2018

On the Growth of Sobolev Norms for the Nonlinear Schrödinger Equation on Tori and Boundary Unique Continuation for Elliptic PDE

Michael Boratko

Follow this and additional works at: https://scholarworks.umass.edu/dissertations_2

Part of the Analysis Commons

Recommended Citation
https://scholarworks.umass.edu/dissertations_2/1325

This Open Access Dissertation is brought to you for free and open access by the Dissertations and Theses at ScholarWorks@UMass Amherst. It has been accepted for inclusion in Doctoral Dissertations by an authorized administrator of ScholarWorks@UMass Amherst. For more information, please contact scholarworks@library.umass.edu.
ON THE GROWTH OF SOBOLEV NORMS FOR THE 
NONLINEAR SCHRÖDINGER EQUATION ON TORI AND 
BOUNDARY UNIQUE CONTINUATION FOR ELLIPTIC PDE

A Dissertation Presented

by

MICHAEL J. BORATKO

Submitted to the Graduate School of the University of Massachusetts Amherst in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

September 2018

Department of Mathematics and Statistics
ON THE GROWTH OF SOBOLEV NORMS FOR THE
NONLINEAR SCHRÖDINGER EQUATION ON TORI AND
BOUNDARY UNIQUE CONTINUATION FOR ELLIPTIC PDE

A Dissertation Presented

by

MICHAEL J. BORATKO

Approved as to style and content by:

Andrea R. Nahmod, Chair

Nestor Guillen, Member

David A. Mix Barrington, Member

Gigliola Staffilani, Member

Nathaniel Whitaker, Department Head
Mathematics and Statistics
ACKNOWLEDGMENTS

I would like to thank my advisor, Professor Andrea R. Nahmod, for her encouragement and support. I would also like to thank Professor Nestor Guillen for not only being on my committee but also proposing and advising me on the problem which constitutes Part II of this thesis.

I would like to thank the other members of my thesis committee: Professor Gigliola Staffilani and Professor David Barrington. I appreciate and take to heart their words of advice and guidance.

I am very fortunate to have academic siblings, and would like to thank Xueying Yu and Haitian Yue for their friendship and support. I would particularly like to thank Haitian Yue for his many helpful discussions regarding Chapter 3.

I would like to thank my parents, who were my first teachers. I learned some of the most important lessons from them.

Finally, I would like to thank my wife, who has been supportive of my graduate education and patient when schoolwork had to take precedence.

I acknowledge support from the National Science Foundation through my advisor Andrea R. Nahmod’s grants NSF-DMS Award 1201443 and NSF-DMS Award 1463714.
This dissertation is composed of two parts. The first part applies techniques from Harmonic and nonlinear Fourier Analysis to the nonlinear Schrödinger equation, and therefore tools from the study of Dispersive Partial Differential Equations (PDEs) will also be employed. The dissertation will apply the $\ell^2$ decoupling conjecture, proved recently by Bourgain and Demeter, to prove polynomial bounds on the growth of Sobolev norms of solutions to polynomial nonlinear Schrödinger equations. The first bound which is obtained applies to the cubic nonlinear Schrödinger equation and yields an improved bound for irrational tori in dimensions 2 and 3. For the 4 dimensional case an argument of Deng that relies on the upside-down I-method and long time Strichartz estimates on generic irrational tori is applied.
to get bounds in for the energy-critical nonlinear Schrödinger equation, assuming small energy.

The second part of the thesis proves unique continuation at the boundary for solutions to a class of degenerate elliptic PDEs. Specifically, there is a weight at each point on the domain which is bounded by some fractional power of the distance to the boundary. Techniques from Calculus of Variations and Geometric PDEs are used to show that solutions to this class of PDEs are uniquely defined simply by specifying their values and some notion of their normal derivative on an open set with positive measure on the boundary.
# Table of Contents

## Acknowledgments

iv

## Abstract

v

## List of Figures

ix

## Part I: Improved Bounds on Sobolev Norms for the $p$-Nonlinear Schrödinger Equation on Tori

### Chapter 1: Introduction

1. The nonlinear Schrödinger equation on tori. 9
2. Preliminaries 9
   1.2.1 Rational and Irrational Tori 10
   1.2.2 Fourier Series Over General Tori 10
   1.2.3 Littlewood-Paley Theory 11
   1.2.4 Sobolev Spaces 12
   1.2.5 $X^{s,b}$ spaces 12
3. Nonlinear Schrödinger on $\mathbb{R}^d$ 14
4. $\ell^2$ decoupling 17
   1.4.1 Applications 18
   1.4.2 Main steps 19
   1.4.3 Multilinear Kakeya 24
   1.4.4 Multilinear Restriction 31
   1.4.5 Using curvature 32

### Chapter 2: Improved Bounds on Sobolev Norms for the Nonlinear Schrödinger Equation on Tori

2. Background 35
3. Main Argument 35
   2.2.1 Leveraging $\ell^2$-decoupling Strichartz 37
3. GROWTH OF SOBOLEV NORMS FOR ENERGY-CRITICAL NLS WITH SMALL ENERGY IN $\mathbb{T}^4_\lambda$ ........................................ 40
3.1 Introduction ........................................... 40
3.1.1 Main Idea ........................................ 42
3.1.2 Notations ......................................... 43
3.1.3 $D$-multiplier (a.k.a. upside-down $I$-operator) .... 44
3.2 Tools .................................................. 45
3.2.1 Definition of norms ................................ 46
3.2.2 Linear estimates ................................... 48
3.2.3 Global well-posedness in $H^1$ ..................... 54
3.3 Proof of main theorem ................................ 55
3.4 Appendix .............................................. 81
3.4.1 Christ-Kiselev ..................................... 81
3.4.2 Derivative of Energy Calculation .................. 82
3.4.3 Summing Dyadic Terms ............................ 83

PART II: UNIQUE CONTINUATION AT THE BOUNDARY

4. INTRODUCTION ......................................... 85
4.1 The Problem .......................................... 89
4.2 The Plan ............................................. 91
4.3 Preliminaries ......................................... 91
5. DOUBLING AND MONOTONICITY ........................ 95
5.1 Extending Almgren’s Monotonicity Formula .......... 99
5.2 Second term in $G(r)$ ............................... 103
5.3 Doubling Implies Reverse Hölder .................... 105
5.4 Appendix ............................................. 121
5.4.1 Cacciopoli’s Inequality ............................ 121
5.4.2 Calculating $H'(r)$ ................................. 123

BIBLIOGRAPHY ............................................. 125
<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Gas at high temperature, treated as a system of billiard balls, with thermal velocity $v$ and density $d^{-3}$, where $d$ is the distance between bosonic particles.</td>
<td>5</td>
</tr>
<tr>
<td>2.</td>
<td>Simplified quantum description of gas at low temperature, in which the particles are regarded as wave packets with a spatial extent of the order of the de Broglie wavelength, $\lambda_{dB}$.</td>
<td>5</td>
</tr>
<tr>
<td>3.</td>
<td>Gas at the transition temperature for Bose-Einstein condensation, when $\lambda_{dB}$ becomes comparable to $d$. The wave packets overlap and a Bose-Einstein condensate forms.</td>
<td>5</td>
</tr>
<tr>
<td>4.</td>
<td>Pure Bose condensate (giant matter wave), which remains as the temperature approaches absolute zero and the thermal cloud disappears.</td>
<td>5</td>
</tr>
<tr>
<td>5.</td>
<td>The setup for the truncated parabola $P^1$. Each dotted region represents a $\theta$ “slab”, and $T$ is the collection of all such $\theta$.</td>
<td>18</td>
</tr>
<tr>
<td>6.</td>
<td>An example of the setup for Multilinear Kakeya.</td>
<td>25</td>
</tr>
<tr>
<td>7.</td>
<td>Zooming in on the square $Q_s$..</td>
<td>26</td>
</tr>
<tr>
<td>8.</td>
<td>How to reduce Multilinear Kakeya to the Nearly Axis Parallel case: We split $S_j$ into pieces, each of which has smaller radius than $\delta$, and then sum over all contributions where the direction of $\ell_{j,a}$ is in $S_{j,\beta}$ (one from each $S_j$). Our angles here are not to scale.</td>
<td>27</td>
</tr>
<tr>
<td>9.</td>
<td>Zooming in</td>
<td>29</td>
</tr>
</tbody>
</table>
PART I: IMPROVED BOUNDS ON SOBOLEV NORMS FOR THE $p$-NONLINEAR SCHRÖDINGER EQUATION ON TORI
The nonlinear Schrödinger equation plays an ubiquitous role as a model for dispersive wave-phenomena in nature. Roughly speaking, dispersion means that when no boundary is present, waves of different wavelengths travel at different phase speeds: long wavelength components propagate faster than short ones. This is the reason why over time dispersive waves spread out in space as they evolve in time, while conserving some form of energy. This phenomenon is called broadening of the wave packet.

Mathematically, the nonlinear Schrödinger equation serves as a model problem for the large class of so-called dispersive partial differential equations [Abl11; Tao06]. It naturally arises in connection to a variety of different physical problems. One is nonlinear optics in a so-called Kerr medium where one considers electromagnetic waves in a material (eg. glass fiber) whose time evolution are governed by Maxwell’s equations. The nonlinear Maxwell equations however have disparate scales and understanding their dynamics is a difficult problem. As a first attempt one looks for further simplifications: asymptotic methods then become useful. A natural ansatz is to write the electric field $E$ as a Taylor series whose leading term is a small amplitude wave packet of the form

$$A(t, x)e^{i(\vec{\xi}_0 \cdot \vec{x} - \omega_0 t)} + \overline{A}(t, x)e^{-i(\vec{\xi}_0 \cdot \vec{x} - \omega_0 t)} \quad (1.1)$$
with the wave vector $\xi_0 \in \mathbb{R}^3$, the frequency $\omega_0 \in \mathbb{R}$ and $A$ is a small amplitude and slowly varying function. After inserting back into the nonlinear Maxwell equations, formal calculations, transformations and multiple scale analysis yield a cubic nonlinear Schrödinger equation where time corresponds to the coordinate of the direction of propagation of the wave along the material for the (transformed) amplitude [Abl11; MN04].

Essentially the same type of approximations can be made in other problems such as water waves. In this context one seeks solutions in which the interface of the fluid region is to leading order a wave packet of the same form as (1.1), i.e. with small $O(\epsilon)$ amplitude and slow spatial variation that are balanced. Lengthy formal calculations then suggest that the envelopes of these wave packets evolve on $O(\epsilon^{-2})$ time scales according to a version of a cubic nonlinear Schrödinger equation [CSS92]. It often turns out that the nonlinear Schrödinger equation (approximately) describes the evolution of envelopes of wave packets on the appropriate NLS time scales; for a precise description, see [TW12; Tot15]. Other examples of cubic NLS arising from other physical situations can be found in [SS07].

The nonlinear Schrödinger equations also arise as the equations governing Bose-Einstein condensates. Bose-Einstein condensation phenomena was predicted in 1925 by A. Einstein [Ein24] using a method introduced by S.N. Bose [Bos24] to derive the black-body spectrum; it is a fascinating phenomena predicted by quantum statistical mechanics. The achievement of Bose-Einstein condensation however was made only in 1995 by Cornell and Wieman who produced the first gaseous condensate. For this they, together with Ketterle received the 2001 Nobel Prize in Physics. A Bose-Einstein condensate (BEC) is the state of matter of a gas of $N$ weakly interacting bosonic atoms confined by an external potential and cooled to
temperatures very near *absolute zero* (0 Kelvin). In his 2001 Nobel lecture, W. S. Ketterle described *how profoundly the properties of a gas of bosonic atoms changes when you cool down the gas. Then the wave nature of matter tells us that the wave packets which describes an atom, this fuzzy object, becomes larger and larger and when the wave packet expand to the size that the waves of neighboring atoms overlap then all atoms start to oscillate in concert and form what you may regard a giant matter wave. And this is the Bose Einstein condensate.* [Ket11]. In other words, all bosons occupy the same quantum state and can thus be described by a single wave function $u(t,x)$. The pointwise density of this gas at time $t$ is represented by $|u(x,t)|^2$. The interactions between the bosons lead to nonlinear contributions to the Schrödinger equation for this quantum system. Considering only binary collisions between the bosons, one sees that $u$ satisfies a cubic nonlinear Schrödinger equation (in this context often called the *Gross-Pitaevski equation*).

Bose-Einstein condensation is based on the wave nature of particles, which is at the heart of quantum mechanics. In a simplified picture, bosonic atoms in a gas may be regarded as quantum-mechanical wave-packets which have an extent on the order of a thermal *de Broglie wavelength* (the position uncertainty associated with the thermal momentum distribution). The lower the temperature, the longer is the de Broglie wavelength. When atoms are cooled to the point where the thermal de Broglie wavelength is comparable to the interatomic separation, then the atomic wave-packets overlap and *the indistinguishability of particles* becomes important. Bosons undergo a phase transition and form a Bose-Einstein condensate, a dense and coherent cloud of atoms all occupying the same quantum mechanical state [Ket02]. Graphically, we can visually this as follows (from D. S. Durfee and W. S. Ketterle paper [DK98]; c.f. W.S. Ketterle [Ket02] (Nobel Lecture).
Figure 1. Gas at high temperature, treated as a system of billiard balls, with thermal velocity $v$ and density $d^{-3}$, where $d$ is the distance between bosonic particles.

Figure 2. Simplified quantum description of gas at low temperature, in which the particles are regarded as wave packets with a spatial extent of the order of the de Broglie wavelength, $\lambda_{dB}$.

Figure 3. Gas at the transition temperature for Bose-Einstein condensation, when $\lambda_{dB}$ becomes comparable to $d$. The wave packets overlap and a Bose-Einstein condensate forms.

Figure 4. Pure Bose condensate (giant matter wave), which remains as the temperature approaches absolute zero and the thermal cloud disappears.
Nonlinear dispersive equations, and in particular the nonlinear Schrödinger equation (NLS) model physical phenomena arising in quantum mechanics, plasma physics, nonlinear optics, oceanography, and more. At the core of the mathematical and scientific understanding the behavior of solutions to NLS is the study of the long-time dynamical behavior of such solutions especially in mass-critical (e.g. the cubic NLS in 2D) and energy-critical (e.g. the cubic NLS in 4D) regimes.

We have by now several examples of nonlinear Schrödinger and wave equations defined $\mathbb{R}^d$ for which it is mathematically proven that dispersion sets in and, after a time long enough, solutions settle into a purely linear behavior. This phenomena is often referred to as scattering (asymptotic stability). For linear solutions, energy at any given frequency does not migrate to higher or lower frequencies, that is there is no forward or backward cascade. As a consequence of scattering then, certain nonlinear solutions in $\mathbb{R}^d$ also will avoid these cascades. We also say that linear solutions are in equilibrium and scattering solutions are asymptotically in equilibrium. The situation is believed to be different on compact domains, where dispersion is weak and does not translate into time decay of the solutions, so much less is known in this regard. For example, on periodic domains $\mathbb{T}^d$, $d \geq 2$, many different long-time behaviors can occur; in particular solutions do not exhibit long-time stability around equilibrium configurations, giving rise to out of equilibrium dynamics. Such a behavior is present in many physically important phenomena, like turbulence in ocean waves or in the transmission of optical signals for instance. While this turbulence\(^1\) is easy to observe and has very strong manifestations in nature, its understanding is rather poor and the mathematical justification of the

\(^1\)Here we refer to wave turbulence, which pertains to similar problems to hydrodynamic turbulence (the phenomenon one observes in daily life, particularly when one travels through fluids; e.g. in a vessel on the ocean or in an airplane in the atmosphere) but for different physical systems involving wave interactions (e.g. ocean or plasma waves).
involved phenomena based on the model equations is rather difficult and is mostly still open. How to analytically describe this expected out-of-equilibrium behavior is one of the most intriguing questions in the study of the long time dynamics of dispersive equations in this case, with only few partial answers given so far.

Wave turbulence theory seeks to obtain a statistical description of the out-of-equilibrium dynamics for Hamiltonian nonlinear dispersive equations by for example deriving effective equations that track the long time evolution for the energy distribution of the system at hand. The overarching problem is that of understanding the long time energy dynamics in an infinite dimensional Hamiltonian systems. Loosely speaking the question is how does the energy of the system gets transferred and redistributed among the different degrees of freedom as time evolves (where energy and degrees of freedom are dependent of the system in question). Will energy keep its original concentration zones or will it cascade to characteristically different scales? The relevant mathematical models are nonlinear dispersive PDE posed on compact domains. An underlying challenge when working on very large domains (e.g. the ocean) is passing suitably to the correct infinite-volume approximation in a manner that captures the energy transfer phenomenon. As mentioned above, putting this theory on a rigorous mathematical foundation is a challenging and difficult proposition. There are two main aspects pertaining a rigorous mathematical study of wave turbulence. One entails deriving the fundamental equations (wave kinetic equations) governing the dynamics and energy distribution of the dispersive equation under consideration. Some work in this direction has been done in recent years by Faou-Germain-Hani [FGH16] and and by Buckmaster-Germain-Hani-Shatah [Buc+18]. The other is that of proving some of its dynamical conclusions –independently of a complete justification of the formalism– such as the energy cascade phenomenon; i.e. the transfer of kinetic energy towards higher and higher
modes—or vice-versa— as the initial data evolves. One approach to the latter is by exhibiting dynamics that reflect the conclusions of the theory. In the mid nineties Bourgain [Bou93] proposed a mechanism to understand this energy transfer which entails proving the growth in time of high Sobolev norms of solutions. Since these norms penalizes large frequencies over low frequencies, a transfer of energy to high frequencies should be accompanied by a growth of such norms since it gives us a quantitative estimate for how much of the support of $|\hat{u}|^2$ has transferred from the low to the high frequencies while maintaining constant mass and energy (forward cascade). This question is known as Bourgain’s unbounded orbits conjecture and relevant work was done by Bourgain [Bou96] and by Kuksin [Kuk97]. More recently, important progress on this conjecture was then made by Colliander-Keel-Staffilani-Takaoka-Tao [Col+10] who constructed solutions exhibiting large but finite growth of Sobolev norms.

However proving the existence of solutions to NLS whose Sobolev norms grow unboundedly in time remained an open conjecture. Germain-Hani-Tzvetkov-Visciglia gave a positive answer to Bourgain’s conjecture for the cubic NLS on product domains $\mathbb{R} \times \mathbb{T}^d$ ($d \geq 2$). Moreover, they proved an explicit growth rate of $\exp (c \log \log t_k)^{1/2}$.

Obtaining results exhibiting and quantifying actual growth (lower bound) of the Sobolev norms of solutions to NLS is an exceedingly hard problem. On the other hand, from the iterative methods used to obtain global well-posedness and propagation of regularity properties of solutions to NLS equations one can at best show an exponential in time upper bound for the Sobolev norms of solutions. Nevertheless one since one expects the growth rate to be very slow, a natural question is whether one can improve the exponential in time upper bound for the Sobolev norms of solutions to be polynomial in time.
1.1 The nonlinear Schrödinger equation on tori.

Let \( d \geq 1 \) and \( \lambda := (\lambda_1, \lambda_2, \ldots, \lambda_d) \) where \( \lambda_j > 0, j = 1, \ldots, d \). We define the domain

\[
T^d_\lambda := (\mathbb{R}/\lambda_1\mathbb{Z}) \times (\mathbb{R}/\lambda_2\mathbb{Z}) \times \cdots \times (\mathbb{R}/\lambda_d\mathbb{Z})
\]

The periodic p-nonlinear Schrödinger equation is given by

\[
\begin{cases}
iu_t + \Delta u = |u|^{p-1}u, \\ u(x, 0) = u_0(x), \quad x \in T^d_\lambda
\end{cases}
\]

(1.2)

where \( u_0(x) \) is the initial profile, \( p > 1 \), \( u : \mathbb{R} \times T^d_\lambda \to \mathbb{C} \). When \( \lambda_j = 1 \), \( T^d_\lambda \) becomes \( T^d \), the square \( d \)-dimensional torus. Equation (1.2) is a good model to illustrate how a partial differential equation that was introduced to model a certain phenomenon in physics may also have structures that touch many different areas of mathematics like: nonlinear Fourier and harmonic analysis, geometry, probability, analytic number theory, and dynamical systems. These connections are very active areas of research at the moment with many open problems.

The Cauchy problem

\[
\begin{cases}
iu_t + \Delta u = \lambda|u|^2u, \\ u(x, 0) = u_0(x), \quad x \in T^3
\end{cases}
\]

(1.3)

is used to describe several phenomena, but in particular is the problem that governs the Bose Einstein Condensate (BEC) we discussed above.

1.2 Preliminaries

In this section we introduce some of the technical background for proving bounds to solutions on NLS.
1.2.1 Rational and Irrational Tori

Many of the results we present will make distinctions between the case of rational and irrational tori. The notation we will use for any tori is

\[ T^d_\lambda = \left( \mathbb{R}/\lambda_1 \mathbb{Z} \right) \times \cdots \left( \mathbb{R}/\lambda_d \mathbb{Z} \right) \quad \text{for} \quad \lambda := (\lambda_1, \lambda_2, \ldots, \lambda_d) \in \mathbb{R}^d. \]

**Definition 1.** \( T^d_\lambda \) is an **irrational tori** if there is some ratio \( \frac{\lambda_i}{\lambda_j} \notin \mathbb{Q} \). Otherwise, we say \( T^d_\lambda \) is a **rational tori**.

1.2.2 Fourier Series Over General Tori

Since we are not working on the square torus, it is useful to redefine the inner product of \( x, y \in T^d_\lambda \) as follows:

\[ \langle x, y \rangle := \sum_{j=1}^{d} \frac{2\pi x_j y_j}{\lambda_j}. \]

We will also take \( \|x\| \) for \( x \in T^d_\lambda \) to be the norm induced by this inner product, that is

\[ \|x\| := \sqrt{\langle x, x \rangle}. \]

For \( f : T^d_\lambda \to \mathbb{C} \), we define

\[ \hat{f}(n) := \frac{1}{|T^d_\lambda|} \int_{T^d_\lambda} f(x) e^{-i\langle x, n \rangle} \, dx \]

where \( |T^d_\lambda| = \prod_{j=1}^{d} \lambda_j. \)

Under this convention, repeated application of the theory for Fourier series in one dimension implies the following relevant results:

1. If \( f \) is integrable on \( T^d \) and differentiable at \( x \in T^d \), then

\[ f(x) = \sum_{n \in \mathbb{Z}^d} \hat{f}(n) e^{i\langle x, n \rangle}. \]
2. \( \frac{\partial}{\partial x_j} f(n) = i2\pi \frac{n_j}{\alpha_j} \hat{f}(n) \).

3. \( \widehat{\Delta f}(n) = \|n\|^2 \hat{f}(n) \).

4. If \( \alpha \in \mathbb{T}^d \) and \( f(x) = e^{i\langle x, \alpha \rangle} \), \( \hat{f}(n) = 1_{\alpha}(n) \).

5. \( \langle f, g \rangle = 0 \) iff \( \text{supp} \hat{f} \cap \text{supp} \hat{g} = \emptyset \).

6. If \( f \in C^k \), then \( \hat{f}(n) = o\left(\frac{1}{\|n\|^k}\right) \) as \( n \to \infty \).

1.2.3 Littlewood-Paley Theory

The last item above suggests that if \( f \) is sufficiently smooth (depending on \( d \)), we have absolute convergence of the Fourier Series of \( f \), and can therefore group and rearrange the terms however we’d like. We will typically rearrange this sum radially and dyadically. For \( N > 1 \) a dyadic number, we denote by \( P_{\leq N} \) the Fourier multiplier:

\[
\widehat{P_{\leq N} f} = 1_{|n| \leq N} \hat{f} \quad \text{that is,} \quad P_{\leq N} f(x) = \sum_{n \in \mathbb{Z}^d : |n| \leq N} \hat{f}(n) e^{in \cdot x}.
\]

Then \( P_N = P_{\leq N} - P_{\leq N/2} \) and hence \( f = \sum_{N \in \mathbb{Z}^d} P_N f \). We may also project over any set \( C \subset \mathbb{Z}^d \), and in that case we write \( P_C \).

The reason the \( P_X \) operators are referred to as multipliers is because they are defined by multiplication on the Fourier side, eg.

\[
P_C f = 1_C \hat{f}.
\]

As a result they commute with any other multipliers, for example \( \partial_{x_i} \).
1.2.4 Sobolev Spaces

We define the **homogeneous Sobolev space** $\dot{H}^s(\mathbb{T}^d)$ as the set of those functions $f$ for which the norm

$$
\|f\|_{\dot{H}^s(\mathbb{T}^d)} := \left( \sum_{k \in \mathbb{Z}^d} |k|^{2s} |\hat{f}(k)|^2 \right)^{\frac{1}{2}}
$$

is bounded, and the **inhomogeneous Sobolev space** $H^s(\mathbb{T}^d)$ as the set of those functions $f$ for which the norm

$$
\|f\|_{H^s(\mathbb{T}^d)} := \left( \sum_{k \in \mathbb{Z}^d} \left(1 + |k|^2\right)^s |\hat{f}(k)|^2 \right)^{\frac{1}{2}}
$$

is bounded. The homogeneous Sobolev norms capture the $L^2$ norm of the $s^{\text{th}}$ derivative via the Fourier transform, where $s$ does not need to be an integer. The inhomogeneous Sobolev spaces capture both the $L^2$ norm of the $s^{\text{th}}$ derivative and the $L^2$ norm of the original function.

We use semigroup notation to denote the solution operator $e^{it\Delta}$ which maps $w_0$ to $w$. That is, given a function $f(x)$, we define

$$
e^{it\Delta} f(x) := \sum_{n \in \mathbb{Z}^d} \hat{f}(n)e^{i(x,n)+t\|n\|^2}.
$$

1.2.5 $X^{s,b}$ spaces

We mentioned previously that we hope to prove (1.8) is a contraction map in some function space. The "right" space to consider is $X^{s,b}$ space, which can be thought of as an adaptation of Sobolev space to our PDE. Given a function $u : \mathbb{R} \times \mathbb{T}^d \to \mathbb{C}$, we want to consider the $H_x^sH_t^b$ norm of the evolution of this function as a free solution to the original PDE.$^2$

---

$^2$ $X^{s,b}$ spaces are always defined in relation to a PDE, and so without context we cannot a priori say if two $X^{s,b}$ spaces are the same. Some authors notate this dependence using something such as $X^{s,b}_{(CNLS)}$, however since in this paper we only use $X^{s,b}$ adapted to the nonlinear Schrödinger equation we do not explicitly indicate this dependence.
Definition 2. Given $u : \mathbb{R} \times \mathbb{T}^d \to \mathbb{C}$ we define the $X^{s,b}$ norm of $u$ to be

$$\|u\|_{X^{s,b}} := \|e^{it\Delta} u(t, x)\|_{H^s_t H^b_x}.$$  

Sobolev spaces are defined more explicitly on the Fourier side, so to make sense of the above definition we take the space-time Fourier Transform, that is given $f$ we define

$$\tilde{f}(\tau, n) := \int_{\mathbb{R}} \hat{f}(t, n)e^{-it\tau} \, dt.$$  

Then

$$\|f\|_{H^s_t H^b_x} := \|\langle n \rangle^s \langle \tau \rangle^b \tilde{f}\|_{l_2^n(Z^d) L_2^2(\mathbb{R})}$$

where $\langle z \rangle$ is the Japanese Bracket of $z$, $^3$

$$\langle z \rangle := \sqrt{1 + \| z \|^2}.$$  

Calculating the space-time Fourier Transform, therefore, we find

$$\|u\|_{X^{s,b}} = \|\langle n \rangle^s \langle \tau \rangle^b \tilde{u}(\tau - \| n \|^2, n)\|_{l_2^n(Z^d) L_2^2(\mathbb{R})},$$

$$= \|\langle n \rangle^s \langle \tau + \| n \|^2 \rangle^b \tilde{u}(\tau, n)\|_{l_2^n(Z^d) L_2^2(\mathbb{R})},$$

where the second line follows from the first from a change of variables.

Definition 3. $X^{s,b}$ is the closure of the set of $u \in C^\infty_0(\mathbb{R} \times \mathbb{T}^d)$ such that $\|u\|_{X^{s,b}} < \infty$.

The following result will also be useful.

Lemma 1. If there exists an $R > 0$ such that $\hat{f}(t, n) = 0$ for all $\| n \| \leq R$, then

$$\|f\|_{X^{0,b}} \leq R^{-s}\|f\|_{X^{s,b}}.$$

$^3$We interpret the norm within the definition of the Japanese Bracket contextually, so if $z \in \mathbb{T}^d$ it is our norm inherited from the adjusted inner product, if $z \in \mathbb{Z}^d$ it is the Euclidean norm, if $z \in \mathbb{R}$ it is simply the absolute value.
Proof. If $\tilde{f}(\tau, n) \neq 0$, $\|n\| \geq R \implies 1 \leq R^{-1}\|n\| \leq R^{-1}\langle n \rangle$, raising both sides to the power $s$ yields $1 \leq R^{-s}\langle n \rangle^s$, and thus

$$\|f\|_{X^{0,b}} = \|\langle \tau + \|n\|^2 \rangle^b \tilde{f}(\tau, n)\|_{L^2(\mathbb{R})} \leq \|R^{-s}\langle n \rangle^s \langle \tau + \|n\|^2 \rangle^b \tilde{f}(\tau, n)\|_{L^2(\mathbb{R})} = R^{-s}\|f\|_{X^{s,b}}.$$  

\[\square\]

### 1.3 Nonlinear Schrödinger on $\mathbb{R}^d$

The linear Schrödinger equation on $\mathbb{R}^d$ is

$$\begin{cases} i u_t + \Delta u = 0, \\ u(0, x) = u_0(x), \end{cases}$$  

(1.4)

where $x \in \mathbb{R}^d$ and $u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$. This can be solved directly by taking the Fourier transform and solving a system of ODEs to find

$$u(t, x) =: e^{it\Delta} u_0(x) = (e^{-it|k|^2} \hat{u}_0(k))^\vee = (4\pi it)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{i|x-y|^2/4t} u_0(y) \, dy$$  

(1.5)

where we have implicitly defined $e^{it\Delta}$ as the solution operator.

In general, we can consider the p-NLS

$$\begin{cases} i u_t + \Delta u = |u|^{p-1}u, \\ u(0, x) = u_0(x). \end{cases}$$  

(1.6)

Duhamel’s principle tells us that, if $u_0$ is regular enough, then

$$u(x, t) = e^{it\Delta} u_0 + c \int_0^t e^{i(t-\tau)\Delta}|u|^{p-1} u(\tau) \, d\tau$$  

(1.7)

solves (1.6) on $\mathbb{R}^d$. 

14
**Definition 4.** We say that the IVP (1.4) is *locally well-posed* in $H^s(\mathbb{R}^d)$ if, for any ball $B \subset H^s(\mathbb{R}^d)$ there exists a time $T$ and a Banach space of functions $X \subseteq L^\infty([-T,T], H^s(\mathbb{R}^d))$ such that for each initial data $u_0 \in B$ there exists a unique solution $u \in X \cap C([-T,T], H^s(\mathbb{R}^d))$ for the integral equation (1.7). Furthermore, the map $u_0 \to u$ is continuous as a map from $H^s$ into $C([-T,T], H^s(\mathbb{R}^d))$. If uniqueness is obtained in $C([-T,T], H^s(\mathbb{R}^d))$ we say that local well-posedness is “unconditional”.

If definition 4 holds for all $T > 0$ we say that the IVP is *globally well-posed*.

The standard way we show that this IVP is well-posed is by showing that

$$u(t) = e^{it\Delta}u_0 - i \int_0^t e^{i(t-\tau)\Delta}|u(\tau)|^2u(\tau)d\tau$$

has a stationary point, which we can do by showing that

$$u \mapsto \int_0^t e^{i(t-\tau)\Delta}|u(\tau)|^2u(\tau)d\tau$$

is a contraction in some appropriate function space, for $T$ sufficiently small. The standard technique in proving such a result comes from first proving a type of Strichartz estimate. In $\mathbb{R}^d$ such Strichartz estimates took the following form:

**Theorem 1 (Strichartz Estimates in $\mathbb{R}^d$).** Fix $n \geq 1$. We call a pair $(q,r)$ of exponents admissible if $2 \leq q,r \leq \infty$, $\frac{2}{q} + \frac{n}{r} = \frac{n}{2}$ and $(q,r,n) \neq (2,\infty,2)$. Then for any admissible exponents $(q,r)$ and $(\tilde{q},\tilde{r})$ we have the homogeneous Strichartz estimate

$$\|e^{it\Delta}u_0\|_{L_t^qL_x^r(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|u_0\|_{L_2^2(\mathbb{R}^d)}$$

and the inhomogeneous Strichartz estimate

$$\left\| \int_0^t e^{i(t-\tau)F(\tau)}d\tau \right\|_{L_t^qL_x^r(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|F\|_{L_t^\infty L_x^{\tilde{r}}(\mathbb{R}\times\mathbb{R}^d)},$$

where $\frac{1}{q} + \frac{1}{q'} = 1$ and $\frac{1}{r} + \frac{1}{r'} = 1$.
As a reference for this result, see [KT98] or [Tao06]. The general idea is to use the form of the solution operator in (1.5) to obtain the dispersive estimate

$$\|e^{it\Delta}u_0\|_{L^\infty} \lesssim \frac{1}{t^{d/2}} \|u_0\|_{L^1}$$

and the conservation of homogeneous Sobolev norms

$$\|e^{it\Delta}u_0\|_{\dot{H}^s} = \|u_0\|_{\dot{H}^s}$$

and then interpolate these two bounds when $s = 0$ using a $TT^*$ argument. Having Strichartz estimates is crucial to proving local well-posedness and bounds on the growth of Sobolev norms to solutions. For rational tori, the first estimates of this form were proved by Bourgain in [Bou93], however they relied on analytic number theory and counting lattice points. By stacking copies of a torus next to one another, we can convert any rational torus into a square one. Over a square torus the Littlewood-Paley decomposition can be handled because the resonant terms are sums taken over intersections of an integer lattice with spheres. Results from analytic number theory provides sufficient bounds to count the terms for this case.

In the case of an irrational torus, however, the counting needs to be taken over intersections of an integer lattice with ellipsoids. A naive bound is to use the area of intersection on the surface of the ellipsoid, however this is too weak to close the argument. Establishing better bounds for the number of integer lattice points within the intersection of ellipsoids is a deep problem in analytic number theory.

Recently, Bourgain-Demeter proved the $\ell^2$-decoupling Theorem in [BD15], which gives us Strichartz bounds over any tori. The result they obtain is as follows:

**Theorem 2** (Strichartz estimates for irrational tori). Let $\phi \in L^2(\mathbb{T}^{n-1})$ with $\text{supp} \widehat{\phi} \subset [-N,N]^{n-1}$. Then for each $\varepsilon > 0$, $p \geq \frac{2(n+1)}{n-1}$ and each interval $I \subset \mathbb{R}$ with $|I| \gtrsim 1$ we have
\[ \|e^{it\Delta} \phi\|_{L^p(\mathbb{R}^{n-1} \times I)} \lesssim \varepsilon N^{\frac{n-1}{2} - \frac{n+1}{p} + \varepsilon} |I|^{1/p} \|\phi\|_2, \quad (1.13) \]

and the implicit constant does not depend on $I$, $N$ or $r_i$.

This result is very significant, and so we will provide an discussion of the proof for the $\ell^2$-decoupling theorem.

1.4 $\ell^2$ decoupling

In this section we provide a brief overview of the proof of the $\ell^2$ decoupling conjecture by J. Bourgain and C. Demeter in [BD15]. We borrow heavily from L. Guth’s notes on the topic [Gut14]. This is a ground-breaking method which has given rise to many substantial results, including Vinogradov’s mean value theorem for degrees higher than three [BDG16], and has connections with incidence geometry and the polynomial method [Gut16].

Throughout this section we take $S$ to be a compact $C^2$ hypersurface in $\mathbb{R}^n$ with positive definite second fundamental form. The typical example we will always refer to is the truncated elliptic paraboloid

\[ P^{n-1} := \{(\xi_1, \ldots, \xi_{n-1}, \xi_1^2 + \cdots + \xi_{n-1}^2) \in \mathbb{R}^n : |\xi_i| \leq 1/2\}. \]

We always assume $n \geq 2$, and we will frequently give examples where $n = 2$. We use the asymptotic notation $A \lesssim B$ to mean that there is a constant $C > 0$ such that $A \leq CB$. For convenience, this $C$ may depend on fixed parameters such as $p$ and $n$, however we record dependence on variable parameters such as $\varepsilon$ using the notation $\lesssim_\varepsilon$. The asymptotic notation $A \gtrsim B$ is defined analogously, and $A \sim B$ will mean $A \lesssim B$ and $B \lesssim A$.

The main result is the following theorem:
Figure 5. The setup for the truncated parabola $P^1$. Each dotted region represents a $\theta$ “slab”, and $T$ is the collection of all such $\theta$.

**Theorem 3** ($\ell^2$ Decoupling$^4$). Let $S$ be a compact $C^2$ hypersurface in $\mathbb{R}^n$ with positive definite second fundamental form. Let $N_\delta$ be the $\delta$-neighborhood of $S$ and let $T$ be a covering of $N_\delta S$ by blocks $\theta$ of dimension $\delta^{1/2} \times \cdots \times \delta^{1/2} \times \delta$. If $\text{supp}(\hat{f}) \subseteq N_\delta S$ then for $p \geq \frac{2(n+1)}{n-1}$ and $\epsilon > 0$,

$$
\|f\|_p \leq C_{p,n,\epsilon} \delta^{-\frac{n+1}{4} + \frac{n+1}{2p} - \epsilon} \left( \sum_{\theta \in T} \|f_\theta\|_2^2 \right)^{1/2}.
$$

Note that we will often switch between using $\delta$ and $R$, where $\delta = R^{-1}$.

1.4.1 Applications

Bourgain and Demeter prove substantial and varied applications of this theorem, including Incidence Geometry and Number Theory, however our current interest lies in establishing Strichartz estimates for the irrational tori. To state the result we seek, we must introduce some notation. We use the abbreviation $e(a) = e^{2\pi i a}$. Fix $\frac{1}{2} < r_1, \ldots, r_{n-1} < 2$. For $\phi \in L^2(\mathbb{T}^{n-1})$ consider the Laplacian on the irrational torus.

$^4$This appears as theorem 1.1 in Bourgain and Demeter's work. [BD15]
torus $\prod_{i=1}^{n-1} \mathbb{R}/(r_i \mathbb{Z})$ which is given by

$$\Delta \phi(x_1, \ldots, x_{n-1}) = \sum_{(\zeta_1, \ldots, \zeta_{n-1}) \in \mathbb{Z}^{n-1}} (\zeta_1^2 r_1 + \cdots + \zeta_{n-1}^2 r_{n-1}) \hat{\phi}(\zeta_1, \ldots, \zeta_{n-1}) e(x_1 \zeta_1 + \cdots + x_{n-1} \zeta_{n-1}).$$

We also have

$$e^{it \Delta \phi}(x_1, \ldots, x_{n-1}) = \sum_{(\zeta_1, \ldots, \zeta_{n-1}) \in \mathbb{Z}^{n-1}} \hat{\phi}(\zeta_1, \ldots, \zeta_{n-1}) e(x_1 \zeta_1 + \cdots + x_{n-1} \zeta_{n-1} + t(\zeta_1^2 r_1 + \cdots + \zeta_{n-1}^2 r_{n-1})).$$

**Theorem 4** (Strichartz estimates for irrational tori). Let $\phi \in L^2(\mathbb{T}^{n-1})$ with $\text{supp} \hat{\phi} \subset [-N, N]^{n-1}$. Then for each $\varepsilon > 0$, $p \geq \frac{2(n+1)}{n-1}$ and each interval $I \subset \mathbb{R}$ with $|I| \gtrsim 1$ we have

$$\|e^{it \Delta \phi}\|_{L^p(\mathbb{R}^{n-1} \times I)} \lesssim \varepsilon N^{\frac{n-1}{2} + \frac{n+1}{p} + \varepsilon} |I|^{1/p} \|\phi\|_2,$$

(1.15)

and the implicit constant does not depend on $I$, $N$ or $r_i$.

The proof of this statement follows rather quickly once theorem 3 is proven. The idea is to use the discrete version of the $\ell^2$ decoupling theorem, as was done in [Bou96], and apply a change of variables to 1.15 which puts us in the perfect position to apply this discrete estimate. We therefore focus on proving theorem 3.

1.4.2 Main steps

Let us first note that the subcritical estimate

$$\|f\|_p \lesssim \delta^{-\varepsilon} \left( \sum_{\theta \in T} \|f_\theta\|_p^2 \right)^{1/2}.$$

will become possible by a localization argument and interpolation between the trivial $p = 2$ case and the endpoint $p = 2(n+1)/(n-1)$ from theorem 3, which
we henceforth refer to as $s$ to distinguish it from general $p$. The endpoint $s$ here is hinted at by a few prior discussions of this topic. First, G. Garrigós and A. Seeger proved in [GS10] that, up to the $\varepsilon$ term, the exponent $-\frac{n-1}{4} + \frac{n+1}{2p} - \varepsilon$ of $\delta$ in theorem 3 is optimal. Thus the clear breaking point for when this exponent is a constraint is precisely $s$. Furthermore, in our argument we will encounter the norm $\|f\|_{L^{2(n+1)/n}}$, and we the algebraic properties of $s$ allow the convenient bound

$$\|f\|_{L^{2(n+1)/n}} \leq \|f\|_{L^2}^{1/2} \|f\|_{L^s}^{1/2}.$$ via the Hölder inequality.

For brevity, we now focus on the endpoint case exclusively, following closely the notes [Gut14] by L. Guth. As mentioned above, proving the endpoint case quickly implies the subcritical estimate as well. The supercritical estimate $p > s$ requires a different argument, but many of the tools introduced here are used in that proof also.

Decoupling Norms

We begin by inspecting the right side of the $\ell^2$ decoupling inequality (1.14) further. For any $f$ such that $\text{supp} \hat{f} \subseteq \mathcal{N}_\delta S$ and $\Omega \subseteq \mathbb{R}^n$ any domain we fix a covering $T$ of $\mathcal{N}_\delta S$ and define

$$\|f\|_{L^p,\delta(\Omega)} := \left( \sum_{\theta \in T} \|f_{\theta}\|_{L^p(\Omega)}^2 \right)^{1/2} = \|\|f_{\theta}\|_{L^p(\Omega)}\|_{\ell^2(T)}.$$ This turns out to be a norm with some similar properties to the $L^p$ norms, in particular it satisfies the Hölder-type inequality

$$\|f\|_{L^q,\delta(\Omega)} \leq \|f\|_{L^p,\delta(\Omega)}^{1-\alpha} \|f\|_{L^{q_1,\delta(\Omega)}}^\alpha \|f\|_{L^{q_2,\delta(\Omega)}}^{\alpha}$$ (1.16)

for $1 \leq q, q_1, q_2 \leq \infty$, $0 < \alpha < 1$, and $\frac{1}{q} = (1 - \alpha)\frac{1}{q_1} + \alpha\frac{1}{q_2}$. 

20
It is useful to establish the following superadditive property, which is proven using the Minkowski inequality for the $\ell^{p/2}$ norm.

**Lemma 2.** If $\Omega$ is a disjoint union of $\Omega_j$ and $p \geq 2$, then for any $\delta$ and any $f$ with $\text{supp}\, \hat{f} \subseteq \mathcal{N}_\delta S$, we have
\[
\sum_j \|f\|_{L_p(\Omega_j)}^p \leq \|f\|_{L_p(\Omega)}^p.
\]

The benefit of this lemma is that it allows us to break $\Omega$ into disjoint pieces, and any decoupling norm which holds on each piece then holds on their union. This is what Guth calls “parallel decoupling”.\footnote{See Guth’s notes [Gut14], page 2.}

**Lemma 3** (Parallel decoupling). Suppose that $\Omega$ is a disjoint union of $\Omega_j$, $\text{supp}\, \hat{f} \subseteq \mathcal{N}_\delta S$, and $p \geq 2$. Suppose that for each $j$ we have the inequality
\[
\|f\|_{L_p(\Omega_j)} \leq M \|f\|_{L_p,\delta(\Omega_j)}.
\]
Then we also have the inequality
\[
\|f\|_{L_p(\Omega)} \leq M \|f\|_{L_p,\delta(\Omega)}.
\]

**Decoupling Constant**

We define the decoupling constant $D_p(R)$ as
\[
D_p(R) := \inf \|f\|_{L_p(B_R)} / \|f\|_{L_p,1/R(B_R)},
\]
where the infimum is taken over all $f$ with $\text{supp}\, \hat{f} \subseteq \mathcal{N}_{1/R} S$. We note that $D_p(R)$ also depends on $S$, but we will ignore this point for now. The claim is that, at the endpoint $s$,
\[
D_p(R) \lesssim R^s
\].
Multiple Scales

We consider the problem at multiple scales in Fourier space. Instead of breaking \( N_{1/R} S \) into pieces at the scale of \( \theta \), what if we first have a function supported in \( \tau \subseteq N_{1/R} S \) and then break \( \tau \) into \( \theta \) caps? The result is the following proposition.

**Proposition 1.** If \( \tau \subseteq N_{1/R} S \) is a \( r^{-1/2} \) cap for some \( r \leq R \), supp \( \hat{f} \subseteq \tau \), and \( \theta \subseteq N_{1/R} S \) are \( R^{-1/2} \) caps as before, then

\[
\|f\|_{L^p(\Omega)} \lesssim D_p(R/r) \left( \sum_{\theta \leq \tau} \|f_\theta\|_{L^p(B_R)}^2 \right)^{1/2}.
\]

The proof of this proposition is based on parabolic rescaling, in which we apply a linear transformation so that the region \( \tau \) has diameter 1 and then use our parallel decoupling lemma from earlier. As a corollary of proposition 1 we get the following estimate:

**Corollary 1.** For any radii \( R_1, R_2 \geq 1 \), we have

\[
D_p(R_1 R_2) \lesssim D_p(R_1) D_p(R_2).
\]

As a result, we see that there is a unique \( \gamma = \gamma(n, p) \) such that for all \( R, \epsilon \) we have

\[
R^{\gamma-\epsilon} \lesssim D_p(R) \lesssim R^{\gamma+\epsilon}.
\]

We want to prove that \( \gamma = 0 \) at the endpoint \( p = \frac{2(n+1)}{n-1} \).

**Multilinear vs. linear decoupling**

Perhaps inspired by the tractability of multilinear Kakeya and restriction over their linear counterparts, which we will discuss in section 1.4.3, we now consider looking at a multilinear version of the problem. We first define the notion of transversality.
**Definition 5.** A collection of $S_j \subseteq \mathbb{R}^n$ hypersurfaces are **transverse** if for any point $\omega \in S_j$, the normal vector $N_{S_j}(\omega)$ obeys

$$\text{Angle}(\nu(\omega), i^t h \text{ coordinate axis}) \leq (10n)^{-1}.$$ 

**Definition 6.** We say that functions $f_1, \ldots, f_n$ on $\mathbb{R}^n$ obey the **multilinear decoupling setup (MDS)** if

- For $i = 1, \ldots, n$, $\text{supp} \widehat{f_i} \subseteq \mathcal{N}_{1/R}S_i$
- $S_i \subseteq \mathbb{R}^n$ are compact positively curved $C^2$ hypersurfaces.
- The surfaces $S_j$ are transverse.

We define $\tilde{D}_{n,p}(R)$ to be the smallest constant so that whenever $f_i$ obey (MDS),

$$\left\| \prod_{i=1}^{n} |f_i|^{1/n} \right\|_{L^p(B_R)} \leq \tilde{D}_{n,p}(R) \prod_{i=1}^{n} \|f_i\|_{L^{p,1/n}(B_R)}.$$ 

Bourgain and Demeter go on to prove the following relationship between linear decoupling and multilinear decoupling:

**Theorem 5.** Suppose that in dimension $n-1$, the decoupling constant $D_{n-1,p}(R) \lesssim R^\epsilon$ for any $\epsilon > 0$. Then for any $\epsilon > 0$,

$$D_{n,p}(R) \lesssim R^\epsilon \tilde{D}_{n,p}(R).$$

The idea of the proof is choose small enough $K^{-1}$ caps $\tau$ such that $|f_\tau|$ is morally constant on cubes of side length $K$ in $B_R$. Then we cover $B_R$ with cubes $Q_K$ and classify them as broad or narrow depending on which $\tau$ make a significant contribution to $f|_{Q_K}$. The broad cubes can be controlled simply by the multilinear decoupling inequality, and the narrow ones are controlled by parallel decoupling and parabolic rescaling.

---

6Here we are directly quoting Guth’s notes [Gut14], page 5.
Note that we always have $\tilde{D}_{n,p}(R) \leq D_{n,p}(R)$ for any $n,p,R$. Then, using induction on the dimension $n$, if the decoupling theorem holds in dimension $n - 1$ for the endpoint $s$, then we have shown that

$$\tilde{D}_{n,p}(R) \sim D_{n,p}(R) \sim R^\gamma.$$ 

In other words, the linear decoupling problem is equivalent to the multilinear decoupling problem. This is quite surprising, as other problems such as linear Kakeya currently seem harder to prove than multilinear Kakeya. Thus, for decoupling, we can attack the problem using multilinear methods, which we will now leverage to our advantage.

### 1.4.3 Multilinear Kakeya

This section is a description of the argument in [Gut15], in which Guth states “the multilinear Kakeya inequality is a geometric estimate about the overlap pattern of cylindrical tubes in $\mathbb{R}^n$ pointing in different directions”. We will use it to prove the multilinear restriction estimate which will then allow us to prove the $\ell^2$ decoupling conjecture.

**Theorem 6** (Multilinear Kakeya). *Suppose that $\ell_{j,a}$ are lines in $\mathbb{R}^n$, and that each line $\ell_{j,a}$ makes an angle of at most $(10n)^{-1}$ with the $x_j$-axis. Let $T_{j,a}$ be the characteristic function of the 1-neighborhood of $\ell_{j,a}$, and let $Q_S$ denote any cube of side length $S$. Then for any $\varepsilon > 0$ and any $S \geq 1$, the following integral inequality holds:*

$$\int_{Q_S} \prod_{j=1}^n \left( \sum_{a=1}^{N_j} T_{j,a} \right)^{\frac{1}{n-1}} \lesssim \varepsilon S^\varepsilon \prod_{j=1}^n N_j^{\frac{1}{n-1}}$$  \hspace{1cm} (1.17)
Figure 6 is an example of a setup for Multilinear Kakeya. The area being considered is simply that within the square $Q_S$. In addition, considering the values in the inequality 1.17 we note that $\sum_{a=1}^{N_j} T_{j,a}$ represents the color density of our overlayed transparencies (see figure 9).

In the darker red portion, $\sum_{a=1}^{2} T_{1,a} = 2$, whereas for the lighter red portion this value will be 1, and for areas with no red the value is 0. Since there is only one line close to the $x_2$-axis in our example, $\sum_{a=1}^{1} T_{2,a} = T_{2,1} = 1$ for the blue areas in figure 9 and 0 otherwise. Since there is a product on the left side of 1.17, the only portion which is being counted at all is the purple region where the 1-neighborhood of $\ell_{2,1}$ intersects the 1-neighborhood of $\ell_{1,1}$ or $\ell_{1,2}$.
The method of proving the Multilinear Kakeya inequality (theorem 6) which was established by Bennett, Carbery, and Tao in [BCT06] and is also followed by Guth in [Gut15], is to first reduce to nearly axis parallel tubes:

Theorem 7. For every $\varepsilon > 0$ there is some $\delta > 0$ so that the following holds.

Suppose that $\ell_{j,a}$ are lines in $\mathbb{R}^n$, and that each line $\ell_{j,a}$ makes an angle of at most $\delta$ with the $x_j$-axis. Then for any $S \geq 1$ and any cube $Q_S$ of side length $S$, the following integral inequality holds:

$$\int_{Q_S} \prod_{j=1}^n \left( \sum_{a=1}^{N_j} T_{j,a} \right)^{\frac{1}{n-1}} \lesssim \varepsilon S^\varepsilon \prod_{j=1}^n N_j^{\frac{1}{n-1}}$$

The claim is that theorem 7 implies theorem 6. Suppose theorem 7 is true, then if $\delta \geq (10n)^{-1}$ for a given $\varepsilon$ then we are easily done. If $\delta < (10n)^{-1}$ however, we would like to stretch along an axis whose lines are not within $\delta$, bringing the lines closer to the axis. The only problem with this idea is that doing so also inevitably pulls other lines away from their axes. Clearly if one axis has lines which make too much of an angle, and the other axes are well within $\delta$, we may be able to stretch...
the space so that all the lines are within delta.

One problem with this idea, however, is that the amount we stretch relies on knowing information about the lines, so the order of quantifiers is incorrect. Obviously the other issue is that this does not help us if more than one set of lines makes an angle of more than $\delta$. The technique to handle both problems will be to split up over all possible contributions from various possible directions of lines and scale them each independently.

Assume that for $\varepsilon > 0$ the corresponding $\delta > 0$ from theorem 7 is less than $(10n)^{-1}$. Then we split the spherical cap $S_j$ of radius $(10n)^{-1}$ into caps $S_{j,\beta}$ of radius $\delta/10$, and then apply a linear change of coordinates to each cap centering it on $e_j$.

![Figure 8. How to reduce Multilinear Kakeya to the Nearly Axis Parallel case: We split $S_j$ into pieces, each of which has smaller radius than $\delta$, and then sum over all contributions where the direction of $\ell_{j,a}$ is in $S_{j,\beta}$ (one from each $S_j$). Our angles here are not to scale.](image)

In this case, the specific angle each $\ell_{j,a}$ makes is not important, as we know it is bounded by $(10n)^{-1}$, and so as we center each $S_{j,\beta}$ this linear change of coordinates has a controlled effect on lengths and areas and we can bound the overall integral by a sum of all combinations of contributions from these transformed systems, each of which is controlled by theorem 7.
Axis parallel (Loomis-Whitney)

The idea of the rest of the argument will be to further simplify our life by zooming in sufficiently close so that nearly axis parallel tubes look almost like axis parallel tubes. In this case we can get the bound we want using the Loomis-Whitney inequality, proven in [LW49], which states

**Theorem 8** (Loomis-Whitney). Suppose that $f_j : \mathbb{R}^{n-1} \to \mathbb{R}$ are measurable functions, and let $\pi_j : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be the linear map that forgets the $j^{th}$ coordinate:

$$\pi_j(x_1, \ldots, x_n) = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n).$$

Then the following inequality holds:

$$\int_{\mathbb{R}^n} \prod_{j=1}^{n} f_j(\pi_j(x)) \frac{1}{n-1} \leq \prod_{j=1}^{n} \|f_j\|_{L_1(\mathbb{R}^{n-1})}.$$

The connection between this theorem and the axis-parallel case is that a line parallel to the $x_j$-axis can be written as $\pi_j(x) = y_a$ for some $y_a \in \mathbb{R}^{n-1}$. Then, as noted in [Gut15] by Guth, $\sum_a T_{j,a}(x) = \sum_{\alpha} \chi_{B(y_a,1)}(\pi_j(x))$, and applying Loomis-Whitney with $f_j = \sum_a \chi_{B(y_a,1)}$ we have

$$\int_{\mathbb{R}^n} \prod_{j=1}^{N_j} \left( \sum_{a=1}^{N_j} T_{j,a} \right)^{1/(n-1)} = \int_{\mathbb{R}^n} \prod_{j=1}^{N_j} \left( f_j(\pi_j(x)) \right)^{1/(n-1)} \leq \prod_{j=1}^{n} \|f_j\|_{L_1(\mathbb{R}^{n-1})} \leq \omega_{n-1} N_j$$

where $\omega_{n-1}$ is the volume of the $n-1$ dimensional unit ball. Therefore the axis parallel case does follow quickly from Loomis-Whitney, so we proceed to describe loosely the “zooming in” part of the argument.

Given a cube $Q_S$, we begin by splitting it up into small enough $Q$ such that each tube $T_{j,a}$ which intersects a small $Q$ can be covered by $\tilde{T}_{j,a,R}$, an axis-parallel
tube with slightly larger radius $R$. Note that, since the $\tilde{T}_{j,a,R}$ actually cover the $T_{j,a}$ within $Q$, we have

$$\int_Q \prod_{j=1}^n \left( \sum_a T_{j,a} \right)^{1/n-1} \leq \int_Q \prod_{j=1}^n \left( \sum_a \tilde{T}_{j,a,R} \right)^{1/n-1} \lesssim R^n \prod_{j=1}^n N_j(Q)^{1/n-1}$$

where the last inequality follows from using Loomis-Whitney, and $N_j(Q)$ indicates the number of tubes $T_{j,a}$ intersecting $Q$. In fact, choosing $Q$ small enough, we can make it so that if the tube $T_{j,a}$ intersects $Q$, the tube $T_{j,a,\delta^{-1}}$ of radius $\delta^{-1}$ around $\ell_{j,a}$ is identically 1 on $Q$. Therefore

$$R^n \prod_{j=1}^n N_j(Q)^{1/n-1} \lesssim \frac{R^n}{|Q|} \int_Q \prod_{j=1}^n \left( \sum_a T_{j,a,\delta^{-1}} \right)^{1/n-1}$$

As Guth shows in [Gut15], with the appropriate choice of $|Q|$ and $R$, we can make $R^n/|Q| \lesssim \delta^n$. Since we can then sum over all $Q$, this proves the following lemma:

**Lemma 4.** Suppose that $\ell_{j,a}$ are lines with angle at most $\delta$ from the $x_j$ axis. Then if $S \geq \delta^{-1}$ and if $Q_S$ is any cube of sidelength $S$, then

---

7Exact details for what constitutes sufficiently small and slightly larger are contained in [Gut15].
\[
\int_{Q_S} \prod_{j=1}^{n} \left( \sum_{a} T_{j,a} \right)^{\frac{1}{n-1}} \lesssim \delta^n \int_{Q_S} \prod_{j=1}^{n} \left( \sum_{a} T_{j,a,\delta^{-1}} \right)^{\frac{1}{n-1}}
\]

We have essentially traded off making the tubes larger for the \(\delta^n\) factor. This can be seen as an exploit of the fact that a naive bound for the integrand is to assume that all tubes are identically 1 on \(Q_S\), which yields \(\prod_{j=1}^{n} N_j^{-\frac{1}{n-1}}\), and so we lose nothing in the trade.

Without loss of generality, assume \(Q_S\) is centered at the origin. Now if \(S \geq \delta^{-M}\) we begin induction on the scales \(\delta^{-1}, \delta^{-2}, \ldots, \delta^{-M}\):

\[
\int_{Q_S} \prod_{j=1}^{n} \left( \sum_{a} T_{j,a}(x) \right)^{\frac{1}{n-1}} \, dx \leq C_n \delta^n \int_{Q_S} \prod_{j=1}^{n} \left( \sum_{a} T_{j,a,\delta^{-1}}(x) \right)^{\frac{1}{n-1}} \, dx \quad (1.18)
\]

\[
= C_n \int_{\delta Q_S} \prod_{j=1}^{n} \left( \sum_{a} T_{j,a,\delta^{-1}}(\delta^{-1}x) \right)^{\frac{1}{n-1}} \, dx \quad (1.19)
\]

where (1.19) follows by a change of variables. In the new coordinates, the functions \(T_{j,a,\delta^{-1}}(\delta^{-1}x)\) are just unit tubes again, and \(\delta Q_S\) is a cube with side lengths \(\geq \delta^{-(M-1)}\), so we repeat the argument. After \(M\) repetitions we arrive at

\[
\int_{Q_S} \prod_{j=1}^{n} \left( \sum_{a} T_{j,a}(x) \right)^{\frac{1}{n-1}} \, dx \leq C_n^M \int_{\delta^M Q_S} \prod_{j=1}^{n} \left( \sum_{a} T_{j,a,\delta^{-M}}(\delta^{-M}x) \right)^{\frac{1}{n-1}} \, dx,
\]

and we can now use our naive bound to find

\[
\int_{Q_S} \prod_{j=1}^{n} \left( \sum_{a} T_{j,a}(x) \right)^{-\frac{1}{n-1}} \, dx \leq C_n^M (\delta^M S)^{n} \prod_{j=1}^{n} N_j^{-\frac{1}{n-1}}.
\]

If we had been working on a cube \(Q_S\) such that \(S = \delta^{-M}\), at this point we would only need that \(C_n^M \leq S^{\epsilon}\) to be done. To accomplish this, we solve \(S = \delta^{-M}\) for \(M = -\log S / \log \delta\), thus we have

\[
C_n^M = S^{-\frac{\log C_n}{\log \delta}}.
\]

Therefore given \(\epsilon > 0\), we choose \(\delta > 0\) such that \(-\frac{\log C_n}{\log \delta} < \epsilon\). Now we have proven the following lemma.
Lemma 5. Given $\epsilon > 0$, there exists $\delta > 0$ such that if $\ell_{j,a}$ are lines in $\mathbb{R}^n$ which make an angle of at most $\delta$ with the $x_j$-axis then, for every cube $Q_S$ such that $S = \delta^{-M}$ for some integer $M$,

$$\int_{Q_S} \prod_{j=1}^n \left( \sum_a T_{j,a} \right)^{\frac{1}{n-1}} \leq S^\epsilon \prod_{j=1}^n N_j^{\frac{1}{n-1}}.$$

This is enough to prove theorem 7, since given $\epsilon > 0$ we take $\delta$ as in the above lemma. Then for any $S$ we take $M$ to be the largest integer such that $S \geq \delta^{-M}$, and then we cover $S$ by at most $C$ cubes of sidelength $\delta^{-M}$ where, apriori, $C$ depends on both $S$ and $\delta$. By proving lemma 5 for all integers $M$, however, we have been able to remove the dependence on $S$, since we can simply cover the cube $Q_S$ with one of side length $\delta^{-(M+1)}$, and then figure out how many cubes of side length $\delta^{-M}$ are needed to cover this cube. Consequently the dependence of $C$ is only on $\delta$, which itself depends only on $\epsilon$. Summing over these cubes yields

$$\int_{Q_S} \prod_{j=1}^n \left( \sum_a T_{j,a} \right)^{\frac{1}{n-1}} \lesssim S^\epsilon \prod_{j=1}^n N_j^{\frac{1}{n-1}}.$$

1.4.4 Multilinear Restriction

At this point we use the multilinear Kakeya inequality to prove multilinear restriction. One way to do this is to use the method of induction on scales, similar to the argument in [BG11], and similar to the argument described above for multilinear Kakeya, which itself was an argument by induction on scales.

A key principle to apply induction on scales is to have some way for bounding a desired quantity of one scale by another. In multilinear Kakeya, this was lemma 4. For multilinear restriction, it will be the following.
Lemma 6. If $\text{supp } f_i \subseteq N_{1/R}S_i$ and $S_i$ are smooth compact transverse hypersurfaces, and if $2 \leq p \leq \frac{2n}{n-1}$, then

$$\text{Avg}_{B_{R/2} \subset BR} \prod_{i=1}^{n} \left\| f_i \right\|_{L^2_{\text{avg}}(B_{R/2})}^{p/n} \lesssim R^\varepsilon \prod_{i=1}^{n} \left\| f_i \right\|_{L^2_{\text{avg}}(BR)}^{p/n}$$

By applying Bernstein’s inequality at a sufficiently small scale $r$ and then inductively working our way up using lemma 6 we move through scales $r^{2^m}$ until we reach $R$. This yields a decoupling estimate for $2 \leq p \leq \frac{2n}{n-1}$, as appeared in [Bou96].

1.4.5 Using curvature

In the above argument we used transversality, but we did not use curvature at all. The introduction of curvature allows a more subtle estimate.

Lemma 7. If $\text{supp } f_i \subseteq N_{1/R}S_i$, and $S_i$ are compact positively curved transverse hypersurfaces, and $s = \frac{2(n+1)}{n-1}$ and $\delta = R^{-1}$, then

$$\text{Avg}_{B_{R/2} \subset BR} \prod_{i=1}^{n} \left\| f_i \right\|_{L^2_{\text{avg}}(B_{R/2})}^{\frac{s}{2}} \lesssim R^\varepsilon \prod_{i=1}^{n} \left\| f_i \right\|_{L^2_{\text{avg}}(BR)}^{\frac{1}{2} + \frac{s}{2n}} \prod_{i=1}^{n} \left\| f_i \right\|_{L^2_{\text{avg}}(BR)}^{\frac{1}{2} + \frac{s}{2n}}$$

Again, note that this estimate is perfectly suited to an induction on scales type argument. In order to prove this lemma, Bourgain and Demeter show that we can reverse the previous Hölder inequality (1.16) if the function can be broken into a small number of “balanced” pieces, where each piece obeys a reverse Hölder inequality, i.e. if $1 \leq q, q_1, q_2 \leq \infty$ and $\frac{1}{q} = (1 - \alpha)\frac{1}{q_1} + \alpha\frac{1}{q_2}$, then

$$\left\| f \right\|_{L^{1-\alpha}(\Omega)}^{1-\alpha} \left\| f \right\|_{L^{\alpha}(\Omega)}^{\alpha} \lesssim \left\| f \right\|_{L^q(\Omega)}.$$  \hfill (1.20)

The proof of this fact relies on a wave packet decomposition of $f$, and essentially allows interpolation with the $L^{p,\delta}$ norms, which in turn facilitates the proof of
lemma 7. Finally, using an induction on scales argument based on lemma 7 as well as parabolic rescaling (discussed in section 1.4.2) Bourgain and Demeter are able to prove $l^2$ decoupling for the endpoint $s$. Some slight adjustments are needed to prove the $p > s$ range for theorem 3 to be complete. In addition, relaxing certain simplifying assumptions we have made requires weights to be brought into the equations, and a certain amount of care is needed to deal with them also.

At this point we will return to the setting of the nonlinear Schrödinger equation, and explore what results we can get now that stronger Strichartz are known.
In what follows, we will consider the cubic nonlinear Schrödinger equation on a torus (CNLS):

\[
\begin{aligned}
&iu_t + \Delta u = |u|^2 u \\
&u(0, x) = u_0(x)
\end{aligned}
\quad \text{where } (t, x) \in \mathbb{R} \times T^d_{\lambda}
\]

(2.1)

Our goal is to prove polynomial bounds on the growth of \(\|u(t)\|_{H^s}\) in time, that is, something of the form

\[
\|u(t, x)\|_{H^s_x} \lesssim \langle t \rangle^{r(s)} \|u_0\|_{H^s_x}.
\]

(2.2)

Achieving such a bound has been the objective of a number of papers. For rational tori, in [Bou96], Bourgain achieved the bound of \(r(s) = 2(s - 1)\) using his “high-low” method. In [Sta+97] Staffilani proved a bound of \(r(s) = (s - 1)\) using multilinear estimates in \(X^{s, b}\)-spaces in the vein of Kenig-Ponce-Vega [KPV96]. For irrational tori in 2-D, Catoire-Wang proved in [CW08] a bound of \(r(s) = \frac{3}{2}(s - 1)\) and Demirbas proved a bound of \(r(s) = \frac{416}{285}(s - 1)\) in [Dem13].

In this chapter we reuse the machinery from Catoire-Wang in [CW08], which itself was synthesized from [Bou96], to prove a stronger bound.
2.1 Background

First we note that, in general, if CNLS is locally-well posed, one can obtain an exponential upper bound for the $H^s$ norm by iterating $H^1$ local theory and using preservation of regularity arguments. Bourgain’s “high-low” method was the first technique to improve the exponential to a polynomial bound for CNLS over rational tori in $d = 2, 3$. The idea is to prove an inequality of the form

$$ \|u(t + 1)\|_{H^s}^2 - \|u(t)\|_{H^s} \lesssim \|u(t)\|_{H^s}^{2(1 - \theta)} \tag{2.3} $$

for some $\theta > 0$. The method involves making a high-low decomposition at a certain threshold frequency $N$, using the Littlewood-Paley decomposition $P_{\leq N}$, and then inspecting the evolution equation obtained by plugging this decomposition into the equation. The resulting interaction between frequencies is bounded using $X^{s,b}$ spaces, for which we can develop bounds from the Strichartz norms.

As mentioned previously, counting arguments allowed Strichartz to be proved much more easily in the rational case. Now that we have a Strichartz estimate from $\ell^2$-decoupling which does not distinguish between the rational and irrational tori, we can leverage this to prove new bounds on the growth.

2.2 Main Argument

In this section we will extract the necessary components of [CW08], the essence of which was originally introduced in [Bou96]. We will then use the structure of their argument to prove stronger polynomial bounds in dimensions $d = 2, 3$.

The crux of the argument in [CW08] relies on proving a bilinear estimate of the following form.
**Conjecture 1.** There exists an $s_0 \in [0, 1)$ such that, for all $f_1, f_2 \in L^2(M)$ with \( \text{supp}(\hat{f}_i) \subset [-N_1, N_1]^d \),

\[
\|e^{it\Delta} f_1 e^{it\Delta} f_2\|_{L^2_{t,loc}L^2_x} \lesssim \min(N_1, N_2)^{s_0} \|f_1\|_{L^2} \|f_2\|_{L^2}, \tag{2.4}
\]

In [CW08], the authors consolidate arguments from [Bou96] and [Zho08] which establish the following:

**Theorem 9.** Suppose Conjecture 1 holds for some $s_0$. Then the CNLS is smoothly locally well posed in $H^s$ for every $s > s_0$. Moreover, in the case that $s \geq 1$, the local time of existence depends only on $\|u\|_{H^1(M)}$, which is uniformly bounded in time.

**Theorem 10.** If Conjecture 1 holds for some $s_0$ and $u$ is the solution to the CNLS with initial data $u_0 \in H^s(\mathbb{T}^d)$ for $s \in \mathbb{N}_{\geq 1}$, and for all $\varepsilon > 0$,

\[
\|u(t)\|_{H^s} \lesssim t^{(\theta(d)+\varepsilon)(s-1)} \tag{2.5}
\]

where

\[
\theta(d)^{-1} := \begin{cases} 
1 - s_0 & \text{if } d = 2, \\
\left(1 - \frac{d-2}{2(d-2s_0)}\right) \left(1 - \frac{d-2}{d-s_0-1}\right) & \text{if } d \geq 3.
\end{cases}
\]

The result obtained in [CW08] is that Conjecture 1 holds with

\[
s_0 = \begin{cases} 
\frac{1}{3}, & \text{if } d = 2, \\
\frac{d}{2} - \frac{d}{d+1} + \varepsilon & \text{if } d \geq 3, d \text{ odd}, \\
\frac{d}{2} - 1 + \varepsilon & \text{if } d \geq 4, d \text{ even},
\end{cases} \tag{2.6}
\]

which leads to local well-posedness in $H^s$ for any $s$ satisfying

\[
s > \begin{cases} 
\frac{1}{3} & \text{if } d = 2, \\
\frac{3}{4} & \text{if } d = 3,
\end{cases} \tag{2.7}
\]
and leads to global well-posedness in $H^s$ for all $s \geq 1$ (by iteration). If $s \in \mathbb{N}_{\geq 1}$, we can also obtain the Sobolev bound with

$$\theta(d) := \begin{cases} \frac{3}{2} & \text{if } d = 2, \\ \frac{15}{2} & \text{if } d = 3. \end{cases}$$ (2.8)

### 2.2.1 Leveraging $\ell^2$-decoupling Strichartz

In [BD15], which makes no distinction between irrational or rational tori, Bourgain and Demeter provide an improved Strichartz estimate which can be used to obtain a larger range of local-well posedness results for general tori. Specifically, they obtain the following:

**Theorem 11.** (Strichartz estimates from $\ell^2$-decoupling). Let $f \in L^2(\mathbb{T}^d)$ with $\text{supp}(\hat{f}) \subset [-N, N]^d$. Then for all $\varepsilon > 0, p \geq \frac{2(d+2)}{d}$ and each interval $I \subset \mathbb{R}$ with $|I| \gtrsim 1$ we have

$$\|e^{it\Delta}f\|_{L^p(\mathbb{T}^d \times I)} \lesssim \varepsilon N^2 \left(\frac{d+2}{p} - \frac{d}{2}\right) + \varepsilon |I|^{1/p} \|f\|_L^2(\mathbb{T}^d)$$ (2.9)

and the implicit constant does not depend on $I, N$, or the irrationality of $\mathbb{T}^d$.

We can apply this in our case to obtain Conjecture 1 for smaller $s_0$.

**Lemma 8.** Given any $\varepsilon > 0$, if $f_1, f_2 \in L^2(\mathbb{T}^d)$ with $\text{supp}(\hat{f}_i) \subset [-N_i, N_i]^d$, then

$$\|e^{it\Delta}f_1e^{it\Delta}f_2\|_{L^2(\mathbb{T}^d \times I)} \lesssim \min(N_1, N_2)^{\frac{d-2}{d} + \varepsilon} \|f_1\|_{L^2(\mathbb{T}^d)} \|f_2\|_{L^2(\mathbb{T}^d)}.$$ (2.10)

**Proof.** Assume $N_1 \leq N_2$, then we decompose $f_2 = \sum_{\alpha \in 2N_i \mathbb{Z}^d} P_\alpha[f_2]$ where

$$P_\alpha[f_2] := 1_{\alpha + [-N_i, N_i]^d} \hat{f}_2.$$
Then, by Littlewood-Payley theory and Hölder we have
\[
\|e^{it\Delta} f_1 e^{it\Delta} f_2\|_{L^2(T^d \times I)} = \left\| e^{it\Delta} f_1 e^{it\Delta} \sum_{\alpha} P_\alpha [f_2] \right\|_{L^2(T^d \times I)} \\
\lesssim \left( \sum_{\alpha} \|e^{it\Delta} f_1 e^{it\Delta} P_\alpha [f_2]\|_{L^2(T^d \times I)}^2 \right)^{1/2} \\
\leq \|e^{it\Delta} f_1\|_{L^4(T^d \times I)} \left( \sum_{\alpha} \|e^{it\Delta} P_\alpha [f_2]\|_{L^4(T^d \times I)}^2 \right)^{1/2}.
\]

We now apply Theorem 11 with \(p = 4\) to each norm, obtaining
\[
\lesssim \varepsilon \, N_1^{\frac{d-2}{2} + \frac{s}{2}} |I|^{\frac{1}{2}} \|f_1\|_{L^2(T^d)} \, N_2^{\frac{d-2}{2} + \frac{s}{2}} |I|^{\frac{1}{2}} \left( \sum_{\alpha} \|P_{\alpha} [f_2]\|_{L^2(T^d)}^2 \right)^{1/2} \\
\lesssim N_1^{\frac{d-2}{2} + \varepsilon} |I|^{\frac{1}{2}} \|f_1\|_{L^2(T^d)} \|f_2\|_{L^2(T^d)}
\]

where the last line follows from typical Littlewood-Payley theory.

This allows us to get local well-posedness in \(H^s\) for any \(s\) satisfying
\[
s \geq \begin{cases} 
0 & \text{if } d = 2, \\
\frac{1}{2} & \text{if } d = 3.
\end{cases}
\]
and bounds on the growth of the Sobolev norms for \(s \in \mathbb{N}_{\geq 1}\)
\[
\|u(t)\|_{H^s(T^d)} \lesssim \begin{cases} 
t^{(1+\varepsilon)(s-1)} & \text{if } d = 2, \\
t^{(4+\varepsilon)(s-1)} & \text{if } d = 3,
\end{cases}
\]
which are strictly better than the current known results.

Ideally, we would like a bound for the \(d = 4\) case as well, however since Lemma 8 would yield \(s_0 = 1 + \varepsilon\) and we need \(s_0 < 1\) in Theorem 9 our bilinear Strichartz is not strong enough. This is not necessarily surprising, as for \(d = 4\) the CNLS is
energy-critical, so we cannot afford any loss of decay. As a result, it is expected that other methods would be necessary, and in the next chapter we will prove a conditional result which yields polynomial bounds for $T^4_\ell$. 
CHAPTER 3

GROWTH OF SOBOLEV NORMS FOR
ENERGY-CRITICAL NLS WITH SMALL ENERGY IN $T_\lambda^4$

3.1 Introduction

In this section we consider the cubic nonlinear Schrödinger equation on $T_\lambda^4$,

$$(i\partial_t + \Delta)u = |u|^2u$$

(3.1)
on $\mathbb{R} \times T_\lambda^4$. For this equation the energy is defined as

$$E_\lambda[u] := \int_{T_\lambda^4} \frac{1}{2} |\nabla u|^2 + \frac{1}{4} |u|^4 \, dx.$$  

(3.2)

Note that the critical scaling value is

$$s_c = \frac{d}{2} - \frac{2}{p-1} = 1.$$  

Since the energy is also at the level of $H^1$, we say that this equation is energy-critical. It was recently proved in [Yue18] that (3.1) is globally well-posed in $H^1(T_\lambda^4)$. As before, we seek bounds in time on the growth of the $H^s$ norm for solutions $u$. This was recently proved in [Den17] for the quintic equation in $T_\lambda^3$, which is also energy-critical, and we follow his method closely. The method in [Den17] relies on the long-time Strichartz norms proved in [DGG17], however, as we will see within the
proof, in our case we will need the following decoupled long-time Strichartz norms to close the argument.

**Definition 7.** We will say a result is true for almost every $T^4_\lambda$ if it is true for almost every $\lambda \in (\mathbb{R}^+)^4$.

**Conjecture 2** (Decoupled long-time Strichartz). For almost every $T^4_\lambda$ we have

$$
\left\| e^{it\Delta} P_N f \right\|_{L_t^q [0,T] L_x^r T^4_\lambda} \lesssim \epsilon N^\epsilon \left( 1 + N^{2-\frac{2}{q} - \frac{4}{q'}} \right) \left[ 1 + \left( \frac{T}{N^{\theta(q,q')}} \right)^{\frac{1}{q'}} \right] \left\| P_N f \right\|_{L^2} \quad (3.3)
$$

and there is some $\kappa > 0$ such that

$$
\min_{5 \leq q' \leq 12} \theta(7/4, q') = \kappa. \quad (3.4)
$$

The main goal in this chapter is the following:

**Theorem 12** (Main Theorem). Given Conjecture (2), there is some constant $\gamma < \kappa$ such that, for almost every $T^4_\lambda$ there is some constant $\eta \in (0,1)$ depending on $\lambda$ such that, for any $s > 1$ and any solution $u$ to (3.1) with energy $E_\lambda[u] \leq \eta^2$ and initial data $u(0) \in H^s(T^4_\lambda)$, one has that

$$
\left\| u(t) \right\|_{H^s(T^4_\lambda)} \lesssim_{\lambda,s} \max \left( \left\| u(0) \right\|_{H^s(T^4_\lambda)}, \left| t \right|^\frac{\epsilon}{\epsilon - 1} \right)
$$

for any time $t \in \mathbb{R}$.

**Remark 1.** The specific $\lambda$ for which Conjecture 2 (and therefore Theorem 12) is expected to hold are those for which we can apply the long-time Strichartz estimates from [DGG17]. In particular, we do not expect the conjecture to hold for rational tori.

**Remark 2.** The form of Conjecture 2 mirrors the form of the long-time Strichartz from [DGG17], where our $N^{2-\frac{2}{q} - \frac{4}{q'}}$ term comes from a guess for the decoupled short-time Strichartz.
Remark 3. Condition (3.4) is exactly what is necessary for our proof, however we expect that

\[ \theta(q, q') \begin{cases} 
0 & \text{for } q \leq \frac{2q}{q-1}, \\
> 0 & \text{for } q > \frac{2q}{q-1}.
\end{cases} \]

Additionally assuming that \( \theta(q, q') \) is continuous or increasing in \( q' \) for fixed \( q \) would also imply (3.4).

3.1.1 Main Idea

As in Deng, we will prove this theorem using the upside-down \( I \)-method to get an almost conserved quantity. To this end, we will define the \( D \) operator such that, if we assume the small energy condition \( E_\lambda[u] \leq \eta^2 \) and choose \( N \) such that \( \|u()\|_{H^s} \sim \eta N^{s-1} \) then \( \|Du(0)\|_{H^s} \lesssim \eta \) and \( \|E[Du(0)]\| \lesssim \eta^2 \). If we can bound the increase in energy for \( t \leq N^\gamma \),

\[ E[Du(t)] - E[Du(0)] \lesssim \eta^2 \]

then

\[ E[Du(t)] \lesssim E[Du(0)] + \eta^2 \lesssim \eta^2. \]

Since the equation is energy-critical this implies a bound

\[ \sup_{0 \leq t \leq N^\gamma} \|Du(t)\|_{H^s} \lesssim \eta, \]

which then implies

\[ \|u(t)\|_{H^s} \lesssim \eta N^{s-1} \quad \text{for} \quad 0 \leq t \leq N^\gamma. \]

Using time translation and scaling, iterating this bound will yield

\[ \|u(t)\|_{H^s} \lesssim t^{\frac{s-1}{\gamma}}. \]
The objective, therefore, is to bound the increase in energy of $Du$. This will be achieved by combining Corollary 3 and Proposition 10, which are analogs of [Den17] Corollary 3.2 and Proposition 3.3.

### 3.1.2 Notations

We use $\chi$ to denote general cutoff functions, and $1_E$ to denote the characteristic function of a set $E$. In this section, $A \lesssim B$ will mean that there exists a constant $C$ (which may depend on $\lambda$ or $s$, but not $\eta$) such that

$$A \leq CB.$$ 

We use $\tilde{u}$ to denote $u$ or $\overline{u}$ in circumstances where the statement applies to both. Capital letters $N, M, K, \cdots$ will denote dyadic numbers in $2^N$, which we will often use in the Littlewood-Paley decomposition

$$1 = \sum_N P_N,$$

where

$$\widehat{P_N f}(k) = \left[ \varphi \left( \frac{k}{N} \right) - \varphi \left( \frac{k}{N^{1/2}} \right) \right] \hat{f}(k) \text{ for } N \geq 2, \quad \widehat{P_1 f}(k) = \varphi(k) \hat{f}(k),$$

and $\varphi(y) = \varphi(|y|)$ is a radial smooth function which is 1 for $|y| \leq 1$ and 0 for $|y| \geq 2$. Define

$$P_{\leq N} := \sum_{M \leq N} P_M \quad \text{and} \quad P_{\geq N} := \sum_{M \geq N} P_M.$$ 

For each fixed $N$ we also define another kind of decomposition,

$$1 = \sum_{B} P_B,$$

where $B$ runs over the collection of balls of radius $N$ centered at points in $(NZ)^4$, and $P_B$ is defined by

$$\widehat{P_B f}(k) := \psi \left( \frac{k-k_0}{N} \right) \hat{f}(k), \quad \text{if } B = B(k_0, N), k_0 \in \mathbb{R}^4.$$
with $\psi$ being a fixed compactly supported smooth radial function which is 1 in a neighborhood of 0 and

$$\sum_{k \in \mathbb{Z}^d} \psi(y - k) = 1.$$  

Note that $P_n$ and $P_B$ are bounded in $L^p$ spaces. We also define sharp cutoff projections, denoted by $P_E$ for any set $E \subset \mathbb{Z}^d$, which are defined by

$$\hat{P}_Ef(k) := 1_E(k) \hat{f}(k).$$

In particular, let $P_0 := P_{\{(0,0,0,0)\}}$ and $P_{\neq 0} := P_{\mathbb{Z}^d \backslash \{(0,0,0,0)\}}$.

### 3.1.3 $D$-multiplier (a.k.a. upside-down $I$-operator)

For each $N$, define the $D$-multiplier by

$$\hat{D}_Nu(k) := \theta \left( \frac{k}{N} \right) \hat{u}(k)$$

where $\theta(y) = \theta(|y|)$ is a radial smooth function such that $\theta(y) = 1$ for $|y| \leq 1$ and $\theta(y) = |y|^{-s-1}$ for $|y| \geq 2$. When $N$ is fixed we will omit the subscript and denote $D_N$ by $D$. Note that for $p, q, r \in [1, \infty]$ with $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$ we have

$$\|D(fg)\|_{L^p} \lesssim \|Df\|_{L^q} \|Dg\|_{L^r}. \quad (3.5)$$

**Lemma 9.**

$$\|f\|_{H^s} \lesssim N^{s-1} \|Df\|_{H^1}$$

**Proof.** We decompose

$$\|f\|_{H^s} \leq \|P_{\leq N} f\|_{H^s} + \|P_{> N} f\|_{H^s}$$

$$= \left( \sum_{k \in \mathbb{Z}^d, |k| \leq N} \langle k \rangle^{2s} |\hat{f}(k)|^2 \right)^{\frac{1}{2}} + \left( \sum_{k \in \mathbb{Z}^d, |k| > N} \langle k \rangle^{2s} |\hat{f}(k)|^2 \right)^{\frac{1}{2}}.$$
Now
\[
\left( \sum_{k \in \mathbb{Z}^4, |k| \leq N} |\langle k \rangle^{2s} \hat{f}(k) |^2 \right)^{\frac{1}{2}} \leq N^{s-1} \left( \sum_{k \in \mathbb{Z}^4, |k| \leq N} |\langle k \rangle^{2} \hat{f}(k) | \right)^{\frac{1}{2}}
\]
\[
\leq N^{s-1} \|Df\|_{H^1}
\]
and
\[
\left( \sum_{k \in \mathbb{Z}^4, |k| > N} |\langle k \rangle^{2s} \hat{f}(k) |^2 \right)^{\frac{1}{2}} \lesssim N^{s-1} \left( \sum_{k \in \mathbb{Z}^4, |k| > N} \left| \frac{1}{N} \langle k \rangle^{s-1} \hat{f}(k) \right| \right)^{\frac{1}{2}}
\]
\[
\lesssim N^{s-1} \|Df\|_{H^1}.
\]

3.2 Tools

To convert the rectangular torus $T^4_\lambda$ to a square torus $T^4 := (\mathbb{R}/\mathbb{Z})^4$ we use a change of variables $x \to (\frac{x_1}{\lambda_1}, \ldots, \frac{x_4}{\lambda_4})$, which transforms (3.1) into
\[
(i\partial_t + \Delta_\beta)u = |u|^2 u \quad \text{on } \mathbb{R} \times T^4,
\]
where $\Delta_\beta$ is the “anisotropic” Laplacian:
\[
\Delta_\beta := \beta_1 \partial_{x_1}^2 + \beta_2 \partial_{x_2}^2 + \beta_3 \partial_{x_3}^2 + \beta_4 \partial_{x_4}^2, \quad \beta_i := \frac{1}{\lambda_i^2}.
\]

In these coordinates, the energy becomes
\[
E_\beta[u] := \int_{T^4} \frac{1}{2} \sum_{i=1}^4 \beta_i |\partial_{x_i} u|^2 + \frac{1}{4} |u|^4 \, dx.
\]
3.2.1 Definition of norms

We recall the definitions of various norms used in [Den17].

**Definition 8** (Atomic Spaces $U^p$ and $V^p$). Define a partition of $\mathbb{R}$ to be a sequence

$$\mathcal{P} := \{t_m\}_{m=0}^M, \quad -\infty < t_0 < t_1 < \cdots < t_M \leq +\infty.$$ 

Given $1 \leq p < \infty$ and a separable Hilbert space $H$, define a $U^p$ atom to be a function $a : \mathbb{R} \to H$ of the form

$$a(t) := \sum_{m=1}^M 1_{(t_{m-1}, t_m)} \phi_{m-1},$$ 

where $\{t_m\}_{m=0}^M$ is a partition of $\mathbb{R}$, $\phi_m \in H$, and

$$\sum_{m=0}^{M-1} \|\phi_m\|^p_H = 1.$$ 

Define the space $U^p(\mathbb{R}; H)$ by the norm

$$\|u\|_{U^p(\mathbb{R}; H)} := \inf \left\{ \sum_{j=1}^{\infty} |\gamma_j| : u = \sum_{j=1}^{\infty} \gamma_j a_j, \text{ each } a_j \text{ is a } U^p \text{ atom} \right\}.$$ 

Define the space $V^p(\mathbb{R}; H)$ by the norm

$$\|u\|_{V^p(\mathbb{R}; H)} := \sup_{\mathcal{P}} \left( \sum_{m=1}^{M} \|u(t_m) - u(t_{m-1})\|_H^p \right)^{\frac{1}{p}},$$ 

where $u(\pm \infty) := \lim_{t \to \pm \infty} u(t)$. We shall restrict to the (closed) subspace of $V^p(\mathbb{R}; H)$ consisting only of right-continuous functions $u$ such that $u(\pm \infty) = 0$.

**Definition 9** (Spacetime norms). Let $s \in \mathbb{R}$ and $u : \mathbb{R} \times T^4 \to \mathbb{C}$. Define the norms

$$\|u\|_{U^p_sH^s} := \|e^{-it\Delta} u\|_{U^p(\mathbb{R}; H^s(T^4))}$$ 

and

$$\|u\|_{V^p_sH^s} := \|e^{-it\Delta} u\|_{V^p(\mathbb{R}; H^s(T^4))}.$$
Moreover, define
\[ \|u\|_{X^s} := \left( \sum_{k \in \mathbb{Z}^4} \langle k \rangle^{2s} \| e^{iQ(k)t} \hat{u}(t,k) \|_{U^2_t}^2 \right)^{\frac{1}{2}} \]
and
\[ \|u\|_{Y^s} := \left( \sum_{k \in \mathbb{Z}^4} \langle k \rangle^{2s} \| e^{iQ(k)t} \hat{u}(t,k) \|_{V^2_t}^2 \right)^{\frac{1}{2}} \]
where \( \hat{u} \) denotes the Fourier transform in space, and
\[ Q(k) := \beta_1 k_1^2 + \beta_2 k_2^2 + b_3 k_3^2 + b_4 k_4^2 \quad \text{for} \quad k = (k_1, k_2, k_3, k_4). \]

For any compact interval \( I \subset \mathbb{R} \), define also the local-in-time spaces \( X^s(I) \) and \( Y^s(I) \) by
\[ \|u\|_{X^s(I)} := \inf_{v \equiv u \text{ on } I} \|v\|_{X^s} \quad \text{and} \quad \|u\|_{Y^s(I)} := \inf_{v \equiv u \text{ on } I} \|v\|_{Y^s}. \]

In addition to the \( X^r \) and \( Y^r \) spaces, we will also use the \( X^{s,b} \) norms, namely
\[ \|u\|_{X^{s,b}} := \left( \sum_{k \in \mathbb{Z}^4} \int \langle k \rangle^{2s} |Q(k)|^{2b} |\mathcal{F}_{t,x} u(k, \xi)|^2 \, d\xi \right)^{\frac{1}{2}}, \]

We also define the following long-time Strichartz norms.

**Definition 10.** For any \( N \geq 1 \) and any finite interval \( J \), define
\[ \|u\|_{S^q_{N,J}} := \left( \sum_{m \in \mathbb{Z}} \left( N^\frac{q-1}{2} \|u\|_{L^q_{t,x}([m,m+1] \cap J)} \right)^q \right)^{\frac{1}{q}} \]
and
\[ \|u\|_{S^q_{N,J}} = \left( \sum_{m \in \mathbb{Z}} \left( \sup_{q \in [5,12]} N^\frac{q-1}{2} \|u\|_{L^q_{t,x}([m,m+1] \cap J)} \right)^q \right)^{\frac{1}{q}}. \]

Note that
\[ \|u\|_{S^q_{N,J}} \sim \max \left( \|u\|_{S^q_{N,J}}, \|u\|_{S^{q,12}_{N,J}} \right) \]
by Hölder, and that \( \|u\|_{S^q_{N,J}} \) and \( \|u\|_{S^{q,12}_{N,J}} \) are decreasing in \( q \), by \( \ell^q \) embedding.

**Remark 4.** These are the same norms as used in [Den17]. These norms are commonly used for energy-critical problems. [Yue18]
3.2.2 Linear estimates

Proposition 2 (Strichartz estimates). Let $I$ be a time interval with length $|I| \lesssim 1$. We have the following estimates.

1. Homogeneous Strichartz estimates:

   $$\left\| e^{it\Delta} P_N f \right\|_{L^q_{t,x}(I)} \lesssim N^{2-\frac{6}{q}} \| P_N f \|_{L^2_x}$$

   for any fixed $q > 3$. The same estimate holds for $P_B f$ for any ball $B$ of radius $N$, that is

   $$\left\| e^{it\Delta} P_B f \right\|_{L^q_{t,x}(I)} \lesssim N^{2-\frac{6}{q}} \| P_B f \|_{L^2_x} .$$

   (3.9)

   In the case of the sharp cutoff projections $\mathbb{P}_C$, we have an improved estimate

   $$\left\| e^{it\Delta} \mathbb{P}_C f \right\|_{L^q_{t,x}(I)} \lesssim N^{2-\frac{6}{q} \left( \# C \right)^{\frac{1}{2} - \frac{3}{2q} + \varepsilon}} \| \mathbb{P}_C f \|_{L^2_x} .$$

   (3.10)

2. Inhomogeneous Strichartz estimates: Let $t_0 \in I$ and

   $$\mathcal{I}_G(t) := \int_{t_0}^{t} e^{i(t-t')\Delta} G(t') \, dt'$$

   be the Duhamel operator. Then for any $q_1 > 3$ and $1 \leq q_2 < \frac{3}{2}$, one has

   $$\left\| \mathcal{I} P_N G \right\|_{L^q_{t,x}(I)} \lesssim N^{\frac{6}{q_2} - \frac{6}{q_1} - 2} \| P_N G \|_{L^2_x} .$$

   (3.13)

Proof. In [KV16] the authors remove the $\varepsilon$ from the $\ell^2$-decoupling Strichartz estimates in [BD15]. Using their result we can obtain the estimate

   $$\left\| e^{it\Delta} P_{\leq 2N} g \right\|_{L^q_{t,x}(I)} \lesssim (2N)^{2-\frac{6}{q}} \| g \|_{L^2_x} \lesssim N^{2-\frac{6}{q}} \| g \|_{L^2_x} .$$

Replacing $g$ by $P_N f$ and noting $P_{\leq 2N} P_N f = P_N f$ implies (3.9). Similarly, replacing $g$ by $P_{\leq N} f$ and using Galilean invariance of the free solution operator yields (3.10). To get (3.11) we note that, for an appropriately chosen $B$, we have $\mathbb{P}_C f = \mathbb{P}_B \mathbb{P}_C f,$
and thus we can apply (3.10) with \( q_0 := 3 + \varepsilon \). Hausdorff-Young and Hölder directly imply
\[
\|P_C f\|_{L^\infty_t \mathbb{R}^d} \lesssim (\# C)^{\frac{1}{2}} \|P_C f\|_{L^2_t \mathbb{R}^d}.
\] (3.14)

Interpolation via log-convexity of \( L^p \) norms between these two yields that for any \( q \in [3 + \varepsilon, \infty] \),
\[
\|P_C f\|_{L^q_t \mathbb{R}^d (I)} \lesssim \left( \frac{N^4}{\# C} \right)^{\frac{\varepsilon}{2q}} (\# C)^{\frac{1}{2} - \frac{3}{2q}} \|P_C f\|_{L^2_t \mathbb{R}^d} \quad (3.15)
\]
\[
= \left( \frac{N^4}{\# C} \right)^{\varepsilon} (\# C)^{\frac{1}{2} - \frac{3}{2q}} \|P_C f\|_{L^2_t \mathbb{R}^d} \quad (3.16)
\]
\[
\leq \left( \frac{N^4}{\# C} \right)^{\varepsilon} (\# C)^{\frac{1}{2} - \frac{3}{2q}} \|P_C f\|_{L^2_t \mathbb{R}^d} \quad (3.17)
\]
\[
= N^{2 - \frac{6}{q}} \left( \frac{\# C}{N^4} \right)^{\frac{3}{2q} + \varepsilon} \|P_C f\|_{L^2_t \mathbb{R}^d} \quad (3.18)
\]
where (3.17) is justified by the fact that \( \# C < N^4 \).

To prove (3.13) we first apply the homogeneous Strichartz estimate
\[
\left\| e^{it \Delta} P_N f \right\|_{L^{q_1}_{t,x}(I)} \lesssim N^{2 - \frac{6}{q_1}} \|P_N f\|_{L^2_t \mathbb{R}^d}.
\] (3.19)

Separately, we obtain the dual Strichartz estimate
\[
\left\| \int_I e^{is \Delta} P_N G(s) \, ds \right\|_{L^2} \lesssim N^{\frac{6}{q_2} - 4} \|P_N G\|_{L^{q_2}_{t,x}(I)},
\] (3.20)
and the conclusion follows from the Christ-Kiselev Lemma (11 in §3.4).

**Proposition 3** (Properties of \( X^s \) and \( Y^s \) spaces). We have the following estimates:

1. Embeddings:
\[
U^p(\mathbb{R}; H) \hookrightarrow V^q(\mathbb{R}; H) \hookrightarrow U^q(\mathbb{R}; H) \hookrightarrow L^\infty(\mathbb{R}; H) \quad (3.21)
\]

for any \( 1 \leq p < q < \infty \), and
\[
U^2_\Delta H^s \hookrightarrow X^s \hookrightarrow Y^s \hookrightarrow V^2_\Delta H^s. \quad (3.22)
\]
In particular, one has
\[ \|u(t)\|_{H^s_x} \lesssim \|u\|_{X^s(I)} \] (3.23)
for any \( t \in I \) if \( u \) is weakly left continuous in \( t \).

2. Linear and Strichartz estimates: For any finite interval \( I \) we have
\[ \|e^{it\Delta}f\|_{X^s(I)} \lesssim \|f\|_{H^s}. \] (3.24)

Moreover, if \( |I| \lesssim 1 \), for all \( q > 3 \) we have the Strichartz estimates
\[ \|P_N u\|_{L_t^q L_x^s(I)} \lesssim N^{2-\frac{6}{q}} \|P_N u\|_{U^0_2 L^2(I)}, \] (3.25)
and the same for \( P_B u \).

3. Duality: For \( s \geq 0, T > 0 \) and any \( u \) we have
\[ \|Iu\|_{X^s([t_0, t_0+T])} \leq \sup \left\{ \|\int u(t, x) \cdot \overline{v(t, x)} \, dx \, dt : \|v\|_{Y^{-s}([t_0, t_0+T])} = 1 \right\} \] (3.26)

**Proof.** These proofs are given in [H+11] and [Den17] for \( \mathbb{T}^3 \), the modifications for \( \mathbb{T}^4 \) are minor. \( \square \)

**Proposition 4** (Basic properties of \( X^s \) and \( Y^s \) space). For any \( s, r, N \), we have

1. \( N^r \|P_N u\|_{X^s} \sim \|P_N u\|_{X^{s+r}} \) and \( N^r \|P_N u\|_{Y^s} \sim \|P_N u\|_{Y^{s+r}} \),

2. \( \|P_N u\|_{X^s} \leq \|u\|_{X^s} \) and \( \|P_N u\|_{Y^s} \leq \|u\|_{Y^s} \).

**Proof.** By definition,
\[ N^r \|P_N u\|_{X^s} = \left( \sum_{k \in \mathbb{Z}^4} N^{2r} \langle k \rangle^{2s} \|\sum_{j=0}^{N} c_j Q(k) \varphi_N(k) \hat{u}(k, t)\|^2_{L^2(\mathbb{R}^4; \mathbb{C})} \right)^{\frac{1}{2}} \] (3.27)
Note that the only indices $k$ which have nonzero summands are those for which

$$
\varphi_N(k) \neq 0,
$$

that is

$$
N \leq \langle k \rangle \leq 2N,
$$

that is, $N \sim \langle k \rangle$,

hence

$$
N^r \| P_N u \|_{X^s} \sim \left( \sum_{k \in \mathbb{Z}^d} \langle k \rangle^{2(s+r)} \| e^{iQ(k)t} \varphi_N(k) \tilde{u}(k,t) \|_{L^2(\mathbb{R};\mathbb{C})}^2 \right)^{1/2} = \| P_N u \|_{X^{s+r} },
$$

which proves the first statement. The second follows from observing $\varphi_N(k) \leq 1$ in the definition of the $X^s$ norm. The proof for $Y^s$ are the same.

**Proposition 5** (Properties of $X^{s,b}$ spaces). We have the following estimates:

1. For any $s$ and $b$, and any fixed smooth cutoff function $\chi$, we have

$$
\| \chi(t) u \|_{X^{s,b}} \lesssim \| u \|_{X^{s,b}} ;
$$

moreover, let

$$
T^G(t) = \chi(t-m) \int_m^t e^{-i(t-s)\Delta} G(s) \, ds
$$

be the smoothly truncated Duhamel operator, then

$$
\| T^G \|_{X^{s,b}} \lesssim \| G \|_{X^{s,b-1}}
$$

for any $s$ and any $b \in (1/2, 1)$.

2. The Strichartz estimate

$$
\| P_N u \|_{L^q_{t,x}(I)} \lesssim N^{2-\frac{6}{q}} \| u \|_{X^{s,b}}
$$

holds for any fixed $b > 1/2$ and $q > 3$, where $I$ is any interval of length $|I| \lesssim 1$.

The same estimate is true for $P_B u$, and we gain the improvement for $P_C$ as in (3.11).
3. For any fixed $\varepsilon > 0$ and any fixed smooth cutoff $\chi$ we have

$$\left\| \chi(t)u \right\|_{X^{s,1/4-\varepsilon}} \lesssim \left\| u \right\|_{X^s}.$$  \hfill (3.33)

Proof. The proofs of (3.29) and (3.31) are standard, see [Tao06] section 2.6, for example. In [Den17] we have the following transfer principle:

If $\left\| e^{it\Delta} Pf \right\|_{L_t^q L_x^{s}(I)} \lesssim \left\| Pf \right\|_{X^s}$, then $\left\| Pu \right\|_{L_t^q L_x^{s,b}(I)} \lesssim \left\| u \right\|_{X^{s,b}}$ when $b > \frac{1}{2}$, \hfill (3.34)

where $P \in \{ P_N, P_B, P_C \}$, which implies the Strichartz estimates (3.32) hold. \hfill \square

**Proposition 6** (Decoupled long-time Strichartz for $S_{K,J}$ norms). Given Conjecture (2), we have

$$\sup_{|J|=N^\nu} \left\| e^{it\Delta} P_N f \right\|_{S_{N,J}^{7/4,q'}} \lesssim N \left\| P_N f \right\|_{L^2}$$  \hfill (3.35)

uniformly for $5 \leq q' \leq 12$, and $0 \leq \nu \leq \kappa$, and any $N$. In particular one has

$$\sup_{|J|=N^\nu} \left\| e^{it\Delta} P_N f \right\|_{S_{N,J}^{7/4}} \lesssim N \left\| P_N f \right\|_{L^2}$$  \hfill (3.36)

uniformly in $\nu$ and $N$.

Proof. Note that

$$\frac{2(7/4)}{7/4-1} = \frac{14}{3} < 5 \leq q'.$$

Furthermore, (3.3) and $\nu \leq \theta \left( \frac{7}{4}, 5 \right) \leq \theta(q,q')$ implies

$$\sup_{|J| \leq 2N^\nu} \left\| e^{it\Delta} P_N f \right\|_{L_t^q(J') L_x^4} \lesssim N^{2 - \frac{2}{q'} - \frac{4}{q} \frac{1}{q}} \left\| P_N f \right\|_{L^2}.$$  \hfill (3.37)

Given $|J| = N^\nu$, among all $m$ such that $[m, m+1] \cap J \neq 0$ there are at most two such that $[m, m+1] \not\subseteq J$. We bound these directly using (3.9). For any other $m$ we have $[m, m+1] \subset J$. Let $I = [m, m+1]$ and let $\chi$ be a cutoff function such that $\chi = 1$ on $I$ and $\chi = 0$ outside $2I$. The Gagliardo-Nirenberg inequality implies, for
any function \(g(t)\), we have

\[
\|g\|_{L^q(I)} \lesssim \|\chi(t)g\|_{L^q(\mathbb{R})} \lesssim \|\chi(t)g\|_{L^q(\mathbb{R})}^{1-\alpha} \|\partial_t(\chi(t)g)\|_{L^q(\mathbb{R})}^{\alpha} \lesssim \|g\|_{L^q(2I)}^{1-\alpha} \|\partial_t g\|_{L^q(2I)}^\alpha + \|g\|_{L^q(2I)}^{1-\alpha},
\]

(3.38)

where \(\alpha = \frac{1}{q} - \frac{1}{q'}\). Let \(g(t) := \|e^{it\Delta} P_N f\|_{L^q_x}'\), then

\[
|\partial_t g(t)| \lesssim \|\partial_t e^{it\Delta} P_N f\|_{L^q_x} \lesssim N^2 |g(t)|
\]

(3.39)

and we find

\[
\|e^{it\Delta} P_N f\|_{L^q_{t,x}(I)} \lesssim N^{2\alpha} \|e^{it\Delta} P_N f\|_{L^q_{t,L^q_x}(2I)}
\]

(3.40)

by using (3.38). If \(J = [b, b + N^\nu]\) we have

\[
\|e^{it\Delta} P_N f\|_{S_N^{q,J}}^q \lesssim N^q \|P_N f\|_{L^2_N}^q + \sum_{b \leq m \leq b + N^\nu - 1} \left(N^{\frac{2}{q} + \frac{4}{q'} - 1} \|e^{it\Delta} P_N f\|_{L^q_{[m-1,m+2]L^q_x}}\right)^q
\]

(3.41)

\[
\lesssim N^q \|P_N f\|_{L^2_N}^q + N^{2 + \frac{4}{q'} - q} \|e^{it\Delta} P_N f\|_{L^q_{[b-1,b+N^\nu+1]L^q_x}}^q
\]

(3.42)

\[
\lesssim N^q \|P_N f\|_{L^2_N}^q
\]

(3.43)

where the last line follows from (3.37).

\[
\square
\]

**Remark 5.** Compare this with [Den17], Proposition 2.7. We can still prove a version of this proposition in our case, however, without Conjecture 2 we can only prove it for \(q > 3 + \varepsilon\). As we will see later in the proof, we will need this to hold true for some \(q < 2\).

**Remark 6.** It may seem surprising that we were able to leverage the decoupled long-time Strichartz of Conjecture 2 without decoupling the \(L^q_{t,x}\) norm in the definition of \(S^q_{K,J}\). A heuristic explanation of this is that the \(q\) in both the conjecture and the definition of \(S^q_{K,J}\) is related to the norm in time - in the conjecture it is an \(L^q_t\) norm, in the definition of \(S^q_{K,J}\) it is an \(\ell^q\) norm over intervals of time. Via this
proposition we “bake” the additional strength of the decoupled long-time Strichartz from Conjecture 2 into an inequality regarding the norm, (3.36). This allows us to exploit the additional strength of Conjecture 2 without decoupling the $S_{K,t}^q$ norms themselves.

**Proposition 7** (Bilinear Strichartz Estimate). Let $I$ be an interval with $|I| \lesssim 1$. Then for $N_1 \geq N_2 \geq 1$ we have

$$\|P_{N_1} u_1 P_{N_2} u_2\|_{L^2_{t,x}(I \times T^4)} \lesssim \left( \frac{N_2}{N_1} + \frac{1}{N_2} \right)^{\frac{1}{8}} N_2 \|P_{N_1} u_1\|_{Y^0(I)} \|P_{N_2} u_2\|_{Y^0(I)}$$  \hspace{1cm} (3.44)

**Proof.** This is Lemma 4.2 in [Yue18], where the exponent $\kappa$ which appears can be shown to be $\frac{1}{8}$ by careful calculation. \qed

3.2.3 Global well-posedness in $H^1$

**Proposition 8.** Let $\lambda$ and $\eta$ be fixed as in the statement of Theorem 12. Given any initial data $f \in H^1(\mathbb{T}^4)$ such that $E[f] \leq \eta^2$, there exists a unique solution

$$u \in \bigcap_{T > 0} X^1([-T,T])$$

to (3.6) with initial data $u(0) = f$. The energy $E[u]$ is conserved, and one has

$$\|u\|_{X^1(I)} \lesssim \eta$$

for any time interval $I$ with $|I| \lesssim 1$. Moreover, for any $s > 1$, if $f \in H^s(\mathbb{T}^4)$ then $u(t) \in H^s(\mathbb{T}^4)$ for all time, and

$$\|u\|_{X^s(I)} \lesssim \|u(t)\|_{H^s(\mathbb{T}^4)}$$

for any time interval $I$ with $|I| \lesssim 1$ and any $t \in I$. 

54
\begin{proof}
Proposition 4.4 from [Yue18] implies the estimate
\begin{equation}
\|I(\tilde{u}_1\tilde{u}_2\tilde{u}_3)\|_{X^1[0,T]} \lesssim \prod_{j=1}^3 \|u_j\|_{X^1[0,T]},
\end{equation}
which is then used in the same paper to prove global well-posedness. \hfill \Box
\end{proof}

\section{Proof of main theorem}

In this section we prove Theorem 12. In Proposition 9 will bound the increase in energy over a fixed time interval using Littlewood-Paley decomposition. We then extend this to a time interval $N^\gamma$ in Corollary 3. In so doing, we will also have a bound which now involves the $\|P_K\mathcal{D}u\|_{S_{K,J}^2}$ norms. In Proposition 10 we bound these norms using a bootstrap argument. The proof uses Littlewood-Payley theory and an appropriate gauge transformation to derive a bound sufficient to leverage a “discrete acausal Gronwall inequality”, Lemma 10.

\begin{prop}
Let $N \geq 1$ be fixed. Suppose $u$ is a solution to (3.6) such that $\|\mathcal{D}u(0)\|_{H^s} \lesssim \eta$ for some $\eta \in (0,1)$, then we have
\begin{equation}
|E[\mathcal{D}u(t)] - E[\mathcal{D}u(0)]| \lesssim \eta^2 \sum_{K_1, K_2 \in 2^\mathbb{N}} \min\left(1, \frac{K_2}{N}\right) \inf_{\frac{1}{p_1} + \frac{1}{p_2} = 1} \left\{ \frac{6}{3 \min \frac{1}{p_1} + \frac{6}{p_2}} \right\} \|P_{K_1} \mathcal{D}\tilde{u}\|_{L^{p_1}} \|P_{K_2} \mathcal{D}\tilde{u}\|_{L^{p_2}}.
\end{equation}
\end{prop}

\begin{proof}
Applying Lemma (9) to $\|\mathcal{D}u(0)\|_{H^s} \lesssim \eta$ lets us conclude
\begin{equation}
\|u(0)\|_{H^s} \lesssim \eta^{N^{s-1}} \forall s \geq 1.
\end{equation}
Choosing $t = 0$ in the last implication of Proposition 8 yields
\begin{align*}
\|u\|_{X^1[-1,2]} &\lesssim \eta, &\|u\|_{X^s[-1,2]} &\lesssim \eta^{N^{s-1}},
\end{align*}

55
which implies $\|\mathcal{D}u\|_{X^1_{[-1,2]}} \lesssim \eta$. By considering a suitable extension of $u$ we may assume
\[
\|\mathcal{D}u\|_{X^1} \lesssim \eta. \tag{3.47}
\]
To compute $E[\mathcal{D}u]$ we note that
\[
(i\partial_t - \Delta)[\mathcal{D}u] = |\mathcal{D}u|^2(\mathcal{D}u) + \mathcal{R}, \tag{3.48}
\]
where
\[
\mathcal{R} = \mathcal{D}(|u|^2u) - |\mathcal{D}u|^2(\mathcal{D}u).
\]
Plugging $\mathcal{D}u$ into the definition for energy we can calculate (see §3.4.2)
\[
\partial_t E[\mathcal{D}u] = \sum_{j=1}^4 \beta_j \Im \int_{T^4} \partial_j\overline{\mathcal{D}u}\partial_j\mathcal{R} \, dx + \Im \int_{T^4} |\mathcal{D}u|^2\overline{\mathcal{D}u}\mathcal{R} \, dx, \tag{3.49}
\]
where we have substituted derivatives in time using (3.48). Therefore, integrating in $t$, it suffices to bound
\[
\int_{[0,T] \times T^4} |\mathcal{D}u|^2\overline{\mathcal{D}u}\mathcal{R} \, dx \, dt \tag{3.50}
\]
and
\[
\int_{[0,T] \times T^4} \partial_j\overline{\mathcal{D}u}\partial_j\mathcal{R} \, dx \, dt. \tag{3.51}
\]
We first consider (3.50). Using Littlewood-Paley decomposition with $P_Nu =: u_N$, this can be written as
\[
\int_{[0,T] \times T^4} |\mathcal{D}u|^2\overline{\mathcal{D}u}\mathcal{R} \, dx \, dt = \sum_{N_1, \ldots, N_6} \mathcal{M}_{N_1, \ldots, N_6}
\]
where
\[
\mathcal{M} := \mathcal{M}_{N_1, \ldots, N_6} := \int_{[0,T] \times T^4} \overline{\mathcal{D}u_{N_1}} D_{u_{N_2}} \overline{D_{u_{N_3}}} \left[ \mathcal{D}(u_{N_4} \overline{u_{N_5}} u_{N_6}) - D_{u_{N_4}} \overline{D_{u_{N_5}}} D_{u_{N_6}} \right] \, dx \, dt. \tag{3.52}
\]
If $N_j \ll N$ for $j \in \{4, 5, 6\}$ then
\[
\mathcal{D}(u_{N_4} \overline{u_{N_5}} u_{N_6}) = u_{N_4} \overline{u_{N_5}} u_{N_6},
\]
56
which implies $\mathcal{M}_{N_1,\ldots,N_6} = 0$. To handle the case where $N_j \gtrsim N$ for some $j \in \{4, 5, 6\}$ we apply Hölder and (3.5) to get

$$|\mathcal{M}_{N_1,\ldots,N_6}| \lesssim \prod_{j=1}^{6} \| P_{N_j} \mathcal{D} \tilde{u} \|_{L^{q_j}_{t,x}[0,T]} \quad \text{where} \quad 1 = \sum_{j=1}^{6} \frac{1}{q_j}. \quad (3.53)$$

At this point the bound is symmetric in the $N_j$, so, without loss of generality, assume $N_1 \leq \cdots \leq N_6$. By using the Strichartz estimate from (3.25) we have

$$\forall M \forall q > 3, \quad \| P_M \mathcal{D} \tilde{u} \|_{L^q_{t,x}[0,T]} \lesssim M^{-\frac{6}{q}} \| P_M \mathcal{D} u \|_{X^0[0,T]}. \quad (3.54)$$

Applying Proposition 4 and leveraging (3.47), the implication of our hypothesis, we can further bound

$$\| P_M \mathcal{D} \tilde{u} \|_{L^q_{t,x}[0,T]} \lesssim M^{-\frac{6}{q}} \| P_M \mathcal{D} u \|_{X^0[0,T]} \lesssim M^{1-\frac{6}{q}} \| \mathcal{D} u \|_{X^1} \lesssim \eta M^{1-\frac{6}{q}}. \quad (3.55)$$

We need to get rid of all but two of the norms (3.53) so we apply this to the smallest $N_j$:

$$|\mathcal{M}_{N_1,\ldots,N_6}| \lesssim \eta^4 \left( \prod_{j=1}^{4} N_j^{\frac{1}{q_j}} \right) \prod_{j=5}^{6} \| P_{N_j} \mathcal{D} \tilde{u} \|_{L^{q_j}_{t,x}[0,T]}. \quad (3.56)$$

This is almost what we need, but we have more than two $N_j$ terms and we have yet to sum over $N_j$. Note that, for $\alpha > 0, K, M \in 2^N$, we have

$$\sum_{K \leq M} K^\alpha \leq \sum_{j=0}^{\infty} M^\alpha 2^{-j\alpha} \lesssim_\alpha M^\alpha.$$

We have assumed $N_1 \leq \cdots \leq N_6$, thus we have

$$\sum_{N_1, N_2, N_3, N_4 \leq N_5} \prod_{j=1}^{4} N_j^{\frac{1}{q_j}} \lesssim N_5^{4-6(\frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} + \frac{1}{q_4})} = N_5^{\frac{6}{q_5} + \frac{6}{q_6} - 2}. \quad (3.57)$$

In this case, the requirement that $q_j > 3$ for $j \in \{1, 2, 3, 4\}$ which came from the Strichartz estimate does not put a restriction on $q_4, q_5$, however if we voluntarily impose that $\frac{1}{q_5} + \frac{1}{q_6} > \frac{1}{3}$ in order to fit the form of the proposition then we can accept the loss

$$N_5^{\frac{6}{q_5} + \frac{6}{q_6} - 2} \leq N_6^{\frac{6}{q_5} + \frac{6}{q_6} - 2}, \quad (3.58)$$
thus taking
\[ K_1 := N_5, \quad p_1 := q_5 \quad K_2 := N_6, \quad p_2 := q_6, \]
we have
\[
\left| \sum_{N_1, \ldots, N_6} M_{N_1, \ldots, N_6} \right| \lesssim \eta^4 \sum_{K_1 \leq N_1} \sum_{K_2 \geq N_2} K_2^{6 p_1} K_2^{6 p_2} - 2 \left\| P_{K_1} D u \right\|_{L^p_1} \left\| P_{K_2} D u \right\|_{L^p_2},
\]
which is enough to close the argument for the first term, (3.50).

To estimate (3.51) we note
\[
\partial_j R = 2 \left[ D((\partial_j u)u) - (D\partial_j u)Du \right] + \left[ D(u^2\partial_j u) - (Du)^2D\partial_j u \right] =: 2R_1 + R_2,
\]
so (3.51) becomes
\[
\int_{[0,T] \times T^4} \partial_j Du \partial_j R \, dx \, dt = 2 \int_{[0,T] \times T^4} \partial_j Du R_1 \, dx \, dt + \int_{[0,T] \times T^4} \partial_j Du R_2 \, dx \, dt. \tag{3.59}
\]
Using Littlewood-Paley decomposition on the first term, we need to estimate
\[
\int_{[0,T] \times T^4} \partial_j Du R_1 \, dx \, dt = \sum_{N_1, \ldots, N_4} N_{N_1, \ldots, N_4}
\]
where
\[
N := N_{N_1, \ldots, N_4} := \int_{[0,T] \times T^4} \partial_j Du N_1 \left[ D((\partial_j u N_2)u N_3 u N_4) - (D\partial_j u N_2)Du N_3 D u N_4 \right] \, dx \, dt. \tag{3.60}
\]
Note that if \( M \leq N/2 \), we have
\[
Du_M = u_M \quad \text{and} \quad D\partial_j u_M = u_M,
\]
and if \( N_i \ll N \) for \( i \in \{2, 3, 4\} \) we have
\[
D((\partial_j u N_2)u N_3 u N_4) = (\partial_j u N_2)u N_3 u N_4,
\]
thus \( N = 0 \) unless one of \( N_i \gtrsim N \) for some \( i \in \{2, 3, 4\} \). Symmetry allows us to assume, without loss of generality, \( N_4 \geq N_3 \).
**Case 1:** Suppose $N_4 \gtrsim \max(N_1, N_2)$. Using (3.5) and H"older we can estimate

$$|N_{N_1, \ldots, N_4}| \lesssim \|P_{N_1} \partial_j \mathcal{D} \tilde{u}\|_{L^{q_1}} \|P_{N_2} \partial_j \mathcal{D} \tilde{u}\|_{L^{q_2}} \|P_{N_3} \mathcal{D} \tilde{u}\|_{L^{q_3}} \|P_{N_4} \mathcal{D} \tilde{u}\|_{L^{q_4}}. \tag{3.61}$$

By (3.25) and (3.47) we know that

$$\forall q > 3, \quad \|P_M \partial_j \mathcal{D} \tilde{u}\|_{L^q[0,T]} \lesssim M^{2-\frac{6}{q}} \|P_M \mathcal{D} u\|_{X^1} \lesssim \eta M^{2-\frac{6}{q}} \tag{3.62}$$

and thus, applying this to the $N_1, N_2$ terms we have

$$|N_{N_1, \ldots, N_4}| \lesssim \eta^2 N_1^{2-\frac{6}{q_1}} N_2^{2-\frac{6}{q_2}} \|P_{N_3} \mathcal{D} u\|_{L^{q_3}} \|P_{N_4} \mathcal{D} u\|_{L^{q_4}} \tag{3.63}$$

Summing this as in the $\mathcal{M}$ case and using $N_4 \gtrsim \max(N_1, N_2)$ we have

$$\sum_{N_1, N_2} N_1^{2-\frac{6}{q_1}} N_2^{2-\frac{6}{q_2}} \lesssim N_4^{4-6 \left(\frac{1}{q_1} + \frac{1}{q_2}\right)} = N_4^{\frac{6}{q_3} + \frac{6}{q_4} - 2} \tag{3.64}$$

where the Strichartz estimate constraint

$$q_1 > 3 \text{ and } q_2 > 3 \quad \text{implies} \quad \frac{1}{q_3} + \frac{1}{q_4} > \frac{1}{3}.$$

Taking

$$K_1 := N_3, \quad p_1 := q_3, \quad K_2 := N_4, \quad p_2 := q_3$$

we have

$$\sum_{N_1, \ldots, N_4 \atop N_4 \gtrsim \max(N_1, N_2)} |N_{N_1, \ldots, N_4}| \lesssim \eta^2 \sum_{K_2 \geq N} \sum_{K_1 \leq K_2} K_2^{\frac{6}{p_1} + \frac{6}{p_2} - 2} \|P_{K_1} \mathcal{D} u\|_{L^{p_1}} \|P_{K_2} \mathcal{D} u\|_{L^{p_2}}.$$

**Remark 7.** Note: In every case but this one, we could have actually proven the slightly stronger bound

$$\eta^2 \sum_{K_2 \geq N} \sum_{K_1 \leq K_2} K_2^{\frac{6}{p_1} + \frac{6}{p_2} - 2} \|P_{K_1} \mathcal{D} u\|_{L^{p_1}} \|P_{K_2} \mathcal{D} u\|_{L^{p_2}},$$

and in fact if $N_3 \gtrsim \min(N_1, N_2)$ we could also prove it here, but if we had something like $N_3 \leq N_1 \leq N_2 \leq N_4$ it is not as straightforward to get this slightly tighter bound.
**Case 2:** Suppose \( N_4 \ll \max(N_1, N_2) \), then \( N_1 \sim N_2 \gg N_3 \). We decompose using balls \( B \) of size \( N_4 \) in Fourier space,

\[
P_{N_2} u = \sum_B P_{N_2} P_B u.
\]

Expanding and taking advantage of almost-orthogonality we have

\[
N_{N_1, \ldots, N_4} = \sum_B \int_{[0,T] \times T^4} P_{N_1} P_{2^{10}B} \partial_j \mathcal{D} u \left[ \mathcal{D} \left( (P_{N_2} P_B \partial_j u) \overline{u_{N_4} u_{N_4}} \right) \right. \\
\left. - \left( \mathcal{D} P_{N_2} P_B \partial_j u \right) \overline{u_{N_3} u_{N_4}} \right] \, dx \, dt. \tag{3.65}
\]

**Subcase 2a:** If \( N_4 \gtrsim N \), then by 3.5 and Hölder we have

\[
|N_{N_1, \ldots, N_4}| \lesssim \sum_B \|P_{N_1} P_{2^{10}B} \partial_j \mathcal{D} u\|_{L^q} \|P_{N_2} P_B \partial_j \mathcal{D} u\|_{L^q} \|P_{N_3} \mathcal{D} u\|_{L^q} \|P_{N_4} \mathcal{D} \tilde{u}\|_{L^q}.
\]  \tag{3.66}

Applying the arguments from Case 1 but commuting the multipliers and focusing on \( P_B \) we have

\[
\forall q > 3, \quad \|P_B P_M \partial_j \mathcal{D} u\|_{L^q[0,T]} \lesssim |B|^{2-\frac{6}{q}} \|P_B P_M \mathcal{D} u\|_{X^1}. \tag{3.67}
\]

Applying this to the \( N_1, N_2 \) terms yields

\[
|N_{N_1, \ldots, N_4}| \lesssim N_4^{\frac{6}{q_0} - \frac{6}{q_1}} \|P_{N_1} \mathcal{D} \tilde{u}\|_{L^q} \|P_{N_4} \mathcal{D} \tilde{u}\|_{L^q} \sum_B \|P_{N_0} P_{2^{10}B} \partial_j \mathcal{D} \tilde{u}\|_{X^1} \|P_{N_1} P_B \partial_j \mathcal{D} \tilde{u}\|_{X^1}. \tag{3.68}
\]

Note that, since \( N_1 \sim N_2 \), given \( N_2 \) there are \( \sim 1 \) choices for \( N_1 \), thus

\[
\sum_{N_1, N_2} \sum_B \|P_{N_2} P_B \mathcal{D} u\|_{X^1}^2 \sim \sum_{N_2} \sum_B \sum_{k \in \mathbb{Z}^4} \langle k \rangle^2 \left\| e^{iQ(k)t} \varphi_{N_2}(k) \psi_B(k) \hat{\mathcal{D}} u \right\|_{U^2(\mathbb{R}^4; \mathbb{C})}^2 \tag{3.69}
\]

\[
= \sum_{k \in \mathbb{Z}^4} \langle k \rangle^2 \sum_{N_2} \varphi_{N_2}^2(k) \sum_B \psi_B^2(k) \left\| e^{iQ(k)t} \hat{\mathcal{D}} u \right\|_{U^2(\mathbb{R}^4; \mathbb{C})}^2 \tag{3.70}
\]

\[
\leq \sum_{k \in \mathbb{Z}^4} \langle k \rangle^2 \sum_{N_2} \varphi_{N_2}^2(k) \sum_B \psi_B^2(k) \left\| e^{iQ(k)t} \hat{\mathcal{D}} u \right\|_{U^2(\mathbb{R}^4; \mathbb{C})}^2 \tag{3.71}
\]

\[
= \sum_{k \in \mathbb{Z}^4} \langle k \rangle^2 \left\| e^{iQ(k)t} \hat{\mathcal{D}} u \right\|_{U^2(\mathbb{R}^4; \mathbb{C})}^2 \tag{3.72}
\]

\[
= \| \mathcal{D} u \|_{X^1}^2 \lesssim \eta^2 \tag{3.73}
\]
Similarly,
\[ \sum_{N_1, N_2} \| P_{N_1} P_{210B} D u \|_{X^1}^2 \lesssim \| D u \|_{X^1}^2 \lesssim \eta^2 \]  
and thus we apply Cauchy-Schwarz when summing (3.68) to find
\[ \sum_{N_1, \ldots, N_4} \| N_{N_1, \ldots, N_4} \| \lesssim \eta^2 \sum_{N_4 \geq N} \sum_{N_2 \leq N_4} N_4^{4 - \frac{6}{q_1} - \frac{6}{q_2}} \| P_{N_3} D \tilde{u} \|_{L^q} \| P_{N_4} D \tilde{u} \|_{L^q}. \]  
(3.75)

Note that
\[ 1 - \frac{1}{q_1} - \frac{1}{q_2} = \frac{1}{q_3} + \frac{1}{q_4} > \frac{1}{3} \]
is imposed on us, in this case, by the previous use of the Strichartz estimates in (3.67), and so we can once again take
\[ K_1 := N_3, \quad p_1 := q_3, \quad K_2 := N_4, \quad p_1 := q_3 \]
to find
\[ \sum_{N_1, \ldots, N_4} \| N_{N_1, \ldots, N_4} \| \lesssim \eta^2 \sum_{K_2 \geq N} \sum_{K_1 \leq K_2} K_2^{\frac{6}{p_1} + \frac{6}{p_2}} \| P_{K_1} D u \|_{L^{p_1}} \| P_{K_2} D u \|_{L^{p_2}}. \]  
(3.76)

**Subcase 2b:** Suppose \( N_4 \ll N \), then we must have
\[ P_{N_3} D u = P_{N_3} u, \quad P_{N_4} D u = P_{N_4} u, \quad \text{and} \quad N_2 \gtrsim N. \]  
(3.77)

For this part we will define, for functions \( G \) and \( F \), the bilinear form
The motivation for this is that with
\[ G := P_{N_3} u P_{N_4} u, \quad F := F_B := P_{N_2} P_B \partial_j D u, \]
equation (3.65) becomes
\[ \mathcal{N}_{N_1, \ldots, N_4} = \sum_B \int_{[0,T] \times T^4} P_{N_1} P_{210B} \partial_j D \tilde{u} [D, G] F_B \, dx \, dt. \]  
(3.78)
Then H"older implies

\[ |N_{N_1, \ldots, N_4}| \leq \sum_B \| P_{N_1} P_{2 \mu B \partial_j \mathcal{D}u} \|_{L^{q_1}} \| [\mathcal{D}, G] F_B \|_{L^{q_1'}}. \] (3.79)

Note that, in general,

\[ F_\mathcal{D} \mathcal{G} F = \sum \theta_k \left( \frac{k + \nu m}{N} \right) \nabla \theta \left( \frac{k - \nu m}{N} \right) \hat{\theta}_k \hat{G}(N) \hat{F}(k + \nu m). \] (3.80)

In our case, \( \text{supp} \hat{G} = \{ |m| \lesssim N_4 \} \). Combining this with the Coifman-Meyer theorem allows us to conclude

\[ \| [\mathcal{D}, G] F_B \|_{L^p} \leq \frac{N_4}{N} \| \hat{G} \|_{L^q} \| F_B \|_{L^{q'}} \quad \text{with} \quad \frac{1}{p} = \frac{1}{q} + \frac{1}{r}. \] (3.82)

Applying this to (3.79) along with H"older for \( \| G \| \) yields

\[ |N_{N_1, \ldots, N_4}| = \frac{N_4}{N} \sum_B \| P_{N_1} P_{2 \mu B \partial_j \mathcal{D}u} \|_{L^{q_1}} \| P_{N_2} P_B \partial_j \mathcal{D}u \|_{L^{q_2}} \| P_{N_3} u \|_{L^{q_3}} \| P_{N_4} u \|_{L^{q_4}}. \] (3.83)

As in case 2a, (3.67), we use Strichartz estimates focused on the \( P_B \) for the \( N_1, N_2 \) terms to get

\[ |N_{N_1, \ldots, N_4}| = \frac{N_4}{N} N_4^4 \sum_B \| P_{N_1} \mathcal{D}u \|_{X^1} \| P_{N_2} P_B \mathcal{D}u \|_{X^1} \| P_{N_3} u \|_{L^{q_3}} \| P_{N_4} u \|_{L^{q_4}}. \] (3.84)

We can sum over \( B, N_1, \) and \( N_2 \) as in case 2a, which implies

\[ \sum_{N_1, N_2} |N_{N_1, \ldots, N_4}| = \eta^2 N_4 \frac{N_4^4}{N_4^4} \sum_B \| P_{N_3} u \|_{L^{q_3}} \| P_{N_4} u \|_{L^{q_4}}. \] (3.85)

Again, taking

\[ K_1 := N_3, \quad p_1 := q_3, \quad K_2 := N_4, \quad p_1 := q_3 \]
yields
\[
\sum_{N_1, \ldots, N_4} |N_{N_1, \ldots, N_4}| \lesssim \eta^2 \sum_{K_2 \leq N} \frac{K_2}{N} \sum_{K_1 \leq K_2} K_2^{p_1} + 6 p_2^{-2} \|P_{K_1} \mathcal{D}u\|_{L^{p_1}} \|P_{K_2} \mathcal{D}u\|_{L^{p_2}}.
\]
(3.86)

The above proposition actually implies something similar to the form in [Den17]:

**Corollary 2.** Let \( N \geq 1 \) be fixed. Suppose \( u \) is a solution to (3.6) such that \( \|\mathcal{D}u(0)\|_{H^1} \lesssim \eta \) for some \( \eta \in (0, 1) \). For any \( \alpha \in [0, 1] \) there is some set \( S_{\alpha} \) such that

\[
|E[\mathcal{D}u(t)] - E[\mathcal{D}u(0)]| \lesssim \\
\eta^2 \sum_{K_1, K_2 \in 2^N} \min \left(1, \frac{K_2}{N}\right) \left(\frac{K_1}{K_2}\right)^\alpha \prod_{j=1}^2 \sup_{p \in S_{\alpha}} K_j^{p-1} \|P_{K_j} \mathcal{D}\tilde{u}\|_{L^p}.
\]
(3.87)

uniformly for all \( 0 \leq t \leq T \leq 1 \). In particular, for \( \alpha = \frac{1}{2} \), \( S_{\alpha} = [5, 12] \).

**Proof.** We have

\[
\inf_{p_1, p_2 \geq 1} K_2^{p_1} + 6 p_2^{-2} \|P_{K_1} \mathcal{D}\tilde{u}\|_{L^{p_1}} \|P_{K_2} \mathcal{D}\tilde{u}\|_{L^{p_2}} \leq K_2^{p_1} + 6 p_2^{-2} \|P_{K_1} \mathcal{D}\tilde{u}\|_{L^{p_1}} \|P_{K_2} \mathcal{D}\tilde{u}\|_{L^{p_2}}
\]
(3.88)

\[
= \left(\frac{K_1}{K_2}\right)^{1-\frac{6 p_1}{p_2}} \prod_{j=1}^2 K_j^{p_j-1} \|P_{K_j} \mathcal{D}\tilde{u}\|_{L^{p_j}}
\]
(3.89)

for any \( p_1, p_2 \geq 1 \) such that \( \frac{1}{3} < \frac{1}{p_1} + \frac{1}{p_2} \leq 1 \). For any given \( \alpha \in [0, 1] \) it is possible to choose \( p_1 \in [6, \infty] \) so that \( \alpha = 1 - \frac{6}{p_1} \) and to choose a corresponding \( p_2 \in [1, 6) \) which satisfies the previous constraint. We then note that

\[
\left(\frac{K_j}{K_2}\right)^{1-\frac{6}{p_1}} \prod_{j=1}^2 K_j^{p_j-1} \|P_{K_j} \mathcal{D}\tilde{u}\|_{L^{p_j}} \leq \left(\frac{K_1}{K_2}\right)^{\alpha} \prod_{j=1}^2 \sup_{p \in S_{\alpha}} K_1^{p-1} \|P_{K_j} \mathcal{D}\tilde{u}\|_{L^p}
\]
(3.90)
where $S_\alpha$ is a set such that $p_1(\alpha), p_2(\alpha) \in S$. For example, if $\alpha = \frac{1}{2}, p_1 = 12$ and $p_2 < 6$, so $S_\alpha = [5, 12]$ would work.

\[ \square \]

**Corollary 3.** Suppose $\gamma < \frac{1}{4}$. Suppose $u$ is a solution to (3.1) on $\mathbb{R} \times T^4$ such that

\[
\sup_{0 \leq t \leq T} \| Du(t) \|_{H^1} \lesssim \eta, \quad T \leq N^\gamma, 
\]

(3.91)

then one has

\[
| E[Du(t)] - E[Du(0)] | \lesssim \eta^2 \sum_K \min \left( 1, \frac{K}{N} \right)^{\frac{1}{4}} \sup_{|J| \leq K^\gamma, J \subseteq [0, T]} \| P_K Du \|_{L^2(I)}^2 
\]

(3.92)

**Proof.** Applying the arithmetic-geometric mean inequality to the product in (3.87) over some time interval $|I| \leq 1$, we have

\[
\sum \min \left( 1, \frac{K_2}{N} \right) \left( \frac{K_1}{K_2} \right)^\alpha \prod_{j=1}^2 \sup_{p \in S_\alpha} K_j^{6p-1} \left\| P_{K_2} Du \right\|_{L^p(I)} \lesssim 
\]

\[
\sum \min \left( 1, \frac{K_2}{N} \right) \left( \frac{K_1}{K_2} \right)^\alpha \sum_{j=1}^2 \left( \sup_{p \in S_\alpha} K_j^{6p-1} \left\| P_{K_2} Du \right\|_{L^p(I)} \right)^2 
\]

(3.93)

We sum carefully, using the results from §3.4.3. The $j = 2$ term is simply

\[
\sum \min \left( 1, \frac{K_2}{N} \right) \left( \frac{K_1}{K_2} \right)^\alpha \left( \sup_{p \in S_\alpha} K_2^{6p-1} \left\| P_{K_2} Du \right\|_{L^p(I)} \right)^2 \lesssim 
\]

\[
\sum \min \left( 1, \frac{K_2}{N} \right) \left( \sup_{p \in S_\alpha} K_2^{6p-1} \left\| P_{K_2} Du \right\|_{L^p(I)} \right)^2 
\]

(3.94)

For the $j = 1$ term, we have

\[
\sum \min \left( 1, \frac{K_2}{N} \right) \left( \frac{K_1}{K_2} \right)^\alpha \left( \sup_{p \in S_\alpha} K_1^{6p-1} \left\| P_{K_1} Du \right\|_{L^p(I)} \right)^2 \lesssim 
\]

\[
\sum \min \left( 1, \frac{K_2}{N^{1-\alpha}} \right) \left( \sup_{p \in S_\alpha} K_1^{6p-1} \left\| P_{K_1} Du \right\|_{L^p(I)} \right)^2 
\]

(3.95)
In either case, if \( \alpha \leq 1/2 \) we can weaken the above inequalities to find that, for any interval \([m, m + T]\) with \( T \leq 1 \) and \( \|D u(m)\|_{H^1} \lesssim \eta \), we have that, for all \( t \in [m, m + T] \),

\[
|E[D u(t)] - E[D u(0)]| \lesssim \sum_{K \in 2^J} \min \left(1, \frac{K}{N} \right)^\alpha \left( \sup_{p \in S_\alpha} K^{\frac{6}{p} - 1} \|P_k D \tilde{u}\|_{L^p(I)} \right)^2.
\]  

(3.96)

By assumption (3.91), we satisfy these constraints on any interval \([m, m + 1] \cap T\). Therefore, taking \( \alpha = 1/2 \) and \( S_\alpha = [5, 12] \), for any \( t \in [0,T] \),

\[
|E[D u(t)] - E[D u(0)]| \lesssim \sum_{K \in 2^J} \min \left(1, \frac{K}{N} \right) \sum_m \left( \sup_{p \in [5,12]} K^{\frac{6}{p} - 1} \|P_k D \tilde{u}\|_{L^p([m,m+1] \cap [0,T])} \right)^2.
\] 

(3.97)

Now divide \([0,T]\) into intervals of length \( \leq K^\gamma \). Since \( T \lesssim N^\gamma \) there are at most \( \lesssim \max(1, (N/K)^\gamma) \) intervals, and therefore we can bound

\[
\sum_m \left( \sup_{p \in [5,12]} K^{\frac{6}{p} - 1} \|P_k D \tilde{u}\|_{L^p([m,m+1] \cap [0,T])} \right)^2 \lesssim \max \left(1, \frac{N}{K} \right)^\gamma \|u\|_{S^2_{R,J}}^2. 
\] 

(3.98)

Since \( \gamma < 1/4 \) this implies the desired result. \( \square \)

We will now proceed with a bootstrap argument as in [Den17]. Note that \( r \) in the following is left free for the moment, and we will choose it based on the requirements within this proof.

**Proposition 10 (Bootstrap).** Suppose Conjecture (2) holds, and let \( \gamma < \min(\kappa, 300) \).

Let \( u \) be a solution to (3.1) on \( \mathbb{R} \times \mathbb{T}^4 \) such that

\[
\sup_{t \in [0,T]} \|D u(t)\|_{H^1} \lesssim \eta,
\]  

(3.99)

where \( T \leq N^\gamma \), and define

\[
A_K = A_K(T) := \sup_{|J| \leq K^\gamma, J \subset [0,T]} \|P_K D u\|_{S^{\gamma/4}_{R,J}}.
\] 

(3.100)

If \( A_K \lesssim 1 \) for any \( K \), we have

\[
A_K \lesssim \eta, \quad \sum_{K \geq N} A_K^2 \lesssim \eta^2.
\]
Proof. Let
\[
a_K := \begin{cases} 
K \|P_K \mathcal{D} u(0)\|_{L^2} & K \geq N, \\
\eta & K < N.
\end{cases}
\]  
(3.101)

We will prove that
\[
A_K \lesssim a_K = \eta K^{-1/2000} + \eta \sum_M \min \left( \frac{K}{M}, \frac{M}{K} \right)^{1/5000} A_M
\]  
(3.102)

for any \( K \). To see that this is sufficient to conclude our desired result, we recall the following "discrete acausal Gronwall inequality". A proof can be found in [Tao06], Theorem 1.19 and Corollary 1.20.

**Lemma 10.** Suppose \( \eta \ll 1 \), \( \{b_K\} \) and \( \{c_k\} \) are two positive sequences satisfying
\[
b_K \lesssim c_K + \eta \sum_M \min \left( \frac{K}{M}, \frac{M}{K} \right)^{1/5000} b_M
\]

and
\[
\sup_K \frac{b_K}{K^{1/6000}} < \infty,
\]

then we have
\[
b_K \lesssim \sup_M \min \left( \frac{K}{M}, \frac{M}{K} \right)^{1/6000} c_M
\]

uniformly for all \( K \).

Thus, taking \( c_K := a_K + \eta K^{-1/2000} \), we use Lemma 10 to conclude
\[
A_M \lesssim \sup_M \min \left( \frac{K}{M}, \frac{M}{K} \right)^{1/6000} (a_M + M^{-1/2000}).
\]  
(3.103)

Since \( a_M \lesssim \eta \) this immediately implies \( A_K \lesssim \eta \). Furthermore, if \( K \geq N \) we have
\[
A_K \lesssim \eta K^{-1/6000} + \sum_M \min \left( \frac{M}{K}, \frac{K}{M} \right)^{1/6000} M \|P_M \mathcal{D} u(0)\|_{L^2} + \eta (N/K)^{1/6000}.
\]  
(3.104)

Since
\[
\sum_M (M \|P_M \mathcal{D} u(0)\|_{L^2})^2 \sim \|\mathcal{D} u(0)\|_{H^1}^2 \lesssim \eta^2,
\]  
(3.105)
Schur’s inequality implies
\[ \sum_{K \geq N} A_K^2 \lesssim \eta^2, \quad (3.106) \]
as desired. Therefore, proving (3.102) is sufficient to conclude the desired bounds on \( A_K \), which we proceed to do in cases.

Note that (3.99) implies
\[ \|D u\|_{X^1(I)} \lesssim \eta \quad (3.107) \]
for any interval \( I \subset [0,T] \) with \( |I| \lesssim 1 \). Let \( J \subset [0,T] \) be any interval with \( |J| \lesssim K^\gamma \). Since \( T \leq N^\gamma \), if \( K \geq N \) we can assume \( J = [0,T] \). If \( K < N \), by translation, we can also assume that the left endpoint of \( J \) is 0. Letting \( J = [0,T'] \) with \( T' \leq T \leq N^\gamma \), we consider the evolution equation satisfied by \( P_K D u \):

\[ (i \partial_t + \Delta) P_K D u = P_K D(|u|^2 u). \]

By Littlewood-Paley decomposition we have
\[ (i \partial_t + \Delta) P_K D u = \sum_{K_1,\ldots,K_3} P_K D(P_{K_1} u \cdot \overline{P_{K_2} u} \cdot P_{K_3} u) \quad (3.108) \]
\[ = \sum_{\text{at least two } K_j \geq K} P_K D(P_{K_1} u \cdot \overline{P_{K_2} u} \cdot P_{K_3} u) \quad (3.109) \]
\[ + \sum_{j=1}^3 \sum_{K_1 \sim K \atop K_i \neq K_j \subset K} P_K D(P_{K_1} u \cdot \overline{P_{K_2} u} \cdot P_{K_3} u). \quad (3.110) \]

We will fix a constant \( \alpha \), to be specified later, and set (3.109) = \( : N_1 \). Then, further
decompose (3.110) as follows:

\[
(3.110) = \left\{ 2 \sum_{K_1 \sim K, K^\alpha \leq \max_{i \neq 1} K_i \ll K} P_K D(P_{K_1} u \cdot \overline{P_{K_2} u} \cdot P_{K_3} u) \\
+ \sum_{K_2 \sim K, K^\alpha \leq \max_{i \neq 2} K_i \ll K} P_K D(P_{K_1} u \cdot \overline{P_{K_2} u} \cdot P_{K_3} u) \\
+ 2 \sum_{K_1 \sim K, \max_{i \neq 1} K_i \ll K^\alpha} P_K (D(P_{K_1} u \cdot \overline{P_{K_2} u} \cdot P_{K_3} u))
\right\}
\]

\[+ 2 P_K D u \cdot \mathbb{P}_0 \left( \sum_{K_2, K_3 \ll K^\alpha} D(\overline{P_{K_2} u} \cdot P_{K_3} u) \right). \] (3.114)

We denote

\[(3.111) =: \mathcal{N}_2, \quad (3.112) =: \mathcal{N}_3, \quad (3.113) =: \mathcal{N}_4, \]
and

\[2 \mathbb{P}_0 \left( \sum_{K_2, K_3 \ll K^\alpha} D(\overline{P_{K_2} u} \cdot P_{K_3} u) \right) =: \omega(t). \]

Then we have

\[(i \partial_t + \Delta) P_K D u = \mathcal{N}_1 + \mathcal{N}_2 + \mathcal{N}_3 + \mathcal{N}_4 + \omega(t) \cdot P_K D u. \] (3.115)

Define

\[v(t) := e^{i \Omega(t)} P_K D u(t), \quad \text{where} \quad \Omega(t) := \int_0^t \omega(s) \, ds. \] (3.116)

Then we have

\[(i \partial_t + \Delta) v(t) = \mathcal{N}'_1 + \mathcal{N}'_2 + \mathcal{N}'_3 + \mathcal{N}'_4 \quad \text{where} \quad \mathcal{N}'_j := e^{i \Omega(t)} \mathcal{N}_j. \] (3.117)

By Duhamel’s formula we have

\[v(t) = e^{i \Delta} P_K D u(0) - i \int_0^t e^{i(t-s) \Delta} (\mathcal{N}'_1(s) + \mathcal{N}'_2(s) + \mathcal{N}'_3(s) + \mathcal{N}'_4(s)) \, ds. \] (3.118)
For $0 \leq t \leq T'$ we will write
\[ v(t) = v_{\text{lin}}(t) - i \sum_{0 \leq m \leq T'} (v'_m + v''_m)(t), \]
where $v_{\text{lin}}(t) := e^{it\Delta} P_K Du(0)$, and
\[
v'_m(t) := 1_{[m,m+1)}(t) \int_m^t e^{i(t-s)\Delta} (N'_1(s) + N'_2(s) + N'_3(s) + N'_4(s)) \, ds \]
and
\[
v''_m(t) := 1_{[m+1,\infty)} \cap J(t) \cdot e^{it\Delta} g_m \]
where
\[
g_m(t) := \int_m^{m+1} e^{-is\Delta} (N'_1(s) + N'_2(s) + N'_3(s) + N'_4(s)) \, ds. \]

Proposition 6 implies
\[
\|v_{\text{lin}}\|_{S^7/4} \lesssim K \|P_K Du(0)\|_{L^2} \lesssim K \|P_K Du(0)\|_{L^2}. \tag{3.119}
\]
We will estimate $v'_m$ and $v''_m$ separately for each fixed $m$. We denote the contributions from $K_1, K_2, K_3$ to a function $f$ as $f_{K_1, K_2, K_3}$.

**Case 1:** $N'_1$, where at least two $K_j \gtrsim K$. We may assume $K_1 \leq K_2 \leq K_3$, and $K \lesssim K_2$. By Proposition 6 we have
\[
\|v''_{m,K_1,K_2,K_3}\|_{S^7/4} \lesssim K \|g_{m,K_1,K_2,K_3}\|_{L^2}. \tag{3.120}
\]
By the dual Strichartz estimate (3.20) we know that for all $p \in [1, \frac{3}{2})$,
\[
K \|g_{m,K_1,K_2,K_3}\|_{L^2} \lesssim K_2^{6/p-3} \|N'_{1,K_1,K_2,K_3}\|_{L^p_{t,x}[m,m+1]} \tag{3.121}
\]
and by the inhomogeneous Strichartz estimate (3.13)
\[
\|v'_{m,K_1,K_2,K_3}\|_{S^7/4} \lesssim \sup_{q \in [5,12]} K^{6/q-1} \|v'_{m,K_1,K_2,K_3}\|_{L^q_{t,x}[m,m+1]} \tag{3.122}
\]
\[
\lesssim K^{6/p-3} \|N'_{1,K_1,K_2,K_3}\|_{L^p_{t,x}[m,m+1]} \tag{3.123}
\]
Note that
\[
\| \mathcal{N}'_{1,K_1,K_2,K_3} \|_{L^p_{t,x}[m,m+1]} = \| \mathcal{N}_{1,K_1,K_2,K_3} \|_{L^p_{t,x}[m,m+1]}
\]
(3.124)
\[
\leq \prod_{j=1}^3 \| P_{K_j} \mathcal{D} \tilde{u} \|_{L^p_{x}[m,m+1]}
\]
(3.125)
where the last line follows by Hölder and (3.5) for \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} \). As before, we have
\[
\forall q > 3, \quad \| P_M \mathcal{D} \tilde{u} \|_{L^q[m,m+1]} \lesssim M^{2 - \frac{6}{p}} \| P_M \mathcal{D} u \|_{X_0} \lesssim \eta M^{1 - \frac{6}{q}}
\]
(3.126)
by (3.25) and (3.47). Applying this to \( K_2 \) and summing over \( K_2 \gtrsim K \) yields
\[
K^\frac{6}{p} - 3 \sum_{K_2 \gtrsim K} \| \mathcal{N}'_{1,K_1,K_2,K_3} \|_{L^p_{t,x}[m,m+1]} \lesssim \eta K^\frac{6}{p} - \frac{6}{p_2} - 2 \| P_{K_1} \mathcal{D} \tilde{u} \|_{L^p_{t,x}[m,m+1]} \| P_{K_3} \mathcal{D} \tilde{u} \|_{L^p_{t,x}[m,m+1]}
\]
(3.127)
Therefore
\[
\sum_{K_2 \gtrsim K} \left( \| v_{m,K_1,K_2,K_3} \| + \| v_{m,K_1,K_2,K_3}'' \| \right) \lesssim \eta K^\frac{6}{p_1} + \frac{6}{p_3} - 2 \| P_{K_1} \mathcal{D} \tilde{u} \|_{L^p_{t,x}[m,m+1]} \| P_{K_3} \mathcal{D} \tilde{u} \|_{L^p_{t,x}[m,m+1]}
\]
(3.128)
where we could take the inf over the same set of \( p_j \) as before, if we wanted. To put this in a similar form to [Den17], we note that, given \( \alpha \), we can take \( p_1, p_3 \) such that
\[
\alpha = -\left( \frac{6}{p_1} - 1 \right) = \frac{1}{3} \left( \frac{6}{p_3} - 1 \right),
\]
and then
\[
K^\frac{6}{p_1} + \frac{6}{p_3} - 2 = K^\alpha K_3^{-3\alpha} K_1^{6\alpha} K_3^{-6\alpha} K_1^{6\alpha} - 1.
\]
Then note that, since \( K_3 \gtrsim K \),
\[
\left( \frac{K}{K_3} \right)^{2\alpha} \left( \frac{K_1}{K_3} \right)^{\alpha} \leq \left( \frac{K}{K_3} \right)^{\alpha} \left( \frac{K_1}{K_3} \right)^{\alpha}
\]
and, if \( K < K_1 \leq K_3 \),
\[
\left( \frac{K}{K_3} \right)^{2\alpha} \left( \frac{K_1}{K_3} \right)^{\alpha} \left( \frac{K_1}{K} \right)^{2\alpha} \left( \frac{K}{K_1} \right)^{\alpha} \leq \left( \frac{K}{K_3} \right)^{\alpha} \left( \frac{K}{K_1} \right)^{\alpha}
\]
thus
\[ \sum_{K_3 \geq K \atop K_1 \leq K_2 \leq K_3} \| \nu'_{m,K_1,K_2,K_3} \| + \| \nu''_{m,K_1,K_2,K_3} \| \] (3.129)
\[ \leq \sum_{K_3 \geq K \atop K_1 \leq K_3} \eta \left( \frac{K}{K_3} \right) \alpha \min \left( \frac{K}{K_1}, \frac{K}{K_3} \right)^{\alpha} \left( K_3^{\frac{6}{p_3} - 1} \| P_{K_3} \tilde{D} \tilde{u} \|_{L^{p_3}} \right) \] (3.130)
\[ \leq \sum_{K_3 \geq K \atop K_1 \leq K_3} \eta \left( \frac{K}{K_3} \right) \alpha \left( K_3^{\frac{6}{p_3} - 1} \| P_{K_3} \tilde{D} \tilde{u} \|_{L^{p_3}} \right)^2, \] (3.131)
the last line following from the arithmetic-mean geometric-mean inequality.

Note
\[ \sum_{K_3 \geq K \atop K_1 \leq K_3} \eta \left( \frac{K}{K_3} \right) \alpha \min \left( \frac{K}{K_1}, \frac{K}{K_3} \right)^{\alpha} \left( K_3^{\frac{6}{p_3} - 1} \| P_{K_3} \tilde{D} \tilde{u} \|_{L^{p_3}} \right)^2 \]
\[ \leq \sum_{K_3 \geq K} \eta \left( \frac{K}{K_3} \right)^{2\alpha} \left( K_3^{\frac{6}{p_3} - 1} \| P_{K_3} \tilde{D} \tilde{u} \|_{L^{p_3}} \right)^2 \]
\[ \leq \sum_{K_3 \geq K} \eta \left( \frac{K}{K_3} \right)^{\alpha} \left( K_3^{\frac{6}{p_3} - 1} \| P_{K_3} \tilde{D} \tilde{u} \|_{L^{p_3}} \right)^2 \]
\[ \leq \sum_{K_3} \eta \min \left( \frac{K}{K_3}, \frac{K_3}{K} \right)^{\alpha} \left( K_3^{\frac{6}{p_3} - 1} \| P_{K_3} \tilde{D} \tilde{u} \|_{L^{p_3}} \right)^2, \]
We also calculate
\[ \sum_{K_1} \sum_{K_3 \geq K_1 \atop K_3 \geq K} \eta \left( \frac{K}{K_3} \right)^{\alpha} \min \left( \frac{K}{K_1}, \frac{K_3}{K} \right)^{\alpha} \left( K_1^{\frac{6}{p_1} - 1} \| P_{K_1} \tilde{D} \tilde{u} \|_{L^{p_1}} \right)^2 \]
\[ \leq \sum_{K_1} \eta \min \left( \frac{K}{K_1}, \frac{K_1}{K} \right)^{\alpha} \left( K_1^{\frac{6}{p_1} - 1} \| P_{K_1} \tilde{D} \tilde{u} \|_{L^{p_1}} \right)^2 \]
In general, therefore, we have
\[ \| \nu'_{m} \| + \| \nu''_{m} \| \lesssim \sum M \eta \min \left( \frac{K}{M}, \frac{M}{K} \right)^{\frac{1}{2}} \left( \sup_{p \in [5,12]} M^{\frac{6}{p} - 1} \| P_{M} \tilde{D} \tilde{u} \|_{L^{p}} \right)^{\frac{1}{2}}. \] (3.132)
Case 2: The term $N'_2$. By Proposition 6 and (3.25) we have that

$$
\left\| v_{m,K_1,K_2,K_3}^{'} \right\|_{S_{K,J}^{7/4}} \lesssim \sup_{q \in [5,12]} K_q^{-1} \left\| v_{m,K_1,K_2,K_3}^{'} \right\|_{L_t^q L_x^{[m,m+1]}} \lesssim \left\| \mathcal{I}(N_2,K_1,K_2,K_3 \cdot e^{i\Omega(t)}) \right\|_{X^{1}[m,m+1]}
$$

(3.133)

and

$$
\left\| v_{m,K_1,K_2,K_3}^{''} \right\|_{S_{K,J}^{7/4}} \lesssim K \left\| g_{m,K_1,K_2,K_3} \right\|_{L^2} \lesssim \left\| \mathcal{I}(N_2,K_1,K_2,K_3 \cdot e^{i\Omega(t)}) \right\|_{X^{1}[m,m+1]}
$$

(3.134)

where $\mathcal{I}$ is the Duhamel operator

$$
IG(t) := \int_t^\tau e^{i(t-s)} \Delta G(s) \, ds.
$$

We will not need to distinguish between $u$ and $\tilde{u}$, so we will only consider the terms

$$
\sum_{K_1 \sim K, K_2 \leq K} N_{2,K_1,K_2,K_3}
$$

from $N_2$, and assume $K_2 \geq K_3$. By (3.26) we have that

$$
\left\| \mathcal{I}(N_{2,K_1,K_2,K_3} \cdot e^{i\Omega(t)}) \right\|_{X^{1}[m,m+1]} \lesssim K \int_m^{m+1} \int_{T^1} e^{i\Omega(t)} N_{2,K_1,K_2,K_3} \cdot \tilde{v} \, dx \, dt,
$$

(3.135)

where $v$ is some function such that $\|v\|_{Y_0[m,m+1]} = 1$. We can extend $v$ such that $\|v\|_{Y_0} \lesssim 1$ and move the factor $P_K \mathcal{D}$ from the $N_{2,K_1,K_2,K_3}$ expression to $v$. We then use Proposition 7 to bound

$$
\left\| \mathcal{I}(N_{2,K_1,K_2,K_3} \cdot e^{i\Omega(t)}) \right\|_{X^{1}[m,m+1]} \lesssim K \left\| e^{i\Omega(t)} \right\|_{L_{t,x}^\infty} \left\| P_{K_1} \tilde{u} P_{K_2} \tilde{v} \right\|_{L^2[m,m+1]} \left\| P_{K_3} \tilde{u} P_K \mathcal{D} \tilde{v} \right\|_{L^2[m,m+1]}
$$

(3.136)

\begin{align*}
&\lesssim K \left( \frac{K_3}{K} + \frac{1}{K_3} \right)^{1/8} K_2 K_3 \left\| P_K \mathcal{D} v \right\|_{Y_0} \prod_{j=1}^{3} \left\| P_{K_j} u \right\|_{Y_0} \\
&\lesssim \eta
\end{align*}

(3.137)

Now using Proposition 4,

$$
K_j \left\| P_{K_j} u \right\|_{Y_0} \lesssim \left\| P_{K_j} u \right\| \lesssim \left\| P_{K_j} u \right\|_{X^1} \lesssim \left\| \mathcal{D} u \right\|_{X^1} \lesssim \eta
$$

72
which we will apply to $j \in \{2, 3\}$. Similarly, inspecting the definition of the $Y^0$ norm and the $D$-multiplier, we have, for $K_1 \sim K$,

$$K \| P_{K_1} u \|_{Y^0} \| P_{K} D v \|_{Y^0} \lesssim K \max(1, (K/N)^{s-1}) \| P_{K_1} u \|_{Y^0} \| P_{K} v \|_{Y^0} \lesssim \| P_{K_1} u \|_{Y^0} \| P_{K} v \|_{Y^0} \lesssim \eta.$$  

Thus, using the fact that $K_3 \gtrsim K^\alpha$,

$$\| I(\mathcal{N}_{2,K_1,K_2,K_3} \cdot e^{\Omega(t)}) \|_{X^1_{\lbrack m,m+1\rbrack}} \lesssim \eta^3 \left( \frac{K_3}{K} + \frac{1}{K_3} \right)^{1/8} \lesssim \eta^3 \left( \frac{K_3}{K} \right)^{1/8} + \eta^3 K^{-\alpha/8}. \quad (3.138)$$

The second term here is raised to a negative power which can absorb the eventual the $\epsilon$ losses which will result when we sum in $K_3 \leq K_2 \lesssim K$ and $K$, however the first term will result in $\eta^3 K^\epsilon$, which won’t be strong enough to close. To fix this, we will interpolate with type of bounds from case 1. For later convenience we weaken this bound as follows:

$$\| I(\mathcal{N}_{2,K_1,K_2,K_3} \cdot e^{\Omega(t)}) \|_{X^1_{\lbrack m,m+1\rbrack}} \lesssim \eta^3 \left( \frac{K_3}{K} \right)^{1/16} \left( \frac{K_2}{K} \right)^{1/16} + \eta^3 K^{-\alpha/16}. \quad (3.139)$$

By the dual Strichartz estimate (3.20) and the inhomogeneous Strichartz estimate (3.13), we know that for all $p \in [1, \frac{3}{2})$,

$$\| I(\mathcal{N}_{2,K_1,K_2,K_3} \cdot e^{\Omega(t)}) \|_{X^1_{\lbrack m,m+1\rbrack}} \lesssim K^\frac{6}{p} - 3 \| \mathcal{N}_{2,K_1,K_2,K_3} \|_{L^p_{t,x}[m,m+1]} \quad (3.140)$$

By Hölder and (3.5) we have, for all $q_j$ such that $\frac{1}{p} = \frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3}$,

$$\| \mathcal{N}_{2,K_1,K_2,K_3} \|_{L^p_{t,x}[m,m+1]} \lesssim \prod_{j=1}^3 \| P_{K_j} \tilde{u} \|_{L^{q_j}_{t,x}[m,m+1]} \quad (3.141)$$

If $q_1 > 3$ we have, by (3.25) and Proposition 4,

$$\| P_{K_1} \tilde{u} \|_{L^{q_1}_{t,x}} \lesssim K_1^{\frac{6}{q_1}} \| P_{K_1} \tilde{u} \|_{X^0} \lesssim K_1^{\frac{1}{q_1}} \| P_{K_1} \tilde{u} \|_{X^1} \lesssim K_1^{\frac{1}{q_1}} \| D u \|_{X^1} \lesssim \eta K_1^{\frac{1}{q_1}} \quad (3.142)$$
Thus

$$\|v_{m,K1,K2,K3}'\|_{S_{K1,j}^{7/4}} + \|v_{m,K1,K2,K3}''\|_{S_{K1,j}^{7/4}} \lesssim \eta K_{p}^{-3} K_{1}^{-\frac{6}{q_1}} \prod_{j=2}^{3} \|P_{K_j} \tilde{u}\|_{L_{t,x}^{q_j}} \quad (3.143)$$

$$\sim \eta K_{q_2}^{6} + \eta K_{q_3}^{-2} \prod_{j=2}^{3} \|P_{K_j} \tilde{u}\|_{L_{t,x}^{q_j}}, \quad (3.144)$$

which, incorporating all earlier constraints, is true for all $q_2, q_3$ such that $\frac{1}{q_2} + \frac{1}{q_3} > \frac{1}{3}$.

To work this into a form more amenable to our end goal, for reasons that will become apparent shortly, note that for any $q_2, q_3 \in [5, 12]$,

$$\eta K_{q_2}^{6} + \eta K_{q_3}^{-6} \prod_{j=2}^{3} \|P_{K_j} \tilde{u}\|_{L_{t,x}^{q_j}} \lesssim \eta \prod_{j=2}^{3} \left( \frac{K}{K_j} \right)^{6} \prod_{j=2}^{3} \sup_{q \in [5,12]} K_{j}^{q} \|P_{K_j} \tilde{u}\|_{L_{t,x}^{q_j}}. \quad (3.145)$$

We can interpolate this with the problematic term in (3.139) to find that, for any $\theta \in [0,1]$,

$$\|v_{m,K1,K2,K3}'\|_{S_{K1,j}^{7/4}} + \|v_{m,K1,K2,K3}''\|_{S_{K1,j}^{7/4}} \lesssim C \eta^{3} K^{-\alpha/16} \lesssim$$

$$\left( \eta^{3} \left( \prod_{j=2}^{3} \frac{K_j}{K} \right)^{1/16} \right)^{1-\theta} \left( \eta \prod_{j=2}^{3} \left( \frac{K}{K_j} \right)^{6} \prod_{j=2}^{3} \sup_{q \in [5,12]} K_{j}^{q} \|P_{K_j} \tilde{u}\|_{L_{t,x}^{q_j}} \right)^{\theta}. \quad (3.146)$$

For reasons that will become clear later, we will take $\theta = 7/8$. Note that we can choose the $q_j$ such that

$$\|v_{m,K1,K2,K3}'\|_{S_{K1,j}^{7/4}} + \|v_{m,K1,K2,K3}''\|_{S_{K1,j}^{7/4}} \lesssim$$

$$\eta^{3} K^{-\alpha/8} + \eta^{3} \prod_{j=2}^{3} \left( \frac{K_j}{K} \right)^{1/256} \prod_{j=2}^{3} \left( \sup_{q \in [5,12]} K_{j}^{q} \|P_{K_j} \tilde{u}\|_{L_{t,x}^{q_j}} \right)^{7/8}. \quad (3.147)$$

We use the arithmetic-geometric mean inequality as before on the last product only:

$$\|v_{m,K1,K2,K3}'\|_{S_{K1,j}^{7/4}} + \|v_{m,K1,K2,K3}''\|_{S_{K1,j}^{7/4}} \lesssim$$

$$\eta^{3} K^{-\alpha} + \eta^{3} \prod_{j=2}^{3} \left( \frac{K_j}{K} \right)^{1/256} \prod_{j=2}^{3} \left( \sup_{q \in [5,12]} K_{j}^{q} \|P_{K_j} \tilde{u}\|_{L_{t,x}^{q_j}} \right)^{7/4}. \quad (3.148)$$
Note that since $K_1 \sim K$ we can sum this in $K_1$ with no loss (there are $O(1)$ frequencies). Summing in $K_1, K_2, K_3$ yields, for any $\varepsilon > 0$,

$$\|v'_m\|_{S_{K,J}^{1/4}} + \|v''_m\|_{S_{K,J}^{1/4}} \lesssim \eta^3 K^{-\alpha \varepsilon} + \eta \sum_{M} \min \left( \frac{M}{K}, \frac{K}{M} \right)^{1/256} \left( \sup_{q \in [5,12]} M^{\frac{6}{q}-1} \|P_M \tilde{u}\|_{L^q_{t,x}} \right)^{7/4}.$$  

(3.149)

**Case 3:** Terms $N'_3$ and $N'_4$. Due to resonance, $N'_4$ is more difficult to bound, so we will focus on that term. Without loss of generality $K_2 \geq K_3$. By (3.29) we have $\|D u\|_{X^{1,1/4-\varepsilon}} \lesssim \eta$ for any fixed $\varepsilon > 0$. We shall prove that

$$\|v'_{m,K_1,K_2,K_3}\|_{S_{K,J}^{1/4}} + \|v''_{m,K_1,K_2,K_3}\|_{S_{K,J}^{1/4}} \lesssim \eta^3 K^{-1/100}$$

for any $K_1, K_2, K_3$, and thus the logarithmic factor coming from summation in $K_0, K_1, K_2, K_3$ can be omitted. Consider the function

$$\rho(t, x) := \mathbb{P}_{\neq 0} \left( P_{K_2} \tilde{u} P_{K_3} \tilde{u} \right).$$

Note that $K_2 \ll K^\alpha$. We use equation (3.6) to compute, for any interval $I$ containing $m$ with $|I| \lesssim 1$, that

$$\|\partial_t \rho\|_{L^2_{t,x}(I)} \lesssim \|P_{\leq K^\alpha} u\|_{L^\infty_{t,x}} \left( K^{2\alpha} \|P_{\leq K^\alpha} u\|_{L^2_{t,x}(I)} + \|P_{\leq K^\alpha} (|u|^2 u)\|_{L^2_{t,x}(I)} \right)$$

$$\lesssim \eta K^{\alpha/2} (K^{2\alpha} \eta + K^{\alpha/2} \|u\|_{L^3_{t,x} L^2_{t,x}(I)}) \lesssim \eta^2 K^{3\alpha}.$$

Therefore $\|\rho\|_{H^1_{t} H^2_{x}} \lesssim K^{2\alpha} \|\partial_t \rho\|_{L^2_{t,x}(I)} \lesssim \eta^2 K^{5\alpha}$. Thus, for a suitable time cutoff $\chi(t - m)$, we can write

$$\chi(t - m) \rho(t - m) = \chi(t - m) \sum_{|k| \leq K^\alpha} \int_{\mathbb{R}} d(k, \xi) e^{i(k \cdot x + t \xi)} \, d\xi$$

with

$$\sum_{|k| \leq K^\alpha} \int_{\mathbb{R}} \langle \xi \rangle^{1/3} |d(k, \varepsilon)| \, d\xi \lesssim \eta^2 K^{7\alpha},$$

75
because
\[
\sum_{|k| \leq K_\alpha} \int_{\mathbb{R}} |\langle \xi \rangle^{1/3} d(k, \xi)| \, d\xi = \sum_{|k| \leq K_\alpha} \int_{\mathbb{R}} \frac{1}{\langle k \rangle^{2/3} \langle \xi \rangle^{2/3}} \langle k \rangle^2 \langle \xi \rangle d(k, \xi) \, d\xi
\]  
(3.150)
\[
\leq \left( \sum_{|k| \leq K_\alpha} \int_{\mathbb{R}} \frac{1}{\langle k \rangle^4 \langle \xi \rangle^{4/3}} \, d\xi \right)^{1/2} \left( \sum_{|k| \leq K_\alpha} \int_{\mathbb{R}} \langle k \rangle^4 \langle \xi \rangle^2 d^2(k, \xi) \, d\xi \right)^{1/2}
\]  
(3.151)
\[
\lesssim K^{4\alpha} \| \rho \|_{H^1_t H^2_x}
\]  
(3.152)
\[
\lesssim \eta^2 \xi K^{7\alpha}.
\]  
(3.153)

Similarly, since we know
\[
\| \partial_t e^{i\Omega(t)} \|_{L^2(I)} = \| \omega(t) \|_{L^2(I)} \lesssim \| P_{\leq K^\alpha} u \|_{L^\infty_t L^2_x} \lesssim \eta^2
\]
we can write
\[
\chi(t - m)e^{i\Omega(t)} = \chi(t - m) \int_{\mathbb{R}} y(\xi)e^{i\xi} \, d\xi,
\]
with
\[
\int_{\mathbb{R}} \langle \xi \rangle^{1/3} |y(\xi)| \, d\xi \lesssim 1
\]
using similar reasoning as before.

Note that, due to the existing sharp cutoff for $v'$, we can introduce cutoff functions as follows:
\[
v'_{m,K_1,K_2,K_3} = 2 \cdot 1_{[m,m+1]}(t) \int_m^t e^{i(t-t')} \Delta \chi(t - m) e^{i\Omega(t')} P_K D u \cdot \chi(t - m) \rho \, dt'.
\]  
(3.154)
We can substitute within the integrand to get

\[
\begin{align*}
\int_m^t e^{i(t-t')\Delta} \chi(t - m) \int_R y(\xi_1) e^{it\xi_1} d\xi_1 P_K \mathcal{D}(P_{K_1} u \cdot \chi(t - m) \sum_{|k| \leq K^\alpha} \int_R d(k, \xi_2) e^{i(kx + t\xi_2)} d\xi_2) dt' \\
= \sum_{|k| \leq K^\alpha} \int_R y(\xi_1) \int_R d(k, \xi_2) \chi(t - m)^2 \int_m^t e^{i(t-t')\Delta} P_K \mathcal{D}(P_{K_1} u \cdot e^{i(kx + (t+\xi_2))}) dt' d\xi_2 d\xi_1.
\end{align*}
\] (3.155)

With the substitution

\[\xi_1 + \xi_2 =: \xi\]

we can therefore reduce to estimating

\[h_{k,\xi}(t) := \langle \xi \rangle^{-1/3} \chi(t - m) \int_m^t e^{i(t-t')\Delta} \mathcal{D}(e^{i(kx + \xi t)} P_{K_1} u(t, x)) dt',\]

since we have

\[
\|v_{m,K_1,K_2,K_3}\|_{S_{K,J}^{s/4}} + \|v''_{m,K_1,K_2,K_3}\|_{S_{K,J}^{s/4}} \leq \eta^2 K^{7\alpha} \sup_{|k| \leq K^\alpha} \|h_{k,\xi}\|_{L_q^q[m,m+1]} + \|e^{i(t-m-1)\Delta} h_{k,\xi}(m + 1)\|_{S_{K,J}^{s/4}}.
\] (3.156)

By (3.23), (3.32), and (3.36) we have

\[
\sup_{q \in [5,12]} \Kappa_{q^{-1}}^{6} \|h_{k,\xi}\|_{L_q^q[m,m+1]} + \|e^{i(t-m-1)\Delta} h_{k,\xi}(m + 1)\|_{S_{K,J}^{s/4}} \lesssim \|h_{k,\xi}\|_{X^{1,3/4}}
\] (3.157)

and by (3.31) we have

\[
\langle \xi \rangle^{1/3} \|h_{k,\xi}\|_{X^{1,3/4}} \lesssim \|\nabla P_{K_1} u\|_{L^2_t L_q^q} \lesssim \|DP_{K_1} u\|_{X^{1,1/5}} \lesssim \eta,
\] (3.158)

thus

\[
\eta^2 K^{7\alpha} \sup_{q \in [5,12]} \Kappa_{q^{-1}}^{6} \|h_{k,\xi}\|_{L_q^q[m,m+1]} + \|e^{i(t-m-1)\Delta} h_{k,\xi}(m + 1)\|_{S_{K,J}^{s/4}} \lesssim \eta^3 K^{-3\alpha}
\] (3.159)
if $|\xi| \gtrsim K^{30\alpha}$.

If $|\xi| \lesssim K^{30\alpha}$ we decompose $P_{K_1} u = \mathbb{P}_C u + P_{K_1} \mathbb{P}' C u$ where

$$C = \bigcup_{0 < |k| \leq K^\alpha} \{ n \in \mathbb{Z}^4 : |n| \sim K_1, |Q(n + k) - Q(n)| \lesssim K^{60\alpha} \}$$

(3.162)

and $C' = \mathbb{Z}^4 \setminus C$. Counting by fixing all but one coordinate of $n$, we see that $\# C \lesssim K^{3 + 64\alpha}$. Denoting the contribution of the $\mathbb{P}_C u$ term as $h'_{k,\xi}$ we have $\| h'_{k,\xi} \|_{X^{1,3/4}} \lesssim \eta$ as above. Since the spatial Fourier transform $\widehat{h'_{k,\xi}}$ is supported in a translate of $C$ we can use the improvement of (3.32) for $\mathbb{P}_C$ to find

$$\eta^2 K^{7\alpha} \sup_{q \in [5,12]} K^{6q - 1} \| h_{k,\xi} \|_{L^q_{t,x}[m,m+1]} + \| e^{i(t-m-1)\Delta} h_{k,\xi}(m + 1) \|_{S^{7/4}_{K,J}}$$

$$\lesssim \eta^2 K^{7\alpha + \gamma} \left( \frac{K^{3 + 64\alpha}}{K^4} \right)^{1/4} \| h'_{k,\xi} \|_{X^{1,3/4}} \lesssim \eta^2 K^{-1/4 + 24\alpha}$$

(3.163)

where we have used that $\gamma < \alpha$.

Finally, for the term $P_{K_1} \mathbb{P}' C u := u^*$, we denote it’s contribution to $h_{k,\xi}$ by $k^*_{k,\xi}$. We know by (3.31) that

$$\| h^*_{k,\xi} \|_{X^{1,3/4}} \lesssim \| \mathcal{D}(e^{i(k \cdot x + \xi t)} u^*) \|_{X^{1,-1/4}}$$

(3.164)

and by duality this is then bounded by

$$\int_{\mathbb{R} \times T^4} K e^{i(k \cdot x + \xi t)} \mathcal{D} \bar{v} \cdot u^* \, dx \, dt,$$

(3.165)

where $\| v \|_{X^{0,1/4}} \lesssim 1$. By translation we can set $m = 0$, and by Plancherel the above can be written

$$K \sum_{|n| \sim K} \int_{\mathbb{R}} \bar{w}^*(n, \zeta) \overline{\mathcal{D} \bar{v}(n + k, \zeta + \xi)} \, d\zeta.$$

(3.166)

Since $|k| \lesssim K^\alpha$ and $|\xi| \lesssim K^{30\alpha}$ we have

$$\max(|\zeta + Q(n)|, |\zeta + \xi + Q(n + k)|) \geq |Q(n + k) - Q(n)| - O(1) K^{30\alpha} \gg K^{60\alpha}$$

(3.167)
since \( n \notin C \). Since \( \|Du^*\|_{X^{0,1/6}} \lesssim K^{-1} \) and thus \( \|u^*\|_{X^{0,1/6}} \lesssim K^{-1} \min(1, (K/N)^{1-s}) \), and \( \|Dv\|_{X^{0,1/4}} \lesssim \max(1, (K/N)^{s-1}) \), we can bound this by extracting a factor of \( |\zeta + Q(n)| \) or \( |\zeta + \xi + Q(n+k)| \) and estimating both factors in \( \ell_n^2 / T^2 \), which yields

\[
\|h_{k, \xi}^*\|_{X^{1/34}} \lesssim \eta K^{-10\alpha}, \quad \text{and thus}
\]

\[
\eta^2 K^{7\alpha} \sup_{q \in [5,12]} K^{q-1} \|h_{k, \xi}\|_{L^q_{m,m+1}} + \|e^{(t-m-1)\Delta} h_{k, \xi}(m+1)\|_{\ell_{m,j}^{1/2}} \lesssim \eta^3 K^{-3\alpha}.
\]

Combining these results we have

\[
\|v_{m,K_1,K_2,K_3}\|_{S^{7/4}_{k,j}} + \|v'_{m,K_1,K_2,K_3}\|_{S^{7/4}_{k,j}} \lesssim \eta^3 \max(K^{-3\alpha}, K^{-1/4+24\alpha}) \leq \eta^3 K^{-1/100},
\]

(3.169)

since \( \alpha = 1/100 \). which completes this case.

Combining the results from all three cases and summing up in \( m \in \{0, \ldots, T'\} \), using \( T' \leq K^{\gamma} \), we have

\[
A_K \lesssim a_K + \eta^3 K^{-1/2000} + \eta \sum_M \min\left(\frac{M}{K}, \frac{K}{M}\right)^{1/256} \sum_0 \left(\sup_{q \in [5,12]} M^{q-1} \|P_M D\tilde{u}\|_{L^q_{m,m+1} \cap \gamma \cap J}\right)^{7/4}.
\]

(3.170)

Since \( |J| \leq K^{\gamma} \) we do not quite have \( A_M \) on the right yet, however for any fixed \( M \) we can divide the summation in \( m \) into \( O(K/M)^\gamma \) terms if \( M \lesssim K \) (covering an interval \( J' \) of size \( |J'| \leq M^{\gamma} \)) and \( O(1) \) terms if \( M \gtrsim K \), such that each term is bounded by \( \|P_M D\tilde{u}\|_{S^{7/4}_{m,J'}} \lesssim A_M^{7/4} \) for some \( J' \subset [0,T] \) such that \( |J'| \leq M^{\gamma} \).

Therefore we have

\[
A_K \lesssim a_K + \eta^3 K^{-1/2000} + \eta \sum_M \min\left(\frac{M}{K}, \frac{K}{M}\right)^{1/256} \max\left(\frac{K}{M}, \frac{M}{K}\right)^{\gamma} A_M^{2\alpha}.
\]

(3.171)

Since \( \gamma = 1/300 < \delta \) and \( A_K \lesssim 1 \) this implies (3.102), and therefore, by our previous argument, this concludes the proof. \( \square \)
Armed with Corollary 3 and Proposition 10 we can finish the proof of Theorem 12 as in [Den17], however for completeness we reproduce it here.

**Proof of Main Theorem.** Let $\|u(0)\|_{H^s} = A$. Fix a large enough constant $D$ not depending on $\eta$. In this proof any implicit constant $C$ appearing in $\lesssim$ will be $\ll D$. Let $u$ be a solution to (3.6) with energy $E[u] \lesssim \eta^2$, and choose $N$ such that $\|u(0)\|_{H^s} \sim \eta N^{s-1}$. By the definition of $\mathcal{D} = \mathcal{D}_N$, we see that $\|\mathcal{D}u(0)\|_{H^1} \lesssim \eta$. By global well-posedness (Proposition 8) and Strichartz, we see that for $T = 1$,

$$
\sup_{0 \leq t \leq T} \|\mathcal{D}u(t)\|_{H^1} \leq D\eta, \quad A_K(T) \leq 1,
$$

with $A_K(T)$ defined in Proposition 10. Suppose (3.172) holds for some $T \leq N^\gamma$. Then Corollary 3 and Proposition 10 implies, for $t \in [0, T]$, we have $A_K(T) \lesssim \eta \ll 1$. Therefore

$$
E[\mathcal{D}u(t)] - E[\mathcal{D}u(0)] \lesssim D \eta^2 \sum_K \min\left(1, \frac{K}{N}\right)^{\frac{1}{6}} \sup_{|J| \leq K^\gamma, J \subseteq [0, T]} \|P_K \mathcal{D}u\|_{S_{K, J}^{1/4}}^2
$$

or

$$
\lesssim D \eta^2 \sum_K \min\left(1, \frac{K}{N}\right)^{\frac{1}{6}} \sup_{|J| \leq K^\gamma, J \subseteq [0, T]} \|P_K \mathcal{D}u\|_{S_{K, J}^{1/4}}^2 \lesssim D \eta^2 \sum_K \min\left(1, \frac{K}{N}\right)^{\frac{1}{6}} A_K^2
$$

or

$$
\lesssim D \eta^2 \sum_{K \leq N} \left(\frac{K}{N}\right)^{\frac{1}{6}} A_K^2 + \eta^2 \sum_{K \geq N} A_K^2 \lesssim D \eta^4.
$$

Since $\eta$ is sufficiently small, this implies that $E[\mathcal{D}u(t)] \leq E[\mathcal{D}u(0)] + O_D(1) \eta^4 \leq C\eta^2$, which gives

$$
\sup_{0 \leq t \leq T} \|\mathcal{D}u(t)\|_{H^1} \ll D\eta.
$$

By bootstrap arguments, this implies 3.172 remains true up to $T = N^\gamma$, which implies $\|u(t)\|_{H^s} \lesssim \eta N^{s-1}$ for $0 \leq t \leq T$. Using time translation and rescaling $N$ by a factor depending on $\eta$, we get the following result for some absolute constant.
$E$ which depends on $\eta$:

If $\|u(t)\|_{H^s} \leq N^{s-1}$, then for $|t' - t| \leq N^\gamma / E$ we have $\|u(t')\|_{H^s} \leq (EN)^{s-1}$.

(3.173)

Now, for any positive integer $m$ such that $E^{m(s-1)} \geq A$, choose the smallest time $t_m > 0$ such that $\|u(t)\|_{H^s} \geq E^{m(s-1)}$. Then by (3.173) we have

$$t_{m+1} - t_m \geq E^{m\gamma - 1},$$

so, in particular, $t_m \gtrsim E^{m\gamma}$. Therefore, for each $t > 0$, if $\|u(t)\|_{H^s} \gg A$, choosing the biggest $m$ such that $\|u(t)\|_{H^s} \geq E^{m(s-1)}$, we get that $t \geq t_m \gtrsim E^{m\gamma}$, thus

$$\|u(t)\|_{H^s} \leq E^{(m+1)(s-1)} \lesssim E^{m(s-1)} \lesssim t^{\frac{s-1}{\gamma}}.$$ 

The negative times are proved in the same way, and this completes the proof of Theorem 12. \qed

3.4 Appendix

3.4.1 Christ-Kiselev

**Lemma 11** (Christ-Kiselev). Let $X, Y$ be Banach spaces, let $I$ be a time interval, and let $K \in C^0(I \times I \to B(X \to Y))$ be a kernel taking values in the space of bounded operators from $X$ to $Y$. Suppose that $1 \leq p < q \leq \infty$ is such that

$$\left\| \int_I K(t, s)f(s)\, ds \right\|_{L^q(I \to Y)} \leq A \|f\|_{L^p(I \to X)}$$

for all $f \in L^p(I \to X)$ and some $A > 0$. Then one also has

$$\left\| \int_{s \in I, s < t} K(t, s)f(s)\, ds \right\|_{L^q(I \to Y)} \lesssim_{p,q} A \|f\|_{L^p(I \to X)}.$$
Remark 8. This is a specific form of a strong result about maximal functions from [CK01]. The statement of Lemma 11 as it appears here is from [Tao06], where it appears as Lemma 2.4 on page 75. A proof can be found in [Tao00].

### 3.4.2 Derivative of Energy Calculation

We start with

\[
E[f] := \int_M \frac{1}{2} \sum_{i=1}^4 \beta_i |\partial_{x_i} f|^2 + \frac{1}{4} |f|^4 \, dx,
\]

then

\[
\partial_t E[f] = \int_M \Re \sum_{i=1}^4 \beta_i \partial_{x_i} \overline{f} \partial_{x_i} f_t + \Re f_t \overline{|f|^2} \, dx.
\]

Then Gauss’ theorem implies

\[
\partial_t E[f] = \Re \int_M (|\overline{f}|^2 - \Delta_{\beta} \overline{f}) f_t \, dx. \tag{3.174}
\]

Let

\[
\omega := |\overline{f}|^2 - \Delta_{\beta} \overline{f},
\]

and let \( R \) be defined by

\[
(i \partial_t - \Delta_{\beta}) f = |f|^2 f + R,
\]

so

\[
f_t = i \Delta_{\beta} f - i |f|^2 f - i R = -i(\omega + R).
\]

Substituting this into (3.174) we have

\[
\partial_t E[f] = \Re \int_M \omega(-i(\omega + R)) \, dx = \Im \int_M \omega R \, dx.
\]
3.4.3 Summing Dyadic Terms

Note that, for $\alpha > 0$, $K, M \in 2^N$, we have

$$\sum_{K \leq M} K^\alpha \leq \sum_{j=0}^\infty M^\alpha 2^{-j\alpha} \lesssim M^\alpha$$

Now if $N_0 \geq \cdots \geq N_5$, we have

$$\sum_{N_2, \ldots, N_5} \left( \frac{N_3N_4N_5}{N_0N_1N_2} \right)^{\frac{1}{2}} = \sum_{N_2, \ldots, N_4} \left( \frac{N_3N_4}{N_0N_1N_2} \right)^{\frac{1}{2}} \sum_{N_5 \leq N_4} N_5^{\frac{1}{2}}$$

$$\lesssim \sum_{N_2, N_3} \left( \frac{N_3}{N_0N_1N_2} \right)^{\frac{1}{2}} \sum_{N_4 \leq N_3} N_4$$

$$\lesssim \sum_{N_2} \left( \frac{1}{N_0N_1N_2} \right)^{\frac{1}{2}} \sum_{N_3 \leq N_2} N_3^{\frac{3}{2}}$$

$$\lesssim \left( \frac{1}{N_0N_1} \right)^{\frac{1}{2}} \sum_{N_2 \leq N_1} N_2$$

$$\lesssim \left( \frac{N_1}{N_0} \right)^{\frac{1}{2}}$$

We also have

$$\sum_{K \leq M} M^{-\alpha} \leq \sum_{j=0}^\infty K^\alpha 2^{-j\alpha} \lesssim K^\alpha.$$
PART II: UNIQUE
CONTINUATION AT THE
BOUNDARY
A bounded domain $\Omega$ in $\mathbb{R}^n$ is said to have the boundary unique continuation property (for the Laplace operator) if a harmonic function $u$ in $\Omega$ that vanishes identically on some open subset $V$ of the boundary and whose normal derivative vanishes identically on some $E \subseteq V$ that has positive surface measure, must be identically zero in $\Omega$. In two dimensions, conformal mappings ensure that the b.u.c.p. always holds, on any domain. Thus, one is interested in $n \geq 3$. That $u$ vanishes identically when $E$ is open follows from elementary considerations, and it is known that $u$ need not vanish identically unless $V$ is open. The counter-examples in this latter case are due to Wolff, and also Bourgain and Wolff [Wol95] [BW90].

This problem is closely related to strong unique continuation properties of elliptic operators in $\mathbb{B}^n$, the unit ball in $\mathbb{R}^n$. Such questions have been studied by Aronszajn, Krzywicki, and Szarski; Garafolo and Lin; Plis; Miller; and others [AKS62] [GL86] [Pli63] [Mil74]. The connection arises because the function $u$ described above must vanish to infinite order at almost every point of the set $E$, and if one now pulls back the problem to the upper half space and reflects across $\mathbb{R}^{n-1}$ the problem reduces to showing that a certain elliptic operator has the strong unique continuation property. This technique actually yields the boundary unique continuation property for $C^{1,1}$ domains [Lin91]. At present, the most general class
of domains that are known to have the boundary unique continuation property are the $C^{1,\alpha}$ domains, where $\alpha$ can be any positive number. This is a result of Adolfsson and Escauriaza [AE97]. The case of convex domains is due to Adolfsson, Escauriaza and Kenig [AEK95]. It follows from Dahlberg’s [Dah77] comparison principle that Lipschitz domains have the boundary unique continuation property for non-negative harmonic functions. However, it is still an open question as to whether the boundary unique continuation property holds for a general harmonic function on a Lipschitz domain.

The Bang-Bang principle is a minimizing problem that is closely connected with boundary unique continuation and that is also of interest because of its applications in control theory. In the treatment of the boundary control problem for parabolic equations, the time-optimal control of heat, for example, one seeks a boundary temperature or control function $f(x,t), x \in \partial \Omega, t \geq 0$ (optimal control) that transfers an initial state $u_0(x), x \in \Omega$ of heat to another one $u_T(x), x \in \Omega$ in the least possible time $T$ subject to the constraint $|f(x,t)| \leq C$ a.e. $x \in \partial \Omega, t \geq 0$. It turns out that if such a control function exists then one that minimizes $T$ can always be constructed from it. However, the uniqueness of the minimizer does not follow automatically and it is harder to establish. Uniqueness can be proven if a type of maximum principle (the “Bang-Bang” principle) can be established. More specifically, if $f_0$ is the minimizer, then $|f_0(x,t)| = C$, a.e. $x \in \partial \Omega, t \geq 0$ [Fat76] [SW92] [SW80].

The boundary control for parabolic equations relies on the boundary behaviour of solutions to the associated elliptic equation. In the elliptic setting the Bang-Bang principle can be stated as follows. Suppose that $\Omega$ is a bounded domain in $\mathbb{R}^n$, and that $X_0$ and $X_1$ are elements of $\Omega$. Among all harmonic functions $u$ in $\Omega$ such that $0 \leq u \leq 1$ and $u(X_0) = \frac{1}{2}$, there is at least one that minimizes the value
of \( u(X_1) \). The problem is to show that there is a unique minimizer \( u^* \) and that \( u^* \) is “Bang-Bang”, i.e., \( u^* \) is the harmonic extension of a characteristic function on \( \partial \Omega \). It so happens, and this is explained below, that this problem can be reduced to showing that \( \Omega \) has the boundary unique continuation for a very special family of harmonic functions. In particular, domains that have the boundary unique continuation property such as \( C^{1,\alpha} \) domains, also satisfy the Bang-Bang principle. Indeed, \( C^{1,\alpha} \) domains are the most general class of domains that are known to satisfy the Bang-Bang principle, and this fact is deduced from boundary unique continuation \([\text{Ken}]\).

The Bang-Bang principle also naturally related to a basic question about harmonic measure. Indeed it is well-known and straightforward to show that the Bang-Bang principle follows from the following separation property \([\text{Ken}]\). For any subset \( F \) of \( \partial \Omega \) with positive surface measure there exists \( f \in L^\infty(\partial \Omega) \) with support in \( F \), such that its harmonic extension \( u_f(X) \) verifies \( u(X_0) = 0 \) and \( u(X_1) > 0 \), i.e., \( u \) separates the given points. The existence of such an \( f \) can be shown by contradiction if the domain \( \Omega \) satisfies the boundary unique continuation property. Now let \( \omega_i \) be harmonic measure based at the point \( X_i \). Then the separation property is equivalent to showing that for any real \( \lambda \) the measure \( \omega_0 - \lambda \omega_1 \) cannot vanish on any subset of the boundary that has positive measure.

When \( \Omega \) is convex, the boundary unique continuation property for harmonic functions was established by Adolfsso, Escauriaza and Kenig. Their methods rely on Rellich identities, Carleman estimates and monotonicity formulas for the corresponding frequency function defined up to the boundary. Convexity is needed to prove the monotonicity of the frequency function, which in turn yields an extension of Dahlberg’s result to harmonic functions with variable sign. More precisely the
monotonicity of the -up to the boundary- frequency function of Almgren, gives the following **doubling property**

Let \( u \) be a harmonic function in your domain \( \Omega \) vanishing continuously on \( \Delta_6(Q_0) \) a surface ball of radius 6 centered at \( Q_0 \) a point on the boundary. Then there exists \( M \) possibly depending on \( u \) such that for all \( Q \in \Delta_3(Q_0) \) and \( 0 < r < 2 \) the following holds,

\[
\int_{\Gamma_{2r}(Q)} u^2 dX \leq M \int_{\Gamma_r(Q)} u^2 dX
\]

where \( \Gamma_r(Q) = B_r(Q) \cap \Omega \) ie. the “bite”.

This doubling property is the key to proving the boundary unique continuation property. Once this property is established, the rest of the proof follows from standard arguments and a Carleman type inequality [Ken].

The point is that if the **doubling property** holds then the absolute value of normal derivative of \( u \) is a \( B_2(d\sigma) \)-weight ( this is Dahlberg’s condition ) and this condition implies that the set \( \{ Q \in \Delta_2(Q_0) : \nabla u(Q) = 0 \} \) has zero surface measure unless the normal derivative of \( u \) at the boundary is zero \( (\partial u / \partial N = 0.) \)

Recall that a **nonnegative** function \( w \) is a \( B_2(d\sigma) \)-weight provided that there is a constant \( C \) such that for all \( Q \in \partial D \) and \( r > 0 \) the following holds,

\[
\left( \frac{1}{\sigma(\Delta_r(Q))} \int_{\Delta_r(Q)} w^2 d\sigma \right)^{1/2} \leq C \frac{1}{\sigma(\Delta_r(Q))} \int_{\Delta_r(Q)} w d\sigma
\]

The proofs of the doubling property in [AEK95] and [AE97] rely on making a careful analysis of the frequency function via the Rellich-Neças identity. The crucial step is choosing a vector field \( \beta \) in the Rellich-Neças identity that fits well with the geometry of the domain. The choice is simple in the case of convex domains. By using a smart type of pullback, Adolfsson and Escauriaza managed to reduce the \( C^{1,\alpha} \) case to a variation of the convex case, where the Laplacian is replaced by a more general elliptic operator \( L \) and the domain is convex with respect to \( L \).
the pullback has been carried out, the choice of $\beta$ is again very straightforward.

In this chapter we prove the boundary unique continuation problem for a special class of second order degenerate elliptic operators on convex domains. These operators were first studied in pioneering work by Fabes, Jerison and Kenig [FJK82; FJK83] and by Fabes, Kenig and Serapioni [FKS82]. More recently these operators naturally arose in the work of Caffarelli and Silvestre [CS] in connection to the extension technique characterizing the fractional Laplacian in $\mathbb{R}^n$ as the Dirichlet-to-Neumann map for a variable depending on one more space dimension. Our proof relies on the approach in the proof of boundary unique continuation problem for the Laplace operator on convex domains by Adolffson Escauriaza and Kenig [AEK95] as well as ideas from Riemannian geometry.

4.1 The Problem

Let $U$ be a domain in $\mathbb{R}^d$, and $u$ be a function on $U$. In what follows, we will consider a partial differential operator $L$ of the form

$$Lu(x) := \text{div}(y(x)^a Du(x))$$

(4.1)

where $a \in (-1, 1)$ and $y(x) := \text{dist}(x, \partial U)$. This operator amounts to the Laplace operator when $a = 0$, and in general it is an elliptic operator which is degenerate near the boundary of $U$. In all cases, there is a Riemannian metric associated to $U$ which is highly relevant to the analysis of $Lu$ (this metric being the standard Euclidean metric when $a = 0$). We can define the Riemannian metric on $U$ as

$$g(x) = y(x)^{\frac{2a}{n-2}} I$$
so that the set of $u$ which solve (4.1) are the same as those which solve

$$\Delta_U u = 0$$

where $\Delta_U$ is the Laplace-Beltrami operator, defined in local coordinates as

$$\Delta_U u = \frac{1}{\sqrt{|g|}} \partial_i \left( \sqrt{|g|} g^{ij} \partial_j u \right) = y^{\frac{n}{n-2}} \sum_{i=1}^{n} \partial_i \left( y^{\frac{n}{n-2}} y^{-\frac{2}{n-2}} \partial_i u \right) = y^{\frac{n}{n-2}} \text{div}(y^a Du).$$

With this choice of metric, the definitions for $D_U$ and $\text{div}_U$ are

$$D_U u := g^{ij} \partial_j u = y^{\frac{2n}{n-2}} Du$$
$$\text{div}_U \beta := \frac{1}{\sqrt{|g|}} \partial_i \left( \sqrt{|g|} \beta_i \right) = y^{\frac{n}{n-2}} \partial_i \left( y^{\frac{n}{n-2}} \beta_i \right) = \frac{an(Dy, \beta)}{(n-2)y} + \text{div} \beta.$$

For a set $V$ compactly contained in $U$, the notion of the normal derivative of $u$ to the boundary of $V$ is defined however if $\partial V \cap \partial U \neq \emptyset$ then it may not be defined. We redefine it as follows:

**Definition 11.** For $x \in \partial U$,

$$\partial_L u(x) := \lim_{h \to 0^+} \langle y(x - h\nu)^a Du(x - h\nu), \nu \rangle$$

where $\nu = \nu(x)$ is the exterior unit normal to $\partial U$ at $x$.

Let $U \subset \mathbb{R}^d$, $V \subset \partial U$ relatively open, and $W \subset V$ such that $d\sigma(W) > 0$. We will consider the equation

$$\begin{cases}
Lu = 0 & \text{in } U, \\
u = 0 & \text{on } V, \\
\partial_L u = 0 & \text{on } W.
\end{cases} 
(4.2)$$

The main result proved in this chapter is the following –see Definition 15 for a description of the notion of solution of (4.2) being used.

**Theorem 13.** If $u$ is a solution to (4.2), then $u = 0$. 

90
4.2 The Plan

We follow at the broadest level the approach in [AEK95], an approach which we review as it also gives us the opportunity to lay out the content of each of the following sections.

1. If $U$ is nice enough, $u$ is doubling.

2. If $u$ is doubling along some open set contained in $V$, then $\partial_L u$ satisfies a reverse Hölder inequality on some open set inside $V$.

3. If $\partial_L u$ satisfies a reverse Hölder inequality, then $\partial_L u = 0$ on $V$.

4. Extending $u$ as zero outside $U$, we can use results about unique continuity on the interior to conclude $u$ is zero in all of $U$.

Although the broad strategy of attack is parallel to that in the case $a = 0$, a number of analytical challenges arise when $a \neq 0$, in particular, one must consider geometrically appropriate energy and monotonicity integrands, derive appropriate Poincaré inequalities, and replace tools used in [AEK95] which are not known when $a \neq 0$, such as certain estimates for the Green function. In dealing with all these matters the Riemannian metric introduced in the previous section provides important guidance on what the appropriate integral quantities and analytical tools are are.

4.3 Preliminaries

We first provide a few results regarding the new operator $\partial_L u$ which extend the idea that it plays the role of the normal derivative to $u$ at the boundary of $U$.
Notation 1. $U_{\delta} := \{ x \in U : \text{dist}(x, \partial U) > \delta \}$

Notation 2. For $q \in \mathbb{R}^d$, define $\Gamma_r(q) := U \cap B_r(q)$ and $\Gamma_{r,\delta}(q) := U_{\delta} \cap B_r(q)$.

Notation 3. For $q \in \mathbb{R}^d$, define $\Delta_r(q) := \partial U \cap B_r(q)$ and $\Delta_{r,\delta}(q) := \partial U_{\delta} \cap B_r(q)$.

The following Sobolev space will be important in all what follows.

Definition 12. Let $U$ be a Lipschitz, bounded domain of $\mathbb{R}^n$ and $a \in (-1, 1)$, then with $y$ denoting the distance function to $\partial U$, we consider the space

$$H^1_a(U) := \{ u \in H^1_{\text{loc}}(U) \mid \int_U y^a u^2 \, dx + \int_U y^a |D u|^2 \, dx < \infty \}.$$  

This is a Hilbert space equipped with inner product

$$(u, v) = \int_U y^a uv \, dx + \int_U y^a Du \cdot Dv \, dx.$$  

The dual space to $H^1_a(U)$ will be denoted by $H^{-1}_a(U)$.

Let us discuss the notion of solution of (4.2) we will be working with. First, we have the notion of weak solutions to $Lu = 0$ in $U$, with $L$ as in (4.1).

Definition 13. A function $u \in H^1_a(U)$ is said to be a weak solution of $Lu = 0$ if

$$\int_U y^a Du \cdot Dv \, dx = 0, \ \forall v \in C^1_c(U).$$

It now comes time to discuss the boundary conditions in (4.2) and in what sense they will be understood. There is, as usual, a “trace theorem” for functions in $H^1_a$, we record it (without proof, as it follows standard ideas) in the following Proposition.

Proposition 11. Assume $U$ has a Lipschitz boundary, there is a linear operator $T : H^1_a(U) \to L^2(\partial U)$ such that if $\phi \in H^1_a(U)$ is a continuous in $\overline{U}$ then

$$T \phi = \phi|_{\partial U}.$$
Moreover, \( T \) is bounded, namely, for some constant \( C = C(a, U) \), we have

\[
\|Tu\|_{L^2(\partial U)} \leq C\|u\|_{H^1_a(U)}.
\]

The limit in Definition 11 might not converge for some points \( x \in \partial U \) for a generic \( u \in H^1_a(U) \). However, when \( u \) is a weak solution it is possible to define \( \partial_L u \) not pointwise but as a distribution belonging to the dual \( H^{-1}_a(U) \).

**Definition 14.** Let \( u \) be a weak solution of \( Lu = 0 \), we define the functional \( \partial_L u \in H^{-1}_a(U) \) by

\[
\langle \partial_L u, v \rangle := \int_U y^a(Du, Dv) \, dx, \quad \forall v \in H^1_a(U).
\]

That \( \partial_L u \) is a bounded linear functional in \( H^1_a(U) \) follows immediately by Cauchy-Schwartz.

Using regularity estimates for (4.1), we will prove the following.

**Lemma 12.** Let \( U \) be a convex, \( C^{1,1} \) domain, and let \( u \) be a weak solution in \( U \) (Definition 13) such that \( u \equiv 0 \) on an open set \( V \subset \subset \partial U \). Then, the restriction of \( \partial_L u \) to \( V \) is a continuous function.

*Proof.* We defer the proof until Section 5.3.

This allows to understand (in a weak sense) the co-normal boundary condition in (4.2). With this at hand, we can introduce a notion of weak solution to the problem (4.2).

**Definition 15.** A function \( u \in H^1_a(U) \) is said to be a weak solution of (4.2) if it is a weak solution in the sense of Definition 13, which vanishes in the trace sense on \( V \), and such that \( \partial_L u|_V \) (as given by Lemma 12) vanishes \( \sigma \)-a.e. in \( W \).
Remark 9. With this definition at hand, Theorem 13 simply states that if $u$ solves (4.2) in the sense of Definition 15 then $u$ is identically zero.

There is one last preliminary definition we will need.

Definition 16 (Reverse Hölder). A nonnegative locally integrable (respect to surface measure) function $w : \partial U \to \mathbb{R}$ is said to satisfy the reverse Hölder property provided there is a constant $C$ such that for all $q \in \partial U$ and $r > 0$,

$$\left( \frac{1}{\sigma(\Delta_r(q))} \int_{\Delta_r(q)} w^2 \, d\sigma \right)^{\frac{1}{2}} \leq C \frac{1}{\sigma(\Delta_r(q))} \int_{\Delta_r(q)} w \, d\sigma$$

Remark 10. It is an elementary fact that a function $w$ that satisfies a reverse Hölder property cannot vanish in a set of positive measure unless it is identically zero, indeed, it follows from the fact that such a $w$ must be an $A_p$ weight, from where Lemma 2.4 from [GD11, Chapter IV] applies. This observation will be important in the latter parts of our argument.
DOUBLING AND MONOTONICITY

In this section we carry out the first part of the argument, as outlined in Section 4.2. We will explain the concept of doubling for a solution \( u \) and motivate how monotonicity formulas are useful for proving such a property for \( u \). The ideas in this section and the following one are partly in the spirit of previous works of [GL86; Gut14; AEK95].

**Definition 17** (Doubling Property). A locally square integrable function \( u : U \rightarrow \mathbb{R} \) is said to have the doubling property at a point \( q \) with radius \( r_0 \) if there exists some \( M > 0 \) such that for all \( 0 < r < r_0 \),

\[
\int_{\Gamma_r(q)} y^n u^2 \, dx \leq M \int_{\Gamma_r(q)} y^n u^2 \, dx.
\]

The condition in this definition means (among other things) that as a point \( x \) near \( \partial U \) begins moving away it, \(|u(x)|\) cannot grow neither too quickly nor too slowly. As we are interested in this condition holding uniformly

**Notation 4.** For \( r_0 > 0 \) we will consider the set

\[
V_{r_0} := \{ q \in V \mid \Delta_r(q) \subset V \ \forall \ r < r_0 \}.
\]

It is worth noting that since \( V \) is nonempty, then \( V_{r_0} \) will always be nonempty if \( r_0 \) is sufficiently small.
**Theorem 14.** If $U$ is convex and $u$ solves (4.2), then $u$ has the doubling property at every point $q \in V_{r_0}$ with radius $r_0$, for some small but positive $r_0$.

To motivate the appearance of the monotonicity formula in our approach to Theorem 14, let us make an elementary computation. Without loss of generality, take $q = 0$. Writing the doubling property by foliating $\Gamma_r(Q)$ by spherical shells, we would like to show that there is some $M$ such that

$$\int_0^{2R} \int_{\partial B_r \cap U} y^a u^2 \, d\sigma \, dr \leq M \int_0^R \int_{\partial B_r \cap U} y^a u^2 \, d\sigma \, dr,$$

therefore we consider the function

$$H(r) := \int_{\partial B_r \cap U} y^a u^2 \, d\sigma.$$

If we can show $H(2r) \leq MH(r)$ for some $M$, then we can integrate from 0 to $R$ and (with a minor change of variables) conclude the doubling property. We therefore seek to maximize

$$\frac{H(2r)}{H(r)}.$$

For simplicity (and since it preserves monotonicity) we will take the natural log:

$$\frac{\partial}{\partial r} \log \left( \frac{H(2r)}{H(r)} \right) = \frac{2H'(2r)}{H(2r)} - \frac{H'(r)}{H(r)}.$$

In our case, calculating $H'(r)$ results in some distracting terms, so it helps to see the argument through in the hypothetical case of integrating over a ball inside $U$, and with $a = 0$. In that case we would have

$$H(r) = \int_{\partial B_r} u^2 \, d\sigma = r^{n-1} \int_{\partial B_1} u^2(r \theta) \, d\theta.$$

and therefore

$$H'(r) = (n-1)r^{n-2} \int_{\partial B_1} u^2(r \theta) \, d\theta + r^{n-1} \int_{\partial B_1} 2u(r \theta) \langle Du(r \theta), \theta \rangle \, d\theta$$

$$= \frac{n-1}{r} H(r) + 2 \int_{\partial B_r} u \frac{\partial u}{\partial \nu} \, d\sigma.$$
If we define
\[ G(r) := H'(r) - \frac{n-1}{r}H(r) = 2 \int_{\partial B_r} u \frac{\partial u}{\partial \nu} \, d\sigma \]
then we find that (5.1) becomes
\[ \frac{\partial}{\partial r} \log \left( \frac{H(2r)}{H(r)} \right) = \frac{2G(2r)}{H(2r)} - \frac{G(r)}{H(r)}. \]
so now we consider the monotonicity properties of \( \frac{G(r)}{H(r)} \). If this quotient is monotonic then we would be done, but this is actually stronger than what we need in order to bound \( \frac{H(2r)}{H(r)} \). Suppose there is some \( C \) such that
\[ \int_{R}^{2R} \frac{G(r)}{H(r)} \, dr \leq C \] for all \( R \in (0, 2) \). Then
\[ \frac{G(r)}{H(r)} = \frac{H'(r)}{H(r)} - \frac{n-1}{r} = \frac{\partial}{\partial r} \log \left( \frac{H(r)}{r^{n-1}} \right). \]
Therefore,
\[ \log \left( \frac{H(2R)}{(2R)^{n-1}} \right) - \log \left( \frac{H(R)}{(R)^{n-1}} \right) \leq C \]
\[ \implies \log \left( \frac{H(2R)}{H(R)} \right) \leq C + \log(2^{n-1}) \]
\[ \implies \frac{H(2R)}{H(R)} \leq 2^{n-1}e^C : M \]
Let us consider the case where \( a = 0 \) and \( B_q(r) \subset U \), that is, let us consider harmonic functions away from the boundary. In this case the estimate in (5.2) follows easily from Almgren’s Monotonicity Formula (and this is one of the first situations were the importance of such formulas was recognized). his formula states that for if \( s \) is smaller than the distance from \( q \) to the boundary of \( U \) then the quantity
\[ r \frac{G(r)}{H(r)} \]
is monotone increasing with respect to $r \in (0, s)$. In particular

$$\frac{G(r)}{H(r)} \leq \frac{1}{r} C(s), \quad C(s) = s \frac{G(s)}{H(s)},$$

from where (5.2) follows by integrating both sides.

We will formulate a similar result for our case where $a \neq 0$ and $q \in \partial U$, where

$$H(r) = \int_{(\partial B_r) \cap U} y^a u^2 d\sigma = r^{n-1} \int_{\partial B_1} 1_U(r\theta) y^a(r\theta) u^2(r\theta) d\theta$$

and therefore (see Appendix for a derivation of this formula)

$$H'(r) = \frac{d - 1 + a}{r} \int_{(\partial B_r(Q)) \cap U} y^a u^2 d\sigma + \frac{a}{r} \int_{(\partial B_r(Q)) \cap U} (Dy, x) - y y^a u^2 d\sigma + \int_{(\partial B_r(Q)) \cap U} y^a 2 uu_\nu d\sigma. \quad (5.3)$$

As before, we will define

$$G(r) := H'(r) - \frac{n - 1}{r} H(r).$$

which, by completing the differentiation, becomes

$$G(r) = \int_{(\partial B_r) \cap U} y^a 2 uu_\nu d\sigma + \int_{(\partial B_r) \cap U} ay^{a-1} y_\nu u^2 d\sigma. \quad (5.4)$$

(Note the presence of the second term in (5.4), which distinguishes this case from the previous one.) We still have

$$\frac{\partial}{\partial r} \log \left( \frac{H(2r)}{H(r)} \right) = \frac{2G(2r)}{H(2r)} - \frac{G(r)}{H(r)},$$

and furthermore

$$\frac{G(r)}{H(r)} = \frac{\partial}{\partial r} \log \left( \frac{H(r)}{r^{n-1}} \right)$$

so our previous analysis holds in this case as well. Thus, we are done if we can find a way to bound $\frac{G(r)}{H(r)}$ as described above.
5.1 Extending Almgren’s Monotonicity Formula

In this section and the next we seek to bound $G(r)/H(r)$ by estimating the rate of change of a quantity that is $rG(r)/H(r)$ save for an extra term. First, note that integration by parts yields

$$\int_{\partial B_r \cap U} y^a u u_\nu \, d\sigma = \int_{\Gamma_r} y^a \|Du\|^2 \, dx,$$

where we have used the fact that $\text{div}(y^a Du) = 0$ and $u = 0$ on $\Delta_r$. Define

$$D(r, Q) := \int_{\Gamma_r(Q)} y^a \|Du\|^2 \, dx, \quad H(r, Q) := \int_{\partial B_r(Q) \cap U} y^a u^2 \, d\sigma$$

and

$$N(r, Q) := \frac{r D(r, Q)}{H(r, Q)} \quad \text{if} \quad H(r, Q) \neq 0.$$

Without loss of generality, we will assume $Q = 0$. The situation in [AEK95] is the case when $a = 0$, and they show that $N(r)$ is non-decreasing via a standard argument involving an identity known as the Rellich-Necas identity (see also Lemma 13, for another generalization). This identity is applied to the vector field $\beta(x) = x$, which yields:

$$\text{div}(x \|Du\|^2) = 2 \text{div}(\langle x, Du \rangle Du) + (d - 2) \|Du\|^2 - 2\Delta u \langle x, Du \rangle.$$

In [AEK95], the last term is zero because $u$ is harmonic. Here is where we take advantage of the Riemannian metric to replace this identity with a proper one in the case $a \neq 0$. Indeed, the analogous identity to Rellich-Necas in the general case follows by using the product rule, and it takes the form

$$\text{div}(xy^a \|Du\|^2) = 2 \text{div}(\langle x, Du \rangle y^a Du) + (d - 2)y^a \|Du\|^2 +$$

$$\langle Dy^a, x \rangle \|Du\|^2 - 2 \text{div}(y^a Du) \langle Du, x \rangle.$$
Applying the Leibnitz rule for the divergence, we have
\[
\text{div}(xy^a \|Du\|^2) = \text{div}(xy^a \|Du\|^2 + \langle xy^a, D \|Du\|^2 \rangle) \\
= (d + ay^{-1}(Dy, x))y^a \|Du\|^2 + 2\langle xy^a, D^2uDu \rangle,
\]
\[
\text{div}(\langle x, Du \rangle y^a Du) = y^a \langle D(\langle x, Du \rangle), Du \rangle + \langle x, Du \rangle \text{div}(y^a Du) \\
= y^a \langle Du, Du \rangle + y^a \langle D^2ux, Du \rangle,
\]
where we made use of \(\text{div}(y^a Du) = 0\) to obtain the second identity. We can combine these formulas to obtain a third one,
\[
\text{div}(xy^a \|Du\|^2) = 2\text{div}(\langle x, Du \rangle y^a Du) + (d - 2 + ay^{-1}(Dy, x))y^a \|Du\|^2
\]
Therefore,
\[
\text{div}(xy^a \|Du\|^2) = 2\text{div}(\langle x, Du \rangle y^a Du) + y^a((d - 2 + ay^{-1}(Dy, x)) \|Du\|^2) \quad (5.5)
\]

**Proposition 12.** If \(U\) is a convex domain, \(0 \in \partial U\), and \(a \in (-1, 0)\), we have
\[
D'(r) \geq 2 \int_{\partial B_r \cap U} y^a(u_r)^2 d\sigma + \frac{d - 2 + a}{r} D(r).
\]

**Proof.** Integrating the identity (5.5) over \(\Gamma_r\) we find
\[
\int_{\Gamma_r} \text{div}(xy^a \|Du\|^2) dx = 2 \int_{\Gamma_r} \text{div}(\langle x, Du \rangle y^a Du) dx \\
+ \int_{\Gamma_r} ((d - 2 + ay^{-1}(Dy, x))y^a \|Du\|^2 dx.
\]
Applying the divergence theorem on the integral on the left hand side (using Lemma 12 to make sense of \(y^a Du\) at the boundary) we have that
\[
\int_{\Gamma_r} \text{div}(xy^a \|Du\|^2) dx = \int_{(\partial U) \cap B_r} y^{-a}(\partial_r u)^2 \langle x, \nu \rangle d\sigma \\
+ \int_{(\partial B_r) \cap U} y^a \|Du\|^2 \langle x, \nu \rangle d\sigma.
\]
Since $\partial_L u$ is bounded (again, by Lemma 12), and we are dealing with the case $a < 0$, we have that the integrand on the first integral on the right is identically zero. Then, using that $\langle x, \nu \rangle = 0$ in the second integral, we obtain the formula

$$r \int_{(\partial B_r) \cap U} y^a \|Du\|^2 \, d\sigma = 2 \int_{\Gamma_r} \text{div}(\langle x, Du \rangle y^a Du) \, dx + (d - 2) D(r)$$

$$+ a \int_{\Gamma_r} y^{-1}(Dy, x) y^a \|Du\|^2 \, dx. \tag{5.6}$$

Now, using the convexity of the domain and $0 \in \partial U$, we can show that

$$\langle Dy(x), x \rangle \leq y(x).$$

Again thanks to $a < 0$, we have that $a \langle Dy(x), x \rangle \geq 0$ whenever $\langle Dy(x), x \rangle \leq 0$. Therefore, for all $x$, we have the inequality

$$a \langle Dy(x), x \rangle \geq ay(x).$$

From here, it follows that

$$r \int_{(\partial B_r) \cap U} y^a \|Du\|^2 \, d\sigma \geq 2 \int_{\Gamma_r} \text{div}(\langle x, Du \rangle y^a Du) \, dx + (d - 2 + a) D(r).$$

We now wish to apply the divergence theorem to the integral on the right hand side. We shall show that

$$\int_{\Gamma_r} \text{div}(\langle x, Du \rangle y^a Du) \, dx \geq \int_{(\partial B_r) \cap U} y^a(x, Du) u_\nu \, d\sigma.$$

To see why, we make use again of Lemma 12 and that $y^a Du$ converges to $(\partial_L u)\nu$ at the boundary, so that applying the divergence theorem,

$$\int_{\Gamma_r} \text{div}(\langle x, Du \rangle y^a Du) \, dx = \int_{(\partial U) \cap B_r} y^{-a}(\partial_L u)^2 \langle x, \nu \rangle \, d\sigma$$

$$+ \int_{(\partial B_r) \cap U} \langle \langle x, Du \rangle y^a Du, \nu \rangle \, d\sigma$$

101
Once again, we have by the boundedness of $\partial L u$ on $(\partial U) \cap B_r$ and because $a < 0$ that the first integrand on the right vanishes. In conclusion,

$$\int_{(\partial B_r) \cap U} y^a \|Du\|^2 d\sigma \geq 2 \int_{(\partial B_r) \cap U} y^a (u_\nu)^2 d\sigma + \frac{d - 2 + a}{r} D(r),$$

recalling that $D'(r) = \int_{(\partial B_r) \cap U} y^a \|Du\|^2 d\sigma$, the proposition is proved.

\[\square\]

**Remark 11.** It is work comparing to what is known when $a = 0$. For instance, in a similar computation in [AEK95], some of the terms that vanished above due to $a < 0$ do not vanish (there, the convexity of $U$ also helps in showing these terms have the right sign). On the other hand, the term

$$\frac{2}{r} \int_{\Gamma_r} \langle D(y^a), x \rangle \|Du\|^2 dx$$

is not present when $a = 0$. In our argument for Proposition 12, we used that $a < 0$ and the convexity of $U$ to guarantee that

$$\frac{2}{r} \int_{\Gamma_r} \langle D(y^a), x \rangle \|Du\|^2 dx \geq \frac{2a}{r} \int_{\Gamma_r} y^a \|Du\|^2 dx$$

which is enough for our purposes.

**Theorem 15.** Let $U$ be a convex domain and $a \in (-1,0)$. Then,

$$N(r) \geq -Cr^{-1+\varepsilon}.$$
Now, on one hand, Proposition 12 says that

\[
\frac{D'(r)}{D(r)} \geq \frac{d - 2 + a}{r} + \frac{2 \int_{(\partial B_r) \cap U} y^a (u_\nu)^2 \, d\sigma}{\int_{(\partial B_r) \cap U} y^a u u_\nu \, d\sigma}
\]

and on the other, the formula (5.3) for \( H'(r) \) says that

\[
\frac{H'(r)}{H(r)} \leq \frac{d - 1 + a}{r} + \frac{2 \int_{(\partial B_r) \cap U} y^a u u_\nu \, d\sigma}{\int_{(\partial B_r) \cap U} y^a u^2 \, d\sigma} + \frac{\alpha}{r} \frac{\int_{(\partial B_r) \cap U} y^{-1}((Dy, x) - y) y^a u^2 \, d\sigma}{\int_{(\partial B_r) \cap U} y^a u^2 \, d\sigma}.
\]

Combining these two formulas, applying Cauchy-Schwartz to handle the main two (just as done for the usual Almgren formula, see also [AEK95, Section 2]), we have

\[
\frac{d}{dr} \log(N(r)) \geq \frac{2 \int_{(\partial B_r) \cap U} y^a (u_\nu)^2 \, d\sigma}{\int_{(\partial B_r) \cap U} y^a u u_\nu \, d\sigma} - \frac{2 \int_{(\partial B_r) \cap U} y^a u u_\nu \, d\sigma}{\int_{(\partial B_r) \cap U} y^a u^2 \, d\sigma} - \frac{\alpha}{r} \frac{\int_{(\partial B_r) \cap U} y^{-1}((Dy, x) - y) y^a u^2 \, d\sigma}{\int_{(\partial B_r) \cap U} y^a u^2 \, d\sigma}
\]

\[
\geq -C r^{-1+\varepsilon}.
\]

The last inequality following as in [Gut14, Lemma 3].$

\left\Box\right.

\[5.2\text{ Second term in } G(r)\]

The last thing we need before proving Theorem 14 is controlling the last term in the expression for \( G/H \). That is, we need to show that

\[
\frac{\int_{\partial B_r \cap U} u^2 \frac{\partial (y^a)}{\partial y} \, d\sigma}{\int_{\partial B_r \cap U} u^2 y^a \, d\sigma} \leq F(r), \quad \text{where} \quad \int_R^{2R} F(r) \, dr = C
\]

for some constant \( C \) independent of \( r \).

**Remark 12.** If \( 0 \in \partial U \), and \( x \in U \) and \( \lambda > 1 \) are such that \( \lambda x \in U \). Then,

\[
y(\lambda x) \leq y\lambda(x).
\]
This follows from the convexity of the domain $U$. In fact, if $U$ is convex then $-y(x)$ is a convex function, in which case we have, for any $x_1, x_2 \in U$ and $t \in [0, 1]$.

$$-y(tx_1 + (1-t)x_2) \leq -ty(x_1) - (1-t)y(x_2)$$

Since $0 \in \partial U$, we may take $x_2 = 0$ in which case $y(x_2) = 0$ and thus

$$ty(x_1) \leq y(tx_1)$$

Letting $x_1 = \lambda x$ and $t = 1/\lambda \in (0, 1)$ (since $\lambda > 1$) the desired inequality follows.

We note that, on $\partial B \cap U$,

$$\frac{\partial (y^a)}{\partial \nu} = \frac{1}{r} ay^{a-1}(Dy(x), x).$$

Then, thanks to Remark 12

$$\langle Dy(x), x \rangle = \lim_{h \to 0} \frac{y((1+h)x) - y(x)}{h} \leq y(x).$$

Multiplying the above inequality by $u^2|a|y^{a-1}$, we have

$$u^2 \frac{1}{r} a|y^{a-1}\langle Dy, x \rangle \leq \frac{1}{r} a|u^2 y^a|.$$

Integrating both sides in $x$ over $(\partial B_r) \cap U$, and using again that $-y(x)$ is convex, we have

$$\int_{\partial B_r \cap U} u^2 \frac{\partial (y^a)}{\partial \nu} d\sigma \leq \frac{a}{r}, \quad \text{and} \quad \int_R^{2R} \frac{a}{r} dr = a \log 2.$$

Proof of Theorem 14. By Theorem 15, there is a constant $C$ such that for all $q \in V_{r_0}$ and any pair $r_1, r_2 < r_0$

$$\int_{r_1}^{r_2} \frac{D(s)}{H(s)} ds \leq C.$$

Then, we have (here $r_2 > r_1$)

$$\log \left( \frac{H(r_2)}{r_2^{n-1}} \right) - \log \left( \frac{H(r_1)}{r_1^{n-1}} \right) \leq C.$$
from where it follows that
\[
\log \left( \frac{H(r_2)}{H(r_1)} \right) \leq C + \log \left( \frac{r_2}{r_1} \right)^{n-1}.
\]

Then, in particular for \( r_2 = 2r_1 \), we have
\[
H(r_2) \leq 2^{n-1}e^C H(r_1),
\]
and we conclude \( u \) is doubling at every \( q \in V_{r_0} \) with \( M := 2^{n-1}e^C \).

\[\square\]

5.3 Doubling Implies Reverse Hölder

For the rest of the chapter we are no longer concerned with monotonicity formulas but focused on showing how a solution \( u \) satisfying a doubling property in turn has a co-normal \( \partial_L u \) which is defined pointwise a.e. on the boundary and satisfies a reverse Hölder inequality. The arguments will involve the use of certain barriers, facilitated by the assumption that \( U \) is convex. It is worth contrasting this with the case \( a = 0 \) where is known, for instance as proved in [AEK95, Theorem 1], that for a domain \( U \) Lipschitz and \( u \) satisfying the doubling property then \( \partial_L u \) satisfies a reverse Hölder inequality. Then, the main result we prove in this section is the following Theorem.

**Theorem 16.** Let \( U \) be a convex, \( C^{1,1} \) domain in \( \mathbb{R}^d \), \( d \geq 2 \), \( q_0 \in \partial U \), and let \( u \) have the doubling property for all \( a \in \Delta_3(q_0) \) and satisfy
\[
\begin{align*}
Lu &= 0 \quad \text{in } U, \\
u &= 0 \quad \text{on } \Delta_6(q_0).
\end{align*}
\]
Then there exists a constant $C$ depending on $u$, the Lipschitz character of $U$, and $d$, such that for all $q \in \Delta_2(q_0)$ and $0 < r < 1$

$$
\left( \frac{1}{\sigma(\Delta_r(q))} \int_{\Delta_r(q)} |\partial_L u|^2 \, d\sigma \right)^{\frac{1}{2}} \leq C \frac{1}{\sigma(\Delta_r(q))} \int_{\Delta_r(q)} |\partial_L u| \, dx.
$$

In particular, the absolute value of $\partial_L u$ satisfies the reverse Hölder property when restricted to $\Delta_2(q_0)$.

For $a = 0$, there are several equivalent approaches to this Theorem, one makes use of the explicit bounds available for the Green function, and another makes use of elliptic estimates. Here, we take a road parallel to the latter approach. The proof will invoke the following technical lemmas:

**Lemma 13** (Rellich-Necas). If $\text{div}(y^a Du) = 0$ and $\beta$ is a $C^1$ vector field,

$$
\text{div}(2\langle \beta, y^a Du \rangle y^a Du - y^{2a} \| Du \|^2 \beta) = 2y^{2a} Du^T (D\beta)^T Du - y^{2a} \| Du \|^2 \text{div} \beta.
$$

**Proof.** A useful vector calculus identity for vector fields $f, g,$ and $h$ is as follows:

$$
\text{div}(f \langle g, h \rangle) = \sum_i \left( \frac{\partial}{\partial x_i} f_i \sum_j g_j h_j \right)
$$

$$
= \sum_{i,j} \left( \frac{\partial f_i}{\partial x_i} g_j h_j + f_i \frac{\partial g_j}{\partial x_i} h_j + f_i g_j \frac{\partial h_j}{\partial x_i} \right)
$$

$$
= (\text{div } f) \langle g, h \rangle + f^T (Dg)^T h + f^T (Dh)^T g.
$$

So we have, for any vector field $\beta$,

$$
\text{div}(\beta \langle y^a Du, y^a Du \rangle) = y^{2a} \| Du \|^2 \text{div } \beta + 2\beta^T [D(y^a Du)]^T y^a Du
$$

and

$$
\text{div}(\langle \beta, y^a Du \rangle y^a Du) = \text{div}(y^a Du) \langle \beta, y^a Du \rangle + y^{2a} Du^T (D\beta)^T Du + y^a Du^T [D(y^a Du)]^T \beta.
$$
Therefore, if $\text{div}(y^a Du) = 0$,

$$\text{div}(2\langle \beta, y^a Du \rangle y^a Du - y^{2a} \| Du \|^2 \beta) = 2y^{2a} Du^T (D\beta)^T Du - y^{2a} \| Du \|^2 \text{div} \beta.$$ 

\[ \square \]

The following Lemmas reproduce part of De Giorgi-Nash-Moser theory, and they lead to a $L^2(y^a dx) \to L^\infty$ local estimate (Lemma 16), following De Giorgi’s method. This will require in particular a weighted Sobolev inequality (Lemma 15), for which we invoke standard results from the theory of $A_p$ weights.

**Lemma 14.** Let $u$ be a subsolution, then

$$\int y^a |D(\eta(u - k)_+)|^2 \, dx \leq \int y^a |D\eta|^2 (u - k)_+^2 \, dx.$$ 

**Proof.** Let $\phi = \eta^2 (u - k)_+$ for some $\eta \in C^1_\infty(B_r(q_0))$ and $k \geq 0$. Then

$$0 \geq \int y^a \langle Du, D\phi \rangle \, dx$$

$$= \int y \langle Du, \eta^2 D(u - k)_+ + 2\eta(u - k)_+ D\eta \rangle \, dx.$$ 

Note that $Du = D(u - k)_+$ in the set where $(u - k)_+ \neq 0$, so it follows that

$$\int y^a \langle \eta D(u - k)_+, \eta D(u - k)_+ \rangle + 2\eta(u - k)_+ y^a \langle D(u - k)_+, D\eta \rangle \, dx \leq 0.$$ 

Adding the integral of $y^a |D\eta|^2 (u - k)_+^2$ to the inequality and simplifying the resulting expression on the left, it follows that

$$\int y^a |D(\eta(u - k)_+)|^2 \, dx \leq \int y^a |D\eta|^2 (u - k)_+^2 \, dx.$$ 

\[ \square \]

**Lemma 15.** For $a \in (-1, 1)$ there is a constant $C = C(n, a, U)$ such that if

$$q = \frac{2(n + a)}{n + a - 2},$$

(5.7)
then for any $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$, smooth and with compact support, we have

$$\left( \int y^a |\phi|^q \, dx \right)^{1/q} \leq C \left( \int y^a |D\phi|^2 \, dx \right)^{1/2}.$$  

Proof. We shall apply the theory of $A_p$ weights, following in particular Sawyer and Wheeden [SW92]. The result in [SW92, Theorem 1] guarantees the validity of the inequality once two things have been established:

1) the functions $y^a$ and $y^{-a}$ satisfy the Reverse Doubling condition, and

2) there is a constant $C > 0$ such that for all cubes $Q \subset \mathbb{R}^n$ we have

$$|Q|^\frac{1}{n-1} \left( \int_Q y^a \, dx \right)^{\frac{1}{q}} \left( \int_Q y^{-a} \, dx \right)^{\frac{1}{2}} \leq C.$$  

These two conditions will follow from the following computations. Consider a cube centered at $x_0$ with side length $2r$ (this cube is thus contained in the ball of radius $r\sqrt{n}$ centered at $x_0$. Suppose first that $y(x_0) \leq 2\sqrt{nr}$, then, since $y$ has Lipschitz norm 1, we have

$$y(x) \leq 3\sqrt{nr} \text{ in } Q.$$  

Therefore,

$$\int_Q y(x)^m \, dx \leq C(n, m)|Q|r^m, \text{ for all } m > 0.$$  

On the other hand, if $y(x_0) > 3\sqrt{nr}$ is empty, again since $y$ is 1-Lipschitz, it follows that

$$|y(x) - y(x_0)| \leq r\sqrt{n}, \text{ for all } x \in Q,$$

and this means that for every $x \in Q$

$$y(x) \leq y(x_0) + r\sqrt{n} \leq 2y(x_0),$$

$$y(x) \geq y(x_0) - r\sqrt{n} \geq \frac{1}{2}y(x_0).$$
for all $x \in Q$, and

$$2^{-|m|} y(x_0) \leq y(x)^m \leq 2^{|m|} y(x_0).$$

It follows that for constants $c(n, m)$ and $C(n, m)$ we have in this case.

$$c(n, m)|Q|y(x_0)^m \leq \int_Q y^m \, dx \leq C(n, m)|Q|y(x_0)^m, \forall m.$$ 

In particular, these inequalities with $m = 1$ show that there are constants $\varepsilon, \delta \in (0, 1)$, depending on $n$ and $a$, such that

$$\int_{\delta Q} y^a \, dx \leq \varepsilon \int_Q y^a \, dx, \quad \int_{\delta Q} y^{-a} \, dx \leq \varepsilon \int_Q y^{-a} \, dx.$$ 

(5.8)

In other words, the functions $y^a$ and $y^{-a}$ satisfy a Reverse Doubling condition, that is the first of the two conditions needed for [SW92, Theorem 5]. This goes to show that, if $m > 0$

$$\left( \frac{1}{|Q|} \int_Q y(x)^m \, dx \right)^{\frac{1}{m}} \leq \frac{1}{|Q|} \int_Q y(x) \, dx$$

and, at the same time

$$\left( \frac{1}{|Q|} \int_Q y^a \, dx \right) \left( \frac{1}{|Q|} \int_Q y^{-a} \, dx \right) \leq C.$$ 

The latter inequality tells us that

$$|Q|^{\frac{1}{n} - 1} \left( \int_Q y^a \, dx \right)^{\frac{1}{q}} \left( \int_Q y^{-a} \, dx \right)^{\frac{1}{2}} \leq C |Q|^{\frac{1}{n} - 1} \left( \int_Q y^a \, dx \right)^{\frac{1}{q} - \frac{1}{2}}.$$ 

Applying the former inequality, and using that $q$ given by (5.7), we can show there is a constant $C = C(n, a, U)$ such that

$$|Q|^{\frac{1}{n} - 1} \left( \int_Q y^a \, dx \right)^{\frac{1}{q}} \left( \int_Q y^{-a} \, dx \right)^{\frac{1}{2}} \leq C,$$

which is the second condition we needed to apply [SW92, Theorem 1], and the lemma is proved.
Lemma 16. Let \( u \) be a subsolution. Then

\[
\sup_{B_r(q_0)} u_+ \leq C \left( \frac{1}{(R - r)^{d+a}} \int_{B_R(q_0)} y^a u_+^2 \, dx \right)^{\frac{1}{2}}.
\]

Proof. Let \( M > 0 \) be a constant which will be determined later, and let

\[
M_k := (1 - 2^{-k})M, \quad r_k = r + 2^{-k}(R - r),
\]

so \( M_k \) goes from 0 \( \nearrow \) \( M \) and \( r_k \) goes from \( R \) to \( r \) dyadically, and define a bump function \( \eta_k \in C^1_c(B_r(q_0)) \) such that

\[
\eta_k \equiv 1 \text{ in } B_{r_k+1}, \quad \eta_k \equiv 0 \text{ outside } B_{r_k}, \quad 0 \leq \eta_k \leq 1, \quad |D\eta_k| \leq C \frac{2^k}{R - r}.
\]

Applying Proposition 14, and writing \( u_k := (u - M_k)_+ \), we have

\[
\int y^a |D(\eta_k u_k)|^2 \, dx \leq \int y^a |D\eta_k|^2 u_k^2 \, dx \leq C \frac{2^k}{(R - r)^2} \int_{\text{supp}(\eta_k)} y^a u_k^2 \, dx.
\]

Now, by the Sobolev inequality in Lemma 15, we have

\[
\int y^a (\eta_k u_k)^2 \, dx \leq \left( \int y^a (\eta_k u_k)^q \, dx \right)^{\frac{2}{q}} \left( \int_{\text{supp}(\eta_k u_k)} y^a \, dx \right)^{1-\frac{2}{q}}
\]

\[
\leq C \int y^a |D(\eta_k u_k)|^2 \, dx \left( \int_{\text{supp}(\eta_k u_k)} y^a \, dx \right)^{1-\frac{2}{q}}
\]

\[
\leq C \frac{2^k}{(R - r)^2} \int_{\text{supp}(\eta_k)} y^a u_k^2 \, dx \left( \int_{\text{supp}(\eta_k u_k)} y^a \, dx \right)^{1-\frac{2}{q}}
\]

(5.9)

Then note that

\[
\eta_{k-1} = 1 \text{ on } B_k = \text{supp} \eta_k,
\]

so

\[
\int_{\text{supp} \eta_k} y^a u_k^2 \, dx \leq \int y^a (\eta_{k-1} u_{k-1})^2 \, dx.
\]

For \( x \in \text{supp} u_k, \ u(x) > M_k \), and so

\[
u_{k-1}(x) = (u(x) - M_{k-1})_+ > M_k - M_{k-1} = 2^{-k}M,
\]
thus

$$\int_{\text{supp}(\eta_k u_k)} y^a \, dx \leq \frac{2^{2k}}{M^2} \int y^a (\eta_{k-1} u_{k-1})^2 \, dx.$$ 

Combining these observations with (5.9), we have arrived at the recursive relation

$$A_k \leq C \frac{b^k}{M^{2\delta}(R - r)^2} A_{k-1}^{1 + \delta}$$  \hspace{1cm} (5.10)

where we have

$$A_k := \int y^a (\eta_k u_k)^2 \, dx$$

$$\delta := 1 - \frac{2}{q}$$

$$b := 2^{2\delta + 1}.$$ 

Now we claim

$$A_k \leq C \frac{b^k}{M^{2\delta}(R - r)^2} A_0^{1 + \delta}.$$ 

To see this, note that we can iterate (5.10) to get

$$A_k \leq \left( \frac{C}{M^{2\delta}(R - r)^2} \right) \varphi b^\psi A_0^{(1 + \delta)^k} = \left( \frac{Cb^{\psi/\varphi} A_0^{(1 + \delta)^k/\varphi}}{M^{2\delta}(R - r)^2} \right) \varphi$$

where

$$\varphi := \sum_{i=0}^{k-1} (1 + \delta)^i, \quad \text{and} \quad \psi := \sum_{i=0}^{k} i(1 + \delta)^{k-i}.$$ 

Since $\varphi \to \infty$ as $k \to \infty$, we would like to control the term inside the parentheses above. We calculate

$$\lim_{k \to \infty} \frac{\psi}{\varphi} = 1 \quad \text{and} \quad \lim_{k \to \infty} \frac{(1 + \delta)^k}{\varphi} = \delta,$$

and thus for $k$ sufficiently large we have

$$\left| \frac{Cb^{\psi/\varphi} A_0^{(1 + \delta)^k/\varphi}}{M^{2\delta}(R - r)^2} - \frac{Cb A_0^\delta}{M^{2\delta}(R - r)^2} \right| < \frac{1}{3}.$$
We can take
\[ M := \frac{(3Cb)^{1/\delta} A_0^{1/2}}{(R - r)^{1/\delta}} \implies \left| \frac{CbA_0^\delta}{M^{2\delta}(R - r)^2} \right| < \frac{1}{3}, \]
and thus for all \( k \) sufficiently large we have
\[ \left| \frac{Cb^{\psi/\varphi} A_0^{(1+\delta)k/\varphi}}{M^{2\delta}(R - r)^2} \right| < \frac{2}{3}. \]
Hence, for \( M \) as defined, we have
\[ \lim_{k \to \infty} A_k = 0 \implies (u - M)_+ = 0 \text{ a.e. in } B_r, \]
which proves the lemma, using that \( q = 2(d + a)/(d + a - 2) \) in the previous lemma.

We now have a \( L^2(y^a dx) \to L^\infty \) estimate near the boundary. There is a well known argument in the divergence theory for elliptic equations that this argument can be upgraded to a \( L^1 \to L^\infty \) estimate (for the right weight in the \( L^1 \) norm). This is the content of the next Lemma.

**Lemma 17.**
\[ \|u\|_{L^\infty(\Gamma_r)} \lesssim \frac{1}{r^{d+a}} \int_{\Gamma_{2r}} y^a |u| \, dx \]

**Proof.** The argument follows by combining lemma 16 and the end of the proof for Theorem 4.1 in [HL11]. From 16 we have that if \( u \) is a subsolution then
\[ \sup_{B_{r/2}(q_0)} u_+ \leq C \left( \frac{1}{r^{d+a}} \int_{B_{r}(q_0)} y^a u_+^2 \right)^{1/2}. \]
Note that for any \( R \in (0, 1] \), \( z \in B_{\theta R} \cap \), Then Hölder’s inequality with \( \frac{2}{p} + \frac{p-2}{p} = 1 \) implies, for any \( p \in [2, \infty) \),
\[ \int_{B_{r}(q_0)} y^a u_+^2 \, dx \leq \left( \int_{B_{r}(q_0)} y^a u_+^p \right)^{2/p} \left( \int_{B_{r}(q_0)} y^a \right)^{p-2/p} \]
\[ \lesssim r^{(d+a)(p-2)/p} \left( \int_{B_{1}(q_0)} y^a u_+^p \right)^{2/p}. \]
(where we have used that \( y^a \leq r^a \) on \( B_\gamma(q_0) \) when \( a > 0 \).) Therefore for \( p \in [2, \infty) \) we have

\[
\sup_{B_{r/2}(q_0)} u_+ \leq C \left( \frac{1}{r^{d+a}} \int_{B_\gamma(q_0)} y^a u_+^p \right)^{1/p}.
\]

For \( p \in (0, 2) \) we have

\[
\int_{B_\gamma(q_0)} y^a u_+^2 \leq \| u_+ \|_{L^\infty(B_\gamma(q_0))}^{2-p} \int_{B_\gamma(q_0)} y^a u_+^p
\]

and hence by Young’s inequality with \( \frac{p}{2} + \frac{2-p}{2} = 1 \),

\[
\| u_+ \|_{L^\infty(B_{r/2}(q_0))} \leq C \left( \frac{1}{r^{d+a}} \| u_+ \|_{L^\infty(B_\gamma(q_0))}^{2-p} \int_{B_\gamma(q_0)} y^a u_+^p \right)^{1/2}
\]

\[
\leq \frac{1}{2} \| u_+ \|_{L^\infty(B_\gamma(q_0))} + C' \left( \frac{1}{r^{d+a}} \int_{B_\gamma(q_0)} y^a u_+^p \right)^{1/p}
\]

\[\square\]

**Corollary 4.**

\[
\left( \int_{\Gamma_r(q)} y^a u^2 \, dx \right)^{1/2} \lesssim \frac{1}{r^{(d+a)/2}} \int_{\Gamma_{2r}} y^a |u| \, dx
\]

**Proof.** Direct calculation and an application of Lemma 17 shows

\[
\int_{\Gamma_r(q)} y^a u^2 \, dx \leq \| u \|_{L^\infty(\Gamma_r)} \int_{\Gamma_r(q)} y^a |u| \, dx
\]

\[
\lesssim \frac{1}{r^{d+a}} \int_{\Gamma_{2r}(q)} y^a |u| \, dx \int_{\Gamma_{2r}(q)} y^a |u| \, dx
\]

\[\square\]

Next, we need to understand the behavior of the derivatives of \( u \) near \( V \), showing in particular that that \( u \) is sufficiently regular for \( \partial_L u \) to be defined pointwise on \( V \) (and in particular, prove Lemma 12).

First, let us first review how elliptic theory can be used to show that \( u \) vanishes like \( y^{1-a} \) in \( \Delta_{2r}(q) \). Concretely, we will achieve this via properly constructed barriers
and the comparison principle – this will use the convexity of $U$ in a crucial way. Let $B > 0$ and consider the function

$$h(x) = y(x)^{1-a} - By(x)^{2-a}$$

Observe that $Dh(x) = (1-a)y(x)^{-a}Dy - (2-a)By(x)^{1-a}Dy$, in which case

$$\text{div}(y^a Dh) = (1-a)\text{div}(Dy) - B(2-a)\text{div}(yDy)$$

$$= (1-a)\Delta y - B(2-a)y\Delta y - B(2-a)|Dy|^2$$

$$= (1-a - (2-a)By)\Delta y - B(2-a).$$

Where we have used that $|Dy| = 1$. Since $U$ is convex, we have $\Delta y \leq 0$ everywhere, therefore in the region $\{0 < y < \delta\}$ for sufficiently small $\delta$ we have

$$\text{div}(y^a Dh) \leq -(2-a)B.$$ 

This mean in particular that $h$ is a supersolution to (4.1). By properly choosing a $C^{1,1}$, function $\phi(x)$ which vanishes at $q$ and only depends on $x - y(x)Dy(x)$ (the closest point to $x$ on $\partial U$) and selecting $B$ according to $r$, we can guarantee that $\Phi := h + \phi$ is a supersolution in $\Gamma_r(q)$.

On the other hand, thanks to the growth of $\phi$, we have that $C(r, \phi)\|u\|_{L^\infty}\Phi$ is always above $u$ on $(\partial B_r) \cap U$, since $u \equiv 0$ on $\Delta_r(q)$ and $\Phi$ is a supersolution, it follows from the comparison principle that $u \leq \Phi$ in $\Gamma_r(q)$, the same argument can be repeated with $-u$ instead of $u$. In summary, we have proved the following.

**Lemma 18.** Assume that $U$ is convex and $C^{1,1}$. Let $u \in H^1_a(U)$ be a weak solution of $Lu = 0$ such that $u \equiv 0$ in $\Delta_r(q)$, for some $q \in \partial U$ and $r$ small enough with respect to $U$. Then,

$$|u(x)| \leq C(r)\|u\|_{L^\infty(\Gamma_2(q))}y(x)^{1-a}, \forall \ x \in \Gamma_r(q).$$
Remark 13. The previous lemma when \( a = 0 \) amounts to the well known Lipschitz regularity of harmonic functions that vanish along part of the boundary, when the boundary is convex.

Using the decay in Lemma 18 together with interior estimates for the equation \( \text{div}(y^a Du) \), it is possible to produce a (weighted) pointwise estimate, as well as a continuity estimate, for the gradient.

Corollary 5. Assume that \( U \) is convex and \( C^{1,1} \). Let \( u \in H^1_a(U) \) be a weak solution of \( Lu = 0 \) such that \( u \equiv 0 \) in \( \Delta_{2r}(q) \), for some \( q \in \partial U \) and \( r \) small enough with respect to \( U \). Then \( y^a Du \) is a bounded, uniformly continuous vector field in \( \Gamma_r(q) \), which is parallel to \( \nu \) along \( \Delta_r(q) \).

We are now ready to prove Lemma 12.

Proof of Lemma 12. Let \( v \in H^1_a(U) \) be a smooth function such that \( v_{|\partial U} \) is supported in some compact set \( V' \subset \subset V \). Then, thanks to Corollary 5 the vector field \( X = y^a Du \) is uniformly continuous in \( V'' \cap U \), where \( V'' \) denotes some tubular neighborhood of \( V \) in \( \mathbb{R}^n \). Therefore,

\[
\langle \partial L u, v \rangle = \int_U \langle X, Dv \rangle \, dx = \int_{V'} v \langle X, \nu \rangle \, d\sigma,
\]

by the divergence theorem. Since \( \langle X, \nu \rangle \) is continuous, it follows that \( \partial L u \) as a distribution agrees in \( V \) with \( \langle X, \nu(x) \rangle \), and the lemma is proved.

Lemma 19. For every \( \varepsilon \) there is a universal constant \( C(\varepsilon) \) such that for any \( Lu = 0 \) in \( \Gamma_{2r} \) vanishing continuously on \( \Delta_{2r} \) we have

\[
\int_{\Gamma_r} y^a |u| \, dx \leq C(\varepsilon) r^2 \int_{\Delta_{2r}} |\partial L u| \, d\sigma + \varepsilon \int_{\Gamma_{2r}} y^a |u| \, dx.
\]
Remark 14. Let \( u : B_{2r} \cap D \mapsto \mathbb{R} \), and define
\[
v(x) = u(rx), \quad v : B_2 \cap D_{1/r} \mapsto \mathbb{R}
\]
Where,
\[
D_{1/r} = \{ x \mid rx \in D \}
\]
Then,
\[
\int_{B_{2r} \cap D} \rho_D(x)^a |u(x)| \, dx = \int_{B_2 \cap D_{1/r}} \rho_{D_{1/r}}(x)^a |v(x)| \, dx
\]
\[
\int_{B_{2r} \cap D} \rho_D(x)^a |u(x)| \, dx = \int_{B_1 \cap D_{1/r}} \rho_{D_{1/r}}(x)^a |v(x)| \, dx
\]
\[
r^{-d-a} \int_{B_{2r} \cap \partial D} |\partial_L u(x)| \, d\sigma(x) = \int_{B_1 \cap \partial D_{1/r}} |\partial_{L_{1/r}} v(x)| \, d\sigma(x)
\]
These identities show that if \( C(\varepsilon) \) is a constant such that for all \( v \) we have,
\[
\int_{B_1 \cap D_{1/r}} \rho_{D_{1/r}}(x)^a |v(x)| \, dx \leq C(\varepsilon) \int_{B_2 \cap \partial D_{1/r}} |\partial_{L_{1/r}} v(x)| \, d\sigma(x)
\]
\[
+ \varepsilon \int_{B_2 \cap D_{1/r}} \rho_{D_{1/r}}(x)^a |v(x)| \, dx.
\]
Then, with that same \( C \) we have,
\[
\int_{B_r \cap D} \rho_D(x)^a |u(x)| \, dx \leq C(\varepsilon) r^2 \int_{B_2 \cap \partial D} |\partial_L u(x)| \, d\sigma(x) + \varepsilon \int_{B_2 \cap D} \rho_D(x)^a |u(x)| \, dx.
\]
Proof. According to the previous remark, it suffices to prove the inequality in the case \( r = 1 \). This in turn will follow from an argument by contradiction. If the inequality did not hold in this case, there would be some \( \varepsilon > 0 \) and sequences \( D_n, u_n \) such that:
1) \( u_n \equiv 0 \) on \( \partial D_n \cap B_2 \)
2) For \( y_n(x) := \text{dist}(x, \partial D_n) \), we have
\[
\text{div}(y_n(x)^a D u_n) = 0 \quad \text{in} \quad D_n \cap B_2
\]
3) \( \int_{D_n \cap B_1} y_n(x)^a |u_n| \, dx = 1 \) and
\[
\varepsilon \int_{D_n \cap B_2} |u_n| \, dx + n \int_{\partial D_n \cap B_2} |\partial L_n u| \, d\sigma \leq 1 \tag{5.11}
\]

By compactness of each of the sequences above, there is a subsequence \( n_k \) such that
\[
D_{n_k} \to D \tag{5.12}
\]
\[
u_{n_k} \to u \tag{5.13}
\]

Then, \( u \) solves
\[
div(y(x)^a Du) = 0 \quad \text{in} \quad D \cap B_2 \tag{5.14}
\]
\[
u = 0 \quad \text{on} \quad \partial D \cap B_2 \tag{5.15}
\]
\[
\partial L u = 0 \quad \text{on} \quad \partial D \cap B_2 \tag{5.16}
\]

Moreover,
\[
\int_{D \cap B_1} y^a |u| \, dx = 1 \tag{5.17}
\]

This means that if we extend \( u \) to the rest of \( B_2 \) by zero, and still writing \( y(x) = d(x, \partial D) \), then we have
\[
div(y(x)^a Du) = 0 \quad \text{in} \quad B_2.
\]

Moreover, \( u \) vanishes in an open subset of \( B_2 \) and is also not identically zero, which contradicts unique continuation in the interior (this follows for instance, arguing via Almgren’s formula as in the previous section to show that \( u \) must be doubling, which means that it can only vanish in a set of positive measure of \( B_2 \) if \( u \) is identically zero).

\[\square\]
Lemma 20. Assume $u$ is a solution which is doubling at $q \in \Delta_r(q)$, then, with a constant depending on the doubling constant of $u$, we have for all sufficiently small $r$

$$\int_{\Gamma_r} y^a |u| \, dx \lesssim r^2 \int_{\Delta_{2r}} |\partial_L u| \, d\sigma.$$

Proof. Let us recall that by assumption (see Definition 17) there is a constant $M > 0$ such that

$$\int_{\Gamma_{2r}} y^a u^2 \, dx \leq M \int_{\Gamma_{r}(q)} y^a u^2 \, dx.$$

Apply Lemma 19 with some $\varepsilon \in (0,1)$ (to be determined later), we have

$$\int_{\Gamma_r} y^a |u| \, dx \leq C(\varepsilon)r^2 \int_{\Delta_{2r}} |\partial_L u| \, d\sigma + \varepsilon \int_{\Gamma_{2r}} y^a |u| \, dx.$$

Then, using Hölder’s inequality in the last integral followed by Corollary 4 it follows that

$$\int_{\Gamma_r} y^a |u| \, dx \leq C(\varepsilon)r^2 \int_{\Delta_{2r}} |\partial_L u| \, d\sigma + \frac{C_{RH}\varepsilon}{r^{(n+a)/2}} \left( \int_{\Gamma_{2r}} y^a |u| \, dx \right)^{1/2} \left( \int_{\Gamma_{2r}} y^a \, dx \right)^{1/2}.$$

Since $\int_{\Gamma_{2r}} y^a \, dx \leq C(n,a,U)(2r)^{n+a}$, this becomes

$$\int_{\Gamma_r} y^a |u| \, dx \leq C(\varepsilon)r^2 \int_{\Delta_{2r}} |\partial_L u| \, d\sigma + C'(n,a,U)C_{RH}\varepsilon \left( \int_{\Gamma_{2r}} y^a |u| \, dx \right),$$

and we conclude that choosing $\varepsilon = \varepsilon_0$ where $\varepsilon_0$ is a small constant depending on $n,a,U$ and the constant $C_{RH}$ we have

$$\frac{1}{2} \int_{\Gamma_r} y^a |u| \, dx \leq C(\varepsilon_0)r^2 \int_{\Delta_{2r}} |\partial_L u| \, d\sigma$$

and the lemma is proved.

Proof of Theorem 16. For every point in $\Gamma_{3r}(q_0)$ and any vector field $\beta$ we have, via the Rellich-Necas identity in Lemma 13,

$$\text{div}(2 \langle \beta, y^a Du \rangle y^a Du - y^{2a} \| Du \|^2 \beta) = 2 y^a Du^T (D\beta)^T Du - y^{2a} \| Du \|^2 \text{div} \beta.$$
Integration yields
\[ \int_{\Gamma_{2r}(q)} \text{div}(2\langle \beta, y^a Du \rangle y^a Du - y^{2^a} \| Du \|^2 \beta) \, dx \leq C \int_{\Gamma_{2r}(q)} y^{2^a} \| Du \|^2 \, dx, \] (5.18)
where \( C(\beta) = C_0 \| D\beta \|_{L^\infty} \). Letting \( \beta = \varphi \nu \) where \( \varphi \) is a smooth bump function chosen so that
\[
0 \leq \varphi \leq 1, \quad \| D\varphi \|_{L^\infty} \leq Cr^{-1},
\]
\[
\varphi \equiv 0, \text{ outside } \Gamma_{3r/2}(q),
\]
\[
\varphi \equiv 1, \text{ inside } \Gamma_r(q),
\]
we have that this is equal to
\[
\int_{\Gamma_{2r}(q)} \text{div}(2\langle \beta, y^a Du \rangle y^a Du - y^{2^a} \| Du \|^2 \beta) \, dx = \int_{\Delta_{2r}(q)} \langle X, \nu \rangle \, d\sigma,
\]
where \( X = 2\langle \beta, y^a Du \rangle y^a Du - y^{2^a} \| Du \|^2 \beta \). Since \( \beta = \varphi \nu \), this vector field \( X \) vanishes on \((\partial B_{2r}) \cap U\). Meanwhile, thanks to Lemma 12 along \((\partial U) \cap B_{2r}\) we have
\[
X = 2\langle \beta, (\partial_L u)\nu \rangle (\partial_L u)\nu - (\partial_L u)^2 \beta = (\partial_L u)^2 \varphi(\nu, \nu) = (\partial_L u)^2 \varphi.
\]
Therefore, we may apply the divergence theorem on the left hand side of (5.18), obtaining
\[
\int_{\Delta_{2r}(q)} (\partial_L u)^2 \varphi \, d\sigma \leq C \| D\beta \|_{L^\infty} \int_{\Gamma_{2r}(q)} y^{2^a} \| Du \|^2 \, dx.
\]
Since \( U \) is of class \( C^{1,1} \), for a sufficiently small \( \delta \) we have that \( \nu \) is a Lipschitz vector field in \( \{ x \mid y(x) \leq \delta \} \). Then, and from the derivative bound on \( \varphi \), we have that
\[
\| D\beta \| \leq Cr^{-1}.
\]
Moreover, since \( \varphi \geq 0 \) everywhere and \( \varphi \equiv 1 \) in \( \Delta_{r}(q) \), we conclude that
\[
\int_{\Delta_{r}(q)} |\partial_L u|^2 \, d\sigma \leq C r \int_{\Gamma_{2r}(q)} y^{2^a} \| Du \|^2 \, dx.
\]
By the gradient estimate, there is another constant (still denoted $C$) such that
\[
\frac{1}{r} \int_{\Gamma_{2r}(q)} y^{2a} \|Du\|^2 \, dx \leq C \frac{1}{r^{3-a}} \int_{\Gamma_{3r}} y^a u^2 \, dx.
\]
Now, we can apply the doubling condition of $u$ repeatedly, to obtain ($k$ to be determined)
\[
\frac{1}{r} \int_{\Gamma_{2r}(q)} y^{2a} \|Du\|^2 \, dx \leq C_k \frac{1}{r^{3-a}} \int_{\Gamma_{3r/2k}(q)} y^a u^2 \, dx.
\]
Since Corollary 4 allows us to bound weighted $L^2$ averages by $L^1$ averages, obtaining
\[
\left( \int_{\Delta_r(q)} |\partial_L u|^2 \, d\sigma \right)^{1/2} \leq C_k \frac{1}{r^{(3+d)/2}} \int_{\Gamma_{6r/2k}} y^a |u| \, dx,
\]
while Lemma 20 implies
\[
\left( \int_{\Delta_r(q)} |\partial_L u|^2 \, d\sigma \right)^{1/2} \leq C_k \frac{1}{r^{(d-1)/2}} \int_{\Gamma_{12r/2k}} |\partial_L u| \, dx.
\]
Thus, taking $k = 4$, we conclude that
\[
\left( \frac{1}{\sigma(\Delta_r(q))} \int_{\Delta_r(q)} |\partial_L u|^2 \, d\sigma \right)^{1/2} \leq C \frac{1}{\sigma(\Delta_r(q))} \int_{\Delta_r(q)} |\partial_L u| \, dx,
\]
and the theorem is proved.

With this last theorem in hand, we are ready to prove the main result.

**Proof of Theorem 13.** By assumption, $\partial_L u$ vanishes in a subset of $V \subset \partial U$ with positive surface measure, this fact together with Theorem 16 guarantees, via Remark 10, that $\partial_L u$ vanishes identically in $V$.

This means, in particular, the following: if we extend the function $u$ to $\mathbb{R}^n \setminus (\partial U \setminus V)$ by defining it as zero in $\mathbb{R}^n \setminus U$, then this new function $u$ solves (in the weak sense) in the open set $\mathbb{R}^n \setminus (\partial U \setminus V)$
\[
\text{div}(y^a Du) = 0,
\]
where we emphasize that, outside $U$, we have $y(x) = \text{dist}(x, \partial U)$. Indeed, this follows from the fact that for any test function $\phi$ with compact support in $\mathbb{R}^n \setminus (\partial U \setminus V)$,

$$
\int_{\mathbb{R}^n} y^a \langle Du, D\phi \rangle \, dx = \int_{U} y^a \langle Du, D\phi \rangle \, dx = \langle \partial_L u, \phi \rangle = 0,
$$
the last equality following from the fact that $\partial_L u$ vanishes in $V$ and $\phi$ being identically zero in $\partial U \setminus V$. Since $u$ vanishes in a non-empty open set, unique continuation (in the interior) implies that $u$ must be zero everywhere in $\mathbb{R}^n \setminus (\partial U \setminus V)$, and the theorem is proved. 

\[ \square \]

5.4 Appendix

5.4.1 Cacciopoli’s Inequality

A fundamental fact about weak solutions to divergence form elliptic equations is the Cacciopoli’s inequality. This is so fundamental that this inequality captures all the relevant information about $u$ solving $Lu = 0$ in all that entails the De Giorgi-Nash-Moser $L^\infty$ and Hölder estimates. Cacciopoli’s inequality can be thought of as a reverse Poincaré inequality, while Poincaré’s inequality holds for all functions, and explains how the $L^2$ norm of $u$ in a ball can be controlled by the $L^2$ norm of $Du$, Cacciopoli’s inequality does the opposite, provided $u$ is a weak solution.

**Theorem 17** (Cacciopoli’s Inequality). *If $u \in W^{1,2}(\Omega)$ is a weak solution of $\text{div}(A(x)Du(x)) = 0$ where $A(x) \geq 0$, then $\forall x_0 \in \Omega$, $0 < r < R \leq |x - \partial\Omega|$ we have

$$
\int_{B_r(x_0)} \langle A(x)Du, Du \rangle \, dx \leq \frac{4}{(R-r)^2} \int_{B_R(x_0) \setminus B_r(x_0)} \|A(x)\|u^2 \, dx. \quad (5.19)
$$

121
Proof. Let $\eta \in C_0^\infty(B_R(x_0))$ be such that
\[ \eta(x) = 1 \text{ on } B_r(x_0) \text{ and } |D\eta(x)| \leq \frac{2}{R-r}. \]
Then $\eta^2 u \in H_0^1(B_R(x_0))$, and since $u$ is a weak solution of $\text{div}(A(x)Du) = 0$ we find
\[
0 = \int_{\Omega} \langle A(x)Du, D(\eta^2 u) \rangle \, dx
= \int_{B_R(x_0)} \langle A(x)Du, \eta^2 Du + 2\eta u D\eta \rangle \, dx
= \int_{B_R(x_0)} \eta^2 \langle A(x)Du, Du \rangle + 2\eta u \langle A(x)Du, D\eta \rangle \, dx.
\]
Taking account of the bilinearity of $\langle A(x)\cdot, \cdot \rangle$, we observe that
\[
\langle A(x)D(\eta u), D(\eta u) \rangle = \langle A(x)(\eta Du + uD\eta), \eta Du + uD\eta \rangle
= \eta^2 \langle A(x)Du, Du \rangle
+ 2\eta u \langle A(x)Du, D\eta \rangle + u^2 \langle A(x)D\eta, D\eta \rangle,
\]
and completing the square in the first identity we obtain
\[
\int_{B_R(x_0)} \langle A(x)(D(\eta u)), D(\eta u) \rangle \, dx = \int_{B_R(x_0)} \langle A(x)(D\eta), D\eta \rangle u^2 \, dx.
\]
Moreover since $\eta(x) = 1$ on $B_r(x_0)$, we find that (5.20) implies
\[ D(\eta u) = Du \text{ on } B_r(x_0). \]

Therefore
\[
\int_{B_r(x_0)} \langle A(x)Du, Du \rangle \, dx = \int_{B_r(x_0)} \langle A(x)D(\eta u), D(\eta u) \rangle \, dx
\leq \int_{B_R(x_0)} \langle A(x)D(\eta u), D(\eta u) \rangle \, dx
= \int_{B_R(x_0)} \|A(x)\| \|D\eta\|^2 u^2 \, dx
\leq \frac{4}{(R-r)^2} \int_{B_R(x_0) \setminus B_r(x_0)} \|A(x)\| u^2 \, dx,
\]
as desired. \qed
Cacciopoli’s inequality sometimes appears in the seemingly more general form: for any $\lambda \in \mathbb{R}$,
\begin{equation}
\int_{B_r(x_0)} \langle A(x) Du, Du \rangle \, dx \leq \frac{4}{(R-r)^2} \int_{B_R(x_0) \setminus B_r(x_0)} \|A(x)\| |u - \lambda|^2 \, dx,
\end{equation}
but this follows from (5.19) with $u$ replaced by $u - \lambda$.

### 5.4.2 Calculating $H'(r)$

Let us justify (5.3). We recall that this formula deals with
\[ H(r) := \int_{\partial B_r \cap U} y^a u^2 \, d\sigma, \]
and states that, for all $r < r_0$, where $r_0$ is such that $\Delta r_0(q) \subset V$, we have
\[ H'(r) = \frac{n-1}{r} H(r) + 2 \int_{(\partial B_r) \cap U} y^a uu_\nu \, d\sigma + \int_{(\partial B_r) \cap U} (y^a)_{\nu} u^2 \, d\sigma. \]
Without loss of generality, let us take $q = 0$. We use spherical coordinates, define
\[ S_U(r) := \{ \theta \in S^{n-1} \mid r\theta \in U \}. \]
Then, any $x \in (\partial B_r) \cap U$ can be written as $x = r\theta$, for some $\theta \in S_U(r)$. Using this change of variables, we write
\[ H(r) = r^{n-1} \int_{S_U(r)} y(r\theta)^a u(r\theta)^2 \, d\sigma. \]
Now, the differentiation is straightforward, indeed, we have
\[ H'(r) = \int_{\partial S_U(r)} r^{n-1} y(r\theta)^a u(r\theta)^2 \, d\sigma + \int_{S_U(r)} \frac{\partial}{\partial r} (r^{n-1} y(r\theta)^a u(r\theta)^2) \, d\sigma, \]
where $\partial S_U(r)$ refers to the boundary of $S_U(r)$ as a subset of $S^{n-1}$. Now, by assumption, $u$ vanishes on $(\partial U) \cap B_r$, and $r\theta \in (\partial U) \cap B_r$ whenever $\theta \in \partial S_U(r)$, so
the integrand for the first integral on the left is identically zero, and we end up with

\[
H'(r) = \int_{S_{U(r)}} \frac{\partial}{\partial r} \left( r^{n-1} y(r\theta)^a u(r\theta)^2 \right) \, d\sigma,
\]

\[
= \frac{n-1}{r} \int_{S_{U(r)}} r^{n-1} y(r\theta)^a u(r\theta)^2 \, d\sigma
\]

\[
+ r^{n-1} \int_{S_{U(r)}} y(r\theta)^a \langle (Dy)(r\theta), \theta \rangle u(r\theta)^2 + 2y(r\theta)^a u(r\theta) \langle (Du)(r\theta), \theta \rangle \, d\sigma.
\]

Reverting the change of variables, we obtain (5.3).
BIBLIOGRAPHY


[TW12] Nathan Totz and Sijue Wu. “A rigorous justification of the modulation approximation to the 2D full water wave problem”. In: *Communications in Mathematical Physics* 310.3 (2012), pp. 817–883.

