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Elliptic Curves And Power Residues

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ELLIPTIC CURVES AND POWER RESIDUES

A Dissertation Presented

by

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DEDICATION

To my Mom.

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ABSTRACT

ELLIPTIC CURVES AND POWER RESIDUES

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Let $E_1 \times E_2$ over $\mathbf{Q}$ be a fixed product of two elliptic curves over $\mathbf{Q}$ with complex multiplication. I compute the probability that the pth Fourier coefficient of $E_1 \times E_2$, denoted as $a_p(E_1) + a_p(E_2)$, is a square modulo p. The results are $1/4$, $7/16$, and $1/2$ for different imaginary quadratic fields, given a technical independence of the twists. The similar prime densities for cubes and 4th power are $19/54$, and $1/4$, respectively. I also compute the probabilities without the technical assumption on the twists in various cases.

Next, I consider the sum of quadratic residue of a_p as primes p and elliptic curves vary. The purpose is to test the conjecture that a_p of an elliptic curve is a square modulo p about half of the time across prime numbers so that the sum is expected to be 0. Although the sum turns out to be positively biased, I show, assuming a natural independence result, that the a_p are evenly distributed between squares and non-squares modulo p asymptotically.

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INTRODUCTION

On the margin of the *Arithmetica*, authored by the 3rd century Greek mathematician *Diophantus* of *Alexandria*, Fermat wrote the famous Fermat's last theorem that there are no positive integers satisfying the equation $x^n + y^n = z^n$ for $n > 2$ [1]. The subject of Diophantine equations, whose solutions of interest are integers, plays an enormous part in the development of algebraic number theory. While linear and quadratic equations of two variables, which define curves of genus 0, are well understood with crucial help from Hasse-Minkowski Theorem, the theorem is false for polynomials of higher degree [10].

An elliptic curve is a smooth plane cubic curve of genus 1; such curves play a central role in modern arithmetic geometry. Understanding the arithmetic of elliptic curves is important in the study of curves of higher genus.

Given an elliptic curve E whose defining equation has rational coefficients, for any prime p, we may consider the Fourier coefficient a_p, which is $p+1$ minus the number of points on the elliptic curve over the finite field F_p. The resulting sequence of integers a_2, a_3, a_5,... encodes a great deal of information about the arithmetic of the elliptic curve E.

Specifically, the power residue problem was motivated by the following observation of Ramakrishna.

Proposition 1. *Let K be an imaginary quadratic field of class number one and let m be a prime relatively prime to $\#\mathcal{O}_K^{\times}$. Let $p \equiv 1 \pmod{m}$ be a prime greater than 3 which splits in $K/\mathbf{Q}$ and let K_p^m be the maximal abelian m-extension of K which is unramified away from p. Then p has inertial degree one in $K_p^m/\mathbf{Q}$ if and only if*

$a_p(E)$ is an m^{th} power modulo p, where E is any rational elliptic curve with complex multiplication by K and good reduction at p.

Ramakrishna then raised the following question: are the Fourier coefficients $a_p(E)$ cubes for infinitely many primes $p \equiv 1 \pmod 3$?

Motivated by this question, Weston [12] [13] considered the problem of how often for a fixed elliptic curve E the number a_p is a square modulo p. He conjectured that the probability of this occurring is usually 1/2, but for certain elliptic curves (those with complex multiplication (CM)) he was able to compute the probability and found it to be 1/4, 1/2 or 3/4 in various cases depending on the twist of the elliptic curve, and the number field related to the curve by complex multiplication. He also computed the probability for cubes and higher powers.

I consider a fixed product of two elliptic curves $E_1 \times E_2$ with CM and compute the probability that the number $a_p(E_1) + a_p(E_2)$ is a square modulo p. Given a technical independence of the twists, the probability is 1/4, 7/16, and 1/2 depending on the imaginary quadratic fields. I also compute similar prime density for cubes, and 4th power. The results are 19/54, and 1/4, respectively. Multiple results are also computed for various special twists.

Next, in testing the conjecture that a_p of an elliptic curve is a square modulo p about half of the time across prime numbers, I consider the sum of quadratic residue of a_p when primes p and elliptic curves vary. Note that quadratic residue is 1 if a_p is a square modulo p, and -1 if a_p is a non-square modulo p. Thus, the sum is expected to be 0 if the probability that a_p is a square modulo p is 1/2. I show, assuming a natural independence result, that when varying elliptic curves as well as the prime p, although the sum is positively biased, the a_p are evenly distributed between squares and non-squares modulo p asymptotically.

I now provide background, and the details of the results.

CHAPTER 1

POWER RESIDUES OF FOURIER COEFFICIENTS OF PRODUCT OF TWO ELLIPTIC CURVES

1.1 Background

1.1.1 Residue Symbol

Fix $m > 1$, and a number field K with residue field of order congruent to 1 modulo m. For a prime ideal $\mathfrak{p}$ of K and $\alpha \in \mathcal{O}_K - \mathfrak{p}$, we write $\left(\frac{\alpha}{\mathfrak{p}}\right)_m$ for the m^{th} power residue symbol modulo $\mathfrak{p}$. So $\left(\frac{\alpha}{\mathfrak{p}}\right)_m \in \mu_m$, and

$$\left(\frac{\alpha}{\mathfrak{p}}\right)_m \equiv \alpha^{\frac{N(\mathfrak{p})-1}{m}} \pmod{\mathfrak{p}}.$$

In particular, $\left(\frac{\alpha}{\mathfrak{p}}\right)_m = 1$ if and only if α is a non-zero m^{th} power residue modulo $\mathfrak{p}$. When $m = 2$, we write $\left(\frac{\alpha}{\mathfrak{p}}\right)$ for short.

1.1.2 Hecke Character and Power Residue

Fix a number field K, and let $\mathbf{A}_K$ denotes the adeles of K. A *Hecke character* over a field K is a continuous homomorphism

$$\chi : \mathbf{A}_K^\times / K^\times \to \bar{\mathbf{Q}}^\times.$$

Let $\mathcal{O}_K$ denotes the ring of integer of the number field K, and let $\mathfrak{p}$ be any prime of $\mathcal{O}_K$. The completion $K_p^\times$ of K at $\mathfrak{p}$ is embeded into $\mathbf{A}_K^\times$. Then χ is *unramified* at

$\mathfrak{p}$ if $\chi(\mathcal{O}_{K_p}^\times) = 1$. We write $\chi(\mathfrak{p})$ for the value of χ on any uniformizer of K_p and χ extends to a character on all unramified frational ideals.

Now we fix an imaginary quadratic field $K = \mathbf{Q}(\sqrt{-d})$, where d is the unique square-free integer, with ring of integer $\mathcal{O}_K$, D for the discriminant of K, and w for the order of $\mathcal{O}_K^\times$. Let H denote the Hilbert class field of K. For simplicity we assume that the class number $h = [H : K]$ of K equals 1, so that d is 1,2, or congruent to 3 mod 4. [12]

By [3] and [12], for $d > 3$ we have a unique *Hecke* character

$$\psi : A_K^\times / K^\times \to K^\times,$$

unramified away from D, and $\psi(\mathfrak{p})$ is the unique generator of $\mathfrak{p}$ which is a square modulo $\sqrt{-d}$, for any prime $\mathfrak{p}$ in $\mathcal{O}_K$ relative prime to D.

By Lemma 2.2 in [12], for $d = 1$ (resp. $d = 3$) $\psi(\mathfrak{p})$ is the unique generator of $\mathfrak{p}$ which is congruent to 1 modulo $2+2i$ (resp. modulo 3). For $d = 2$, $\psi(\mathfrak{p})$ is the unique generator of $\mathfrak{p}$ which is congruent to one of $\{1, 3, 5+\sqrt{-2}, 7+\sqrt{-2}, 5+2\sqrt{-2}, 7+2\sqrt{-2}, 5+3\sqrt{-2}, 7+3\sqrt{-2}\}$ modulo $4\sqrt{-2}$.

Then, by [12], for any $\alpha \in \mathbf{Q}^\times$, we define the *Hecke* character ψ_α unramified away from D and α:

$$\psi_\alpha(\mathfrak{p}) = \psi(\mathfrak{p}) \cdot \left(\frac{\epsilon}{\mathfrak{p}}\right)_w \cdot \left(\frac{\alpha}{\mathfrak{p}}\right)_w$$

for any $\mathfrak{p}$ relatively prime to D and α, where we set $\epsilon = 2$ (resp. $\epsilon = -1$, resp. $\epsilon = 1$) for $d = 1$ (resp. $d \equiv 3 \pmod 8$, resp. $d \equiv 7 \pmod 8$ or $d = 2$). Note that we include the extra twist ϵ to simplify the following statements.

By [8] and [12], there exists a cusp form of weight 2 for $\Gamma_1(\mathbf{N}\mathfrak{m})$, where $\mathfrak{m}$ is the conductor of ψ_α, which is an eigenform for the *Hecke* operators T_n with n prime to D and α, written as the Fourier series:

$$g_\alpha := \sum_{(\mathfrak{a},D)=(\mathfrak{a},\alpha)=1} \psi_\alpha(\mathfrak{a}) q^{N\mathfrak{a}},$$

where $\mathfrak{a}$ are prime ideals in $\mathcal{O}_K$ prime to D and α. By [12], the normalized newform associated to g_α:

$$f_\alpha = \sum a_n(f_\alpha) q^n,$$

which has $a_n(f_\alpha) = a_n(g_\alpha)$ for n prime to D and α, has rational Fourier coefficients. Particularly, for p relatively prime to D and α, Lemma 2.1 and 2.2 in [12] shows that

$$a_p(f_\alpha) = \psi_\alpha(\mathfrak{p}) + \psi_\alpha(\bar{\mathfrak{p}}) \in \mathbf{Q}.$$

And, since $\psi_\alpha(\bar{\mathfrak{p}})$ is divisible by $\bar{\mathfrak{p}}$ by [12, Lemma 2.1, and 2.2], for fixed $m \geq 1$ and $\alpha \in \mathbf{Q}^\times$, we have

$$\left(\frac{a_p(f_\alpha)}{p}\right)_m = \left(\frac{\psi_\alpha(\mathfrak{p}) + \psi_\alpha(\bar{\mathfrak{p}})}{\bar{\mathfrak{p}}}\right)_m = \left(\frac{\psi_\alpha(\mathfrak{p})}{\bar{\mathfrak{p}}}\right)_m,$$

for any rational prime $p \equiv 1 \bmod m$, relatively prime to α, and splits as $p = \mathfrak{p}\bar{\mathfrak{p}}$ in $K/\mathbf{Q}$. Then, by [12],

$$\left(\frac{a_p(f_\alpha)}{p}\right)_m = \left(\frac{a_p(E)}{p}\right)_m,$$

where E is a rational elliptic curve corresponding to *Hecke* character ψ over K. We also note that $a_p(E_1 \times E_2) = a_p(E_1) + a_p(E_2)$.

1.1.3 Density

Given a set of positive rational primes $\mathcal{P}$, we define the *zeta function* $\zeta(s;\mathcal{P})$ of $\mathcal{P}$ as in [13]:

$$\zeta(s;\mathcal{P}) = \sum_{p\in\mathcal{P}} p^{-s},$$

which converges for $\mathrm{Re}(s) > 1$, and define the *relative density* $\rho_{\mathcal{P}}(\mathcal{P}')$ of $\mathcal{P}'$ in $\mathcal{P}$ as:

$$\rho_{\mathcal{P}}(\mathcal{P}') = \lim_{s \to 1^+} \frac{\zeta(s; \mathcal{P} \cap \mathcal{P}')}{\zeta(s; \mathcal{P})}$$

(assuming it exists), where $\mathcal{P}$ and $\mathcal{P}'$ are sets of primes and $\mathcal{P}$ has positive density ($\lim_{s \to 1^+} \zeta(s; \mathcal{P})$ diverges).

We introduce some notation for sets of primes.

For a finite Galois extension $K/\mathbf{Q}$ and a union S of conjugacy classes in $\mathrm{Gal}(K/\mathbf{Q})$, we write:

$\mathcal{C}_K^S$ for the set of rational primes p, unramified in $K/\mathbf{Q}$, with Frobenius over K lying in S.

For relatively prime integers a and b, we write:

$\mathcal{C}_b^a$ for the set of primes congruent to a modulo b.

For $t \in \mathbf{Q}^\times$, $m \geq 1$ and $\zeta \in \mu_m$, we write:

$\mathcal{C}_{\sqrt[m]{t}}^{\zeta}$ for the *Chebotarev* set of primes $p \equiv 1 \pmod{m}$ such that $\left(\frac{t}{p}\right)_m$ is conjugate to ζ,

$\mathcal{C}_{\sqrt[m]{t}}^{+}$ (resp. $\mathcal{C}_{\sqrt[m]{t}}^{-}$) for $\mathcal{C}_{\sqrt[m]{t}}^{1}$ (resp. $\mathcal{C}_{\sqrt[m]{t}}^{-1}$).

A set $\mathcal{P}$ is *Chebotarev* if it agrees with some $\mathcal{C}_K^S$ up to finite sets. The absolute density of the set $\mathcal{C}_K^S$ is $\frac{\#S}{[K:\mathbf{Q}]}$.

For any $m \geq 1$ and set of primes $\mathcal{C}$, which is contained in the set of primes congruent to one modulo m, we define

$$\delta_m^1(E_1 \times E_2; \mathcal{C})$$

as the relative density of primes $p \in \mathcal{C}$ for which the p^{th} Fourier coefficient of $E_1 \times E_2$ is an m^{th} power modulo p.

We use the following result from [13, Lemma 1.1] to compute prime densities throughout the paper:

Lemma 1.1.1. *Let K_1, K_2 be finite Galois extensions of $\mathbf{Q}$ and fix subsets $S_i \subseteq$ Gal($K_i/\mathbf{Q}$) stable under conjugation. Then*

$$\rho_{\mathcal{C}_{K_1}^{S_1}}(\mathcal{C}_{K_2}^{S_2}) = \frac{\#\{(\sigma_1, \sigma_2) \subseteq S_1 \times S_2; \sigma_1|_{K_1 \cap K_2} = \sigma_2|_{K_1 \cap K_2}\}}{\#S_1 \cdot [K_2 : K_1 \cap K_2]}$$

Remark 1.1.1. Assumption on Technical Independence of the Twists

To use the above formula, we need specific fields K_1, K_2, so we assume that the twists α_i are those which make K_2 as large as possible, and $K_1 \cap K_2$ as small as possible.

Let $D(\alpha_i)$ denotes the discriminant of the field $K(\sqrt{\alpha_i})$. The twists which cause "trouble" include, but are not necessarily limited to, those with discriminant $D(\alpha_i) \in \{1, D, -4, 4d, \pm 8, \pm 8d\}$.

1.2 Power Residues of Fourier Coefficients of Product of two Elliptic Curves

1.2.1 Theorem Statement and Idea of the Proof

Theorem 1.2.1. *Fix an imaginary quadratic field $K = \mathbf{Q}(\sqrt{-d})$ of class number one. Let $E^d_{\alpha_1} \times E^d_{\alpha_2}$ be a product of two rational elliptic curves with twist α_1, α_2, respectively, and with complex multiplication by the ring of integers of $\mathbf{Q}(\sqrt{-d})$. Fix $m = 2, 3, 4$. Let $\mathcal{C}$ be the Chebotarev set of rational primes p congruent to 1 mod m respectively, relatively prime to α_i, which splits in $K/\mathbf{Q}$. Assuming a technical independence of the twists, we have the relative density of primes p such that $a_p(E^d_{\alpha_1}) + a_p(E^d_{\alpha_2})$ is a m^{th} power residue modulo p as follows:*

$$\delta^1_m(E^d_{\alpha_1} \times E^d_{\alpha_2}; \mathcal{C}) = \begin{cases} 1/4 & m=2,\ d \neq 1,3 \\ 7/16 & m=2,\ d=1 \\ 1/2 & m=2,\ d=3 \\ 19/54 & m=3,\ d=3 \\ 1/4 & m=4,\ d=1 \end{cases}$$

The idea of the proof is that first we find how $a_p(E^d_{\alpha_1})+a_p(E^d_{\alpha_2})$ being, for example, a square modulo p depends on d and α_i. For example, when $p \equiv 3 \bmod 4$, if both α_1, α_2 are squares mod p, and 2 is a square mod p, then $a_p(E^d_{\alpha_1}) + a_p(E^d_{\alpha_2})$ is a square modulo p. Then we apply Lemma 1.1.1 by [13] to translate this case into density language and compute the frequency of primes p for which the above case happens. Summing up all the probability for each case yields the result.

To double check the results using numerical data, I wrote a program in Sage-Math to analyze primes from 20 to 500000 and found, for example, 35.19 percents ($\approx 19/54$) of primes p for which a_p satisfies the cubic residue modulo p condition.

These are the most general cases in which I assume a technical independence of the twists of the elliptic curves. There are special cases in which the probabilities depend on the twists. For example, when $d = 1$, if α_1, $\alpha_2 = \pm 2^k$, where k is an odd integer, then the probability is 1.

1.2.2 Proof for Squares

Lemma 1.2.2. *Let $E^d_{\alpha_1} \times E^d_{\alpha_2}$ be a fixed product of two rational elliptic curves with Hecke character $\psi^d_{\alpha_i}$ over $K = \mathbf{Q}(\sqrt{-d})$ with twist α_1 and α_2 respectively. Let p be a rational prime relatively prime to α_i which splits in $K/\mathbf{Q}$.*

(i) For $d \neq 1, 3$

$$\left(\frac{a_p(E^d_{\alpha_1}) + a_p(E^d_{\alpha_2})}{p}\right) = \begin{cases} \left(\frac{-d}{p}\right)_4 \cdot \left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right) + \left(\frac{\alpha_2}{\mathfrak{p}}\right)}{\bar{\mathfrak{p}}}\right) & p \equiv 1 \ (mod\ 4) \\ \left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right) + \left(\frac{\alpha_2}{\mathfrak{p}}\right)}{\bar{\mathfrak{p}}}\right) & p \equiv 3 \ (mod\ 4) \end{cases}$$

(ii) For $d = 3$

$$\left(\frac{a_p(E^3_{\alpha_1}) + a_p(E^3_{\alpha_2})}{p}\right) = \begin{cases} \left(\frac{-3}{p}\right)_4 \cdot \left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6 + \left(\frac{\alpha_2}{\mathfrak{p}}\right)_6}{\bar{\mathfrak{p}}}\right) & p \equiv 1 \ (mod\ 12) \\ \left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6 + \left(\frac{\alpha_2}{\mathfrak{p}}\right)_6}{\bar{\mathfrak{p}}}\right) & p \equiv 7 \ (mod\ 12) \end{cases}$$

(iii) For $d = 1$

$$\left(\frac{a_p(E^1_{\alpha_1}) + a_p(E^1_{\alpha_2})}{p}\right) = \left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_4 + \left(\frac{\alpha_2}{\mathfrak{p}}\right)_4}{\bar{\mathfrak{p}}}\right)$$

Proof. Noting earlier that $\psi(\mathfrak{p}) \equiv 0 \ (\mathfrak{p})$, we compute

$$\begin{aligned}
\left(\frac{a_p(E^d_{\alpha_1})+a_p(E^d_{\alpha_2})}{p}\right) &= \left(\frac{\psi^d_{\alpha_1}(\mathfrak{p})+\psi^d_{\alpha_1}(\bar{\mathfrak{p}})+\psi^d_{\alpha_2}(\mathfrak{p})+\psi^d_{\alpha_2}(\bar{\mathfrak{p}})}{\bar{\mathfrak{p}}}\right)\\
&= \left(\frac{\psi^d_{\alpha_1}(\mathfrak{p})+\psi^d_{\alpha_2}(\mathfrak{p})}{\bar{\mathfrak{p}}}\right)\\
&= \left(\frac{\psi^d(\mathfrak{p})\cdot\left(\frac{\epsilon}{\mathfrak{p}}\right)_w\cdot\left(\frac{\alpha_1}{\mathfrak{p}}\right)_w+\psi^d(\mathfrak{p})\cdot\left(\frac{\epsilon}{\mathfrak{p}}\right)_w\cdot\left(\frac{\alpha_2}{\mathfrak{p}}\right)_w}{\bar{\mathfrak{p}}}\right)\\
&= \left(\frac{\psi^d(\mathfrak{p})\cdot\left(\frac{\epsilon}{\mathfrak{p}}\right)_w\cdot\left(\left(\frac{\alpha_1}{\mathfrak{p}}\right)_w+\left(\frac{\alpha_2}{\mathfrak{p}}\right)_w\right)}{\bar{\mathfrak{p}}}\right)\\
&= \left(\frac{\psi^d(\mathfrak{p})}{\bar{\mathfrak{p}}}\right)\cdot\left(\frac{\left(\frac{\epsilon}{\mathfrak{p}}\right)_w}{\bar{\mathfrak{p}}}\right)\cdot\left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_w+\left(\frac{\alpha_2}{\mathfrak{p}}\right)_w}{\bar{\mathfrak{p}}}\right)
\end{aligned}$$

(i) For $d \neq 1, 3$, then $\epsilon = 1$ or -1, and $w = 2$, we have:

When $p \equiv 1 \pmod 4$: $\left(\frac{\psi^d(\mathfrak{p})}{\bar{\mathfrak{p}}}\right) = \left(\frac{-d}{p}\right)_4$, and $\left(\frac{\left(\frac{\epsilon}{\mathfrak{p}}\right)}{\bar{\mathfrak{p}}}\right) = 1$, by [12, Lemma 3.1].

When $p \equiv 3 \pmod 4$: $\left(\frac{\psi^d(\mathfrak{p})}{\bar{\mathfrak{p}}}\right) = \epsilon$, and $\left(\frac{\left(\frac{\epsilon}{\mathfrak{p}}\right)}{\bar{\mathfrak{p}}}\right) = \epsilon$, by [12, Lemma 2.1, 2.2, 3.1].

(ii) For $d = 3$, then $\epsilon = -1$, and $w = 6$, we proceed similarly, and use

$$\left(\frac{\left(\frac{\epsilon}{\mathfrak{p}}\right)_6}{\bar{\mathfrak{p}}}\right) = \begin{cases} 1 & p \equiv 1 \pmod{12} \\ \left(\frac{\epsilon}{\mathfrak{p}}\right) & p \equiv 7 \pmod{12} \end{cases}$$

So that we have:

When $p \equiv 1 \pmod{12}$: $\left(\frac{\psi^3(\mathfrak{p})}{\bar{\mathfrak{p}}}\right) = \left(\frac{-3}{p}\right)_4$, and $\left(\frac{\left(\frac{-1}{\mathfrak{p}}\right)_6}{\bar{\mathfrak{p}}}\right) = 1$, by [12, Lemma 3.1].

When $p \equiv 7 \pmod{12}$: $\left(\frac{\psi^3(\mathfrak{p})}{\bar{\mathfrak{p}}}\right) = \epsilon = -1$, and $\left(\frac{\left(\frac{-1}{\mathfrak{p}}\right)_6}{\bar{\mathfrak{p}}}\right) = \left(\frac{-1}{p}\right) = -1$, by [12, Lemma 2.1, 2.2, 3.1].

(iii) For $d = 1$, then $\epsilon = 2$, and $w = 4$, we proceed similarly, and use $\left(\frac{-1}{p}\right)_4 = 1$ for $p \equiv 1 \pmod 8$, and that

$$\left(\frac{\left(\frac{\epsilon}{\mathfrak{p}}\right)_4}{\bar{\mathfrak{p}}}\right) = \begin{cases} 1 & p \equiv 1 \pmod 8 \\ \left(\frac{\epsilon}{\mathfrak{p}}\right) & p \equiv 5 \pmod 8 \end{cases}$$

So that we have:

When $p \equiv 1 \pmod 8$: $\left(\frac{\psi^1(\mathfrak{p})}{\bar{\mathfrak{p}}}\right) = \left(\frac{-1}{p}\right)_4 = 1$, and $\left(\frac{\left(\frac{2}{\mathfrak{p}}\right)_4}{\bar{\mathfrak{p}}}\right) = 1$, by [12, Lemma 3.1].

When $p \equiv 5 \pmod 8$: $\left(\frac{\psi^1(\mathfrak{p})}{\bar{\mathfrak{p}}}\right) = \left(\frac{-1}{p}\right)_4 = -1$, and $\left(\frac{\left(\frac{2}{\mathfrak{p}}\right)_4}{\bar{\mathfrak{p}}}\right) = \left(\frac{2}{p}\right) = -1$, by [12, Lemma 2.1, 2.2, 3.1]. □

For the next Proposition, we let $\mathcal{C} = \mathcal{C}_m^1 \cap \mathcal{C}_{\sqrt{-d}}^+ = \mathcal{C}_{\mathbf{Q}(\zeta_m, \sqrt{-d})}^{\{1\}}$, which is the set of primes congruent to 1 modulo m which splits in $K = \mathbf{Q}\sqrt{-d}$. Since we are considering squares, $m = 2$, so we have $\mathcal{C} = \mathcal{C}_{\mathbf{Q}(\sqrt{-d})}^{\{1\}}$.

Recalling that $\rho_{\mathcal{P}}(\mathcal{P}')$ denotes the *relative density* of $\mathcal{P}'$ in $\mathcal{P}$, we will use the following [13, Lemma 1.1]:

Let K_1, K_2 be finite Galois extensions of $\mathbf{Q}$ and fix subsets $S_i \subseteq Gal(K_i/\mathbf{Q})$ stable under conjugation. Then

$$\rho_{\mathcal{C}_{K_1}^{S_1}}(\mathcal{C}_{K_2}^{S_2}) = \frac{\#\{(\sigma_1, \sigma_2) \subseteq S_1 \times S_2; \sigma_1|_{K_1 \cap K_2} = \sigma_2|_{K_1 \cap K_2}\}}{\#S_1 \cdot [K_2 : K_1 \cap K_2]}$$

with $\mathcal{C}_{K_1}^{S_1} = \mathcal{C} = \mathcal{C}_{\mathbf{Q}(\sqrt{-d})}^{\{1\}}$.

Remark 1.2.1. Let $\alpha = \left(\frac{\alpha_1}{\mathfrak{p}}\right) + \left(\frac{\alpha_2}{\mathfrak{p}}\right)$. We assume $K_2 = \mathbf{Q}(i, \sqrt[4]{-d}, \sqrt{\alpha}, \sqrt[w]{\alpha_1}, \sqrt[w]{\alpha_2})$ is as large as possible, and $K_1 \cap K_2$ is as small as possible. This means that we assume that the twists α_i and their residue sum α do not "collapse" K_2 into a smaller field, or "enlarge" the field $K_1 \cap K_2$. This is the assumed "technical independence of the twists" as in Remark 1.1.1. We will consider special cases in which those conditions are not met in Chapter 2.

Proposition 2. *With the technical independence of the twists, we have the relative density of primes $p \in \mathcal{C}$, the set of odd primes which splits in $\mathbf{Q}\sqrt{-d}$, for which the p^{th} Fourier coefficient of $E^d_{\alpha_1} \times E^d_{\alpha_2}$ is an square power modulo p:*

$$\delta^1_2(E^d_{\alpha_1} \times E^d_{\alpha_2}, \mathcal{C}) = \begin{cases} 1/4 & d \neq 1, 3 \\ 7/16 & d = 1 \\ 1/2 & d = 3 \end{cases}$$

.

Proof. We assume $d \neq 1, 2, 3$, so that $w = 2$. Recall that by the lemma 1.2.2 above, we have the formula

$$\left(\frac{a_p(E^d_{\alpha_1}) + a_p(E^d_{\alpha_2})}{p}\right) = \begin{cases} \left(\frac{-d}{p}\right)_4 \cdot \left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right) + \left(\frac{\alpha_2}{\mathfrak{p}}\right)}{\bar{\mathfrak{p}}}\right) & p \equiv 1 \pmod 4 \\ \left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right) + \left(\frac{\alpha_2}{\mathfrak{p}}\right)}{\bar{\mathfrak{p}}}\right) & p \equiv 3 \pmod 4 \end{cases}$$

For $\left(\frac{a_p(E^d_{\alpha_1}) + a_p(E^d_{\alpha_2})}{p}\right) = 1$, we have the following cases:

Case 1: When $p \equiv 1 \pmod 4$, $\left(\frac{-d}{p}\right)_4 = 1$, and $\left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right) + \left(\frac{\alpha_2}{\mathfrak{p}}\right)}{\bar{\mathfrak{p}}}\right) = \left(\frac{\alpha}{\bar{\mathfrak{p}}}\right) = 1$, the density of the primes satisfying this condition is:

$\rho_{\mathcal{C}}(\mathcal{C}_4^1 \cap \mathcal{C}^+_{\sqrt[4]{-d}} \cap \mathcal{C}^+_{\sqrt{\alpha=2}} \cap \mathcal{C}^+_{\sqrt{\alpha_1}} \cap \mathcal{C}^+_{\sqrt{\alpha_2}})$ (1)

$+$

$\rho_{\mathcal{C}}(\mathcal{C}_4^1 \cap \mathcal{C}^+_{\sqrt[4]{-d}} \cap \mathcal{C}^+_{\sqrt{\alpha=-2}} \cap \mathcal{C}^-_{\sqrt{\alpha_1}} \cap \mathcal{C}^-_{\sqrt{\alpha_2}})$ (2).

Note that (1) means the density of primes which is 1 mod 4, and for which $-d$ is a 4-*th* power mod p, α_1, α_2, and 2 is a square mod p. Now since $\mathcal{C}_4^1 \cap \mathcal{C}^+_{\sqrt[4]{-d}} \cap \mathcal{C}^+_{\sqrt{\alpha=2}} \cap \mathcal{C}^+_{\sqrt{\alpha_1}} \cap \mathcal{C}^+_{\sqrt{\alpha_2}} = \mathcal{C}^{\{1\}}_{\mathbf{Q}(i, \sqrt[4]{-d}, \sqrt{\alpha}, \sqrt{\alpha_1}, \sqrt{\alpha_2})}$, we can use Lemma 1.1.1 by [13] with $\mathcal{C}^{S_2}_{K_2} = \mathcal{C}^1_{\mathbf{Q}(i, \sqrt[4]{-d}, \sqrt{\alpha=2}, \sqrt{\alpha_1}, \sqrt{\alpha_2})}$. Together with $\mathcal{C}^{S_1}_{K_1} = \mathcal{C} = \mathcal{C}^{\{1\}}_{\mathbf{Q}(\sqrt{-d})}$ and $K_1 \cap K_2 = \mathbf{Q}(\sqrt{-d})$, we have density (1) as follows:

$$\begin{aligned}\rho_{\mathcal{C}}(\mathcal{C}_4^1 \cap \mathcal{C}^+_{\sqrt[4]{-d}} \cap \mathcal{C}^+_{\sqrt{\alpha=2}} \cap \mathcal{C}^+_{\sqrt{\alpha_1}} \cap \mathcal{C}^+_{\sqrt{\alpha_2}}) &= \frac{1}{[\mathbf{Q}(i, \sqrt[4]{-d}, \sqrt{2}, \sqrt{\alpha_1}, \sqrt{\alpha_2}) : \mathbf{Q}(\sqrt{-d})]} \\ &= \frac{1}{32}.\end{aligned}$$

Similarly, we also have density (2), $\rho_{\mathcal{C}}(\mathcal{C}_4^1 \cap \mathcal{C}^+_{\sqrt[4]{-d}} \cap \mathcal{C}^+_{\sqrt{\alpha=-2}} \cap \mathcal{C}^-_{\sqrt{\alpha_1}} \cap \mathcal{C}^-_{\sqrt{\alpha_2}}) = \frac{1}{32}$.

Case 2: When $p \equiv 1 \pmod 4$, $\left(\frac{-d}{p}\right)_4 = -1$, and $\left(\frac{\alpha}{\bar{\mathfrak{p}}}\right) = -1$, the density of the primes satisfying this condition is:

$\rho_{\mathcal{C}}(\mathcal{C}_4^1 \cap \mathcal{C}^-_{\sqrt[4]{-d}} \cap \mathcal{C}^-_{\sqrt{\alpha=2}} \cap \mathcal{C}^+_{\sqrt{\alpha_1}} \cap \mathcal{C}^+_{\sqrt{\alpha_2}})$ (3)

$+$

$\rho_{\mathcal{C}}(\mathcal{C}_4^1 \cap \mathcal{C}^-_{\sqrt[4]{-d}} \cap \mathcal{C}^-_{\sqrt{\alpha=-2}} \cap \mathcal{C}^-_{\sqrt{\alpha_1}} \cap \mathcal{C}^-_{\sqrt{\alpha_2}})$ (4).

We proceed similarly and also get a density of $2 \cdot \frac{1}{32}$.

Case 3: When $p \equiv 3 \pmod 4$, and $\left(\frac{\alpha}{\bar{\mathfrak{p}}}\right) = 1$, the density of the primes satisfying this condition is:

$\rho_{\mathcal{C}}(\mathcal{C}_4^3 \cap \mathcal{C}^+_{\sqrt{\alpha=2}} \cap \mathcal{C}^+_{\sqrt{\alpha_1}} \cap \mathcal{C}^+_{\sqrt{\alpha_2}})$ (5)

$+$

$\rho_{\mathcal{C}}(\mathcal{C}_4^3 \cap \mathcal{C}^+_{\sqrt{\alpha=-2}} \cap \mathcal{C}^-_{\sqrt{\alpha_1}} \cap \mathcal{C}^-_{\sqrt{\alpha_2}})$ (6).

We proceed similarly with $K_2 = \mathbf{Q}(i, \sqrt{\alpha}, \sqrt{\alpha_1}, \sqrt{\alpha_2})$, and and $K_1 \cap K_2 = \mathbf{Q}$, so the density when $p \equiv 3 \pmod 4$ is $2 \cdot \frac{1}{16}$.

Thus, when $d \neq 1, 2, 3$, we have $\delta_2^1(E^d_{\alpha_1} \times E^d_{\alpha_2}, \mathcal{C}) = 4 \cdot \frac{1}{32} + 2 \cdot \frac{1}{16} = \frac{1}{4}$.

For $d = 2$, the proof is similar. Note that $\left(\frac{-8}{p}\right) = 1$ if and only if $p \equiv 1$ or $3 \pmod 8$, so we only consider $p \equiv 1$ or $3 \pmod 8$.

For $p \equiv 1 \pmod 8$, $K_2 = \mathbf{Q}(\zeta_8, \sqrt[4]{-2}, \sqrt{\alpha_1}, \sqrt{\alpha_2})$ (note that $\sqrt{2} = \zeta_8 + \frac{1}{\zeta_8}$), and $K_1 \cap K_2 = \mathbf{Q}(\sqrt{-2})$, so density (1) = density (2) = $\frac{1}{16}$. And using the fact that

$$\left(\frac{2}{\bar{\mathfrak{p}}}\right) = \begin{cases} 1 & p \equiv 1, 7 \pmod 8 \\ -1 & p \equiv 3, 5 \pmod 8 \end{cases}$$

and that

$$\left(\frac{-1}{\bar{\mathfrak{p}}}\right) = \begin{cases} 1 & p \equiv 1 \pmod 4 \\ -1 & p \equiv 3 \pmod 4 \end{cases}$$

we have $\left(\frac{\alpha}{\bar{\mathfrak{p}}}\right) = \left(\frac{\pm 2}{\bar{\mathfrak{p}}}\right) = 1$ for $p \equiv 1 \bmod 8$, so density (3) = density (4) = 0.

For $p \equiv 3 \pmod 8$, $K_2 = \mathbf{Q}(\zeta_8, \sqrt{\alpha_1}, \sqrt{\alpha_2})$, and $K_1 \cap K_2 = \mathbf{Q}(\sqrt{-2})$, we easily have density (6) = $\frac{1}{8}$. Density (5) requires 2 is a square mod p, but 2 is not square for $p \equiv 3 \bmod 8$, so density (5) = 0. Thus, when $d = 2$, $\delta_2^1(E_{\alpha_1}^2 \times E_{\alpha_2}^2, \mathcal{C}) = \frac{1}{4}$.

For $d = 1$, we have $w = 4$. Recall that by Lemma 1.2.2 above, we have

$$\left(\frac{a_p(E_{\alpha_1}^d) + a_p(E_{\alpha_2}^d)}{p}\right) = \left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_4 + \left(\frac{\alpha_2}{\mathfrak{p}}\right)_4}{\bar{\mathfrak{p}}}\right)$$

We notice that, for $b = 1, 2, 3, 4$

$$\left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_4 + \left(\frac{\alpha_2}{\mathfrak{p}}\right)_4}{\bar{\mathfrak{p}}}\right) = \begin{cases} \left(\frac{2}{\bar{\mathfrak{p}}}\right) \cdot \left(\frac{i}{\bar{\mathfrak{p}}}\right)^b & \text{(4 of 16 cases)} \\ \left(\frac{1+i}{\bar{\mathfrak{p}}}\right) \cdot \left(\frac{i}{\bar{\mathfrak{p}}}\right)^b & \text{(8 of 16 cases), (2 for each } b\text{)} \\ 0 & \text{(4 of 16 cases)} \end{cases}$$

Using the fact that

$$\left(\frac{i}{\bar{\mathfrak{p}}}\right) = \begin{cases} 1 & p \equiv 1 \pmod 8 \\ \\ -1 & p \equiv 5 \pmod 8 \end{cases}$$

we have for $p \equiv 1 \pmod 8$

$$\left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_4 + \left(\frac{\alpha_2}{\mathfrak{p}}\right)_4}{\bar{\mathfrak{p}}}\right) = \begin{cases} \left(\frac{2}{\bar{\mathfrak{p}}}\right) = 1 & \text{(4 cases)} \\ \left(\frac{1+i}{\bar{\mathfrak{p}}}\right) & \text{(8 cases)} \end{cases}$$

So the density for $p \equiv 1 \pmod 8$ is
$\left(4 \cdot \frac{1}{[\mathbf{Q}(\zeta_8, \sqrt[4]{\alpha_1}, \sqrt[4]{\alpha_2}) : \mathbf{Q}(i)]} + 8 \cdot \frac{1}{[\mathbf{Q}(\zeta_8, \sqrt[4]{\alpha_1}, \sqrt[4]{\alpha_2}, \sqrt{1+i}) : \mathbf{Q}(i)]}\right) = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$.

Note that $\mathbf{Q}(\zeta_8, \sqrt[4]{\alpha_1}, \sqrt[4]{\alpha_2}, \sqrt{2}) = \mathbf{Q}(\zeta_8, \sqrt[4]{\alpha_1}, \sqrt[4]{\alpha_2})$ since $\sqrt{2} = \zeta_8 + \frac{1}{\zeta_8}$. We define $\sqrt{1+i}$ as $\zeta_{16}\sqrt[4]{2} = \sqrt[4]{2i}$, and note that $2i$ is a square mod p.

For $p \equiv 5 \pmod 8$, we have

$$\left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_4 + \left(\frac{\alpha_2}{\mathfrak{p}}\right)_4}{\bar{\mathfrak{p}}}\right) = \begin{cases} \left(\frac{2}{\bar{\mathfrak{p}}}\right) & \text{(for b = 2,4)} \\ -\left(\frac{2}{\bar{\mathfrak{p}}}\right) & \text{(for b = 1,3)} \\ \left(\frac{i+1}{\bar{\mathfrak{p}}}\right) & \text{(for b = 2,4), (2 cases for each b)} \\ -\left(\frac{i+1}{\bar{\mathfrak{p}}}\right) & \text{(for b = 1,3), (2 cases or each b)} \end{cases}$$

Using the fact that

$$\left(\frac{\pm 2}{\bar{\mathfrak{p}}}\right) = \begin{cases} 1 & p \equiv 1 \pmod 8 \\ \\ -1 & p \equiv 5 \pmod 8 \end{cases}$$

We have the density for $p \equiv 5 \pmod 8$ is
$\left(2 \cdot \frac{1}{[\mathbf{Q}(\zeta_8, \sqrt[4]{\alpha_1}, \sqrt[4]{\alpha_2}) : \mathbf{Q}(i)]} + 8 \cdot \frac{1}{[\mathbf{Q}(\zeta_8, \sqrt[4]{\alpha_1}, \sqrt[4]{\alpha_2}, \sqrt{1+i}) : \mathbf{Q}(i)]}\right) = 2 \cdot \frac{1}{2^5} + 8 \cdot \frac{1}{2^6} = \frac{3}{16}$.

Adding up the density for $d=1$, we have $\delta_2^1(E_{\alpha_1}^1 \times E_{\alpha_2}^1, \mathcal{C}) = \frac{1}{4} + \frac{3}{16} = \frac{7}{16}$.

For $d=3$, we have $w=6$. Recall that by Lemma 1.2.2 above, we have

$$\left(\frac{a_p(E_{\alpha_1}^d) + a_p(E_{\alpha_2}^d)}{p}\right) = \begin{cases} \left(\frac{-3}{p}\right)_4 \cdot \left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6 + \left(\frac{\alpha_2}{\mathfrak{p}}\right)_6}{\bar{\mathfrak{p}}}\right) & p \equiv 1 \pmod{12} \\ \left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6 + \left(\frac{\alpha_2}{\mathfrak{p}}\right)_6}{\bar{\mathfrak{p}}}\right) & p \equiv 7 \pmod{12} \end{cases}$$

We notice that, for $b = 1,2,3,4,5,6$

$$\left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6 + \left(\frac{\alpha_2}{\mathfrak{p}}\right)_6}{\bar{\mathfrak{p}}}\right) = \begin{cases} \left(\frac{\zeta_6}{\bar{\mathfrak{p}}}\right)^b & \text{(12 of 36 cases)} \\ \left(\frac{2}{\bar{\mathfrak{p}}}\right) \cdot \left(\frac{\zeta_6}{\bar{\mathfrak{p}}}\right)^b & \text{(6 of 36 cases)} \\ \left(\frac{\sqrt{-3}}{\bar{\mathfrak{p}}}\right) \cdot \left(\frac{\zeta_6}{\bar{\mathfrak{p}}}\right)^b & \text{(12 of 36 cases)} \\ 0 & \text{(6 of 36 cases)} \end{cases}$$

Using the fact that

$$\left(\frac{\zeta_6}{\bar{\mathfrak{p}}}\right) = \begin{cases} 1 & p \equiv 1 \pmod{12} \\ -1 & p \equiv 7 \pmod{12} \end{cases}$$

and that $\left(\frac{-3}{p}\right)_4 = \left(\frac{\sqrt{-3}}{\bar{\mathfrak{p}}}\right)$ for $p \equiv 1 \pmod{12}$.

We have for $p \equiv 1 \pmod{12}$

$$\left(\frac{-3}{p}\right)_4 \cdot \left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6 + \left(\frac{\alpha_2}{\mathfrak{p}}\right)_6}{\bar{\mathfrak{p}}}\right) = \begin{cases} \left(\frac{-3}{p}\right)_4 \cdot 1 & \text{(12 cases)} \\ \left(\frac{-3}{p}\right)_4 \cdot \left(\frac{2}{\bar{\mathfrak{p}}}\right) & \text{(6 cases)} \\ \left(\frac{-3}{p}\right)_4 \cdot \left(\frac{\sqrt{-3}}{\bar{\mathfrak{p}}}\right) & \text{(12 cases)} \end{cases}$$

And the density for $p \equiv 1 \pmod{12}$ is

$$12 \cdot \frac{1}{[\mathbf{Q}(\zeta_{12}, \sqrt[6]{\alpha_1}, \sqrt[6]{\alpha_2}, \sqrt[4]{-3}) : \mathbf{Q}(\sqrt{-3})]} + 2 \cdot 6 \cdot \frac{1}{[\mathbf{Q}(\zeta_{12}, \sqrt[6]{\alpha_1}, \sqrt[6]{\alpha_2}, \sqrt[4]{-3}, \sqrt{2}) : \mathbf{Q}(\sqrt{-3})]} + \frac{1}{6} = \frac{1}{12} + \frac{1}{24} + \frac{1}{6} = \frac{7}{24}$$

Note that $\sqrt{-3} = 2 \cdot \zeta_{12}^4 + 1$.

For $p \equiv 7 \pmod{12}$, we have

$$\left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6 + \left(\frac{\alpha_2}{\mathfrak{p}}\right)_6}{\bar{\mathfrak{p}}} \right) = \begin{cases} (-1)^b & \text{(12 cases)} \\ (-1)^b \cdot \left(\frac{2}{\bar{\mathfrak{p}}}\right) & \text{(6 cases)} \\ (-1)^b \cdot \left(\frac{\sqrt{-3}}{\bar{\mathfrak{p}}}\right) & \text{(12 cases)} \end{cases}$$

We have the density for $p \equiv 7 \pmod{12}$ is

$$\frac{1}{2} \cdot 12 \cdot \frac{1}{[\mathbf{Q}(\zeta_{12}, \sqrt[6]{\alpha_1}, \sqrt[6]{\alpha_2}) : \mathbf{Q}(\sqrt{-3})]} + 6 \cdot \frac{1}{[\mathbf{Q}(\zeta_{12}, \sqrt[6]{\alpha_1}, \sqrt[6]{\alpha_2}, \sqrt{2}) : \mathbf{Q}(\sqrt{-3})]} + 12 \cdot \frac{1}{[\mathbf{Q}(\zeta_{12}, \sqrt[6]{\alpha_1}, \sqrt[6]{\alpha_2}, \sqrt[4]{-3}) : \mathbf{Q}(\sqrt{-3})]} = \frac{1}{12} + \frac{1}{24} + \frac{1}{12} = \frac{5}{24}$$

Adding up the density for $d = 3$, we have $\delta_2^1(E_{\alpha_1}^3 \times E_{\alpha_2}^3, \mathcal{C}) = \frac{7}{24} + \frac{5}{24} = \frac{1}{2}$. □

1.2.3 Proof for Cubes

For the next Proposition, we let $\mathcal{C} = \mathcal{C}_m^1 \cap \mathcal{C}_{\sqrt{-d}}^+ = \mathcal{C}_{\mathbf{Q}(\zeta_m, \sqrt{-d})}^{\{1\}}$, which is the set of primes congruent to 1 modulo m which splits in $K = \mathbf{Q}(\sqrt{-d})$. Since we are considering cubes, $m = 3$, and $K = \mathbf{Q}(\sqrt{-3})$. Note that $\sqrt{-3} = 2 \cdot \zeta_3 + 1$. So we have $\mathcal{C} = \mathcal{C}_3^1 \cap \mathcal{C}_{\sqrt{-3}}^+ = \mathcal{C}_{\mathbf{Q}(\zeta_3, \sqrt{-3})}^{\{1\}} = \mathcal{C}_{\mathbf{Q}(\zeta_3)}^{\{1\}}$.

Recalling that $\rho_{\mathcal{P}}(\mathcal{P}')$ denotes the *relative density* of $\mathcal{P}'$ in $\mathcal{P}$, we will use the following [13, Lemma 1.1]:

Let K_1, K_2 be finite Galois extensions of $\mathbf{Q}$ and fix subsets $S_i \subseteq Gal(K_i/\mathbf{Q})$ stable under conjugation. Then

$$\rho_{\mathcal{C}_{K_1}^{S_1}}(\mathcal{C}_{K_2}^{S_2}) = \frac{\#\{(\sigma_1, \sigma_2) \subseteq S_1 \times S_2; \sigma_1|_{K_1 \cap K_2} = \sigma_2|_{K_1 \cap K_2}\}}{\#S_1 \cdot [K_2 : K_1 \cap K_2]}$$

with $\mathcal{C}_{K_1}^{S_1} = \mathcal{C} = \mathcal{C}_{\mathbf{Q}(\zeta_3)}^{\{1\}}$.

Remark 1.2.2. Let $\alpha = \left(\frac{\alpha_1}{\mathfrak{p}}\right)_6 + \left(\frac{\alpha_2}{\mathfrak{p}}\right)_6$. We assume $K_2 = \mathbf{Q}(\zeta_9, \sqrt[6]{\alpha_1}, \sqrt[6]{\alpha_2}, \sqrt[3]{\alpha})$ is as large as possible, and $K_1 \cap K_2$ is as small as possible. This is the assumed "technical independence of the twists" as in Remark 1.1.1. We will consider special cases in which those conditions are not met in Chapter 2.

Recall that the notation

$$\delta_m^1(E_1 \times E_2; \mathcal{C})$$

denotes the relative density of primes $p \in \mathcal{C}$ for which the p^{th} Fourier coefficient of $E_1 \times E_2$ is an m^{th} power modulo p.

Proposition 3. *With the assumption above, we have* $\delta_3^1(E_{\alpha_1}^3 \times E_{\alpha_2}^3, \mathcal{C}) = \frac{19}{54}$

Proof. Let π be a prime divisor of p which is congruent to 1 modulo 3. For $d = 3$, we have $w = 6$, and $\epsilon = -1$. Using the fact that $\left(\frac{\psi(\pi)}{\bar{\pi}}\right)_3 = \left(\frac{\pi}{\bar{\pi}}\right)_3 = 1$, and $\left(\frac{\left(\frac{-1}{\pi}\right)_6}{\bar{\pi}}\right)_3 = 1$ by [12, Proposition 4.1], we have

$$\left(\frac{a_p(E_{\alpha_1}^3) + a_p(E_{\alpha_2}^3)}{p}\right)_3 = \left(\frac{\psi(\pi)}{\bar{\pi}}\right)_3 \cdot \left(\frac{\left(\frac{-1}{\pi}\right)_6}{\bar{\pi}}\right)_3 \cdot \left(\frac{\left(\frac{\alpha_1}{\pi}\right)_6 + \left(\frac{\alpha_2}{\pi}\right)_6}{\bar{\pi}}\right)_3$$
$$= \left(\frac{\left(\frac{\alpha_1}{\pi}\right)_6 + \left(\frac{\alpha_2}{\pi}\right)_6}{\bar{\pi}}\right)_3$$

We notice that, for $b = 1, 2, 3, 4, 5, 6$

$$\left(\frac{\left(\frac{\alpha_1}{\pi}\right)_6 + \left(\frac{\alpha_2}{\pi}\right)_6}{\bar{\pi}} \right)_3 = \begin{cases} \left(\frac{\zeta_6}{\bar{\pi}}\right)_3^b & \text{(12 of 36 cases)} \\ \left(\frac{2}{\bar{\pi}}\right)_3 \cdot \left(\frac{\zeta_6}{\bar{\pi}}\right)_3^b & \text{(6 of 36 cases)} \\ \left(\frac{\sqrt{-3}}{\bar{\pi}}\right)_3 \cdot \left(\frac{\zeta_6}{\bar{\pi}}\right)_3^b & \text{(12 of 36 cases)} \\ 0 & \text{(6 of 36 cases)} \end{cases}$$

Using the fact that

$$\left(\frac{\zeta_6}{\bar{\pi}}\right)_3 = \begin{cases} 1 & p \equiv 1 \pmod 9 \\ \zeta_3^2 & p \equiv 4 \pmod 9 \\ \zeta_3 & p \equiv 7 \pmod 9 \end{cases}$$

we have for $p \equiv 1 \pmod 9$

$$\left(\frac{\left(\frac{\alpha_1}{\pi}\right)_6 + \left(\frac{\alpha_2}{\pi}\right)_6}{\bar{\pi}} \right) = \begin{cases} 1 & \text{(12 cases)} \\ \left(\frac{2}{\bar{\pi}}\right)_3 & \text{(6 cases)} \\ \left(\frac{\sqrt{-3}}{\bar{\pi}}\right)_3 & \text{(12 cases)} \end{cases}$$

And the density for $p \equiv 1 \pmod 9$ is

$$12 \cdot \frac{1}{[\mathbf{Q}(\zeta_9, \sqrt[6]{\alpha_1}, \sqrt[6]{\alpha_2}) : \mathbf{Q}(\zeta_3)]} + 6 \cdot \frac{1}{[\mathbf{Q}(\zeta_9, \sqrt[6]{\alpha_1}, \sqrt[6]{\alpha_2}, \sqrt[3]{2}) : \mathbf{Q}(\zeta_3)]}$$
$$+ 12 \cdot \frac{1}{[\mathbf{Q}(\zeta_9, \sqrt[6]{\alpha_1}, \sqrt[6]{\alpha_2}, , \sqrt[6]{-3}) : \mathbf{Q}(\zeta_3)]}$$
$$= \frac{1}{9} + \frac{1}{54} + \frac{1}{27} = \frac{1}{6}$$

Note that $\mathbf{Q}(\zeta_9, \sqrt[6]{\alpha_1}, \sqrt[6]{\alpha_2}, \zeta_6) = \mathbf{Q}(\zeta_9, \sqrt[6]{\alpha_1}, \sqrt[6]{\alpha_2})$ since $\zeta_6 = \zeta_9^3 + 1$.

For $p \equiv 4 \pmod 9$, we have

$$\left(\frac{\left(\frac{\alpha_1}{\pi}\right)_6 + \left(\frac{\alpha_2}{\pi}\right)_6}{\bar{\pi}}\right) = \begin{cases} (\zeta_3^2)^b & \text{(12 cases)} \\ \left(\frac{2}{\bar{\pi}}\right)_3 \cdot (\zeta_3^2)^b & \text{(6 cases)} \\ \left(\frac{\sqrt{-3}}{\bar{\pi}}\right)_3 \cdot (\zeta_3^2)^b & \text{(12 cases)} \end{cases}$$

We have the density for $p \equiv 4 \pmod 9$ is

$$\frac{1}{3} \cdot 12 \cdot \frac{1}{[\mathbf{Q}(\zeta_9, \sqrt[6]{\alpha_1}, \sqrt[6]{\alpha_2}) : \mathbf{Q}(\zeta_3)]} + 6 \cdot \frac{1}{[\mathbf{Q}(\zeta_9, \sqrt[6]{\alpha_1}, \sqrt[6]{\alpha_2}, \sqrt[3]{2}) : \mathbf{Q}(\zeta_3)]}$$
$$+ 12 \cdot \frac{1}{[\mathbf{Q}(\zeta_9, \sqrt[6]{\alpha_1}, \sqrt[6]{\alpha_2}, , \sqrt[6]{-3}) : \mathbf{Q}(\zeta_3)]}$$
$$= \frac{1}{27} + \frac{1}{54} + \frac{1}{27} = \frac{5}{54}$$

The density for $p \equiv 7 \pmod 9$ is similar. Adding up the density for $d = 3$, we have $\delta_3^1(E_{\alpha_1}^3 \times E_{\alpha_2}^3, \mathcal{C}) = \frac{1}{6} + 2 \cdot \frac{5}{54} = \frac{19}{54}$. □

1.2.4 Proof for Fourth Powers

In this setting, let E_α^1 denotes the elliptic curve $y^2 = x^3 - \alpha x$ with complex multiplication by $\mathbf{Z}[i]$.

Again assuming the technical independence of the twists as in Remark 1.1.1, I will show that the density of primes $p \equiv 1 \bmod 4$, split in $\mathbf{Q}(i)$, for which the p^{th} Fourier coefficients a_p is a fourth power, is $\frac{1}{4}$.

We let $K = \mathbf{Q}(i)$. Let $\mathcal{C} = \mathcal{C}_4^1 \cap \mathcal{C}_i^+ = \mathcal{C}_{\mathbf{Q}(i)}^1$, which denotes the set of primes 1 mod 4 which splits in $\mathbf{Q}(i)$.

Lemma 1.2.3. $\left(\frac{a_p(E_{\alpha_1}^1 \times E_{\alpha_2}^1)}{\pi}\right)_4 = \left(\frac{\bar{\pi}}{\pi}\right)_4 \left(\frac{\left(\frac{\alpha_1}{\pi}\right)_4 + \left(\frac{\alpha_2}{\pi}\right)_4}{\pi}\right)_4$

Proof. By [9], for $p \equiv 1 \bmod 4$ and relative prime to α_i, we have $a_p(E^1_{\alpha_i}) = \left(\frac{\alpha_i}{\bar\pi}\right)_4 \pi + \left(\frac{\alpha_i}{\pi}\right)_4 \bar\pi$ where $p = \pi\bar\pi$ with π, $\bar\pi$ primary irreducibles in $\mathbf{Z}[i]$. Note that $a_p(E^1_{\alpha_i}) = 0$ otherwise. So we have $a_p(E^1_{\alpha_1} \times E^1_{\alpha_2}) = \pi\left(\left(\frac{\alpha_1}{\bar\pi}\right)_4 + \left(\frac{\alpha_2}{\bar\pi}\right)_4\right) + \bar\pi\left(\left(\frac{\alpha_1}{\pi}\right)_4 + \left(\frac{\alpha_2}{\pi}\right)_4\right)$. So, $a_p(E^1_{\alpha_1} \times E^1_{\alpha_2}) \equiv \bar\pi\left(\left(\frac{\alpha_1}{\bar\pi}\right)_4 + \left(\frac{\alpha_2}{\bar\pi}\right)_4\right) \pmod{\pi}$, and the result follows. □

By [13], I considered primes 5 mod 8, and the four subsets which partition primes 1 mod 8. Those are sets of prime 1 mod 8 of which the primary divisors are congruent to one of $\{1, 1+4i, 5, 5+4i\}$ mod 8. Primary divisors are those congruent to 1 mod $(2+2i)$. Those four subsets were denoted $\mathcal{G}_8^1, \mathcal{G}_8^{1+4i}, \mathcal{G}_8^5$, and $\mathcal{G}_8^{5+4i}$.

We see that

$$\left(\frac{\left(\frac{\alpha_1}{\pi}\right)_4 + \left(\frac{\alpha_2}{\pi}\right)_4}{\pi}\right)_4 = \begin{cases} \left(\frac{2}{\pi}\right)_4 \left(\frac{i}{\pi}\right)_4^b & b = 1,2,3,4 \\ \left(\frac{\pm i \pm 1}{\pi}\right)_4 & \text{(8 cases)} \\ 0 & \text{(4 cases)} \end{cases}$$

Therefore we compute

Table 1.1. Key Values in Fourth Power Formula

	$\left(\frac{\bar\pi}{\pi}\right)_4$	$\left(\frac{2}{\pi}\right)_4$	$\left(\frac{i}{\pi}\right)_4$	note
$\mathcal{G}_8^1$	1	1	1	$p \equiv 1 \bmod 16$
$\mathcal{G}_8^{1+4i}$	-1	-1	1	$p \equiv 1 \bmod 16$
$\mathcal{G}_8^5$	-1	1	-1	$p \equiv 9 \bmod 16$
$\mathcal{G}_8^{5+4i}$	1	-1	-1	$p \equiv 9 \bmod 16$
$\mathcal{C}_8^5$	$\pm i$	$\pm i$	$\pm i$	

Using the residues in the table above by [13], Lemma 1.1.1 by [12], we compute the density for $p \equiv 1 \bmod 8$ as follows:

$$4 \cdot \frac{1}{2} \cdot \frac{1}{[\mathbf{Q}(\zeta_{16}, \sqrt[4]{\alpha_1}, \sqrt[4]{\alpha_2}) : \mathbf{Q}(i)]} + 8 \cdot \frac{1}{2} \cdot \frac{1}{[\mathbf{Q}(\zeta_{16}, \sqrt[4]{\alpha_1}, \sqrt[4]{\alpha_2}, \sqrt[4]{i+1}) : \mathbf{Q}(i)]}$$
$$+ 4 \cdot \frac{1}{2} \cdot \frac{1}{[\mathbf{Q}(\zeta_{16}, \sqrt[4]{\alpha_1}, \sqrt[4]{\alpha_2}) : \mathbf{Q}(i)]}$$
$$+ 2 \cdot \frac{1}{2} \cdot \frac{1}{[\mathbf{Q}(\zeta_{16}, \sqrt[4]{\alpha_1}, \sqrt[4]{\alpha_2}) : \mathbf{Q}(i)]}$$
$$+ 2 \cdot \frac{1}{2} \cdot \frac{1}{[\mathbf{Q}(\zeta_{16}, \sqrt[4]{\alpha_1}, \sqrt[4]{\alpha_2}) : \mathbf{Q}(i)]} + 8 \cdot \frac{1}{2} \cdot \frac{1}{[\mathbf{Q}(\zeta_{16}, \sqrt[4]{\alpha_1}, \sqrt[4]{\alpha_2}, \sqrt[4]{i+1}) : \mathbf{Q}(i)]}$$
$$= \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{3}{64} = \frac{10}{64}$$

Note that we define $\sqrt[4]{1+i} = \sqrt[8]{2}\zeta_{32} = \sqrt[8]{2i}$ and $2i$ is 4-th power mod p when $p \in \mathcal{G}_8^1 \cup \mathcal{G}_8^{5+4i}$.

When $p \equiv 5 \bmod 8$, by [13], we have $\left(\frac{\bar{\pi}}{\pi}\right)_4 = \left(\frac{2}{\pi}\right)_4 \left(\frac{2}{a}\right)(-1)$, where a is the real part of π. So we have

$$\left(\frac{\bar{\pi}}{\pi}\right)_4 \left(\frac{\left(\frac{\alpha_1}{\pi}\right)_4 + \left(\frac{\alpha_2}{\pi}\right)_4}{\pi}\right)_4 = \begin{cases} \left(\frac{\bar{\pi}}{\pi}\right)_4 \left(\frac{2}{\pi}\right)_4 \left(\frac{i}{\pi}\right)_4^b & (b = 1, 2, 3, 4) \\ \left(\frac{\bar{\pi}}{\pi}\right)_4 \left(\frac{\pm i \pm 1}{\pi}\right)_4 & (8 \text{ cases}) \\ 0 & (4 \text{ cases}) \end{cases}$$
$$= \begin{cases} \left(\frac{2}{\pi}\right)_4 \left(\frac{2}{a}\right)(-1) \left(\frac{2}{\pi}\right)_4 \left(\frac{i}{\pi}\right)_4^b & (b = 1, 2, 3, 4) \\ \left(\frac{2}{\pi}\right)_4 \left(\frac{2}{a}\right)(-1) \left(\frac{\pm i \pm 1}{\pi}\right)_4 & (8 \text{ cases}) \\ 0 & (4 \text{ cases}) \end{cases}$$
$$= \begin{cases} \left(\frac{2}{a}\right) \left(\frac{i}{\pi}\right)_4^b & (b = 1, 2, 3, 4) \\ \left(\frac{2}{\pi}\right)_4 \left(\frac{2}{a}\right) \left(\frac{\pm i \pm 1}{\pi}\right)_4 & (8 \text{ cases}) \\ 0 & (4 \text{ cases}) \end{cases}$$

Note that $\left(\frac{2}{a}\right) = \begin{cases} 1 & p \equiv 5 \pmod{16} \\ -1 & p \equiv 13 \pmod{16} \end{cases}$

So for $p \equiv 5$ mod 16, the density is

$$\frac{1}{[\mathbf{Q}(\zeta_{16}, \sqrt[4]{\alpha_1}, \sqrt[4]{\alpha_2}) : \mathbf{Q}(i)]} + 8 \cdot 2 \cdot \frac{1}{[\mathbf{Q}(\zeta_{16}, \sqrt[4]{\alpha_1}, \sqrt[4]{\alpha_2}, \sqrt[4]{2}, \sqrt[4]{1+i}) : \mathbf{Q}(i)]} = \frac{3}{64}$$

The density for $p \equiv 13$ mod 16 is similarly $3/64$.

Thus, under the assumption on the technical independence of the twist as in Remark 1.1.1, the density of primes $p \equiv 1$ mod 4, split in $\mathbf{Q}(i)$, for which the p^{th} Fourier coefficients a_p is a fourth power is $\frac{1}{4}$. Special cases considered without the above assumption are computed in Chapter 2.

CHAPTER 2

POWER RESIDUE SPECIAL CASES

2.1 Special cases for Squares

Next we do cases when K_2 is not maximal or $K_1 \cap K_2$ is not minimal by varying α_1 and α_2. First let us introduce a lemma which is helpful in some cases.

Lemma 2.1.1. *Let $\alpha_3 = \alpha_1/\alpha_2$, and $d \neq 1$, then $\delta_2^1(E_{\alpha_1}^d \times E_{\alpha_2}^d, \mathcal{C})$ only depends on $-d$ and α_3 for $p \equiv 1 \pmod 4$, and on α_3 and the quadratic residue of α_2 for $p \equiv 3 \pmod 4$; for $d = 1$, $\delta_2^1(E_{\alpha_1}^d \times E_{\alpha_2}^d, \mathcal{C})$ depends on α_3 and the quadratic residue of α_2.*

Proof. For $d \neq 1$, we have

$$\left(\frac{a_p(E^d_{\alpha_1}) + a_p(E^d_{\alpha_2})}{p}\right) = \begin{cases} \left(\frac{-d}{p}\right)_4 \cdot \left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_w + \left(\frac{\alpha_2}{\mathfrak{p}}\right)_w}{\bar{\mathfrak{p}}}\right) & \text{p} \equiv 1 (\text{mod } 4) \\ \left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_w + \left(\frac{\alpha_2}{\mathfrak{p}}\right)_w}{\bar{\mathfrak{p}}}\right) & \text{p} \equiv 3 (\text{mod } 4) \end{cases}$$

$$= \begin{cases} \left(\frac{-d}{p}\right)_4 \cdot \left(\frac{\left(\frac{\alpha_2}{\mathfrak{p}}\right)_w}{\bar{\mathfrak{p}}}\right) \cdot \left(\frac{\left(\frac{\alpha_3}{\mathfrak{p}}\right)_w + 1}{\bar{\mathfrak{p}}}\right) & \text{p} \equiv 1 (\text{mod } 4) \\ \left(\frac{\left(\frac{\alpha_2}{\mathfrak{p}}\right)_w}{\bar{\mathfrak{p}}}\right) \cdot \left(\frac{\left(\frac{\alpha_3}{\mathfrak{p}}\right)_w + 1}{\bar{\mathfrak{p}}}\right) & \text{p} \equiv 3 (\text{mod } 4) \end{cases}$$

$$= \begin{cases} \left(\frac{-d}{p}\right)_4 \cdot \left(\frac{\left(\frac{\alpha_3}{\mathfrak{p}}\right)_w + 1}{\bar{\mathfrak{p}}}\right) & \text{p} \equiv 1 (\text{mod } 4) \\ \left(\frac{\alpha_2}{\mathfrak{p}}\right) \cdot \left(\frac{\left(\frac{\alpha_3}{\mathfrak{p}}\right)_w + 1}{\bar{\mathfrak{p}}}\right) & \text{p} \equiv 3 (\text{mod } 4) \end{cases}$$

For $d = 1$, $\left(\frac{a_p(E^d_{\alpha_1}) + a_p(E^d_{\alpha_2})}{p}\right) = \left(\frac{\alpha_2}{\mathfrak{p}}\right) \cdot \left(\frac{\left(\frac{\alpha_3}{\mathfrak{p}}\right)_4 + 1}{\bar{\mathfrak{p}}}\right)$ similarly.

Note that we used the fact that for $w = 4, 6$ in the case $d = 1, 3$ respectively

$$\left(\frac{\left(\frac{\alpha_2}{\mathfrak{p}}\right)_w}{\bar{\mathfrak{p}}}\right) = \begin{cases} 1 & \text{if } p \equiv 1 \pmod{2w} \\ \left(\frac{\alpha_2}{\mathfrak{p}}\right) & \text{if } p \equiv 1 + w \pmod{2w} \end{cases}$$

□

2.1.1 When $\mathbf{d \geq 7}$:

Let $D(\alpha_i)$ denotes the discriminant of $K(\sqrt{\alpha_i})$. We recognize differences among 3 sets of values of α_i

Set 1 (squares): $D(\alpha_i) \in \{1, D\}$, such that $\left(\frac{\alpha_i}{\mathfrak{p}}\right) = 1$.

Set 2: (negative set 1): $D(\alpha_i) \in \{-4, 4d\}$ such that

$$\left(\frac{\alpha_i}{\mathfrak{p}}\right) = \begin{cases} 1 & \text{if } p \equiv 1 \pmod 4 \\ -1 & \text{if } p \equiv 3 \pmod 4 \end{cases}$$

Set 3: $\alpha_i \in \{\pm 2\}$, or $D(\alpha_i) = \pm 8$.

Set 4: other cases.

For the following cases involving the above sets, we compute $\delta_2^1(E^d_{\alpha_1} \times E^d_{\alpha_2}, \mathcal{C})$:

Case 1: α_1 and $\alpha_2 \in$ Set 1: 1/2

Case 2: α_1 and $\alpha_2 \in$ Set 2: 1/2

Case 3: $\alpha_1, \alpha_2 = \pm 2 : 1/4$ or $3/4$

Case 4: $\alpha_1 = \alpha_2$, α_1 and $\alpha_2 \in$ Set 4: 1/2

Case 5: $\alpha_1 \in$ Set 1, and $\alpha_2 \in$ Set 2: 1/4

Case 6: $\alpha_1 \in$ Set 1, and $\alpha_2 \in$ Set 3: 1/8 or 3/8.

Case 7: $\alpha_1 \in$ Set 1, and $\alpha_2 \in$ Set 4: 1/4

Case 8: $\alpha_1 \in$ Set 2, and $\alpha_2 \in$ Set 3: 1/8 or 3/8

Case 9: $\alpha_1 \in$ Set 2, and $\alpha_2 \in$ Set 4: 1/4

Case 10: $\alpha_1 \in$ Set 3, and $\alpha_2 \in$ Set 4: 1/8 or 3/8

Case 11: $\alpha_1/\alpha_2 = \alpha_3 \in$ Set 1: 1/2

Case 12: $\alpha_1/\alpha_2 = \alpha_3 \in$ Set 2: 1/4

Case 13: $\alpha_1/\alpha_2 = \alpha_3 \in$ Set 3: 1/4

Proof. **Case 1**: α_1 and $\alpha_2 \in$ Set 1: 1/2

Recall that by Lemma 1.2.2, we have

$$\left(\frac{a_p(E^d_{\alpha_1})+a_p(E^d_{\alpha_2})}{p}\right)=\begin{cases}\left(\frac{-d}{p}\right)_4\cdot\left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)+\left(\frac{\alpha_2}{\mathfrak{p}}\right)}{\bar{\mathfrak{p}}}\right) & p\equiv 1 \pmod 4\\ \left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)+\left(\frac{\alpha_2}{\mathfrak{p}}\right)}{\bar{\mathfrak{p}}}\right) & p\equiv 3 \pmod 4\end{cases}$$

Now that $\left(\frac{\alpha_i}{\mathfrak{p}}\right)=1$. We have

$$\left(\frac{a_p(E^d_{\alpha_1})+a_p(E^d_{\alpha_2})}{p}\right)=\begin{cases}\left(\frac{-d}{p}\right)_4\cdot\left(\frac{2}{\bar{\mathfrak{p}}}\right) & p\equiv 1 \pmod 4\\ \left(\frac{2}{\bar{\mathfrak{p}}}\right) & p\equiv 3 \pmod 4\end{cases}$$

So the density is

$$\begin{aligned}2\cdot\frac{1}{[\mathbf{Q}(i,\sqrt{\alpha_1},\sqrt{\alpha_2},\sqrt[4]{-2},\sqrt{2}):\mathbf{Q}(\sqrt{-d})]}&+\frac{1}{[\mathbf{Q}(i,\sqrt{\alpha_1},\sqrt{\alpha_2},\sqrt{2}):\mathbf{Q}]}\\ &=2\cdot\frac{1}{[\mathbf{Q}(i,\sqrt[4]{-2},\sqrt{2}):\mathbf{Q}(\sqrt{-d})]}+\frac{1}{[\mathbf{Q}(i,\sqrt{2}):\mathbf{Q}]}=\frac{1}{2}.\end{aligned}$$

Case 2: α_1 and $\alpha_2\in$ Set 2: 1/2

Now that

$$\left(\frac{\alpha_i}{\mathfrak{p}}\right)=\begin{cases}1 & \text{if } p\equiv 1 \pmod 4\\ -1 & \text{if } p\equiv 3 \pmod 4\end{cases}$$

We have

$$\left(\frac{a_p(E^d_{\alpha_1})+a_p(E^d_{\alpha_2})}{p}\right)=\begin{cases}\left(\frac{-d}{p}\right)_4\cdot\left(\frac{2}{\bar{\mathfrak{p}}}\right) & \text{if } p\equiv 1 \pmod 4\\ \left(\frac{-2}{\bar{\mathfrak{p}}}\right) & \text{if } p\equiv 3 \pmod 4\end{cases}$$

Similar to Case 1, the density is 1/2.

Case 3: $\alpha_1, \alpha_2 = \pm 2 : 1/4$ or $3/4$

If $\alpha_1 = \alpha_2$, then

$$\left(\frac{a_p(E^d_{\alpha_1}) + a_p(E^d_{\alpha_2})}{p}\right) = \begin{cases} \left(\frac{-d}{p}\right)_4 \cdot \left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right) + \left(\frac{\alpha_2}{\mathfrak{p}}\right)}{\bar{\mathfrak{p}}}\right) & p \equiv 1 \pmod 4 \\ \left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right) + \left(\frac{\alpha_2}{\mathfrak{p}}\right)}{\bar{\mathfrak{p}}}\right) & p \equiv 3 \pmod 4 \end{cases}$$

$$= \begin{cases} \left(\frac{-d}{p}\right)_4 \cdot \left(\frac{\left(\frac{2}{\mathfrak{p}}\right) \cdot \left(\frac{\alpha_2}{\mathfrak{p}}\right)}{\bar{\mathfrak{p}}}\right) & p \equiv 1 \pmod 4 \\ \left(\frac{\left(\frac{2}{\mathfrak{p}}\right) \cdot \left(\frac{\alpha_2}{\mathfrak{p}}\right)}{\bar{\mathfrak{p}}}\right) & p \equiv 3 \pmod 4 \end{cases}$$

$$= \begin{cases} \left(\frac{-d}{p}\right)_4 & p \equiv 1 \pmod 4 \\ \left(\frac{2}{\mathfrak{p}}\right) \cdot \left(\frac{\alpha_2}{\mathfrak{p}}\right) & p \equiv 3 \pmod 4 \end{cases}$$

So if $\alpha_1 = \alpha_2 = 2$, then the density is $3/4$. If $\alpha_1 = \alpha_2 = -2$, then the density is $1/4$. And since $\left(\frac{-1}{\mathfrak{p}}\right) = \begin{cases} 1 & \text{if } p \equiv 1 \pmod 4 \\ -1 & \text{if } p \equiv 3 \pmod 4 \end{cases}$, the density is $1/4$ if $\alpha_1 = -\alpha_2$.

Case 4: $\alpha_1 = \alpha_2$, α_1 and $\alpha_2 \in$ Set 4: $1/2$

The proof proceeds similarly to Case 3 when $\alpha_1 = \alpha_2$. The density is easily $1/2$.

Case 5: $\alpha_1 \in$ Set 1, and $\alpha_2 \in$ Set 2: $1/4$

Since $\left(\frac{\alpha_1}{\mathfrak{p}}\right) = 1$. and $\left(\frac{\alpha_2}{\mathfrak{p}}\right) = \begin{cases} 1 & \text{if } p \equiv 1 \pmod 4 \\ -1 & \text{if } p \equiv 3 \pmod 4 \end{cases}$

we have

$$\left(\frac{a_p(E^d_{\alpha_1})+a_p(E^d_{\alpha_2})}{p}\right)=\begin{cases}\left(\frac{-d}{p}\right)_4\cdot\left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)+\left(\frac{\alpha_2}{\mathfrak{p}}\right)}{\bar{\mathfrak{p}}}\right) & p\equiv 1 \pmod 4\\ \left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)+\left(\frac{\alpha_2}{\mathfrak{p}}\right)}{\bar{\mathfrak{p}}}\right) & p\equiv 3 \pmod 4\end{cases}$$

$$=\begin{cases}\left(\frac{-d}{p}\right)_4\cdot\left(\frac{2}{\bar{\mathfrak{p}}}\right) & p\equiv 1 \pmod 4\\ 0 & p\equiv 3 \pmod 4\end{cases}$$

So the density is $1/4$

Case 6: $\alpha_1\in$ Set 1, and $\alpha_2\in$ Set 3: $1/8$ or $3/8$.
We have $\left(\frac{\alpha_1}{\mathfrak{p}}\right)=1$, let $\alpha_2=2$

$$\left(\frac{a_p(E^d_{\alpha_1})+a_p(E^d_{\alpha_2})}{p}\right)=\begin{cases}\left(\frac{-d}{p}\right)_4\cdot\left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)+\left(\frac{\alpha_2}{\mathfrak{p}}\right)}{\bar{\mathfrak{p}}}\right) & p\equiv 1 \pmod 4\\ \left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)+\left(\frac{\alpha_2}{\mathfrak{p}}\right)}{\bar{\mathfrak{p}}}\right) & p\equiv 3 \pmod 4\end{cases}$$

$$=\begin{cases}\left(\frac{-d}{p}\right)_4,\text{ or } 0, & p\equiv 1 \pmod 4\\ 1 \text{ or } 0 & p\equiv 3 \pmod 4\end{cases}$$

So the density $=\frac{3}{8}$, if $\alpha_1\in$ Set 1, and $\alpha_2=2$.
If $\alpha_2=-2$, then

$$\left(\frac{a_p(E^d_{\alpha_1})+a_p(E^d_{\alpha_2})}{p}\right)=\begin{cases}\left(\frac{-d}{p}\right)_4\cdot\left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)+\left(\frac{\alpha_2}{\mathfrak{p}}\right)}{\bar{\mathfrak{p}}}\right) & p\equiv 1 \pmod 4\\ \left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)+\left(\frac{\alpha_2}{\mathfrak{p}}\right)}{\bar{\mathfrak{p}}}\right) & p\equiv 3 \pmod 4\end{cases}$$

$$=\begin{cases}\left(\frac{-d}{p}\right)_4\text{, or } 0, & p\equiv 1 \pmod 4\\ -1 \text{ or } 0 & p\equiv 3 \pmod 4\end{cases}$$

So the density $=\frac{1}{8}$, if $\alpha_1\in$ Set 1, and $\alpha_2=-2$.

Case 7: $\alpha_1\in$ Set 1, and $\alpha_2\in$ Set 4: 1/4

We have $\left(\frac{\alpha_1}{\mathfrak{p}}\right)=1$, and

$$\left(\frac{a_p(E^d_{\alpha_1})+a_p(E^d_{\alpha_2})}{p}\right)=\begin{cases}\left(\frac{-d}{p}\right)_4\cdot\left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)+\left(\frac{\alpha_2}{\mathfrak{p}}\right)}{\bar{\mathfrak{p}}}\right) & p\equiv 1 \pmod 4\\ \left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)+\left(\frac{\alpha_2}{\mathfrak{p}}\right)}{\bar{\mathfrak{p}}}\right) & p\equiv 3 \pmod 4\end{cases}$$

$$=\begin{cases}\left(\frac{-d}{p}\right)_4\cdot\left(\frac{1\pm 1}{\bar{\mathfrak{p}}}\right) & p\equiv 1 \pmod 4\\ \left(\frac{1\pm 1}{\bar{\mathfrak{p}}}\right) & p\equiv 3 \pmod 4\end{cases}$$

So the density is 1/8+1/8=1/4.

Case 8: $\alpha_1\in$ Set 2, and $\alpha_2\in$ Set 3: 1/8 or 3/8

We have $\left(\frac{\alpha_1}{\mathfrak{p}}\right)=\begin{cases}1 & \text{if } p\equiv 1 \pmod 4\\ -1 & \text{if } p\equiv 3 \pmod 4\end{cases}$, and let $\alpha_2=2$. Then

$$\left(\frac{a_p(E^d_{\alpha_1})+a_p(E^d_{\alpha_2})}{p}\right)=\begin{cases}\left(\frac{-d}{p}\right)_4\cdot\left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)+\left(\frac{\alpha_2}{\mathfrak{p}}\right)}{\bar{\mathfrak{p}}}\right) & p\equiv 1 \pmod 4\\ \left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)+\left(\frac{\alpha_2}{\mathfrak{p}}\right)}{\bar{\mathfrak{p}}}\right) & p\equiv 3 \pmod 4\end{cases}$$

$$=\begin{cases}\left(\frac{-d}{p}\right)_4, \text{or } 0, & p\equiv 1 \pmod 4\\ 1 \text{ or } 0 & p\equiv 3 \pmod 4\end{cases}$$

So the density is $3/8$, if $\alpha_1 \in$ Set 2, and $\alpha_2 = 2$. Similar to Case 6, the density is $1/8$ if if $\alpha_1 \in$ Set 2, and $\alpha_2 = -2$.

Case 9: $\alpha_1 \in$ Set 2, and $\alpha_2 \in$ Set 4: $1/4$

We have $\left(\frac{\alpha_1}{\mathfrak{p}}\right) = \begin{cases}1 & \text{if } p\equiv 1 \pmod 4\\ -1 & \text{if } p\equiv 3 \pmod 4\end{cases}$, so

$$\left(\frac{a_p(E^d_{\alpha_1})+a_p(E^d_{\alpha_2})}{p}\right)=\begin{cases}\left(\frac{-d}{p}\right)_4\cdot\left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)+\left(\frac{\alpha_2}{\mathfrak{p}}\right)}{\bar{\mathfrak{p}}}\right) & p\equiv 1 \pmod 4\\ \left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)+\left(\frac{\alpha_2}{\mathfrak{p}}\right)}{\bar{\mathfrak{p}}}\right) & p\equiv 3 \pmod 4\end{cases}$$

$$=\begin{cases}\left(\frac{-d}{p}\right)_4\cdot\left(\frac{1\pm 1}{\bar{\mathfrak{p}}}\right) & p\equiv 1 \pmod 4\\ \left(\frac{-1\pm 1}{\bar{\mathfrak{p}}}\right) & p\equiv 3 \pmod 4\end{cases}$$

So the density is $1/4$.

Case 10: $\alpha_1 \in$ Set 3, and $\alpha_2 \in$ Set 4: $1/8$ or $3/8$

If $\alpha_1 = 2$, then

$$\left(\frac{a_p(E^d_{\alpha_1})+a_p(E^d_{\alpha_2})}{p}\right)=\begin{cases}\left(\frac{-d}{p}\right)_4\cdot\left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)+\left(\frac{\alpha_2}{\mathfrak{p}}\right)}{\bar{\mathfrak{p}}}\right) & p\equiv 1 \pmod 4\\ \left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)+\left(\frac{\alpha_2}{\mathfrak{p}}\right)}{\bar{\mathfrak{p}}}\right) & p\equiv 3 \pmod 4\end{cases}$$

$$=\begin{cases}\pm\left(\frac{-d}{p}\right)_4, & \text{2 cases, } p\equiv 1 \pmod 4\\ 0, & \text{2 cases, } p\equiv 1 (\bmod 4)\\ 1 \text{ or } 0 & p\equiv 3 \pmod 4\end{cases}$$

So the density is 3/8, if $\alpha_1 \in$ Set 3, and $\alpha_2 = 2$. Similar to Case 6, the density is 1/8 if if $\alpha_1 \in$ Set 3, and $\alpha_2 = -2$.

Case 11: $\alpha_1/\alpha_2 = \alpha_3 \in$ Set 1: 1/2

By Lemma 2.1.1:

$$\left(\frac{a_p(E^d_{\alpha_1})+a_p(E^d_{\alpha_2})}{p}\right)=\begin{cases}\left(\frac{-d}{p}\right)_4\cdot\left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)+\left(\frac{\alpha_2}{\mathfrak{p}}\right)}{\bar{\mathfrak{p}}}\right) & p\equiv 1 \pmod 4\\ \left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)+\left(\frac{\alpha_2}{\mathfrak{p}}\right)}{\bar{\mathfrak{p}}}\right) & p\equiv 3 \pmod 4\end{cases}$$

$$=\begin{cases}\left(\frac{-d}{p}\right)_4\cdot\left(\frac{\left(\frac{\alpha_2}{\mathfrak{p}}\right)}{\bar{\mathfrak{p}}}\right)\cdot\left(\frac{\left(\frac{\alpha_3}{\mathfrak{p}}\right)+1}{\bar{\mathfrak{p}}}\right) & p\equiv 1 \pmod 4\\ \left(\frac{\left(\frac{\alpha_2}{\mathfrak{p}}\right)}{\bar{\mathfrak{p}}}\right)\cdot\left(\frac{\left(\frac{\alpha_3}{\mathfrak{p}}\right)+1}{\bar{\mathfrak{p}}}\right) & p\equiv 3 \pmod 4\end{cases}$$

$$=\begin{cases}\left(\frac{-d}{p}\right)_4\cdot\left(\frac{\left(\frac{\alpha_3}{\mathfrak{p}}\right)+1}{\bar{\mathfrak{p}}}\right) & p\equiv 1 \pmod 4\\ \left(\frac{\alpha_2}{\mathfrak{p}}\right)\cdot\left(\frac{\left(\frac{\alpha_3}{\mathfrak{p}}\right)+1}{\bar{\mathfrak{p}}}\right) & p\equiv 3 \pmod 4\end{cases}$$

$$=\begin{cases}\left(\frac{-d}{p}\right)_4\cdot\left(\frac{2}{\bar{\mathfrak{p}}}\right) & p\equiv 1 \pmod 4\\ \left(\frac{\alpha_2}{\mathfrak{p}}\right)\cdot\left(\frac{2}{\bar{\mathfrak{p}}}\right) & p\equiv 3 \pmod 4\end{cases}$$

So the density equals $\frac{1}{2}$.

Case 12: $\alpha_1/\alpha_2=\alpha_3\in$ Set 2: $1/4$

We have $\left(\frac{\alpha_3}{\mathfrak{p}}\right)=\begin{cases}1 & \text{if } p\equiv 1 \pmod 4\\ -1 & \text{if } p\equiv 3 \pmod 4\end{cases}$, so

By Lemma 2.1.1:

$$\left(\frac{a_p(E^d_{\alpha_1})+a_p(E^d_{\alpha_2})}{p}\right) = \begin{cases} \left(\frac{-d}{p}\right)_4 \cdot \left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)+\left(\frac{\alpha_2}{\mathfrak{p}}\right)}{\bar{\mathfrak{p}}}\right) & p \equiv 1 \pmod 4 \\ \left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)+\left(\frac{\alpha_2}{\mathfrak{p}}\right)}{\bar{\mathfrak{p}}}\right) & p \equiv 3 \pmod 4 \end{cases}$$

$$= \begin{cases} \left(\frac{-d}{p}\right)_4 \cdot \left(\frac{\left(\frac{\alpha_3}{\mathfrak{p}}\right)+1}{\bar{\mathfrak{p}}}\right) & p \equiv 1 \pmod 4 \\ \left(\frac{\alpha_2}{\mathfrak{p}}\right) \cdot \left(\frac{\left(\frac{\alpha_3}{\mathfrak{p}}\right)+1}{\bar{\mathfrak{p}}}\right) & p \equiv 3 \pmod 4 \end{cases}$$

$$= \begin{cases} \left(\frac{-d}{p}\right)_4 \cdot \left(\frac{2}{\bar{\mathfrak{p}}}\right) & p \equiv 1 \pmod 4 \\ 0 & p \equiv 3 \pmod 4 \end{cases}$$

So the density equals $\frac{1}{4}$.

Case 13: $\alpha_1/\alpha_2 = \alpha_3 \in$ Set 3: 1/4

By Lemma 2.1.1:

$$\left(\frac{a_p(E^d_{\alpha_1})+a_p(E^d_{\alpha_2})}{p}\right) = \begin{cases} \left(\frac{-d}{p}\right)_4 \cdot \left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)+\left(\frac{\alpha_2}{\mathfrak{p}}\right)}{\bar{\mathfrak{p}}}\right) & p \equiv 1 \pmod 4 \\ \left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)+\left(\frac{\alpha_2}{\mathfrak{p}}\right)}{\bar{\mathfrak{p}}}\right) & p \equiv 3 \pmod 4 \end{cases}$$

$$= \begin{cases} \left(\frac{-d}{p}\right)_4 \cdot \left(\frac{\left(\frac{\alpha_3}{\mathfrak{p}}\right)+1}{\bar{\mathfrak{p}}}\right) & p \equiv 1 \pmod 4 \\ \left(\frac{\alpha_2}{\mathfrak{p}}\right) \cdot \left(\frac{\left(\frac{\alpha_3}{\mathfrak{p}}\right)+1}{\bar{\mathfrak{p}}}\right) & p \equiv 3 \pmod 4 \end{cases}$$

$$= \begin{cases} \left(\frac{-d}{p}\right)_4 \text{ or } 0 & p \equiv \ \pmod 4 \\ \pm\left(\frac{\alpha_2}{\mathfrak{p}}\right) \text{ or } 0 & p \equiv 3 \pmod 4 \end{cases}$$

So the density is $1/4$.

$\square$

2.1.2 When $\mathbf{d = 1}$:

We recognize differences among 5 sets of values of α_i:

Set 1 (quartic): $\alpha_i \in \{1, \text{-}4, 16, \text{etc.,}\}$ such that $\left(\frac{\alpha_i}{\mathfrak{p}}\right)_4 = 1$. Note that $\left(\frac{-4}{\mathfrak{p}}\right)_4 = \left(\frac{2i}{\mathfrak{p}}\right) = 1$.

Set 2: (negative set 1): $\alpha_i \in \{\text{-}1.\ 4.\ \text{-}16, \text{etc.,}\}$ such that

$$\left(\frac{\alpha_i}{\mathfrak{p}}\right)_4 = \begin{cases} 1, & p \equiv 1 \pmod 8 \\ -1, & p \equiv 5 \pmod 8 \end{cases}$$

Set 3: $\alpha_i \in \mathbf{Q}^{\times 2}$, such that $\left(\frac{\alpha_i}{\mathfrak{p}}\right)_4 = \pm 1$

Set 4: $\alpha_i = \pm 2^c$ for c some odd integer (or $D(\alpha_i) = \pm 8$), such that

$$\left(\frac{\alpha_i}{\mathfrak{p}}\right)_4 = \begin{cases} \pm 1, & p \equiv 1 \pmod 8 \\ \pm i, & p \equiv 5 \pmod 8 \end{cases}$$

Set 5: other cases.

For the following cases involving the above sets, we compute the $\delta_2^1(E^1_{\alpha_1} \times E^1_{\alpha_2}, \mathcal{C})$:

Case 1: $\alpha_1 = \alpha_2 \in$ Set 1,2,3: 1/2

Case 2: $\alpha_1 = \alpha_2 \in$ Set 4: 1

Case 3: $\alpha_1 = \alpha_2 \in$ Set 5: 3/4

Case 4: $\alpha_1 \neq \alpha_2$, α_1 and $\alpha_2 \in$ Set 1, or α_1 and $\alpha_2 \in$ Set 2: 1/2

Case 5: $\alpha_1 \neq \alpha_2$, α_1 and $\alpha_2 \in$ Set 3: 1/4

Case 6: $\alpha_1 \neq \alpha_2$, α_1 and $\alpha_2 \in$ Set 4: 1.

Case 7: $\alpha_1 \neq \alpha_2$, $\alpha_1 \in$ Set 1, and $\alpha_2 \in$ Set 2: 1/2

Case 8: $\alpha_1 \neq \alpha_2$, $\alpha_1 \in$ Set 1, and $\alpha_2 \in$ Set 3: 1/4

Case 9: $\alpha_1 \neq \alpha_2$, $\alpha_1 \in$ Set 1, and $\alpha_2 \in$ Set 4: 1/2

Case 10: $\alpha_1 \neq \alpha_2$, $\alpha_1 \in$ Set 1, and $\alpha_2 \in$ Set 5: 3/8

Case 11: $\alpha_1 \neq \alpha_2$, $\alpha_1 \in$ Set 2, and $\alpha_2 \in$ Set 3: 1/4

Case 12: $\alpha_1 \neq \alpha_2$, $\alpha_1 \in$ Set 2, and $\alpha_2 \in$ Set 4: 1/2

Case 13: $\alpha_1 \neq \alpha_2$, $\alpha_1 \in$ Set 2, and $\alpha_2 \in$ Set 5: 3/8

Case 14: $\alpha_1 \neq \alpha_2$, $\alpha_1 \in$ Set 3, and $\alpha_2 \in$ Set 4: 1/2

Case 15: $\alpha_1 \neq \alpha_2$, $\alpha_1 \in$ Set 3, and $\alpha_2 \in$ Set 5: 3/8

Case 16: $\alpha_1 \neq \alpha_2$, $\alpha_1 \in$ Set 4, and $\alpha_2 \in$ Set 5: 1/2

Case 17: $\alpha_1/\alpha_2 \in$ Set 1: $\begin{cases} 1/2 & \text{if } \alpha_2 \in \text{Set 1,2,3} \\ 1 & \text{if } \alpha_2 \in \text{Set 4} \\ 3/4 & \text{if } \alpha_2 \in \text{Set 5} \end{cases}$

Case 18: $\alpha_1/\alpha_2 \in$ Set 2: 1/2

Case 19: $\alpha_1/\alpha_2 \in$ Set 3: $\begin{cases} 1/4 & \text{if } \alpha_2 \in \text{Set 1,2,3} \\ 1/2 & \text{if } \alpha_2 \in \text{Set 4} \\ 3/8 & \text{if } \alpha_2 \in \text{Set 5} \end{cases}$

Case 20: $\alpha_1/\alpha_2 \in$ Set 4: 1/2

Case 21: $\alpha_1/\alpha_2 \in$ Set 5: $\alpha_1/\alpha_2 \in$ Set 5: $\begin{cases} 1/2 & \text{if } \alpha_2 \in \text{Set 1,2,3,4} \\ 7/16 & \text{if } \alpha_2 \in \text{Set 5} \end{cases}$

Proof. **Case 1**: $\alpha_1 = \alpha_2 \in$ Set 1,2, or 3: 1/2.

We have $\left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_4 + \left(\frac{\alpha_2}{\mathfrak{p}}\right)_4}{\bar{\mathfrak{p}}}\right) = \left(\frac{\pm 2}{\bar{\mathfrak{p}}}\right)$, which equals $\left(\frac{2}{\bar{\mathfrak{p}}}\right)$ since $\left(\frac{-1}{\bar{\mathfrak{p}}}\right) = 1$ when $p \equiv 1 \pmod 4$. Also, using the fact that

$$\left(\frac{2}{\bar{\mathfrak{p}}}\right) = \begin{cases} 1 & p \equiv 1 \pmod 8 \\ -1 & p \equiv 5 \pmod 8 \end{cases}$$

We have that the density is 1/2 when $p \equiv 1 \pmod 8$, and zero when $p \equiv 5 \pmod 8$.

Case 2: $\alpha_1 = \alpha_2 \in$ Set 4: 1

We will let $\alpha_1 = \alpha_2 = 2$ to simplify the case. Since

$$\left(\frac{2}{\mathfrak{p}}\right) = \begin{cases} 1 & p \equiv 1 \pmod 8 \\ -1 & p \equiv 5 \pmod 8 \end{cases}$$

we have

$$\left(\frac{2}{\mathfrak{p}}\right)_4 = \begin{cases} \pm 1 & p \equiv 1 \pmod 8 \\ \pm i & p \equiv 5 \pmod 8 \end{cases}$$

Also, using the fact that

$$\left(\frac{i}{\bar{\mathfrak{p}}}\right) = \begin{cases} 1 & p \equiv 1 (\text{mod } 8) \\ -1 & p \equiv 5 (\text{mod } 8) \end{cases}$$

we have:

$$\left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_4 + \left(\frac{\alpha_2}{\mathfrak{p}}\right)_4}{\bar{\mathfrak{p}}}\right) = \begin{cases} \left(\frac{2}{\bar{\mathfrak{p}}}\right) \cdot \left(\frac{\pm 1}{\bar{\mathfrak{p}}}\right) = \left(\frac{2}{\bar{\mathfrak{p}}}\right) = 1, & p \equiv 1 \pmod 8 \\ \left(\frac{2}{\bar{\mathfrak{p}}}\right) \cdot \left(\frac{\pm i}{\bar{\mathfrak{p}}}\right) = \left(\frac{2}{\bar{\mathfrak{p}}}\right) \cdot \left(\frac{i}{\bar{\mathfrak{p}}}\right) = 1, & p \equiv 5 \pmod 8 \end{cases}$$

So the density is 1.

Case 3: $\alpha_1 = \alpha_2 \in$ Set 5: 3/4

We have 4 cases: $\left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_4 + \left(\frac{\alpha_2}{\mathfrak{p}}\right)_4}{\bar{\mathfrak{p}}}\right) = \left(\frac{2}{\bar{\mathfrak{p}}}\right) \cdot \left(\frac{i}{\bar{\mathfrak{p}}}\right)^b$ $(b = 1, 2, 3, 4)$.

Using the fact that

$$\left(\frac{i}{\bar{\mathfrak{p}}}\right) = \begin{cases} 1 & p \equiv 1 \pmod 8 \\ -1 & p \equiv 5 \pmod 8 \end{cases}$$

and,

$$\left(\frac{2}{\bar{\mathfrak{p}}}\right) = \begin{cases} 1 & p \equiv 1 \pmod 8 \\ -1 & p \equiv 5 \pmod 8 \end{cases}$$

we have the density $= \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

Case 4: $\alpha_1 \neq \alpha_2$, α_1 and $\alpha_2 \in$ Set 1, or α_1 and $\alpha_2 \in$ Set 2: 1/2

Similar to Case 1.

Case 5: $\alpha_1 \neq \alpha_2$, α_1 and $\alpha_2 \in$ Set 3: 1/4

Similar to Case 1 but now $\left(\frac{\alpha_1}{\mathfrak{p}}\right)_4 + \left(\frac{\alpha_2}{\mathfrak{p}}\right)_4 = 0$ half of the time.

Case 6: $\alpha_1 \neq \alpha_2$, α_1 and $\alpha_2 \in$ Set 4: 1.

Similar to the case $\alpha_1 = \alpha_2 = \pm 2^c$ for c any odd integer.

Case 7: $\alpha_1 \neq \alpha_2$, $\alpha_1 \in$ Set 1, and $\alpha_2 \in$ Set 2: 1/2

$$\left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_4 + \left(\frac{\alpha_2}{\mathfrak{p}}\right)_4}{\bar{\mathfrak{p}}}\right) = \begin{cases} \left(\frac{2}{\bar{\mathfrak{p}}}\right) = 1 & \text{if } p \equiv 1 \pmod 8 \\ 0 & \text{if } p \equiv 5 \pmod 8 \end{cases}$$

So the density is 1/2.

Case 8: $\alpha_1 \neq \alpha_2$, $\alpha_1 \in$ Set 1, and $\alpha_2 \in$ Set 3: 1/4
$\left(\dfrac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_4 + \left(\frac{\alpha_2}{\mathfrak{p}}\right)_4}{\bar{\mathfrak{p}}}\right) = \left(\frac{2}{\bar{\mathfrak{p}}}\right)$ or 0. So the density is $1/4 + 0 = 1/4$

Case 9: $\alpha_1 \neq \alpha_2$, $\alpha_1 \in$ Set 1, and $\alpha_2 \in$ Set 4: 1/2

$$\left(\dfrac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_4 + \left(\frac{\alpha_2}{\mathfrak{p}}\right)_4}{\bar{\mathfrak{p}}}\right) = \begin{cases} \left(\dfrac{\pm 1 \pm 1}{\bar{\mathfrak{p}}}\right), & p \equiv 1 \pmod 8 \\ \left(\dfrac{\pm 1 \pm i}{\bar{\mathfrak{p}}}\right), & p \equiv 5 \pmod 8 \end{cases}$$

So we have the density: $1/4 + 1/4 = 1/2$.

Case 10: $\alpha_1 \neq \alpha_2$, $\alpha_1 \in$ Set 1, and $\alpha_2 \in$ Set 5: 3/8

We have: $\left(\dfrac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_4 + \left(\frac{\alpha_2}{\mathfrak{p}}\right)_4}{\bar{\mathfrak{p}}}\right) = \left(\dfrac{\pm 1 \pm 1}{\bar{\mathfrak{p}}}\right), \left(\dfrac{\pm 1 \pm i}{\bar{\mathfrak{p}}}\right)$, and use the fact that $\left(\frac{2}{\bar{\mathfrak{p}}}\right) = -1$ when $p \equiv 5 \pmod 8$. So we have the density: $1/4 + 1/8 = 3/8$.

Case 11: $\alpha_1 \neq \alpha_2$, $\alpha_1 \in$ Set 2, and $\alpha_2 \in$ Set 3: 1/4

$$\left(\dfrac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_4 + \left(\frac{\alpha_2}{\mathfrak{p}}\right)_4}{\bar{\mathfrak{p}}}\right) = \begin{cases} \left(\dfrac{2}{\bar{\mathfrak{p}}}\right), 0 & \text{if } p \equiv 1 \pmod 8 \\ \left(\dfrac{-2}{\bar{\mathfrak{p}}}\right), 0 & \text{if } p \equiv 5 \pmod 8 \end{cases}$$

So the density is $1/4 + 0 = 1/4$

Case 12: $\alpha_1 \neq \alpha_2$, $\alpha_1 \in$ Set 2, and $\alpha_2 \in$ Set 4: 1/2

$$\left(\dfrac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_4 + \left(\frac{\alpha_2}{\mathfrak{p}}\right)_4}{\bar{\mathfrak{p}}}\right) = \begin{cases} \left(\dfrac{2}{\bar{\mathfrak{p}}}\right), 0 & \text{if } p \equiv 1 \pmod 8 \\ \left(\dfrac{-1 \pm i}{\bar{\mathfrak{p}}}\right) & \text{if } p \equiv 5 \pmod 8 \end{cases}$$

So we have the density: $1/4 + 1/4 = 1/2$.

Case 13: $\alpha_1 \neq \alpha_2$, $\alpha_1 \in$ Set 2, and $\alpha_2 \in$ Set 5: 3/8

We have:

$$\left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_4+\left(\frac{\alpha_2}{\mathfrak{p}}\right)_4}{\bar{\mathfrak{p}}}\right)=\begin{cases}\left(\frac{1\pm 1}{\bar{\mathfrak{p}}}\right),\left(\frac{1\pm i}{\bar{\mathfrak{p}}}\right), & p\equiv 1 \pmod 8\\ \left(\frac{-1\pm 1}{\bar{\mathfrak{p}}}\right),\left(\frac{-1\pm i}{\bar{\mathfrak{p}}}\right), & p\equiv 5 \pmod 8\end{cases}$$

So we have the density: $1/4 + 1/8 = 3/8$.

Case 14: $\alpha_1 \neq \alpha_2$, $\alpha_1 \in$ Set 3, and $\alpha_2 \in$ Set 4: 1/2

We have:

$$\left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_4+\left(\frac{\alpha_2}{\mathfrak{p}}\right)_4}{\bar{\mathfrak{p}}}\right)=\begin{cases}\left(\frac{\pm 1\pm 1}{\bar{\mathfrak{p}}}\right) & \text{if } p\equiv 1 \pmod 8\\ \left(\frac{\pm 1\pm i}{\bar{\mathfrak{p}}}\right) & \text{if } p\equiv 5 \pmod 8\end{cases}$$

So we have the density: $1/4 + 1/4 = 1/2$

Case 15: $\alpha_1 \neq \alpha_2$, $\alpha_1 \in$ Set 3, and $\alpha_2 \in$ Set 5: 3/8

We have:

$$\left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_4+\left(\frac{\alpha_2}{\mathfrak{p}}\right)_4}{\bar{\mathfrak{p}}}\right)=\begin{cases}\left(\frac{\pm 1\pm 1}{\bar{\mathfrak{p}}}\right),\left(\frac{\pm 1\pm i}{\bar{\mathfrak{p}}}\right), & p\equiv 1 \pmod 8\\ \left(\frac{\pm 1\pm 1}{\bar{\mathfrak{p}}}\right),\left(\frac{\pm 1\pm i}{\bar{\mathfrak{p}}}\right), & p\equiv 5 \pmod 8\end{cases}$$

So we have the density: $1/4 + 1/8 = 3/8$.

Case 16: $\alpha_1 \neq \alpha_2$, $\alpha_1 \in$ Set 4, and $\alpha_2 \in$ Set 5: 1/2

We have:

$$\left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_4+\left(\frac{\alpha_2}{\mathfrak{p}}\right)_4}{\bar{\mathfrak{p}}}\right)=\begin{cases}\left(\frac{\pm 1\pm 1}{\bar{\mathfrak{p}}}\right),\left(\frac{\pm 1\pm i}{\bar{\mathfrak{p}}}\right), & p\equiv 1 \pmod 8\\ \left(\frac{\pm i\pm 1}{\bar{\mathfrak{p}}}\right),\left(\frac{\pm i\pm i}{\bar{\mathfrak{p}}}\right), & p\equiv 5 \pmod 8\end{cases}$$

So we have the density: $1/4 + 1/4 = 1/2$.

Case 17: $\alpha_1/\alpha_2 \in$ Set 1: $= \begin{cases} 1/2 & \text{if } \alpha_2 \in \text{Set 1,2,3} \\ 1 & \text{if } \alpha_2 \in \text{Set 4} \\ 3/4 & \text{if } \alpha_2 \in \text{Set 5} \end{cases}$

We let $\alpha_1/\alpha_2 = -4$ to simplify the case.

$$\left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_4 + \left(\frac{\alpha_2}{\mathfrak{p}}\right)_4}{\bar{\mathfrak{p}}}\right) = \left(\frac{\left(\frac{\alpha_2}{\mathfrak{p}}\right)_4 \cdot \left(\frac{-4}{\mathfrak{p}}\right)_4 + \left(\frac{\alpha_2}{\mathfrak{p}}\right)_4}{\bar{\mathfrak{p}}}\right)$$
$$= \left(\frac{\left(\frac{\alpha_2}{\mathfrak{p}}\right)_4 \cdot \left(\left(\frac{-4}{\mathfrak{p}}\right)_4 + 1\right)}{\bar{\mathfrak{p}}}\right)$$
$$= \left(\frac{\left(\frac{\alpha_2}{\mathfrak{p}}\right)_4 \cdot 2}{\bar{\mathfrak{p}}}\right)$$
$$= \left(\frac{\left(\frac{\alpha_2}{\mathfrak{p}}\right)_4}{\bar{\mathfrak{p}}}\right) \cdot \left(\frac{2}{\bar{\mathfrak{p}}}\right)$$

Using the fact that

$$\left(\frac{\left(\frac{\alpha_2}{\mathfrak{p}}\right)_4}{\bar{\mathfrak{p}}}\right) = \begin{cases} 1 & \text{if } p \equiv 1 \pmod 8 \\ \left(\frac{\alpha_2}{\bar{\mathfrak{p}}}\right) & \text{if } p \equiv 5 \pmod 8 \end{cases}$$

we have

$$\left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_4 + \left(\frac{\alpha_2}{\mathfrak{p}}\right)_4}{\bar{\mathfrak{p}}}\right) = \begin{cases} 1 & \text{if } p \equiv 1 \pmod 8 \\ -\left(\frac{\alpha_2}{\bar{\mathfrak{p}}}\right) & \text{if } p \equiv 5 \pmod 8 \end{cases}$$

$$\text{So the density is } = \begin{cases} 1/2 & \text{if } \alpha_2 \in \text{Set 1,2,3} \\ 1 & \text{if } \alpha_2 \in \text{Set 4} \\ 3/4 & \text{if } \alpha_2 \in \text{Set 5} \end{cases}$$

Case 18: $\alpha_1/\alpha_2 = 4$ (Set 2):

$$\left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_4 + \left(\frac{\alpha_2}{\mathfrak{p}}\right)_4}{\bar{\mathfrak{p}}}\right) = \left(\frac{\left(\frac{\alpha_2}{\mathfrak{p}}\right)_4 \cdot \left(\frac{4}{\mathfrak{p}}\right)_4 + \left(\frac{\alpha_2}{\mathfrak{p}}\right)_4}{\bar{\mathfrak{p}}}\right)$$
$$= \left(\frac{\left(\frac{\alpha_2}{\mathfrak{p}}\right)_4 \cdot \left(\left(\frac{4}{\mathfrak{p}}\right)_4 + 1\right)}{\bar{\mathfrak{p}}}\right)$$

Using the fact that

$$\left(\frac{4}{\mathfrak{p}}\right)_4 = \begin{cases} 1 & \text{if } p \equiv 1 \pmod 8 \\ -1 & \text{if } p \equiv 5 \pmod 8 \end{cases}$$

we have

$$\left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_4 + \left(\frac{\alpha_2}{\mathfrak{p}}\right)_4}{\bar{\mathfrak{p}}}\right) = \begin{cases} 1 & \text{if } p \equiv 1 \pmod 8 \\ 0 & \text{if } p \equiv 5 \pmod 8 \end{cases}$$

So the density is 1/2.

$$\textbf{Case 19}: \alpha_1/\alpha_2 \in \text{Set 3: } = \begin{cases} 1/4 & \text{if } \alpha_2 \in \text{Set 1,2,3} \\ 1/2 & \text{if } \alpha_2 \in \text{Set 4} \\ 3/8 & \text{if } \alpha_2 \in \text{Set 5} \end{cases}$$

$$\left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_4 + \left(\frac{\alpha_2}{\mathfrak{p}}\right)_4}{\bar{\mathfrak{p}}}\right) = \left(\frac{\left(\frac{\alpha_2}{\mathfrak{p}}\right)_4 \cdot \left(\left(\frac{9}{\mathfrak{p}}\right)_4 + 1\right)}{\bar{\mathfrak{p}}}\right) = \left(\frac{\left(\frac{\alpha_2}{\mathfrak{p}}\right)_4 \cdot (\pm 1 + 1)}{\bar{\mathfrak{p}}}\right)$$

So this case is similar to the case $\alpha_1/\alpha_2 \in$ Set 1 except that the density is cut in half.

Case 20: $\alpha_1/\alpha_2 \in$ Set 4: 1/2

$$\left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_4 + \left(\frac{\alpha_2}{\mathfrak{p}}\right)_4}{\bar{\mathfrak{p}}}\right) = \left(\frac{\left(\frac{\alpha_2}{\mathfrak{p}}\right)_4 \cdot \left(\frac{2}{\mathfrak{p}}\right)_4 + \left(\frac{\alpha_2}{\mathfrak{p}}\right)_4}{\bar{\mathfrak{p}}}\right)$$
$$= \left(\frac{\left(\frac{\alpha_2}{\mathfrak{p}}\right)_4 \cdot \left(\left(\frac{2}{\mathfrak{p}}\right)_4 + 1\right)}{\bar{\mathfrak{p}}}\right)$$
$$= \begin{cases} 1 \cdot \left(\frac{(\pm 1 + 1)}{\bar{\mathfrak{p}}}\right) & \text{if } p \equiv 1 \pmod 8 \\ \left(\frac{\alpha_2}{\mathfrak{p}}\right) \cdot \left(\frac{(\pm i + 1)}{\bar{\mathfrak{p}}}\right) & \text{if } p \equiv 5 \pmod 8 \end{cases}$$

So we have that the density is 1/4+1/4=1/2.

Case 21: $\alpha_1/\alpha_2 \in$ Set 5:= $\begin{cases} 1/2 & \text{if } \alpha_2 \in \text{Set 1,2,3,4} \\ 7/16 & \text{if } \alpha_2 \in \text{Set 5} \end{cases}$

The proof proceed similarly except we have more sub-cases.

□

2.1.3 When $\mathbf{d = 2}$:

Let $D(\alpha_i)$ denotes the discriminant of $K(\sqrt{\alpha_i})$. We recognize differences among 3 sets of values of α_i:

Set 1 (squares): $D(\alpha_i) \in \{1, D\}$, such that $\left(\frac{\alpha_i}{\mathfrak{p}}\right) = 1$.

Set 2: (negative set 1): $D(\alpha_i) \in \{-4, 4d\}$ such that

$$\left(\frac{\alpha_i}{\mathfrak{p}}\right) = \begin{cases} 1 & \text{if } p \equiv 1 \pmod 8 \\ -1 & \text{if } p \equiv 3 \pmod 8 \end{cases}$$

Set 3: other cases.

For the following cases involving the above sets, we compute $\delta_2^1(E_{\alpha_1}^2 \times E_{\alpha_1}^2, \mathcal{C})$:

Case 1: α_1 and $\alpha_2 \in$ Set 1: 1/4

Case 2: α_1 and $\alpha_2 \in$ Set 2: 3/4

Case 3: $\alpha_1 = \alpha_2$, α_1 and $\alpha_2 \in$ Set 5: 1/2

Case 4: $\alpha_1 \in$ Set 1, and $\alpha_2 \in$ Set 2: 1/4

Case 5: $\alpha_1 \in$ Set 1, and $\alpha_2 \in$ Set 3: 1/8

Case 6: $\alpha_1 \in$ Set 2, and $\alpha_2 \in$ Set 3: 3/8

Case 7: $\alpha_1/\alpha_2 \in$ Set 1: 1/2

Case 8: $\alpha_1/\alpha_2 \in$ Set 2: 1/4

Case 9: $\alpha_1/\alpha_2 \in$ Set 3: 1/4

Proof. **Case 1**: α_1 and $\alpha_2 \in$ Set 1: 1/4

Recall that by Lemma 1.2.2, we have

$$\left(\frac{a_p(E_{\alpha_1}^2) + a_p(E_{\alpha_2}^2)}{p}\right) = \begin{cases} \left(\frac{-2}{p}\right)_4 \cdot \left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right) + \left(\frac{\alpha_2}{\mathfrak{p}}\right)}{\bar{\mathfrak{p}}}\right) & p \equiv 1 \pmod 8 \\ \left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right) + \left(\frac{\alpha_2}{\mathfrak{p}}\right)}{\bar{\mathfrak{p}}}\right) & p \equiv 3 \pmod 8 \end{cases}$$

Now that $\left(\frac{\alpha_i}{\mathfrak{p}}\right) = 1$. We have

$$\left(\frac{a_p(E_{\alpha_1}^2) + a_p(E_{\alpha_2}^2)}{p}\right) = \begin{cases} \left(\frac{-2}{p}\right)_4 \cdot \left(\frac{2}{\bar{\mathfrak{p}}}\right) & p \equiv 1 \pmod 8 \\ \left(\frac{2}{\bar{\mathfrak{p}}}\right) & p \equiv 3 \pmod 8 \end{cases}$$

Using the fact that

$$\left(\frac{2}{\mathfrak{p}}\right) = \begin{cases} 1 & \text{if } p \equiv 1 \pmod 8 \\ -1 & \text{if } p \equiv 3 \pmod 8 \end{cases}$$

we have

$$\left(\frac{a_p(E^2_{\alpha_1}) + a_p(E^2_{\alpha_2})}{p}\right) = \begin{cases} \left(\frac{-2}{p}\right)_4 & p \equiv 1 \pmod 8 \\ -1 & p \equiv 3 \pmod 8 \end{cases}$$

So the density equals $\dfrac{1}{[\mathbf{Q}(\zeta_8, \sqrt[4]{-2}) : \mathbf{Q}(\sqrt{-2})]} = \dfrac{1}{4}$.

Case 2: α_1 and $\alpha_2 \in$ Set 2: 3/4

Now that

$$\left(\frac{\alpha_i}{\mathfrak{p}}\right) = \begin{cases} 1 & \text{if } p \equiv 1 \pmod 8 \\ -1 & \text{if } p \equiv 3 \pmod 8 \end{cases}$$

We have

$$\left(\frac{a_p(E^2_{\alpha_1}) + a_p(E^2_{\alpha_2})}{p}\right) = \begin{cases} \left(\frac{-2}{p}\right)_4 \cdot \left(\frac{2}{\bar{\mathfrak{p}}}\right) = \left(\frac{-2}{p}\right)_4 & \text{if } p \equiv 1 \pmod 8 \\ \left(\frac{-2}{\bar{\mathfrak{p}}}\right) = 1 & \text{if } p \equiv 3 \pmod 8 \end{cases}$$

So the density is 1/4+1/2=3/4.

Case 3: $\alpha_1 = \alpha_2$, α_1 and $\alpha_2 \in$ Set 5: 1/2

$$\left(\frac{a_p(E^2_{\alpha_1})+a_p(E^2_{\alpha_2})}{p}\right)=\begin{cases}\left(\frac{-2}{p}\right)_4\cdot\left(\frac{\left(\frac{2}{\mathfrak{p}}\right)\cdot\left(\frac{\alpha_2}{\mathfrak{p}}\right)}{\bar{\mathfrak{p}}}\right) & p\equiv 1 \pmod 8\\ \left(\frac{\left(\frac{2}{\mathfrak{p}}\right)\cdot\left(\frac{\alpha_2}{\mathfrak{p}}\right)}{\bar{\mathfrak{p}}}\right) & p\equiv 3 \pmod 8\end{cases}$$

Using the fact that

$$\left(\frac{\left(\frac{\alpha_2}{\mathfrak{p}}\right)}{\bar{\mathfrak{p}}}\right)=\begin{cases}1 & \text{if } p\equiv 1 \pmod 8\\ \left(\frac{\alpha_2}{\mathfrak{p}}\right) & \text{if } p\equiv 3 \pmod 8\end{cases}$$

we have

$$\left(\frac{a_p(E^2_{\alpha_1})+a_p(E^2_{\alpha_2})}{p}\right)=\begin{cases}\left(\frac{-2}{p}\right)_4 & p\equiv 1 \pmod 8\\ -\left(\frac{\alpha_2}{\mathfrak{p}}\right) & p\equiv 3 \pmod 8\end{cases}$$

So the density equals $\frac{1}{4}+\frac{1}{[\mathbf{Q}(\zeta_8,\sqrt{\alpha_2}):\mathbf{Q}(\sqrt{-2})]}=\frac{1}{2}$.

Case 4: $\alpha_1\in$ Set 1, and $\alpha_2\in$ Set 2: 1/4

Since $\left(\frac{\alpha_1}{\mathfrak{p}}\right)=1$. and $\left(\frac{\alpha_2}{\mathfrak{p}}\right)=\begin{cases}1 & \text{if } p\equiv 1 \pmod 8\\ -1 & \text{if } p\equiv 3 \pmod 8\end{cases}$

we have

$$\left(\frac{a_p(E_1)+a_p(E_2)}{p}\right)=\begin{cases}\left(\frac{-2}{p}\right)_4 & p\equiv 1 \pmod 8\\ 0 & p\equiv 3 \pmod 8\end{cases}$$

So the density equals $\frac{1}{4}$.

Case 5: $\alpha_1\in$ Set 1, and $\alpha_2\in$ Set 3: 1/8

We have:

$$\left(\frac{a_p(E^2_{\alpha_1})+a_p(E^2_{\alpha_2})}{p}\right)=\begin{cases}\left(\frac{-2}{p}\right)_4\cdot\left(\frac{1\pm 1}{\bar{\mathfrak{p}}}\right) & p\equiv 1 \pmod 8\\ \left(\frac{1\pm 1}{\bar{\mathfrak{p}}}\right) & p\equiv 3 \pmod 8\end{cases}$$

So the density equals $\frac{1}{8}$.

Case 6: $\alpha_1 \in$ Set 2, and $\alpha_2 \in$ Set 3: 3/8

We have:

$$\left(\frac{a_p(E^2_{\alpha_1})+a_p(E^2_{\alpha_2})}{p}\right)=\begin{cases}\left(\frac{-2}{p}\right)_4\cdot\left(\frac{1\pm 1}{\bar{\mathfrak{p}}}\right) & p\equiv 1 \pmod 8\\ \left(\frac{-1\pm 1}{\bar{\mathfrak{p}}}\right) & p\equiv 3 \pmod 8\end{cases}$$

So the density is 1/8+1/4=3/8.

Case 7: $\alpha_1/\alpha_2=\beta \in$ Set 1: $=1/2$

$$\left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)+\left(\frac{\alpha_2}{\mathfrak{p}}\right)}{\bar{\mathfrak{p}}}\right)=\left(\frac{\left(\frac{\alpha_2}{\mathfrak{p}}\right)\cdot\left(\frac{\beta}{\mathfrak{p}}\right)+\left(\frac{\alpha_2}{\mathfrak{p}}\right)}{\bar{\mathfrak{p}}}\right)=\left(\frac{\left(\frac{\alpha_2}{\mathfrak{p}}\right)\cdot\left(\left(\frac{\beta}{\mathfrak{p}}\right)+1\right)}{\bar{\mathfrak{p}}}\right)$$
$$=\left(\frac{\left(\frac{\alpha_2}{\mathfrak{p}}\right)\cdot 2}{\bar{\mathfrak{p}}}\right)=\left(\frac{\left(\frac{\alpha_2}{\mathfrak{p}}\right)}{\bar{\mathfrak{p}}}\right)\cdot\left(\frac{2}{\bar{\mathfrak{p}}}\right)$$

Using the fact that

$$\left(\frac{\left(\frac{\alpha_2}{\mathfrak{p}}\right)}{\bar{\mathfrak{p}}}\right)=\begin{cases}1 & \text{if } p\equiv 1 \pmod 8\\ \left(\frac{\alpha_2}{\bar{\mathfrak{p}}}\right) & \text{if } p\equiv 3 \pmod 8\end{cases}$$

we have

$$\left(\frac{a_p(E_{\alpha_1}^2)+a_p(E_{\alpha_2}^2)}{p}\right)=\begin{cases}\left(\frac{-2}{p}\right)_4 & p\equiv 1 \pmod 8\\ -\left(\frac{\alpha_2}{\mathfrak{p}}\right) & p\equiv 3 \pmod 8\end{cases}$$

So the density equals $\dfrac{1}{4}+\dfrac{1}{[\mathbf{Q}(\zeta_8,\sqrt{\alpha_2}):\mathbf{Q}(\sqrt{-2})]}=\dfrac{1}{2}.$

Case 8: $\alpha_1/\alpha_2=\beta\in$ Set 2: $1/4$

$$\left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)+\left(\frac{\alpha_2}{\mathfrak{p}}\right)}{\bar{\mathfrak{p}}}\right)=\left(\frac{\left(\frac{\alpha_2}{\mathfrak{p}}\right)\cdot\left(\frac{\beta}{\mathfrak{p}}\right)+\left(\frac{\alpha_2}{\mathfrak{p}}\right)}{\bar{\mathfrak{p}}}\right)=\left(\frac{\left(\frac{\alpha_2}{\mathfrak{p}}\right)\cdot\left(\left(\frac{\beta}{\mathfrak{p}}\right)+1\right)}{\bar{\mathfrak{p}}}\right)$$

Using the fact that

$$\left(\frac{\beta}{\mathfrak{p}}\right)=\begin{cases}1 & \text{if } p\equiv 1 \pmod 8\\ -1 & \text{if } p\equiv 3 \pmod 8\end{cases}$$

and that

$$\left(\frac{\left(\frac{\alpha_2}{\mathfrak{p}}\right)}{\bar{\mathfrak{p}}}\right)=\begin{cases}1 & \text{if } p\equiv 1 \pmod 8\\ \left(\frac{\alpha_2}{\bar{\mathfrak{p}}}\right) & \text{if } p\equiv 3 \pmod 8\end{cases}$$

we have

$$\left(\frac{a_p(E_{\alpha_1}^2)+a_p(E_{\alpha_2}^2)}{p}\right)=\begin{cases}\left(\frac{-2}{p}\right)_4 & p\equiv 1 \pmod 8\\ 0 & p\equiv 3 \pmod 8\end{cases}$$

So the density is $1/4$.

Case 9: $\alpha_1/\alpha_2 = \beta \in$ Set 3: $1/4$

$$\left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)+\left(\frac{\alpha_2}{\mathfrak{p}}\right)}{\bar{\mathfrak{p}}}\right) = \left(\frac{\left(\frac{\alpha_2}{\mathfrak{p}}\right)\cdot\left(\frac{\beta}{\mathfrak{p}}\right)+\left(\frac{\alpha_2}{\mathfrak{p}}\right)}{\bar{\mathfrak{p}}}\right) = \left(\frac{\left(\frac{\alpha_2}{\mathfrak{p}}\right)\cdot\left(\left(\frac{\beta}{\mathfrak{p}}\right)+1\right)}{\bar{\mathfrak{p}}}\right)$$

$$= \left(\frac{\left(\frac{\alpha_2}{\mathfrak{p}}\right)\cdot(\pm 1+1)}{\bar{\mathfrak{p}}}\right) = \left(\frac{\left(\frac{\alpha_2}{\mathfrak{p}}\right)}{\bar{\mathfrak{p}}}\right)\cdot\left(\frac{\pm 1+1}{\bar{\mathfrak{p}}}\right)$$

So it ends up similar to Case 7 except that we get zero half of the time. Thus the density is $1/4$. □

2.1.4 When $\mathbf{d = 3}$:

We recognize differences among the following sets of values of α_i:

Set 1 (sextic): $\alpha_i \in \mathbf{Q}^{\times 6} \cup \{1, -27,$ Set 2 cubed, Set 3 squared, etc.,$\}$ such that $\left(\frac{\alpha_i}{\mathfrak{p}}\right)_6 = 1$.

Set 2 (square): $\alpha_i \in \mathbf{Q}^{\times 2} \cup \{-3, 4, 9\}$, such that $\left(\frac{\alpha_i}{\mathfrak{p}}\right)_6 = \zeta_3^{1,2,3}$

Set 3 (cubic): $\alpha_i \in \mathbf{Q}^{\times 3}$ that does not contain 2 or 3 , etc., such that $\left(\frac{\alpha_i}{\mathfrak{p}}\right)_6 = \pm 1$

Set 4 (negative Set 1): $\alpha_i \in \{-1, 27,$ etc.,$\}$, such that

$$\left(\frac{\alpha_i}{\mathfrak{p}}\right)_6 = \begin{cases} 1 & \text{if } p \equiv 1 \pmod{12} \\ -1 & \text{if } p \equiv 7 \pmod{12} \end{cases}$$

Set 5 (negative Set 2): $\alpha_i \in \{3, -4, -9,$ etc., and their multiplication by Set 2 e.g., $12\}$, such that

$$\left(\frac{\alpha_i}{\mathfrak{p}}\right)_6 = \begin{cases} \zeta_3^{1,2,3} & \text{if } p \equiv 1 \pmod{12} \\ -\zeta_3^{1,2,3} & \text{if } p \equiv 7 \pmod{12} \end{cases}$$

Set 6: $\{2^a \cdot (-3)^b\}$ that are not already contained in previous sets.

Set 7: other cases.

For the following cases involving the above sets, we compute $\delta_2^1(E^3_{\alpha_1} \times E^3_{\alpha_1}, \mathcal{C})$:

Case 1: $\alpha_1 = \alpha_2 \in$ Set 1,2,3,4,5,7: 1/2

Case 2: $\alpha_1 = \alpha_2 \in$ Set 6: various values, e.g., 3/4, 1/4.

Case 3: $\alpha_1 \neq \alpha_2$, α_1 and $\alpha_2 \in$ Set 1: 1/2

Case 4: $\alpha_1 \neq \alpha_2$, α_1 and $\alpha_2 \in$ Set 2: 1/3

Case 5: $\alpha_1 \neq \alpha_2$, α_1 and $\alpha_2 \in$ Set 3: 1/4

Case 6: $\alpha_1 \neq \alpha_2$, α_1 and $\alpha_2 \in$ Set 4: 1/2

Case 7: $\alpha_1 \neq \alpha_2$, α_1 and $\alpha_2 \in$ Set 5: 2/3

Case 8: $\alpha_1 \neq \alpha_2$, α_1 and $\alpha_2 \in$ Set 6: various values, e.g., 5/12, 7/12, etc.

Case 9: $\alpha_1 \in$ Set 1, and $\alpha_2 \in$ Set 2: 1/3

Case 10: $\alpha_1 \in$ Set 1, and $\alpha_2 \in$ Set 3: 1/4

Case 11: $\alpha_1 \in$ Set 1, and $\alpha_2 \in$ Set 4: 1/4

Case 12: $\alpha_1 \in$ Set 1, and $\alpha_2 \in$ Set 5: 5/12

Case 13: $\alpha_1 \in$ Set 1, and $\alpha_2 \in$ Set 7: 5/12

Case 14: $\alpha_1 \in$ Set 2, and $\alpha_2 \in$ Set 3: 5/12

Case 15: $\alpha_1 \in$ Set 2, and $\alpha_2 \in$ Set 4: 5/12

Case 16: $\alpha_1 \in$ Set 2, and $\alpha_2 \in$ Set 5: 5/12

Case 17: $\alpha_1 \in$ Set 2, and $\alpha_2 \in$ Set 7: 5/12

Case 18: $\alpha_1 \in$ Set 3, and $\alpha_2 \in$ Set 4: 1/4

Case 19: $\alpha_1 \in$ Set 3, and $\alpha_2 \in$ Set 5: 7/12

Case 20: $\alpha_1 \in$ Set 3, and $\alpha_2 \in$ Set 7: 1/2

Case 21: $\alpha_1 \in$ Set 4, and $\alpha_2 \in$ Set 5: 2/3

Case 22: $\alpha_1 \in$ Set 4, and $\alpha_2 \in$ Set 7: 7/12

Case 23: $\alpha_1 \in$ Set 5, and $\alpha_2 \in$ Set 7: 7/12

Case 24: $\alpha_1/\alpha_2 \in$ Set 1: 1/2

Case 25: $\alpha_1/\alpha_2 \in$ Set 2: 1/2

Case 26: $\alpha_1/\alpha_2 \in$ Set 3: $1/4$

Case 27: $\alpha_1/\alpha_2 \in$ Set 4: $1/4$

Case 28: $\alpha_1/\alpha_2 \in$ Set 5: $5/12$

Proof. **Case 1**: $\alpha_1 = \alpha_2 \in$ Set 1,2,3,4,5,10,11,12: $1/2$

If $\alpha_1 = \alpha_2 \in$ Set 1:

Recall that by Lemma 1.2.2, we have

$$\left(\frac{a_p(E^3_{\alpha_1}) + a_p(E^3_{\alpha_2})}{p}\right) = \begin{cases} \left(\frac{-3}{p}\right)_4 \cdot \left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6 + \left(\frac{\alpha_2}{\mathfrak{p}}\right)_6}{\bar{\mathfrak{p}}}\right) & p \equiv 1 (\text{mod } 12) \\ \left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6 + \left(\frac{\alpha_2}{\mathfrak{p}}\right)_6}{\bar{\mathfrak{p}}}\right) & p \equiv 7 (\text{mod } 12) \end{cases}$$

Since $\alpha_1 = \alpha_2$, we have

$$\left(\frac{a_p(E^3_{\alpha_1}) + a_p(E^3_{\alpha_2})}{p}\right) = \begin{cases} \left(\frac{-3}{p}\right)_4 \cdot \left(\frac{\left(\frac{2}{\mathfrak{p}}\right)_6 \cdot \left(\frac{\alpha_2}{\mathfrak{p}}\right)_6}{\bar{\mathfrak{p}}}\right) & p \equiv 1 \pmod{12} \\ \left(\frac{\left(\frac{2}{\mathfrak{p}}\right)_6 \cdot \left(\frac{\alpha_2}{\mathfrak{p}}\right)_6}{\bar{\mathfrak{p}}}\right) & p \equiv 7 \pmod{12} \end{cases}$$

Using the fact that

$$\left(\frac{\left(\frac{\alpha_2}{\mathfrak{p}}\right)_6}{\bar{\mathfrak{p}}}\right) = \begin{cases} 1 & \text{if } p \equiv 1 \pmod{12} \\ \left(\frac{\alpha_2}{\mathfrak{p}}\right) & \text{if } p \equiv 7 \pmod{12} \end{cases}$$

we have

$$\left(\frac{a_p(E^3_{\alpha_1}) + a_p(E^3_{\alpha_2})}{p}\right) = \begin{cases} \left(\frac{-3}{p}\right)_4 & p \equiv 1 (\text{mod } 12) \\ \left(\frac{2}{\mathfrak{p}}\right) \cdot \left(\frac{\alpha_2}{\mathfrak{p}}\right) & p \equiv 7 \ (\text{mod } 12) \end{cases}$$

Now if $\left(\frac{\alpha_i}{\mathfrak{p}}\right)_6 = 1$, we have

$$\left(\frac{a_p(E^3_{\alpha_1}) + a_p(E^3_{\alpha_2})}{p}\right) = \begin{cases} \left(\frac{-3}{p}\right)_4 & p \equiv 1 (\text{mod } 12) \\ \left(\frac{2}{\bar{\mathfrak{p}}}\right) & p \equiv 7 \ (\text{mod } 12) \end{cases}$$

So the density is $1/2 = \frac{1}{[\mathbf{Q}(\zeta_{12}, \sqrt[4]{-3}) : \mathbf{Q}(\sqrt{-3})]} + \frac{1}{[\mathbf{Q}(\zeta_{12}, \sqrt{2}) : \mathbf{Q}(\sqrt{-3})]}$. Note that $\sqrt{-3} = 2 \cdot \zeta_{12}^4 + 1$, and α_2 is a square.

If $\alpha_1 = \alpha_2 \in$ Set 2, then similarly $\left(\frac{\alpha_i}{\mathfrak{p}}\right) = 1$, or by the formula, the density is 1/2.

If $\alpha_1 = \alpha_2 \in$ Set 3: $\left(\frac{\alpha_i}{\mathfrak{p}}\right)_6 = \pm 1$, so $\left(\frac{\left(\frac{\alpha_2}{\mathfrak{p}}\right)_6}{\bar{\mathfrak{p}}}\right) = \left(\frac{\pm 1}{\mathfrak{p}}\right) = \pm 1$ when $p \equiv 7$ (mod 12).

So the density is also

$$\frac{1}{[\mathbf{Q}(\zeta_{12}, \sqrt[4]{-3}) : \mathbf{Q}(\sqrt{-3})]} + 2 \cdot \frac{1}{[\mathbf{Q}(\zeta_{12}, \sqrt{2}, \sqrt{\alpha_2}) : \mathbf{Q}(\sqrt{-3})]} = 1/2.$$

The case $\alpha_1 = \alpha_2 \in$ Set 7 is similar because $\left(\frac{\alpha_2}{\mathfrak{p}}\right) = \pm 1$.

If $\alpha_1 = \alpha_2 \in$ Set 4, then

$$\left(\frac{\alpha_i}{\mathfrak{p}}\right)_6 = \begin{cases} 1 & \text{if } p \equiv 1 \ (\text{mod } 12) \\ -1 & \text{if } p \equiv 7 \ (\text{mod } 12) \end{cases}$$

we have

$$\left(\frac{a_p(E^3_{\alpha_1})+a_p(E^3_{\alpha_2})}{p}\right)=\begin{cases}\left(\dfrac{-3}{p}\right)_4 & p\equiv 1 \pmod{12}\\ -\left(\dfrac{2}{\bar{\mathfrak{p}}}\right) & p\equiv 7 \pmod{12}\end{cases}$$

So again the density is $1/2$. The case $\alpha_1=\alpha_2\in$ Set 5 is similar.

Case 2: $\alpha_1=\alpha_2\in$ Set 6: $3/4$ or $1/4$

Recall that when $\alpha_1=\alpha_2$, we have

$$\left(\frac{a_p(E^3_{\alpha_1})+a_p(E^3_{\alpha_2})}{p}\right)=\begin{cases}\left(\dfrac{-3}{p}\right)_4 & p\equiv 1 \pmod{12}\\ \left(\dfrac{2}{\mathfrak{p}}\right)\cdot\left(\dfrac{\alpha_2}{\mathfrak{p}}\right) & p\equiv 7 \pmod{12}\end{cases}$$

If $\left(\dfrac{\alpha_2}{\mathfrak{p}}\right)=\left(\dfrac{2}{\mathfrak{p}}\right)$, then $\left(\dfrac{a_p(E^3_{\alpha_1})+a_p(E^3_{\alpha_2})}{p}\right)=1$ for $p\equiv 7 \pmod{12}$. The density is $1/4+1/2=3/4$.

If $\left(\dfrac{\alpha_2}{\mathfrak{p}}\right)=\left(\dfrac{-2}{\mathfrak{p}}\right)$, then $\left(\dfrac{a_p(E^3_{\alpha_1})+a_p(E^3_{\alpha_2})}{p}\right)=-1$ for $p\equiv 7 \pmod{12}$. The density is $1/4$.

If $\left(\dfrac{\alpha_2}{\mathfrak{p}}\right)=\left(\dfrac{2}{\mathfrak{p}}\right)\cdot\left(\dfrac{3}{\mathfrak{p}}\right)$, then $\left(\dfrac{a_p(E^3_{\alpha_1})+a_p(E^3_{\alpha_2})}{p}\right)=-1$ for $p\equiv 7 \pmod{12}$. The density is $1/4$.

If $\left(\dfrac{\alpha_2}{\mathfrak{p}}\right)=\left(\dfrac{2}{\mathfrak{p}}\right)\cdot\left(\dfrac{-3}{\mathfrak{p}}\right)$, then $\left(\dfrac{a_p(E^3_{\alpha_1})+a_p(E^3_{\alpha_2})}{p}\right)=1$ for $p\equiv 7 \pmod{12}$. The density is $3/4$.

Case 3: $\alpha_1\neq\alpha_2$, α_1 and $\alpha_2\in$ Set 1: $1/2$

$$\left(\frac{a_p(E^3_{\alpha_1})+a_p(E^3_{\alpha_2})}{p}\right)=\begin{cases}\left(\dfrac{-3}{p}\right)_4\cdot\left(\dfrac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6+\left(\frac{\alpha_2}{\mathfrak{p}}\right)_6}{\bar{\mathfrak{p}}}\right) & p\equiv 1 \pmod{12}\\ \left(\dfrac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6+\left(\frac{\alpha_2}{\mathfrak{p}}\right)_6}{\bar{\mathfrak{p}}}\right) & p\equiv 7 \pmod{12}\end{cases}$$

$$=\begin{cases}\left(\dfrac{-3}{p}\right)_4\cdot\left(\dfrac{2}{\bar{\mathfrak{p}}}\right) & p\equiv 1 \pmod{12}\\ \left(\dfrac{2}{\bar{\mathfrak{p}}}\right) & p\equiv 7 \pmod{12}\end{cases}$$

So the density is $2 \cdot \dfrac{1}{[\mathbf{Q}(\zeta_{12}, \sqrt[4]{-3}, \sqrt{2}) : \mathbf{Q}(\sqrt{-3})]} + \dfrac{1}{[\mathbf{Q}(\zeta_{12}, \sqrt{2}) : \mathbf{Q}(\sqrt{-3})]} = 1/2$

Case 4: $\alpha_1 \neq \alpha_2$, α_1 and $\alpha_2 \in$ Set 2: 1/3

Since $\left(\dfrac{\alpha_i}{\mathfrak{p}}\right)_6 = \zeta_3^{1,2,3}$, we have

$$\left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6 + \left(\frac{\alpha_2}{\mathfrak{p}}\right)_6}{\bar{\mathfrak{p}}}\right) = \begin{cases} \left(\frac{\zeta_6}{\bar{\mathfrak{p}}}\right)^{1,3,5} & \text{(6 of 9 cases)} \\ \left(\frac{2}{\bar{\mathfrak{p}}}\right) \cdot \left(\frac{\zeta_6}{\bar{\mathfrak{p}}}\right)^{2,4,6} & \text{(3 of 9 cases)} \end{cases}$$

Using the fact that

$$\left(\frac{\zeta_6}{\bar{\mathfrak{p}}}\right) = \begin{cases} 1 & p \equiv 1 \pmod{12} \\ -1 & p \equiv 7 \pmod{12} \end{cases}$$

we have for $p \equiv 1 \pmod{12}$

$$\left(\frac{-3}{p}\right)_4 \cdot \left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6 + \left(\frac{\alpha_2}{\mathfrak{p}}\right)_6}{\bar{\mathfrak{p}}}\right) = \begin{cases} \left(\frac{-3}{p}\right)_4 \cdot 1 & \text{(6 cases)} \\ \left(\frac{-3}{p}\right)_4 \cdot \left(\frac{2}{\bar{\mathfrak{p}}}\right) & \text{(3 cases)} \end{cases}$$

And the density for $p \equiv 1 \pmod{12}$ is

$$6 \cdot \frac{1}{[\mathbf{Q}(\zeta_{12}, \sqrt[6]{\alpha_1}, \sqrt[6]{\alpha_2}, \sqrt[4]{-3}) : \mathbf{Q}(\sqrt{-3})]} + 2 \cdot 3 \cdot \frac{1}{[\mathbf{Q}(\zeta_{12}, \sqrt[6]{\alpha_1}, \sqrt[6]{\alpha_2}, \sqrt[4]{-3}, \sqrt{2}) : \mathbf{Q}(\sqrt{-3})]} = \frac{1}{6} + \frac{1}{12} = \frac{1}{4}$$

For $p \equiv 7 \pmod{12}$, we have

$$\left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6 + \left(\frac{\alpha_2}{\mathfrak{p}}\right)_6}{\bar{\mathfrak{p}}}\right) = \begin{cases} (-1) & \text{(6 cases)} \\ \left(\frac{2}{\bar{\mathfrak{p}}}\right) & \text{(3 cases)} \end{cases}$$

We have the density for $p \equiv 7 \pmod{12}$ is

$$3 \cdot \frac{1}{[\mathbf{Q}(\zeta_{12}, \sqrt[6]{\alpha_1}, \sqrt[6]{\alpha_2}, \sqrt{2}) : \mathbf{Q}(\sqrt{-3})]} = \frac{1}{12}.$$

Adding up the density we have $\frac{1}{4} + \frac{1}{12} = \frac{1}{3}$.

Case 5: $\alpha_1 \neq \alpha_2$, α_1 and $\alpha_2 \in$ Set 3: 1/4

Since $\left(\frac{\alpha_i}{\mathfrak{p}}\right)_6 = \pm 1$, we have

$$\left(\frac{a_p(E^3_{\alpha_1}) + a_p(E^3_{\alpha_2})}{p}\right) = \begin{cases} \left(\frac{-3}{p}\right)_4 \cdot \left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6 + \left(\frac{\alpha_2}{\mathfrak{p}}\right)_6}{\bar{\mathfrak{p}}}\right) & p \equiv 1 \pmod{12} \\ \left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6 + \left(\frac{\alpha_2}{\mathfrak{p}}\right)_6}{\bar{\mathfrak{p}}}\right) & p \equiv 7 \pmod{12} \end{cases}$$

$$= \begin{cases} \left(\frac{-3}{p}\right)_4 \cdot \left(\frac{\pm 1 \pm 1}{\bar{\mathfrak{p}}}\right) & p \equiv 1 \pmod{12} \\ \left(\frac{\pm 1 \pm 1}{\bar{\mathfrak{p}}}\right) & p \equiv 7 \pmod{12} \end{cases}$$

So the density is

$$2 \cdot 2 \cdot \frac{1}{[\mathbf{Q}(\zeta_{12}, \sqrt[6]{\alpha_1}, \sqrt[6]{\alpha_2}, \sqrt[4]{-3}, \sqrt{2}) : \mathbf{Q}(\sqrt{-3})]} + \frac{1}{[\mathbf{Q}(\zeta_{12}, \sqrt[6]{\alpha_1}, \sqrt[6]{\alpha_2}, \sqrt{\pm 2}) : \mathbf{Q}(\sqrt{-3})]} = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$$

Case 6: $\alpha_1 \neq \alpha_2$, α_1 and $\alpha_2 \in$ Set 4: 1/2

Since

$$\left(\frac{\alpha_i}{\mathfrak{p}}\right)_6 = \begin{cases} 1 & \text{if } p \equiv 1 \pmod{12} \\ -1 & \text{if } p \equiv 7 \pmod{12} \end{cases}$$

we have

$$\left(\frac{a_p(E^3_{\alpha_1})+a_p(E^3_{\alpha_2})}{p}\right)=\begin{cases}\left(\frac{-3}{p}\right)_4\cdot\left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6+\left(\frac{\alpha_2}{\mathfrak{p}}\right)_6}{\bar{\mathfrak{p}}}\right) & p\equiv 1 \pmod{12}\\ \left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6+\left(\frac{\alpha_2}{\mathfrak{p}}\right)_6}{\bar{\mathfrak{p}}}\right) & p\equiv 7 \pmod{12}\end{cases}$$

$$=\begin{cases}\left(\frac{-3}{p}\right)_4\cdot\left(\frac{2}{\bar{\mathfrak{p}}}\right) & p\equiv 1 \pmod{12}\\ \left(\frac{-2}{\bar{\mathfrak{p}}}\right) & p\equiv 7 \pmod{12}\end{cases}$$

So the density is $2\cdot\frac{1}{[\mathbf{Q}(\zeta_{12},\sqrt[4]{-3},\sqrt{2}):\mathbf{Q}(\sqrt{-3})]}+\frac{1}{[\mathbf{Q}(\zeta_{12},\sqrt{-2}):\mathbf{Q}(\sqrt{-3})]}=1/2$

Case 7: $\alpha_1\neq\alpha_2$, α_1 and $\alpha_2\in$ Set 5: $2/3$

Since

$$\left(\frac{\alpha_i}{\mathfrak{p}}\right)_6=\begin{cases}\zeta_3^{1,2,3} & \text{if } p\equiv 1 \pmod{12}\\ -\zeta_3^{1,2,3} & \text{if } p\equiv 7 \pmod{12}\end{cases}$$

we have the density is $1/4$ for $p\equiv 1 \pmod{12}$ since it's similar to case 5 above.

For $p\equiv 7 \pmod{12}$, we have

$$\left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6+\left(\frac{\alpha_2}{\mathfrak{p}}\right)_6}{\bar{\mathfrak{p}}}\right)=\begin{cases}\left(\frac{\zeta_6}{\bar{\mathfrak{p}}}\right)^{2,4,6} & \text{(6 in 9 cases)}\\ \left(\frac{2}{\bar{\mathfrak{p}}}\right)\cdot\left(\frac{\zeta_6}{\bar{\mathfrak{p}}}\right)^{1,3,5} & \text{(3 in 9 cases)}\end{cases}$$

$$=\begin{cases}(1) & \text{(6 cases)}\\ \left(\frac{-2}{\bar{\mathfrak{p}}}\right) & \text{(3 cases)}\end{cases}$$

We have the density for $p\equiv 7 \pmod{12}$ is

$$6 \cdot \frac{1}{[\mathbf{Q}(\zeta_{12}, \sqrt[6]{\alpha_1}, \sqrt[6]{\alpha_2}) : \mathbf{Q}(\sqrt{-3})]} + 3 \cdot \frac{1}{[\mathbf{Q}(\zeta_{12}, \sqrt[6]{\alpha_1}, \sqrt[6]{\alpha_2}, \sqrt{-2}) : \mathbf{Q}(\sqrt{-3})]}$$
$$= \frac{1}{3} + \frac{1}{12} = 5/12.$$

Adding up the density we have $\frac{1}{4} + \frac{5}{12} = \frac{2}{3}$.

Case 8: $\alpha_1 \neq \alpha_2$, α_1 and $\alpha_2 \in$ Set 6: various values

Subcase 1: $\alpha_1 = 2, \alpha_2 = 6$

$$\left(\frac{a_p(E^3_{\alpha_1})+a_p(E^3_{\alpha_2})}{p}\right)=\begin{cases}\left(\frac{-3}{p}\right)_4\cdot\left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6+\left(\frac{\alpha_2}{\mathfrak{p}}\right)_6}{\bar{\mathfrak{p}}}\right) & p\equiv 1 \pmod{12}\\ \left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6+\left(\frac{\alpha_2}{\mathfrak{p}}\right)_6}{\bar{\mathfrak{p}}}\right) & p\equiv 7 \pmod{12}\end{cases}$$

$$=\begin{cases}\left(\frac{-3}{p}\right)_4\cdot\left(\frac{\left(\frac{2}{\mathfrak{p}}\right)_6\cdot\left(1+\left(\frac{3}{\mathfrak{p}}\right)_6\right)}{\bar{\mathfrak{p}}}\right) & p\equiv 1 \pmod{12}\\ \left(\frac{\left(\frac{2}{\mathfrak{p}}\right)_6\cdot\left(1+\left(\frac{3}{\mathfrak{p}}\right)_6\right)}{\bar{\mathfrak{p}}}\right) & p\equiv 7 \pmod{12}\end{cases}$$

$$=\begin{cases}\left(\frac{-3}{p}\right)_4\cdot\left(\frac{(\zeta_6,\zeta_6^5,2)}{\bar{\mathfrak{p}}}\right) & p\equiv 1 \pmod{12}\\ \left(\frac{\left(\frac{2}{\mathfrak{p}}\right)_6\cdot(\sqrt{-3})\cdot(\zeta_6^5,\zeta_6^4,0)}{\bar{\mathfrak{p}}}\right) & p\equiv 7 (\mathrm{mod}\ 12)\end{cases}$$

$$=\begin{cases}\left(\frac{-3}{p}\right)_4 & (2\text{ cases}),\ p\equiv 1(\mathrm{mod}\ 12)\\ \left(\frac{-3}{p}\right)_4\cdot\left(\frac{2}{\bar{\mathfrak{p}}}\right) & (1\text{ case}),\ p\equiv 1 \pmod{12}\\ -\left(\frac{2}{\mathfrak{p}}\right)\cdot\left(\frac{\sqrt{-3}}{\mathfrak{p}}\right) & (1\text{ case}),\ p\equiv 7 \pmod{12}\\ \left(\frac{2}{\mathfrak{p}}\right)\cdot\left(\frac{\sqrt{-3}}{\mathfrak{p}}\right) & (1\text{ case}),\ p\equiv 7 \pmod{12}\\ 0 & (1\text{ case}),\ p\equiv 7 \pmod{12}\end{cases}$$

By Lemma 2.1.1 above, we use $\sqrt[6]{\frac{\alpha_2}{\alpha_1}}=\sqrt[6]{3}$ in the computation. So we have

$$2\cdot\frac{1}{[\mathbf{Q}(\zeta_{12},\sqrt[6]{3},\sqrt[4]{-3}):\mathbf{Q}(\sqrt{-3})]}+3\cdot 2\cdot\frac{1}{[\mathbf{Q}(\zeta_{12},\sqrt[6]{3},\sqrt{2},\sqrt[4]{-3}):\mathbf{Q}(\sqrt{-3})]}$$
$$=\frac{1}{6}+\frac{1}{4}=\frac{5}{12}.$$

Subcase 2: $\alpha_1 = 2, \alpha_2 = -6$

The proof proceed similarly, we have

$$\left(\frac{a_p(E^3_{\alpha_1}) + a_p(E^3_{\alpha_2})}{p}\right) = \begin{cases} \left(\frac{-3}{p}\right)_4 \cdot \left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6 + \left(\frac{\alpha_2}{\mathfrak{p}}\right)_6}{\bar{\mathfrak{p}}}\right) & p \equiv 1 \pmod{12} \\ \left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6 + \left(\frac{\alpha_2}{\mathfrak{p}}\right)_6}{\bar{\mathfrak{p}}}\right) & p \equiv 7 \pmod{12} \end{cases}$$

$$= \begin{cases} \left(\frac{-3}{p}\right)_4 \cdot \left(\frac{\left(\frac{2}{\mathfrak{p}}\right)_6 \cdot \left(1 + \left(\frac{-3}{\mathfrak{p}}\right)_6\right)}{\bar{\mathfrak{p}}}\right) & p \equiv 1 \pmod{12} \\ \left(\frac{\left(\frac{2}{\mathfrak{p}}\right)_6 \cdot \left(1 + \left(\frac{-3}{\mathfrak{p}}\right)_6\right)}{\bar{\mathfrak{p}}}\right) & p \equiv 7 \pmod{12} \end{cases}$$

$$= \begin{cases} \left(\frac{-3}{p}\right)_4 \cdot \left(\frac{(\zeta_6, \zeta_6^5, 2)}{\bar{\mathfrak{p}}}\right) & p \equiv 1 \pmod{12} \\ \left(\frac{\left(\frac{2}{\mathfrak{p}}\right)_6 \cdot (\zeta_6, \zeta_6^5, 2)}{\bar{\mathfrak{p}}}\right) & p \equiv 7 \pmod{12} \end{cases}$$

$$= \begin{cases} \left(\frac{-3}{p}\right)_4 \cdot \left(\frac{(\zeta_6, \zeta_6^5, 2)}{\bar{\mathfrak{p}}}\right) & p \equiv 1 \pmod{12} \\ \left(\frac{2}{\mathfrak{p}}\right) \cdot \left(\frac{(\zeta_6, \zeta_6^5, 2)}{\bar{\mathfrak{p}}}\right) & p \equiv 7 \pmod{12} \end{cases}$$

$$= \begin{cases} \left(\frac{-3}{p}\right)_4 & (2 \text{ cases}),\ p \equiv 1 \pmod{12} \\ \left(\frac{-3}{p}\right)_4 \cdot \left(\frac{2}{\mathfrak{p}}\right) & (1 \text{ case}),\ p \equiv 1 \pmod{12} \\ -\left(\frac{2}{\mathfrak{p}}\right) & (2 \text{ cases}),\ p \equiv 7 \pmod{12} \\ 1 & (1 \text{ case}),\ p \equiv 7(\bmod 12) \end{cases}$$

So we have

$$2 \cdot \frac{1}{[\mathbf{Q}(\zeta_{12}, \sqrt[6]{-3}, \sqrt[4]{-3}) : \mathbf{Q}(\sqrt{-3})]} + 2 \cdot \frac{1}{[\mathbf{Q}(\zeta_{12}, \sqrt[6]{-3}, \sqrt{2}, \sqrt[4]{-3}) : \mathbf{Q}(\sqrt{-3})]}$$
$$+ 2 \cdot \frac{1}{[\mathbf{Q}(\zeta_{12}, \sqrt[6]{-3}, \sqrt{2}) : \mathbf{Q}(\sqrt{-3})]} + \frac{1}{[\mathbf{Q}(\zeta_{12}, \sqrt[6]{-3}) : \mathbf{Q}(\sqrt{-3})]}$$
$$= \frac{1}{6} + \frac{1}{12} + \frac{1}{6} + \frac{1}{6} = \frac{7}{12}.$$

Case 9: $\alpha_1 \in$ Set 1, and $\alpha_2 \in$ Set 2: 1/3

Since $\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6 = 1$ and $\left(\frac{\alpha_2}{\mathfrak{p}}\right)_6 = \zeta_3^{1,2,3}$, we have

$$\left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6 + \left(\frac{\alpha_2}{\mathfrak{p}}\right)_6}{\bar{\mathfrak{p}}}\right) = \begin{cases} \left(\frac{\zeta_6}{\bar{\mathfrak{p}}}\right)^{1,5} & \text{(2 in 3 cases)} \\ \left(\frac{2}{\bar{\mathfrak{p}}}\right) & \text{(1 in 3 cases)} \end{cases}$$

So we have

$$\left(\frac{a_p(E^3_{\alpha_1}) + a_p(E^3_{\alpha_2})}{p}\right) = \begin{cases} \left(\frac{-3}{p}\right)_4 \cdot \left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6 + \left(\frac{\alpha_2}{\mathfrak{p}}\right)_6}{\bar{\mathfrak{p}}}\right) & p \equiv 1 \pmod{12} \\ \left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6 + \left(\frac{\alpha_2}{\mathfrak{p}}\right)_6}{\bar{\mathfrak{p}}}\right) & p \equiv 7 \pmod{12} \end{cases}$$
$$= \begin{cases} \left(\frac{-3}{p}\right)_4 & \text{(2 cases), } p \equiv 1 \pmod{12} \\ \left(\frac{-3}{p}\right)_4 \cdot \left(\frac{2}{\bar{\mathfrak{p}}}\right) & \text{(1 case), } p \equiv 1 \pmod{12} \\ -1 & \text{(2 cases), } p \equiv 7 \pmod{12} \\ \left(\frac{2}{\bar{\mathfrak{p}}}\right) & \text{(1 case), } p \equiv 7 \pmod{12} \end{cases}$$

So the density is

$$2\cdot\frac{1}{[\mathbf{Q}(\zeta_{12},\sqrt[6]{\alpha_2},\sqrt[4]{-3}):\mathbf{Q}(\sqrt{-3})]}+2\cdot\frac{1}{[\mathbf{Q}(\zeta_{12},\sqrt[6]{\alpha_2},\sqrt[4]{-3},\sqrt{2}):\mathbf{Q}(\sqrt{-3})]}+\frac{1}{[\mathbf{Q}(\zeta_{12},\sqrt[6]{\alpha_2},\sqrt{2}):\mathbf{Q}(\sqrt{-3})]}=\frac{1}{6}+\frac{1}{12}+\frac{1}{12}=\frac{1}{3}$$

Case 10: $\alpha_1\in$ Set 1, and $\alpha_2\in$ Set 3: 1/4

Since $\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6=1$ and $\left(\frac{\alpha_2}{\mathfrak{p}}\right)_6=\zeta_3^{3,6}$, we have

$$\left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6+\left(\frac{\alpha_2}{\mathfrak{p}}\right)_6}{\bar{\mathfrak{p}}}\right)=\begin{cases}0 & (1\text{ in }2\text{ cases})\\ \left(\frac{2}{\bar{\mathfrak{p}}}\right) & (1\text{ in }2\text{ cases})\end{cases}$$

So we have

$$\left(\frac{a_p(E^3_{\alpha_1})+a_p(E^3_{\alpha_2})}{p}\right)=\begin{cases}\left(\frac{-3}{p}\right)_4\cdot\left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6+\left(\frac{\alpha_2}{\mathfrak{p}}\right)_6}{\bar{\mathfrak{p}}}\right) & p\equiv 1\pmod{12}\\ \left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6+\left(\frac{\alpha_2}{\mathfrak{p}}\right)_6}{\bar{\mathfrak{p}}}\right) & p\equiv 7\pmod{12}\end{cases}$$

$$=\begin{cases}0 & (1\text{ case}),\ p\equiv 1\pmod{12}\\ \left(\frac{-3}{p}\right)_4\cdot\left(\frac{2}{\bar{\mathfrak{p}}}\right) & (1\text{ case}),\ p\equiv 1\pmod{12}\\ 0 & (1\text{ case}),\ p\equiv 7\pmod{12}\\ \left(\frac{2}{\bar{\mathfrak{p}}}\right) & (1\text{ case}),\ p\equiv 7\pmod{12}\end{cases}$$

So the density is

$$2\cdot\frac{1}{[\mathbf{Q}(\zeta_{12},\sqrt[6]{\alpha_2},\sqrt[4]{-3},\sqrt{2}):\mathbf{Q}(\sqrt{-3})]}+\frac{1}{[\mathbf{Q}(\zeta_{12},\sqrt[6]{\alpha_2},\sqrt{2}):\mathbf{Q}(\sqrt{-3})]}=\frac{1}{8}+\frac{1}{8}=\frac{1}{4}$$

Case 11: $\alpha_1\in$ Set 1, and $\alpha_2\in$ Set 4: 1/4

Since $\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6 = 1$, and $\left(\frac{\alpha_2}{\mathfrak{p}}\right)_6 = \begin{cases} 1 & \text{if } p \equiv 1 \pmod{12} \\ -1 & \text{if } p \equiv 7 \pmod{12} \end{cases}$

we have

$$\left(\frac{a_p(E^3_{\alpha_1}) + a_p(E^3_{\alpha_2})}{p}\right) = \begin{cases} \left(\frac{-3}{p}\right)_4 \cdot \left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6 + \left(\frac{\alpha_2}{\mathfrak{p}}\right)_6}{\bar{\mathfrak{p}}}\right) & p \equiv 1 \pmod{12} \\ \left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6 + \left(\frac{\alpha_2}{\mathfrak{p}}\right)_6}{\bar{\mathfrak{p}}}\right) & p \equiv 7 \pmod{12} \end{cases}$$

$$= \begin{cases} \left(\frac{-3}{p}\right)_4 \cdot \left(\frac{2}{\bar{\mathfrak{p}}}\right) & \text{(1 case), } p \equiv 1 \pmod{12} \\ 0 & \text{(1 case), } p \equiv 7 \pmod{12} \end{cases}$$

So the density is $2 \cdot \frac{1}{[\mathbf{Q}(\zeta_{12}, \sqrt[4]{-3}, \sqrt{2}) : \mathbf{Q}(\sqrt{-3})]} = \frac{1}{4}$

Case 12: $\alpha_1 \in$ Set 1, and $\alpha_2 \in$ Set 5: 5/12

Since $\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6 = 1$, and $\left(\frac{\alpha_2}{\mathfrak{p}}\right)_6 = \begin{cases} \zeta_3^{1,2,3} & \text{if } p \equiv 1 \pmod{12} \\ -\zeta_3^{1,2,3} & \text{if } p \equiv 7 \pmod{12} \end{cases}$, we have

$$\left(\frac{a_p(E^3_{\alpha_1})+a_p(E^3_{\alpha_2})}{p}\right)=\begin{cases}\left(\frac{-3}{p}\right)_4\cdot\left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6+\left(\frac{\alpha_2}{\mathfrak{p}}\right)_6}{\bar{\mathfrak{p}}}\right) & p\equiv 1 \pmod{12}\\ \left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6+\left(\frac{\alpha_2}{\mathfrak{p}}\right)_6}{\bar{\mathfrak{p}}}\right) & p\equiv 7 \pmod{12}\end{cases}$$

$$=\begin{cases}\left(\frac{-3}{p}\right)_4\cdot\left(\frac{\zeta_6,\zeta_6^5,2}{\bar{\mathfrak{p}}}\right) & \text{if } p\equiv 1 \pmod{12}\\ \left(\frac{\sqrt{-3}}{\bar{\mathfrak{p}}}\right)\cdot\left(\frac{\zeta_6^4,\zeta_6^5,0}{\bar{\mathfrak{p}}}\right) & \text{if } p\equiv 7 \pmod{12}\end{cases}$$

$$=\begin{cases}\left(\frac{-3}{p}\right)_4 & (2\text{ cases}),\ p\equiv 1 \pmod{12}\\ \left(\frac{-3}{p}\right)_4\cdot\left(\frac{2}{\bar{\mathfrak{p}}}\right) & (1\text{ case}),\ p\equiv 1 (\mathrm{mod}\ 12)\\ 0 & (1\text{ case}),\ p\equiv 7 \pmod{12}\\ \left(\frac{\sqrt{-3}}{\bar{\mathfrak{p}}}\right) & (1\text{ case}),\ p\equiv 7 \pmod{12}\\ -\left(\frac{\sqrt{-3}}{\bar{\mathfrak{p}}}\right) & (1\text{ case}),\ p\equiv 7 \pmod{12}\end{cases}$$

So the density is

$$2\cdot\frac{1}{[\mathbf{Q}(\zeta_{12},\sqrt[6]{\alpha_2},\sqrt[4]{-3}):\mathbf{Q}(\sqrt{-3})]}+2\cdot\frac{1}{[\mathbf{Q}(\zeta_{12},\sqrt[6]{\alpha_2},\sqrt[4]{-3},\sqrt{2}):\mathbf{Q}(\sqrt{-3})]}$$
$$+2\cdot\frac{1}{[\mathbf{Q}(\zeta_{12},\sqrt[6]{\alpha_2},\sqrt[4]{-3}):\mathbf{Q}(\sqrt{-3})]}$$
$$=\frac{1}{6}+\frac{1}{12}+\frac{1}{6}=\frac{5}{12}$$

Case 13: $\alpha_1\in$ Set 1, and $\alpha_2\in$ Set 7: $5/12$

We have

$$\left(\frac{a_p(E^3_{\alpha_1})+a_p(E^3_{\alpha_2})}{p}\right)=\begin{cases}\left(\frac{-3}{p}\right)_4\cdot\left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6+\left(\frac{\alpha_2}{\mathfrak{p}}\right)_6}{\bar{\mathfrak{p}}}\right) & \text{p} \equiv 1(\text{mod } 12)\\ \left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6+\left(\frac{\alpha_2}{\mathfrak{p}}\right)_6}{\bar{\mathfrak{p}}}\right) & \text{p} \equiv 7(\text{mod } 12)\end{cases}$$

$$=\begin{cases}\left(\frac{-3}{p}\right)_4\cdot\left(\frac{0,2,\zeta_6,\zeta_6^5,\sqrt{-3}\cdot\zeta_6^4,\sqrt{-3}\cdot\zeta_6^5}{\bar{\mathfrak{p}}}\right) & \text{if p} \equiv 1(\text{mod } 12)\\ \left(\frac{0,2,\zeta_6,\zeta_6^5,\sqrt{-3}\cdot\zeta_6^4,\sqrt{-3}\cdot\zeta_6^5}{\bar{\mathfrak{p}}}\right) & \text{if p} \equiv 7(\text{mod } 12)\end{cases}$$

$$=\begin{cases}\left(\frac{-3}{p}\right)_4 & \text{(2 cases), p} \equiv 1 \text{ (mod 12)}\\ \left(\frac{-3}{p}\right)_4\cdot\left(\frac{2}{\bar{\mathfrak{p}}}\right) & \text{(1 case), p} \equiv 1(\text{mod 12})\\ 1 & \text{(2 cases), p} \equiv 1(\text{mod 12})\\ \left(\frac{2}{\bar{\mathfrak{p}}}\right) & \text{(1 case), p} \equiv 7(\text{mod 12})\\ \pm\left(\frac{\sqrt{-3}}{\bar{\mathfrak{p}}}\right) & \text{(2 cases), p} \equiv 7(\text{mod 12})\end{cases}$$

So the density is

$$2\cdot\frac{1}{[\mathbf{Q}(\zeta_{12},\sqrt[6]{\alpha_2},\sqrt[4]{-3}):\mathbf{Q}(\sqrt{-3})]}+2\cdot\frac{1}{[\mathbf{Q}(\zeta_{12},\sqrt[6]{\alpha_2},\sqrt[4]{-3},\sqrt{2}):\mathbf{Q}(\sqrt{-3})]}$$
$$+2\cdot\frac{1}{[\mathbf{Q}(\zeta_{12},\sqrt[6]{\alpha_2}):\mathbf{Q}(\sqrt{-3})]}+\frac{1}{[\mathbf{Q}(\zeta_{12},\sqrt[6]{\alpha_2},\sqrt{2}):\mathbf{Q}(\sqrt{-3})]}$$
$$+2\cdot\frac{1}{[\mathbf{Q}(\zeta_{12},\sqrt[6]{\alpha_2},\sqrt[4]{-3}):\mathbf{Q}(\sqrt{-3})]}$$
$$=\frac{1}{12}+\frac{1}{24}+\frac{1}{6}+\frac{1}{24}+\frac{1}{12}=\frac{5}{12}$$

Case 14: $\alpha_1\in$ Set 2, and $\alpha_2\in$ Set 3: $5/12$

Since $\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6=\zeta_3^{1,2,3}$ and $\left(\frac{\alpha_2}{\mathfrak{p}}\right)_6=\pm1$, we have

$$\left(\frac{a_p(E^3_{\alpha_1})+a_p(E^3_{\alpha_2})}{p}\right) = \begin{cases} \left(\frac{-3}{p}\right)_4 \cdot \left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6 + \left(\frac{\alpha_2}{\mathfrak{p}}\right)_6}{\bar{\mathfrak{p}}}\right) & p \equiv 1 \pmod{12} \\ \left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6 + \left(\frac{\alpha_2}{\mathfrak{p}}\right)_6}{\bar{\mathfrak{p}}}\right) & p \equiv 7 \pmod{12} \end{cases}$$

$$= \begin{cases} \left(\frac{-3}{p}\right)_4 \cdot \left(\frac{0, 2, \zeta_6, \zeta_6^5, \sqrt{-3}\cdot\zeta_6, \sqrt{-3}\cdot\zeta_6^2}{\bar{\mathfrak{p}}}\right) & \text{if } p \equiv 1 \pmod{12} \\ \left(\frac{0, 2, \zeta_6, \zeta_6^5, \sqrt{-3}\cdot\zeta_6, \sqrt{-3}\cdot\zeta_6^2}{\bar{\mathfrak{p}}}\right) & \text{if } p \equiv 7 \pmod{12} \end{cases}$$

$$= \begin{cases} \left(\frac{-3}{p}\right)_4 & \text{(2 cases), } p \equiv 1 \pmod{12} \\ \left(\frac{-3}{p}\right)_4 \cdot \left(\frac{2}{\bar{\mathfrak{p}}}\right) & \text{(1 case), } p \equiv 1 \pmod{12} \\ 1 & \text{(2 cases), } p \equiv 1 (\text{mod } 12) \\ \left(\frac{2}{\bar{\mathfrak{p}}}\right) & \text{(1 case), } p \equiv 7 \pmod{12} \\ \pm\left(\frac{\sqrt{-3}}{\bar{\mathfrak{p}}}\right) & \text{(2 cases), } p \equiv 7 \pmod{12} \end{cases}$$

So the density is

$$2\cdot\frac{1}{[\mathbf{Q}(\zeta_{12}, \sqrt[6]{\alpha_1}, \sqrt[6]{\alpha_2}, \sqrt[4]{-3}) : \mathbf{Q}(\sqrt{-3})]} + 2\cdot\frac{1}{[\mathbf{Q}(\zeta_{12}, \sqrt[6]{\alpha_1}, \sqrt[6]{\alpha_2}, \sqrt[4]{-3}, \sqrt{2}) : \mathbf{Q}(\sqrt{-3})]}$$
$$+ 2\cdot\frac{1}{[\mathbf{Q}(\zeta_{12}, \sqrt[6]{\alpha_1}, \sqrt[6]{\alpha_2}) : \mathbf{Q}(\sqrt{-3})]} + \frac{1}{[\mathbf{Q}(\zeta_{12}, \sqrt[6]{\alpha_1}, \sqrt[6]{\alpha_2}, \sqrt{2}) : \mathbf{Q}(\sqrt{-3})]}$$
$$+ 2\cdot\frac{1}{[\mathbf{Q}(\zeta_{12}, \sqrt[6]{\alpha_1}, \sqrt[6]{\alpha_2}, \sqrt[4]{-3}) : \mathbf{Q}(\sqrt{-3})]}$$
$$= \frac{1}{12} + \frac{1}{24} + \frac{1}{6} + \frac{1}{24} + \frac{1}{12} = \frac{5}{12}$$

Case 15: $\alpha_1 \in$ Set 2, and $\alpha_2 \in$ Set 4: $5/12$

Since $\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6 = \zeta_3^{1,2,3}$, and $\left(\frac{\alpha_2}{\mathfrak{p}}\right)_6 = \begin{cases} 1 & \text{if } p \equiv 1 \pmod{12} \\ -1 & \text{if } p \equiv 7 \pmod{12} \end{cases}$

we have

$$\left(\frac{a_p(E^3_{\alpha_1})+a_p(E^3_{\alpha_2})}{p}\right) = \begin{cases} \left(\frac{-3}{p}\right)_4 \cdot \left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6 + \left(\frac{\alpha_2}{\mathfrak{p}}\right)_6}{\bar{\mathfrak{p}}}\right) & p \equiv 1 \pmod{12} \\ \left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6 + \left(\frac{\alpha_2}{\mathfrak{p}}\right)_6}{\bar{\mathfrak{p}}}\right) & p \equiv 7 \pmod{12} \end{cases}$$

$$= \begin{cases} \left(\frac{-3}{p}\right)_4 \cdot \left(\frac{2, \zeta_6, \zeta_6^5}{\bar{\mathfrak{p}}}\right) & \text{if } p \equiv 1 \pmod{12} \\ \left(\frac{0, \sqrt{-3}\cdot\zeta_6, \sqrt{-3}\cdot\zeta_6^2}{\bar{\mathfrak{p}}}\right) & \text{if } p \equiv 7 \pmod{12} \end{cases}$$

$$= \begin{cases} \left(\frac{-3}{p}\right)_4 & \text{(2 cases), } p \equiv 1 \pmod{12} \\ \left(\frac{-3}{p}\right)_4 \cdot \left(\frac{2}{\bar{\mathfrak{p}}}\right) & \text{(1 case), } p \equiv 1 \pmod{12} \\ \pm\left(\frac{\sqrt{-3}}{\bar{\mathfrak{p}}}\right) & \text{(2 cases), } p \equiv 7 \pmod{12} \end{cases}$$

So the density is

$$2\cdot\frac{1}{[\mathbf{Q}(\zeta_{12}, \sqrt[6]{\alpha_1}, \sqrt[4]{-3}) : \mathbf{Q}(\sqrt{-3})]} + 2\cdot\frac{1}{[\mathbf{Q}(\zeta_{12}, \sqrt[6]{\alpha_1}, \sqrt[4]{-3}, \sqrt{2}) : \mathbf{Q}(\sqrt{-3})]}$$
$$+ 2\cdot\frac{1}{[\mathbf{Q}(\zeta_{12}, \sqrt[6]{\alpha_1}, \sqrt[4]{-3}) : \mathbf{Q}(\sqrt{-3})]}$$
$$= \frac{1}{6}+\frac{1}{12}+\frac{1}{6} = \frac{5}{12}$$

Case 16: $\alpha_1 \in$ Set 2, and $\alpha_2 \in$ Set 5: $5/12$

Since $\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6 = \zeta_3^{1,2,3}$, and $\left(\frac{\alpha_i}{\mathfrak{p}}\right)_6 = \begin{cases} \zeta_3^{1,2,3} & \text{if } p \equiv 1 \pmod{12} \\ -\zeta_3^{1,2,3} & \text{if } p \equiv 7 \pmod{12} \end{cases}$,

we have

$$\left(\frac{a_p(E^3_{\alpha_1}) + a_p(E^3_{\alpha_2})}{p}\right) = \begin{cases} \left(\frac{-3}{p}\right)_4 \cdot \left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6 + \left(\frac{\alpha_2}{\mathfrak{p}}\right)_6}{\bar{\mathfrak{p}}}\right) & p \equiv 1 \pmod{12} \\ \left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6 + \left(\frac{\alpha_2}{\mathfrak{p}}\right)_6}{\bar{\mathfrak{p}}}\right) & p \equiv 7 \pmod{12} \end{cases}$$

$$= \begin{cases} \left(\frac{-3}{p}\right)_4 \cdot \left(\frac{2 \cdot \zeta_6^{2,4,6}, \zeta_6^{1,3,5}}{\bar{\mathfrak{p}}}\right) & \text{if } p \equiv 1 \pmod{12} \\ \left(\frac{0, \sqrt{-3} \cdot \zeta_6^{1,2,3,4,5,6}}{\bar{\mathfrak{p}}}\right) & \text{if } p \equiv 7 \pmod{12} \end{cases}$$

$$= \begin{cases} \left(\frac{-3}{p}\right)_4 & \text{(6 cases), } p \equiv 1 \pmod{12} \\ \left(\frac{-3}{p}\right)_4 \cdot \left(\frac{2}{\bar{\mathfrak{p}}}\right) & \text{(3 case), } p \equiv 1 \pmod{12} \\ \pm\left(\frac{\sqrt{-3}}{\bar{\mathfrak{p}}}\right) & \text{(6 cases), } p \equiv 7 \pmod{12} \end{cases}$$

So the density is

$$6 \cdot \frac{1}{[\mathbf{Q}(\zeta_{12}, \sqrt[6]{\alpha_1}, \sqrt[6]{\alpha_2}, \sqrt[4]{-3}) : \mathbf{Q}(\sqrt{-3})]} + 3 \cdot 2 \cdot \frac{1}{[\mathbf{Q}(\zeta_{12}, \sqrt[6]{\alpha_1}, \sqrt[6]{\alpha_2}, \sqrt[4]{-3}, \sqrt{2}) : \mathbf{Q}(\sqrt{-3})]} + 6 \cdot \frac{1}{[\mathbf{Q}(\zeta_{12}, \sqrt[6]{\alpha_1}, \sqrt[6]{\alpha_2}, \sqrt[4]{-3}) : \mathbf{Q}(\sqrt{-3})]} = \frac{1}{6} + \frac{1}{12} + \frac{1}{6} = \frac{5}{12}$$

Case 17: $\alpha_1 \in$ Set 2, and $\alpha_2 \in$ Set 7: $5/12$

Since $\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6 = \zeta_3^{1,2,3}$, we have

$$\left(\frac{a_p(E^3_{\alpha_1})+a_p(E^3_{\alpha_2})}{p}\right)=\begin{cases}\left(\frac{-3}{p}\right)_4\cdot\left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6+\left(\frac{\alpha_2}{\mathfrak{p}}\right)_6}{\bar{\mathfrak{p}}}\right) & p\equiv 1 \pmod{12}\\ \left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6+\left(\frac{\alpha_2}{\mathfrak{p}}\right)_6}{\bar{\mathfrak{p}}}\right) & p\equiv 7 \pmod{12}\end{cases}$$

$$=\begin{cases}\left(\frac{-3}{p}\right)_4\cdot\left(\frac{0,\zeta_6^{1,2,3},2\cdot\zeta_6^{2,4,6},\sqrt{-3}\cdot\zeta_6^{1,2,3,4,5,6}}{\bar{\mathfrak{p}}}\right) & \text{if } p\equiv 1 \pmod{12}\\ \left(\frac{0,\zeta_6^{1,2,3},2\cdot\zeta_6^{2,4,6},\sqrt{-3}\cdot\zeta_6^{1,2,3,4,5,6}}{\bar{\mathfrak{p}}}\right) & \text{if } p\equiv 7 \pmod{12}\end{cases}$$

$$=\begin{cases}\left(\frac{-3}{p}\right)_4 & (6\text{ cases}),\ p\equiv 1 \pmod{12}\\ \left(\frac{-3}{p}\right)_4\cdot\left(\frac{2}{\bar{\mathfrak{p}}}\right) & (3\text{ cases}),\ p\equiv 1 \pmod{12}\\ \left(\frac{-3}{p}\right)_4\cdot\left(\frac{\sqrt{-3}}{\bar{\mathfrak{p}}}\right)=1 & (6\text{ case}),\ p\equiv 1 \pmod{12}\\ \left(\frac{2}{\bar{\mathfrak{p}}}\right) & (3\text{ cases}),\ p\equiv 7 \pmod{12}\\ \pm\left(\frac{\sqrt{-3}}{\bar{\mathfrak{p}}}\right) & (6\text{ cases}),\ p\equiv 7 \pmod{12}\end{cases}$$

So the density is

$$\begin{aligned}&6\cdot\frac{1}{[\mathbf{Q}(\zeta_{12},\sqrt[6]{\alpha_1},\sqrt[6]{\alpha_2},\sqrt[4]{-3}):\mathbf{Q}(\sqrt{-3})]}\\&+3\cdot 2\cdot\frac{1}{[\mathbf{Q}(\zeta_{12},\sqrt[6]{\alpha_1},\sqrt[6]{\alpha_2},\sqrt[4]{-3},\sqrt{2}):\mathbf{Q}(\sqrt{-3})]}+6\cdot\frac{1}{[\mathbf{Q}(\zeta_{12},\sqrt[6]{\alpha_1},\sqrt[6]{\alpha_2}):\mathbf{Q}(\sqrt{-3})]}\\&+3\cdot\frac{1}{[\mathbf{Q}(\zeta_{12},\sqrt[6]{\alpha_1},\sqrt[6]{\alpha_2},\sqrt{2}):\mathbf{Q}(\sqrt{-3})]}+6\cdot\frac{1}{[\mathbf{Q}(\zeta_{12},\sqrt[6]{\alpha_1},\sqrt[6]{\alpha_2},\sqrt[4]{-3}):\mathbf{Q}(\sqrt{-3})]}\\&=\frac{1}{12}+\frac{1}{24}+\frac{1}{6}+\frac{1}{24}+\frac{1}{12}=\frac{5}{12}\end{aligned}$$

Case 18: $\alpha_1\in$ Set 3, and $\alpha_2\in$ Set 4: $1/4$

Since $\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6=\pm 1$, and $\left(\frac{\alpha_2}{\mathfrak{p}}\right)_6=\begin{cases}1 & \text{if } p\equiv 1 \pmod{12}\\ -1 & \text{if } p\equiv 7 \pmod{12}\end{cases}$, we have

$$\left(\frac{a_p(E^3_{\alpha_1}) + a_p(E^3_{\alpha_2})}{p}\right) = \begin{cases} \left(\frac{-3}{p}\right)_4 \cdot \left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6 + \left(\frac{\alpha_2}{\mathfrak{p}}\right)_6}{\bar{\mathfrak{p}}}\right) & p \equiv 1 \pmod{12} \\ \left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6 + \left(\frac{\alpha_2}{\mathfrak{p}}\right)_6}{\bar{\mathfrak{p}}}\right) & p \equiv 7 \pmod{12} \end{cases}$$

$$= \begin{cases} \left(\frac{-3}{p}\right)_4 \cdot \left(\frac{2}{\bar{\mathfrak{p}}}\right) & \text{(1 case)}, p \equiv 1 \pmod{12} \\ \left(\frac{-2}{\bar{\mathfrak{p}}}\right) & \text{(1 case)}, p \equiv 7 \pmod{12} \end{cases}$$

So the density is

$$2 \cdot \frac{1}{[\mathbf{Q}(\zeta_{12}, \sqrt{\alpha_1}, \sqrt[4]{-3}, \sqrt{2}) : \mathbf{Q}(\sqrt{-3})]} + \frac{1}{[\mathbf{Q}(\zeta_{12}, \sqrt{\alpha_1}, \sqrt{-2}) : \mathbf{Q}(\sqrt{-3})]} = \frac{1}{8} + \frac{1}{8} = 1/4.$$

Case 19: $\alpha_1 \in$ Set 3, and $\alpha_2 \in$ Set 5: 7/12

Since $\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6 = \pm 1$, and $\left(\frac{\alpha_i}{\mathfrak{p}}\right)_6 = \begin{cases} \zeta_3^{1,2,3} & \text{if } p \equiv 1 \pmod{12} \\ -\zeta_3^{1,2,3} & \text{if } p \equiv 7 \pmod{12} \end{cases}$, we have

$$\left(\frac{a_p(E^3_{\alpha_1})+a_p(E^3_{\alpha_2})}{p}\right)=\begin{cases}\left(\frac{-3}{p}\right)_4\cdot\left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6+\left(\frac{\alpha_2}{\mathfrak{p}}\right)_6}{\bar{\mathfrak{p}}}\right) & p\equiv 1 \pmod{12}\\ \left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6+\left(\frac{\alpha_2}{\mathfrak{p}}\right)_6}{\bar{\mathfrak{p}}}\right) & p\equiv 7 \pmod{12}\end{cases}$$

$$=\begin{cases}\left(\frac{-3}{p}\right)_4\cdot\left(\frac{0,2,\zeta_6,\zeta_6^5,\sqrt{-3}\cdot\zeta_6,\sqrt{-3}\cdot\zeta_6^2}{\bar{\mathfrak{p}}}\right) & \text{if } p\equiv 1 \pmod{12}\\ \left(\frac{0,-2,\zeta_6^2,\zeta_6^4,\sqrt{-3}\cdot\zeta_6^5,\sqrt{-3}\cdot\zeta_6^4}{\bar{\mathfrak{p}}}\right) & \text{if } p\equiv 7 \pmod{12}\end{cases}$$

$$=\begin{cases}\left(\frac{-3}{p}\right)_4 & (2 \text{ cases}),\ p\equiv 1 (\text{mod } 12)\\ \left(\frac{-3}{p}\right)_4\cdot\left(\frac{2}{\bar{\mathfrak{p}}}\right) & (1 \text{ case}),\ p\equiv 1 \pmod{12}\\ \left(\frac{-3}{p}\right)_4\cdot\left(\frac{\sqrt{-3}}{\bar{\mathfrak{p}}}\right)=1 & (2 \text{ cases}),\ p\equiv 1 \pmod{12}\\ \left(\frac{-2}{\bar{\mathfrak{p}}}\right) & (1 \text{ case}),\ p\equiv 7 \pmod{12}\\ 1 & (2 \text{ cases}),\ p\equiv 7 \pmod{12}\\ \pm\left(\frac{\sqrt{-3}}{\bar{\mathfrak{p}}}\right) & (2 \text{ cases}),\ p\equiv 7 \pmod{12}\end{cases}$$

So the density is

$$2\cdot\frac{1}{[\mathbf{Q}(\zeta_{12},\sqrt[6]{\alpha_1},\sqrt[6]{\alpha_2},\sqrt[4]{-3}):\mathbf{Q}(\sqrt{-3})]}+2\cdot\frac{1}{[\mathbf{Q}(\zeta_{12},\sqrt[6]{\alpha_1},\sqrt[6]{\alpha_2},\sqrt[4]{-3},\sqrt{2}):\mathbf{Q}(\sqrt{-3})]}$$
$$+4\cdot\frac{1}{[\mathbf{Q}(\zeta_{12},\sqrt[6]{\alpha_1},\sqrt[6]{\alpha_2}):\mathbf{Q}(\sqrt{-3})]}+\frac{1}{[\mathbf{Q}(\zeta_{12},\sqrt[6]{\alpha_1},\sqrt[6]{\alpha_2},\sqrt{2}):\mathbf{Q}(\sqrt{-3})]}$$
$$+2\cdot\frac{1}{[\mathbf{Q}(\zeta_{12},\sqrt[6]{\alpha_1},\sqrt[6]{\alpha_2},\sqrt[4]{-3}):\mathbf{Q}(\sqrt{-3})]}$$
$$=\frac{1}{12}+\frac{1}{24}+\frac{1}{3}+\frac{1}{24}+\frac{1}{12}=\frac{7}{12}$$

Case 20: $\alpha_1\in$ Set 3, and $\alpha_2\in$ Set 7: 1/2

Since $\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6=\pm1$, we have

$$\left(\frac{a_p(E^3_{\alpha_1}) + a_p(E^3_{\alpha_2})}{p}\right) = \begin{cases} \left(\frac{-3}{p}\right)_4 \cdot \left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6 + \left(\frac{\alpha_2}{\mathfrak{p}}\right)_6}{\bar{\mathfrak{p}}}\right) & p \equiv 1 \pmod{12} \\ \left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6 + \left(\frac{\alpha_2}{\mathfrak{p}}\right)_6}{\bar{\mathfrak{p}}}\right) & p \equiv 7 \pmod{12} \end{cases}$$

$$= \begin{cases} \left(\frac{-3}{p}\right)_4 \cdot \left(\frac{0, 2, \zeta_6^{1,2,4,5}, 2 \cdot \zeta_6^{3,6}, \sqrt{-3} \cdot \zeta_6^{1,2,4,5}}{\bar{\mathfrak{p}}}\right) & \text{if } p \equiv 1 \pmod{12} \\ \left(\frac{0, 2, \zeta_6^{1,2,4,5}, 2 \cdot \zeta_6^{3,6}, \sqrt{-3} \cdot \zeta_6^{1,2,4,5}}{\bar{\mathfrak{p}}}\right) & \text{if } p \equiv 7 \pmod{12} \end{cases}$$

$$= \begin{cases} \left(\frac{-3}{p}\right)_4 & (4 \text{ cases}),\ p \equiv 1 \pmod{12} \\ \left(\frac{-3}{p}\right)_4 \cdot \left(\frac{2}{\bar{\mathfrak{p}}}\right) & (2 \text{ cases}),\ p \equiv 1 \pmod{12} \\ \left(\frac{-3}{p}\right)_4 \cdot \left(\frac{\sqrt{-3}}{\bar{\mathfrak{p}}}\right) = 1 & (4 \text{ cases}),\ p \equiv 1 \pmod{12} \\ \pm\left(\frac{2}{\bar{\mathfrak{p}}}\right) & (2 \text{ cases}),\ p \equiv 7 \pmod{12} \\ 1 & (2 \text{ cases}),\ p \equiv 7 \pmod{12} \\ \pm\left(\frac{\sqrt{-3}}{\bar{\mathfrak{p}}}\right) & (4 \text{ cases}),\ p \equiv 7 \pmod{12} \end{cases}$$

So the density is

$$4 \cdot \frac{1}{[\mathbf{Q}(\zeta_{12}, \sqrt[6]{\alpha_1}, \sqrt[6]{\alpha_2}, \sqrt[4]{-3}) : \mathbf{Q}(\sqrt{-3})]} + 2 \cdot 2 \cdot \frac{1}{[\mathbf{Q}(\zeta_{12}, \sqrt[6]{\alpha_1}, \sqrt[6]{\alpha_2}, \sqrt[4]{-3}, \sqrt{2}) : \mathbf{Q}(\sqrt{-3})]}$$

$$+ 6 \cdot \frac{1}{[\mathbf{Q}(\zeta_{12}, \sqrt[6]{\alpha_1}, \sqrt[6]{\alpha_2}) : \mathbf{Q}(\sqrt{-3})]} + 2 \cdot \frac{1}{[\mathbf{Q}(\zeta_{12}, \sqrt[6]{\alpha_1}, \sqrt[6]{\alpha_2}, \sqrt{2}) : \mathbf{Q}(\sqrt{-3})]}$$

$$+ 4 \cdot \frac{1}{[\mathbf{Q}(\zeta_{12}, \sqrt[6]{\alpha_1}, \sqrt[6]{\alpha_2}, \sqrt[4]{-3}) : \mathbf{Q}(\sqrt{-3})]}$$

$$= \frac{1}{12} + \frac{1}{24} + \frac{1}{4} + \frac{1}{24} + \frac{1}{12} = \frac{1}{2}$$

Case 21: $\alpha_1 \in$ Set 4, and $\alpha_2 \in$ Set 5: 2/3

Since $\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6 = \begin{cases} 1 & \text{if } p \equiv 1 \pmod{12} \\ -1 & \text{if } p \equiv 7 \pmod{12} \end{cases}$,

and $\left(\frac{\alpha_2}{\mathfrak{p}}\right)_6 = \begin{cases} \zeta_3^{1,2,3} & \text{if } p \equiv 1 \pmod{12} \\ -\zeta_3^{1,2,3} & \text{if } p \equiv 7 \pmod{12} \end{cases}$,

we have

$$\left(\frac{a_p(E^3_{\alpha_1}) + a_p(E^3_{\alpha_2})}{p}\right) = \begin{cases} \left(\frac{-3}{p}\right)_4 \cdot \left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6 + \left(\frac{\alpha_2}{\mathfrak{p}}\right)_6}{\bar{\mathfrak{p}}}\right) & p \equiv 1 \pmod{12} \\ \left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6 + \left(\frac{\alpha_2}{\mathfrak{p}}\right)_6}{\bar{\mathfrak{p}}}\right) & p \equiv 7 \pmod{12} \end{cases}$$

$$= \begin{cases} \left(\frac{-3}{p}\right)_4 \cdot \left(\frac{2, \zeta_6, \zeta_6^5}{\bar{\mathfrak{p}}}\right) & \text{if } p \equiv 1 \pmod{12} \\ \left(\frac{-2, \zeta_6^2, \zeta_6^4}{\bar{\mathfrak{p}}}\right) & \text{if } p \equiv 7 \pmod{12} \end{cases}$$

$$= \begin{cases} \left(\frac{-3}{p}\right)_4 & (2 \text{ cases}),\ p \equiv 1 \pmod{12} \\ \left(\frac{-3}{p}\right)_4 \cdot \left(\frac{2}{\bar{\mathfrak{p}}}\right) & (1 \text{ case}),\ p \equiv 1 \pmod{12} \\ 1 & (2 \text{ cases}),\ p \equiv 7 \pmod{12} \\ \left(\frac{-2}{\bar{\mathfrak{p}}}\right) & (1 \text{ case}),\ p \equiv 7 \pmod{12} \end{cases}$$

So the density is

$$2 \cdot \frac{1}{[\mathbf{Q}(\zeta_{12}, \sqrt[6]{\alpha_2}, \sqrt[4]{-3}) : \mathbf{Q}(\sqrt{-3})]} + 2 \cdot \frac{1}{[\mathbf{Q}(\zeta_{12}, \sqrt[6]{\alpha_2}, \sqrt[4]{-3}, \sqrt{2}) : \mathbf{Q}(\sqrt{-3})]}$$
$$+ 2 \cdot \frac{1}{[\mathbf{Q}(\zeta_{12}, \sqrt[6]{\alpha_2} : \mathbf{Q}(\sqrt{-3})]} + \frac{1}{[\mathbf{Q}(\zeta_{12}, \sqrt[6]{\alpha_2}, \sqrt{-2}) : \mathbf{Q}(\sqrt{-3})]}$$
$$= \frac{1}{6} + \frac{1}{12} + \frac{1}{3} + \frac{1}{12} = \frac{2}{3}$$

Case 22: $\alpha_1 \in$ Set 4, and $\alpha_2 \in$ Set 7: 7/12

Since $\left(\dfrac{\alpha_1}{\mathfrak{p}}\right)_6 = \begin{cases} 1 & \text{if } p \equiv 1 \pmod{12} \\ -1 & \text{if } p \equiv 7 \pmod{12} \end{cases}$, we have

$$\left(\frac{a_p(E^3_{\alpha_1}) + a_p(E^3_{\alpha_2})}{p}\right) = \begin{cases} \left(\dfrac{-3}{p}\right)_4 \cdot \left(\dfrac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6 + \left(\frac{\alpha_2}{\mathfrak{p}}\right)_6}{\bar{\mathfrak{p}}}\right) & p \equiv 1 \pmod{12} \\ \left(\dfrac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6 + \left(\frac{\alpha_2}{\mathfrak{p}}\right)_6}{\bar{\mathfrak{p}}}\right) & p \equiv 7 \pmod{12} \end{cases}$$

$$= \begin{cases} \left(\dfrac{-3}{p}\right)_4 \cdot \left(\dfrac{0, 2, \zeta_6^{1,5}, \sqrt{-3} \cdot \zeta_6^{4,5}}{\bar{\mathfrak{p}}}\right) & \text{if } p \equiv 1 \pmod{12} \\ \left(\dfrac{0, -2, \zeta_6^{2,4}, \sqrt{-3} \cdot \zeta_6^{1,2}}{\bar{\mathfrak{p}}}\right) & \text{if } p \equiv 7 \pmod{12} \end{cases}$$

$$= \begin{cases} \left(\dfrac{-3}{p}\right)_4 & \text{(2 cases), } p \equiv 1 \pmod{12} \\ \left(\dfrac{-3}{p}\right)_4 \cdot \left(\dfrac{2}{\bar{\mathfrak{p}}}\right) & \text{(1 case), } p \equiv 1 \pmod{12} \\ \left(\dfrac{-3}{p}\right)_4 \cdot \left(\dfrac{\sqrt{-3}}{\bar{\mathfrak{p}}}\right) = 1 & \text{(2 cases), } p \equiv 1 \pmod{12} \\ \left(\dfrac{-2}{\bar{\mathfrak{p}}}\right) & \text{(1 case), } p \equiv 7 \pmod{12} \\ 1 & \text{(2 cases), } p \equiv 7 \pmod{12} \\ \pm\left(\dfrac{\sqrt{-3}}{\bar{\mathfrak{p}}}\right) & \text{(2 cases), } p \equiv 7 \pmod{12} \end{cases}$$

So the density is

$$2\cdot\frac{1}{[\mathbf{Q}(\zeta_{12},\sqrt[6]{\alpha_2},\sqrt[4]{-3}):\mathbf{Q}(\sqrt{-3})]}+2\cdot\frac{1}{[\mathbf{Q}(\zeta_{12},\sqrt[6]{\alpha_2},\sqrt[4]{-3},\sqrt{2}):\mathbf{Q}(\sqrt{-3})]}$$

$$+4\cdot\frac{1}{[\mathbf{Q}(\zeta_{12},\sqrt[6]{\alpha_2}):\mathbf{Q}(\sqrt{-3})]}+\frac{1}{[\mathbf{Q}(\zeta_{12},\sqrt[6]{\alpha_2},\sqrt{-2}):\mathbf{Q}(\sqrt{-3})]}$$

$$+2\cdot\frac{1}{[\mathbf{Q}(\zeta_{12},\sqrt[6]{\alpha_2},\sqrt[4]{-3}):\mathbf{Q}(\sqrt{-3})]}$$

$$=\frac{1}{12}+\frac{1}{24}+\frac{1}{3}+\frac{1}{24}+\frac{1}{12}=\frac{7}{12}$$

Case 23: $\alpha_1\in$ Set 5, and $\alpha_2\in$ Set 7: 7/12

Since $\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6=\begin{cases}\zeta_3^{1,2,3} & \text{if } p\equiv 1 \pmod{12}\\ -\zeta_3^{1,2,3} & \text{if } p\equiv 7 \pmod{12}\end{cases}$, we have

$$\left(\frac{a_p(E^3_{\alpha_1})+a_p(E^3_{\alpha_2})}{p}\right)=\begin{cases}\left(\frac{-3}{p}\right)_4\cdot\left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6+\left(\frac{\alpha_2}{\mathfrak{p}}\right)_6}{\bar{\mathfrak{p}}}\right) & p\equiv 1 \pmod{12}\\ \left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6+\left(\frac{\alpha_2}{\mathfrak{p}}\right)_6}{\bar{\mathfrak{p}}}\right) & p\equiv 7 \pmod{12}\end{cases}$$

$$=\begin{cases}\left(\frac{-3}{p}\right)_4\cdot\left(\frac{0,\zeta_6^{1,3,5},2\cdot\zeta_6^{2,4,6},\sqrt{-3}\cdot\zeta_6^{1,2,3,4,5,6}}{\bar{\mathfrak{p}}}\right) & \text{if } p\equiv 1 \pmod{12}\\ \left(\frac{-3}{p}\right)_4\cdot\left(\frac{0,\zeta_6^{2,4,6},2\cdot\zeta_6^{1,3,5},\sqrt{-3}\cdot\zeta_6^{1,2,3,4,5,6}}{\bar{\mathfrak{p}}}\right) & \text{if } p\equiv 7 \pmod{12}\end{cases}$$

$$=\begin{cases}\left(\frac{-3}{p}\right)_4 & \text{(6 cases), } p\equiv 1 \pmod{12}\\ \left(\frac{-3}{p}\right)_4\cdot\left(\frac{2}{\bar{\mathfrak{p}}}\right) & \text{(3 cases), } p\equiv 1 \pmod{12}\\ \left(\frac{-3}{p}\right)_4\cdot\left(\frac{\sqrt{-3}}{\bar{\mathfrak{p}}}\right)=1 & \text{(6 cases), } p\equiv 1 \pmod{12}\\ \left(\frac{-2}{\bar{\mathfrak{p}}}\right) & \text{(3 cases), } p\equiv 7 \pmod{12}\\ 1 & \text{(6 cases), } p\equiv 7 \pmod{12}\\ \pm\left(\frac{\sqrt{-3}}{\bar{\mathfrak{p}}}\right) & \text{(6 cases), } p\equiv 7 \pmod{12}\end{cases}$$

So the density is

$$6\cdot\frac{1}{[\mathbf{Q}(\zeta_{12},\sqrt[6]{\alpha_1},\sqrt[6]{\alpha_2},\sqrt[4]{-3}):\mathbf{Q}(\sqrt{-3})]}$$
$$+3\cdot 2\cdot\frac{1}{[\mathbf{Q}(\zeta_{12},\sqrt[6]{\alpha_1},\sqrt[6]{\alpha_2},\sqrt[4]{-3},\sqrt{2}):\mathbf{Q}(\sqrt{-3})]}$$
$$+12\cdot\frac{1}{[\mathbf{Q}(\zeta_{12},\sqrt[6]{\alpha_1},\sqrt[6]{\alpha_2}):\mathbf{Q}(\sqrt{-3})]}+3\cdot\frac{1}{[\mathbf{Q}(\zeta_{12},\sqrt[6]{\alpha_1},\sqrt[6]{\alpha_2},\sqrt{-2}):\mathbf{Q}(\sqrt{-3})]}$$
$$+6\cdot\frac{1}{[\mathbf{Q}(\zeta_{12},\sqrt[6]{\alpha_1},\sqrt[6]{\alpha_2},\sqrt[4]{-3}):\mathbf{Q}(\sqrt{-3})]}$$
$$=\frac{1}{12}+\frac{1}{24}+\frac{1}{3}+\frac{1}{24}+\frac{1}{12}=\frac{7}{12}$$

Case 24: $\alpha_1/\alpha_2\in$ Set 1: 1/2

Let $\alpha_3=\alpha_1/\alpha_2$, we have $\left(\frac{\alpha_3}{\mathfrak{p}}\right)_6=1$, and by Lemma 2.1.1

$$\left(\frac{a_p(E^3_{\alpha_1})+a_p(E^3_{\alpha_2})}{p}\right)=\begin{cases}\left(\frac{-3}{p}\right)_4\cdot\left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6+\left(\frac{\alpha_2}{\mathfrak{p}}\right)_6}{\bar{\mathfrak{p}}}\right) & p\equiv 1 \pmod{12}\\ \left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6+\left(\frac{\alpha_2}{\mathfrak{p}}\right)_6}{\bar{\mathfrak{p}}}\right) & p\equiv 7 \pmod{12}\end{cases}$$
$$=\begin{cases}\left(\frac{-3}{p}\right)_4\cdot\left(\frac{\left(\frac{\alpha_3}{\mathfrak{p}}\right)_6+1}{\bar{\mathfrak{p}}}\right) & p\equiv 1 \pmod{12}\\ \left(\frac{\alpha_2}{\mathfrak{p}}\right)\cdot\left(\frac{\left(\frac{\alpha_3}{\mathfrak{p}}\right)_6+1}{\bar{\mathfrak{p}}}\right) & p\equiv 7 \pmod{12}\end{cases}$$
$$=\begin{cases}\left(\frac{-3}{p}\right)_4\cdot\left(\frac{2}{\bar{\mathfrak{p}}}\right) & p\equiv 1 \pmod{12}\\ \left(\frac{\alpha_2}{\mathfrak{p}}\right)\cdot\left(\frac{2}{\bar{\mathfrak{p}}}\right) & p\equiv 7 \pmod{12}\end{cases}$$

So the density is

$$2\cdot\frac{1}{[\mathbf{Q}(\zeta_{12},\sqrt[4]{-3},\sqrt{2}):\mathbf{Q}(\sqrt{-3})]}+2\cdot\frac{1}{[\mathbf{Q}(\zeta_{12},\sqrt{\alpha_2},\sqrt{2}):\mathbf{Q}(\sqrt{-3})]}=\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$$

Case 25: $\alpha_1/\alpha_2 \in$ Set 2: 1/2

Let $\alpha_3 = \alpha_1/\alpha_2$. We have $\left(\frac{\alpha_3}{\mathfrak{p}}\right)_6 = \zeta_3^{1,2,3}$, and by Lemma 2.1.1

$$\left(\frac{a_p(E^3_{\alpha_1}) + a_p(E^3_{\alpha_2})}{p}\right) = \begin{cases} \left(\frac{-3}{p}\right)_4 \cdot \left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6 + \left(\frac{\alpha_2}{\mathfrak{p}}\right)_6}{\bar{\mathfrak{p}}}\right) & p \equiv 1 \pmod{12} \\ \left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6 + \left(\frac{\alpha_2}{\mathfrak{p}}\right)_6}{\bar{\mathfrak{p}}}\right) & p \equiv 7 \pmod{12} \end{cases}$$

$$= \begin{cases} \left(\frac{-3}{p}\right)_4 \cdot \left(\frac{\left(\frac{\alpha_3}{\mathfrak{p}}\right)_6 + 1}{\bar{\mathfrak{p}}}\right) & p \equiv 1 \pmod{12} \\ \left(\frac{\alpha_2}{\mathfrak{p}}\right) \cdot \left(\frac{\left(\frac{\alpha_3}{\mathfrak{p}}\right)_6 + 1}{\bar{\mathfrak{p}}}\right) & p \equiv 7 \pmod{12} \end{cases}$$

$$= \begin{cases} \left(\frac{-3}{p}\right)_4 \cdot \left(\frac{2, \zeta_6, \zeta_6^5}{\bar{\mathfrak{p}}}\right) & p \equiv 1 \pmod{12} \\ \left(\frac{\alpha_2}{\mathfrak{p}}\right) \cdot \left(\frac{2, \zeta_6, \zeta_6^5}{\bar{\mathfrak{p}}}\right) & p \equiv 7 \pmod{12} \end{cases}$$

$$= \begin{cases} \left(\frac{-3}{p}\right)_4 & \text{(2 cases)}, p \equiv 1 \pmod{12} \\ \left(\frac{-3}{p}\right)_4 \cdot \left(\frac{2}{\bar{\mathfrak{p}}}\right) & \text{(1 case)}, p \equiv 1 \pmod{12} \\ \left(\frac{\alpha_2}{p}\right) \cdot \left(\frac{2}{\bar{\mathfrak{p}}}\right) & \text{(1 case)}, p \equiv 7 \pmod{12} \\ -\left(\frac{\alpha_2}{\bar{\mathfrak{p}}}\right) & \text{(2 cases)}, p \equiv 7 \pmod{12} \end{cases}$$

So the density is

$$2 \cdot \frac{1}{[\mathbf{Q}(\zeta_{12}, \sqrt[6]{\alpha_3}, \sqrt[4]{-3}) : \mathbf{Q}(\sqrt{-3})]} + 2 \cdot \frac{1}{[\mathbf{Q}(\zeta_{12}, \sqrt[6]{\alpha_3}, \sqrt[4]{-3}, \sqrt{2}) : \mathbf{Q}(\sqrt{-3})]}$$
$$2 \cdot \frac{1}{[\mathbf{Q}(\zeta_{12}, \sqrt[6]{\alpha_3}, \sqrt{\alpha_2}, \sqrt{2}) : \mathbf{Q}(\sqrt{-3})]} + 2 \cdot \frac{1}{[\mathbf{Q}(\zeta_{12}, \sqrt[6]{\alpha_3}, \sqrt{\alpha_2}) : \mathbf{Q}(\sqrt{-3})]}$$
$$= \frac{1}{6} + \frac{1}{12} + \frac{1}{12} + \frac{1}{6} = \frac{1}{2}$$

Case 26: $\alpha_1/\alpha_2 \in$ Set 3: $1/4$

Let $\alpha_3 = \alpha_1/\alpha_2$. We have $\left(\frac{\alpha_3}{\mathfrak{p}}\right)_6 = \pm 1$, and by Lemma 2.1.1

$$\left(\frac{a_p(E^3_{\alpha_1}) + a_p(E^3_{\alpha_2})}{p}\right) = \begin{cases} \left(\frac{-3}{p}\right)_4 \cdot \left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6 + \left(\frac{\alpha_2}{\mathfrak{p}}\right)_6}{\bar{\mathfrak{p}}}\right) & p \equiv 1 \pmod{12} \\ \left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6 + \left(\frac{\alpha_2}{\mathfrak{p}}\right)_6}{\bar{\mathfrak{p}}}\right) & p \equiv 7 \pmod{12} \end{cases}$$
$$= \begin{cases} \left(\frac{-3}{p}\right)_4 \cdot \left(\frac{\left(\frac{\alpha_3}{\mathfrak{p}}\right)_6 + 1}{\bar{\mathfrak{p}}}\right) & p \equiv 1 \pmod{12} \\ \left(\frac{\alpha_2}{\mathfrak{p}}\right) \cdot \left(\frac{\left(\frac{\alpha_3}{\mathfrak{p}}\right)_6 + 1}{\bar{\mathfrak{p}}}\right) & p \equiv 7 \pmod{12} \end{cases}$$
$$= \begin{cases} \left(\frac{-3}{p}\right)_4 \cdot \left(\frac{0, 2}{\bar{\mathfrak{p}}}\right) & p \equiv 1 \pmod{12} \\ \left(\frac{\alpha_2}{\mathfrak{p}}\right) \cdot \left(\frac{0, 2}{\bar{\mathfrak{p}}}\right) & p \equiv 7 \pmod{12} \end{cases}$$

So the density is

$$2 \cdot \frac{1}{[\mathbf{Q}(\zeta_{12}, \sqrt[6]{\alpha_3}, \sqrt[4]{-3}, \sqrt{2}) : \mathbf{Q}(\sqrt{-3})]} + 2 \cdot \frac{1}{[\mathbf{Q}(\zeta_{12}, \sqrt[6]{\alpha_3}, \sqrt{\alpha_2}, \sqrt{2}) : \mathbf{Q}(\sqrt{-3})]}$$
$$= \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$$

Case 27: $\alpha_1/\alpha_2 \in$ Set 4: $1/4$

Let $\alpha_3 = \alpha_1/\alpha_2$. We have $\left(\frac{\alpha_i}{\mathfrak{p}}\right)_6 = \begin{cases} 1 & \text{if } p \equiv 1 \pmod{12} \\ -1 & \text{if } p \equiv 7 \pmod{12} \end{cases}$, and by Lemma 2.1.1

$$\left(\frac{a_p(E^3_{\alpha_1}) + a_p(E^3_{\alpha_2})}{p}\right) = \begin{cases} \left(\frac{-3}{p}\right)_4 \cdot \left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6 + \left(\frac{\alpha_2}{\mathfrak{p}}\right)_6}{\bar{\mathfrak{p}}}\right) & p \equiv 1 \pmod{12} \\ \left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6 + \left(\frac{\alpha_2}{\mathfrak{p}}\right)_6}{\bar{\mathfrak{p}}}\right) & p \equiv 7 \pmod{12} \end{cases}$$

$$= \begin{cases} \left(\frac{-3}{p}\right)_4 \cdot \left(\frac{\left(\frac{\alpha_3}{\mathfrak{p}}\right)_6 + 1}{\bar{\mathfrak{p}}}\right) & p \equiv 1 \pmod{12} \\ \left(\frac{\alpha_2}{\mathfrak{p}}\right) \cdot \left(\frac{\left(\frac{\alpha_3}{\mathfrak{p}}\right)_6 + 1}{\bar{\mathfrak{p}}}\right) & p \equiv 7 \pmod{12} \end{cases}$$

$$= \begin{cases} \left(\frac{-3}{p}\right)_4 \cdot \left(\frac{2}{\bar{\mathfrak{p}}}\right) & p \equiv 1 \pmod{12} \\ \left(\frac{\alpha_2}{\mathfrak{p}}\right) \cdot \left(\frac{0}{\bar{\mathfrak{p}}}\right) & p \equiv 7 \pmod{12} \end{cases}$$

So the density is $2 \cdot \frac{1}{[\mathbf{Q}(\zeta_{12}, \sqrt[4]{-3}, \sqrt{2}) : \mathbf{Q}(\sqrt{-3})]} = \frac{1}{4}$

Case 28: $\alpha_1/\alpha_2 \in$ Set 5: 5/12

Let $\alpha_3 = \alpha_1/\alpha_2$. We have $\left(\frac{\alpha_i}{\mathfrak{p}}\right)_6 = \begin{cases} \zeta_3^{1,2,3} & \text{if } p \equiv 1 \pmod{12} \\ -\zeta_3^{1,2,3} & \text{if } p \equiv 7 \pmod{12} \end{cases}$, and by Lemma 2.1.1

$$\left(\frac{a_p(E^3_{\alpha_1}) + a_p(E^3_{\alpha_2})}{p}\right) = \begin{cases} \left(\frac{-3}{p}\right)_4 \cdot \left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6 + \left(\frac{\alpha_2}{\mathfrak{p}}\right)_6}{\bar{\mathfrak{p}}}\right) & p \equiv 1 \pmod{12} \\ \left(\frac{\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6 + \left(\frac{\alpha_2}{\mathfrak{p}}\right)_6}{\bar{\mathfrak{p}}}\right) & p \equiv 7 \pmod{12} \end{cases}$$

$$= \begin{cases} \left(\frac{-3}{p}\right)_4 \cdot \left(\frac{\left(\frac{\alpha_3}{\mathfrak{p}}\right)_6 + 1}{\bar{\mathfrak{p}}}\right) & p \equiv 1 \pmod{12} \\ \left(\frac{\alpha_2}{\mathfrak{p}}\right) \cdot \left(\frac{\left(\frac{\alpha_3}{\mathfrak{p}}\right)_6 + 1}{\bar{\mathfrak{p}}}\right) & p \equiv 7 \pmod{12} \end{cases}$$

$$= \begin{cases} \left(\frac{-3}{p}\right)_4 \cdot \left(\frac{2, \zeta_6, \zeta_6^5}{\bar{\mathfrak{p}}}\right) & p \equiv 1 \pmod{12} \\ \left(\frac{\alpha_2}{\mathfrak{p}}\right) \cdot \left(\frac{0, \sqrt{-3}\cdot\zeta_6^4, \sqrt{-3}\cdot\zeta_6^5}{\bar{\mathfrak{p}}}\right) & p \equiv 7 \pmod{12} \end{cases}$$

$$= \begin{cases} \left(\frac{-3}{p}\right)_4 & \text{(2 cases), } p \equiv 1 \pmod{12} \\ \left(\frac{-3}{p}\right)_4 \cdot \left(\frac{2}{\bar{\mathfrak{p}}}\right) & \text{(1 case), } p \equiv 1 \pmod{12} \\ \pm\left(\frac{\alpha_2}{p}\right) \cdot \left(\frac{\sqrt{-3}}{\bar{\mathfrak{p}}}\right) & \text{(2 cases), } p \equiv 7 \pmod{12} \end{cases}$$

So the density is

$$2 \cdot \frac{1}{[\mathbf{Q}(\zeta_{12}, \sqrt[6]{\alpha_3}, \sqrt[4]{-3}) : \mathbf{Q}(\sqrt{-3})]} + 2 \cdot \frac{1}{[\mathbf{Q}(\zeta_{12}, \sqrt[6]{\alpha_3}, \sqrt[4]{-3}, \sqrt{2}) : \mathbf{Q}(\sqrt{-3})]}$$
$$2 \cdot 2 \cdot \frac{1}{[\mathbf{Q}(\zeta_{12}, \sqrt[6]{\alpha_3}, \sqrt{\alpha_2}, \sqrt[4]{-3}) : \mathbf{Q}(\sqrt{-3})]} = \frac{1}{6} + \frac{1}{12} + \frac{1}{6} = \frac{5}{12}$$

□

2.2 Special Cases for Cubes

We recognize differences among the following sets of values of α_i

Set 1 (sextic): $\alpha_i \in \mathbf{Q}^{\times 6} \cup \{1, -27,$ Set 2 cubed, Set 3 squared, etc.,$\}$ such that $\left(\frac{\alpha_i}{\mathfrak{p}}\right)_6 = 1$.

Set 2 (square): $\alpha_i \in \mathbf{Q}^{\times 2} \cup \{-3, 4, 9\}$, such that $\left(\frac{\alpha_i}{\mathfrak{p}}\right)_6 = \zeta_3^{1,2,3}$

Set 3 (cubic): $\alpha_i \in \mathbf{Q}^{\times 3}$ that does not contain 2 or 3 , etc., such that $\left(\frac{\alpha_i}{\mathfrak{p}}\right)_6 = \pm 1$

Set 4 (negative Set 1): $\alpha_i \in \{-1, 27,$ etc.,$\}$, such that

$$\left(\frac{\alpha_i}{\mathfrak{p}}\right)_6 = \begin{cases} 1 & \text{if } p \equiv 1 \pmod{12} \\ -1 & \text{if } p \equiv 7 \pmod{12} \end{cases}$$

Set 5 (negative Set 2): $\alpha_i \in \{3, -4, -9,,$ etc., and their multiplication by Set 2 e.g., $12\}$, such that

$$\left(\frac{\alpha_i}{\mathfrak{p}}\right)_6 = \begin{cases} \zeta_3^{1,2,3} & \text{if } p \equiv 1 \pmod{12} \\ -\zeta_3^{1,2,3} & \text{if } p \equiv 7 \pmod{12} \end{cases}$$

Set 6: $\{2^a \cdot (-3)^b\}$ that are not already contained in previous sets.

Set 7: other cases.

For the following cases involving the above sets, we compute the $\delta_2^1(E_{\alpha_1}^3 \times E_{\alpha_1}^3, \mathcal{C})$:

Case 1: $\alpha_1 = \alpha_2 \in$ Set 1,2,3,4,5,7: 1/3

Case 2: $\alpha_1 = \alpha_2 \in$ Set 6: various values, e.g., 5/9, 1/3.

Case 3: $\alpha_1 \neq \alpha_2$, α_1 and $\alpha_2 \in$ Set 1: 1/3

Case 4: $\alpha_1 \neq \alpha_2$, α_1 and $\alpha_2 \in$ Set 2: 13/27

Case 5: $\alpha_1 \neq \alpha_2$, α_1 and $\alpha_2 \in$ Set 3: 1/6

Case 6: $\alpha_1 \neq \alpha_2$, α_1 and $\alpha_2 \in$ Set 4: 1/3

Case 7: $\alpha_1 \neq \alpha_2$, α_1 and $\alpha_2 \in$ Set 5: 13/27

Case 8: $\alpha_1 \neq \alpha_2$, α_1 and $\alpha_2 \in$ Set 6: various values, e.g., 5/9, 19/54.

Case 9: $\alpha_1 \in$ Set 1, and $\alpha_2 \in$ Set 2: 1/3

Case 10: $\alpha_1 \in$ Set 1, and $\alpha_2 \in$ Set 3: $1/6$

Case 11: $\alpha_1 \in$ Set 1, and $\alpha_2 \in$ Set 4: $1/6$

Case 12: $\alpha_1 \in$ Set 1, and $\alpha_2 \in$ Set 5: $5/18$

Case 13: $\alpha_1 \in$ Set 1, and $\alpha_2 \in$ Set 7: $5/18$

Case 14: $\alpha_1 \in$ Set 2, and $\alpha_2 \in$ Set 3: $5/18$

Case 15: $\alpha_1 \in$ Set 2, and $\alpha_2 \in$ Set 4: $5/18$

Case 16: $\alpha_1 \in$ Set 2, and $\alpha_2 \in$ Set 5: $19/54$

Case 17: $\alpha_1 \in$ Set 2, and $\alpha_2 \in$ Set 7: $19/54$

Case 18: $\alpha_1 \in$ Set 3, and $\alpha_2 \in$ Set 4: $1/6$

Case 19: $\alpha_1 \in$ Set 3, and $\alpha_2 \in$ Set 5: $5/18$

Case 20: $\alpha_1 \in$ Set 3, and $\alpha_2 \in$ Set 7: $5/18$

Case 21: $\alpha_1 \in$ Set 4, and $\alpha_2 \in$ Set 5: $1/3$

Case 22: $\alpha_1 \in$ Set 4, and $\alpha_2 \in$ Set 7: $5/18$

Case 23: $\alpha_1 \in$ Set 5, and $\alpha_2 \in$ Set 7: $19/54$

Case 24: $\alpha_1/\alpha_2 \in$ Set 1: $1/3$

Case 25: $\alpha_1/\alpha_2 \in$ Set 2: $\begin{cases} \dfrac{13}{27} & \text{if } \alpha_2 \in \text{ Set 2,5,7}, i.e., \left(\dfrac{\alpha_2}{\pi}\right)_3 = \zeta_3^{1,2,3} \\ \dfrac{1}{3} & \text{if } \alpha_2 \in \text{ Set 1,3,4}, i.e., \left(\dfrac{\alpha_2}{\pi}\right)_3 = 1 \end{cases}$

Case 26: $\alpha_1/\alpha_2 \in$ Set 3: $1/6$

Case 27: $\alpha_1/\alpha_2 \in$ Set 4: $1/6$

Case 28: $\alpha_1/\alpha_2 \in$ Set 5: $\begin{cases} \dfrac{19}{54} & \text{if } \alpha_2 \in \text{ Set 2,5,7}, i.e., \left(\dfrac{\alpha_2}{\pi}\right)_3 = \zeta_3^{1,2,3} \\ \dfrac{5}{18} & \text{if } \alpha_2 \in \text{ Set 1,3,4}, i.e., \left(\dfrac{\alpha_2}{\pi}\right)_3 = 1 \end{cases}$

Proof. **Case 1**: $\alpha_1 = \alpha_2 \in$ Set 1,2,3,4,5,7: $1/3$

If $\alpha_1 = \alpha_2 \in$ Set 1: $\left(\dfrac{\alpha_i}{\mathfrak{p}}\right)_6 = 1$, the density is $1/3$:

$$\left(\frac{a_p(E^3_{\alpha_1})+a_p(E^3_{\alpha_2})}{p}\right)_3 = \left(\frac{\left(\frac{\alpha_1}{\pi}\right)_6+\left(\frac{\alpha_2}{\pi}\right)_6}{\bar{\pi}}\right)_3$$
$$= \left(\frac{2}{\bar{\pi}}\right)_3$$

If $\alpha_1 = \alpha_2 \in$ Set 2: $\left(\frac{\alpha_i}{\mathfrak{p}}\right)_6 = \zeta_3^{1,2,3}$, then the density is also $1/3$.

$$\left(\frac{a_p(E^3_{\alpha_1})+a_p(E^3_{\alpha_2})}{p}\right)_3 = \left(\frac{\left(\frac{\alpha_1}{\pi}\right)_6+\left(\frac{\alpha_2}{\pi}\right)_6}{\bar{\pi}}\right)_3$$
$$= \left(\frac{2}{\bar{\pi}}\right)_3 \cdot \left(\frac{\zeta_6}{\bar{\pi}}\right)_3^{2,4,6}$$

The rest of the sub-cases are similar.

Case 2: $\alpha_1 = \alpha_2 \in$ Set 6: various values. Sub-case 2.1: $\alpha_1 = \alpha_2 = 2$

$$\left(\frac{\left(\frac{\alpha_1}{\pi}\right)_6+\left(\frac{\alpha_2}{\pi}\right)_6}{\bar{\pi}}\right)_3 = \left(\frac{(2)\left(\frac{\alpha_2}{\pi}\right)_6}{\bar{\pi}}\right)_3 = \left(\frac{2}{\bar{\pi}}\right)_3 \left(\frac{\left(\frac{\alpha_2}{\pi}\right)_6}{\bar{\pi}}\right)_3$$

Also, by [12],

$$\left(\frac{\left(\frac{\alpha_2}{\pi}\right)_6}{\bar{\pi}}\right)_3 = \begin{cases} 1 & p \equiv 1 \bmod 9 \\ \left(\frac{\alpha_2}{\pi}\right)_3^2 & p \equiv 4 \bmod 9 \\ \\ \left(\frac{\alpha_2}{\pi}\right)_3 & p \equiv 7 \bmod 9 \end{cases}$$

So if $\alpha_2 = 2$,

$$\left(\frac{\left(\frac{\alpha_1}{\pi}\right)_6+\left(\frac{\alpha_2}{\pi}\right)_6}{\bar{\pi}}\right)_3=\begin{cases}\left(\frac{2}{\bar{\pi}}\right)_3 & p\equiv 1 \bmod 9\\ \left(\frac{\alpha_2}{\pi}\right)_3^2\left(\frac{2}{\bar{\pi}}\right)_3 & p\equiv 4 \bmod 9\\ \left(\frac{\alpha_2}{\pi}\right)_3\left(\frac{2}{\bar{\pi}}\right)_3 & p\equiv 7 \bmod 9\end{cases}$$

$$=\begin{cases}\left(\frac{2}{\bar{\pi}}\right)_3 & p\equiv 1 \bmod 9\\ 1 & p\equiv 4 \bmod 9\\ \left(\frac{2}{\pi}\right)_3^2 & p\equiv 7 \bmod 9\end{cases}$$

So the density is

$$\frac{1}{[\mathbf{Q}(\zeta_9,\sqrt[3]{2}):\mathbf{Q}(\zeta_3)]}+\frac{1}{[\mathbf{Q}(\zeta_9):\mathbf{Q}(\zeta_3)]}+\frac{1}{[\mathbf{Q}(\zeta_9,\sqrt[3]{2}):\mathbf{Q}(\zeta_3)]}=\frac{1}{9}+\frac{1}{3}+\frac{1}{9}=\frac{5}{9}$$

Sub-case 2.2: $\alpha_1=\alpha_2=6$

$$\left(\frac{\left(\frac{\alpha_1}{\pi}\right)_6+\left(\frac{\alpha_2}{\pi}\right)}{\bar{\pi}}\right)_3=\left(\frac{(2)\left(\frac{\alpha_2}{\pi}\right)_6}{\bar{\pi}}\right)_3=\left(\frac{2}{\bar{\pi}}\right)_3\left(\frac{\left(\frac{\alpha_2}{\pi}\right)_6}{\bar{\pi}}\right)_3$$

Also, by [12],

$$\left(\frac{\left(\frac{\alpha_2}{\pi}\right)_6}{\bar{\pi}}\right)_3=\begin{cases}1 & p\equiv 1 \bmod 9\\ \left(\frac{\alpha_2}{\pi}\right)_3^2 & p\equiv 4 \bmod 9\\ \\ \left(\frac{\alpha_2}{\pi}\right)_3 & p\equiv 7 \bmod 9\end{cases}$$

So if $\alpha_2=6$,

$$\left(\frac{\left(\frac{\alpha_1}{\pi}\right)_6+\left(\frac{\alpha_2}{\pi}\right)_6}{\bar{\pi}}\right)_3 = \begin{cases} \left(\frac{2}{\bar{\pi}}\right)_3 & p \equiv 1 \bmod 9 \\ \left(\frac{3}{\bar{\pi}}\right)_3^2 & p \equiv 4 \bmod 9 \\ \left(\frac{2}{\bar{\pi}}\right)_3^2 \left(\frac{3}{\bar{\pi}}\right)_3 & p \equiv 7 \bmod 9 \end{cases} \qquad (2.1)$$

So the density is

$$\frac{1}{[\mathbf{Q}(\zeta_9, \sqrt[3]{2}) : \mathbf{Q}(\zeta_3)]} + \frac{1}{[\mathbf{Q}(\zeta_9, \sqrt[3]{3}) : \mathbf{Q}(\zeta_3)]} + 3 \cdot \frac{1}{[\mathbf{Q}(\zeta_9, \sqrt[3]{3}, \sqrt[3]{2}) : \mathbf{Q}(\zeta_3)]} = \frac{1}{3}$$

Case 3: $\alpha_1 \neq \alpha_2$, α_1 and $\alpha_2 \in$ Set 1: 1/3 Similar to the first sub-case of Case 1.

Case 4: $\alpha_1 \neq \alpha_2$, α_1 and $\alpha_2 \in$ Set 2: 13/27
Since $\left(\frac{\alpha_i}{\mathfrak{p}}\right)_6 = \zeta_3^{1,2,3}$, we have

$$\left(\frac{\left(\frac{\alpha_1}{\pi}\right)_6+\left(\frac{\alpha_2}{\pi}\right)_6}{\bar{\pi}}\right)_3 = \begin{cases} \left(\frac{\zeta_6}{\bar{\pi}}\right)_3^{1,3,5} & \text{(6 in 9 cases)} \\ \left(\frac{2}{\bar{\pi}}\right)_3 \cdot \left(\frac{\zeta_6}{\bar{\pi}}\right)_3^{2,4,6} & \text{(3 in 9 cases)} \end{cases}$$

Using the fact that

$$\left(\frac{\zeta_6}{\bar{\pi}}\right)_3 = \begin{cases} 1 & p \equiv 1 \pmod 9 \\ \zeta_3^2 & p \equiv 4 \pmod 9 \\ \zeta_3 & p \equiv 7 \pmod 9 \end{cases}$$

we have

$$\left(\frac{\left(\frac{\alpha_1}{\pi}\right)_6+\left(\frac{\alpha_2}{\pi}\right)_6}{\bar{\pi}}\right)_3=\begin{cases}1 & \text{(6 cases), } p\equiv 1 \bmod 9\\ \left(\frac{2}{\bar{\pi}}\right)_3 & \text{(3 cases), } p\equiv 1 \bmod 9\\ & \\ (\zeta_3^2)^{1,3,5} & \text{(6 cases), } p\equiv 4 \bmod 9\\ \left(\frac{2}{\bar{\pi}}\right)_3\cdot(\zeta_3^2)^{2,4,6} & \text{(3 cases), } p\equiv 4 \bmod 9\\ & \\ (\zeta_3)^{1,3,5} & \text{(6 cases), } p\equiv 7 \bmod 9\\ \left(\frac{2}{\bar{\pi}}\right)_3\cdot(\zeta_3)^{2,4,6} & \text{(3 cases), } p\equiv 7 \bmod 9\end{cases} \tag{2.2}$$

The density for $p\equiv 1 \pmod 9$ is

$$6\cdot\frac{1}{[\mathbf{Q}(\zeta_9,\sqrt[6]{\alpha_1},\sqrt[6]{\alpha_2}):\mathbf{Q}(\zeta_3)]}+3\cdot\frac{1}{[\mathbf{Q}(\zeta_9,\sqrt[6]{\alpha_1},\sqrt[6]{\alpha_2},\sqrt[3]{2}):\mathbf{Q}(\zeta_3)]}=\frac{2}{9}+\frac{1}{27}=\frac{7}{27}$$

The density for $p\equiv 4 \pmod 9$ is

$$\frac{1}{3}\cdot 6\cdot\frac{1}{[\mathbf{Q}(\zeta_9,\sqrt[6]{\alpha_1},\sqrt[6]{\alpha_2}):\mathbf{Q}(\zeta_3)]}+3\cdot\frac{1}{[\mathbf{Q}(\zeta_9,\sqrt[6]{\alpha_1},\sqrt[6]{\alpha_2},\sqrt[3]{2}):\mathbf{Q}(\zeta_3)]}=\frac{2}{27}+\frac{1}{27}=\frac{3}{27}$$

The density for $p\equiv 7 \pmod 9$ is, similarly, $3/27$. Thus, the density is $13/27$.

Case 5: $\alpha_1\neq\alpha_2$, α_1 and $\alpha_2\in$ Set 3: $1/6$

Since $\left(\frac{\alpha_i}{\mathfrak{p}}\right)_6=\pm 1$, we have

$$\left(\frac{\left(\frac{\alpha_1}{\pi}\right)_6+\left(\frac{\alpha_2}{\pi}\right)_6}{\bar{\pi}}\right)_3=\left(\frac{\pm 1\pm 1}{\bar{\pi}}\right)_3$$

So the density is $1/6$

Case 6: $\alpha_1\neq\alpha_2$, α_1 and $\alpha_2\in$ Set 4: $1/3$

Since

$$\left(\frac{\alpha_i}{\pi}\right)_6 = \begin{cases} 1 & \text{if } p \equiv 1 \pmod{12} \\ -1 & \text{if } p \equiv 7 \pmod{12} \end{cases}$$

we have

$$\left(\frac{\left(\frac{\alpha_1}{\pi}\right)_6 + \left(\frac{\alpha_2}{\pi}\right)_6}{\bar{\pi}}\right)_3 = \begin{cases} \left(\frac{2}{\bar{\pi}}\right)_3 & p \equiv 1 \pmod{12} \\ \left(\frac{-2}{\bar{\pi}}\right)_3 & p \equiv 7 \pmod{12} \end{cases}$$

So the density is $1/3$

Case 7: $\alpha_1 \neq \alpha_2$, α_1 and $\alpha_2 \in$ Set 5: $13/27$

Since

$$\left(\frac{\alpha_i}{\pi}\right)_6 = \begin{cases} \zeta_3^{1,2,3} & \text{if } p \equiv 1 \pmod{12} \\ -\zeta_3^{1,2,3} & \text{if } p \equiv 7 \pmod{12} \end{cases}$$

we have the density is $13/27$ since it's similar to case 4 above.

Case 8: $\alpha_1 \neq \alpha_2$, α_1 and $\alpha_2 \in$ Set 6: various values

Suppose $\alpha_1 = 2$, $\alpha_2 = 6$, then

$$\left(\frac{\left(\frac{\alpha_1}{\pi}\right)_6 + \left(\frac{\alpha_2}{\pi}\right)_6}{\bar{\pi}}\right)_3 = \left(\frac{\left(1 + \left(\frac{3}{\pi}\right)_6\right)\left(\frac{2}{\pi}\right)_6}{\bar{\pi}}\right)_3$$

Note that $\left(\frac{3}{\pi}\right)_6 = \begin{cases} \zeta_3^{1,2,3} & \text{if } p \equiv 1 \pmod{12} \\ -\zeta_3^{1,2,3} & \text{if } p \equiv 7 \pmod{12} \end{cases}$, so that

$$\left(\frac{1 + \left(\frac{3}{\pi}\right)_6}{\bar{\pi}}\right)_3 = \begin{cases} \left(\frac{2, \zeta_6, \zeta_6^5}{\bar{\pi}}\right)_3 & \text{if } p \equiv 1 \pmod{12} \\ \left(\frac{0, \zeta_6^4\sqrt{-3}, \zeta_6^5\sqrt{-3}}{\bar{\pi}}\right)_3 & \text{if } p \equiv 7 \pmod{12} \end{cases}$$

Also, by [12],

$$\left(\frac{\left(\frac{2}{\pi}\right)_6}{\bar{\pi}}\right)_3 = \begin{cases} 1 & p \equiv 1 \bmod 9 \\ \left(\frac{2}{\pi}\right)_3^2 & p \equiv 4 \bmod 9 \\ \left(\frac{2}{\pi}\right)_3 & p \equiv 7 \bmod 9 \end{cases}$$

So for $p \equiv 1 \bmod 12$, we have

$$\left(\frac{\left(\frac{2}{\pi}\right)_6}{\bar{\pi}}\right)_3 \left(\frac{1+\left(\frac{3}{\pi}\right)_6}{\bar{\pi}}\right)_3 = \begin{cases} \left(\frac{2,\zeta_6,\zeta_6^5}{\bar{\pi}}\right)_3 & p \equiv 1 \bmod 9 \\ \left(\frac{2}{\pi}\right)_3^2 \left(\frac{2,\zeta_6,\zeta_6^5}{\bar{\pi}}\right)_3 & p \equiv 4 \bmod 9 \\ \left(\frac{2}{\pi}\right)_3 \left(\frac{2,\zeta_6,\zeta_6^5}{\bar{\pi}}\right)_3 & p \equiv 7 \bmod 9 \end{cases}$$

Using the fact that

$$\left(\frac{\zeta_6}{\bar{\pi}}\right)_3 = \begin{cases} 1 & p \equiv 1 \pmod 9 \\ \zeta_3^2 & p \equiv 4 \pmod 9 \\ \zeta_3 & p \equiv 7 \pmod 9 \end{cases}$$

we have for $p \equiv 1 \bmod 12$,

$$\left(\frac{\left(\frac{2}{\pi}\right)_6}{\bar{\pi}}\right)_3 \left(\frac{1+\left(\frac{3}{\pi}\right)_6}{\bar{\pi}}\right)_3 = \begin{cases} \left(\frac{2}{\bar{\pi}}\right)_3 & \text{1 case, } p \equiv 1 \bmod 9 \\ 1 & \text{2 cases, } p \equiv 1 \bmod 9 \\ 1 & \text{1 case, } p \equiv 4 \bmod 9 \\ \left(\frac{2}{\pi}\right)_3^2 \zeta_3^{1,2} & \text{2 cases, } p \equiv 4 \bmod 9 \\ \left(\frac{2}{\pi}\right)_3^2 & \text{1 case, } p \equiv 7 \bmod 9 \\ \left(\frac{2}{\pi}\right)_3 \zeta_3^{1,2} & \text{2 cases, } p \equiv 7 \bmod 9 \end{cases}$$

so the density for $p \equiv 1 \bmod 12$ is

$$\frac{1}{[\mathbf{Q}(\zeta_{36},\sqrt[6]{3},\sqrt[3]{2}):\mathbf{Q}(\zeta_3)]}+2\cdot\frac{1}{[\mathbf{Q}(\zeta_{36},\sqrt[6]{3}:\mathbf{Q}(\zeta_3)]}+\frac{1}{[\mathbf{Q}(\zeta_{36},\sqrt[6]{3}):\mathbf{Q}(\zeta_3)]}$$
$$+2\cdot\frac{1}{[\mathbf{Q}(\zeta_{36},\sqrt[6]{3},\sqrt[3]{2}):\mathbf{Q}(\zeta_3)]}+\frac{1}{[\mathbf{Q}(\zeta_{36},\sqrt[6]{3},\sqrt[3]{2}):\mathbf{Q}(\zeta_3)]}+2\cdot\frac{1}{[\mathbf{Q}(\zeta_{36},\sqrt[6]{3},\sqrt[3]{2}):\mathbf{Q}(\zeta_3)]}$$
$$=\frac{1}{54}+\frac{2}{18}+\frac{1}{18}+\frac{2}{54}+\frac{1}{54}+\frac{2}{54}=\frac{15}{54}$$

Now for $p\equiv 7 \bmod 12$, we have

$$\left(\frac{\left(\frac{2}{\pi}\right)_6}{\bar{\pi}}\right)_3\left(\frac{1+\left(\frac{3}{\pi}\right)_6}{\bar{\pi}}\right)_3=\begin{cases}\left(\frac{0,\zeta_6^4\sqrt{-3},\zeta_6^5\sqrt{-3}}{\bar{\pi}}\right)_3 & p\equiv 1 \bmod 9\\ \left(\frac{2}{\pi}\right)_3^2\left(\frac{0,\zeta_6^4\sqrt{-3},\zeta_6^5\sqrt{-3}}{\bar{\pi}}\right)_3 & p\equiv 4 \bmod 9\\ \left(\frac{2}{\pi}\right)_3\left(\frac{0,\zeta_6^4\sqrt{-3},\zeta_6^5\sqrt{-3}}{\bar{\pi}}\right)_3 & p\equiv 7 \bmod 9\end{cases}$$

Using the fact that

$$\left(\frac{\zeta_6}{\bar{\pi}}\right)_3=\begin{cases}1 & p\equiv 1 \pmod 9\\ \zeta_3^2 & p\equiv 4 \pmod 9\\ \zeta_3 & p\equiv 7 \pmod 9\end{cases}$$

we have for $p\equiv 7 \bmod 12$,

$$\left(\frac{\left(\frac{2}{\pi}\right)_6}{\bar{\pi}}\right)_3\left(\frac{1+\left(\frac{3}{\pi}\right)_6}{\bar{\pi}}\right)_3=\begin{cases}0 & 1\text{ case, } p\equiv 1 \bmod 9\\ \left(\frac{\sqrt{-3}}{\bar{\pi}}\right)_3 & 2\text{ cases, } p\equiv 1 \bmod 9\\ 0 & 1\text{ case, } p\equiv 4 \bmod 9\\ \left(\frac{2}{\pi}\right)_3^2\left(\frac{\sqrt{-3}}{\bar{\pi}}\right)_3\zeta_3^{1,2} & 2\text{ cases, } p\equiv 4 \bmod 9\\ 0 & 1\text{ case, } p\equiv 7 \bmod 9\\ \left(\frac{2}{\pi}\right)_3\left(\frac{\sqrt{-3}}{\bar{\pi}}\right)_3\zeta_3^{1,2} & 2\text{ cases, } p\equiv 7 \bmod 9\end{cases}$$

Note that $\left(\frac{\sqrt{-3}}{\bar{\pi}}\right)_3 = -\left(\frac{3}{\bar{\pi}}\right)_6$ when $p \equiv 7 \bmod 12$.

$$\text{So, } \left(\frac{\left(\frac{2}{\pi}\right)_6}{\bar{\pi}}\right)_3 \left(\frac{1+\left(\frac{3}{\pi}\right)_6}{\bar{\pi}}\right)_3 = \begin{cases} 0 & 1 \text{ case, } p \equiv 1 \bmod 9 \\ \zeta_3, \zeta_3^2 & 2 \text{ cases, } p \equiv 1 \bmod 9 \\ 0 & 1 \text{ case, } p \equiv 4 \bmod 9 \\ \left(\frac{2}{\pi}\right)_3^2 & 2 \text{ cases, } p \equiv 4 \bmod 9 \\ & \\ 0 & 1 \text{ case, } p \equiv 7 \bmod 9 \\ \left(\frac{2}{\pi}\right)_3 \zeta_3^{1,2} & 2 \text{ cases, } p \equiv 7 \bmod 9 \end{cases}$$

so the density for $p \equiv 7 \bmod 12$ is

$$2 \cdot \frac{1}{[\mathbf{Q}(\zeta_{36}, \sqrt[6]{3}, \sqrt[3]{2}) : \mathbf{Q}(\zeta_3)]} + 2 \cdot \frac{1}{[\mathbf{Q}(\zeta_{36}, \sqrt[6]{3}, \sqrt[3]{2}) : \mathbf{Q}(\zeta_3)]} = \frac{2}{54} + \frac{2}{54} = \frac{4}{54}.$$

So the sum density is 19/54.

Suppose $\alpha_1 = 2$, $\alpha_2 = -6$, then

$$\left(\frac{\left(\frac{\alpha_1}{\pi}\right)_6 + \left(\frac{\alpha_2}{\pi}\right)_6}{\bar{\pi}}\right)_3 = \left(\frac{\left(1+\left(\frac{-3}{\pi}\right)_6\right)\left(\frac{2}{\pi}\right)_6}{\bar{\pi}}\right)_3$$

Note that $\left(\frac{-3}{\pi}\right)_6 = \zeta_3^{1,2,3}$, so that $\left(\frac{1+\left(\frac{3}{\pi}\right)_6}{\bar{\pi}}\right)_3 = \left(\frac{2, \zeta_6, \zeta_6^5}{\bar{\pi}}\right)_3$

It's similar to the sub-case above when $p \equiv 1 \bmod 12$, so density is $2 \cdot \frac{15}{54} = \frac{5}{9}$.

Case 9: $\alpha_1 \in$ Set 1, and $\alpha_2 \in$ Set 2: 1/3

Since $\left(\frac{\alpha_1}{\pi}\right)_6 = 1$ and $\left(\frac{\alpha_2}{\pi}\right)_6 = \zeta_3^{1,2,3}$, we have

$$\left(\frac{\left(\frac{\alpha_1}{\pi}\right)_6 + \left(\frac{\alpha_2}{\pi}\right)_6}{\bar{\pi}}\right)_3 = \begin{cases} \left(\frac{\zeta_6}{\bar{\pi}}\right)_3^{1,5} & \text{(2 in 3 cases)} \\ \left(\frac{2}{\bar{\pi}}\right)_3 & \text{(1 in 3 cases)} \end{cases}$$

Using the fact that

$$\left(\frac{\zeta_6}{\bar{\pi}}\right)_3 = \begin{cases} 1 & p \equiv 1 \pmod 9 \\ \zeta_3^2 & p \equiv 4 \pmod 9 \\ \zeta_3 & p \equiv 7 \pmod 9 \end{cases}$$

we have

$$\left(\frac{\left(\frac{\alpha_1}{\pi}\right)_6 + \left(\frac{\alpha_2}{\pi}\right)_6}{\bar{\pi}}\right)_3 = \begin{cases} \left(\frac{2}{\bar{\pi}}\right)_3 & \text{1 case, } p \equiv 1 \text{ mod } 9 \\ 1 & \text{2 cases, } p \equiv 1 \text{ mod } 9 \\ \left(\frac{2}{\bar{\pi}}\right)_3 & \text{1 case, } p \equiv 4 \text{ mod } 9 \\ \zeta_3^{1,2} & \text{2 cases, } p \equiv 4 \text{ mod } 9 \\ \left(\frac{2}{\pi}\right)_3 & \text{1 case, } p \equiv 7 \text{ mod } 9 \\ \zeta_3^{1,2} & \text{2 cases, } p \equiv 7 \text{ mod } 9 \end{cases}$$

So the density is

$$\frac{1}{[\mathbf{Q}(\zeta_9, \sqrt[6]{\alpha_2}, \sqrt[3]{2}) : \mathbf{Q}(\zeta_3)]} + 2 \cdot \frac{1}{[\mathbf{Q}(\zeta_9, \sqrt[6]{\alpha_2} : \mathbf{Q}(\zeta_3)]} + \frac{1}{[\mathbf{Q}(\zeta_9, \sqrt[6]{\alpha_2}, \sqrt[3]{2}) : \mathbf{Q}(\zeta_3)]}$$
$$+ \frac{1}{[\mathbf{Q}(\zeta_9, \sqrt[6]{\alpha_2}, \sqrt[3]{2}) : \mathbf{Q}(\zeta_3)]} = \frac{1}{27} + \frac{2}{9} + \frac{1}{27} + \frac{1}{27} = \frac{1}{3}$$

Case 10: $\alpha_1 \in$ Set 1, and $\alpha_2 \in$ Set 3: $1/6$

Since $\left(\frac{\alpha_1}{\pi}\right)_6 = 1$ and $\left(\frac{\alpha_2}{\pi}\right)_6 = \zeta_3^{3,6}$, we have

$$\left(\frac{\left(\frac{\alpha_1}{\pi}\right)_6 + \left(\frac{\alpha_2}{\pi}\right)_6}{\bar{\pi}}\right)_3 = \begin{cases} 0 & \text{(1 in 2 cases)} \\ \left(\frac{2}{\bar{\pi}}\right)_3 & \text{(1 in 2 cases)} \end{cases}$$

So the density is $1/6$.

Case 11: $\alpha_1 \in$ Set 1, and $\alpha_2 \in$ Set 4: $1/6$

Since $\left(\frac{\alpha_1}{\pi}\right)_6 = 1$, and $\left(\frac{\alpha_2}{\pi}\right)_6 = \begin{cases} 1 & \text{if } p \equiv 1 \pmod{12} \\ -1 & \text{if } p \equiv 7 \pmod{12} \end{cases}$

we have

$$\left(\frac{\left(\frac{\alpha_1}{\pi}\right)_6 + \left(\frac{\alpha_2}{\pi}\right)_6}{\bar{\pi}}\right)_3 = \begin{cases} \left(\frac{2}{\bar{\pi}}\right)_3 & p \equiv 1 \pmod{12} \\ 0 & p \equiv 7 \pmod{12} \end{cases}$$

So the density is $1/6$.

Case 12: $\alpha_1 \in$ Set 1, and $\alpha_2 \in$ Set 5: $5/18$

Since $\left(\frac{\alpha_1}{\pi}\right)_6 = 1$, and $\left(\frac{\alpha_2}{\pi}\right)_6 = \begin{cases} \zeta_3^{1,2,3} & \text{if } p \equiv 1 \pmod{12} \\ -\zeta_3^{1,2,3} & \text{if } p \equiv 7 \pmod{12} \end{cases}$, we have

$$\left(\frac{\left(\frac{\alpha_1}{\pi}\right)_6 + \left(\frac{\alpha_2}{\pi}\right)_6}{\bar{\pi}}\right)_3 = \begin{cases} \left(\frac{\zeta_6, \zeta_6^5, 2}{\bar{\pi}}\right)_3 & \text{if } p \equiv 1 \pmod{12} \\ \left(\frac{\sqrt{-3}}{\bar{\pi}}\right)_3 \cdot \left(\frac{\zeta_6^4, \zeta_6^5, 0}{\bar{\mathfrak{p}}}\right)_3 & \text{if } p \equiv 7 \pmod{12} \end{cases}$$

For $p \equiv 1 \bmod 12$, it's similar to Case 9, so the density is $1/6$.

Now for $p \equiv 7 \bmod 12$, using the fact that

$$\left(\frac{\zeta_6}{\bar{\pi}}\right)_3 = \begin{cases} 1 & p \equiv 1 \pmod 9 \\ \zeta_3^2 & p \equiv 4 \pmod 9 \\ \zeta_3 & p \equiv 7 \pmod 9 \end{cases}$$

we have

$$\left(\frac{\left(\frac{\alpha_1}{\pi}\right)_6+\left(\frac{\alpha_2}{\pi}\right)_6}{\bar{\pi}}\right)_3=\begin{cases}0 & \text{1 case, } p\equiv 1 \bmod 9\\ \left(\frac{\sqrt{-3}}{\bar{\pi}}\right)_3 & \text{2 cases, } p\equiv 1 \bmod 9\\ 0 & \text{1 case, } p\equiv 4 \bmod 9\\ \left(\frac{\sqrt{-3}}{\bar{\pi}}\right)_3\zeta_3^{1,2} & \text{2 cases, } p\equiv 4 \bmod 9\\ \\ 0 & \text{1 case, } p\equiv 7 \bmod 9\\ \left(\frac{\sqrt{-3}}{\bar{\pi}}\right)_3\zeta_3^{1,2} & \text{2 cases, } p\equiv 7 \bmod 9\end{cases}$$

So the density for $p\equiv 7 \bmod 12$ is

$6\cdot\dfrac{1}{[\mathbf{Q}(\zeta_{36},\sqrt[6]{\alpha_2},\sqrt[6]{3}):\mathbf{Q}(\zeta_3)]}=\dfrac{1}{9}$. So the sum density is 1/6+1/9=5/18.

Case 13: $\alpha_1\in$ Set 1, and $\alpha_2\in$ Set 7: 5/18

We have $\left(\dfrac{\left(\frac{\alpha_1}{\pi}\right)_6+\left(\frac{\alpha_2}{\pi}\right)_6}{\bar{\pi}}\right)_3=\left(\dfrac{0,2,\zeta_6,\zeta_6^5,\zeta_6^4\sqrt{-3},\zeta_6^5\sqrt{-3}}{\bar{\pi}}\right)_3$

Using the fact that

$$\left(\frac{\zeta_6}{\bar{\pi}}\right)_3=\begin{cases}1 & p\equiv 1 \pmod 9\\ \zeta_3^2 & p\equiv 4 \pmod 9\\ \zeta_3 & p\equiv 7 \pmod 9\end{cases}$$

we have

$$\left(\frac{\left(\frac{\alpha_1}{\pi}\right)_6 + \left(\frac{\alpha_2}{\pi}\right)_6}{\bar{\pi}}\right)_3 = \begin{cases} 0 & 1 \text{ case, } p \equiv 1 \bmod 9 \\ 1 & 2 \text{ cases, } p \equiv 1 \bmod 9 \\ \left(\frac{2}{\bar{\pi}}\right)_3 & 1 \text{ case, } p \equiv 1 \bmod 9 \\ \left(\frac{\sqrt{-3}}{\bar{\pi}}\right)_3 & 2 \text{ cases, } p \equiv 1 \bmod 9 \\ 0 & 1 \text{ case, } p \equiv 4 \bmod 9 \\ \left(\frac{2}{\bar{\pi}}\right)_3 & 1 \text{ case, } p \equiv 4 \bmod 9 \\ \zeta_3^{1,2} & 2 \text{ cases, } p \equiv 4 \bmod 9 \\ \left(\frac{\sqrt{-3}}{\bar{\pi}}\right)_3 \zeta_3^{1,2} & 2 \text{ cases, } p \equiv 4 \bmod 9 \\ 0 & 1 \text{ case, } p \equiv 7 \bmod 9 \\ \left(\frac{2}{\bar{\pi}}\right)_3 & 1 \text{ case, } p \equiv 7 \bmod 9 \\ \zeta_3^{1,2} & 2 \text{ cases, } p \equiv 7 \bmod 9 \\ \left(\frac{\sqrt{-3}}{\bar{\pi}}\right)_3 \zeta_3^{1,2} & 2 \text{ cases, } p \equiv 7 \bmod 9 \end{cases}$$

So the density is

$$2 \cdot \frac{1}{[\mathbf{Q}(\zeta_9, \sqrt[6]{\alpha_2} : \mathbf{Q}(\zeta_3)]} + \frac{1}{[\mathbf{Q}(\zeta_9, \sqrt[6]{\alpha_2}, \sqrt[3]{2}) : \mathbf{Q}(\zeta_3)]} + 2 \cdot \frac{1}{[\mathbf{Q}(\zeta_9, \sqrt[6]{\alpha_2}, \sqrt[6]{-3}) : \mathbf{Q}(\zeta_3)]}$$
$$+ \left(\frac{1}{[\mathbf{Q}(\zeta_9, \sqrt[6]{\alpha_2}, \sqrt[3]{2}) : \mathbf{Q}(\zeta_3)]} + 2 \cdot \frac{1}{[\mathbf{Q}(\zeta_9, \sqrt[6]{\alpha_2}, \sqrt[6]{-3}) : \mathbf{Q}(\zeta_3)]}\right) \cdot 2$$
$$= \frac{1}{9} + \frac{1}{54} + \frac{2}{54} + 2\left(\frac{1}{54} + \frac{2}{54}\right) = \frac{5}{18}$$

Case 14: $\alpha_1 \in$ Set 2, and $\alpha_2 \in$ Set 3: 5/18

Since $\left(\frac{\alpha_1}{\pi}\right)_6 = \zeta_3^{1,2,3}$ and $\left(\frac{\alpha_2}{\pi}\right)_6 = \pm 1$, we have

$$\left(\frac{\left(\frac{\alpha_1}{\pi}\right)_6 + \left(\frac{\alpha_2}{\pi}\right)_6}{\bar{\pi}}\right)_3 = \left(\frac{0, 2, \zeta_6, \zeta_6^5, \sqrt{-3}\cdot\zeta_6, \sqrt{-3}\cdot\zeta_6^2}{\bar{\pi}}\right)_3$$

Similar to Case 13, the density is 5/18.

Case 15: $\alpha_1 \in$ Set 2, and $\alpha_2 \in$ Set 4: 5/18

Since $\left(\frac{\alpha_1}{\pi}\right)_6 = \zeta_3^{1,2,3}$, and $\left(\frac{\alpha_2}{\pi}\right)_6 = \begin{cases} 1 & \text{if } p \equiv 1 \pmod{12} \\ -1 & \text{if } p \equiv 7 \pmod{12} \end{cases}$

we have

$$\left(\frac{\left(\frac{\alpha_1}{\pi}\right)_6 + \left(\frac{\alpha_2}{\pi}\right)_6}{\bar{\pi}}\right)_3 = \begin{cases} \left(\frac{2, \zeta_6, \zeta_6^5}{\bar{\pi}}\right)_3 & \text{if } p \equiv 1 \pmod{12} \\ \left(\frac{0, \zeta_6\sqrt{-3}, \zeta_6^2\sqrt{-3}}{\bar{\pi}}\right)_3 & \text{if } p \equiv 7 \pmod{12} \end{cases}$$

Similar to Case 12, the density is 5/18.

Case 16: $\alpha_1 \in$ Set 2, and $\alpha_2 \in$ Set 5: 19/54

Since $\left(\frac{\alpha_1}{\pi}\right)_6 = \zeta_3^{1,2,3}$, and $\left(\frac{\alpha_i}{\pi}\right)_6 = \begin{cases} \zeta_3^{1,2,3} & \text{if } p \equiv 1 \pmod{12} \\ -\zeta_3^{1,2,3} & \text{if } p \equiv 7 \pmod{12} \end{cases}$,

we have

$$\left(\frac{\left(\frac{\alpha_1}{\pi}\right)_6 + \left(\frac{\alpha_2}{\pi}\right)_6}{\bar{\pi}}\right)_3 = \begin{cases} \left(\frac{2\cdot\zeta_6^{2,4,6}, \zeta_6^{1,3,5} \text{ (6 cases)}}{\bar{\pi}}\right)_3 & \text{if } p \equiv 1 \pmod{12} \\ \left(\frac{0 \text{ (3 cases) }, \sqrt{-3}\cdot\zeta_6^{1,2,3,4,5,6}}{\bar{\pi}}\right)_3 & \text{if } p \equiv 7 \pmod{12} \end{cases}$$

Using the fact that

$$\left(\frac{\zeta_6}{\bar{\pi}}\right)_3 = \begin{cases} 1 & p \equiv 1 \pmod 9 \\ \zeta_3^2 & p \equiv 4 \pmod 9 \\ \zeta_3 & p \equiv 7 \pmod 9 \end{cases}$$

for $p \equiv 1 \bmod 12$ we have

$$\left(\frac{\left(\frac{\alpha_1}{\pi}\right)_6 + \left(\frac{\alpha_2}{\pi}\right)_6}{\bar{\pi}}\right)_3 = \begin{cases} 1 & 6 \text{ cases, } p \equiv 1 \bmod 9 \\ \left(\frac{2}{\bar{\pi}}\right)_3 & 3 \text{ cases, } p \equiv 1 \bmod 9 \\ \\ 1 & 2 \text{ cases, } p \equiv 4 \bmod 9 \\ \left(\frac{2}{\bar{\pi}}\right)_3 \zeta_3^{1,2,3} & 3 \text{ cases, } p \equiv 4 \bmod 9 \\ \\ 1 & 2 \text{ cases, } p \equiv 7 \bmod 9 \\ \left(\frac{2}{\bar{\pi}}\right)_3 \zeta_3^{1,2,3} & 3 \text{ cases, } p \equiv 7 \bmod 9 \end{cases}$$

So the density is

$$6 \cdot \frac{1}{[\mathbf{Q}(\zeta_{36}, \sqrt[6]{\alpha_1}, \sqrt[6]{\alpha_2} : \mathbf{Q}(\zeta_3)]} + 3 \cdot \frac{1}{[\mathbf{Q}(\zeta_{36}, \sqrt[6]{\alpha_1}, \sqrt[6]{\alpha_2}, \sqrt[3]{2}) : \mathbf{Q}(\zeta_3)]}$$
$$+ 2 \cdot \frac{1}{[\mathbf{Q}(\zeta_{36}, \sqrt[6]{\alpha_1}, \sqrt[6]{\alpha_2} : \mathbf{Q}(\zeta_3)]} + 3 \cdot \frac{1}{[\mathbf{Q}(\zeta_{36}, \sqrt[6]{\alpha_1}, \sqrt[6]{\alpha_2}, \sqrt[3]{2}) : \mathbf{Q}(\zeta_3)]}$$
$$+ 2 \cdot \frac{1}{[\mathbf{Q}(\zeta_{36}, \sqrt[6]{\alpha_1}, \sqrt[6]{\alpha_2} : \mathbf{Q}(\zeta_3)]} + 3 \cdot \frac{1}{[\mathbf{Q}(\zeta_{36}, \sqrt[6]{\alpha_1}, \sqrt[6]{\alpha_2}, \sqrt[3]{2}) : \mathbf{Q}(\zeta_3)]}$$
$$= \frac{1}{9} + \frac{1}{54} + 2\left(\frac{2}{54} + \frac{1}{54}\right) = \frac{13}{54}$$

For $p \equiv 7 \bmod 12$ we have

$$\left(\frac{\left(\frac{\alpha_1}{\pi}\right)_6+\left(\frac{\alpha_2}{\pi}\right)_6}{\bar{\pi}}\right)_3=\begin{cases}0 & 3\text{ cases}, p\equiv 1 \bmod 9\\ \left(\frac{\sqrt{-3}}{\bar{\pi}}\right)_3 & 6\text{ cases}, p\equiv 1 \bmod 9\\ \\ 0 & 3\text{ cases}, p\equiv 4 \bmod 9\\ \left(\frac{\sqrt{-3}}{\bar{\pi}}\right)_3\zeta_3^{1,2,3} & 6\text{ cases}, p\equiv 4 \bmod 9\\ \\ 0 & 3\text{ cases}, p\equiv 7 \bmod 9\\ \left(\frac{\sqrt{-3}}{\bar{\pi}}\right)_3\zeta_3^{1,2,3} & 6\text{ cases}, p\equiv 7 \bmod 9\end{cases}$$

So the density is

$$18\cdot\frac{1}{[\mathbf{Q}(\zeta_{36},\sqrt[6]{\alpha_1},\sqrt[6]{\alpha_2},\sqrt[6]{-3}):\mathbf{Q}(\zeta_3)]}=\frac{1}{9}.$$

So the sum density is $13/54+1/9=19/54$.

Case 17: $\alpha_1\in$ Set 2, and $\alpha_2\in$ Set 7: 19/54

Since $\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6=\zeta_3^{1,2,3}$, we have

$$\left(\frac{\left(\frac{\alpha_1}{\pi}\right)_6+\left(\frac{\alpha_2}{\pi}\right)_6}{\bar{\pi}}\right)_3=\begin{cases}0 & 3\text{ cases}\\ \left(\frac{\zeta_6^{1,3,5}}{\bar{\mathfrak{p}}}\right)_3 & 6\text{ cases}\\ \left(\frac{2\cdot\zeta_6^{2,4,6}}{\bar{\mathfrak{p}}}\right)_3 & 3\text{ cases}\\ \left(\frac{\sqrt{-3}\cdot\zeta_6^{1,2,3,4,5,6}}{\bar{\mathfrak{p}}}\right)_3 & 6\text{ cases}\end{cases}$$

Using the fact that

$$\left(\frac{\zeta_6}{\bar{\pi}}\right)_3=\begin{cases}1 & p\equiv 1 \pmod 9\\ \zeta_3^2 & p\equiv 4 \pmod 9\\ \zeta_3 & p\equiv 7 \pmod 9\end{cases}$$

we have

$$\left(\frac{\left(\frac{\alpha_1}{\pi}\right)_6+\left(\frac{\alpha_2}{\pi}\right)_6}{\bar{\pi}}\right)_3=\begin{cases} 0 & 3\text{ cases, } p\equiv 1 \bmod 9\\ 1 & 6\text{ cases, } p\equiv 1 \bmod 9\\ \left(\frac{2}{\bar{\pi}}\right)_3 & 3\text{ cases, } p\equiv 1 \bmod 9\\ \\ \left(\frac{\sqrt{-3}}{\bar{\pi}}\right)_3 & 6\text{ cases, } p\equiv 1 \bmod 9\\ \\ 0 & 3\text{ cases, } p\equiv 4 \bmod 9\\ 1 & 2\text{ cases, } p\equiv 4 \bmod 9\\ \left(\frac{2}{\bar{\pi}}\right)_3 \zeta_3^{1,2,3} & 3\text{ cases, } p\equiv 4 \bmod 9\\ \left(\frac{\sqrt{-3}}{\bar{\pi}}\right)_3 \zeta_3^{1,2,3} & 6\text{ cases, } p\equiv 4 \bmod 9\\ \\ 0 & 3\text{ cases, } p\equiv 7 \bmod 9\\ 1 & 2\text{ cases, } p\equiv 7 \bmod 9\\ \left(\frac{2}{\bar{\pi}}\right)_3 \zeta_3^{1,2,3} & 3\text{ cases, } p\equiv 7 \bmod 9\\ \left(\frac{\sqrt{-3}}{\bar{\pi}}\right)_3 \zeta_3^{1,2,3} & 6\text{ cases, } p\equiv 7 \bmod 9 \end{cases}$$

So the density is

$$\begin{aligned} 6\cdot\frac{1}{[\mathbf{Q}(\zeta_9,\sqrt[6]{\alpha_1},\sqrt[6]{\alpha_2}:\mathbf{Q}(\zeta_3)]}&+3\cdot\frac{1}{[\mathbf{Q}(\zeta_9,\sqrt[6]{\alpha_1},\sqrt[6]{\alpha_2},\sqrt[3]{2}):\mathbf{Q}(\zeta_3)]}\\ +6\cdot\frac{1}{[\mathbf{Q}(\zeta_9,\sqrt[6]{\alpha_1},\sqrt[6]{\alpha_2},\sqrt[6]{-3}):\mathbf{Q}(\zeta_3)]}&+4\cdot\frac{1}{[\mathbf{Q}(\zeta_9,\sqrt[6]{\alpha_1},\sqrt[6]{\alpha_2}:\mathbf{Q}(\zeta_3)]}\\ +6\cdot\frac{1}{[\mathbf{Q}(\zeta_9,\sqrt[6]{\alpha_1},\sqrt[6]{\alpha_2},\sqrt[3]{2}):\mathbf{Q}(\zeta_3)]}&+12\cdot\frac{1}{[\mathbf{Q}(\zeta_9,\sqrt[6]{\alpha_1},\sqrt[6]{\alpha_2},\sqrt[6]{-3}):\mathbf{Q}(\zeta_3)]}\\ &=\frac{1}{9}+\frac{1}{54}+\frac{1}{27}+\frac{2}{27}+\frac{1}{27}+\frac{2}{27}=\frac{19}{54} \end{aligned}$$

Case 18: $\alpha_1 \in$ Set 3, and $\alpha_2 \in$ Set 4: 1/6

Since $\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6 = \pm 1$, and $\left(\frac{\alpha_2}{\mathfrak{p}}\right)_6 = \begin{cases} 1 & \text{if } p \equiv 1 \pmod{12} \\ -1 & \text{if } p \equiv 7 \pmod{12} \end{cases}$, we have

So the density is 1/6.

Case 19: $\alpha_1 \in$ Set 3, and $\alpha_2 \in$ Set 5: 5/18

Since $\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6 = \pm 1$, and $\left(\frac{\alpha_i}{\mathfrak{p}}\right)_6 = \begin{cases} \zeta_3^{1,2,3} & \text{if } p \equiv 1 \pmod{12} \\ -\zeta_3^{1,2,3} & \text{if } p \equiv 7 \pmod{12} \end{cases}$, we have

$$\left(\frac{\left(\frac{\alpha_1}{\pi}\right)_6 + \left(\frac{\alpha_2}{\pi}\right)_6}{\bar{\pi}}\right)_3 = \begin{cases} \left(\frac{0, 2, \zeta_6, \zeta_6^5, \zeta_6\sqrt{-3}, \zeta_6^2\sqrt{-3}}{\bar{\pi}}\right)_3 & \text{if } p \equiv 1 \pmod{12} \\ \left(\frac{0, -2, \zeta_6^2, \zeta_6^4, \zeta_6^5\sqrt{-3}, \zeta_6^4\sqrt{-3}}{\bar{\pi}}\right)_3 & \text{if } p \equiv 7 \pmod{12} \end{cases}$$

Similarly to Case 14, the density is 5/18.

Case 20: $\alpha_1 \in$ Set 3, and $\alpha_2 \in$ Set 7: 5/18

Since $\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6 = \pm 1$, we have

$$\left(\frac{\left(\frac{\alpha_1}{\pi}\right)_6 + \left(\frac{\alpha_2}{\pi}\right)_6}{\bar{\pi}}\right)_3 = \begin{cases} 0 & \text{2 cases} \\ \left(\frac{\zeta_6^{1,2,4,5}}{\bar{\mathfrak{p}}}\right)_3 & \text{4 cases} \\ \left(\frac{2 \cdot \zeta_6^{3,6}}{\bar{\mathfrak{p}}}\right)_3 & \text{2 cases} \\ \left(\frac{\sqrt{-3} \cdot \zeta_6^{1,2,4,5}}{\bar{\mathfrak{p}}}\right)_3 & \text{4 cases} \end{cases}$$

Using the fact that

$$\left(\frac{\zeta_6}{\bar{\pi}}\right)_3 = \begin{cases} 1 & p \equiv 1 \pmod 9 \\ \zeta_3^2 & p \equiv 4 \pmod 9 \\ \zeta_3 & p \equiv 7 \pmod 9 \end{cases}$$

we have

$$\left(\frac{\left(\frac{\alpha_1}{\pi}\right)_6 + \left(\frac{\alpha_2}{\pi}\right)_6}{\bar{\pi}}\right)_3 = \begin{cases} 0 & \text{2 cases, } p \equiv 1 \bmod 9 \\ 1 & \text{4 cases, } p \equiv 1 \bmod 9 \\ \left(\frac{2}{\bar{\pi}}\right)_3 & \text{2 cases, } p \equiv 1 \bmod 9 \\ & \\ \left(\frac{\sqrt{-3}}{\bar{\pi}}\right)_3 & \text{4 cases, } p \equiv 1 \bmod 9 \\ & \\ 0 & \text{2 cases, } p \equiv 4 \bmod 9 \\ \zeta_3, \zeta_3^2 & \text{4 cases, } p \equiv 4 \bmod 9 \\ \left(\frac{2}{\bar{\pi}}\right)_3 & \text{2 cases, } p \equiv 4 \bmod 9 \\ \left(\frac{\sqrt{-3}}{\bar{\pi}}\right)_3 \zeta_3^{1,2} & \text{4 cases, } p \equiv 4 \bmod 9 \\ & \\ 0 & \text{2 cases, } p \equiv 7 \bmod 9 \\ \zeta_3, \zeta_3^2 & \text{4 cases, } p \equiv 7 \bmod 9 \\ \left(\frac{2}{\bar{\pi}}\right)_3 & \text{2 cases, } p \equiv 7 \bmod 9 \\ \left(\frac{\sqrt{-3}}{\bar{\pi}}\right)_3 \zeta_3^{1,2} & \text{4 cases, } p \equiv 7 \bmod 9 \end{cases}$$

So the density is

$$4 \cdot \frac{1}{[\mathbf{Q}(\zeta_9, \sqrt[6]{\alpha_1}, \sqrt[6]{\alpha_2} : \mathbf{Q}(\zeta_3)]} + 2 \cdot \frac{1}{[\mathbf{Q}(\zeta_9, \sqrt[6]{\alpha_1}, \sqrt[6]{\alpha_2}, \sqrt[3]{2}) : \mathbf{Q}(\zeta_3)]}$$
$$+ 4 \cdot \frac{1}{[\mathbf{Q}(\zeta_9, \sqrt[6]{\alpha_1}, \sqrt[6]{\alpha_2}, \sqrt[6]{-3}) : \mathbf{Q}(\zeta_3)]} + 4 \cdot \frac{1}{[\mathbf{Q}(\zeta_9, \sqrt[6]{\alpha_1}, \sqrt[6]{\alpha_2}, \sqrt[3]{2}) : \mathbf{Q}(\zeta_3)]}$$
$$+ 8 \cdot \frac{1}{[\mathbf{Q}(\zeta_9, \sqrt[6]{\alpha_1}, \sqrt[6]{\alpha_2}, \sqrt[6]{-3}) : \mathbf{Q}(\zeta_3)]}$$
$$= \frac{1}{9} + \frac{1}{54} + \frac{1}{27} + \frac{1}{27} + \frac{2}{27} = \frac{5}{18}.$$

Case 21: $\alpha_1 \in$ Set 4, and $\alpha_2 \in$ Set 5: $1/3$

Since $\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6 = \begin{cases} 1 & \text{if } p \equiv 1 \pmod{12} \\ -1 & \text{if } p \equiv 7 \pmod{12} \end{cases}$,

and $\left(\frac{\alpha_2}{\pi}\right)_6 = \begin{cases} \zeta_3^{1,2,3} & \text{if } p \equiv 1 \pmod{12} \\ -\zeta_3^{1,2,3} & \text{if } p \equiv 7 \pmod{12} \end{cases}$,

we have

$$\left(\frac{\left(\frac{\alpha_1}{\pi}\right)_6 + \left(\frac{\alpha_2}{\pi}\right)_6}{\bar{\pi}}\right)_3 = \begin{cases} \left(\frac{2, \zeta_6, \zeta_6^5}{\bar{\pi}}\right) & \text{if } p \equiv 1 \pmod{12} \\ \left(\frac{-2, \zeta_6^2, \zeta_6^4}{\bar{\pi}}\right) & \text{if } p \equiv 7 \pmod{12} \end{cases}$$

Similar to Case 9, the density is $1/3$.

Case 22: $\alpha_1 \in$ Set 4, and $\alpha_2 \in$ Set 7: $5/18$

Since $\left(\frac{\alpha_1}{\mathfrak{p}}\right)_6 = \begin{cases} 1 & \text{if } p \equiv 1 \pmod{12} \\ -1 & \text{if } p \equiv 7 \pmod{12} \end{cases}$, we have

$$\left(\frac{\left(\frac{\alpha_1}{\pi}\right)_6 + \left(\frac{\alpha_2}{\pi}\right)_6}{\bar{\pi}}\right)_3 = \begin{cases} \left(\frac{0, 2, \zeta_6^{1,5}, \sqrt{-3} \cdot \zeta_6^{4,5}}{\bar{\mathfrak{p}}}\right) & \text{if } p \equiv 1 \pmod{12} \\ \left(\frac{0, -2, \zeta_6^{2,4}, \sqrt{-3} \cdot \zeta_6^{1,2}}{\bar{\mathfrak{p}}}\right) & \text{if } p \equiv 7 \pmod{12} \end{cases}$$

Similar to Case 13, the density is $5/18$.

Case 23: $\alpha_1 \in$ Set 5, and $\alpha_2 \in$ Set 7: $19/54$

Since $\left(\dfrac{\alpha_1}{\mathfrak{p}}\right)_6 = \begin{cases} \zeta_3^{1,2,3} & \text{if } p \equiv 1 \pmod{12} \\ -\zeta_3^{1,2,3} & \text{if } p \equiv 7 \pmod{12} \end{cases}$, we have the density is $19/54$, similar to Case 17.

Case 24: $\alpha_1/\alpha_2 \in$ Set 1: $1/3$

Let $\alpha_3 = \alpha_1/\alpha_2$, we have $\left(\dfrac{\alpha_3}{\mathfrak{p}}\right)_6 = 1$, and by Lemma 2.1.1

$$\left(\frac{\left(\frac{\alpha_1}{\pi}\right)_6 + \left(\frac{\alpha_2}{\pi}\right)_6}{\bar{\pi}}\right)_3 = \left(\frac{\left(1 + \left(\frac{\alpha_3}{\pi}\right)_6\right)\left(\frac{\alpha_2}{\pi}\right)_6}{\bar{\pi}}\right)_3 = \left(\frac{(2)\left(\frac{\alpha_2}{\pi}\right)_6}{\bar{\pi}}\right)_3$$

Also, by [12],

$$\left(\frac{\left(\frac{\alpha_2}{\pi}\right)_6}{\bar{\pi}}\right)_3 = \begin{cases} 1 & p \equiv 1 \bmod 9 \\ \left(\frac{\alpha_2}{\pi}\right)_3^2 & p \equiv 4 \bmod 9 \\ \left(\frac{\alpha_2}{\pi}\right)_3 & p \equiv 7 \bmod 9 \end{cases}$$

So,

$$\left(\frac{\left(\frac{\alpha_1}{\pi}\right)_6 + \left(\frac{\alpha_2}{\pi}\right)_6}{\bar{\pi}}\right)_3 = \begin{cases} \left(\frac{2}{\bar{\pi}}\right)_3 & p \equiv 1 \bmod 9 \\ \left(\frac{\alpha_2}{\pi}\right)_3^2 \left(\frac{2}{\bar{\pi}}\right)_3 & p \equiv 4 \bmod 9 \\ \left(\frac{\alpha_2}{\pi}\right)_3 \left(\frac{2}{\bar{\pi}}\right)_3 & p \equiv 7 \bmod 9 \end{cases}$$

If $\left(\dfrac{\alpha_2}{\pi}\right)_3 = \zeta_3^{1,2,3}$, then the density is

$$\frac{1}{[\mathbf{Q}(\zeta_9, \sqrt[3]{2}) : \mathbf{Q}(\zeta_3)]} + 3 \cdot \frac{1}{[\mathbf{Q}(\zeta_9, \sqrt[3]{\alpha_2}, \sqrt[3]{2}) : \mathbf{Q}(\zeta_3)]} + 3 \cdot \frac{1}{[\mathbf{Q}(\zeta_9, \sqrt[3]{\alpha_2}, \sqrt[3]{2}) : \mathbf{Q}(\zeta_3)]}.$$

If $\left(\dfrac{\alpha_2}{\pi}\right)_3 = 1$, then we have

$$\frac{1}{[\mathbf{Q}(\zeta_9, \sqrt[3]{2}) : \mathbf{Q}(\zeta_3)]} + \frac{1}{[\mathbf{Q}(\zeta_9, \sqrt[3]{2}) : \mathbf{Q}(\zeta_3)]} + \frac{1}{[\mathbf{Q}(\zeta_9, \sqrt[3]{2}) : \mathbf{Q}(\zeta_3)]}.$$

So the density is

$$\begin{cases} \frac{1}{9}+\frac{1}{9}+\frac{1}{9}=\frac{1}{3} & \text{if } \alpha_2 \in \text{ Set 2,5,7} \\ \\ \frac{1}{9}+\frac{1}{9}+\frac{1}{9}=\frac{1}{3} & \text{if } \alpha_2 \in \text{ Set 1,3,4} \end{cases}$$

Case 25: $\alpha_1/\alpha_2 \in$ Set 2: $\begin{cases} \frac{13}{27} & \text{if } \alpha_2 \in \text{ Set } 2,5,7, \textit{i.e.}, \left(\frac{\alpha_2}{\pi}\right)_3 = \zeta_3^{1,2,3} \\ \\ \frac{1}{3} & \text{if } \alpha_2 \in \text{ Set } 1,3,4, \textit{i.e.}, \left(\frac{\alpha_2}{\pi}\right)_3 = 1 \end{cases}$

Let $\alpha_3 = \alpha_1/\alpha_2$. We have $\left(\frac{\alpha_3}{\mathfrak{p}}\right)_6 = \zeta_3^{1,2,3}$, and by Lemma 2.1.1

$$\left(\frac{\left(\frac{\alpha_1}{\pi}\right)_6+\left(\frac{\alpha_2}{\pi}\right)_6}{\bar{\pi}}\right)_3 = \left(\frac{\left(1+\left(\frac{\alpha_3}{\pi}\right)_6\right)\left(\frac{\alpha_2}{\pi}\right)_6}{\bar{\pi}}\right)_3 = \left(\frac{2,\zeta_6,\zeta_6^5}{\bar{\pi}}\right)_3 \left(\frac{\left(\frac{\alpha_2}{\pi}\right)_6}{\bar{\pi}}\right)_3$$

Also, by [12],

$$\left(\frac{\left(\frac{\alpha_2}{\pi}\right)_6}{\bar{\pi}}\right)_3 = \begin{cases} 1 & p \equiv 1 \bmod 9 \\ \left(\frac{\alpha_2}{\pi}\right)_3^2 & p \equiv 4 \bmod 9 \\ \\ \left(\frac{\alpha_2}{\pi}\right)_3 & p \equiv 7 \bmod 9 \end{cases}$$

So,

$$\left(\frac{\left(\frac{\alpha_1}{\pi}\right)_6+\left(\frac{\alpha_2}{\pi}\right)_6}{\bar{\pi}}\right)_3 = \begin{cases} \left(\frac{2,\zeta_6,\zeta_6^5}{\bar{\pi}}\right)_3 & p \equiv 1 \bmod 9 \\ \left(\frac{\alpha_2}{\pi}\right)_3^2 \left(\frac{2,\zeta_6,\zeta_6^5}{\bar{\pi}}\right)_3 & p \equiv 4 \bmod 9 \\ \\ \left(\frac{\alpha_2}{\pi}\right)_3 \left(\frac{2,\zeta_6,\zeta_6^5}{\bar{\pi}}\right)_3 & p \equiv 7 \bmod 9 \end{cases}$$

Using the fact that

$$\left(\frac{\zeta_6}{\bar{\pi}}\right)_3 = \begin{cases} 1 & p \equiv 1 \pmod 9 \\ \zeta_3^2 & p \equiv 4 \pmod 9 \\ \zeta_3 & p \equiv 7 \pmod 9 \end{cases}$$

we have

$$\left(\frac{\left(\frac{\alpha_1}{\pi}\right)_6 + \left(\frac{\alpha_2}{\pi}\right)_6}{\bar{\pi}}\right)_3 = \begin{cases} \left(\frac{2}{\bar{\pi}}\right)_3 & \text{1 case, } p \equiv 1 \bmod 9 \\ 1 & \text{2 cases, } p \equiv 1 \bmod 9 \\ & \\ \left(\frac{\alpha_2}{\pi}\right)_3^2 \left(\frac{2}{\bar{\pi}}\right)_3 & \text{1 case, } p \equiv 4 \bmod 9 \\ \left(\frac{\alpha_2}{\pi}\right)_3^2 \zeta_3^{1,2} & \text{2 cases, } p \equiv 4 \bmod 9 \\ & \\ \left(\frac{\alpha_2}{\pi}\right)_3 \left(\frac{2}{\bar{\pi}}\right)_3 & \text{1 case, } p \equiv 7 \bmod 9 \\ \left(\frac{\alpha_2}{\pi}\right)_3 \zeta_3^{1,2} & \text{2 cases, } p \equiv 7 \bmod 9 \end{cases}$$

If $\left(\frac{\alpha_2}{\pi}\right)_3 = \zeta_3^{1,2,3}$, then the density is

$$\frac{1}{[\mathbf{Q}(\zeta_9, \sqrt[6]{\alpha_3}, \sqrt[3]{2}) : \mathbf{Q}(\zeta_3)]} + 2 \cdot \frac{1}{[\mathbf{Q}(\zeta_9, \sqrt[6]{\alpha_3}) : \mathbf{Q}(\zeta_3)]} + 6 \cdot \frac{1}{[\mathbf{Q}(\zeta_9, \sqrt[6]{\alpha_3}, \sqrt[3]{\alpha_2}, \sqrt[3]{2}) : \mathbf{Q}(\zeta_3)]} + 4 \cdot \frac{1}{[\mathbf{Q}(\zeta_9, \sqrt[6]{\alpha_3}, \sqrt[3]{\alpha_2}) : \mathbf{Q}(\zeta_3)]}$$

If $\left(\frac{\alpha_2}{\pi}\right)_3 = 1$, then we have

$$\frac{1}{[\mathbf{Q}(\zeta_9, \sqrt[6]{\alpha_3}, \sqrt[3]{2}) : \mathbf{Q}(\zeta_3)]} + 2 \cdot \frac{1}{[\mathbf{Q}(\zeta_9, \sqrt[6]{\alpha_3}) : \mathbf{Q}(\zeta_3)]} + 2 \cdot \frac{1}{[\mathbf{Q}(\zeta_9, \sqrt[6]{\alpha_3}, \sqrt[3]{2}) : \mathbf{Q}(\zeta_3)]}.$$

So the density is

$$\begin{cases} \frac{1}{27}+\frac{2}{9}+\frac{2}{27}+\frac{4}{27}=\frac{13}{27} & \text{if } \alpha_2 \in \text{ Set 2,5,7} \\ \\ \frac{1}{27}+\frac{2}{9}+\frac{2}{27}=\frac{1}{3} & \text{if } \alpha_2 \in \text{ Set 1,3,4} \end{cases}$$

Case 26: $\alpha_1/\alpha_2 \in$ Set 3: 1/6

Let $\alpha_3 = \alpha_1/\alpha_2$. We have $\left(\frac{\alpha_3}{\mathfrak{p}}\right)_6 = \pm 1$, and by Lemma 2.1.1

$$\left(\frac{\left(\frac{\alpha_1}{\pi}\right)_6 + \left(\frac{\alpha_2}{\pi}\right)_6}{\bar{\pi}}\right)_3 = \left(\frac{\left(1+\left(\frac{\alpha_3}{\pi}\right)_6\right)\left(\frac{\alpha_2}{\pi}\right)_6}{\bar{\pi}}\right)_3 = \left(\frac{(0,2)\left(\frac{\alpha_2}{\pi}\right)_6}{\bar{\pi}}\right)_3$$

The density is a half of the density in Case 24. So it's 1/6.

Case 27: $\alpha_1/\alpha_2 \in$ Set 4: 1/6

Let $\alpha_3 = \alpha_1/\alpha_2$. We have $\left(\frac{\alpha_3}{\mathfrak{p}}\right)_6 = \begin{cases} 1 & \text{if } p \equiv 1 \pmod{12} \\ -1 & \text{if } p \equiv 7 \pmod{12} \end{cases}$, and by Lemma 2.1.1

Similar to Case 26, the density is 1/6.

Case 28: $\alpha_1/\alpha_2 \in$ Set 5: $\begin{cases} \frac{19}{54} & \text{if } \alpha_2 \in \text{ Set 2,5,7}, i.e., \left(\frac{\alpha_2}{\pi}\right)_3 = \zeta_3^{1,2,3} \\ \\ \frac{5}{18} & \text{if } \alpha_2 \in \text{ Set 1,3,4}, i.e., \left(\frac{\alpha_2}{\pi}\right)_3 = 1 \end{cases}$

Let $\alpha_3 = \alpha_1/\alpha_2$. We have $\left(\frac{\alpha_3}{\mathfrak{p}}\right)_6 = \begin{cases} \zeta_3^{1,2,3} & \text{if } p \equiv 1 \pmod{12} \\ -\zeta_3^{1,2,3} & \text{if } p \equiv 7 \pmod{12} \end{cases}$, and by Lemma 2.1.1

$$\left(\frac{\left(\frac{\alpha_1}{\pi}\right)_6+\left(\frac{\alpha_2}{\pi}\right)_6}{\bar{\pi}}\right)_3 = \left(\frac{\left(1+\left(\frac{\alpha_3}{\pi}\right)_6\right)\left(\frac{\alpha_2}{\pi}\right)_6}{\bar{\pi}}\right)_3$$

$$= \begin{cases} \left(\frac{2,\zeta_6,\zeta_6^5}{\bar{\pi}}\right)_3 \left(\frac{\left(\frac{\alpha_2}{\pi}\right)_6}{\bar{\pi}}\right)_3 & \text{if } p \equiv 1 \pmod{12} \\ \left(\frac{0,\zeta_6^4\sqrt{-3},\zeta_6^5\sqrt{-3}}{\bar{\pi}}\right)_3 \left(\frac{\left(\frac{\alpha_2}{\pi}\right)_6}{\bar{\pi}}\right)_3 & \text{if } p \equiv 7 \pmod{12} \end{cases}$$

Also, by [12],

$$\left(\frac{\left(\frac{\alpha_2}{\pi}\right)_6}{\bar{\pi}}\right)_3 = \begin{cases} 1 & p \equiv 1 \bmod 9 \\ \left(\frac{\alpha_2}{\pi}\right)_3^2 & p \equiv 4 \bmod 9 \\ \left(\frac{\alpha_2}{\pi}\right)_3 & p \equiv 7 \bmod 9 \end{cases}$$

So for $p \equiv 1 \bmod 12$, it's similar to Case 25.

Specifically, $\begin{cases} \frac{13}{54} & \text{if } \alpha_2 \in \text{ Set 2,5,7}, i.e., \left(\frac{\alpha_2}{\pi}\right)_3 = \zeta_3^{1,2,3} \\ \frac{1}{6} & \text{if } \alpha_2 \in \text{ Set 1,3,4}, i.e., \left(\frac{\alpha_2}{\pi}\right)_3 = 1 \end{cases}$

Now for $p \equiv 7 \bmod 12$, we have

$$\left(\frac{\left(\frac{\alpha_1}{\pi}\right)_6+\left(\frac{\alpha_2}{\pi}\right)_6}{\bar{\pi}}\right)_3 = \begin{cases} \left(\frac{0,\zeta_6^4\sqrt{-3},\zeta_6^5\sqrt{-3}}{\bar{\pi}}\right)_3 & p \equiv 1 \bmod 9 \\ \left(\frac{\alpha_2}{\pi}\right)_3^2 \left(\frac{0,\zeta_6^4\sqrt{-3},\zeta_6^5\sqrt{-3}}{\bar{\pi}}\right)_3 & p \equiv 4 \bmod 9 \\ \left(\frac{\alpha_2}{\pi}\right)_3 \left(\frac{0,\zeta_6^4\sqrt{-3},\zeta_6^5\sqrt{-3}}{\bar{\pi}}\right)_3 & p \equiv 7 \bmod 9 \end{cases}$$

Using the fact that

$$\left(\frac{\zeta_6}{\bar{\pi}}\right)_3 = \begin{cases} 1 & p \equiv 1 \pmod{9} \\ \zeta_3^2 & p \equiv 4 \pmod{9} \\ \zeta_3 & p \equiv 7 \pmod{9} \end{cases}$$

we have

$$\left(\frac{\left(\frac{\alpha_1}{\pi}\right)_6 + \left(\frac{\alpha_2}{\pi}\right)_6}{\bar{\pi}}\right)_3 = \begin{cases} 0 & \text{1 case, } p \equiv 1 \bmod 9 \\ \left(\frac{\sqrt{-3}}{\bar{\pi}}\right)_3 & \text{2 cases, } p \equiv 1 \bmod 9 \\ & \\ 0 & \text{1 case, } p \equiv 4 \bmod 9 \\ \left(\frac{\alpha_2}{\pi}\right)_3^2 \left(\frac{\sqrt{-3}}{\bar{\pi}}\right)_3 \zeta_3^{1,2} & \text{2 cases, } p \equiv 4 \bmod 9 \\ & \\ 0 & \text{1 case, } p \equiv 7 \bmod 9 \\ \left(\frac{\alpha_2}{\pi}\right)_3^2 \left(\frac{\sqrt{-3}}{\bar{\pi}}\right)_3 \zeta_3^{1,2} & \text{2 cases, } p \equiv 7 \bmod 9 \end{cases}$$

If $\left(\frac{\alpha_2}{\pi}\right)_3 = \zeta_3^{1,2,3}$, then the density is

$$2 \cdot \frac{1}{[\mathbf{Q}(\zeta_{36}, \sqrt[6]{\alpha_3}, \sqrt[6]{-3}) : \mathbf{Q}(\zeta_3)]} + 6 \cdot \frac{1}{[\mathbf{Q}(\zeta_{36}, \sqrt[6]{\alpha_3}, \sqrt[3]{\alpha_2}, \sqrt[6]{-3}) : \mathbf{Q}(\zeta_3)]} + 6 \cdot \frac{1}{[\mathbf{Q}(\zeta_{36}, \sqrt[6]{\alpha_3}, \sqrt[3]{\alpha_2}, \sqrt[6]{-3}) : \mathbf{Q}(\zeta_3)]}$$

If $\left(\frac{\alpha_2}{\pi}\right)_3 = 1$, then we have

$$2 \cdot \frac{1}{[\mathbf{Q}(\zeta_{36}, \sqrt[6]{\alpha_3}, \sqrt[6]{-3}) : \mathbf{Q}(\zeta_3)]} + 2 \cdot \frac{1}{[\mathbf{Q}(\zeta_{36}, \sqrt[6]{\alpha_3}, \sqrt[6]{-3}) : \mathbf{Q}(\zeta_3)]} + 2 \cdot \frac{1}{[\mathbf{Q}(\zeta_{36}, \sqrt[6]{\alpha_3}, \sqrt[6]{-3}) : \mathbf{Q}(\zeta_3)]}.$$

So the density for $p \equiv 7 \bmod 12$ is

$$\begin{cases} \frac{1}{27}+\frac{1}{27}+\frac{1}{27}=\frac{1}{9} & \text{if } \alpha_2 \in \text{ Set 2,5,7} \\ \\ \frac{1}{27}+\frac{1}{27}+\frac{1}{27}=\frac{1}{9} & \text{if } \alpha_2 \in \text{ Set 1,3,4} \end{cases}$$

$$\text{Thus, the sum density is } \begin{cases} \frac{13}{54}+\frac{1}{9}=\frac{19}{54} & \text{if } \alpha_2 \in \text{ Set } 2,5,7, i.e., \left(\frac{\alpha_2}{\pi}\right)_3 = \zeta_3^{1,2,3} \\ \\ \frac{1}{6}+\frac{1}{9}=\frac{5}{18} & \text{if } \alpha_2 \in \text{ Set } 1,3,4, i.e., \left(\frac{\alpha_2}{\pi}\right)_3 = 1 \end{cases}$$

□

2.3 Special Cases for Fourth Power

We recognize differences among 5 sets of values of α_i

Set 1 (quartic): $\alpha_i \in$ {1, -4, 16, etc.,} such that $\left(\frac{\alpha_i}{\mathfrak{p}}\right)_4 = 1$.

Note that $\left(\frac{-4}{\mathfrak{p}}\right)_4 = \left(\frac{2i}{\mathfrak{p}}\right) = 1$, by [12].

Set 2: (negative set 1): $\alpha_i \in$ {-1. 4. -16, etc.,} such that

$$\left(\frac{\alpha_i}{\mathfrak{p}}\right)_4 = \begin{cases} 1, & p \equiv 1 \pmod 8 \\ \\ -1, & p \equiv 5 \pmod 8 \end{cases}$$

Set 3: $\alpha_i \in \mathbf{Q}^{\times 2}$, such that $\left(\frac{\alpha_i}{\mathfrak{p}}\right)_4 = \pm 1$

Set 4: $\alpha_i = \pm 2^{odd}$, such that

$$\left(\frac{\alpha_i}{\mathfrak{p}}\right)_4 = \begin{cases} \pm 1, & p \equiv 1 \pmod 8 \\ \\ \pm i, & p \equiv 5 \pmod 8 \end{cases}$$

Set 5: other cases.

For the following cases involving the above sets, we compute $\delta_2^1(E^1_{\alpha_1} \times E^1_{\alpha_2}, \mathcal{C})$:

Case 1: $\alpha_1 = \alpha_2 \in$ Set 1,2,3: 1/2

Case 2: $\alpha_1 = \alpha_2 \in$ Set 4: 1/4

Case 3: $\alpha_1 = \alpha_2 \in$ Set 5: 1/2

Case 4: $\alpha_1 \neq \alpha_2$, α_1 and $\alpha_2 \in$ Set 1, or α_1 and $\alpha_2 \in$ Set 2: 1/2

Case 5: $\alpha_1 \neq \alpha_2$, α_1 and $\alpha_2 \in$ Set 3: 1/4

Case 6: $\alpha_1 \neq \alpha_2$, α_1 and $\alpha_2 \in$ Set 4: 1/4

Case 7: $\alpha_1 \neq \alpha_2$, $\alpha_1 \in$ Set 1, and $\alpha_2 \in$ Set 2: 1/4

Case 8: $\alpha_1 \neq \alpha_2$, $\alpha_1 \in$ Set 1, and $\alpha_2 \in$ Set 3: 1/4

Case 9: $\alpha_1 \neq \alpha_2$, $\alpha_1 \in$ Set 1, and $\alpha_2 \in$ Set 4: 1/4

Case 10: $\alpha_1 \neq \alpha_2$, $\alpha_1 \in$ Set 1, and $\alpha_2 \in$ Set 5: 1/4

Case 11: $\alpha_1 \neq \alpha_2$, $\alpha_1 \in$ Set 2, and $\alpha_2 \in$ Set 3: 1/4

Case 12: $\alpha_1 \neq \alpha_2$, $\alpha_1 \in$ Set 2, and $\alpha_2 \in$ Set 4: 1/4

Case 13: $\alpha_1 \neq \alpha_2$, $\alpha_1 \in$ Set 2, and $\alpha_2 \in$ Set 5: 1/4

Case 14: $\alpha_1 \neq \alpha_2$, $\alpha_1 \in$ Set 3, and $\alpha_2 \in$ Set 4: 1/4

Case 15: $\alpha_1 \neq \alpha_2$, $\alpha_1 \in$ Set 3, and $\alpha_2 \in$ Set 5: 1/4

Case 16: $\alpha_1 \neq \alpha_2$, $\alpha_1 \in$ Set 4, and $\alpha_2 \in$ Set 5: 3/16

Case 17: $\alpha_1/\alpha_2 \in \text{Set 1:} = \begin{cases} 1/2 & \text{if } \alpha_2 \in \text{Set 1,2,3,5} \\ 1/4 & \text{if } \alpha_2 \in \text{Set 4} \end{cases}$

Case 18: $\alpha_1/\alpha_2 \in \text{Set 2:} = \begin{cases} 1/4 & \text{if } \alpha_2 \in \text{Set 1,2,3,4} \\ 3/8 & \text{if } \alpha_2 \in \text{Set 5} \end{cases}$

Case 19: $\alpha_1/\alpha_2 \in \text{Set 3:} = \begin{cases} 1/4 & \text{if } \alpha_2 \in \text{Set 1,2,3,5} \\ 1/8 & \text{if } \alpha_2 \in \text{Set 4} \end{cases}$

Case 20: $\alpha_1/\alpha_2 \in \text{Set 4:} \begin{cases} 1/4 & \alpha_2 \in \text{ Set 1,2,3,4} \\ 5/16 & \alpha_2 \in \text{ Set 5} \end{cases}$

Case 21: $\alpha_1/\alpha_2 \in$ Set 5: $\alpha_1/\alpha_2 \in$ Set 5: $= \begin{cases} 1/4 & \text{if } \alpha_2 \in \text{Set 1,2,3} \\ 3/16 & \text{if } \alpha_2 \in \text{Set 4} \end{cases}$

Proof. **Case 1**: $\alpha_1 = \alpha_2 \in$ Set 1,2, or 3: 1/2

If $\alpha_1 = \alpha_2 \in$ Set 1, which means $\left(\frac{\alpha_i}{\mathfrak{p}}\right)_4 = 1$, then we have

$$\left(\frac{a_p(E^1_{\alpha_1} \times E^1_{\alpha_2})}{\pi}\right)_4 = \left(\frac{\bar{\pi}}{\pi}\right)_4 \left(\frac{\left(\frac{\alpha_1}{\pi}\right)_4 + \left(\frac{\alpha_2}{\pi}\right)_4}{\pi}\right)_4 = \left(\frac{\bar{\pi}}{\pi}\right)_4 \left(\frac{2}{\pi}\right)_4$$

By the table 1.2.4, we have

$$\left(\frac{a_p(E^1_{\alpha_1} \times E^1_{\alpha_2})}{\pi}\right)_4 = \begin{cases} 1 & p \in \mathcal{G}_8^1, \mathcal{G}_8^{1+4i} \\ -1 & p \in \mathcal{G}_8^5, \mathcal{G}_8^{5+4i} \\ \left(\frac{\bar{\pi}}{\pi}\right)_4 \left(\frac{2}{\pi}\right)_4 = \left(\frac{2}{\pi}\right)_4 \left(\frac{2}{a}\right)(-1)\left(\frac{2}{\pi}\right)_4 & p \in \mathcal{C}_8^5 \end{cases}$$

$$= \begin{cases} 1 & p \in \mathcal{G}_8^1, \mathcal{G}_8^{1+4i} \\ -1 & p \in \mathcal{G}_8^5, \mathcal{G}_8^{5+4i} \\ \left(\frac{2}{a}\right) & p \in \mathcal{C}_8^5 \end{cases}$$

$$= \begin{cases} 1 & p \in \mathcal{G}_8^1, \mathcal{G}_8^{1+4i} \\ -1 & p \in \mathcal{G}_8^5, \mathcal{G}_8^{5+4i} \\ 1 & p \in \mathcal{C}_{16}^5 \\ -1 & p \in \mathcal{C}_{16}^{13} \end{cases}$$

So the density is

$\frac{1}{[\mathbf{Q}(\zeta_{16}) : \mathbf{Q}(i)]} + \frac{1}{[\mathbf{Q}(\zeta_{16}) : \mathbf{Q}(i)]} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$

When $\alpha_1 = \alpha_2 \in$ Case 2, we have

$$\left(\frac{a_p(E^1_{\alpha_1}\times E^1_{\alpha_2})}{\pi}\right)_4 = \left(\frac{\bar\pi}{\pi}\right)_4 \left(\frac{\left(\frac{\alpha_1}{\pi}\right)_4 + \left(\frac{\alpha_2}{\pi}\right)_4}{\pi}\right)_4$$

$$= \begin{cases} \left(\frac{\bar\pi}{\pi}\right)_4 \left(\frac{2}{\pi}\right)_4 & p \in \mathcal{C}_8^1 \\ \left(\frac{\bar\pi}{\pi}\right)_4 \left(\frac{-2}{\pi}\right)_4 & p \in \mathcal{C}_8^5 \end{cases}$$

Using the fact that

$$\left(\frac{-1}{\pi}\right)_4 = \begin{cases} 1 & p \in \mathcal{C}_8^1 \\ -1 & p \in \mathcal{C}_8^5 \end{cases}$$

we have

$$\left(\frac{a_p(E^1_{\alpha_1}\times E^1_{\alpha_2})}{\pi}\right)_4 = \begin{cases} 1 & p \in \mathcal{G}_8^1, \mathcal{G}_8^{1+4i} \\ -1 & p \in \mathcal{G}_8^5, \mathcal{G}_8^{5+4i} \\ \left(\frac{\bar\pi}{\pi}\right)_4 \left(\frac{-2}{\pi}\right)_4 = \left(\frac{2}{\pi}\right)_4 \left(\frac{2}{a}\right) (-1) \left(\frac{-2}{\pi}\right)_4 & p \in \mathcal{C}_8^5 \end{cases}$$

$$= \begin{cases} 1 & p \in \mathcal{G}_8^1, \mathcal{G}_8^{1+4i} \\ -1 & p \in \mathcal{G}_8^5, \mathcal{G}_8^{5+4i} \\ -\left(\frac{2}{a}\right) & p \in \mathcal{C}_8^5 \end{cases}$$

$$= \begin{cases} 1 & p \in \mathcal{G}_8^1, \mathcal{G}_8^{1+4i} \\ -1 & p \in \mathcal{G}_8^5, \mathcal{G}_8^{5+4i} \\ -1 & p \in \mathcal{C}_{16}^5 \\ 1 & p \in \mathcal{C}_{16}^{13} \end{cases}$$

So the density is $1/2$.

When $\alpha_1 = \alpha_2 \in$ Case 3, we have

$$\left(\frac{a_p(E^1_{\alpha_1} \times E^1_{\alpha_2})}{\pi}\right)_4 = \left(\frac{\bar{\pi}}{\pi}\right)_4 \left(\frac{\left(\frac{\alpha_1}{\pi}\right)_4 + \left(\frac{\alpha_2}{\pi}\right)_4}{\pi}\right)_4 \left(\frac{\bar{\pi}}{\pi}\right)_4 \left(\frac{\pm 2}{\pi}\right)_4$$

The proof proceeds similarly.

Case 2: $\alpha_1 = \alpha_2 \in$ Set 4: 1/4

Since $\left(\frac{\alpha_i}{\mathfrak{p}}\right)_4 = \begin{cases} \pm 1, & p \equiv 1 \pmod 8 \\ \pm i, & p \equiv 5 \pmod 8 \end{cases}$

we have

$$\begin{aligned}\left(\frac{a_p(E^1_{\alpha_1} \times E^1_{\alpha_2})}{\pi}\right)_4 &= \left(\frac{\bar{\pi}}{\pi}\right)_4 \left(\frac{\left(\frac{\alpha_1}{\pi}\right)_4 + \left(\frac{\alpha_2}{\pi}\right)_4}{\pi}\right)_4 \\ &= \begin{cases} \left(\frac{\bar{\pi}}{\pi}\right)_4 \left(\frac{2}{\pi}\right)_4 \left(\frac{i}{\pi}\right)_4^{2,4} & p \in \mathcal{C}_8^1 \\ \left(\frac{\bar{\pi}}{\pi}\right)_4 \left(\frac{2}{\pi}\right)_4 \left(\frac{i}{\pi}\right)_4^{1,3} & p \in \mathcal{C}_8^5 \end{cases} \\ &= \begin{cases} 1 & p \in \mathcal{G}_8^1, \mathcal{G}_8^{1+4i} \\ -1 & p \in \mathcal{G}_8^5, \mathcal{G}_8^{5+4i} \\ \pm i & p \in \mathcal{C}_8^5 \end{cases}\end{aligned}$$

So the density is $\frac{1}{[\mathbf{Q}(\zeta_{16}) : \mathbf{Q}(i)]} = \frac{1}{4}$.

Case 3: $\alpha_1 = \alpha_2 \in$ Set 5: 1/2

We have

$$\left(\frac{a_p(E^1_{\alpha_1} \times E^1_{\alpha_2})}{\pi}\right)_4 = \left(\frac{\bar{\pi}}{\pi}\right)_4 \left(\frac{\left(\frac{\alpha_1}{\pi}\right)_4 + \left(\frac{\alpha_2}{\pi}\right)_4}{\pi}\right)_4 = \left(\frac{\bar{\pi}}{\pi}\right)_4 \left(\frac{2}{\pi}\right)_4 \left(\frac{i}{\pi}\right)_4^{1,2,3,4}$$

By the table 1.2.4 we have the density as follows:

$$4\cdot\frac{1}{2}\cdot\frac{1}{[\mathbf{Q}(\zeta_{16},\sqrt[4]{\alpha_1}):\mathbf{Q}(i)]}+4\cdot\frac{1}{2}\cdot\frac{1}{[\mathbf{Q}(\zeta_{16},\sqrt[4]{\alpha_1}):\mathbf{Q}(i)]}+2\cdot\frac{1}{2}\cdot\frac{1}{[\mathbf{Q}(\zeta_{16},\sqrt[4]{\alpha_1}):\mathbf{Q}(i)]}$$
$$+2\cdot\frac{1}{2}\cdot\frac{1}{[\mathbf{Q}(\zeta_{16},\sqrt[4]{\alpha_1}):\mathbf{Q}(i)]}+2\cdot\frac{1}{[\mathbf{Q}(\zeta_{16},\sqrt[4]{\alpha_1}):\mathbf{Q}(i)]}=\frac{1}{8}+\frac{1}{8}+\frac{1}{16}+\frac{1}{16}+\frac{1}{8}=\frac{1}{2}.$$

Case 4: $\alpha_1 \neq \alpha_2$, α_1 and $\alpha_2 \in$ Set 1, or α_1 and $\alpha_2 \in$ Set 2: 1/2

Similar to Case 1.

Case 5: $\alpha_1 \neq \alpha_2$, α_1 and $\alpha_2 \in$ Set 3: 1/4

Similar to Case 1 but the density is 0 half of the time when $\left(\frac{\alpha_1}{\pi}\right)_4 = -\left(\frac{\alpha_2}{\pi}\right)_4$.

Case 6: $\alpha_1 \neq \alpha_2$, α_1 and $\alpha_2 \in$ Set 4: 1/4

Similar to Case 2

Case 7: $\alpha_1 \neq \alpha_2$, $\alpha_1 \in$ Set 1, and $\alpha_2 \in$ Set 2: 1/4

Since $\left(\frac{\alpha_1}{\mathfrak{p}}\right)_4 = 1$, and $\left(\frac{\alpha_i}{\mathfrak{p}}\right)_4 = \begin{cases} 1, & p \equiv 1 \pmod 8 \\ -1, & p \equiv 5 \pmod 8 \end{cases}$, we have

$$\left(\frac{a_p(E^1_{\alpha_1}\times E^1_{\alpha_2})}{\pi}\right)_4 = \left(\frac{\bar{\pi}}{\pi}\right)_4\left(\frac{\left(\frac{\alpha_1}{\pi}\right)_4+\left(\frac{\alpha_2}{\pi}\right)_4}{\pi}\right)_4$$
$$= \begin{cases} \left(\frac{\bar{\pi}}{\pi}\right)_4\left(\frac{2}{\pi}\right)_4 & p \in \mathcal{C}^1_8 \\ 0 & p \in \mathcal{C}^5_8 \end{cases}$$
$$= \begin{cases} 1 & p \in \mathcal{G}^1_8, \mathcal{G}^{1+4i}_8 \\ -1 & p \in \mathcal{G}^5_8, \mathcal{G}^{5+4i}_8 \\ 0 & p \in \mathcal{C}^5_8 \end{cases}$$

So the density is $\frac{1}{4}$.

Case 8: $\alpha_1 \neq \alpha_2$, $\alpha_1 \in$ Set 1, and $\alpha_2 \in$ Set 3: 1/4.

Since $\left(\frac{\alpha_1}{\mathfrak{p}}\right)_4 = 1$, and $\left(\frac{\alpha_2}{\mathfrak{p}}\right)_4 = \pm 1$, we have

$$\left(\frac{a_p(E^1_{\alpha_1} \times E^1_{\alpha_2})}{\pi}\right)_4 = \left(\frac{\bar{\pi}}{\pi}\right)_4 \left(\frac{\left(\frac{\alpha_1}{\pi}\right)_4 + \left(\frac{\alpha_2}{\pi}\right)_4}{\pi}\right)_4 = \left(\frac{\bar{\pi}}{\pi}\right)_4 \left(\frac{2}{\pi}\right)_4 \text{ or } 0.$$

So the density is half of that in Case 1, 1/4.

Case 9: $\alpha_1 \neq \alpha_2$, $\alpha_1 \in$ Set 1, and $\alpha_2 \in$ Set 4: 1/4

Since $\left(\frac{\alpha_1}{\mathfrak{p}}\right)_4 = 1$, and $\left(\frac{\alpha_2}{\mathfrak{p}}\right)_4 = \begin{cases} \pm 1, & p \equiv 1 \pmod 8 \\ \pm i, & p \equiv 5 \pmod 8 \end{cases}$, we have

$$\begin{aligned}
\left(\frac{a_p(E^1_{\alpha_1} \times E^1_{\alpha_2})}{\pi}\right)_4 &= \left(\frac{\bar{\pi}}{\pi}\right)_4 \left(\frac{\left(\frac{\alpha_1}{\pi}\right)_4 + \left(\frac{\alpha_2}{\pi}\right)_4}{\pi}\right)_4 \\
&= \begin{cases} \left(\frac{\bar{\pi}}{\pi}\right)_4 \left(\frac{0,2}{\pi}\right)_4 & p \in \mathcal{C}_8^1 \\ \left(\frac{\bar{\pi}}{\pi}\right)_4 \left(\frac{1 \pm i}{\pi}\right)_4 & p \in \mathcal{C}_8^5 \end{cases} \\
&= \begin{cases} 1 \text{ or } 0 & p \in \mathcal{G}_8^1, \mathcal{G}_8^{1+4i} \\ -1 \text{ or } 0 & p \in \mathcal{G}_8^5, \mathcal{G}_8^{5+4i} \\ \left(\frac{2}{\pi}\right)_4 \left(\frac{2}{a}\right) (-1) \left(\frac{1 \pm i}{\pi}\right)_4 & p \in \mathcal{C}_8^5 \end{cases} \\
&= \begin{cases} 1 \text{ or } 0 & p \in \mathcal{G}_8^1, \mathcal{G}_8^{1+4i} \\ -1 \text{ or } 0 & p \in \mathcal{G}_8^5, \mathcal{G}_8^{5+4i} \\ \left(\frac{2}{\pi}\right)_4 \left(\frac{-1 \pm i}{\pi}\right)_4 & p \in \mathcal{C}_{16}^5 \\ \left(\frac{2}{\pi}\right)_4 \left(\frac{1 \pm i}{\pi}\right)_4 & p \in \mathcal{C}_{16}^{13} \end{cases}
\end{aligned}$$

where a is the real part of π, and $\left(\frac{2}{a}\right) = \begin{cases} 1 & p \equiv 5 \pmod{16} \\ -1 & p \equiv 13 \pmod{16} \pmod{16} \end{cases}$. So the density is

$$\frac{1}{2} \cdot \frac{1}{[\mathbf{Q}(\zeta_{16}, \sqrt[4]{\alpha_2}) : \mathbf{Q}(i)]} + \frac{1}{2} \cdot \frac{1}{[\mathbf{Q}(\zeta_{16}, \sqrt[4]{\alpha_2}) : \mathbf{Q}(i)]} + 4 \cdot 2 \cdot 2 \cdot \frac{1}{[\mathbf{Q}(\zeta_{16}, \sqrt[4]{\alpha_2}, \sqrt[4]{2}, \sqrt[4]{1+i}) : \mathbf{Q}(i)]} = \frac{1}{16} + \frac{1}{16} + \frac{1}{8} = \frac{1}{4}.$$

Case 10: $\alpha_1 \neq \alpha_2$, $\alpha_1 \in$ Set 1, and $\alpha_2 \in$ Set 5: $1/4$

$$\left(\frac{a_p(E^1_{\alpha_1} \times E^1_{\alpha_2})}{\pi}\right)_4 = \left(\frac{\bar{\pi}}{\pi}\right)_4 \left(\frac{\left(\frac{\alpha_1}{\pi}\right)_4 + \left(\frac{\alpha_2}{\pi}\right)_4}{\pi}\right)_4 = \begin{cases} \left(\frac{\bar{\pi}}{\pi}\right)_4 \left(\frac{2}{\pi}\right)_4 & 1 \text{ case} \\ \left(\frac{\bar{\pi}}{\pi}\right)_4 \left(\frac{1 \pm i}{\pi}\right)_4 & 2 \text{ cases} \\ 0 & 1 \text{ case} \end{cases}$$

Note that $\left(\frac{\bar{\pi}}{\pi}\right)_4 = \begin{cases} \left(\frac{2}{\pi}\right)_4 (-1) & p \equiv 5 \pmod{16} \\ \left(\frac{2}{\pi}\right)_4 & p \equiv 13 \pmod{16} \end{cases}$

So the density is

$$\frac{1}{2}\cdot\frac{1}{[\mathbf{Q}(\zeta_{16},\sqrt[4]{\alpha_1},\sqrt[4]{\alpha_2}):\mathbf{Q}(i)]}+2\cdot\frac{1}{2}\cdot\frac{1}{[\mathbf{Q}(\zeta_{16},\sqrt[4]{\alpha_1},\sqrt[4]{\alpha_2},\sqrt[4]{1+i}):\mathbf{Q}(i)]}$$
$$+\frac{1}{2}\cdot\frac{1}{[\mathbf{Q}(\zeta_{16},\sqrt[4]{\alpha_1},\sqrt[4]{\alpha_2}):\mathbf{Q}(i)]}+2\cdot\frac{1}{2}\cdot\frac{1}{[\mathbf{Q}(\zeta_{16},\sqrt[4]{\alpha_1},\sqrt[4]{\alpha_2},\sqrt[4]{1+i}):\mathbf{Q}(i)]}$$
$$+\frac{1}{[\mathbf{Q}(\zeta_{16},\sqrt[4]{\alpha_1},\sqrt[4]{\alpha_2}):\mathbf{Q}(i)]}+2\cdot 2\cdot 2\cdot\frac{1}{[\mathbf{Q}(\zeta_{16},\sqrt[4]{\alpha_1},\sqrt[4]{\alpha_2},\sqrt[4]{1+i},\sqrt[4]{2}):\mathbf{Q}(i)]}$$
$$=\frac{1}{32}+\frac{1}{32}+\frac{1}{32}+\frac{1}{32}+\frac{1}{16}+\frac{1}{16}=\frac{1}{4}.$$

Case 11: $\alpha_1 \neq \alpha_2$, $\alpha_1 \in$ Set 2, and $\alpha_2 \in$ Set 3: 1/4

Since $\left(\frac{\alpha_1}{\mathfrak{p}}\right)_4 = \begin{cases} 1, & p \equiv 1 \pmod 8 \\ -1, & p \equiv 5 \pmod 8 \end{cases}$, and $\left(\frac{\alpha_2}{\mathfrak{p}}\right)_4 = \pm 1$, we have

$$\left(\frac{a_p(E^1_{\alpha_1}\times E^1_{\alpha_2})}{\pi}\right)_4 = \left(\frac{\bar{\pi}}{\pi}\right)_4 \left(\frac{\left(\frac{\alpha_1}{\pi}\right)_4 + \left(\frac{\alpha_2}{\pi}\right)_4}{\pi}\right)_4$$
$$= \begin{cases} \left(\frac{\bar{\pi}}{\pi}\right)_4 \left(\frac{0,2}{\pi}\right)_4 & p \in \mathcal{C}_8^1 \\ \left(\frac{\bar{\pi}}{\pi}\right)_4 \left(\frac{0,-2}{\pi}\right)_4 & p \in \mathcal{C}_8^5 \end{cases}$$

Similar to Case 1 but the density is 0 half of the time. So the density is 1/4.

Case 12: $\alpha_1 \neq \alpha_2$, $\alpha_1 \in$ Set 2, and $\alpha_2 \in$ Set 4: 1/4.

Similar to Case 9.

Case 13: $\alpha_1 \neq \alpha_2$, $\alpha_1 \in$ Set 2, and $\alpha_2 \in$ Set 5: 1/4.

Similar to Case 10

Case 14: $\alpha_1 \neq \alpha_2$, $\alpha_1 \in$ Set 3, and $\alpha_2 \in$ Set 4: 1/4

Since $\left(\frac{\alpha_1}{\mathfrak{p}}\right)_4 = \pm 1$, and $\left(\frac{\alpha_2}{\mathfrak{p}}\right)_4 = \begin{cases} \pm 1, & p \equiv 1 \pmod 8 \\ \pm i, & p \equiv 5 \pmod 8 \end{cases}$, we have

$$\left(\frac{a_p(E^1_{\alpha_1}\times E^1_{\alpha_2})}{\pi}\right)_4 = \left(\frac{\bar{\pi}}{\pi}\right)_4 \left(\frac{\left(\frac{\alpha_1}{\pi}\right)_4 + \left(\frac{\alpha_2}{\pi}\right)_4}{\pi}\right)_4$$

$$= \begin{cases} \left(\frac{\bar{\pi}}{\pi}\right)_4 \left(\frac{\pm 2}{\pi}\right)_4 & 2 \text{ cases, } p \in \mathcal{C}_8^1 \\ 0 & 2 \text{ cases, } p \in \mathcal{C}_8^1 \\ \left(\frac{\bar{\pi}}{\pi}\right)_4 \left(\frac{\pm 1 \pm i}{\pi}\right)_4 & 4 \text{ cases, } p \in \mathcal{C}_8^5 \end{cases}$$

So the density is

$$2\cdot\frac{1}{2}\cdot\frac{1}{[\mathbf{Q}(\zeta_{16},\sqrt[4]{\alpha_1},\sqrt[4]{\alpha_2}):\mathbf{Q}(i)]} + 2\cdot\frac{1}{2}\cdot\frac{1}{[\mathbf{Q}(\zeta_{16},\sqrt[4]{\alpha_1},\sqrt[4]{\alpha_2}):\mathbf{Q}(i)]}$$
$$+4\cdot 2\cdot 2\cdot\frac{1}{[\mathbf{Q}(\zeta_{16},\sqrt[4]{\alpha_1},\sqrt[4]{\alpha_2},\sqrt[4]{2},\sqrt[4]{1+i}):\mathbf{Q}(i)]}$$
$$=\frac{1}{16}+\frac{1}{16}+\frac{1}{8}=\frac{1}{4}.$$

Case 15: $\alpha_1 \neq \alpha_2$, $\alpha_1 \in$ Set 3, and $\alpha_2 \in$ Set 5: $1/4$

Since $\left(\frac{\alpha_1}{\mathfrak{p}}\right)_4 = \pm 1$, we have

$$\left(\frac{a_p(E^1_{\alpha_1}\times E^1_{\alpha_2})}{\pi}\right)_4 = \left(\frac{\bar{\pi}}{\pi}\right)_4 \left(\frac{\left(\frac{\alpha_1}{\pi}\right)_4 + \left(\frac{\alpha_2}{\pi}\right)_4}{\pi}\right)_4$$

$$= \begin{cases} \left(\frac{\bar{\pi}}{\pi}\right)_4 \left(\frac{\pm 2}{\pi}\right)_4 & 2 \text{ cases} \\ \left(\frac{\bar{\pi}}{\pi}\right)_4 \left(\frac{\pm 1 \pm i}{\pi}\right)_4 & 4 \text{ cases} \\ 0 & 2 \text{ cases} \end{cases}$$

So the density is

$$
\begin{aligned}
&2\cdot\frac{1}{2}\cdot\frac{1}{[\mathbf{Q}(\zeta_{16},\sqrt[4]{\alpha_1},\sqrt[4]{\alpha_2}):\mathbf{Q}(i)]}+4\cdot\frac{1}{2}\cdot\frac{1}{[\mathbf{Q}(\zeta_{16},\sqrt[4]{\alpha_1},\sqrt[4]{\alpha_2},\sqrt[4]{1+i}):\mathbf{Q}(i)]}\\
&+2\cdot\frac{1}{2}\cdot\frac{1}{[\mathbf{Q}(\zeta_{16},\sqrt[4]{\alpha_1},\sqrt[4]{\alpha_2}):\mathbf{Q}(i)]}+4\cdot\frac{1}{2}\cdot\frac{1}{[\mathbf{Q}(\zeta_{16},\sqrt[4]{\alpha_1},\sqrt[4]{\alpha_2},\sqrt[4]{1+i}):\mathbf{Q}(i)]}\\
&+2\cdot\frac{1}{[\mathbf{Q}(\zeta_{16},\sqrt[4]{\alpha_1},\sqrt[4]{\alpha_2}):\mathbf{Q}(i)]}+4\cdot 2\cdot 2\cdot\frac{1}{[\mathbf{Q}(\zeta_{16},\sqrt[4]{\alpha_1},\sqrt[4]{\alpha_2},\sqrt[4]{2},\sqrt[4]{1+i}):\mathbf{Q}(i)]}\\
&=\frac{1}{32}+\frac{1}{32}+\frac{1}{32}+\frac{1}{32}+\frac{1}{16}+\frac{1}{16}=\frac{1}{4}
\end{aligned}
$$

Case 16: $\alpha_1 \neq \alpha_2$, $\alpha_1 \in$ Set 4, and $\alpha_2 \in$ Set 5: 3/16.

Since $\left(\frac{\alpha_1}{\mathfrak{p}}\right)_4 = \begin{cases} \pm 1, & p \equiv 1 \pmod 8 \\ \pm i, & p \equiv 5 \pmod 8 \end{cases}$, we have

$$
\left(\frac{a_p(E^1_{\alpha_1}\times E^1_{\alpha_2})}{\pi}\right)_4 = \left(\frac{\bar\pi}{\pi}\right)_4\left(\frac{\left(\frac{\alpha_1}{\pi}\right)_4+\left(\frac{\alpha_2}{\pi}\right)_4}{\pi}\right)_4
$$

$$
= \begin{cases}
\left(\frac{\bar\pi}{\pi}\right)_4\left(\frac{\pm 2}{\pi}\right)_4 & 2 \text{ cases, } p\in\mathcal{C}_8^1\\
0 & 2 \text{ cases, } p\in\mathcal{C}_8^1\\
\left(\frac{\bar\pi}{\pi}\right)_4\left(\frac{\pm 1\pm i}{\pi}\right)_4 & 4 \text{ cases, } p\in\mathcal{C}_8^1\\
\\
\left(\frac{\bar\pi}{\pi}\right)_4\left(\frac{\pm 2}{\pi}\right)_4\left(\frac{i}{\pi}\right)_4 & 2 \text{ cases, } p\in\mathcal{C}_8^5\\
0 & 2 \text{ cases, } p\in\mathcal{C}_8^5\\
\left(\frac{\bar\pi}{\pi}\right)_4\left(\frac{\pm 1\pm i}{\pi}\right)_4 & 4 \text{ cases, } p\in\mathcal{C}_8^5
\end{cases}
$$

So the density is

$$2\cdot\frac{1}{2}\cdot\frac{1}{[\mathbf{Q}(\zeta_{16},\sqrt[4]{\alpha_1},\sqrt[4]{\alpha_2}):\mathbf{Q}(i)]}+4\cdot\frac{1}{2}\cdot\frac{1}{[\mathbf{Q}(\zeta_{16},\sqrt[4]{\alpha_1},\sqrt[4]{\alpha_2},\sqrt[4]{1+i}):\mathbf{Q}(i)]}$$
$$+2\cdot\frac{1}{2}\cdot\frac{1}{[\mathbf{Q}(\zeta_{16},\sqrt[4]{\alpha_1},\sqrt[4]{\alpha_2}):\mathbf{Q}(i)]}+4\cdot\frac{1}{2}\cdot\frac{1}{[\mathbf{Q}(\zeta_{16},\sqrt[4]{\alpha_1},\sqrt[4]{\alpha_2},\sqrt[4]{1+i}):\mathbf{Q}(i)]}$$
$$+4\cdot 2\cdot 2\cdot\frac{1}{[\mathbf{Q}(\zeta_{16},\sqrt[4]{\alpha_1},\sqrt[4]{\alpha_2},\sqrt[4]{2},\sqrt[4]{1+i}):\mathbf{Q}(i)]}$$
$$=\frac{1}{32}+\frac{1}{32}+\frac{1}{32}+\frac{1}{32}+\frac{1}{16}=\frac{3}{16}.$$

Case 17: $\alpha_1/\alpha_2 \in$ Set 1: $= \begin{cases} 1/2 & \text{if } \alpha_2 \in \text{Set 1,2,3,5} \\ 1/4 & \text{if } \alpha_2 \in \text{Set 4} \end{cases}$

Let $\alpha_3 = \alpha_1/\alpha_2 \in$ Set 1. So that $\left(\frac{\alpha_3}{\mathfrak{p}}\right)_4 = 1$. We have

$$\left(\frac{a_p(E^1_{\alpha_1}\times E^1_{\alpha_2})}{\pi}\right)_4 = \left(\frac{\bar\pi}{\pi}\right)_4\left(\frac{\left(\frac{\alpha_1}{\pi}\right)_4+\left(\frac{\alpha_2}{\pi}\right)_4}{\pi}\right)_4$$
$$= \left(\frac{\bar\pi}{\pi}\right)_4\left(\frac{\left(\left(\frac{\alpha_3}{\pi}\right)_4+1\right)\cdot\left(\frac{\alpha_2}{\pi}\right)_4}{\pi}\right)_4$$
$$= \left(\frac{\bar\pi}{\pi}\right)_4\left(\frac{2}{\pi}\right)_4\left(\frac{\left(\frac{\alpha_2}{\pi}\right)_4}{\pi}\right)_4$$

If $\alpha_2 \in$ Set 1,2,3, then it's similar to Case 1, so the density is $1/2$. If $\alpha_2 \in$ Set 4, then it's similar to Case 2, so the density is $1/4$. If $\alpha_2 \in$ Set 5, then it's similar to Case 3, so the density is $1/2$.

Case 18: $\alpha_1/\alpha_2 \in$ Set 2: $= \begin{cases} 1/4 & \text{if } \alpha_2 \in \text{Set 1,2,3,4} \\ 3/8 & \text{if } \alpha_2 \in \text{Set 5} \end{cases}$

Let $\alpha_3 = \alpha_1/\alpha_2 \in$ Set 2. So that $\left(\frac{\alpha_3}{\mathfrak{p}}\right)_4 = \begin{cases} 1, & p\equiv 1 \pmod 8 \\ -1, & p\equiv 5 \pmod 8 \end{cases}$. We have

$$\left(\frac{a_p(E^1_{\alpha_1}\times E^1_{\alpha_2})}{\pi}\right)_4 = \left(\frac{\bar{\pi}}{\pi}\right)_4 \left(\frac{\left(\frac{\alpha_1}{\pi}\right)_4 + \left(\frac{\alpha_2}{\pi}\right)_4}{\pi}\right)_4$$

$$= \left(\frac{\bar{\pi}}{\pi}\right)_4 \left(\frac{\left(\left(\frac{\alpha_3}{\pi}\right)_4 + 1\right)\cdot\left(\frac{\alpha_2}{\pi}\right)_4}{\pi}\right)_4$$

$$= \begin{cases} \left(\frac{\bar{\pi}}{\pi}\right)_4 \left(\frac{2}{\pi}\right)_4 \left(\frac{\left(\frac{\alpha_2}{\pi}\right)_4}{\pi}\right)_4 & p \in \mathcal{C}^1_8 \\ 0 & p \in \mathcal{C}^5_8 \end{cases}$$

If $\alpha_2 \in$ Set 1,2, then it's similar to Case 7, so the density is $1/4$. If $\alpha_2 \in$ Set 3,4, then it's similar to Case 2, so the density is $1/4$.

If $\alpha_2 \in$ Set 5, then it's similar to Case 3 when $p \equiv 1 \bmod 8$, so the density is $3/8$.

Case 19: $\alpha_1/\alpha_2 \in$ Set 3: $= \begin{cases} 1/4 & \text{if } \alpha_2 \in \text{Set 1,2,3,5} \\ 1/8 & \text{if } \alpha_2 \in \text{Set 4} \end{cases}$

Let $\alpha_3 = \alpha_1/\alpha_2 \in$ Set 3. So that $\left(\frac{\alpha_i}{\mathfrak{p}}\right)_4 = \pm 1$. We have

$$\left(\frac{a_p(E^1_{\alpha_1}\times E^1_{\alpha_2})}{\pi}\right)_4 = \left(\frac{\bar{\pi}}{\pi}\right)_4 \left(\frac{\left(\frac{\alpha_1}{\pi}\right)_4 + \left(\frac{\alpha_2}{\pi}\right)_4}{\pi}\right)_4$$

$$= \left(\frac{\bar{\pi}}{\pi}\right)_4 \left(\frac{\left(\left(\frac{\alpha_3}{\pi}\right)_4 + 1\right)\cdot\left(\frac{\alpha_2}{\pi}\right)_4}{\pi}\right)_4$$

$$= \left(\frac{\bar{\pi}}{\pi}\right)_4 \left(\frac{2,0}{\pi}\right)_4 \left(\frac{\left(\frac{\alpha_2}{\pi}\right)_4}{\pi}\right)_4$$

The density in this case is a half of the density in Case 17.

Case 20: $\alpha_1/\alpha_2 \in$ Set 4: $\begin{cases} 1/4 & \alpha_2 \in \text{ Set 1,2,3,4} \\ 5/16 & \alpha_2 \in \text{ Set 5} \end{cases}$

Let $\alpha_3 = \alpha_1/\alpha_2 \in$ Set 4. So that $\left(\frac{\alpha_i}{\mathfrak{p}}\right)_4 = \begin{cases} \pm 1, & p \equiv 1 \pmod 8 \\ \pm i, & p \equiv 5 \pmod 8 \end{cases}$.

We have

$$\begin{aligned}
\left(\frac{a_p(E^1_{\alpha_1} \times E^1_{\alpha_2})}{\pi}\right)_4 &= \left(\frac{\bar{\pi}}{\pi}\right)_4 \left(\frac{\left(\frac{\alpha_1}{\pi}\right)_4 + \left(\frac{\alpha_2}{\pi}\right)_4}{\pi}\right)_4 \\
&= \left(\frac{\bar{\pi}}{\pi}\right)_4 \left(\frac{\left(\left(\frac{\alpha_3}{\pi}\right)_4 + 1\right) \cdot \left(\frac{\alpha_2}{\pi}\right)_4}{\pi}\right)_4 \\
&= \begin{cases} \left(\frac{\bar{\pi}}{\pi}\right)_4 \left(\frac{2,0}{\pi}\right)_4 \left(\frac{\left(\frac{\alpha_2}{\pi}\right)_4}{\pi}\right)_4 & p \in \mathcal{C}^1_8 \\ \left(\frac{\bar{\pi}}{\pi}\right)_4 \left(\frac{1 \pm i}{\pi}\right)_4 \left(\frac{\left(\frac{\alpha_2}{\pi}\right)_4}{\pi}\right)_4 & p \in \mathcal{C}^5_8 \end{cases}
\end{aligned}$$

If $\alpha_2 \in$ Set 1, then it's similar to Case 9, so the density is $1/4$. If $\alpha_2 \in$ Set 2, then it's similar to Case 12, so the density is $1/4$. If $\alpha_2 \in$ Set 3,4, then it's similar to Case 14, so the density is $1/4$.

If $\alpha_2 \in$ Set 5, then

$$\left(\frac{a_p(E^1_{\alpha_1} \times E^1_{\alpha_2})}{\pi}\right)_4 = \begin{cases} \left(\frac{\bar{\pi}}{\pi}\right)_4 \left(\frac{2}{\pi}\right)_4 \left(\frac{i}{\pi}\right)_4^{1,2,3,4} & \text{4 cases, } p \in \mathcal{C}^1_8 \\ 0 & \text{4 cases, } p \in \mathcal{C}^1_8 \\ \left(\frac{\bar{\pi}}{\pi}\right)_4 \left(\frac{\pm 1 \pm i}{\pi}\right)_4 & \text{8 cases, } p \in \mathcal{C}^5_8 \end{cases}$$

So the density is

$$4 \cdot \frac{1}{2} \cdot \frac{1}{[\mathbf{Q}(\zeta_{16}, \sqrt[4]{\alpha_2}, \sqrt[4]{\alpha_3}) : \mathbf{Q}(i)]} + 4 \cdot \frac{1}{2} \cdot \frac{1}{[\mathbf{Q}(\zeta_{16}, \sqrt[4]{\alpha_2}, \sqrt[4]{\alpha_3}) : \mathbf{Q}(i)]}$$
$$2 \cdot \frac{1}{2} \cdot \frac{1}{[\mathbf{Q}(\zeta_{16}, \sqrt[4]{\alpha_2}, \sqrt[4]{\alpha_3}) : \mathbf{Q}(i)]} + 2 \cdot \frac{1}{2} \cdot \frac{1}{[\mathbf{Q}(\zeta_{16}, \sqrt[4]{\alpha_2}, \sqrt[4]{\alpha_3}) : \mathbf{Q}(i)]}$$
$$+ 8 \cdot 2 \cdot 2 \cdot \frac{1}{[\mathbf{Q}(\zeta_{16}, \sqrt[4]{\alpha_2}, \sqrt[4]{\alpha_3}, \sqrt[4]{2}, \sqrt[4]{1+i}) : \mathbf{Q}(i)]}$$
$$= \frac{1}{16} + \frac{1}{16} + \frac{1}{32} + \frac{1}{32} + \frac{1}{8} = \frac{5}{16}$$

Case 21: $\alpha_1/\alpha_2 \in$ Set 5: $\alpha_1/\alpha_2 \in$ Set 5:$= \begin{cases} 1/4 & \text{if } \alpha_2 \in \text{Set 1,2,3} \\ 3/16 & \text{if } \alpha_2 \in \text{Set 4} \end{cases}$

Let $\alpha_3 = \alpha_1/\alpha_2 \in$ Set 5. We have

$$\left(\frac{a_p(E^1_{\alpha_1} \times E^1_{\alpha_2})}{\pi}\right)_4 = \left(\frac{\bar{\pi}}{\pi}\right)_4 \left(\frac{\left(\frac{\alpha_1}{\pi}\right)_4 + \left(\frac{\alpha_2}{\pi}\right)_4}{\pi}\right)_4$$
$$= \left(\frac{\bar{\pi}}{\pi}\right)_4 \left(\frac{\left(\left(\frac{\alpha_3}{\pi}\right)_4 + 1\right) \cdot \left(\frac{\alpha_2}{\pi}\right)_4}{\pi}\right)_4$$
$$= \left(\frac{\bar{\pi}}{\pi}\right)_4 \left(\frac{2, 0, 1 \pm i}{\pi}\right)_4 \left(\frac{\left(\frac{\alpha_2}{\pi}\right)_4}{\pi}\right)_4$$

If $\alpha_2 \in$ Set 1, then it's similar to Case 10, so the density is $1/4$. If $\alpha_2 \in$ Set 2, then it's similar to Case 13, so the density is $1/4$. If $\alpha_2 \in$ Set 3, then it's similar to Case 15, so the density is $1/4$. If $\alpha_2 \in$ Set 4, then it's similar to Case 16, so the density is $3/16$. □

CHAPTER 3

QUADRATIC POWER RESIDUE DISTRIBUTION

3.1 Introduction

We wanted to test the conjecture that a_p of non-CM elliptic curves is a square modulo p about half of the time across prime numbers. The quadratic residue of a_p is 1 if a_p is a square modulo p, and -1 if a_p is a non-square modulo p. So if the probability that a_p is a square modulo p is $1/2$, we would expect $\sum \left(\frac{a_p}{p}\right)$ across all primes p to be small. The sum turned out to be positively biased. However, as more primes are considered, the bias becomes insignificant compared with the sum. We show, assuming a natural independence result, that when varying elliptic curves as well as the prime p, the a_p are evenly distributed between squares and non-squares modulo p, asymptotically.

We are computing the sum of the power residue of isomorphic classes of elliptic curves $E : y^2 = x^3 + ax + b$. Fix bounds M, N with M much smaller than N. We let $|a|, |b| \leq N, p \leq M$ and look for a good upper bound of

$$\sum_{a,b} \sum_{p} \left(\frac{a_p}{p}\right)$$

with $\left(\frac{a_p}{p}\right)$ the Legendre symbol.

Remark 3.1.1. The obvious bound is the number of terms

$$\sum_{a,b} \sum_{p} \left(\frac{a_p}{p}\right) \leq (4N^2 - 1)\pi(M)$$

with π the prime counting function. Note that any improvement of this bound implies even distribution between squares and non-squares in the limit. Specifically, if $\sum_{a,b}\sum_p \left(\frac{a_p}{p}\right) \leq B$, and $\mathcal{O}(B) < N^2\pi(M)$, then $\lim_{p\to\infty} \frac{B}{N^2\pi(M)} = 0$, which implies $\left(\frac{a_p}{p}\right) = 1$ half of the time.

The idea of the proof is that if we fix a prime p, as a, b vary, each possible value of a_p reoccurs a certain number of times which turns out to be a familiar function of $(4p - (a_p)^2)$ [7, Theorem 3.2], and we can easily find the upper bound of this function in terms of p. We can also bound the frequency of a_p being a square modulo p given a fix prime p using Burgess' bound [4]. Then, assuming the independence of the function and the frequency mentioned above, we obtain the result using partial summation to sum over primes.

3.2 Sum of Quadratic Residues over Isomorphic Classes of Elliptic Curves

We begin by reversing the order of the summation.

$$\sum_{a,b}\sum_{p}\left(\frac{a_p}{p}\right) = \sum_{p}\sum_{a,b}\left(\frac{a_p}{p}\right)$$

For each prime p, let $N = pn_p + r_p$, so

$$\sum_{p}\sum_{a,b}\left(\frac{a_p}{p}\right) =$$

$$\sum_{p}\left(\sum_{\substack{|a|,|b|\leq pn_p\\ p\nmid(4a^3+27b^2)}}\left(\frac{a_p}{p}\right) + \sum_{\substack{|a|\leq N,\\ pn_p<|b|\leq N\\ p\nmid(4a^3+27b^2)}}\left(\frac{a_p}{p}\right) + \sum_{\substack{|b|\leq N,\\ pn_p<|a|\leq N\\ p\nmid(4a^3+27b^2)}}\left(\frac{a_p}{p}\right) + \sum_{\substack{pn_p<|a|,|b|\leq N\\ p\nmid(4a^3+27b^2)}}\left(\frac{a_p}{p}\right)\right)$$

For the first summation, n_p represents the number of times a, b goes over a complete set of isomorphism classes of elliptic curves. Each isomorphism class has at most

$(p-1)/2$ members. For each integer $|t| < 2\sqrt{p}$, we learn from [7, Menezes, theorem 3.2, page 36] that the Kronecker class number of $(4p - t^2)$, denoted as $H(4p - t^2)$, represents the number of isomorphism classes of elliptic curves of which $a_p = t$. Thus,

$$\begin{aligned}
\sum_{\substack{|a|,|b|\le pn_p \\ p\nmid(4a^3+27b^2)}} \left(\frac{a_p}{p}\right) &\le 4(n_p)^2 \cdot \frac{p-1}{2} \sum_{|t|<2\sqrt{p}} H(4p-t^2)\left(\frac{t}{p}\right) \\
&< 4(n_p)^2 \cdot \frac{p}{2} \sum_{|t|<2\sqrt{p}} H(4p-t^2)\left(\frac{t}{p}\right) \\
&= 2\frac{(N-r_p)^2}{p} \sum_{|t|<2\sqrt{p}} H(4p-t^2)\left(\frac{t}{p}\right) \\
&\le 2\frac{N^2}{p} \sum_{|t|<2\sqrt{p}} H(4p-t^2)\left(\frac{t}{p}\right)
\end{aligned}$$

Remark 3.2.1. Assumption on the Kronecker class number

In the following theorem, for any fixed rational prime p, and integer t with $|t| < 2\sqrt{p}$, we assume that $H(4p - t^2)$ is independent of $\left(\frac{t}{p}\right)$. In other words, we assume that neither $\left(\frac{t}{p}\right) = 1$ nor $\left(\frac{t}{p}\right) = -1$ is systematically associated with larger values of $H(4p - t^2)$.

Theorem 3.2.1. *Let M, N be integers such that $N \le \dfrac{M^{17/16}}{2Log^2(M)}$. For integers a, b such that $|a|, |b| \le N$, consider elliptic curves $E : y^2 = x^3 + ax + b$. Under the assumption on the Kronecker class number as in Remark 3.2.1, we have*

$$\sum_{|a|,|b|\le N} \sum_{p\le M} \left(\frac{a_p}{p}\right) \ll_\epsilon N^2 M^{15/16+\epsilon} Log(M) + \frac{NM^2}{2Log(M)} + \frac{M^3}{3Log(M)}$$

Proof. If we fix a prime p, as a, b vary, we have

$$\sum_{\substack{|a|,|b|\le pn_p \\ p\nmid(4a^3+27b^2)}} \left(\frac{a_p}{p}\right) < 2\frac{N^2}{p} \sum_{|t|<2\sqrt{p}} H(4p-t^2)\left(\frac{t}{p}\right)$$

where $H(4p-t^2)$ represents the number of isomorphic classes of elliptic curves of which $a_p = t$, which we assume is independent of $\left(\frac{t}{p}\right)$, or whether or not t is square modulo p. We know that the class number formula for a fundamental $D < 0$ is

$$h(D) = \frac{w\sqrt{|D|}}{2\pi} L(1,\chi)$$

where D denotes the discriminant of a imaginary quadratic field K, w is the number of roots of unity in K, $L(1,\chi)$ is the L-function of K, and χ is the quadratic character of K.

Using Hua's bound of the L-function [5, chapter 2] as in [6, Louboutin, page 214], we have:

$$L(1,\chi) \leq Log(\sqrt{(|D|}) + 1$$

So we have

$$\begin{aligned} h(D) &= \frac{w}{2\pi}|D|^{1/2}L(1,\chi) \\ &\leq \frac{w}{2\pi}|D|^{1/2}(Log(\sqrt{(|D|}) + 1) \\ &\leq C_1 \cdot |D|^{1/2} Log(D) \end{aligned}$$

where $C_1 \leq \frac{3}{\pi}$ is a constant.

Now for non-fundamental discriminant, which is the discriminant of a unique order $\mathcal{O}$ of index f in the ring of integer $\mathcal{O}_K$ of K. From [2, Cox, Theorem 7.24] we have the following formula for $h(D)$:

$$h(D) = \frac{h(d_K)f}{[\mathcal{O}_K^* : \mathcal{O}^*]} \prod_{p|f} \left(1 - \left(\frac{d_K}{p}\right)\frac{1}{p}\right)$$

where d_K denotes the discriminant of the field K, and $\left(\frac{d_K}{p}\right)$ denotes the Legendre symbol. We note that $D = f^2 d_K$.

Now using the formula above for $h(d_K)$, and the fact that $D = f^2 d_K$, we have:

$$h(D) = \frac{w}{2\pi} \frac{|D|^{1/2} L(1, \chi)}{[\mathcal{O}_K^* : \mathcal{O}^*]} \prod_{p|f} \left(1 - \left(\frac{d_K}{p}\right) \frac{1}{p}\right)$$

Now

$$\begin{aligned}
\prod_{p|f} \left(1 - \left(\frac{d_K}{p}\right) \frac{1}{p}\right) &\leq \prod_{p|f} \left(1 + \frac{1}{p}\right) \\
&= \prod_{p|f} \left(\frac{p+1}{p}\right) \\
&< \prod_{p \leq f} \left(\frac{p}{p-1}\right) \\
&= \frac{Log(f)}{e^{-\gamma} + o(1)}
\end{aligned}$$

The last equality is from Mertens' third theorem by Tao [11, theorem 3], with γ being the Euler-Mascheroni constant.

Since $f \leq |D|^{1/2}$, together with the upper bound above, we have

$$\begin{aligned}
h(D) &< C_1 \cdot |D|^{1/2} Log(D) \frac{Log(|D|^{1/2})}{e^{-\gamma} + o(1)} \\
&\leq C_2 \cdot |D|^{1/2} (Log(D))^2
\end{aligned}$$

where $C_2 = \dfrac{C_1}{2e^{-\gamma} + o(1)} \leq \dfrac{3}{2\pi e^{-\gamma} + o(1)}$.

Let $D = 4p - t^2 \geq 0$. Then we have

$$H(D) = h(-D) < C_2 \cdot |D|^{1/2} (Log(D))^2$$

More specifically,

$$\begin{aligned}
H(4p - t^2) &< C_2 \cdot \sqrt{4p - t^2} Log^2(4p - t^2) \\
&\leq C_2 \cdot \sqrt{4p} Log^2(4p)
\end{aligned}$$

Under the assumption on the independence between the Kronecker class number $H(4p-t^2)$ and $\left(\frac{t}{p}\right)$ as in Remark 3.2.1, we now have

$$\sum_{\substack{|a|,|b|\leq pn_p \\ p\nmid(4a^3+27b^2)}} \left(\frac{a_p}{p}\right) < 2\frac{N^2}{p} \sum_{0<t<2\sqrt{p}} H(4p-t^2)\left(\frac{t}{p}\right)$$
$$< \frac{2N^2}{p} \cdot C_2\sqrt{4p}Log^2(4p) \cdot \sum_{0<t<2\sqrt{p}} \left(\frac{t}{p}\right)$$

Now we let χ be the character modulus $p : \left(\frac{\mathbb{Z}}{p}\right)^* \to \mathbb{C}^*$. By Burgess's bound on character sums from [4], we have

$$\sum_{0<t<2\sqrt{p}} \left(\frac{t}{p}\right) = \sum_{0<t\leq 2\sqrt{p}} \chi(t) \ll_{\epsilon,r} (2\sqrt{p})^{1-1/r} \cdot p^{\frac{r+1}{4r^2}+\epsilon}$$

The bound is minimized when $r=2$, so specifically

$$\sum_{0<t<2\sqrt{p}} \left(\frac{t}{p}\right) \ll_\epsilon p^{7/16+\epsilon}$$

so

$$\sum_{0<t<2\sqrt{p}} H(4p-t^2)\left(\frac{t}{p}\right) \ll_\epsilon p^{15/16+\epsilon}Log^2(p).$$

Thus,

$$\sum_{\substack{|a|,|b|\leq pn_p \\ p\nmid(4a^3+27b^2)}} \left(\frac{a_p}{p}\right) < 2\frac{N^2}{p} \sum_{0<t<2\sqrt{p}} H(4p-t^2)\left(\frac{t}{p}\right)$$
$$\ll_\epsilon \frac{N^2}{p} p^{15/16+\epsilon}Log^2(p)$$
$$= N^2 p^{-1/16+\epsilon}Log^2(p).$$

Now we will sum over primes

$$\sum_{p\leq M}\sum_{\substack{|a|,|b|\leq pn_p\\ p\nmid(4a^3+27b^2)}}\left(\frac{a_p}{p}\right)\ll_\epsilon N^2\sum_{p\leq M}p^{-1/16+\epsilon}Log^2(p$$

We will use partial summation for the sum $\sum_{p\leq M}p^{-1/16+\epsilon}Log^2(p)$.

Let $f(x)=x^{-1/16+\epsilon}Log^2(x)$, and $c_n=\mathbb{1}_{n\in P}$ where P is the set of primes.

Let $\pi(x)$ be the prime counting function. By partial summation we have

$$\sum_{p\leq M}p^{-1/16+\epsilon}Log^2(p)$$

$$\begin{aligned}
&= f(M)\sum_{n\leq M}c_n-\int_2^M\left(\sum_{n\leq x}c_n\right)\cdot f'(x)\mathrm{d}x\\
&= M^{-1/16+\epsilon}Log^2(M)\pi(M)-\int_2^M\pi(x)\cdot f'(x)\mathrm{d}x\\
&\approx M^{-1/16+\epsilon}Log^2(M)\frac{M}{Log(M)}-\int_2^M\pi(x)\cdot f'(x)\mathrm{d}x\\
&\ll M^{15/16+\epsilon}Log(M)
\end{aligned}$$

Thus, we have the upper bound for the dominant term

$$\sum_{p\leq M}\sum_{\substack{|a|,|b|\leq pn_p\\ p\nmid(4a^3+27b^2)}}\left(\frac{a_p}{p}\right)\ll_\epsilon N^2M^{15/16+\epsilon}Log(M)$$

For the middle sums, we have

$$\sum_{p\leq M}\sum_{\substack{|a|\leq N,\\ pn_p<|b|\leq N\\ p\nmid(4a^3+27b^2)}}\left(\frac{a_p}{p}\right)\leq\sum_{p\leq M}N\cdot p$$

We again use partial summation with $c_n=\mathbb{1}_{n\in P}$ where P is the set of primes, and $f(x)=x$.

We have

$$\sum_{p\leq M}\sum_{\substack{|a|\leq N,\\ pn_p<|b|\leq N\\ p\nmid(4a^3+27b^2)}}\left(\frac{a_p}{p}\right)\leq N\sum_{p\leq M}p$$

$$\begin{aligned}
&= N\left(f(M)\pi(M)-\int_2^M \pi(x)\cdot f'(x)\mathrm{d}x\right)\\
&\approx N\left(M\frac{M}{Log(M)}-\int_2^M \frac{x}{Log(x)}\mathrm{d}x\right)\\
&\leq N\left(M\frac{M}{Log(M)}-\frac{1}{Log(M)}\int_2^M x\mathrm{d}x\right)\\
&\leq N\left(\frac{M^2+4}{2Log(M)}\right)\approx\frac{NM^2}{2Log(M)}
\end{aligned}$$

Similarly, using partial summation with $f(x)=x^2$ for the last term we have

$$\begin{aligned}
\sum_{p\leq M}\sum_{\substack{pn_p<|a|,|b|\leq N\\ p\nmid(4a^3+27b^2)}}\left(\frac{a_p}{p}\right)&\leq\sum_{p\leq M}p^2\\
&\leq\frac{M^3+16}{3Log(M)}\\
&\approx\frac{M^3}{3Log(M)}
\end{aligned}$$

To make sure the dominant sum is the larger than the other sums, we need

$$N\leq max\{\frac{M^{17/16}}{2Log^2(M)},\frac{M^{31/32}}{\sqrt{3}Log(M)}\}$$

□

The result shows that while it was conjectured that the quadratic residue of a_p modulo p is 1 or -1 with the chance 50:50, the sum of quadratic residues of a_p modulo p is not zero as expected. However, it also shows that as N, M increase, the bias becomes insignificant compared with the number of terms in the sum, which means that the a_p are evenly distributed between squares and non-squares modulo p asymptotically.

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