Bruhat-Tits Buildings and a Characteristic p Unimodular Symbol Algorithm

Matthew Bates

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BRUHAT-TITS BUILDINGS AND A CHARACTERISTIC $p$ UNIMODULAR SYMBOL ALGORITHM

A Dissertation Presented
by
MATTHEW BATES

Submitted to the Graduate School of the University of Massachusetts Amherst in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

September 2019

Department of Mathematics and Statistics
BRUHAT-TITS BUILDINGS AND A CHARACTERISTIC $P$ UNIMODULAR SYMBOL ALGORITHM

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ACKNOWLEDGMENTS

I would like to thank my advisor Paul Gunnells for his guidance and advice during the last five years. I would also like to thank Tom Weston, Siman Wong, and David Barrington for agreeing to be on my committee. Finally I would like to thank Jane Wang for her support and patience.
ABSTRACT

BRUHAT-TITS BUILDINGS AND A CHARACTERISTIC \( p \) UNIMODULAR SYMBOL ALGORITHM

SEPTEMBER 2019

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Let \( k \) be the finite field with \( q \) elements, let \( \mathcal{F} \) be the field of Laurent series in the variable \( t^{-1} \) with coefficients in \( k \), and let \( A \) be the polynomial ring in the variable \( t \) with coefficients in \( k \). Let \( \text{SL}_n(\mathcal{F}) \) be the ring of \( n \times n \)-matrices with entries in \( \mathcal{F} \), and determinant 1. Given a polynomial \( g \in A \), let \( \Gamma(g) \subseteq \text{SL}_n(\mathcal{F}) \) be the full congruence subgroup of level \( g \).

In this thesis we examine the action of \( \Gamma(g) \) on the Bruhat-Tits building \( \mathfrak{X}_n \) associated to \( \text{SL}_n(\mathcal{F}) \) for \( n = 2 \) and \( n = 3 \). Our first main result gives an explicit formula for the homology of the quotient space \( \Gamma(g) \backslash \mathfrak{X}_2 \) (Theorem 1.5.1). We also give a complete description of \( \text{SL}_n(A) \backslash \mathfrak{X}_3 \) (Theorem 2.2.5), and explicitly compute the stabiliser groups of the cells therein (Theorem 2.2.3). Using this data we derive information about the topology and simplicial structure of the quotient space \( \Gamma(g) \backslash \mathfrak{X}_3 \) (Theorem 2.3.7), and explicitly compute the homology groups (Theorem 2.4.4). We
also define an appropriate generalisation of unimodular symbols for $\text{SL}_3(\mathcal{F})$ (Definition 3.3.1), and prove that a continued fraction type algorithm exists (in the sense of [AR79]), thus showing any modular symbol can be written a sum of unimodular symbols (Theorem 3.3.2).
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CHAPTER I
INTRODUCTION

In this section we introduce the main objects of study and give a summary of our results. In Chapter I we establish some notation and shorthand which will be used throughout this thesis. Chapter I introduces buildings and modular symbols, and give some relevant background information. Finally, Chapter I contains a summary of our main results.

I.1 Notation

Throughout this thesis we will use the following notation:

\( q = p^r \), a prime power

\( \mathbb{F}_q = \) the finite field with \( q \) elements

\( \mathbb{F}_q[t] = \) the polynomial ring in the variable \( t \), with coefficients in \( \mathbb{F}_q \)

\( \mathbb{F}_q[[t^{-1}]] = \) the field of Taylor series in the variable \( 1/t \), with coefficients in \( \mathbb{F}_q \)

\( \mathbb{F}_q((t^{-1})) = \) the field of Laurent series in the variable \( 1/t \), with coefficients in \( \mathbb{F}_q \)

\( \text{SL}_n(\mathbb{R}) = \) the group of \((n \times n)\)-matrices with determinant 1 and entries in the ring \( \mathbb{R} \).

We will use the following shorthand:

\( A = \mathbb{F}_q[t] \)

\( \mathcal{O} = \mathbb{F}_q[[t^{-1}]] \)
\[ \mathcal{F} = \mathbb{F}_q((t^{-1})) \]
\[ \pi = t^{-1} \]
\[ \Gamma(g) = \text{full congruence subgroup of } \text{SL}_n(A) \text{ of level } g \]
\[ \mathcal{X}_n = \text{the Bruhat-Tits buildings associated to } \text{SL}_n(\mathcal{F}) \]
\[ \mathcal{X}_n(g) = \text{the quotient space } \Gamma(g) \backslash \mathcal{X}_n \]
\[ \{\mathcal{X}_n(g)\}_\sigma = \text{the preimage of } \sigma \text{ in the quotient map } \rho: \mathcal{X}_n(g) \longrightarrow \text{SL}_n(A) \backslash \mathcal{X}_n \]
\[ \text{Sim}_k(\Delta) = \text{the set of } k\text{-simplices in a simplicial complex } \Delta \]

\section*{I.2 Background Material}

\subsection*{I.2.1 Buildings}

The notion of a building was first introduced in the 1950’s by Jacques Tits [Tit74] as a means of understanding algebraic groups over an arbitrary field. The general idea is to construct a space upon which the group acts in a nice manner, and to use information about this space and the action to learn about the group itself. Tits described how, given a simple algebraic group \( G \), one can construct an associated simplicial complex \( \Delta(G) \) with a canonical \( G \)-action. The simplicial complex \( \Delta(G) \) is called the \emph{spherical building} of \( G \). Furthermore, the construction is natural in the sense that a group homomorphism \( G \to H \) induces a morphism of simplicial complexes \( \Delta(G) \to \Delta(H) \).

The spherical building of a semi-simple algebraic group \( \Delta(G) \) is defined as follows:

1. The 0-cells are indexed by the maximal parabolic subgroups of \( G \).

2. The vertices \( P_0, P_1, \ldots, P_k \) form an \( k \)-simplex if and only if \( P_0 \cap P_1 \cap \cdots \cap P_k \) is a parabolic subgroup of \( G \).
The spherical building $\Delta(G)$ of a semi-simple Lie group $G$ can also be realized in terms of the geometry at infinity of $G$. Let $K \subseteq G$ be a maximal compact subgroup, and $X = G/K$ be the associated Riemannian symmetric space with the invariant metric. We say that two geodesics in $X$ are equivalent if they remain a bounded distance apart. Denote the set of equivalence classes of geodesics in $X$ by $X(\infty)$. Endowed with the appropriate topology, the space $X \cup X(\infty)$ is a compactification of $X$, called the geodesic compactification. The action of $G$ on $X$ extends to an action on $X \cup X(\infty)$. It can be shown that the stabilizer of any $\gamma \in X(\infty)$ is a parabolic subgroup of $G$. Given a parabolic subgroup $P \subseteq G$, we let $\sigma_P \subseteq X(\infty)$ denote the set of points in $X(\infty)$ with stabilizer exactly equal to $P$. The following theorem essentially says that spherical buildings arise as the boundaries of symmetric spaces.

**Theorem I.2.1 ([BJ06]).**

The disjoint decomposition $X(\infty) = \bigcup_{P \subseteq G} \sigma_P$ gives a simplicial complex structure to $X(\infty)$, that is isomorphic (as simplicial complexes) to the spherical building $\Delta(G)$.

There is a very rigid relationship between a group $G$, and its associated spherical building $\Delta(G)$. In fact, Tits showed that often the group structure of $G$ can be completely recovered from $\Delta(G)$ [Tit74, Theorem 5.8].

The simplicial complexes which arise as the spherical buildings of algebraic groups share a number of combinatorial and topological properties. By considering these properties abstractly, Tits arrived at the general definition of an abstract building. Roughly speaking, a building is a simplicial complex formed by gluing together multiple copies of a Coxeter complex, in a highly regular and symmetric way. The Coxeter complexes are called the apartments of the building. When the Coxeter complex is topologically a sphere we say the building is of spherical type, and refer to it as a spherical building; otherwise we say the building is of Euclidean type, or an Euclidean building.
The relevant Coxeter complex for the spherical building $\Delta(G)$ is the one coming from the root system of the algebraic group $G$. When $G$ is an algebraic group this complex is topologically a sphere, which explains the terminology “spherical”. Although not all spherical buildings are associated to simple algebraic groups, Tits proved that those of dimension at least two arise as $\Delta(G)$ for some algebraic group $G$.

In the special case where $G$ is a semi-simple algebraic group defined over a non-archimedean field, one can construct an associated Euclidean building $\Delta^{BT}(G)$ called the Bruhat-Tits building. The main objects of study in this thesis are the Bruhat-Tits buildings associated to the groups $\text{SL}_2(F)$ and $\text{SL}_3(F)$.

The construction of the Bruhat-Tits building associated to a general semi-simple algebraic group defined over a non-archimedean field is technical, relying on knowledge of the structure theory of algebraic groups. Broadly speaking, construction is analogous to that of the spherical building, but with parabolic subgroups replaced by parahoric subgroups. We will not go into any more detail on this construction since in the special case of the Bruhat-Tits building associated to $\text{SL}_n(F)$, there is a simpler construction. See Chapter I for more details.

In [Ser03, Chapter II.1], Serre gives a detailed description of the Bruhat-Tits building associated to $\text{SL}_2(F)$. In particular he shows that $\Delta^{BT}(\text{SL}_2(F))$ is a $(q+1)$-regular tree. He then describes a canonical (up to a choice of basis) 2-colouring and numbering of the building. Serre also examines in detail a canonical action of $\text{SL}_2(A)$ on the building. It is shown that this action is simplicial, acts without inversions, and preserves the 2-colouring and the numbering. A fundamental domain is then computed together with all relevant stabilisers. Using Bass-Serre theory, the structure of congruence subgroups of $\text{SL}_2(A)$ are determined by studying how these groups acts on the tree (of particular importance is the fundamental domain and the stabilisers of the vertices and edges in the fundamental domain).
One reason to study Bruhat-Tits buildings is because they are a characteristic–p replacement of Riemannian symmetric spaces. The Riemannian symmetric spaces associated to a Lie group $G$ is the quotient space $G/K$, where $K \subseteq G$ is a maximal compact subgroup. There is a strong analogy between Bruhat-Tits buildings and symmetric spaces associated to Lie groups. We list some of the similarities in Table I.1.

<table>
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<th>$G(\mathbb{R})/K(\mathbb{R})$</th>
<th>$\Delta^{BT}(G(K))$</th>
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<tr>
<td>Homogeneous manifold</td>
<td>Homogeneous simplicial complex</td>
</tr>
<tr>
<td>Constant curvature $-1$</td>
<td>Non-positive curvature (CAT(0)-space)</td>
</tr>
<tr>
<td>Contractable</td>
<td>Contractable</td>
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<td>Geodesically complete</td>
<td>Geodesically complete</td>
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<tr>
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<td>$\partial(G(\mathbb{R})/K(\mathbb{R})) \cong \Delta(G(\mathbb{R}))$</td>
<td>$\partial(\Delta^{BT}(G(K))) \cong \Delta(G(K))$</td>
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Table I.1: Table comparing Riemannian symmetric spaces with Bruhat-Tits buildings

I.2.2 Construction of the Building Associated to $\text{SL}_n(F)$ via Lattices

In this section we will describe an alternate construction of the Bruhat-Tits building $X_n$ associated to $\text{SL}_n(F)$. The construction is based on lattices as opposed to the more general definition which is based on parahoric subgroups. The primary reference for this section is [AB08].

Let $W = F^n$, and define a lattice in $W$ to be a free $O$-sub-module of $W$ of rank $n$. Denote the set of all such lattices by $\text{Lat}$. We say that two lattices $\mathcal{L}, \mathcal{L}' \in \text{Lat}$ are equivalent if they are equivalent up to homothety, i.e. $\mathcal{L} \sim \mathcal{L}'$ if $\mathcal{L} = x\mathcal{L}'$ for some $x \in F^\times$. Denote the equivalence class of a lattice $\mathcal{L}$ by $[\mathcal{L}]$.

We form a simplicial complex $\Delta$, using $\text{Lat}/\sim$ as the set of vertices. We say that a collection of $(k + 1)$ vertices, $\Lambda_0, \Lambda_1, \ldots, \Lambda_k \in \text{Lat}/\sim$ form a $k$-simplex in $\Delta$ if there exists lattice representatives $[\mathcal{L}_i] = \Lambda_i$, such that $\pi \mathcal{L}_0 \subsetneq \mathcal{L}_k \subsetneq \cdots \subsetneq \mathcal{L}_1 \subsetneq \mathcal{L}_0$. If
\[ \Lambda_0, \Lambda_1, \ldots, \Lambda_k \in \text{Lat}/\sim \] are the vertices of a \( k \)-simplex in \( \Delta \), we denote the \( k \)-simplex they span by \( \Lambda_0\Lambda_1 \ldots \Lambda_k \).

**Definition I.2.2.**

Let \( \Delta \) be the abstract simplicial complex with the following simplices

\[
\begin{align*}
\text{Sim}_0(\Delta) &= \{ \Lambda_i \mid \Lambda_i \in \text{Lat}/\sim \} \\
\text{Sim}_k(\Delta) &= \left\{ \Lambda_0\Lambda_1 \ldots \Lambda_k \right. \\
&\quad \left. \mid \Lambda_i \in \text{Lat}/\sim, \text{ such that there exists } \mathcal{L}_i \in \text{Lat} \\
&\quad \text{with } [\mathcal{L}_i] = \Lambda_i, \text{ and } \pi \mathcal{L}_0 \subsetneq \mathcal{L}_k \subsetneq \cdots \subsetneq \mathcal{L}_1 \subsetneq \mathcal{L}_0 \right\}.
\end{align*}
\]

Where \( \Lambda'_0\Lambda'_1 \ldots \Lambda'_{k'} \) is a face of \( \Lambda_0\Lambda_1 \ldots \Lambda_k \) iff \( \{ \Lambda'_0, \Lambda'_1, \ldots, \Lambda'_{k'} \} \subsetneq \{ \Lambda_0, \Lambda_1, \ldots, \Lambda_k \} \)

**Proposition I.2.3** ([AB08, Chapter 6.9]).

The simplicial complex \( \Delta \) is isomorphic to the Euclidean Bruhat-Tits building associated to \( \text{SL}_n(\mathcal{F}) \), i.e. \( \Delta \cong \mathcal{X}_n \) as simplicial complexes.

### I.2.3 Group Cohomology

Let \( \Gamma \) be a discrete group, and \( M \) a \( \Gamma \)-module. The cohomology groups \( \{ H^n(\Gamma; M) \}_{n=0}^\infty \) are defined to be the right derived functors of the Hom-functor, i.e.

\[
H^*(\Gamma; M) = R^*(\text{Hom}_{\mathbb{Z}[\Gamma]}(\mathbb{Z}, -))(M).
\]

Group cohomology arises in many situations, most commonly as classifying objects or as obstructions. Many theorems and definitions in number theory can be rephrased in terms of group cohomology. Some important examples of group cohomology include: classifying group extensions, modular and automorphic forms, Tate-Shafarevich groups, Brauer groups, K-theory, Galois cohomology, and class-field theory.

Unfortunately, it is often too difficult to calculate \( H^n(\Gamma; M) \) directly from its definition. Fortunately, there are other methods to calculate \( H^n(\Gamma; M) \). One method
which is especially useful when calculating the group cohomology of an arithmetic
group \( \Gamma \) is to relate \( H^n(\Gamma; \mathcal{M}) \) to the singular cohomology groups of a \( K(\Gamma, 1) \)-space.

A \( K(\Gamma, 1) \)-space is a connected CW-space \( X_\Gamma \), such that

\[
\pi_1(X_\Gamma) = \Gamma, \quad \text{and} \quad \pi_n(X_\Gamma) = 0 \quad \text{for} \quad n \geq 2.
\]

Such spaces always exist, and they are unique up to weak homotopy. In an abuse
of notation, we denote any such space \( X_\Gamma \) by \( K(\Gamma, 1) \), and call it a \( K(\Gamma, 1) \)-space.

**Theorem I.2.4.**

If \( \Gamma \) is a discrete group then we have

\[
H^n(\Gamma; \mathcal{M}) \cong H^n(K(\Gamma, 1); \mathcal{M}),
\]

where \( H^n(K(\Gamma, 1); \mathcal{M}) \) is the singular cohomology of \( K(\Gamma, 1) \), with twisted coeffi-
cients in \( \mathcal{M} \).

For more details on group cohomology see [Wei94] Chapter 6, and for an intro-
duction to group cohomology from a topological point of view see [Löh10].

Although there exists an explicit method to construct \( K(\Gamma, 1) \)-spaces,\(^1\) the re-
sulting spaces are often extremely complicated and useless for explicit cohomology
calculations. For example, spaces constructed in this manner are often cohomolog-
ically inefficient, in the sense that they are always infinite dimensional even when
they only have non-trivial cohomology in bounded degrees. If one intends to use
Theorem I.2.4 to calculate the group cohomology of \( \Gamma \), then it is essential to have
a \( K(\Gamma, 1) \)-space which is geometrically/topologically simple enough to allow coho-

---

\(^1\)Let \( E \) be the \( \Delta \)-complex whose \( n \)-simplices are indexed by ordered \((n + 1)\)-tuples of elements of
\( \Gamma \), with the obvious attaching maps. There is a natural action of \( \Gamma \) on \( E \), and the quotient \( E/\Gamma \) is
a \( K(\Gamma, 1) \)-space. For more details see [Hat02, p. 89].
mology calculations. In general, finding such a nice $K(\Gamma, 1)$-space for a given $\Gamma$ is a difficult problem.

### I.2.4 Nice $K(\Gamma, 1)$-Spaces

When $\Gamma$ acts freely on a contractable topological space $X$, the quotient space $\Gamma \backslash X$ is a $K(\Gamma, 1)$-space. An important example of this construction arises in the theory of Lie groups and their associated Riemannian symmetric spaces. If $G = G(\mathbb{R})$ is a Lie group, and $K \subseteq G$ a maximal compact subgroup, then $G/K$ is homeomorphic to a Euclidean space and thus contractable. There is a natural action of $G(\mathbb{Z})$ on $G/K$, where all non-torsion elements act freely. Thus when $\Gamma \subseteq G(\mathbb{Z})$ is torsion free, the quotient space $\Gamma \backslash (G/K)$ is a $K(\Gamma, 1)$-space.

A classic example of the above ideas is in the study of modular curves and modular forms. A modular curve is a quotient space $\Gamma \backslash \mathbb{H}$, where $\Gamma \subseteq \text{SL}_2(\mathbb{Z})$ is a discrete subgroup and $\mathbb{H} \cong \text{SL}_2(\mathbb{R})/\text{SO}(2)$ is the upper half plane. If $\Gamma \subseteq \text{SL}_2(\mathbb{Z})$ is torsion free (e.g. $\Gamma = \Gamma(n)$ with $n \geq 3$) then the action of $\Gamma$ on $\mathbb{H}$ is free, and thus $\Gamma \backslash \mathbb{H}$ is a $K(\Gamma, 1)$-space. When $\Gamma$ is a classic congruence subgroup—i.e. $\Gamma(n)$, $\Gamma_0(n)$, or $\Gamma_1(n)$—then the topology of $\Gamma \backslash \mathbb{H}$ is well understood: it is always a compact Riemann surface with a finite number of punctures, and there exists exact formulas for the genus and number of punctures. There is a direct relationship between the first group cohomology of $\Gamma \subseteq \text{SL}_2(\mathbb{Z})$, and modular forms for $\Gamma$. This allows one to learn about modular forms by studying the topology of the modular curve $\Gamma \backslash \mathbb{H}$. For more details see [DS05].

When $G$ is a semi-simple algebraic group defined over a non-archimedean field, it does not have an associated contractable Riemannian symmetric space upon which to act. As we highlighted in Table I.1, a suitable replacement for the associated Riemannian symmetric space is the associated Bruhat-Tits building. For example, if $\Gamma \subseteq \text{SL}_n(\mathbb{A})$ is torsion-free, then we have
\[ H^* (\Gamma; M) \cong H^* (\Gamma \backslash \Delta^{BT} (SL_n(F)); \mathcal{M}) \].

Of particular arithmetic interest is the cohomology groups of \( \Gamma (g) \backslash X_n \), where \( \Gamma (g) \) is the full congruence subgroup of level \( g \). Some of these groups can be determined using elementary topological considerations. For example, the 0-th cohomology groups are determined since \( X_2 \) and \( X_3 \) are both connected and thus so are the quotients \( \Gamma \backslash X_2 \) and \( \Gamma \backslash X_3 \). Furthermore, since \( \dim(X_2) = 1 \) and \( \dim(X_3) = 2 \), we know that \( H^* (X_2; M) \) is only supported in dimensions 0 and 1, similarly \( H^* (X_3; M) \) is only supported in dimensions 0, 1, and 2. Finally, since \( X_2 \) is a tree, the quotient \( \Gamma \backslash X_2 \) is a graph, thus homotopy equivalent to a wedge of \( r \) circles for some integer \( r \geq 0 \); thus \( H^1 (\Gamma \backslash X_2) \cong M^r \). These remarks are summarised in Table I.2 below.

<table>
<thead>
<tr>
<th>( H^0 (\Gamma \backslash X_2; M) )</th>
<th>( H^0 (\Gamma \backslash X_3; M) )</th>
<th>( H^1 (\Gamma \backslash X_2; M) )</th>
<th>( H^1 (\Gamma \backslash X_3; M) = ? )</th>
<th>( H^n (\Gamma \backslash X_2; M) = 0 ) for all ( n \geq 2 )</th>
<th>( H^2 (\Gamma \backslash X_3; M) = ? )</th>
<th>( H^n (\Gamma \backslash X_3; M) = 0 ) for all ( n \geq 3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( M )</td>
<td>( M )</td>
<td>( M^r ) for some ( r \geq 0 )</td>
<td>( ? )</td>
<td>( 0 ) for all ( n \geq 2 )</td>
<td>( ? )</td>
<td>( 0 ) for all ( n \geq 3 )</td>
</tr>
</tbody>
</table>

Table I.2: Partially complete table of cohomology groups of \( \Gamma \backslash X_2 \) and \( \Gamma \backslash X_3 \)

In this thesis we will study the above mentioned cohomology groups by studying the induced covering maps \( \rho: X_n(g) \to SL_n(A) \backslash X_n \) for \( n = 2 \) and \( n = 3 \). To do this we must first have a good understanding of the spaces \( SL_2(A) \backslash X_2 \) and \( SL_3(A) \backslash X_3 \). The first space was examined in detail by Serre in [Ser03]. we will examine the structure of \( SL_3(A) \backslash X_3 \) by explicitly describing a fundamental domain for the action of \( SL_3(A) \) on \( X_3 \).

I.2.5 Modular Symbols

Modular symbols were invented by Yuri Manin in [Man72], as a tool for studying the arithmetic of modular forms for congruence subgroups \( \Gamma \subseteq SL_2(\mathbb{Z}) \). Since their introduction they have been used for a variety of purposes. Notably, John Cre-
Mona used modular symbols to perform large scale computations of Hecke eigenvalues [Cre97].

Abstractly, a modular symbol for \( \Gamma \) is an ordered pair \( \{ \alpha, \beta \} \in \mathbb{P}^1(\mathbb{Q}) \times \mathbb{P}^1(\mathbb{Q}) \), considered as an element of the first relative homology group of \( \Gamma \backslash \mathbb{H} \), that is, \( H_1(\Gamma \backslash \mathbb{H}, \text{cusps}; M) \). There is a very nice geometric interpretation of modular symbols: If \( \alpha, \beta \) are rational cusps\(^2\) of \( \mathbb{H} \), then there is a unique geodesic path in \( \mathbb{H} \) from \( \alpha \) to \( \beta \), we denote this path by \( \{ \alpha, \beta \} \). The image of \( \{ \alpha, \beta \} \) under the quotient map \( \mathbb{H} \mapsto \Gamma \backslash \mathbb{H} \) can be considered as an element of the relative homology group \( H_1(\Gamma \backslash \mathbb{H}, \text{cusps}; M) \). By an abuse of notation we write \( \{ \alpha, \beta \} \in H_1(\Gamma \backslash \mathbb{H}, \text{cusps}; M) \).

Thus, a modular symbol for \( \Gamma \) can be thought of as a geodesic path in \( \mathbb{H} \) joining two rational cusps \( \alpha, \beta \in (\mathbb{Q} \cup \{ \infty \}) \), which we consider as an element of the first relative homology group of \( \Gamma \backslash \mathbb{H} \), i.e. \( H_1(\Gamma \backslash \mathbb{H}, \text{cusps}; M) \). It can be shown that all elements of the relative homology group arise in this way. For a geometric illustration of a modular symbol see Figure I.1a.

Modular symbols relate to the above discussions in Chapter I and Chapter I about cohomology groups since one can show \( H_1(\Gamma \backslash \mathbb{H}, \text{cusps}; M) \cong H_1(\Gamma \backslash \mathbb{H}; M) \).\(^3\)

In [Man72], Manin gives an explicit finite set of generators and relations for the group of modular symbols, making it relatively easy to calculate \( H_1(\Gamma \backslash \mathbb{H}, \text{cusps}; M) \). Using the duality between modular forms and \( H_1(\Gamma \backslash \mathbb{H}, \text{cusps}; M) \) one can use modular symbols to calculate the structure of the space of modular forms. Manin also describes how the Hecke operators act on modular symbols.

---

\(^2\)The rational cusps of \( \mathbb{H} \) are \( \mathbb{Q} \cup \{ \infty \} \cong \mathbb{P}^1(\mathbb{Q}) \).

\(^3\)This follows directly from the Lefschetz duality theorem:

**Theorem I.2.5 (Lefschetz Duality Theorem).**

Let \( X \) be a compact, triangulated \( n \)-dimensional manifold with boundary. Then

\[ H_k(X, \partial X) \cong H^{n-k}(X) \text{ for all } k. \]
A unimodular symbol is a modular symbol which corresponds to an edge of the Farey tessellation\(^4\) of \(\mathbb{H}\). Manin showed that any modular symbol can be written as a sum of unimodular symbols by describing an explicit algorithm to do so. The algorithm is often referred to as Manin’s trick, or the continued fraction algorithm. Since any congruence subgroups \(\Gamma \subseteq \text{SL}_2(\mathbb{Z})\) is of finite index, there are only finitely many unimodular symbols modulo \(\Gamma\). Thus the set of unimodular symbols modulo \(\Gamma\), is a finite generating set for the set of all modular symbols.

The notion of modular symbols have since been generalised to groups other than \(\text{SL}_2(\mathbb{R})\). In the 70’s Barry Mazur described a generalisation of modular symbols for an arbitrary reductive \(\mathbb{Q}\)-group \([\text{Maz}]\). The \(\text{SL}_n(\mathbb{R})\)-case was then studied in more detail by Ash and Rudolph in \([\text{AR79}]\). Another way in which modular symbols have been generalised is to groups over non-archimedean field. In \([\text{Tei92}]\), Teitelbaum described a theory of modular symbols for \(\text{SL}_2(\mathbb{A})\). Geometrically, the modular symbols of Teitelbaum are analogous to those of Manin: instead of considering geodesics between cusps in \(\mathbb{H}\), one now considers geodesics between cusps in the Bruhat-Tits building \(\Delta^{BT}(\text{SL}_2(\mathcal{F}))\). For a geometric illustration of a Teitelbaum modular symbol see Figure I.1b.

I.3 Overview of Results

In Chapter 1 we examine the structure of \(\mathcal{X}_2\), and the action of \(\text{SL}_2(\mathbb{A})\) on \(\mathcal{X}_2\). We start by reviewing some results of Serre on the structure of \(\mathcal{X}_2\). We then describe the boundary of \(\mathcal{X}_2\) explicitly (Section 1.2). After which we examine the quotient map 
\[
\rho: \Gamma(g)\backslash\mathcal{X}_2 \to \text{SL}_2(\mathbb{A})\backslash\mathcal{X}_2,
\]
and derive a formula for the number of simplices lying above a given simplex in \(\text{SL}_2(\mathbb{A})\backslash\mathcal{X}_2\) (Theorem 1.4.7). Using this formula, we go on to derive a formula for the homology groups of the quotient space \(\Gamma(g)\backslash\mathcal{X}_2\), for a general \(g \in \mathbb{A}\) (Theorem 1.5.2).

\(^4\)The Farey tessellation is the ideal triangulation of \(\mathbb{H}\) with edges given by the \(\text{SL}_2(\mathbb{Z})\)-orbit of the geodesic \(\{0, i\infty\}\).
In Chapter 2 we carry out a similar program for $X_3$. We make basic observations on the structure of $X_3$, describe how $\text{SL}_3(F)$ acts on $X_3$, and show that there exists a $\text{SL}_3(F)$ invariant 3-colouring of $X_3$. We then examine the action of $\text{SL}_3(A)$ on $X_3$, and give a complete description of a fundamental domain (Theorem 2.2.5), and calculate all relevant stabiliser subgroups (Theorem 2.2.3). Finally, we examine the quotient spaces $\Gamma(g)\backslash X_3$, for $\Gamma(g) \subseteq \text{SL}_3(A)$ a full congruence subgroup. Using the covering map $\rho: \Gamma(g)\backslash X_3 \longrightarrow \text{SL}_3(A)\backslash X_3$ we calculate the cardinality of the set of simplices which lie above any given simplex of $\text{SL}_2(A)\backslash X_2$ (Example 2.3.6).

In Chapter 3 we define an appropriate generalisation of unimodular symbols for $\text{SL}_3((t^{-1}))$, and prove that a continued fraction type algorithm exists (in the sense of [AR79]), thus showing any modular symbol can be written a sum of unimodular symbols.
CHAPTER 1

THE BUILDING ASSOCIATED TO SL$_2$(\mathcal{F})

In this section we will examine the Bruhat-Tits building associated to SL$_2$(\mathcal{F}), which we denote by \mathcal{X}_2. The main objectives of this section are:

1. Make basic observations on the structure of \mathcal{X}_2, describe how SL$_2$(\mathcal{F}) acts on \mathcal{X}_2, and show that there exists a SL$_2$(\mathcal{F}) invariant 2-colouring of \mathcal{X}_2.

2. Discuss distance between vertices in \mathcal{X}_2, index all vertices of a given distance from some fixed vertex, define and examine the boundary of \mathcal{X}_2.

3. Examine the quotient SL$_2$(A)\backslash\mathcal{X}_2, describe a fundamental domain, and calculate all relevant stabiliser subgroups.

4. Examine the quotients \Gamma(g)\backslash\mathcal{X}_2, for \Gamma(g) \subseteq SL_2(A) a full congruence subgroup. We use the covering map \rho: \Gamma(g)\backslash\mathcal{X}_2 \longrightarrow SL_2(A)\backslash\mathcal{X}_2 to calculate the cardinality of the set of simplices which lie above any given simplex of SL$_2$(A)\backslash\mathcal{X}_2.

5. Calculate the homology of the quotients \Gamma(g)\backslash\mathcal{X}_2.

Throughout this section we will be considering \mathcal{X}_2 from the point of view of lattices, which was discussed in Chapter I. i.e. Let \mathcal{W} = \mathcal{F}^2, and define a lattice in \mathcal{W} to be a free \mathcal{O}-sub-module of \mathcal{W} of rank 2. Let \text{Lat}/\sim denote the set of all lattices up to \mathcal{F}^\times-homothety, and denote the equivalence class of a lattice \mathcal{L} by [\mathcal{L}]. Then the Bruhat-Tits building \mathcal{X}_2 is defined as follows:
Definition 1.0.1.

Let $\mathcal{X}_2$ be the abstract simplicial complex with the following simplices

$\text{Sim}_0(\mathcal{X}_2) = \{ \Lambda_i \mid \Lambda_i \in \text{Lat}/\sim \}$

$\text{Sim}_1(\mathcal{X}_2) = \left\{ \Lambda_1 \Lambda_2 \mid \begin{array}{c}
\Lambda_1, \Lambda_2 \in \text{Lat}/\sim, \text{ such that there exists} \\
\mathcal{L}_1, \mathcal{L}_2 \in \text{Lat} \text{ with } [\mathcal{L}_i] = \Lambda_i, \text{ and } \pi \mathcal{L}_1 \subsetneq \mathcal{L}_2 \subsetneq \mathcal{L}_1
\end{array} \right\}$

and the obvious attaching maps.

1.1 Basic Properties of $\mathcal{X}_2$

The primary reference for this section is [Ser03, Chapter II].

**Theorem 1.1.1** ([Ser03, Chapter II.1, Theorem 1.]).

*The building $\mathcal{X}_2$ is simplicially isomorphic to the infinite $(q + 1)$-regular tree.*

![Figure 1.1: The Bruhat-Tits building associated to $\text{SL}_2(\mathbb{F}_2((t^{-1})))$ with a 2-colouring of the vertices](image-url)
Definition 1.1.2.
Given \(a_{i,j} \in F\), let \(\mathcal{L} = \langle \begin{pmatrix} a_{1,1} & a_{1,2} & \ldots & a_{1,n} \\ a_{2,1} & a_{2,2} & \ldots & a_{2,n} \end{pmatrix} \rangle\) be the \(O\)-lattice spanned by the columns of the matrix, with respect to the standard basis of \(F^2\).

Proposition 1.1.3.
The group \(GL_2(F)\) acts transitively on the set of all rank 2 lattices, and thus on \(\text{Sim}_0(X_2)\).

Proof. Fix \(\mathcal{L}_0 = \langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \rangle\), and let \(\mathcal{L} = \langle \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rangle\) be an arbitrary rank-2 lattice. Since \(\begin{pmatrix} a \\ c \end{pmatrix}\) and \(\begin{pmatrix} b \\ d \end{pmatrix}\) are \(F\)-linearly-independent, \(\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(F)\). Finally note that \(\alpha \mathcal{L}_0 = \mathcal{L}\). \(\square\)

Given any two lattices, \(\mathcal{L}\) and \(\mathcal{L}'\), let \(\alpha \in GL(W)\) be such that \(\alpha \mathcal{L} = \mathcal{L}'\). There exists bases \(\{e_1, e_2\}\) of \(\mathcal{L}\), and \(\{f_1, f_2\}\) of \(\mathcal{L}'\), such that the matrix of \(A\) with respect to these bases is
\[
\alpha = \begin{pmatrix} \pi^{n_1} & 0 \\ 0 & \pi^{n_2} \end{pmatrix},
\]
were \(n_1, n_2 \in Z\).\(^{1}\) Thus there exists an \(O\)-basis of \(\mathcal{L}\), \(\{e_1, e_2\}\), such that \(\{\pi^{n_1}e_1, \pi^{n_2}e_2\}\) is an \(O\)-basis for \(\mathcal{L}'\). Moreover, it can be shown that the set, \(\{n_1, n_2\}\) is independent of the choice of such a basis.

Proposition 1.1.4.
Let \(\mathcal{L}, \mathcal{M}, \mathcal{L}', \mathcal{M}'\) be lattices such that \(\mathcal{L} \sim \mathcal{L}'\) and \(\mathcal{M} \sim \mathcal{M}'\). Let \(\alpha, \beta \in SL_2(F)\) be such that \(\alpha \mathcal{L} = \mathcal{M}\) and \(\beta \mathcal{L}' = \mathcal{M}'\). If \(\{n_1, n_2\}\) are the integers associated to \(\alpha\) and \(\{n_1', n_2'\}\) those associated to \(\beta\), then
\[
n_1 + n_2 = n_1' + n_2' \pmod{2}, \quad \text{and} \quad |n_1 - n_2| = |n_1' - n_2'|.
\]

Proof. If \(\mathcal{L} \sim \mathcal{L}'\) then \(\mathcal{L}' = \pi^c \mathcal{L}\) for some \(c \in Z\), similarly \(\mathcal{M}' = \pi^d \mathcal{M}\) for some \(d \in Z\). Thus \(\beta (\pi^c \mathcal{L}) = (\pi^d \mathcal{M})\), so \((\beta \pi^{c-d}) \mathcal{L} = \mathcal{M}\). Hence the map \((\beta \pi^{c-d})\)

\(^{1}\)This is called the Smith Normal form of \(\alpha\).
has the same associated integers as $\alpha$ does, thus the integers associated to $\beta$ are 
\[ \{n'_1, n'_2\} = \{n_1 - c + d, n_2 - c + d\}. \]

**Corollary 1.1.5** (Colouring of vertices).

Choosing a distinguished vertex $\Lambda$ induces a 2-colouring on the set of all vertices.

There is a natural notion of distance between two vertices of $X_2$. We say that every edge has length 1, and the distance between two vertices is the length of the shortest path between the two vertices.

**Theorem 1.1.6** (Distance between vertices [Ser03, Remark 1, p71]).

Let $\Lambda, \Lambda'$ be two vertices of $X_2$, and let $\alpha \in \mathrm{SL}_2(\mathcal{F})$ be such that $\alpha \Lambda = \Lambda'$. If $\{n_1, n_2\}$ are the integers associated to $\alpha$ then the distance between $\Lambda$ and $\Lambda'$ is $|n_1 - n_2|$.

**Definition 1.1.7.**

Let $\mathcal{L}_1$, and $\mathcal{L}_2$ be any two lattices, then we define

\[ \chi(\mathcal{L}_1, \mathcal{L}_2) = l(\mathcal{L}_1/\mathcal{L}_1 \cap \mathcal{L}_2) - l(\mathcal{L}_2/\mathcal{L}_1 \cap \mathcal{L}_2). \]

This can be thought of as a generalised notion of index, where we no longer require one lattice to contain the other.

**Theorem 1.1.8** ([Ser03, Proposition 1. p75]).

Let $\mathcal{L}$ be a lattice and $s \in \mathrm{SL}(\mathcal{W})$, then $\chi(\mathcal{L}, s\mathcal{L}) = 0$.

**Corollary 1.1.9** ([Ser03, Corollary. p75]).

If $\Lambda$ is a vertex of $X_2$ and $s \in \mathrm{SL}(\mathcal{W})$, then $d(\Lambda, s\Lambda) = 0 \pmod{2}$.

Since we know that $X_2$ is a tree, Corollary 1.1.9 implies the following proposition.

---

2Where $l(\mathcal{L}/\mathcal{L}')$ is the length of $\mathcal{L}/\mathcal{L}'$, i.e. the longest chain of submodules of $\mathcal{L}/\mathcal{L}'$. 

---

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Proposition 1.1.10.

The group \( \text{SL}(W) \) acts simplicially (i.e. without inversions), and thus preserves the colouring.

Definition 1.1.11 (Stabiliser Subgroups).

Let \( G \subseteq \text{SL}(W) \) be a subgroup, \( L \in \text{Lat} \), \( \Lambda \in \text{Sim}_0(X_2) \), and \( \Lambda\Lambda' \in \text{Sim}_1(X_2) \), then

\[
G_L \overset{\text{def}}{=} \text{Stab}_G(L) = \{ g \in G \mid g \cdot L = L \},
\]

\[
G_\Lambda \overset{\text{def}}{=} \text{Stab}_G(\Lambda) = \{ g \in G \mid g \cdot \Lambda = \Lambda \},
\]

\[
G_{\Lambda\Lambda'} \overset{\text{def}}{=} \text{Stab}_G(\Lambda\Lambda') = \{ g \in G \mid g \cdot \Lambda\Lambda' = \Lambda\Lambda' \}.
\]

Proposition 1.1.12.

a) If \( G \subseteq \text{SL}(W) \), then \( G_{\Lambda\Lambda'} = G_\Lambda \cap G_{\Lambda'} \).

b) If \([L] = \Lambda\), then \( G_\Lambda = G_L \).

Proof. Part a) follows directly form Proposition 1.1.10. Part b) is a consequence of the fact that all linear transformations in \( \text{SL}(W) \) have determinant equal to 1.

Example 1.1.13. Let \( G = \text{SL}(W) \). Given \( \Lambda\Lambda' \in \text{Sim}_1(X_2) \), one can choose representative lattices \([L] = \Lambda\) and \([L'] = \Lambda'\) with basis, \( L = \langle e_1, e_2 \rangle \) and \( L' = \langle e_1, \pi e_2 \rangle \). With respect to these bases the stabilisers are:

\[
G_\Lambda = \text{SL}_2(\mathcal{O})
\]

\[
G_{\Lambda'} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathcal{F}) \mid \pi|c, \text{ and } \pi b, a, d \in \mathcal{O} \right\}
\]

\[
G_{\Lambda\Lambda'} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathcal{O}) \mid \pi|c \right\}.
\]
1.2 Distance and the boundary of $\mathcal{X}_2$

In this subsection we will examine the set of all vertices in $\mathcal{X}_2$ which are a fixed distance from a given vertex. We will then describe the boundary of $\mathcal{X}_2$. But first we need to recall the definition of projective space.

**Definition 1.2.1** ($n$-Dimensional Projective Space).

Let $R$ be a ring, and $V \cong R^{n+1}$. The projective space $\mathbb{P}(V)$ is the set of full lines in $V$, i.e.

$$\mathbb{P}(V) = \{ (a_0 : a_1 : \ldots : a_n) \mid \gcd(a_0, a_1, \ldots, a_n) \in R^\times \} / R^\times.$$  

This is also sometimes written as $\mathbb{P}^n(R)$, and called $n$-dimensional projective space over $R$.

Let $L_0 = \langle (1\ 0\ 0\ 1) \rangle$, and $\Lambda_0 = [L_0]$. Each vertex of $\mathcal{X}_2$ can be represented by a unique sub-lattice, $L \subseteq L_0$, such that $L_0/L \cong \mathcal{O}/\pi^n$ where $d(\Lambda_0, \Lambda) = n$. Note, $L_0/\pi^nL_0$ is a free $\mathcal{O}/\pi^n$ module of rank 2, and $L/\pi^nL_0$ is a direct factor of rank 1. i.e.

$$\frac{\mathcal{O}}{\pi^n} \cong \frac{L}{\pi^nL_0} \hookrightarrow \frac{L_0}{\pi^nL_0} \cong \frac{\mathcal{O}}{\pi^n} \oplus \frac{\mathcal{O}}{\pi^n}.$$  

We will use this observation to index the vertices of distance $n$ from $\Lambda_0$.

**Proposition 1.2.2** (Indexing Vertices of Distance $n$ [Ser03, p72]).

Vertices of distance $n$ from $\Lambda_0$ correspond bijectively to lines in $L_0/\pi^nL_0$, i.e. points in $\mathbb{P}(L_0/\pi^nL_0) \cong \mathbb{P}^1(\mathcal{O}/\pi^n)$.

We will now describe the boundary of $\mathcal{X}_2$.

**Definition 1.2.3** (Boundary Points).

The boundary of $\mathcal{X}_2$ is the set of equivalence classes of geodesic rays (non-backtracking half lines) in $\mathcal{X}_2$ starting at $\Lambda_0$. We denote the boundary by $\partial \mathcal{X}_2$.  

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This is the Gromov boundary of $\mathcal{X}_2$ with the natural metric. One can show that the above construction of $\partial \mathcal{X}_2$ doesn’t depend on the starting vertex $\Lambda_0$. We call the points on the boundary boundary points or points at infinity.

**Proposition 1.2.4** (Indexing Points at Infinity).

Points at infinity correspond bijectively to lines in $L_0$, i.e. points in $\mathbb{P}(L_0) \cong \mathbb{P}^1(O)$.

**Proof.** A geodesic ray emanating from $\Lambda_0$ can be thought of as a consistent choice of vertices $\Lambda_0, \Lambda_1, \Lambda_2, \ldots$, where $d(\Lambda_0, \Lambda_n) = n$ and $d(\Lambda_n, \Lambda_{n+1}) = 1$. From Proposition 1.2.2 we know that this corresponds to a collection of lines, $\{l_n \subset L_0/\pi^n L_0\}_{n=0}^\infty$, such that $l_{n+1} = l_n \pmod{\pi^n L_0}$. This is the same as an element of the inverse limit

$$\lim_\leftarrow \mathbb{P}\left(\frac{L_0}{\pi^n L_0}\right) \cong \mathbb{P}\left(\lim_\leftarrow \frac{L_0}{\pi^n L_0}\right) \cong \mathbb{P}(L_0),$$

where the last isomorphism follows from the fact that $\mathcal{L} \cong O^2$ and $O$ is complete and profinite thus $\lim_\leftarrow \frac{O}{\pi^n O} \cong O$. \hfill $\square$

**Proposition 1.2.5.**

If $D \in \mathbb{P}(L_0)$, then the sequence of lattices

$$L_n = \pi^n L_0 + D,$$  \hspace{1cm} (1.1)

forms a geodesic ray based at $L_0$, which represents the point at infinity $D$.

Conversely, given an infinite sequence of lattices that form a geodesic ray $\{L_n\}_{n=0}^\infty$, such that $\pi L_n \subsetneq L_{n+1} \subsetneq L_n$, the corresponding point at infinity corresponds to the line

$$D = \lim_{n \to \infty} L_n = \bigcap_{n=0}^\infty L_n \in \mathbb{P}(L_0).$$  \hspace{1cm} (1.2)

\[3\] The usual definition of the Gromov boundary uses an equivalence relation on the set of all geodesic rays starting at a given point, where two rays are equivalent if they stay a bounded distance apart. Since $\mathcal{X}_2$ is a tree, this equivalence relation becomes trivial in our case.
Example 1.2.6. Let $D = (1 : 0) = \langle e_1 \rangle$. The geodesic ray starting at $L_0$, which corresponds to $D$ is given by: $L_n = \pi^n L_0 + \langle e_1 \rangle = \langle \pi^n e_1, \pi^n e_2, e_1 \rangle = \langle e_1, \pi^n e_2 \rangle$.

Example 1.2.7. Let $D = (1 : 1) = \langle e_1 + e_2 \rangle$. The geodesic ray starting at $L_0$, which corresponds to $D$ is given by: $L_n = \pi^n L_0 + \langle e_1 + e_2 \rangle = \langle \pi^n e_1, \pi^n e_2 \rangle = \langle (\pi^n 0 1) \rangle$.

Example 1.2.8. Let $D = (1 : \pi) = \langle e_1 + \pi e_2 \rangle$. The geodesic ray starting at $L_0$, which corresponds to $D$ is given by:

$$L_n = \pi^n L_0 + \langle e_1 + \pi e_2 \rangle = \langle (\pi^n 0 1) \rangle = \begin{cases} \langle (0 0 1) \rangle & n \leq 1 \\ \langle (\pi^{n-1} 0 1) \rangle & n > 1 \end{cases}.$$

We summarise these examples and a few others in Table 1.1. Notice that if $D \equiv D' \pmod{\pi^k}$ then the first $k$ lattices in the geodesic ray corresponding to $D$ coincide with those in the geodesic ray corresponding to $D'$. See Figure 1.2 for a visual representation of this.
Table 1.1: A table of geodesic rays in $\mathfrak{X}_2$ and their corresponding points at infinity.

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<th>$D \in \mathbb{P}(\mathcal{L}_0)$</th>
<th>$\mathcal{L}_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(1 : 0)$</td>
<td>$\langle (\frac{1}{0} \pi^n) \rangle$</td>
</tr>
<tr>
<td>$(1 : 1)$</td>
<td>$\langle (\frac{\pi^n}{0} 1) \rangle$</td>
</tr>
<tr>
<td>$(0 : 1)$</td>
<td>$\langle (\frac{\pi^n}{0} 0) \rangle$</td>
</tr>
<tr>
<td>$(1 : \pi)$</td>
<td>$\begin{cases} \langle (\frac{1}{0} \pi^n) \rangle &amp; n = 0, 1 \ \langle (\frac{\pi^{n-1}}{0} 1) \rangle &amp; n &gt; 1 \end{cases}$</td>
</tr>
<tr>
<td>$(1 : \pi^2)$</td>
<td>$\begin{cases} \langle (\frac{1}{0} \pi^n) \rangle &amp; n = 0, 1, 2 \ \langle (\frac{\pi^{n-2}}{0} 1) \rangle &amp; n &gt; 2 \end{cases}$</td>
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<tr>
<td>$(1 + \pi : \pi)$</td>
<td>$\begin{cases} \langle (\frac{1}{0} \pi^n) \rangle &amp; n = 0, 1 \ \langle (\frac{\pi^{n-1}}{0} 1+\pi) \rangle &amp; n &gt; 1 \end{cases}$</td>
</tr>
<tr>
<td>$(1 + \pi^2 : \pi)$</td>
<td>$\begin{cases} \langle (\frac{1}{0} \pi^n) \rangle &amp; n = 0, 1 \ \langle (\frac{\pi^{n}}{0} 1) \rangle &amp; n = 2, 3 \ \langle (\frac{\pi^{n-1}}{0} 1+\pi^2) \rangle &amp; n &gt; 3 \end{cases}$</td>
</tr>
<tr>
<td>$(1 : \pi + \pi^2)$</td>
<td>$\begin{cases} \langle (\frac{1}{0} \pi^n) \rangle &amp; n = 0, 1 \ \langle (\frac{0}{\pi^n} 0) \rangle &amp; n &gt; 1 \end{cases}$</td>
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1.3 The Quotient $\text{SL}_2(\mathbb{A}) \backslash \mathfrak{X}_2$

In this section we will examine the quotient space $\text{SL}_2(\mathbb{A}) \backslash \mathfrak{X}_2$. We will describe a fundamental domain for the action of $\text{SL}_2(\mathbb{A})$ on $\mathfrak{X}_2$, and determine the quotient space $\text{SL}_2(\mathbb{A}) \backslash \mathfrak{X}_2$ up to simplicial isomorphism. For the remainder of this section we denote $\text{SL}_2(\mathbb{A})$ by $\Gamma$.

**Definition 1.3.1** (Fundamental Lattices).

For $n \in \mathbb{Z}_{\geq 0}$ let $\mathcal{L}_n = \langle (t^n 0 \ 0) \rangle$, and let $\Lambda_n = [\mathcal{L}_n]$ be the corresponding vertex in $\mathfrak{X}_2$.

**Proposition 1.3.2.**

The set of vertices $\{\Lambda_n\}_{n=0}^{\infty}$ form a non-backtracking path in $\mathfrak{X}_2$. 
Proof. Since \( \Lambda_n \subseteq \Lambda_{n+1} \subseteq t\Lambda_n \), the vertices \( \Lambda_n \) and \( \Lambda_{n+1} \), are adjacent. It is clear that \( t^k\Lambda_n \neq \Lambda_m \) for any \( k, n, m \in \mathbb{Z} \). Thus the vertices \( \Lambda_n \) and \( \Lambda_m \) are distinct for all \( n, m \in \mathbb{Z} \), hence the path is non-backtracking.

Let \( \mathcal{P} \) denote the subcomplex of \( \mathcal{X}_2 \) which is spanned by the vertices \( \{\Lambda_n\}_{n=0}^{\infty} \). We will show that \( \mathcal{P} \) is a fundamental domain for the action of \( SL_2(A) \) on \( \mathcal{X}_2 \).

**Theorem 1.3.3.**

*The \( \Lambda_n \) are pairwise inequivalent modulo \( SL_2(A) \).*

**Proof.** Let \( s = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \in SL_2(A) \), and suppose that for some \( n \in \mathbb{Z}_{\geq 0}, m \in \mathbb{Z}_{\geq -n} \), we have \( s\Lambda_n = \Lambda_{n+m} \). We will show that \( m = 0 \). By assumption we have that \( s\mathcal{L}_n = t^{-h}\mathcal{L}_{n+m} \) for some \( h \in \mathbb{Z} \). By Theorem 1.1.8 we have \( \chi(\mathcal{L}_n, s\mathcal{L}_n) = \chi(\mathcal{L}_n, t^{-h}\mathcal{L}_{n+m}) = 0 \), i.e.

\[
0 = \chi(\mathcal{L}_n, s\mathcal{L}_n) = \chi(\mathcal{L}_n, t^{-h}\mathcal{L}_{n+m}) = l \left( \frac{\mathcal{L}_n}{\mathcal{L}_n \cap t^{-h}\mathcal{L}_{n+m}} \right) - l \left( \frac{t^{-h}\mathcal{L}_{n+m}}{\mathcal{L}_n \cap t^{-h}\mathcal{L}_{n+m}} \right) = l \left( \left\langle \begin{pmatrix} t^n & 0 \\ 0 & 1 \end{pmatrix} \right\rangle - \left\langle \begin{pmatrix} t^{n+m-h} & 0 \\ 0 & t^{-h} \end{pmatrix} \right\rangle \right) - l \left( \left\langle \begin{pmatrix} t^{\min(n, n+m-h)} & 0 \\ 0 & t^{\min(0, -h)} \end{pmatrix} \right\rangle - \left\langle \begin{pmatrix} t^{\min(n, n+m-h)} & 0 \\ 0 & t^{\min(0, -h)} \end{pmatrix} \right\rangle \right) = [n - (\min(n, n+ m - h) + \min(0, -h))] - [n + m - 2h - (\min(n, n+ m - h) + \min(0, -h))] = -m + 2h.
\]

Thus,

\[ m = 2h. \quad (1.3) \]
Hence $sL_n = t^{-h}L_{n+h}$, i.e. $\langle \left( \begin{array}{cc} a_{1,1}t^n & a_{1,2} \\ a_{2,1}t^n & a_{2,2} \end{array} \right) \rangle = \langle \left( \begin{array}{cc} t^{n+h} & 0 \\ 0 & t^{-h} \end{array} \right) \rangle$. Therefore, there exists, $\alpha, \beta, \gamma, \delta \in O$, such that:

$$a_{1,1}t^n = \alpha t^{n+h}, \quad a_{1,2} = \gamma t^{n+h},$$

$$a_{2,1}t^n = \beta t^{-h}, \quad a_{2,2} = \delta t^{-h}.$$ 

By rearranging we get the following degree conditions:

$$\deg_t(a_{1,1}) \leq h, \quad \deg_t(a_{1,2}) \leq n+h,$$

$$\deg_t(a_{2,1}) \leq -n-h, \quad \deg_t(a_{2,2}) \leq -h. \quad (1.4)$$

We now use these degree constraints to show that $h = 0$.

$$h > 0 \implies \begin{cases} 
\deg_t(a_{2,1}) \leq -n-h < 0 \implies a_{2,1} = 0 \\
\deg_t(a_{2,2}) \leq -h < 0 \implies a_{2,2} = 0 
\end{cases}$$

$$\implies \det(s) = 0$$

$$\implies \bot,$$

$$h < 0 \implies \begin{cases} 
\deg_t(a_{1,1}) \leq h < 0 \implies a_{1,1} = 0 \\
\deg_t(a_{2,1}) \leq -n-h < 0 \implies a_{2,1} = 0 
\end{cases}$$

$$\implies \det(s) = 0$$

$$\implies \bot .$$

Where $-n-h < 0$ when $h < 0$ because, $-n-h = -(n+m) + h \leq h$.

Thus $h = 0$. By Equation (1.3) we have $m = 2h = 0$. $\square$
Theorem 1.3.4.

The vertex and edge stabilisers of the cells in $\mathcal{P}$ are given by:

\[
\Gamma_{\Lambda_n} = \begin{cases} 
\text{SL}_2(\mathbb{F}_q) & n = 0 \\
\left\{ \begin{pmatrix} a & b \\
0 & a^{-1} \end{pmatrix} \bigg| a \in \mathbb{F}_q^\times, b \in \mathbb{F}_q[t], \deg_t(b) \leq n \right\} & n > 0
\end{cases}
\] (1.5a)

\[
\Gamma_{\Lambda_n}\Lambda_{n+1} = \left\{ \begin{pmatrix} a & b \\
0 & a^{-1} \end{pmatrix} \bigg| a \in \mathbb{F}_q^\times, b \in \mathbb{F}_q[t], \deg_t(b) \leq n \right\}.
\] (1.5b)

Proof. Suppose $s\Lambda_n = \Lambda_n$ for some $s \in \text{SL}_2(\mathbb{A})$. Then the entries of $s$ must satisfy Equation (1.4) with $h = 0$. Thus $s \in \Gamma_{\Lambda_n}$. It is a straightforward calculation to show that if $s \in \Gamma_{\Lambda_n}$ then $s\Lambda_n = \Lambda_n$.

The edge stabiliser groups follow immediately from $\Gamma_{\Lambda_n}\Lambda_{n+1} = \Gamma_{\Lambda_n} \cap \Gamma_{\Lambda_{n+1}}$. \qed

Proposition 1.3.5.

a) $\Gamma_{\Lambda_0}$ acts transitively on the edges of $\Lambda_0$.

b) For $n \geq 1$, $\Gamma_{\Lambda_n}$ fixes $\Lambda_n\Lambda_{n+1}$, and acts transitively on the remaining edges.

Proof. Recall that the vertices adjacent to a given vertex $\Lambda$ correspond to lines in $\mathbb{F}_q$. The group $\Gamma_{\Lambda_0}$ acts transitively on the set of such lines. Similarly, for $n > 0$ the group $\Gamma_{\Lambda_n}$ fixes one line, and acts transitively on the remaining lines. \qed

We are now ready to show that $\mathcal{P}$ is a fundamental domain for the action of $\text{SL}(\mathbb{A})$ on $\mathcal{X}_2$.

Theorem 1.3.6 (Quotient space $\text{SL}_2(\mathbb{A}) \backslash \mathcal{X}_2$).

The subcomplex $\mathcal{P} \in \mathcal{X}_2$ is a fundamental domain for the action of $\text{SL}_2(\mathbb{A})$ on $\mathcal{X}_2$. Furthermore, the quotient space $\text{SL}_2(\mathbb{A}) \backslash \mathcal{X}_2$ is simplicially isomorphic to $\mathcal{P}$. 24
Proof. By Theorem 1.3.3 we know that the $\Lambda_n$ are pairwise inequivalent modulo $\text{SL}_2(A)$. It remains to show that if $\Lambda \not\in \{\Lambda_n\}_{n=0}^{\infty}$ is a vertex in $\mathcal{X}_2$, then it is equivalent to some $\Lambda_n$ modulo $\text{SL}_2(A)$. This follows from Proposition 1.3.5.

As a consequence of Theorem 1.3.6 there is a natural numbering of the vertices of $\mathcal{X}_2$, where the vertex $\Lambda$ is numbered $n$ if and only if $\Lambda$ maps to $\Lambda_n$ via the quotient map $\mathcal{X}_2 \rightarrow \text{SL}_2(A) \backslash \mathcal{X}_2$. See Figure 1.3 for an illustration of this numbering.

![Figure 1.3: The Bruhat-Tits building associated to SL$_2(F_2((t^{-1})))$ with a 2-colouring and numbering of the vertices](image)

1.4 Quotients by full congruence subgroups $\Gamma(g) \subseteq \text{SL}_2(A)$

Let $g \in A$ be non-zero. In this section we will examine the quotient $\Gamma(g) \backslash \mathcal{X}_2$, where

$$\Gamma(g) \overset{\text{def}}{=} \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(A) \left| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{g} \right. \right\}.$$

We call $\Gamma(g)$ the full congruence subgroup of $\Gamma$ of level $g$. Note that $\Gamma(g)$ is a normal subgroup of $\Gamma$, thus there exists a quotient map $\rho: \Gamma(g) \backslash \mathcal{X}_2 \rightarrow \Gamma \backslash \mathcal{X}_2$. We will study $\Gamma(g) \backslash \mathcal{X}_2$ by studying this quotient map. We denote $\Gamma(g) \backslash \mathcal{X}_2$ by $\mathcal{X}_2(g)$.
Definition 1.4.1.

Identify $\Gamma \backslash X_2$ with $\mathcal{P}$, and define

$$\{\mathcal{X}_2(g)\}_{\Lambda_n} = \{\text{Vertices in } \mathcal{X}_2(g) \text{ lying above } \Lambda_n \},$$

$$\{\mathcal{X}_2(g)\}_{\Lambda_n, \Lambda_{n+1}} = \{\text{Edges in } \mathcal{X}_2(g) \text{ lying above } \Lambda_{n,n+1} \}.$$ 

We will calculate the size of the sets $\{\mathcal{X}_2(g)\}_{\Lambda_n}$, and $\{\mathcal{X}_2(g)\}_{\Lambda_n, \Lambda_{n+1}}$, using the following identity.

Theorem 1.4.2.

The set of vertices (resp. edges) in $\mathcal{X}_2(g)$ which lie over $\Lambda_n$ (resp. $\Lambda_{n,n+1}$), is given by

$$\{\mathcal{X}_2(g)\}_{\Lambda_n} \simeq \frac{\Gamma / \Gamma(g)}{\Gamma_{\Lambda_n} / \Gamma(g)_{\Lambda_n}},$$

(1.6a)

$$\{\mathcal{X}_2(g)\}_{\Lambda_n, \Lambda_{n+1}} \simeq \frac{\Gamma / \Gamma(g)}{\Gamma_{\Lambda_n, \Lambda_{n+1}} / \Gamma(g)_{\Lambda_n, \Lambda_{n+1}}}. $$

(1.6b)

In particular the cardinality is given by

$$\# \{\mathcal{X}_2(g)\}_{\Lambda_n} = \frac{[\Gamma : \Gamma(g)]}{[\Gamma_{\Lambda_n} : \Gamma(g)_{\Lambda_n}]},$$

(1.7a)

$$\# \{\mathcal{X}_2(g)\}_{\Lambda_n, \Lambda_{n+1}} = \frac{[\Gamma : \Gamma(g)]}{[\Gamma_{\Lambda_n, \Lambda_{n+1}} : \Gamma(g)_{\Lambda_n, \Lambda_{n+1}}]}.$$ 

(1.7b)

Proof. We will only show Equation (1.6a), the proof of Equation (1.6b) is directly analogous. The proof is essentially a series of simple isomorphisms. Since $\Gamma$ acts transitively on the set of all vertices in $\mathcal{X}_2$ which lie over $\Lambda_n$, by the orbit-stabiliser theorem we have that $\Gamma / \Gamma_{\Lambda_n} \cong \{\text{Vertices in } \mathcal{X}_2 \text{ which lie over } \Lambda_n \}$. Thus
\[ \{ X_2(g) \}_\Lambda_n \cong \Gamma(g) \backslash \Gamma / \Gamma_{\Lambda_n} \]

\[ \cong \frac{\Gamma / \Gamma(g) \Gamma_{\Lambda_n}}{\Gamma(g) \Gamma_{\Lambda_n} / \Gamma(g)} \]

Since both \( \Gamma(g) \) and \( \Gamma_{\Lambda_n} \) are normal in \( \Gamma \)

\[ \cong \frac{\Gamma / \Gamma(g)}{\Gamma_{\Lambda_n} / \Gamma_{\Lambda_n} \cap \Gamma(g)} \]

By the third isomorphism theorem

\[ = \frac{\Gamma / \Gamma(g)}{\Gamma_{\Lambda_n} / \Gamma(g) \Lambda_n} \]

Since \( \Gamma_{\Lambda_n} \cap \Gamma(g) = \Gamma(g)_{\Lambda_n} \).

\[ \square \]

Before we use Theorem 1.4.2 to calculate \( \{ X_2(g) \}_\Lambda_n \) and \( \{ X_2(g) \}_{\Lambda_n \Lambda_{n+1}} \), we first derive a general expression for the index \([ \Gamma : \Gamma(g) \]).

**Theorem 1.4.3.**

*Let \( g \in A \) with \( \text{deg}_t(g) = N > 0 \), and assume that \( g \) factors as \( g = \prod_{i=1}^{k} g_i^{e_i} \) where the \( g_i \in A \) are distinct, irreducible, and \( \text{deg}_t(g_i) = d_i \). Then*

\[ [ \Gamma : \Gamma(g) ] = q^{3N} \prod_{i=1}^{k} \left( 1 - \frac{1}{q^{2d_i}} \right). \]  

(1.8)

**Proof.** We break the proof up into multiple steps:

**Step 1.** Show that \([ \Gamma : \Gamma(g) ] = \# \text{SL}_2 \left( A / g \right) \]

**Step 2.** Reduce to the case \( \# \text{SL}_2 \left( A / g^e \right) \) for \( g \) irreducible

**Step 3.** Show that \( \# \text{SL}_2 \left( A / g^e \right) = \frac{\# \text{GL}_2 \left( A / g^e \right)}{\left( A / g^e \right)^x} \)
Step 4. Show that \[ \# \left( \frac{A}{g^e} \right)^x = q^{ed} - q^{(e-1)d} \]

Step 5. Show that \[ \# \text{GL}_2 \left( \frac{A}{g^e} \right) = q^{4(e-1)d}(q^{2d} - 1)(q^{2d} - q^d) \]

Step 6. Conclude that \[ \# \text{SL}_2 \left( \frac{A}{g^e} \right) = q^{3ed} \left( 1 - \frac{1}{q^{2d}} \right) \]

Step 1. This is a direct consequence of the following short exact sequence

\[ 1 \rightarrow \Gamma(g) \rightarrow \Gamma \rightarrow \text{SL}_2 \left( \frac{A}{g^e} \right) \rightarrow 1. \]

Step 2. This is a consequence of the Chinese Remainder Theorem for \text{SL}_2, i.e.

\[ \text{SL}_2 \left( \frac{A}{g} \right) \cong \prod_{i=1}^k \text{SL}_2 \left( \frac{A}{g_i^e} \right). \]

Step 3. This is a direct consequence of the following short exact sequence

\[ 1 \rightarrow \text{SL}_2 \left( \frac{A}{g^e} \right) \rightarrow \text{GL}_2 \left( \frac{A}{g^e} \right) \rightarrow \left( \frac{A}{g^e} \right)^x \rightarrow 1. \]

Step 4. It is straightforward to see that \[ \left( \frac{A}{g^e} \right)^x = \{ a \in \frac{A}{g^e} \mid a \not\equiv 0 \pmod{g} \}. \]

Thus \[ \# \left( \frac{A}{g^e} \right)^x = q^{ed} - q^{(e-1)d}. \]

Step 5. The reduction map \[ \rho: f \pmod{g^e} \mapsto f \pmod{g} \] induces a surjective map \[ \tilde{\rho}: \text{GL}_2 \left( \frac{A}{g^e} \right) \rightarrow \text{GL}_2 \left( \frac{A}{g} \right). \] Thus \[ \# \text{GL}_2 \left( \frac{A}{g^e} \right) = \# \ker(\tilde{\rho}) \times \# \text{GL}_2 \left( \frac{A}{g} \right). \]

The kernel of the map is \[ \ker(\tilde{\rho}) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + A \mid A \in g \text{M}_{2,2} \left( \frac{A}{g^e} \right) \right\}, \] which has cardinality \[ \# \ker(\tilde{\rho}) = q^{4(e-1)d}. \]

Thus \[ \# \text{GL}_2 \left( \frac{A}{g} \right) = q^{4(e-1)d}(q^{2d} - 1)(q^{2d} - q^d). \]
Step 6. From Step 3., Step 4., and Step 5. we have that

\[
\# \mathrm{SL}_2 \left( \mathbb{F}_q \right) = \frac{q^{4(e-1)d}(q^{2d} - 1)(q^{2d} - q^d)}{q^{ed} - q^{(e-1)d}} \\
= q^{3(e-1)d}(q^{2d} - 1)q^d \\
= q^{3ed} \left( 1 - \frac{1}{q^{2d}} \right).
\]

\[\Box\]

**Proposition 1.4.4.**

Let \( g \in A \) with \( \deg_t(g) = N > 0 \), and assume that \( g \) factors as \( g = \prod_{i=1}^{k} g_i^{e_i} \) where the \( g_i \in A \) are distinct, irreducible, and \( \deg_t(g_i) = d_i \). Then

\[
[\Gamma_{\Lambda_n} : \Gamma(g)_{\Lambda_n}] = \begin{cases} 
q(q^2 - 1) & \text{if } n = 0 \\
q^{\min(n+1, N)}(q - 1) & \text{if } n > 0 
\end{cases}
\]

\[
[\Gamma_{\Lambda_n, \Lambda_{n+1}} : \Gamma(g)_{\Lambda_n, \Lambda_{n+1}}] = q^{\min(n+1, N)}(q - 1) \quad \text{if } n \geq 0.
\]

**Proof.** Using Equation (1.5a) and Equation (1.5b) we calculate the cardinality of the stabiliser subgroups of \( \Gamma \), i.e.

\[
\# \Gamma_{\Lambda_n} = \begin{cases} 
q(q^2 - 1) & \text{if } n = 0 \\
q^{n+1}(q - 1) & \text{if } n > 0 
\end{cases}
\]

\[
\# \Gamma_{\Lambda_n, \Lambda_{n+1}} = q^{n+1}(q - 1).
\]

The vertex stabiliser subgroups of \( \Gamma(g) \) are given by

\[
\Gamma(g)_{\Lambda_0} = \mathrm{SL}_2(\mathbb{F}_q) \cap \Gamma(g) = \{ \text{Id} \},
\]
and for \( n > 0 \) then we have

\[
\Gamma(g)_{\Lambda_n} = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid a \in \mathbb{F}_q^\times, b \in \mathbb{F}_q[t], \text{ and } \deg_t(b) \leq n \right\} \cap \Gamma(g)
\]

\[
= \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in \mathbb{F}_q[t], \deg_t(b) \leq n, \text{ and } b \equiv 0 \pmod{g} \right\}
\]

\[
= \left\{ \begin{pmatrix} 1 & g \cdot b \\ 0 & 1 \end{pmatrix} \mid b \in \mathbb{F}_q[t], \text{ and } \deg_t(b) \leq n - N \right\}.
\]

Similarly, the edge stabilisers are,

\[
\Gamma(g)_{\Lambda_n \Lambda_{n+1}} = \left\{ \begin{pmatrix} 1 & g \cdot b \\ 0 & 1 \end{pmatrix} \mid b \in \mathbb{F}_q[t], \text{ and } \deg_t(b) \leq n - N \right\}.
\]

Taking the cardinality of these groups gives

\[
\#\Gamma(g)_{\Lambda_n} = \begin{cases} 
1 & \text{if } n < N \\
q^{n-N+1} & \text{if } n \geq N
\end{cases}
\]

\[
\#\Gamma(g)_{\Lambda_n \Lambda_{n+1}} = \begin{cases} 
1 & \text{if } n < N \\
q^{n-N+1} & \text{if } n \geq N
\end{cases}.
\]

Taking the appropriate quotients gives:

\[
[\Gamma_{\Lambda_n} : \Gamma(g)_{\Lambda_n}] = \begin{cases} 
q(q^2 - 1) & \text{if } n = 0 \\
q^{n+1}(q - 1) & \text{if } 0 < n < N \\
q^N(q - 1) & \text{if } n \geq N
\end{cases}
\]

\[
[\Gamma_{\Lambda_n \Lambda_{n+1}} : \Gamma(g)_{\Lambda_n \Lambda_{n+1}}] = \begin{cases} 
q^{n+1}(q - 1) & \text{if } n < N \\
q^N(q - 1) & \text{if } n \geq N
\end{cases}.
\]
We are now ready to calculate some examples.

Example 1.4.5 ($X_2(t^N)$).

We will calculate the number of vertices and edges in $X_2(t^N)$ that lie over $\Lambda_n$ and $\Lambda_{n,n+1}$. By Equation (1.8) we have that $[\Gamma : \Gamma(t^N)] = q^{3N}(1 - \frac{1}{q^2}) = q^{3N-2}(q^2 - 1)$. By Proposition 1.4.4 we have that

$$[\Gamma : \Gamma(t^N)_{\Lambda_n}] = \begin{cases} q(q^2 - 1) & \text{if } n = 0 \\ q^{\min(n+1,N)}(q - 1) & \text{if } n > 0 \end{cases}$$

$$[\Gamma_{\Lambda_n\Lambda_{n+1}} : \Gamma(t^N)_{\Lambda_n\Lambda_{n+1}}] = q^{\min(n+1,N)}(q - 1) \text{ if } n \geq 0.$$

By Theorem 1.4.2 we have

$$\#\{X_2(t^N)\}_{\Lambda_n} = \frac{[\Gamma : \Gamma(t^N)_{\Lambda_n}]}{[\Gamma : \Gamma(t^N)_{\Lambda_n}]} = \begin{cases} q^{2(N-1)} & n = 0 \\ q^{2(N-1)-n}(q + 1) & 0 < n < N \\ q^{2(N-1)}(q + 1) & n \geq N \end{cases}$$

and

$$\#\{X_2(t^N)\}_{\Lambda_n\Lambda_{n+1}} = \frac{[\Gamma : \Gamma(t^N)]}{[\Gamma_{\Lambda_n\Lambda_{n+1}} : \Gamma(t^N)_{\Lambda_n\Lambda_{n+1}}]} = \begin{cases} q^{2(N-1)-n}(q + 1) & n < N \\ q^{2(N-1)}(q + 1) & n \geq N \end{cases}.$$

The results of the above example are summarised in Table 1.2.

Example 1.4.6 ($X_2(t^2 + 1)$).

Let $g = t^2 + 1$. There are three different cases to consider here, depending on the characteristic of the field $F_q((t^{-1}))$:

**Case I:** char$(F) = 2$, the ramified case.
Table 1.2: A table of $\# \{ X_2(t^N) \}_{\Lambda_n}$ and $\# \{ X_2(t^N) \}_{\Lambda_n\Lambda_{n+1}}$, and some low degree examples.

CASE II: $\text{char}(\mathcal{F}) \equiv 1 \pmod{4}$, the split case.

CASE III: $\text{char}(\mathcal{F}) = 3 \pmod{4}$, the non-split case.

One thing which is common to all the cases is the index of the stabiliser subgroups, i.e. by Proposition 1.4.4 we have that

$$[\Gamma_{\Lambda_n} : \Gamma(t^2 + 1)_{\Lambda_n}] = \begin{cases} q(q^2 - 1) & n = 0, \\ q^2(q - 1) & n \geq 1. \end{cases}$$

$$[\Gamma_{\Lambda_n\Lambda_{n+1}} : \Gamma(t^2 + 1)_{\Lambda_n\Lambda_{n+1}}] = \begin{cases} q(q - 1) & n = 0, \\ q^2(q - 1) & n \geq 1. \end{cases}$$

CASE I: $\text{char}(\mathcal{F}) = 2$.

In this case $(t^2 + 1)$ ramifies, i.e. $(t^2 + 1) = (t + 1)^2$. Thus by Equation (1.8) we have that $[\Gamma : \Gamma(g)] = q^6 \left( 1 - \frac{1}{q^2} \right) = q^4(q^2 - 1)$. By Theorem 1.4.2 we have
\[
\# \{\mathcal{X}_2(g)\}_{\Lambda_n} = \begin{cases} 
q^3 & \text{if } n = 0 \\
q^2(q+1) & \text{if } n \geq 1
\end{cases}
\]

and

\[
\# \{\mathcal{X}_2(g)\}_{\Lambda_n \Lambda_{n+1}} = \begin{cases} 
q^3(q+1) & \text{if } n = 0 \\
q^2(q+1) & \text{if } n \geq 1
\end{cases}
\]

**Case II :** \(\text{char}(\mathcal{F}) \equiv 1 \pmod{4}\).

In this case \((t^2 + 1)\) splits, i.e. \((t^2 + 1) = (t - a)(t + a)\) for some \(a \in \mathbb{F}_q\). Thus by Equation (1.8) we have that

\[
[\Gamma : \Gamma(g)] = q^6 \left(1 - \frac{1}{q^2}\right) \left(1 - \frac{1}{q^2}\right) = q^2(q^2 - 1)(q^2 - 1).
\]

By Theorem 1.4.2 we have that

\[
\# \{\mathcal{X}_2(g)\}_{\Lambda_n} = \begin{cases} 
q(q^2 - 1) & \text{if } n = 0 \\
(q+1)(q^2 - 1) & \text{if } n \geq 1
\end{cases}
\]

and

\[
\# \{\mathcal{X}_2(g)\}_{\Lambda_n \Lambda_{n+1}} = \begin{cases} 
q(q+1)(q^2 - 1) & \text{if } n = 0 \\
(q+1)(q^2 - 1) & \text{if } n \geq 1
\end{cases}
\]

**Case III :** \(\text{char}(\mathcal{F}) \equiv 3 \pmod{4}\).

In this case \((t^2 + 1)\) is irreducible. Thus by Equation (1.8) we have that

\[
[\Gamma : \Gamma(g)] = q^6 \left(1 - \frac{1}{q^2}\right) = q^2(q^2 - 1)(q^2 + 1).
\]

By Theorem 1.4.2 we have that
\[ \#\{\mathcal{X}_2(g)\}_{\Lambda_n} = \begin{cases} 
q(q^2 + 1) & \text{if } n = 0 \\
(q + 1)(q^2 + 1) & \text{if } n \geq 1 
\end{cases} \]

\[ \#\{\mathcal{X}_2(g)\}_{\Lambda_n,\Lambda_{n+1}} = \begin{cases} 
q(q + 1)(q^2 + 1) & \text{if } n = 0 \\
(q + 1)(q^2 + 1) & \text{if } n \geq 1 
\end{cases}. \]

The results of the above example are summarised in Table 1.3.

<table>
<thead>
<tr>
<th>(g)</th>
<th>(#{\mathcal{X}<em>2(g)}</em>{\Lambda_n})</th>
<th>(#{\mathcal{X}<em>2(g)}</em>{\Lambda_n,\Lambda_{n+1}})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(t^2 + 1 = (t + 1)^2)</td>
<td>(q^3)</td>
<td>(q^3(q + 1))</td>
</tr>
<tr>
<td></td>
<td>(q^2(q + 1))</td>
<td>(q^2(q + 1))</td>
</tr>
<tr>
<td>(t^2 + 1 = (t - a)(t + a))</td>
<td>(q(q^2 - 1))</td>
<td>(q(q + 1)(q^2 - 1))</td>
</tr>
<tr>
<td></td>
<td>((q + 1)(q^2 - 1))</td>
<td>((q + 1)(q^2 - 1))</td>
</tr>
<tr>
<td>(t^2 + 1) irreducible</td>
<td>(q(q^2 + 1))</td>
<td>(q(q + 1)(q^2 + 1))</td>
</tr>
<tr>
<td></td>
<td>((q + 1)(q^2 + 1))</td>
<td>((q + 1)(q^2 + 1))</td>
</tr>
</tbody>
</table>

Table 1.3: A table of \(\#\{\mathcal{X}_2(t^2 + 1)\}_{\Lambda_n}\) and \(\#\{\mathcal{X}_2(t^2 + 1)\}_{\Lambda_n,\Lambda_{n+1}}\).

It turns out that \(\#\{\mathcal{X}_2(g)\}_{\Lambda_n}\) and \(\#\{\mathcal{X}_2(g)\}_{\Lambda_n,\Lambda_{n+1}}\) only depend on the splitting type of \(g\).

**Theorem 1.4.7** \((\mathcal{X}_2(g)\) for a general \(g \in A\)).

Let \(g \in A\) with \(\deg_t(g) = N > 0\), and assume that \(g\) factors as \(g = \prod_{i=1}^{k} g_i^{e_i}\), where the \(g_i \in A\) are distinct, irreducible, and \(\deg_t(g_i) = d_i\).
\[
\#\{X_2(t^2 + 1)\}_n = \begin{cases} 
q^{3N} \prod_{i=1}^{k} \left(1 - \frac{1}{q^{2^n_i}}\right) & \text{if } n = 0, \\
q^{n+1}(q - 1) & \text{if } 0 < n < N - 1, \\
q^n(q - 1) & \text{if } n \geq N - 1.
\end{cases}
\]

\[
\#\{X_2(t^2 + 1)\}_{n+1} = \begin{cases} 
q^{3N} \prod_{i=1}^{k} \left(1 - \frac{1}{q^{2^n_i}}\right) & \text{if } n = 0, \\
q(q - 1) & \text{if } 0 < n < N - 1, \\
q^N(q - 1) & \text{if } n \geq N - 1.
\end{cases}
\]

In the following table (Table 1.4), we summarise the results of Theorem 1.4.7 as well as completely classify \#\{X_2(g)\}_n as well as \#\{X_2(g)\}_{n+1} for all \(g \in A\) with \(\deg_t(g) \leq 3\).
\[
\begin{array}{ccc}
g & \#\{\mathcal{X}_2(g)\}_{\Lambda_n} & \#\{\mathcal{X}_2(g)\}_{\Lambda_n\Lambda_{n+1}} \\
\prod_{i=1}^{k} g_i^{e_i} & \frac{q^{3N} \prod_{i=0}^{k} (1 - q^{-2d_i})}{q(q^2 - 1)} & \frac{q^{3N} \prod_{i=0}^{k} (1 - q^{-2d_i})}{q(q - 1)} \\
& \frac{q^{3N} \prod_{i=0}^{k} (1 - q^{-2d_i})}{q^{\min(n+1,N)}(q - 1)} & \frac{q^{3N} \prod_{i=0}^{k} (1 - q^{-2d_i})}{q^{\min(n+1,N)}(q - 1)} \\
(1) & 1 & q + 1 \\
& q + 1 & q + 1 \\
(2) & q(q^2 + 1) & q(q^2 + 1)(q + 1) \\
& (q^2 + 1)(q + 1) & (q^2 + 1)(q + 1) \\
(1)(1) & q(q^2 - 1) & q(q^2 - 1)(q + 1) \\
& (q^2 - 1)(q + 1) & (q^2 - 1)(q + 1) \\
(1)^2 & q^3 & q^3(q + 1) \\
& q^2(q + 1) & q^2(q + 1) \\
(3) & q^2(q^4 + q^2 + 1) & q^2(q^4 + q^2 + 1)(q + 1) \\
& q(q^4 + q^2 + 1)(q + 1) & q(q^4 + q^2 + 1)(q + 1) \\
& (q^4 + q^2 + 1)(q + 1) & (q^4 + q^2 + 1)(q + 1) \\
(2)(1) & q^2(q^4 - 1) & q^2(q^4 - 1)(q + 1) \\
& q(q^4 - 1)(q + 1) & q(q^4 - 1)(q + 1) \\
& (q^4 - 1)(q + 1) & (q^4 - 1)(q + 1) \\
(1)(1)(1) & q^2(q^2 - 1)^2 & q^2(q^2 - 1)^2(q + 1) \\
& q(q^2 - 1)^2(q + 1) & q(q^2 - 1)^2(q + 1) \\
& (q^2 - 1)^2(q + 1) & (q^2 - 1)^2(q + 1) \\
(1)^2(1) & q^4(q^2 - 1) & q^4(q^2 - 1)(q + 1) \\
& q^3(q^2 - 1)(q + 1) & q^3(q^2 - 1)(q + 1) \\
& q^2(q^2 - 1)(q + 1) & q^2(q^2 - 1)(q + 1) \\
\end{array}
\]
Table 1.4: A table of \( \#\{\mathcal{X}_2(g)\}_{\Lambda_n} \) and \( \#\{\mathcal{X}_2(g)\}_{\Lambda_n\Lambda_{n+1}} \) for a general \( g \in A \), and some low degree examples.

We can use what we know about \( \#\{\mathcal{X}_2(g)\}_{\Lambda_n} \) and \( \#\{\mathcal{X}_2(g)\}_{\Lambda_n\Lambda_{n+1}} \) to say something about the structure of the quotient space \( \mathcal{X}_2(g) \).

**Theorem 1.4.8.**

The quotient space \( \mathcal{X}_2(g) \) is a union of a finite graph \( \mathcal{X}_2(g)_{\text{finite}} \), and a finite collection of cusps \( \mathcal{X}_2(g)_{\text{cusps}} \). Moreover, if \( \deg_\tau(g) = N \), then \( \mathcal{X}_2(g)_{\text{finite}} \) is given by the subcomplex spanned by the vertices of type \( \leq N \).

**Proof.** By Theorem 1.4.7 we know that

\[
\#\{\mathcal{X}_2(g)\}_{\Lambda_n} = \#\{\mathcal{X}_2(g)\}_{\Lambda_n\Lambda_{n+1}} = \#\{\mathcal{X}_2(g)\}_{\Lambda_{n+1}}
\]

for all \( n \geq N - 1 \). The result follows since the quotient \( \mathcal{X}_2(g) \) is connected. \( \square \)

Figure 1.4 highlights the general structure of \( \mathcal{X}_2(g) \), and how it lies above \( \Gamma \backslash \mathcal{X}_2 \).
Figure 1.4: A figure showing the general structure of $\mathcal{X}_2(g)$, for $\deg_2(g) = N$, and how it lies over $\Gamma \backslash \mathcal{X}_2$. 
Figure 1.5: A figure showing $\Gamma(t^N) \setminus \mathcal{X}_2$ for $N \in \{1, 2, 3, 4\}$, and how they lie over $\Gamma \setminus \mathcal{X}_2$. 
1.5 Homology of $\Gamma(g) \setminus \mathcal{X}_2$

Let $g \in A$ with $\deg_t(g) = N > 0$, and assume that $g$ factors as $g = \prod_{i=1}^{k} g_i^{e_i}$, where the $g_i \in A$ are distinct, irreducible, and $\deg_t(g_i) = d_i$. To simplify notation we denote the quotient space $\Gamma(g) \setminus \mathcal{X}_2$ by $\mathcal{X}_2(g)$.

The homology of $\mathcal{X}_2(g)$ is only supported in dimensions 0 and 1, since it is a graph. Moreover, the homology groups of $\mathcal{X}_2(g)$ are free, since any graph is homotopy equivalent to a wedge of circles. Thus

$$H_0(\mathcal{X}_2(g); \mathbb{Z}) \cong \mathbb{Z}$$
$$H_1(\mathcal{X}_2(g); \mathbb{Z}) \cong \mathbb{Z}^r,$$ for some $r$
$$H_k(\mathcal{X}_2(g); \mathbb{Z}) = 0,$$ for all $k > 1$.

We will use our knowledge of $\#\{\mathcal{X}_2(g)\}_\Lambda_n$ and $\#\{\mathcal{X}_2(g)\}_\Lambda_n\Lambda_{n+1}$, to calculate $r$.

**Theorem 1.5.1.**

Let $\mathcal{X}_2(g)_{\text{finite}}$ be as in Theorem 1.4.8. Then

$$\text{rank}_\mathbb{Z}(H_1(\mathcal{X}_2(g); \mathbb{Z})) = \text{rank}_\mathbb{Z}(H_1(\mathcal{X}_2(g)_{\text{finite}}; \mathbb{Z}))$$

$$= \# \text{Sim}_1(\mathcal{X}_2(g)_{\text{finite}}) - \# \text{Sim}_0(\mathcal{X}_2(g)_{\text{finite}}) + 1.$$ 

Moreover,

$$\text{rank}_\mathbb{Z}(H_1(\mathcal{X}_2(g), \partial \mathcal{X}_2(g); \mathbb{Z})) = \text{rank}_\mathbb{Z}(H_1(\mathcal{X}_2(g); \mathbb{Z})) + \#\text{cusps} - 1.$$ 

**Proof.** This follows from the Euler characteristic.  

Note that since $\#\{\mathcal{X}_2(g)\}_\Lambda_n$ and $\#\{\mathcal{X}_2(g)\}_\Lambda_n\Lambda_{n+1}$ only depended on the splitting type of $g \in A$, the same is true for the homology.
Theorem 1.5.2 (Homology of $X_2(g)$).

Let $g \in A$ with $\deg_t(g) = N$, and assume that $g$ factors as $g = \prod_{i=1}^{k} g_i^{e_i}$, where the $g_i \in A$ are distinct, irreducible, and $\deg_t(g_i) = d_i$. Then

$$\text{rank}_Z(H_1(X_2(g); \mathbb{Z})) = \frac{(q^N - q - 1)q^{2N} \prod_{i=1}^{k} (1 - q^{-2d_i})}{(q - 1)(q + 1)} + 1$$

and

$$\text{rank}_Z(H_1(X_2(g), \partial X_2(g); \mathbb{Z})) = \frac{q^{3N} \prod_{i=1}^{k} (1 - q^{-2d_i})}{(q - 1)(q + 1)}$$

Proof. By Theorem 1.5.1 we have

$$\text{rank}_Z(H_1(X_2(g); \mathbb{Z})) = \# \text{Sim}_1(X_2(g)_{\text{finite}}) - \# \text{Sim}_0(X_2(g)_{\text{finite}}) + 1$$

$$= \sum_{i=0}^{N-1} \# \{X_2(g)\}_{\Lambda_i \Lambda_{i+1}} - \sum_{i=0}^{N} \# \{X_2(g)\}_{\Lambda_i} + 1$$

$$= \# \{X_2(g)\}_{\Lambda_0 \Lambda_1} - \# \{X_2(g)\}_{\Lambda_0} - \# \{X_2(g)\}_{\Lambda_N}$$

$$+ \sum_{i=1}^{N-1} \left( \# \{X_2(g)\}_{\Lambda_i \Lambda_{i+1}} - \# \{X_2(g)\}_{\Lambda_i} \right) + 1$$

$$= \# \{X_2(g)\}_{\Lambda_0 \Lambda_1} - \# \{X_2(g)\}_{\Lambda_0} - \# \{X_2(g)\}_{\Lambda_N} + 1.$$
\[
\text{rank}_\mathbb{Z}(H_1(\mathcal{X}_2(g), \mathbb{Z})) \]
\[
= \frac{q^{3N} \prod_i (1 - q^{-2d_i})}{q(q - 1)} - \frac{q^{3N} \prod_i (1 - q^{-2d_i})}{q(q^2 - 1)} - \frac{q^{3N} \prod_i (1 - q^{-2d_i})}{q^N(q - 1)} + 1
\]
\[
= q^{3N} \prod_i (1 - q^{-2d_i}) \left( \frac{1}{q(q - 1)} - \frac{1}{q(q - 1)(q + 1)} - \frac{1}{q^N(q - 1)} \right) + 1
\]
\[
= q^{3N} \prod_i (1 - q^{-2d_i}) \left( \frac{q^{N-1}(q + 1) - q^{N-1} - q - 1}{q^N(q - 1)(q + 1)} \right) + 1
\]
\[
= \left( \frac{q^N - q - 1}{q - 1} \right)^2 \frac{q^{2N} \prod_i (1 - q^{-2d_i})}{(q - 1)(q + 1)} + 1.
\]

The number of cusps of \( \mathcal{X}_2(g) \) is \( \# \{ \mathcal{X}_2(g) \}_\Lambda_N \). Thus

\[
\text{rank}_\mathbb{Z}(H_1(\mathcal{X}_2(g), \partial \mathcal{X}_2(g); \mathbb{Z}))
\]
\[
= \frac{(q^N - q - 1)q^{2N} \prod_i (1 - q^{-2d_i})}{(q - 1)(q + 1)} - \frac{q^{3N} \prod_i (1 - q^{-2d_i})}{q^N(q - 1)}
\]
\[
= \frac{(q^N - q - 1)q^{2N} \prod_i (1 - q^{-2d_i})}{(q - 1)(q + 1)} - \frac{(q + 1)q^{2N} \prod_i (1 - q^{-2d_i})}{(q - 1)(q + 1)}
\]
\[
= \frac{q^{3N} \prod_i (1 - q^{-2d_i})}{(q - 1)(q + 1)}.
\]

\[\square\]

In the following table (Table 1.5) we determine \( \text{rank}_\mathbb{Z}(H_1(\mathcal{X}_2(g); \mathbb{Z})) \) as well as \( \text{rank}_\mathbb{Z}(H_1(\mathcal{X}_2(g), \partial \mathcal{X}_2(g); \mathbb{Z})) \) for all \( g \in \mathcal{A} \) with \( \deg_t(g) \leq 3 \).
Example 1.5.3 (Homology of $X_2(t^N)$).

By Theorem 1.5.2 we have

$$\text{rank}_\mathbb{Z}(H_1(X_2(t^N); \mathbb{Z})) = q^{2(N-1)}(q^N - q - 1) + 1,$$

and

$$\text{rank}_\mathbb{Z}(H_1(X_2(t^N), \partial X_2(t^N); \mathbb{Z})) = q^{3N-2}.$$
CHAPTER 2
THE BUILDING ASSOCIATED TO $\text{SL}_3(\mathcal{F})$

In this chapter we will examine the Bruhat-Tits building associated to $\text{SL}_3(\mathcal{F})$, which we denote by $\mathfrak{X}_3$. The main objectives of this section are:

1. Make basic observations on the structure of $\mathfrak{X}_3$, describe how $\text{SL}_3(\mathcal{F})$ acts on $\mathfrak{X}_3$, and show that there exists a $\text{SL}_3(\mathcal{F})$ invariant 3-colouring of $\mathfrak{X}_3$.

2. Examine the quotient $\text{SL}_3(\mathcal{A}) \backslash \mathfrak{X}_3$, describe a fundamental domain, and calculate all relevant stabiliser subgroups.

3. Examine the quotients $\Gamma(g) \backslash \mathfrak{X}_3$, for $\Gamma(g) \subseteq \text{SL}_3(\mathcal{A})$ a full congruence subgroup.

We use the covering map $\rho: \Gamma(g) \backslash \mathfrak{X}_3 \longrightarrow \text{SL}_3(\mathcal{A}) \backslash \mathfrak{X}_3$ to calculate the cardinality of the set of simplices which lie above any given simplex of $\text{SL}_3(\mathcal{A}) \backslash \mathfrak{X}_3$.

Similarly to Chapter 1, in this chapter we will be considering $\mathfrak{X}_3$ from the point of view of lattices, as discussed in Chapter I. i.e. Let $\mathcal{W} = \mathcal{F}^3$, and define a lattice in $\mathcal{W}$ to be a free $\mathcal{O}$-sub-module of $\mathcal{W}$ of rank 3. Let $\text{Lat}/\sim$ denote the set of all lattices up to $\mathcal{F}^\times$-homothety, and denote the equivalence class of a lattice $\mathcal{L}$ by $[\mathcal{L}]$. Then the Bruhat-Tits building $\mathfrak{X}_3$ is defined as follows:

**Definition 2.0.1.**
Let $\mathfrak{X}_3$ be the abstract simplicial complex with the following simplices
Sim₀(𝓥₃) = \{ Λᵢ | Λᵢ ∈ \text{Lat}/∼ \}

Sim₁(𝓥₃) = \left\{ Λ₁Λ₂ \middle| Λ₁, Λ₂ ∈ \text{Lat}/∼, \text{ such that there exists } \mathcal{L}_1, \mathcal{L}_2 ∈ \text{Lat} \text{ with } [\mathcal{L}_i] = Λᵢ, \text{ and } \pi \mathcal{L}_1 ⊊ \mathcal{L}_2 ⊊ \mathcal{L}_1 \right\}

Sim₂(𝓥₃) = \left\{ Λ₁Λ₂Λ₃ \middle| Λ₁, Λ₂, Λ₃ ∈ \text{Lat}/∼, \text{ such that there exists } \mathcal{L}_1 ∈ \text{Lat} \text{ with } [\mathcal{L}_i] = Λᵢ, \text{ and } \pi \mathcal{L}_1 ⊊ \mathcal{L}_2 ⊊ \mathcal{L}_3 ⊊ \mathcal{L}_1 \right\}

and the obvious attaching maps.

2.1 Basic Properties of 𝓥₃

**Theorem 2.1.1** (Structure of the Apartments of 𝓥₃).

The apartments of 𝓥₃ are copies of two-dimensional Euclidean space, tiled by equilateral triangles. See Figure 2.1.

![Figure 2.1: An apartment of 𝓥₃](image)

**Definition 2.1.2.**

Given \( a_{i,j} ∈ \mathcal{F} \), let \( \mathcal{L} = \begin{pmatrix} a_{1,1} & a_{1,2} & \ldots & a_{1,n} \\ a_{2,1} & a_{2,2} & \ldots & a_{2,n} \\ a_{3,1} & a_{3,2} & \ldots & a_{3,n} \end{pmatrix} \) be the \( \mathcal{O} \)-lattice spanned by the columns of the matrix, with respect to the standard basis of \( \mathcal{F}^₃ \).
Proposition 2.1.3.

The group \( \text{GL}_3(\mathcal{F}) \) acts transitively on the set of all rank 3 lattices, and thus on \( \text{Sim}_{0}(\mathcal{X}_3) \).

Proof. The proof is directly analogous to the proof of Proposition 1.1.3

Given any two lattices, \( \mathcal{L} \) and \( \mathcal{L}' \), let \( \alpha \in \text{GL}(\mathcal{W}) \) be such that \( \alpha \mathcal{L} = \mathcal{L}' \). There exists bases \( \{e_1, e_2, e_3\} \) of \( \mathcal{L} \), and \( \{f_1, f_2, f_3\} \) of \( \mathcal{L}' \), such that the matrix of \( \alpha \) with respect to these bases is

\[
\alpha = \begin{pmatrix}
\pi^{n_1} & 0 & 0 \\
0 & \pi^{n_2} & 0 \\
0 & 0 & \pi^{n_3}
\end{pmatrix}
\]

were \( n_1, n_2, n_3 \in \mathbb{Z} \).^1 Thus there exists an \( \mathcal{O} \)-basis of \( \mathcal{L} \), \( \{e_1, e_2, e_3\} \), such that \( \{\pi^{n_1}e_1, \pi^{n_2}e_2, \pi^{n_3}e_3\} \) is an \( \mathcal{O} \)-basis for \( \mathcal{L}' \). Moreover, it can be shown that the set, \( \{n_1, n_2, n_3\} \) is independent of the choice of such a basis.

Proposition 2.1.4.

Let \( \mathcal{L}, \mathcal{M}, \mathcal{L}', \mathcal{M}' \) be lattices such that \( \mathcal{L} \sim \mathcal{L}' \) and \( \mathcal{M} \sim \mathcal{M}' \). Let \( \alpha, \beta \in \text{SL}_2(\mathcal{F}) \) be such that \( \alpha \mathcal{L} = \mathcal{M} \) and \( \beta \mathcal{L}' = \mathcal{M}' \). If \( \{n_1, n_2, n_3\} \) are the integers associated to \( \alpha \) and \( \{n'_1, n'_2, n'_3\} \) those associated to \( \beta \), then \( n_1 + n_2 + n_3 = n'_1 + n'_2 + n'_3 \pmod{3} \).

Proof. The proof is directly analogous to the proof of Proposition 1.1.4.

Corollary 2.1.5 (Colouring of vertices).

Choosing a distinguished vertex \( \Lambda \) induces a 3-colouring on the set of all vertices.

Theorem 2.1.6.

Let \( \mathcal{L} \) be a lattice and \( s \in \text{SL}(\mathcal{W}) \), then \( \chi(\mathcal{L}, s\mathcal{L}) = 0 \).

Proposition 2.1.7.

a) If \( G \subseteq \text{SL}(\mathcal{W}) \), then \( G_{\Lambda \Lambda'} = G_{\Lambda} \cap G_{\Lambda'} \), and \( G_{\Lambda \Lambda' \Lambda''} = G_{\Lambda} \cap G_{\Lambda'} \cap G_{\Lambda''} \).

---

^1This is called the Smith Normal form of \( \alpha \).
b) If $[\mathcal{L}] = \Lambda$, then $G_{\Lambda} = G_{\mathcal{L}}$.

2.2 The Quotient $\text{SL}_3(\mathbb{A}) \backslash \mathcal{X}_3$

In this section we will examine the quotient space $\text{SL}_3(\mathbb{A}) \backslash \mathcal{X}_3$. We will describe a fundamental domain for the action of $\text{SL}_3(\mathbb{A})$ on $\mathcal{X}_3$, and determine the quotient space $\text{SL}_3(\mathbb{A}) \backslash \mathcal{X}_3$ up to simplicial isomorphism. For the remainder of this section we denote $\text{SL}_3(\mathbb{A})$ by $\Gamma$.

Definition 2.2.1 (Fundamental Lattices).

For $n, m \in \mathbb{Z}_{\geq 0}$, let $\mathcal{L}_{n,m}$ be the lattice $\mathcal{L}_{n,m} = \langle \begin{pmatrix} t^{n+m} & 0 & 0 \\ 0 & t^n & 0 \\ 0 & 0 & 1 \end{pmatrix} \rangle$, and $\Lambda_{n,m} = [\mathcal{L}_{n,m}]$ the corresponding vertex in $\mathcal{X}_3$.

Let $\mathcal{D}$ denote the subcomplex of $\mathcal{X}_3$ which is spanned by the vertices $\{\Lambda_{n,m}\}_{n, m \geq 0}$. We will show that $\mathcal{D}$ is a fundamental domain for the action of $\text{SL}_3(\mathbb{A})$ on $\mathcal{X}_3$.

Theorem 2.2.2.

The $\Lambda_{n,m}$ are pairwise inequivalent modulo $\text{SL}_3(\mathbb{A})$.

Proof. Let $s = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} \in \text{SL}_3(\mathbb{A})$, and suppose that $s\Lambda_{n,m} = \Lambda_{n+h_1,m+h_2}$, for some $n, m, n+h_1, m+h_2 \geq 0$. We will show that $h_1 = h_2 = 0$. By assumption we have that $s\mathcal{L}_{n,m} = t^{-h} \mathcal{L}_{n+h_1,m+h_2}$ for some $h \in \mathbb{Z}$. By Theorem 2.1.6 we know that $\chi(\mathcal{L}_{n,m}, s\mathcal{L}_{n,m}) = \chi(\mathcal{L}_{n,m}, t^{-h} \mathcal{L}_{n+h_1,m+h_2}) = 0$, i.e.
0 = \chi(\mathcal{L}_{n,m}, s\mathcal{L}_{n,m})
= \chi(\mathcal{L}_{n,m}, t^{-h}\mathcal{L}_{n+1,m+h_2})
= l \left( \frac{\mathcal{L}_{n,m}}{\mathcal{L}_{n,m} \cap t^{-h}\mathcal{L}_{n+1,m+h_2}} \right) - l \left( \frac{t^{-h}\mathcal{L}_{n+1,m+h_2}}{\mathcal{L}_{n,m} \cap t^{-h}\mathcal{L}_{n+1,m+h_2}} \right)
= [2n + m - (\min(n + m, n + h_1 + m + h_2 - h) + \min(n, n + h_1 - h) + \min(0, -h))] - [2n + 2h_1 + m + h_2 - 3h - (\min(n + m, n + h_1 + m + h_2 - h) + \min(n, n + h_1 - h) + \min(0, -h))] 
= -2h_1 - h_2 + 3h.

Thus,

\[2h_1 + h_2 = 3h. \tag{2.1}\]

We now find degree restraints of the entries of the matrix \( s \). By assumption we have

\[
\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} \left\langle \begin{pmatrix} t^{n+m} & 0 & 0 \\ 0 & t^n & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle = t^{-h} \left\langle \begin{pmatrix} t^{n+h_1+m+h_2} & 0 & 0 \\ 0 & t^{n+h_2} & 0 \\ 0 & 0 & t^{-h} \end{pmatrix} \right\rangle,
\]

i.e.

\[
\left\langle \begin{pmatrix} a_{1,1}t^{n+m} & a_{1,2}t^n & a_{1,3} \\ a_{2,1}t^{n+m} & a_{2,2}t^n & a_{2,3} \\ a_{3,1}t^{n+m} & a_{3,2}t^n & a_{3,3} \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} t^{n+h_1+m+h_2-h} & 0 & 0 \\ 0 & t^{n+h_2-h} & 0 \\ 0 & 0 & t^{-h} \end{pmatrix} \right\rangle.
\]

Thus, there exists some \( P = (\alpha_{i,j}) \in \text{SL}_3(A) \), such that

\[
\begin{pmatrix} a_{1,1}t^{n+m} & a_{1,2}t^n & a_{1,3} \\ a_{2,1}t^{n+m} & a_{2,2}t^n & a_{2,3} \\ a_{3,1}t^{n+m} & a_{3,2}t^n & a_{3,3} \end{pmatrix} = \begin{pmatrix} t^{n+h_1+m+h_2-h} & 0 & 0 \\ 0 & t^{n+h_2-h} & 0 \\ 0 & 0 & t^{-h} \end{pmatrix} P
\]

\[
= \begin{pmatrix} \alpha_{1,1}t^{n+h_1+m+h_2-h} & \alpha_{1,2}t^{n+h_1+m+h_2-h} & \alpha_{1,3}t^{n+h_1+m+h_2-h} \\ \alpha_{2,1}t^{n+h_2-h} & \alpha_{2,2}t^{n+h_2-h} & \alpha_{2,3}t^{n+h_2-h} \\ \alpha_{3,1}t^{-h} & \alpha_{3,2}t^{-h} & \alpha_{3,3}t^{-h} \end{pmatrix},
\]

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thus

\[
\begin{pmatrix}
a_{1,1} & a_{1,2} & a_{1,3} \\
a_{2,1} & a_{2,2} & a_{2,3} \\
a_{3,1} & a_{3,2} & a_{3,3}
\end{pmatrix} =
\begin{pmatrix}
\alpha_{1,1} t^{h_2+h_1-h} & \alpha_{1,2} t^{h_1+m+h_2-h} & \alpha_{1,3} t^{n+h_1+m+h_2-h} \\
\alpha_{2,1} t^{h_1-m-h} & \alpha_{2,2} t^{h_1-h} & \alpha_{2,3} t^{n+h_1-h} \\
\alpha_{3,1} t^{-n-m-h} & \alpha_{3,2} t^{-n-h} & \alpha_{3,3} t^{-h}
\end{pmatrix}.
\] (2.2)

We will now show that

\[
\begin{align*}
h_2 + h_1 - h & \geq 0 \\
h_1 - h & \geq 0 \\
-h & \geq 0.
\end{align*}
\]

We then use the fact that

\[
(h_2 + h_1 - h) + (h_1 - h) + (-h) = 2h_1 + h_2 - 3h = 0,
\]

to conclude that \( h = h_1 = h_2 = 0 \).

Note, a non-singular \( n \times n \)-matrix cannot contain a \((n - k) \times (k + 1)\) sub-matrix of all zeros.\(^2\) We now consider the all \((3 - k) \times (k + 1)\) sub-matrices of \( s = (a_{i,j}) \) which contain \( a_{3,1} \).

The \(3 \times 1\) sub-matrix:

\[
\begin{pmatrix}
a_{1,1} \\
a_{2,1} \\
a_{3,1}
\end{pmatrix} =
\begin{pmatrix}
\alpha_{1,1} t^{h_2+h_1-h} \\
\alpha_{2,1} t^{-(m+h_2)+h_2+h_1-h} \\
\alpha_{3,1} t^{-(n+h_1)-(m+h_2)+h_2+h_1-h}
\end{pmatrix} \neq \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}.
\]

At least one of the \(a_{i,1} \neq 0\). For such an \( a_{i,1} \) we have

\[
0 \leq \deg_t a_{i,1} \leq \deg_t \alpha_{i,1} + (h_2 + h_1 - h) \leq (h_2 + h_1 - h),
\]

\(^2\)If a \( n \times n \)-matrix contained a \((n - k) \times (k + 1)\)-sub-matrix of all zeros, then it would have \((k + 1)\) columns whose span is at most \( k \)-dimensional, and thus not be invertible.
thus $h_2 + h_1 - h \geq 0$.

The $2 \times 2$ sub-matrix:

$$
\begin{pmatrix}
  a_{2,1} & a_{2,2} \\
  a_{3,1} & a_{3,2}
\end{pmatrix}
\begin{pmatrix}
  \alpha_{2,1} t^{-m+h_1-h} & \alpha_{2,2} t^{h_1-h} \\
  \alpha_{3,1} t^{-(n+h_1)-m+h_1-h} & \alpha_{3,2} t^{-(n+h_1)+h_1-h}
\end{pmatrix}
\neq
\begin{pmatrix}
  0 & 0 \\
  0 & 0
\end{pmatrix}.
$$

At least one of $a_{2,1}, a_{2,2}, a_{3,1}, a_{3,2}$ is not 0. For such an $a_{i,j}$ we have

$$
0 \leq \deg_t a_{i,j} \leq \deg_t \alpha_{i,j} + (h_1 - h) \leq (h_1 - h),
$$

thus $h_1 - h \geq 0$.

The $1 \times 3$ sub-matrix:

$$
\begin{pmatrix}
  a_{3,1} & a_{3,2} & a_{3,3}
\end{pmatrix}
\begin{pmatrix}
  \alpha_{3,1} t^{-n-m-h} & \alpha_{3,2} t^{-n-h} & \alpha_{3,3} t^{-h}
\end{pmatrix}
\neq
\begin{pmatrix}
  0 & 0 & 0
\end{pmatrix}.
$$

At least one of the $a_{3,j} \neq 0$. For such an $a_{3,j}$ we have

$$
0 \leq \deg_t a_{3,j} \leq \deg_t \alpha_{3,j} + (-h) \leq -h,
$$

and so $-h \geq 0$. Thus, $h = h_1 = h_2 = 0$, as required.

\begin{flushright}
\Box
\end{flushright}

**Theorem 2.2.3.**

*The vertex, edge, and face stabilisers of the simplices in $D$ are given by:*
\[
\Gamma_{\Lambda_{n,m}} = \begin{cases} 
\text{SL}_3(A) & \text{if } n = 0, \text{ and } m = 0, \\
\begin{cases} 
\begin{pmatrix} \mathbb{F}_q & \mathbb{F}_q & \leq n \\
\mathbb{F}_q & \mathbb{F}_q & \leq n \\
0 & 0 & \mathbb{F}_q 
\end{pmatrix} \subset \text{SL}_3(A) 
\end{cases} & \text{if } n > 0, \text{ and } m = 0, \\
\begin{cases} 
\begin{pmatrix} \mathbb{F}_q & \leq m & \leq m \\
0 & \mathbb{F}_q & \mathbb{F}_q \\
0 & \mathbb{F}_q & \mathbb{F}_q 
\end{pmatrix} \subset \text{SL}_3(A) 
\end{cases} & \text{if } n = 0, \text{ and } m > 0, \\
\begin{cases} 
\begin{pmatrix} \mathbb{F}_q & \leq m & \leq n + m \\
0 & \mathbb{F}_q & \leq n \\
0 & 0 & \mathbb{F}_q 
\end{pmatrix} \subset \text{SL}_3(A) 
\end{cases} & \text{if } n > 0, \text{ and } m > 0.
\end{cases}
\]
\[ \Gamma_{\Lambda_n, m \Lambda_{n+1}} = \left\{ \begin{array}{l}
\left( \begin{array}{c}
\mathbb{F}_q 
\leq m 
\leq m
\end{array} \right) 
\subset SL_3(A)
\end{array} \right\} \text{ if } n = 0
\]
\[ \Gamma_{\Lambda_n, m \Lambda_{n+1}} = \left\{ \begin{array}{l}
\left( \begin{array}{c}
\mathbb{F}_q 
\leq m 
\leq n + m
\end{array} \right) 
\subset SL_3(A)
\end{array} \right\} \text{ if } n > 0.
\]
\[ \Gamma_{\Lambda_n, m \Lambda_{n+1}} = \left\{ \begin{array}{l}
\left( \begin{array}{c}
\mathbb{F}_q 
\leq m 
\leq n
\end{array} \right) 
\subset SL_3(A)
\end{array} \right\} \text{ if } m = 0.
\]
\[ \Gamma_{\Lambda_n, m \Lambda_{n+1}} = \left\{ \begin{array}{l}
\left( \begin{array}{c}
\mathbb{F}_q 
\leq m 
\leq n + m + 1
\end{array} \right) 
\subset SL_3(A)
\end{array} \right\} \text{ if } m > 0.
\]

\[ \Gamma_{\Lambda_n, m \Lambda_{n+1}} = \left\{ \begin{array}{l}
\left( \begin{array}{c}
0
0
0
\end{array} \right) 
\subset SL_3(A)
\end{array} \right\} \text{ if } m = 0.
\]

\[ \Gamma_{\Lambda_n, m \Lambda_{n+1}} = \left\{ \begin{array}{l}
\left( \begin{array}{c}
0
0
0
\end{array} \right) 
\subset SL_3(A)
\end{array} \right\} \text{ if } m > 0.
\]

**Proof.** Suppose \( s = (a_{i,j}) \in SL_2(A), \) is such that \( s\Lambda_n = \Lambda_{n,m}. \) Then \( s \) must satisfy Equation (2.2) with \( h_1 = h_2 = h = 0. \) This gives the following degree constraints on the \( a_{i,j}: \)

\[ \Gamma_{\Lambda_n, m \Lambda_{n+1}} = \left\{ \begin{array}{l}
\left( \begin{array}{c}
\mathbb{F}_q 
\leq m 
\leq n + m
\end{array} \right) 
\subset SL_3(A)
\end{array} \right\} \text{ if } n = 0
\]
\[ \Gamma_{\Lambda_n, m \Lambda_{n+1}} = \left\{ \begin{array}{l}
\left( \begin{array}{c}
0
0
0
\end{array} \right) 
\subset SL_3(A)
\end{array} \right\} \text{ if } m = 0.
\]
Thus \( s \in \Gamma_{n,m} \). Conversely, one can show that every element of \( \Gamma_{n,m} \) is an element of the stabiliser by explicit calculation. For example, if \( n > 0 \), and \( m > 0 \):

\[
\begin{pmatrix}
  a_{1,1} & a_{1,2} & a_{1,3} \\
  0 & a_{2,2} & a_{2,3} \\
  0 & 0 & a_{3,3}
\end{pmatrix}
\begin{pmatrix}
  t^{n+m} & 0 & 0 \\
  0 & t^n & 0 \\
  0 & 0 & 1
\end{pmatrix}
= \begin{pmatrix}
  a_{1,1} t^{n+m} & a_{1,2} t^n & a_{1,3} \\
  0 & a_{2,2} t^n & a_{2,3} \\
  0 & 0 & a_{3,3}
\end{pmatrix}
\]

The other cases are similar so we omit them.

The edge and face stabilisers follow directly from the vertex stabilisers and Proposition 2.1.7.

**Definition 2.2.4** (Fundamental Domain).

Let \( \Delta \) be a simplicial complex of dimension \( n \), and let \( G \) be a group which acts on \( \Delta \). A fundamental domain for the action of \( G \) on \( \Delta \) is a subcomplex \( D \subseteq \Delta \) such that:

**Condition 1:** If \( \sigma \in \text{Sim}_i(D) \), then for all \( g \in G \), \( g\sigma \in \text{Sim}_i(D) \) \( \implies \sigma = g\sigma \)

**Condition 2:** For every \( \sigma \in \text{Sim}_i(D) \), there exits a \( g \in G \) such that \( g\sigma \in \text{Sim}_i(D) \)
for $i = 0, 1, 2, \ldots, n$.

We will show that $\mathcal{D} \subset \mathfrak{X}_3$ is a fundamental domain for the action of $\Gamma$ on $\mathfrak{X}_3$. But first note that since $\mathfrak{X}_3$ is homogeneous and $\Gamma$ acts simplicially, to show that $\mathcal{D} \subseteq \mathfrak{X}$ is a fundamental domain for $\Gamma$ it suffices to check the conditions of Definition 2.2.4 for $i = 2$ only.

**Theorem 2.2.5 (Fundamental Domain).**

The subcomplex $\mathcal{D} \subset \mathfrak{X}_3$ is a fundamental domain for the action of $\Gamma = \text{SL}_3(\mathbb{A})$ on $\mathfrak{X}_3$. Furthermore, the quotient space $\Gamma \backslash \mathfrak{X}_3$ is simplicially isomorphic to $\mathcal{D}$.

**Proof.** It suffices to show that $\mathcal{D}$ satisfies both the conditions of Definition 2.2.4 for 2-simplices.

**CONDITION 1:** This follows directly from Theorem 2.2.2.

**CONDITION 2:** We will show that one can “fold-up” $\mathfrak{X}_3$ onto $\mathcal{D}$. More specifically, we will show that every 2-simplex which has an edge in $\mathcal{D}$ can be “folded”, via an element of $\Gamma$, onto a 2-simplex in $\mathcal{D}$. This is sufficient to show that $\mathcal{D}$ satisfies CONDITION 2 since any 2-simplex can be taken to a 2-simplex in $\mathcal{D}$ by an appropriate finite series of “foldings”.

**CASE 1:** Edges of the form $\Lambda_{0,m}\Lambda_{0,m+1}$, with $m \geq 0$

The 2-simplices containing the edge $\Lambda_{0,m}\Lambda_{0,m+1}$, for $m \geq 0$, correspond to vertices $\Lambda$ such that

$$\Lambda_{0,m} \subsetneq \Lambda_{0,m+1} \subsetneq \Lambda \subsetneq t\Lambda_{0,m}.$$  

Such vertices come from lattices of the form,

$$\begin{pmatrix} tm+1 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} tm+1 & 0 & 0 \\ 0 & 1 & \lambda t \\ 0 & 0 & t \end{pmatrix}, \quad \text{with } \lambda \in \mathbb{F}_q.$$
It is clear that $\Gamma_{A_0,A_0+1} = \left( \begin{array}{ccc} \mathbb{F}_q & \leq m & \leq m \\ 0 & \mathbb{F}_q & \mathbb{F}_q \\ 0 & \mathbb{F}_q & \mathbb{F}_q \end{array} \right) \subset SL_3(A)$ acts transitively on these lattices.

**Case 2: Edges of the form $\Lambda_n,0 \Lambda_{n+1,0}$, with $n \geq 0$**

The 2-simplices containing the edge $\Lambda_n,0 \Lambda_{n+1,0}$, for $n \geq 0$, correspond to vertices $\Lambda$, such that,

$$\Lambda_n,0 \varsubsetneq \Lambda \varsubsetneq \Lambda_{n+1,0} \varsubsetneq t\Lambda_n,0.$$  

Such vertices come from lattices of the form,

$$\left( \begin{array}{ccc} t^{n+1} & 0 & 0 \\ 0 & t^n & 0 \\ 0 & 0 & 1 \end{array} \right) \text{ and } \left( \begin{array}{ccc} t^n & \lambda t^{n+1} & 0 \\ 0 & t^{n+1} & 0 \\ 0 & 0 & 1 \end{array} \right), \quad \text{with } \lambda \in \mathbb{F}_q.$$  

It is clear that $\Gamma_{A_0,A_n+1} = \left( \begin{array}{ccc} \mathbb{F}_q & \mathbb{F}_q & \leq n \\ 0 & \mathbb{F}_q & \mathbb{F}_q \leq n \\ 0 & 0 & \mathbb{F}_q \end{array} \right) \subset SL_3(A)$ acts transitively on these lattices.

**Case 3: Edges of the form $\Lambda_{n,m} \Lambda_{n,m+1}$, with $n > 0$, and $m \geq 0$**

The 2-simplices containing the edge $\Lambda_{n,m} \Lambda_{n,m+1}$, for $n > 0$, $m \geq 0$, correspond to vertices $\Lambda$ such that,

$$\Lambda_n,0 \varsubsetneq \Lambda \varsubsetneq \Lambda_{n+1,0} \varsubsetneq t\Lambda_n,0.$$  

Such vertices come from lattices of the form,

$$\Lambda_{n+1,m} = \left\langle \begin{array}{ccc} t^{n+m+1} & 0 & 0 \\ 0 & t^{n+1} & 0 \\ 0 & 0 & 1 \end{array} \right\rangle \text{ or } \left\langle \begin{array}{ccc} t^{n+m+1} & 0 & 0 \\ 0 & t^n & \lambda t^{n+1} \\ 0 & 0 & t \end{array} \right\rangle, \quad \text{with } \lambda \in \mathbb{F}_q.$$  

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The edge stabiliser $\Gamma_{n,m,\Lambda_{n,m+1}}$ stabilises the 2-simplex, $\Lambda_{n,m}\Lambda_{n,m+1}\Lambda_{n+1,m}$; and acts transitively on the set of all other 2-simplices with edge $\Lambda_{n,m}\Lambda_{n,m+1}$. To see this note that a general element of the stabiliser acts as follows,

$$
\begin{pmatrix}
a_{1,1} & a_{1,2} & a_{1,3} \\
0 & a_{2,2} & a_{2,3} \\
0 & 0 & a_{3,3}
\end{pmatrix}
\begin{pmatrix}
t^{n+m+1} & 0 & 0 \\
0 & t^n & 0 \\
0 & 0 & t
\end{pmatrix}
= 
\begin{pmatrix}
a_{1,1}t^{n+m+1} & a_{1,2}t^n & a_{1,3}t \\
0 & a_{2,2}t^n & a_{2,3}t \\
0 & 0 & a_{3,3}t
\end{pmatrix}.
$$

In particular,

$$
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & \lambda t^n \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
t^{n+m+1} & 0 & 0 \\
0 & t^n & 0 \\
0 & 0 & t
\end{pmatrix}
= 
\begin{pmatrix}
t^{n+m+1} & 0 & 0 \\
0 & t^n & \lambda t^{n+1} \\
0 & 0 & t
\end{pmatrix}.
$$
The 2-simplices containing the edge $\Lambda_{n,m+1} \Lambda_{n+1,m}$, $n \geq 0$, $m \geq 0$, correspond to vertices $\Lambda$ such that,

$$t^{-1} \Lambda_{n+1,m} \subseteq \Lambda \subseteq \Lambda_{n,m+1} \subseteq \Lambda_{n+1,m}.$$  

Such vertices come from lattices of the form,

$$\Lambda_{n+1,m+1} = \begin{pmatrix} t^{n+m+1} & 0 & 0 \\ 0 & t^n & 0 \\ 0 & 0 & t^{-1} \end{pmatrix} \text{ or } \begin{pmatrix} t^{n+m} & 0 & \lambda t^{n+m+1} \\ 0 & t^n & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{with } \lambda \in \mathbb{F}_q.$$  

The edge stabiliser $\Gamma_{\Lambda_{n,m+1} \Lambda_{n+1,m}}$ stabilises the 2-simplex $\Lambda_{n,m+1} \Lambda_{n+1,m} \Lambda_{n+1,m+1}$, and acts transitively on the set of all other 2-simplices with edge $\Lambda_{n,m+1} \Lambda_{n+1,m}$. To
see this note that,

\[
\begin{pmatrix}
1 & 0 & \lambda t^{n+m+1} \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
t^{n+m} & 0 & 0 \\
0 & t^n & 0 \\
0 & 0 & 1
\end{pmatrix}
= 
\begin{pmatrix}
t^{n+m} & 0 & \lambda t^{n+m+1} \\
0 & t^n & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

**CASE 5:** EDGES OF THE FORM \( \Lambda_{n,m} \Lambda_{n+1,m} \), WITH \( n \geq 0 \), AND \( m > 0 \)

A similar argument to above shows that, for \( m > 0 \), the stabiliser of the edge \( \Lambda_{n,m} \Lambda_{n+1,m} \), fixes the 2-simplex, \( \Lambda_{n,m} \Lambda_{n+1,m} \Lambda_{n,m+1} \), and acts transitively on all other 2-simplices containing \( \Lambda_{n,m} \Lambda_{n+1,m} \).

Thus, \( D \) is a fundamental domain for the action of \( \Gamma \) on \( X_3 \).  

\( \square \)
2.3 Quotients by full congruence subgroups \( \Gamma(g) \subseteq \text{SL}_3(A) \)

Let \( g \in A \) be non-zero, and let \( \Gamma(g) \) denote the full congruence subgroup of \( \text{SL}_2(A) \) of level \( g \) i.e.

\[
\Gamma(g) \overset{\text{def}}{=} \left\{ \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} \in \text{SL}_2(A) \mid \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \pmod{g} \right\}.
\]

Similarly to what we did in Section 1.4, we will study \( \Gamma(g) \setminus \mathcal{X}_3 \) by studying the quotient map \( \rho: \Gamma(g) \setminus \mathcal{X}_3 \to \mathcal{X}_3 \). We denote the quotient \( \Gamma(g) \setminus \mathcal{X}_3 \) by \( \mathcal{X}_3(g) \).

**Definition 2.3.1.**

Identify \( \Gamma \setminus \mathcal{X}_3 \) with \( \mathcal{D} \), and let \( \sigma \) be a simplex of \( \mathcal{D} \). Define

\[
\{ \mathcal{X}_3(g) \}_\sigma = \{ \text{Simplices in } \mathcal{X}_3(g), \text{ lying above the simplex } \sigma \}.
\]

We will calculate the size of the sets \( \{ \mathcal{X}_3(g) \}_\sigma \), for \( \sigma \) the vertices, edges, and faces of \( \mathcal{D} \). The following theorem is a direct analogy of Theorem 1.4.2. We omit the proof as it is essentially exactly the same as the proof of Theorem 1.4.2.

**Theorem 2.3.2.**

*Let \( \sigma \) be a simplex of \( \mathcal{D} \). Then*

\[
\{ \mathcal{X}_3(g) \}_\sigma \overset{\text{def}}{=} \frac{\Gamma}{\Gamma(g)} \frac{\Gamma}{\Gamma(g) \sigma},
\]

*in particular we have that*

\[
\# \{ \mathcal{X}_3(g) \}_\sigma = \frac{[\Gamma : \Gamma(g)]}{[\Gamma_\sigma : \Gamma(g) \sigma]}.
\]
Before we use Theorem 2.3.2 to calculate \( \{X_3(g)\}_\sigma \) for \( \sigma \in \mathcal{D} \), we first derive a general expression for the index \([\Gamma: \Gamma(g)]\).

**Theorem 2.3.3.**

Let \( g \in A \) with \( \deg_t(g) = N \), and assume that \( g \) factors as \( g = \prod_{i=1}^{k} g_i^{e_i} \) where the \( g_i \in A \) are distinct, irreducible, and \( \deg_t(g_i) = d_i \). Then

\[
[\Gamma: \Gamma(g)] = q^{8N} \prod_{i=1}^{k} \left(1 - \frac{1}{q^{3d_i}}\right) \left(1 - \frac{1}{q^{2d_i}}\right).
\]

(2.8)

**Proof.** We break the proof up into multiple steps:

**Step 1.** Show that \([\Gamma: \Gamma(g)] = \# \text{SL}_3\left(\mathbb{F}_q g\right)\)

**Step 2.** Reduce to the case \( \# \text{SL}_3\left(\mathbb{F}_q g^e\right) \) for \( g \) irreducible

**Step 3.** Show that \( \# \text{SL}_3\left(\mathbb{F}_q g^e\right) = \frac{\# \text{GL}_3\left(\mathbb{F}_q g^e\right)}{\# \left(\mathbb{F}_q g^e\right)^{\times}} \)

**Step 4.** Show that \( \left(\mathbb{F}_q g^e\right)^{\times} = q^{(e-1)d}(q^d - 1) \)

**Step 5.** Show that \( \# \text{GL}_3\left(\mathbb{F}_q g^e\right) = q^{9(e-1)d}(q^{3d} - 1)(q^{3d} - q^d)(q^{3d} - q^{2d}) \)

**Step 6.** Conclude that \( \# \text{SL}_3\left(\mathbb{F}_q g^e\right) = q^{8ed} \left(1 - \frac{1}{q^{3d}}\right) \left(1 - \frac{1}{q^{2d}}\right) \)

**Step 1.** This is a direct consequence of the following short exact sequence

\[
1 \longrightarrow \Gamma(g) \longrightarrow \Gamma \longrightarrow \text{SL}_3\left(\mathbb{F}_q g\right) \longrightarrow 1.
\]

**Step 2.** This is a consequence of the Chinese Remainder Theorem for \( \text{SL}_3 \), i.e.

\[
\text{SL}_3\left(\mathbb{F}_q\right) \cong \prod_{i=1}^{k} \text{SL}_3\left(\mathbb{F}_q g_i^{e_i}\right).
\]

**Step 3.** This is a direct consequence of the following short exact sequence

\[
1 \longrightarrow \text{SL}_3\left(\mathbb{F}_q g^e\right) \longrightarrow \text{GL}_3\left(\mathbb{F}_q g^e\right) \longrightarrow \left(\mathbb{F}_q g^e\right)^{\times} \longrightarrow 1.
\]
STEP 4. It is straightforward to see that \( \left( \mathbb{A}/g^e \right)^{\times} = \{ a \in \mathbb{A}/g^e \mid a \not\equiv 0 \pmod{g} \} \).

Thus \( \# \left( \mathbb{A}/g^e \right)^{\times} = q^{ed} - q^{(e-1)d} = q^{(e-1)d}(q^d-1) \).

STEP 5. The reduction map \( \rho: f \pmod{g^e} \longmapsto f \pmod{g} \) induces a surjection

\[
\hat{\rho}: \text{GL}_3 \left( \mathbb{A}/g^e \right) \longrightarrow \text{GL}_3 \left( \mathbb{A}/g \right).
\]

Thus \( \# \text{GL}_3 \left( \mathbb{A}/g^e \right) = \# \ker(\hat{\rho}) \times \# \text{GL}_3 \left( \mathbb{A}/g \right) \). It is straightforward to see that

\( \# \text{GL}_3 \left( \mathbb{A}/g \right) = (q^{3d} - 1)(q^{3d} - q^d)(q^{3d} - q^{2d}) \). The kernel of the map is \( \ker(\hat{\rho}) = \{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + A \mid A \in g \text{M}_{3,3} \left( \mathbb{A}/g^e \right) \} \), which has cardinality \( \# \ker(\hat{\rho}) = q^{9(e-1)d} \). Thus

\( \# \text{GL}_3 \left( \mathbb{A}/g^e \right) = q^{9(e-1)d}(q^{3d} - 1)(q^{3d} - q^d)(q^{3d} - q^{2d}) \).

STEP 6. From STEP 3., STEP 4., and STEP 5. we have that

\[
\# \text{SL}_3 \left( \mathbb{A}/g^e \right) = \frac{q^{9(e-1)d}(q^{3d} - 1)(q^{3d} - q^d)(q^{3d} - q^{2d})}{q^{(e-1)d}(q^d - 1)}
\]

\[
= q^{8(e-1)d}(q^{3d} - 1)(q^{3d} - q^d)q^{2d}
\]

\[
= q^{3ed} \left( 1 - \frac{1}{q^{3d}} \right) \left( 1 - \frac{1}{q^{2d}} \right).
\]

We now calculate the stabiliser indices \( [\Gamma_\sigma : \Gamma(g)_\sigma] \) for \( \sigma \in \mathcal{D} \). To simplify the formulas we define the following functions:

\[
\phi(n, m, N) = q^{\min(n+1, N)}q^{\min(m+1, N)}q^{\min(n+m+1, N)}
\]

\[
\psi(n, m, N) = q^{\min(n+1, N)}q^{\min(m+1, N)}q^{\min(n+m+2, N)}.
\]

**Proposition 2.3.4.**

Let \( g \in \mathbb{A} \) with \( \deg_t(g) = N > 0 \), and assume that \( g \) factors as \( g = \prod_{i=1}^k g_i^{e_i} \) where the \( g_i \in \mathbb{A} \) are distinct, irreducible, and \( \deg_t(g_i) = d_i \). Let \( \sigma \in \mathcal{D} \) be a simplex, then the stabiliser indices \( [\Gamma_\sigma : \Gamma(g)_\sigma] \) are given in Table 2.1\(^3\).

\(^3\) The term “xor” used in the table means “exclusive or”.

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Table 2.1: A table of the indices of the stabiliser subgroups of $\Gamma$ and $\Gamma(g)$

<table>
<thead>
<tr>
<th>$\sigma \in \mathcal{D}$</th>
<th>$[\Gamma_{\sigma} : \Gamma(g)_{\sigma}]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Lambda_{n,m}$</td>
<td>$q^3(q^3 - 1)(q^2 - 1)$</td>
</tr>
<tr>
<td></td>
<td>$\phi(n, m, N)(q^2 - 1)(q - 1)$</td>
</tr>
<tr>
<td></td>
<td>$\phi(n, m, N)(q^2 - 1)(q - 1)^2$</td>
</tr>
<tr>
<td></td>
<td>if $n = 0$, and $m = 0$</td>
</tr>
<tr>
<td></td>
<td>if $n &gt; 0$, xor $m &gt; 0$</td>
</tr>
<tr>
<td></td>
<td>if $n &gt; 0$, and $m &gt; 0$</td>
</tr>
<tr>
<td>$\Lambda_{n,m}(\Lambda_{n,m+1}$</td>
<td>$\phi(n, m, N)(q^2 - 1)(q^2 - 1)(q - 1)$</td>
</tr>
<tr>
<td></td>
<td>$\phi(n, m, N)(q^2 - 1)(q - 1)^2$</td>
</tr>
<tr>
<td></td>
<td>if $n = 0$</td>
</tr>
<tr>
<td></td>
<td>if $n &gt; 0$</td>
</tr>
<tr>
<td>$\Lambda_{n,m}(\Lambda_{n+1,m}$</td>
<td>$\phi(n, m, N)(q^2 - 1)(q^2 - 1)(q - 1)$</td>
</tr>
<tr>
<td></td>
<td>$\phi(n, m, N)(q^2 - 1)(q - 1)^2$</td>
</tr>
<tr>
<td></td>
<td>if $m = 0$</td>
</tr>
<tr>
<td></td>
<td>if $m &gt; 0$</td>
</tr>
<tr>
<td>$\Lambda_{n+1,m}(\Lambda_{n,m+1,m}$</td>
<td>$\psi(n, m, N)(q^2 - 1)(q - 1)$</td>
</tr>
<tr>
<td></td>
<td>$\psi(n, m, N)(q^2 - 1)(q - 1)^2$</td>
</tr>
</tbody>
</table>

Proof. Using Equation (2.3), Equation (2.4), and Equation (2.5) we calculate the cardinality of the stabiliser subgroups of $\Gamma$, as seen in Table 2.2

We now calculate the stabiliser subgroups of $\Gamma(g)$. The vertex stabiliser subgroups of $\Gamma(g)$ are given by:
\[
\begin{array}{ll}
\#\Gamma_{\Lambda_{n,m}} & q^3(q^3 - 1)(q^2 - 1) \quad \text{if } n = 0, \text{ and } m = 0 \\
& q^n q^m q^{n+m} q^3(q^2 - 1)(q - 1) \quad \text{if } n > 0, \text{ xor } m > 0 \\
& q^n q^m q^{n+m} q^3(q - 1)^2 \quad \text{if } n = 0, \text{ and } m > 0 \\
\#\Gamma_{\Lambda_{n,m}\Lambda_{n,m+1}} & q^n q^m q^{n+m} q^3(q^2 - 1)(q - 1) \quad \text{if } n = 0 \\
& q^n q^m q^{n+m} q^3(q - 1)^2 \quad \text{if } n > 0 \\
\#\Gamma_{\Lambda_{n+1,m}\Lambda_{n+1,m}} & q^n q^m q^{n+m} q^3(q - 1)^2 \quad \text{if } n > 0 \\
\#\Gamma_{\Lambda_{n,m+1}\Lambda_{n,m+1}\Lambda_{n+1,m}} & q^n q^m q^{n+m} q^3(q - 1)^2 \\
\#\Gamma_{\Lambda_{n+1,m+1}\Lambda_{n,m+1}\Lambda_{n+1,m}} & q^n q^m q^{n+m} q^4(q - 1)^2 \\
\end{array}
\]

Table 2.2: A table of the cardinalities of the stabiliser subgroups of $\Gamma$

\[
\Gamma(g)_{\Lambda_{n,m}} = \begin{cases}
\{ \text{Id} \} & \text{if } n = 0, \text{ } m = 0 \\
\begin{cases}
\begin{pmatrix} 1 & 0 & g\beta \\ 0 & 1 & g\gamma \\ 0 & 0 & 1 \end{pmatrix} & \text{deg}_t(\beta) \leq n - N \\
\begin{pmatrix} 1 & g\alpha & g\beta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{deg}_t(\gamma) \leq n - N \\
\end{cases} & \text{if } n > 0, \text{ } m = 0 \\
\begin{cases}
\begin{pmatrix} 1 & g\alpha & g\beta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{deg}_t(\alpha) \leq m - N \\
\begin{pmatrix} 1 & g\alpha & g\beta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{deg}_t(\beta) \leq m - N \\
\end{cases} & \text{if } n = 0, \text{ } m > 0 \\
\begin{cases}
\begin{pmatrix} 1 & g\alpha & g\beta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{deg}_t(\alpha) \leq m - N \\
\begin{pmatrix} 1 & g\alpha & g\beta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{deg}_t(\beta) \leq n + m - N \\
\end{cases} & \text{if } n > 0, \text{ } m > 0 \\
\begin{cases}
\begin{pmatrix} 1 & g\alpha & g\beta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{deg}_t(\alpha) \leq m - N \\
\begin{pmatrix} 1 & g\alpha & g\beta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{deg}_t(\beta) \leq n + m - N \\
\end{cases} & \text{if } n = 0, \text{ } m > 0 \\
\end{cases}
\]

Taking the cardinality of these groups gives:
\[
\#\Gamma(g)_{\Lambda_{n,m}} = \begin{cases} 
1 & \text{if } n + m < N \\
q^{n+m-N+1} & \text{if } n + m \geq N, \ n < N, \ m < N \\
q^{n+m-N+1}q^{-N+1} & \text{if } n \geq N, \ m < N \\
q^{n+m-N+1}q^{m-N+1} & \text{if } n < N, \ m \geq N \\
q^{n+m-N+1}q^{-N+1}q^{m-N+1} & \text{if } n \geq N, \ m \geq N 
\end{cases}
\]

\[
= q^{\max(n+1-N,0)}q^{\max(m+1-N,0)}q^{\max(n+m+1-N,0)}
\]

\[
= q^nq^m q^{\max(1-N, -n)}q^{\max(1-N, -m)}q^{\max(1-N, -n-m)}
\]

\[
= q^nq^m q^{-\min(n, N-1)}q^{-\min(m, N-1)}q^{-\min(n+m, N-1)}.
\]

Taking the quotient \(\#\Gamma_{\Lambda_{n,m}}/\#\Gamma(g)_{\Lambda_{n,m}}\) gives:

\[
[\Gamma_{\Lambda_{n,m}} : \Gamma(g)_{\Lambda_{n,m}}] = \begin{cases} 
q^3(q^3 - 1)(q^2 - 1) & \text{if } n = 0, \text{ and } m = 0 \\
q^{\min(n, N-1)}q^{\min(m, N-1)}q^{\min(n+m, N-1)}q^3(q^2 - 1)(q - 1) & \text{if } n > 0, \text{ xor } m > 0 \\
q^{\min(n, N-1)}q^{\min(m, N-1)}q^{\min(n+m, N-1)}q^3(q - 1)^2 & \text{if } n > 0, \text{ and } m > 0.
\end{cases}
\]

A similar calculation for the edge stabilisers gives:
#Γ(\(g\))_{\Lambda_{n,m}, \Lambda_{n,m+1}} =\begin{cases} 
1 & \text{if } n + m < N \\
q^{n+m-N+1} & \text{if } n + m \geq N, n < N, m < N \\
q^{n+m-N+1}q^{n-N+1} & \text{if } n \geq N, m < N \\
q^{n+m-N+1}q^{m-N+1} & \text{if } n < N, m \geq N \\
q^{n+m-N+1}q^{n-N+1}q^{m-N+1} & \text{if } n \geq N, m \geq N 
\end{cases}
= q^n q^m q^{n+m - \min(n, N-1)} q^{\min(m, N-1)} q^{-\min(n+m, N-1)}

#Γ(\(g\))_{\Lambda_{n,m}, \Lambda_{n+1,m}} =\begin{cases} 
1 & \text{if } n + m < N \\
q^{n+m-N+1} & \text{if } n + m \geq N, n < N, m < N \\
q^{n+m-N+1}q^{n-N+1} & \text{if } n \geq N, m < N \\
q^{n+m-N+1}q^{m-N+1} & \text{if } n < N, m \geq N \\
q^{n+m-N+1}q^{n-N+1}q^{m-N+1} & \text{if } n \geq N, m \geq N 
\end{cases}
= q^n q^m q^{n+m - \min(n, N-1)} q^{\min(m, N-1)} q^{-\min(n+m, N-1)}

#Γ(\(g\))_{\Lambda_{n+1,m}, \Lambda_{n+1,m+1}} =\begin{cases} 
1 & \text{if } n + m + 1 < N \\
q^{n+m+2-N} & \text{if } n + m + 1 \geq N, n < N, m < N \\
q^{n+m+2-N}q^{n-N+1} & \text{if } n + m + 1 \geq N, n \geq N, m < N \\
q^{n+m+2-N}q^{m-N+1} & \text{if } n + m + 1 \geq N, n < N, m \geq N \\
q^{n+m+2-N}q^{n-N+1}q^{m-N+1} & \text{if } n + m + 1 \geq N, n \geq N, m \geq N 
\end{cases}
= q^n q^m q^{n+m - \min(n, N-1)} q^{\min(m, N-1)} q^{-\min(n+m, N-2)}.

Taking the appropriate quotients gives:
\[
\Gamma_{n,m} \Lambda_{n,m+1} : \Gamma(g) \Lambda_{n,m} \Lambda_{n,m+1} = \begin{cases} 
q^{\min(n, N-1)} q^{\min(m, N-1)} q^{\min(n+m, N-1)} q^3(q^2 - 1)(q - 1) & \text{if } n = 0 \\
q^{\min(n, N-1)} q^{\min(m, N-1)} q^{\min(n+m, N-1)} q^3(q - 1)^2 & \text{if } n > 0 
\end{cases}
\]

\[
\Gamma_{n,m} \Lambda_{n+1,m} \Lambda_{n+1,m} : \Gamma(g) \Lambda_{n,m} \Lambda_{n+1,m} = \begin{cases} 
q^{\min(n, N-1)} q^{\min(m, N-1)} q^{\min(n+m, N-1)} q^3(q - 1)^2 & \text{if } m = 0 \\
q^{\min(n, N-1)} q^{\min(m, N-1)} q^{\min(n+m, N-1)} q^3(q - 1)^2 & \text{if } m > 0 
\end{cases}
\]

\[
\Gamma_{n,m+1} \Lambda_{n,m+1} : \Gamma(g) \Lambda_{n,m} \Lambda_{n,m+1} = q^{\min(n, N-1)} q^{\min(m, N-1)} q^{\min(n+m, N-2)} q^4(q - 1)^2.
\]

A similar calculation for the face stabilisers gives:

\[
\#\Gamma_{n,m} \Lambda_{n,m+1} \Lambda_{n+1,m} = q^n q^m q^{n+m} q^{\min(n, N-1)} q^{\min(m, N-1)} q^{\min(n+m, N-1)}
\]

\[
\#\Gamma_{n+1,m} \Lambda_{n,m+1} \Lambda_{n+1,m} = q^n q^m q^{n+m} q^{\min(n, N-1)} q^{\min(m, N-1)} q^{\min(n+m, N-2)}.
\]

Taking the appropriate quotients gives:

\[
\Gamma_{n,m} \Lambda_{n,m+1} \Lambda_{n+1,m} : \Gamma(g) \Lambda_{n,m} \Lambda_{n,m+1} \Lambda_{n+1,m} = q^{\min(n, N-1)} q^{\min(m, N-1)} q^{\min(n+m, N-1)} q^3(q - 1)^2
\]

\[
\Gamma_{n+1,m} \Lambda_{n+1,m} \Lambda_{n+1,m} : \Gamma(g) \Lambda_{n+1,m} \Lambda_{n+1,m} = q^{\min(n, N-1)} q^{\min(m, N-1)} q^{\min(n+m, N-2)} q^4(q - 1)^2
\]

We are now ready to calculate some examples.
Example 2.3.5 \((\mathcal{X}_3(t))\). Let \(\sigma \in \mathcal{D}\). We will consider \(#\{\mathcal{X}_3(t)\}_\sigma\). By Equation (2.8) the index is \([\Gamma: \Gamma(t)] = q^8(1 - q^{-3})(1 - q^{-2}) = q^3(q^3 - 1)(q^2 - 1)\). The number of simplices in \(\mathcal{X}_3(t)\) which lie above \(\sigma\) is given in Table 2.3.

<table>
<thead>
<tr>
<th>(\sigma \in \mathcal{D})</th>
<th>(#{\mathcal{X}<em>3(t)}</em>\sigma)</th>
<th>Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\Lambda_{n,m})</td>
<td>1</td>
<td>if (n = 0), and (m = 0)</td>
</tr>
<tr>
<td>(\Lambda_{n,m}\Lambda_{n,m+1})</td>
<td>((q^2 + q + 1))</td>
<td>if (n &gt; 0), xor (m &gt; 0)</td>
</tr>
<tr>
<td>(\Lambda_{n,m}\Lambda_{n+1,m})</td>
<td>((q^2 + q + 1)(q + 1))</td>
<td>if (m &gt; 0)</td>
</tr>
<tr>
<td>(\Lambda_{n,m+1}\Lambda_{n+1,m})</td>
<td>((q^2 + q + 1))</td>
<td>if (n = 0)</td>
</tr>
<tr>
<td>(\Lambda_{n,m+1}\Lambda_{n,m+1}\Lambda_{n+1,m})</td>
<td>((q^2 + q + 1)(q + 1))</td>
<td></td>
</tr>
</tbody>
</table>

Table 2.3: A table of the number of simplices in \(\mathcal{X}_3(t)\) which lie above a given simplex in \(\mathcal{D}\)

Example 2.3.6 \((\mathcal{X}_3(t^N))\). Let \(\sigma \in \mathcal{D}\). We will consider \(#\{\mathcal{X}_3(t^N)\}_\sigma\) for a general \(N > 0\). By Equation (2.8) the index is \([\Gamma: \Gamma(t^N)] = q^{8N}(1 - q^{-3})(1 - q^{-2}) = q^{8N-5}(q^3 - 1)(q^2 - 1)\). The number of simplices in \(\mathcal{X}_3(t^N)\) which lie above \(\sigma\) is given in Table 2.4\(^4\).

\(^4\)The term “xor” used in the table means “exclusive or”.

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Table 2.4: A table of the number of simplices in $\mathcal{X}_3(t^N)$ which lie above a given simplex in $D$
Theorem 2.3.7.

Let $g \in A$ with $\deg_t(g) = N$, and assume that $g$ factors as $g = \prod_{i=1}^{k} g_i^{e_i}$, where the $g_i \in A$ are distinct, irreducible, and $\deg_t(g_i) = d_i$. Let $\sigma \in D$, and

\[
\phi(n, m, N) = q^{\min(n+1, N)} q^{\min(m+1, N)} q^{\min(n+m+1, N)} \\
\psi(n, m, N) = q^{\min(n+1, N)} q^{\min(m+1, N)} q^{\min(n+m+2, N)}.
\]

Then the number of simplices in $X_3(g)$ lying the simplices $\sigma \in D$ are given in Table 2.5.

Proof. The proof is directly analogous to the $SL_2$ case, just the equations are bigger and there are more cases to consider. \qed
\[
\begin{array}{ccc}
\sigma \in \mathcal{D} & \# \{ \mathcal{X}_3(g) \}_\sigma \\
\Lambda_{n, m} & \frac{q^N \prod_{i=1}^{k} (1 - q^{-3d_i})(1 - q^{-2d_i})}{q^3(q^2 - 1)(q - 1)} & \text{if } n = 0, \text{ and } m = 0 \\
\Lambda_{n, m} \Lambda_{n+1, m} & \frac{q^N \prod_{i=1}^{k} (1 - q^{-3d_i})(1 - q^{-2d_i})}{\phi(n, m, N)(q^2 - 1)(q - 1)} & \text{if } n > 0, \text{ xor } m > 0 \\
\Lambda_{n+1, m} & \frac{q^N \prod_{i=1}^{k} (1 - q^{-3d_i})(1 - q^{-2d_i})}{q^3(q^2 - 1)(q - 1)} & \text{if } n > 0, \text{ and } m > 0 \\
\Lambda_{n+1, m} \Lambda_{n, m+1} \Lambda_{n+1, m} & \frac{q^N \prod_{i=1}^{k} (1 - q^{-3d_i})(1 - q^{-2d_i})}{\psi(n, m, N)(q - 1)^2} & \\
\Lambda_{n, m+1} \Lambda_{n+1, m} & \frac{q^N \prod_{i=1}^{k} (1 - q^{-3d_i})(1 - q^{-2d_i})}{\psi(n, m, N)(q - 1)^2} & \\
\Lambda_{n+1, m+1} & \frac{q^N \prod_{i=1}^{k} (1 - q^{-3d_i})(1 - q^{-2d_i})}{\psi(n, m, N)(q - 1)^2} & \\
\Lambda_{n+1, m} \Lambda_{n, m+1} & \frac{q^N \prod_{i=1}^{k} (1 - q^{-3d_i})(1 - q^{-2d_i})}{\psi(n, m, N)(q - 1)^2} & \\
\end{array}
\]

Table 2.5: A table of the number of simplices in \( \mathcal{X}_3(g) \) which lie above a given simplex in \( \mathcal{D} \)
We can use Theorem 2.3.7 to examine the structure of $\mathcal{X}_3(g)$. Since it is unfeasible to draw $\mathcal{X}_3(g)$ in any capacity, instead we consider $\mathcal{D}$ and colour code the simplices to represent how many simplices in $\mathcal{X}_3(g)$ lie over them.

![Figure 2.2: A figure highlighting the number of simplices in $\mathcal{X}_3(t^3)$ which lie over the simplices in $\mathcal{D}$.
Purple have the most simplices over them, and red have the least.](image)

We see that there is some stabilisation that happens as $n \geq N$ and $m \geq N$. The slices $n + m = l$ for $l \geq 2N$ are a disjoint union of graphs, see Figure 2.2 above.

**Conjecture 2.3.8.**

The quotient $\mathcal{X}_3(g)$ is a union of a finite simplicial complex $\mathcal{X}_3(g)_{\text{finite}}$, and a finite collection of cusps $\mathcal{X}_3(g)_{\infty}$. The cusps are topologically equivalent to the slice in Figure 2.2 times a half-line.

At this point in time we are unable to prove the above conjecture. A large obstacle to proving the conjecture is that there currently isn’t a general theory of compactifications of arithmetic quotients of affine buildings. Although this is under development
by various groups, and it is expected that such a theory exists with analogous properties to those in the symmetric space case.

2.4 Homology of $\Gamma(g)\setminus X_3$

Let $g \in A$ with $\deg_t(g) = N$, and assume that $g$ factors as $g = \prod_{i=1}^{k} g_i^{e_i}$, where the $g_i \in A$ are distinct, irreducible, and $\deg_t(g_i) = d_i$. To simplify notation we denote the quotient space $\Gamma(g)\setminus X_3$ by $X_3(g)$.

We will calculate the Euler characteristic of $X_3(g)$. First we use the quotient map $\rho: \Gamma(g)\setminus X_3 \longrightarrow \Gamma\setminus X_3$ to examine the Euler characteristic of a subcomplex of $X_3(g)$.

Definition 2.4.1.

Let $D \subset X_3$ be the fundamental domain spanned by the vertices $\{ \Lambda_{n,m} | n, m \geq 0 \}$, define $D_l \subset D$ to be the subcomplex of $D$ spanned by the vertices

$$\{ \Lambda_{n,m} | n, m \geq 0, \text{ and } n + m \leq l \}.$$

Let $\chi_l(g)$ denote the Euler characteristic of $\rho^{-1}(D_l)$.

Theorem 2.4.2.

For a fixed $g \in A$, the Euler characteristic $\chi_l(g)$ is a constant function of $l$ for all $l \geq 2N$. Specifically,

$$\chi_l(g) = \frac{(q^{2N+1} - q^3 + 1)q^6N^k \prod_{i=0}^{k} \left(1 - \frac{1}{q^{3d_i}}\right) \left(1 - \frac{1}{q^{2d_i}}\right)}{q(q^3 - 1)(q^2 - 1)} \text{ for all } l \geq 2N.$$
Proof. Let \( l \geq 2N \). Recall that the Euler characteristic of a 2-dimensional simplicial complex is given by \( \chi = \#\text{Vertices} - \#\text{Edges} + \#\text{Faces} \). Thus,

\[
\chi_l(g) = \sum_{n,m \geq 0} \#\{X_3(g)\}_{\Lambda_n,m} \cdot \Lambda_n,m + \sum_{n,m \geq 0} \#\{X_3(g)\}_{\Lambda_n,m,\Lambda_n+1,m} \cdot \Lambda_n,m,\Lambda_n+1,m
\]

We simplify this sum by exploiting some cancellations, i.e. we have the following cancellations between vertices and edges:

\[
\#\{X_3(g)\}_{\Lambda_0,m} = \#\{X_3(g)\}_{\Lambda_0,m,\Lambda_0,m+1} \quad \text{for } 0 < m < l
\]

\[
\#\{X_3(g)\}_{\Lambda_n,0} = \#\{X_3(g)\}_{\Lambda_n,0,\Lambda_n+1,0} \quad \text{for } 0 < n < l
\]

\[
\#\{X_3(g)\}_{\Lambda_n,m} = \#\{X_3(g)\}_{\Lambda_n,m,\Lambda_n+1,m} \quad \text{for } 0 < n, 0 < m, n + m < l
\]

\[
\#\{X_3(g)\}_{\Lambda_n,m} = \#\{X_3(g)\}_{\Lambda_n,m,\Lambda_n+1,m} \quad \text{for } 0 < n, 0 < m, n + m = l
\]

and we have the following cancellations between faces and edges:

\[
\#\{X_3(g)\}_{\Lambda_{n+1,m+1},\Lambda_{n,m+1,\Lambda_{n+1,m}}} = \#\{X_3(g)\}_{\Lambda_{n,m+1,\Lambda_{n+1,m}}} \quad \text{for } n + m < l
\]

\[
\#\{X_3(g)\}_{\Lambda_n,m,\Lambda_{n,m+1,\Lambda_{n+1,m}}} = \#\{X_3(g)\}_{\Lambda_n,m,\Lambda_{n+1,m}} \quad \text{for } 0 < m < l
\]

\[
\#\{X_3(g)\}_{\Lambda_n,m,\Lambda_{n,m+1,\Lambda_{n+1,m}}} = \#\{X_3(g)\}_{\Lambda_n,m,\Lambda_{n,m+1}} \quad \text{for } n > 0, n + m < l.
\]

See Figure 2.3 for a visual representation of this cancellation. Thus,
\[ \chi_l(g) = \# \{ \mathcal{X}_3(g) \} \Lambda_{0,0} + \# \{ \mathcal{X}_3(g) \} \Lambda_{0,l} + \# \{ \mathcal{X}_3(g) \} \Lambda_{1,0} \\
- \# \{ \mathcal{X}_3(g) \} \Lambda_{0,0,1} - \# \{ \mathcal{X}_3(g) \} \Lambda_{0,0,1} - \# \{ \mathcal{X}_3(g) \} \Lambda_{0,0,1} + \# \{ \mathcal{X}_3(g) \} \Lambda_{0,0,1} \Lambda_{1,0} \\
+ \# \{ \mathcal{X}_3(g) \} \Lambda_{0,0,1} \Lambda_{1,0} \\
= \# \{ \mathcal{X}_3(g) \} \Lambda_{0,0} - \# \{ \mathcal{X}_3(g) \} \Lambda_{0,0,1} - \# \{ \mathcal{X}_3(g) \} \Lambda_{0,0,1} + \# \{ \mathcal{X}_3(g) \} \Lambda_{0,0,1} \Lambda_{1,0} \\
+ \# \{ \mathcal{X}_3(g) \} \Lambda_{0,l} + \# \{ \mathcal{X}_3(g) \} \Lambda_{1,0} - \# \{ \mathcal{X}_3(g) \} \Lambda_{0,0,1} \Lambda_{1,0,1} \\
= q^{8N} \prod_{i=0}^k \left( 1 - \frac{1}{q^{3d_i}} \right) \left( 1 - \frac{1}{q^{2d_i}} \right) \left[ \frac{1}{q^3(q^3-1)(q^2-1)} - \frac{2}{q^3(q^2-1)(q-1)} \right] \\
+ \frac{1}{q^3(q-1)^2} + \frac{1}{q^{2N+1}(q^2-1)(q-1)} + \frac{1}{q^{2N+1}(q^2-1)(q-1)} \\
- \frac{1}{q^{2N+1}(q-1)^2} \right] \\
(q^{N+1} - q^3 + 1) q^{6N} \prod_{i=0}^k \left( 1 - \frac{1}{q^{3d_i}} \right) \left( 1 - \frac{1}{q^{2d_i}} \right) \\
= \frac{q(q^3-1)(q^2-1)}{q^2 - 1}.
\]
Proposition 2.4.3.
\[ \chi(\mathfrak{X}_3(g)) = \chi_{2N}(\mathfrak{X}_3(g)). \]

Proof. By Conjecture 2.3.8 we have that \( \mathfrak{X}_3(g) \) is homotopy equivalent to \( \mathfrak{X}_3(g)_{\text{finite}} \). So \( \chi(\mathfrak{X}_3(g)) = \chi(\mathfrak{X}_3(g)_{\text{finite}}) = \chi_{2N}(\mathfrak{X}_3(g)). \]

We now compute the homology of \( \mathfrak{X}_3(g) \). A priori, since \( \dim(\mathfrak{X}_3) = 2 \) the homology of \( \mathfrak{X}_3(g) \) is supported in dimensions 0, 1, and 2. But, by a theorem of Harder [Har77] we know that the homology is in fact only supported in dimension 0 and 2, and moreover it is free abelian, i.e.

\[
\begin{align*}
H_0(\mathfrak{X}_3(g); \mathbb{Z}) & \cong \mathbb{Z} \\
H_1(\mathfrak{X}_3(g); \mathbb{Z}) & = 0 \\
H_2(\mathfrak{X}_3(g); \mathbb{Z}) & \cong \mathbb{Z}^r \text{ for some } r \\
H_i(\mathfrak{X}_3(g); \mathbb{Z}) & = 0 \text{ for all } i > 2.
\end{align*}
\]

Thus the Euler characteristic is sufficient to calculate the homology of \( \mathfrak{X}_3(g) \).

Theorem 2.4.4 (Homology of \( \mathfrak{X}_3(g) \) for general \( g \)).

Let \( g \in A \) with \( \deg_t(g) = N \), and assume that \( g \) factors as \( g = \prod_{i=1}^{k} g_i^{e_i} \), where the \( g_i \in A \) are distinct, irreducible, and \( \deg_t(g_i) = d_i \). Then

\[
\text{rank}_\mathbb{Z}(H_2(\mathfrak{X}_3(g); \mathbb{Z})) = \frac{(q^{2N+1} - q^3 + 1)q^{6N}\prod_{i=0}^{k} \left(1 - \frac{1}{q^{3d_i}}\right)(1 - \frac{1}{q^{2d_i}})}{q(q^3 - 1)(q^2 - 1)} - 1.
\]
CHAPTER 3

UNIMODULAR SYMBOL ALGORITHM FOR $\text{SL}_n(\mathcal{F})$

In this section we prove that every modular symbol for $\text{SL}_n(\mathcal{F})$ can be written as a sum of unimodular symbols, we also describe an algorithm for finding such a decomposition. The proofs rely heavily on an analogue of Minkowski’s theorem on lattices in $\mathbb{R}^n$, to the function field setting.

In Section 3.1 we review some preliminary material on modular symbols. Section 3.2 gives a quick overview of the theory of volume and convex bodies in the function field setting, and concludes with the statement of Minkowski’s theorem for function fields Theorem 3.2.8. Finally, in Section 3.3.1 we define unimodular symbols for $\text{SL}_3(\mathcal{F})$, and show that every modular symbol can be written as a sum of unimodular symbols by describing an algorithm to do so.

3.1 Preliminaries on Modular Symbols

Modular symbols were invented by Yuri Manin in [Man72], as a tool for studying modular forms for congruence subgroups $\Gamma \subseteq \text{SL}_2(\mathbb{Z})$. They have since been generalised in many different directions. In the 70’s Barry Mazur described a generalisation of modular symbols for an arbitrary reductive $\mathbb{Q}$-group[Maz]. There has also been some work in generalising modular symbols to groups over non-archimedean fields.
3.1.1 Modular Symbols for $\text{SL}_2(\mathbb{R})$

Let $\Gamma \subseteq \text{SL}_2(\mathbb{Z})$, and let $\mathbb{H}$ be the complex upper-half plane. We define a $\Gamma$-modular symbol to be an ordered pair of rational cusps\(^1\) $\{\alpha, \beta\} \in \mathbb{P}^1(\mathbb{Q}) \times \mathbb{P}^1(\mathbb{Q})$, considered as an element of $H_1(\Gamma \backslash \mathbb{H}, \text{cusps}; M)$. We called modular symbols like this, modular symbols for $\text{SL}_2(\mathbb{R})$.

The geometric interpretation of these modular symbols is as follows: If $\alpha, \beta$ are rational cusps of $\mathbb{H}$, then let $\{\alpha, \beta\}$ denote the unique geodesic path in $\mathbb{H}$ from $\alpha$ to $\beta$. The image of $\{\alpha, \beta\}$ under the quotient map $\mathbb{H} \to \Gamma \backslash \mathbb{H}$ can be considered as an element of the relative homology group $H_1(\Gamma \backslash \mathbb{H}, \text{cusps}; M)$. By an abuse of notation we write $\{\alpha, \beta\} \in H_1(\Gamma \backslash \mathbb{H}, \text{cusps}; M)$. For a geometric illustration of a modular symbol see Figure I.1a. It can be shown that all elements of the relative homology group arise in this way.

A unimodular symbol is a modular symbol which is in the $\text{SL}_2(\mathbb{Z})$-orbit of the modular symbol $\{0, i\infty\}$. Manin showed that (up to homology) any modular symbol can be written as a sum of unimodular symbols by describing an explicit algorithm to do so. The algorithm is often referred to as Manin’s trick, or the continued fraction algorithm. This algorithm is essential to the usefulness of modular symbols for $\Gamma \subseteq \text{SL}_2(\mathbb{Z})$ since it allows one to find a finite presentation for the space of modular symbols.\(^2\)

3.1.2 Modular Symbols for $\text{SL}_2(\mathcal{F})$

In [Tei92], Teitelbaum described a theory of modular symbols for $\text{SL}_2(\mathcal{F})$. Geometrically, the modular symbols of Teitelbaum are analogous to those of Manin for $\text{SL}_2(\mathbb{R})$: instead of considering geodesics between cusps in $\mathbb{H}$, one now considers

\(^{1}\)The rational cusps of $\mathbb{H}$ are $\partial \mathbb{H} \overset{\text{def}}{=} \mathbb{Q} \cup \{\infty\} \cong \mathbb{P}^1(\mathbb{Q})$.

\(^{2}\)Since any congruence subgroups $\Gamma \subseteq \text{SL}_2(\mathbb{Z})$ is of finite index, there are only finitely many unimodular symbols modulo $\Gamma$. Thus the set of unimodular symbols modulo $\Gamma$, is a finite generating set for the set of all modular symbols.
geodesics between cusps in the Bruhat-Tits building $\Delta^{\text{BT}}(\text{SL}_2(\mathcal{F}))$. For a geometric illustration of a Teitelbaum modular symbol see Figure I.1b.

3.1.3 Modular Symbols for $\text{SL}_n(\mathbb{R})$, $n \geq 3$

In [AR79], Ash and Rudolph study modular symbols for $\text{SL}_n(\mathbb{R})$ in detail. Let $\Gamma \subseteq \text{SL}_n(\mathbb{Z})$, $X = \text{SL}_n(\mathbb{R})/\text{SO}(n)$, and $\overline{X}$ be the Borel-Serre compactification of $X$. Define $M = \Gamma \backslash \overline{X}$, then $M$ is a manifold with boundary, and $M$ is homotopy equivalent to $\Gamma \backslash X$.

**Definition 3.1.1** ([AR79]).

A modular symbol is an $n$-tuple of non-zero column vectors in $\mathbb{Q}^n$, $[q_1, \ldots, q_n]$ modulo the following relations:

1. It is anti-symmetric, i.e. $[q_{\sigma(1)}, \ldots, q_{\sigma(n)}] = \text{sign}(\sigma)[q_1, \ldots, q_n]$ for $\sigma \in S_n$
2. It is homogeneous of degree zero, i.e. $[aq_1, \ldots, q_n] = [q_1, \ldots, q_n]$ for all $a \in \mathbb{Q}^\times$
3. If $\det(Q) = 0$, then $[Q] = 0$
4. If $q_1, \ldots, q_{n+1}$ are all non-zero, then $\sum_{i=1}^{n+1} (-1)^{i+1}[q_1, \ldots, \hat{q_i}, \ldots, q_{n+1}] = 0$
5. If $A \in \text{GL}_n(\mathbb{R})$, then $[AQ] = A \cdot [Q]$ considered as an element of $H_{n-1}(M, \partial M; \mathbb{Z})$.

A modular symbol $[Q]$ is called a unimodular symbol if $\det(Q) = 1$.

Ash and Rudolph then prove that the above defined modular symbols generate all of $H_{n-1}(M, \partial M; \mathbb{Z})$[AR79, Proposition 3.2]. Finally they describe an algorithm which reduces any modular symbol to a sum of unimodular symbols, thus proving the following theorem:

---

$^3$Note, the Poincare dual of $H_{n-1}(M, \partial M; \mathbb{Z})$ is $H^{2n(n+1)}(M; \mathbb{Z})$, which is isomorphic to $H^{2n(n+1)}(\Gamma; \mathbb{Z})$. Thus the modular symbols could be considered as elements of $H^{2n(n+1)}(\Gamma; \mathbb{Z})$. 

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Theorem 3.1.2 ([AR79, Theorem 4.1]).

As $A$ runs over $\text{SL}_n(\mathbb{Z})$, the modular symbols $[A]$ generate $H_{n-1}(M, \partial M; \mathbb{Z})$.

Proof. The general idea of the algorithm is as follows:

INPUT: $[A] = [u_1, \ldots, u_n]$

STEP 1: If $\det(A) < 0$ then swap the first two columns of $A$ so that $\det(A) > 0$

STEP 2: If $\det(A) = 1$ then STOP, else $\det(A) \geq 2$ and go to STEP 3:

STEP 3: Find $B_i \in M_n(\mathbb{Z})$ such that $[A] = \sum_{i=1}^{n}[B_i]$ and $0 < \det(B_i) < \det(A)$ for all $i$

STEP 4: Repeat this process for each of the $B_i$’s until all the modular symbols are unimodular or zero

We now explain how to do STEP 4: i.e. finding the $B_i$.

If $A = (u_1, \ldots, u_n)$ is an $n \times n$-matrix and $v$ is a column vector, then we denote by $A_i\{v\}$ the matrix $(u_1, \ldots, u_{i-1}, v, u_{i+1}, \ldots, u_n)$. By property 4. of modular symbols we have the following homology, $[A] = \sum_{i=1}^{n}(-1)^i[A_i\{v\}]$. Consider the closed parallelepiped

$$P_\epsilon = \left\{ \sum_{i=1}^{n} t_i u_i \left| |t_i| \leq 1 - \epsilon \right. \right\}.$$

Note that $\text{Vol}(P_0) = 2^n \det(A) \geq 2^{n+1}$, so for sufficiently small $\epsilon > 0$ we have $\text{Vol}(P_\epsilon) > 2^n$. My Minkowski’s theorem there exists some $v' \neq 0$ in $P_\epsilon \cap \mathbb{Z}^n$. If $v' = \sum_{i=1}^{n} t'_i u_i$ then let

$$t_i = \begin{cases} 
  t'_i & \text{when } 0 \leq t'_i \leq 1 - \epsilon \\
  1 + t'_i & \text{when } -1 + \epsilon \leq t'_i \leq 0
\end{cases}$$

and let $v = \sum_{i=1}^{n} t_i u_i$. By a basic property of determinants we have that

$$\det(A_i\{v\}) = t_i \det(A),$$

thus $0 \leq \det(A_i\{v\}) \leq \det(A)$, as required. \qed
3.2 Minkowski’s Theorem for Function Fields

In this section we will briefly state the main results from the work of Kurt Mahler on the generalisation of Minkowski’s theorem to function fields. We only mention material needed for our purposes and omit all proofs, for full details and proofs see [Mah41].

3.2.1 Notation

For the rest of this chapter we will use the following notation:

\( A = \mathbb{F}_q[t] \)

\( F = \mathbb{F}_q((t^{-1})) \)

\( \mathcal{W}_n = \mathcal{F}^n \), the \( n \)-dimensional vector space over \( \mathcal{F} \)

\( \mathcal{L}_n = \mathbb{A}^n \subseteq \mathcal{F}^n \), the set of all lattice points in \( \mathcal{W}_n \)

\( |\cdot| = \) the absolute value on \( \mathcal{F} \) coming from \( 1/t \) (thus, \( |t^n| = e^n > 1 \))

\( F_\infty(X) = |X| \) the \( L^\infty \)-norm on \( \mathcal{W}_n \), i.e. \( |X| = \max(|x_1|, |x_2|, \ldots, |x_n|) \)

3.2.2 Convex Bodies

**Definition 3.2.1** (Distance Function).

A function \( F: \mathcal{W}_n \rightarrow \mathbb{R}_{\geq 0} \) is called a distance function if

1. \( F(X) > 0 \) for all \( X \neq 0 \)
2. \( F(0) = 0 \)
3. \( F(\lambda X) = |\lambda|F(X) \) for all \( \lambda \in \mathcal{F} \)
4. \( F(X + Y) \leq \max(F(X), F(Y)) \)

The standard distance function to keep in mind is the \( L^\infty \) norm, i.e. \( F_\infty(X) = |X| \).
Definition 3.2.2 (Ball of Radius $\tau$).

The ball of radius $\tau$ associated to the distance function $F$, is the set $C_F(\tau) = \{ X \in W_n \mid F(X) \leq \tau \}$. We also call $C_F(\tau)$ the convex body associated to $F$.

Theorem 3.2.3 ([Mah41, Part 3]).

For any distance function $F$, the associated convex body $C_F(\tau)$ is a parallelepiped. Thus there exists $\Omega \in \text{GL}_n(\mathcal{F})$ such that $F(X) = |\Omega X|$.

3.2.3 Volume of a Convex Body

In this section we will define the notion of volume of a convex body. Let $F$ be a distance function. We will only consider convex bodies of the form $C_F(e^l)$ where $l \in \mathbb{Z}$.

Definition 3.2.4.

Given a distance function $F$,

- Let $m_F(l)$ be the set of lattice points in $C_F(e^l)$, i.e. $m_F(l) = C(e^l) \cap \mathbb{L}_n$.
- Let $M_F(l)$ be the maximal number of $F_q$-independent lattice vectors in $m_F(l)$.
- In the special case $F = F_\infty$ we denote this by $m_\infty(l)$ and $M_\infty(l)$.

Proposition 3.2.5 ([Mah41, Part 7]).

Let $F$ be a distance function, then

1. For all $l$, $M_\infty(l) = n(l + 1)$ and thus $M_\infty(l + 1) = M_\infty(l) + n$.
2. For $l$ big enough, $M_F(l + 1) = M_F(l) + n$.

Thus, for $l$ big enough $M_F(l) - M_\infty(l)$ is independent of $l$. We can use this fact to define the volume of $C_F(1)$.

Definition 3.2.6 (Volume).

The volume of the convex body $C_F(1)$ is defined to be

$$\text{Vol}(C_F(1)) = \lim_{l \to \infty} e^{M_F(l) - M_\infty(l)}.$$
In particular we have that, $\Vol(C_\infty(1)) = 1$.

**Theorem 3.2.7** ([Mah41, Part 8]).

If $\Omega \in \text{GL}_n(\mathcal{F})$ then $\Vol(\Omega C_F(1)) = |\det(\Omega)|\Vol(C_F(1))$.

### 3.2.4 Minkowski’s Theorems for Function Fields

We now state Minkowski’s theorem for function fields. Later we will use this theorem to show the existence of a lattice point inside a certain parallelepiped.

**Theorem 3.2.8** (Minkowski’s Theorem for Function Fields, [Mah41, Part 9]).

To any special distance function $F$, there exists $n$ $\mathcal{F}$-independent lattice points,

$$X^{(m)} = (x^{(m)}_1, x^{(m)}_2, \ldots, x^{(m)}_n), \quad \text{for } m = 1, 2, \ldots, n$$

such that

- $F(X^{(1)}) = \sigma_1 = e^{g_1}$ is the minimum of $F(X)$ over all non-zero lattice points.
- $F(X^{(m)}) = \sigma_m = e^{g_m}$ is the minimum of $F(X)$ over all non-zero lattice points which are $\mathcal{F}$-independent of $X^{(1)}, \ldots, X^{(m-1)}$.

The $\sigma_i$ are called the successive minima of $F$. They satisfy the following equality,

$$\sigma_1\sigma_2\ldots\sigma_n = \frac{1}{\Vol(C_F(1))}.$$

Moreover, the determinant of the matrix spanned by the $X^{(m)}$, satisfies

$$|D| = |\det(x^{m}_i)| = 1.$$
3.3 Modular Symbols for $\text{SL}_n(\mathcal{F})$, $n \geq 3$

In this section we define modular symbols for $\text{SL}_n(\mathcal{F})$, define unimodular symbols, and show that every modular symbol can be written as a sum of unimodular symbols.

By a direct analogue of Ash and Rudolph’s definition Definition 3.1.1 we define modular symbols for $\text{SL}_n(\mathcal{F})$ as follows.

**Definition 3.3.1.**
A modular symbol is an ordered $n$-tuple of non-zero column vectors in $(\mathbb{F}_q(t))^n$, $\Omega = [c_1, \ldots, c_n]$ modulo the following relations:

1. It is anti-symmetric, i.e. $[c_{\sigma(1)}, \ldots, c_{\sigma(n)}] = \text{sign}(\sigma) [c_1, \ldots, c_n]$ for $\sigma \in S_n$
2. It is homogeneous of degree zero, i.e. $[\lambda c_1, \ldots, c_n] = [c_1, \ldots, c_n]$, for all $\lambda \in \mathbb{F}_q^\times$
3. If $\det(\Omega) = 0$ then $[\Omega] = 0$
4. If $c_1, \ldots, c_{n+1}$ are all non-zero, then $\sum_{i=1}^{n+1} (-1)^{i+1} [c_1, \ldots, \hat{c_i}, \ldots, c_{n+1}] = 0$
5. If $A \in \text{GL}_n(\mathcal{F})$, then $[A \Omega] = A \cdot [\Omega]$

considered as an element of $H_{n-1}(\mathcal{X}_n, \partial \mathcal{X}_n; \mathbb{Z})$.

A modular symbol $[\Omega]$ is called a unimodular symbol if $|\det(\Omega)| = 1$.

Note that by property 2. of Definition 3.3.1 we can assume that the columns of the modular symbols have entries in $\mathbb{F}_q[t]$.

**3.3.1 Unimodular Symbol Algorithm for $\text{SL}_n(\mathcal{F})$**

In this section we will show that unimodular symbols for $\text{SL}_n(\mathcal{F})$ generate all modular symbols for $\text{SL}_n(\mathcal{F})$.

**Theorem 3.3.2.**
As $\Omega$ runs over $\text{SL}_n(\mathbb{F}_q[t])$, the modular symbols $[\Omega]$ generate $H_{n-1}(\mathcal{X}_n, \partial \mathcal{X}_n, \mathbb{Z})$.  

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Proof. The theorem is equivalent to saying that every modular symbol for \( \text{SL}_n(F) \) can be written as a sum of unimodular symbols for \( \text{SL}_n(F) \). To prove this we will adapt the algorithm of Ash and Rudolph given in the proof of Theorem 3.1.2, i.e. we will show that if \( |\det(\Omega)| > 1 \) then there exists a vector \( v \in L_n \) such that \( |\det(\Omega_i\{v\})| < |\det(\Omega)| \).

Let \( \Omega = [c_1, c_2, \ldots, c_n] \in \text{GL}_n(F_q[t]) \) be a matrix with columns \( c_i \), and assume \( |\det(\Omega)| > 1 \). Let \( C_\Omega \) be the parallelepiped in \( W_n \) spanned by the columns of \( \Omega \), i.e.

\[
C_\Omega = \left\{ \sum_{i=1}^{n} \lambda_i c_i \mid |\lambda_i| \leq 1 \right\}.
\]

Then \( C_\Omega = \Omega C_\infty(1) \), and by Theorem 3.2.7 we have that

\[
\text{Vol}(C_\Omega) = |\det(\Omega)|\text{Vol}(C_\infty(1)) = |\det(\Omega)|.
\]

Since \( |\det(\Omega)| > 1 \), Minkowski’s theorem for function fields (Theorem 3.2.8) implies that \( \sigma_1 < 1 \). Thus \( C_\Omega \) contains a non-zero lattice point, and since \( \sigma_1 \) is strictly less than 1, we can assume the lattice point is not on \( \partial C_\Omega \), i.e. \( 0 \neq v \in C_\Omega \cap L_n \). If \( v = \sum_{i=1}^{n} \lambda_i c_i \) then \( |\det(\Omega_i\{v\})| = |\lambda_i \det(\Omega)| < |\det(\Omega)| \), as required. \( \square \)

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BIBLIOGRAPHY


