}{2}} K.$$

## 9.2 Impossibility of quasi-soliton solutions

We first proof a concentration lemma:

**Lemma 9.7.** *There is an* $R_0 = R_0(T) > 0$

$$\int_{|x-x(t)|\leq \frac{R_0}{N(t)}} |P_{\geq \varepsilon_3 K^{-1}} u(t,x)|^4 \, dx \gtrsim 1,$$

*uniformly for any* $t \in [0,T]$

*Proof.* Following the proof of Lemma 4.2 in [29], we can show that $u$ concentrates a nontrivial portion of its $L_x^4$ norm on some region, that is, there exists a positive number $\alpha_0$ satisfying

$$\|u(t)\|_{L_x^4(|x-x(t)|\leq \frac{R_0}{N(t)})} \geq \alpha_0 > 0.$$

On the other hand, for any $\eta_0$ fixed, we can always find $c(\eta_0)$ in Definition 4.5 such that

$$\left\|P_{\leq c(\eta_0)N(t)} u\right\|_{L_t^\infty L_x^4}^2 \leq \eta_0,$$

and

$$\left|\int_{|x-x(t)|\leq \frac{R_0}{N(t)}} |u(t,x)|^4 - |P_{\geq \varepsilon_3 K^{-1}} u(t,x)|^4 \, dx\right| \lesssim \|P_{\leq \varepsilon_3 K^{-1}} u\|_{L_t^\infty L_x^4} \|u\|_{L_t^\infty L_x^4}^3 \lesssim \eta_0^{\frac{1}{2}}$$

for $t \in [0,T]$. This inequality above is true since we may choose $\varepsilon_3$ small enough such that $\varepsilon_3 K^{-1} \leq c(\eta_0)N(t)$.

Thus for $\eta_0 = \eta_0(u)$ sufficiently small, we find

$$\int_{|x-x(t)|\leq \frac{R_0}{N(t)}} |P_{\geq \varepsilon_3 K^{-1}} u(t,x)|^4 \, dx \gtrsim 1.$$

$\square$

**Theorem 9.8** (Impossibility of quasi-soliton)**.** *If $u$ is an almost periodic solution to* (1.6.1) *and* $\int_0^\infty N(t)\, dt = \infty$*, then* $u \equiv 0$.

*Proof.* Recall $K = \int_0^T N(t)\,dt$. By Lemma 9.7, the frequency-localized interaction Morawetz estimates and Hölder's inequality, we have

$$\begin{aligned}
1 &= \lim_{K \nearrow \infty} \frac{1}{K} \int_0^T N(t)\,dt \\
&\lesssim \lim_{k \nearrow \infty} \frac{1}{K} \int_0^T N(t) \left( \int_{|x-x(t)| \leq \frac{\eta(K)}{N(t)}} |P_{\geq \varepsilon_3 K^{-1}} u(t,x)|^4 \, dx \right) dt \\
&\lesssim \lim_{K \nearrow \infty} \frac{1}{K} \int_0^T N(t) \left\| \chi_{|x-x(t)| \leq \frac{\eta(K)}{N(t)}} \right\|_{L_x^8(\mathbb{R}^2)}^4 \|P_{\geq \varepsilon_3 K^{-1}} u(t)\|_{L_x^8(\mathbb{R}^2)}^4 \, dt \\
&= \lim_{K \nearrow \infty} \frac{1}{K} \int_0^T N(t) \frac{\eta(K)}{N(t)} \|P_{\geq \varepsilon_3 K^{-1}} u(t)\|_{L_x^8(\mathbb{R}^2)}^4 \, dt \\
&= \lim_{k \nearrow \infty} \frac{\eta(K)}{K} \|P_{\geq \varepsilon_3 K^{-1}} u\|_{L_t^4 L_x^8([0,T]\times\mathbb{R}^2)}^4 \\
&\lesssim \lim_{K \nearrow \infty} \eta(K)^{3/2} = 0.
\end{aligned}$$

Therefore, $u \equiv 0$, this contradicts the fact that $u$ is a blow-up solution. $\square$

At this point, we have ruled out the existence of both finite-time blow-up solutions and quasi-soliton solutions, hence we complete the proof of Theorem 1.5.

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