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Continua

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CONTINUA

A Dissertation Presented

by

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DEDICATION

I dedicate this work to everyone who has shed light on my path.

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ABSTRACT

CONTINUA

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The subject of my dissertation is the structure of continua and, in particular, of physical space and time. Consider the region of space you occupy: is it composed of indivisible parts? Are the indivisible parts, if any, extended? Are there infinitesimal parts? The standard view that space is composed of unextended points faces both *a priori* and empirical difficulties. In my dissertation, I develop and evaluate several novel approaches to these questions based on metaphysical, mathematical and physical considerations. In particular, I develop and evaluate two infinitesimal theories of space based on Robinson's nonstandard analysis. I argue that *Infinitesimal Gunk*, according to which every region is further divisible and some regions have infinitesimal sizes, has distinct advantages over alternative gunky views. I also advance a new account of distance for atomistic space, *the mixed account*, in response to Weyl's tile argument, which is an influential argument against the view that space is composed of indivisible regions. Having these theories in stock, we make progress in discovering the best theory of continua.

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CHAPTER 1

METAPHYSICS OF CONTINUA

Space and time play important roles in metaphysics, physics, and in daily life. In daily life, spatial and temporal coordinates are essential ways to locate and order things and events. I am writing this very sentence at the latitude of 42.393706 and the longitude of -72.531586 at 6:50 pm on Oct 22, 2019. With this information, you can easily compare this event with any other event (perhaps you are reading this sentence ten days later). In physics, space and time are ubiquitous. Contemporary physics is all about fields, which are physical quantities that have a value at every point of spacetime. Field theories consist of mathematical equations of how field values evolve in spacetime. In general relativity, the metric structure of spacetime itself is treated as a physical field that interacts with other fields. The great challenge of unifying quantum theories and general relativity also centers around the structure of spacetime. It is considered received wisdom among physicists that a certain kind of discrete spacetime is required for meeting that challenge. In metaphysics, we use space and time to sort things into different categories: simply put, those that are located in spacetime are concrete and the rest are abstract. We use the structure of spacetime to discern various structures of reality. For example, some philosophers argue that a thing A is a part of a thing B iff the region A occupies is a part of the region B occupies. Space and time also pose particularly challenging questions for metaphysics: what sort of things are space and time? Are they real, or mere projections of the human mind? Are they fundamental or emergent? If they are real, what structure do they have?

It is the last question that I am going to investigate in this dissertation as part of the bigger project of understanding space and time. More generally, I will study the structure of physical continua, of which space and time are primary examples. A continuum, intuitively speaking, is something that is uninterrupted and has no gap. We can mentally picture such entities: a line, a surface, or a three-dimensional block. But the nature of a continuum is puzzling. Consider the region you occupy. Is it infinitely divisible? Can it be decomposed into its ultimate parts, or "atoms"? If so, what is the size of an atom? Is it perhaps infinitesimally small, that is, smaller than any finite size?

Zeno's famous paradox of measure is based on "infinite divisibility": a continuum can be divided into smaller and smaller parts without limit. The paradox can be reconstructed as this: if a finite line segment is infinitely divisible, then it can be divided into infinitely many parts with the same size. If every part has a finite size, then the whole line segment would have an infinite size. But if every part has zero size, then the whole would have zero size. Either way, the whole segment would not have a finite length, which contradicts the assumption. (Skyrms 1983; see also Furley 1967)

At its core, Zeno's paradox reveals the tension between two intuitive claims on the composition of a continuum. On the one hand, it is intuitive that every extended part of a continuum is further divisible, and therefore a complete decomposition of a continuum can only end up with unextended points. On the other hand, a continuum cannot be exhaustively composed of unextended points because zero sizes cannot add up to a finite size.

The atomists of ancient Greece, such as Leucippus and Democritus, rejected infinite divisibility of matter. They held that all matter was composed of indivisible extended atoms below the level of visibility. Anaxagoras, on the other hand, maintained that there are no indivisible magnitudes:

> Nor of the small is there a smallest, but always a smaller. . . (Curd 2007, B3)

Aristotle agreed with Anaxagoras and famously held that a continuum is infinitely divisible, in the sense that every part of a continuum is further divisible. Since an unextended point is not further divisible, a continuum is not composed of points. At one point, this was even considered the definition of a continuum.

> . . . no continuous thing is divisible into things without parts. Nor can there be anything of any other kind between; for it would be either indivisible or divisible, and if it is divisible, divisible either into indivisibles, or into divisibles that are always divisible, in which case it is continuous. (Physics VI, 231b11–15)

In response to Zeno's paradox, Aristotle famously distinguished between potential infinity and actual infinity, and rejected the existence of actual infinity. While the process of division can go on without limit, a continuum cannot be divided into infinitely many parts. Thus, there is no dilemma as for what sizes those infinitely many parts have. It also follows that there are no infinitesimal parts.

Many philosophers after Aristotle agreed with him on the nature of continua, and in particular, that a continuum is not composed of unextended points. For example, Leibniz held that a point should not be considered a part of the line (Rescher 1967, 109). Kant also wrote:

> Space and time are quantum continua... points and instants mere positions... and out of mere positions viewed as constituents capable of being given prior to space and time neither space nor time can be constructed. (Kant 1964[1781], 204)

Let's call Aristotle's view of continua "the old orthodoxy": every part of a continuum is further divisible; in particular, points are not the ultimate parts of a continuum; and there are no infinite or infinitesimal parts. Even Descartes, who, as a mathematician, is known for his Cartesian coordinate system (which involve analyzing geometry using numbers), held in *Meditations* (1641) that matter and space are divisible without limit.

In the nineteenth century, Dedekind and Cantor respectively made a breakthrough in real analysis. In *Continuity and Irrational Numbers* (1872[1963]), Dedekind demonstrated that "there are infinitely many numbers to which no rational numbers correspond." Furthermore, while rational numbers are gappy and "cut" by real numbers, the real number line is gapless and continuous in the sense that every number that can be constructed from rational numbers through "Dedekind cut" is a real number. Cantor further postulated that there is a one-to-one correspondence between the set of all real numbers and the set of points on a geometric line, which led to the full arithmetization of the linear continuum. In his famous series of papers *On Infinite, Linear Point Manifold* (1883), Cantor aimed at giving a rigorous definition of continua, which he complained as never having been "subject to any thorough inspection." According to Cantor, an N-dimenisional space is a set of points represented by n-tuples of real numbers equipped with a metric defined by real numbers in the usual way. It is well known that Cantor developed a rigorous theory of different infinite cardinalities of sets, which is a generalization of Dedekind's comparison between rational numbers and real numbers. As a result of the arithmetization of continua, we now have a precise way to compare the cardinalities of points in different parts of continua. For example, the points in the linear continuum are as many as the points in the interval (0,1), and are as many as the points in a three-dimensional space.

The Dedekind-Cantor continuum soon became the new orthodoxy and part of the theory that I call *the standard view*, according to which a line is exhaustively composed of uncountably many unextended points, and those points can be algebraically represented by real numbers (see Appendix A for details). It sharply contrasts with the old orthodoxy, according to which a point is not a part of the line, and a line cannot be analyzed into a set of points. Mathematically rigorous and structurally rich, the Dedekind-Cantor continuum has been a solid foundation for calculus and modern physics.

How does the standard view solve Zeno's paradox of measure? According to the standard measure on the real line ("the Lebesgue measure"), the interval [0,1] has a measure of one, and a singleton set of a point such as $\{0\}$ has a measure of zero. A line segment that is entirely composed of unextended points can still have a finite length. This violates the principle of *additivity* implicitly assumed by Zeno's paradox, which says that the measure of a whole is the sum of the measures of any disjoint parts that compose the whole. In particular, it violates *uncountable additivity*, the principle that the measure of a whole composed of uncountably many disjoint parts is equal to the sum of the measures of those parts. The arithmetic sum of zeros, even if uncountably many, is still zero (see 2.2). But the fusion of uncountably many disjoint regions of zero size can have a non-zero measure. This approach was classically defended by Grünbaum (1973) and has become the standard solution to Zeno's paradox.

Although the view is considered "standard," the complaints it faces have never subsided. For one thing, it is still counterintuitive that an extended line can be composed of unextended points. The standard measure theory also leads to other undesirable results. For example, assuming the axiom of choice, Vitali (1905) has shown that a unit circle can be divided into *countably* many parts of "the same size" in the sense that they can be turned into one another through rigid transformation. These parts are called "the Vitali sets." Since the standard measure is countably additive and translation invariant, the Vitali sets cannot have any measure: if they have measure zero, then the unit circle would have measure zero, and if they have any finite measure, then the circle would have an infinite measure. Either way, it's a contradiction.

For another example, according to the standard view, every bounded part of a continuum has a boundary, and a closed region includes its boundary. Suppose that two rigid bodies come into perfect contact, namely that there is no gap between them. When they come into contact, the boundaries of the two bodies overlap, since we cannot put two boundaries side to side in the pointy space without leaving a gap in-between. Consider this region of overlap. Is this region occupied? If not, then there would be a gap between the two bodies, which contradicts that they are in perfect contact. If it is occupied, then either it is occupied by one of the bodies, or by both. If it is occupied by only one of the bodies, strange asymmetry follows. If it is occupied by both bodies, then partial colocation would occur. Since the bodies are rigid and not interpenetrable, colocation cannot happen. So if the standard view is true, then perfect contact would be impossible. (Russell 2008; see also Zimmerman 1996b and Artnzenius 2008)

The standard view does not only face conceptual difficulties, but also worries on empirical grounds. For example, Geroch (1972) pointed out that, because a metric is crucial for identifying events (namely spacetime points), quantum fluctuations of a metric suggest "smearing out of events." Arntzenius (2004, 2012) argue that spacetime points are redundant for the standard approach to quantum theories. In quantum mechanics, wavefunctions that differ only on regions of measure zero give identical results for all probabilities of a particle being found in a region upon measurement. Moreover, in the standard approach to relativistic quantum field theory, there are no field operators defined on individual spacetime points but rather on open regions of spacetime. These results suggest that spacetime points may not have physical reality.

Moreover, as Baez (2018) pointed out, the standard view causes troubles even for classical physics. For example, in classical electromagnetism, the problem of unifying Maxwell's equations and the Lorentz force law in order to predict the behavior of the electric field and the magnetic field of a charged particle still has no definite solution. An immediate difficulty is that an electric field of a charged particle is "infinite" or ill-defined at the particle's location, and therefore it's unclear how the electric field affects the particle. Various proposed solutions to this problem also bring unexpected complications. Of course, classical physics is wrong about our world. But if we cannot formulate attractive laws that fully describe a classical world, it's more likely that the framework we use to formulate laws is inadequate than that a classical world is impossible.

None of these problems is a knock-down argument against the standard view, but taken together, they give good reasons for exploring alternatives. The gunky approach to continua is one main alternative to the standard view. It is a revival of the old orthodoxy, according to which a continuum cannot be decomposed into ultimate parts. The modern development of this approach is often associated with Whitehead (1919, 1920, 1929). Like Aristotle, Whitehead held that all regions have at least a finite size. So, it avoids the counterintuitive feature that an extended region is composed of unextended points. Moreover, the contact puzzle can be avoided by denying the existence of boundaries. However, the task of developing this view further is not straightforward. As Arntzenius (2008) and Russell (2008) pointed out, a natural development of this view is inconsistent with countable additivity, a standard measure-theoretic principle. Both authors have proposed their solutions, but at the cost of attractive principles.

Another main alternative is *finite atomism*, according to which a continuum is composed of finitely extended indivisible parts. This revives the atomist view from ancient Greece, which was criticized by Aristotle because it violates infinite divisibility, the intuitive principle that every extended part can be further divided. But this view is starting to be taken more seriously because of favorable considerations from physics. As I mentioned at the beginning, the unification of quantum theories and general relativity seems to require a certain sort of discrete structure. Along this line, the size of a smallest region is often considered to be at the Planck level (Hogan 2012). However, finite atomism faces a simple but influential objection from Weyl (1949), which purports to show that there are no natural metrics on atomistic space that approximate Euclidean geometry at any scale. Since our space is approximately Euclidean at least at the ordinary scale, it follows that our space is not atomistic. More generally, atomistic models face "the anisotropic problem": there are privileged directions in those models, but our space seems to be isotropic and rotationally invariant. Moreover, even though finite atomism can solve some empirical problems that trouble the standard view, it is still very unclear whether the atomistic framework is adequate for formulating and advancing our physical theories. For example, it is unclear how "immutable" atoms of space and time fit in with length contraction and time dilation in special relativity (Crouse and Skufca 2018).

In the meantime, the exclusive role of real numbers in arithmetizing geometry began to be challenged as alternative number theories come into light. In particular, modeling the linear continuum on the real number line has the consequence that there are no infinitesimal or infinite regions. In the past, many people have found the notion of an infinitesimal size incoherent. The notion of infinitesimals appeared as early as in the work of Democritus (450 BC). (The ancient Greeks used the same word for "infinitesimal" as for "indivisible," which adds difficulties to a definite interpretation—the word "infinitesimal" is a seventeenth-century coinage.) Archimedes (287 BC–212 BC) was perhaps the first to give an explicit definition for infinitesimals (or "indivisibles"). In *The Method of Mechanical Theorems*, he wrote that a number is infinitesimal iff it is smaller than any $1/n$ for any natural number n (Archimedes 1912). On the other hand, infinitesimals were criticized by Eudoxus (350 BC) and banished from mainstream mathematics. Infinitesimals reappeared during the invention of modern calculus. Newton and Leibniz both employed the method of infinitesimals in developing calculus. But both of them considered infinitesimals as merely heuristic rather than real physical quantities. The notion was further attacked by Berkeley (1734) as incoherent and illusory. As the standard view became popular, infinitesimals were condemned as "cholera-bacilli" (Cantor 1887) and "unnecessary, erroneous, and self-contradictory" (Russell 1903).

However, the situation has changed. There are now several rigorous theories of infinitesimals, among which are Robinson's (1966) *nonstandard analysis*, and *smooth infinitesimal analysis* developed by Lawvere (1980), Kock (1981) and many others. Nonstandard analysis features the hyperreal line, which augments the familiar real numbers with infinite and infinitesimal numbers. Those nonstandard numbers behave very similarly as real numbers. For example, for any (non-zero) infinitesimal δ and an infinite number N, we have $\sqrt{\delta}$, δ^2, $1/\delta$, $\delta \cdot N$ and so on. The inverse of an infinitesimal is an infinite number. If you square an infinitesimal, the result is a smaller infinitesimal. More generally, a hyperreal system is an alternative model for the first-order truths of standard analysis that is strictly larger than the real number system. There are many non-isomorphic hyperreal models. For example, Conway's (1976) surreal number system was found to be isomorphic to the largest hyperreal system (Ehrlich 2012).

Smooth infinitesimal analysis is a very different approach to infinitesimals. It features *nilpotent infinitesimals*, quantities so small that their squares equal zero. It is directly motivated by the infinitesimal method in geometry. The idea that a circle is a regular polygon with infinitesimal sides has a long tradition—for example, it was posited by Bryson of Heraclea (late 5th-century BC), Kepler, Galileo, Leibniz, and de l'Hôpital. This enables us to calculate the area of a circle straightforwardly. Just as the area of a regular polygon with finitely many sides is equal to half of the product of its apothem (the distance from the center to a side) and its perimeter, the area of a circle as an infinilateral regular polygon is equal to half of the product of its radius and its perimeter. This idea is also connected to the idea of the Greek atomists that indivisibles are qualitatively different from the whole they compose—for example, the indivisibles of a curve are straight (Bell 2017). Smooth infinitesimal analysis is the most developed system that realizes this idea. However, it requires intuitionistic logic and is classically inconsistent. So it's unclear whether classical logicians can share the insight of this novel system of infinitesimals. (In my recent work, I suggest a way to overcome this difficulty.) But there are also classical systems of nilpotent infinitesimals under development. For example, see Giordano (2010).

With these new theories at hand, time is ripe for developing infinitesimal theories of continua. Indeed, there have been attempts in using infinitesimals to solve conceptual puzzles. In "Zeno's Arrow, Divisible Infinitesimals, and Chrysippus," White (1982) suggested that *the present* has a duration of an infinitesimal value, where infinitesimals are understood in the framework of nonstandard analysis. Skyrms (1983) also gestured towards an infinitesimal approach to Zeno's paradox of measure based on nonstandard analysis. For infinitely many parts of a finite region that have equal size, the size of those parts is not finite but infinitesimal. Reeder (2015), on the other hand, suggested using Giordano's nilpotent infinitesimals as the duration of "the present," and argued that this approach is less arbitrary than White's. Hellman (2006) discussed smooth infinitesimal analysis but didn't take it as a realistic approach to space because of its classical inconsistency.

As I stated at the beginning, my aim is to study the structure of space and time as physical continua. I aim to make progress by developing alternative theories of continua that are technically rigorous, mathematically grounded, and constrained by our empirical data. By expanding our stock of coherent and attractive theories of continua, we have a better chance of nailing down the best approach. In developing such theories, I employ conceptual tools from metaphysics and engage with contemporary mathematics and science seriously. The standard Dedekind-Cantor continuum was never motivated on grounds other than its mathematical rigor and scientific fruitfulness. My aim is to develop theories with comparable rigor and fruitfulness in light of updated mathematics and science, but more sensitive to philosophical considerations and with an openness to more possibilities.

In this dissertation, I enrich our stock with two infinitesimal theories of continua in the framework of nonstandard analysis, which are respectively *infinitesimal atomism* (Chapter 2) and *Infinitesimal Gunk* (Chapter 3). According to infinitesimal atomism, space is composed of indivisible atoms that have an infinitesimal size. But unlike the approach Skyrms (1983) gestured at, I do not merely assign an infinitesimal size to points or atoms of space that are represented by familiar real numbers. Instead, I represent atoms by infinitesimal intervals on the *hyperreal* line. This has several distinct advantages. Notably, the theory satisfies *weak additivity*: for any measurable region, and for any disjoint measurable parts of the region, the measure of the whole is the sum of the measure of those parts. For example, you can add infinitely many atoms together and get a finite number (though technically, it's called a *hyperfinite sum*). Both the standard view and Skryms's suggestion violate this principle. In Skryms's suggested model, the measure of the interval [0,1] (which is one) is not the sum of the measures of the atoms in the interval (represented by real numbers), because—figuratively speaking—you could squeeze all those atoms within an infinitesimal interval on the hyperreal line! In other words, there are vastly more atoms in my model than real number points. Relatedly, another advantage of infinitesimal atomism is that it has a rich measure theory. For any Lebesgue measurable region on the real line, the measure can be approximated by the hyperreal measure of its corresponding region on the hyperreal line up to infinitesimal differences. This is not surprising—after all, it's like measuring space with an infinitesimally long rod.

However, infinitesimal atomism does not only deliver good news. In fact, it faces *both* the main objection to the standard view, which is the problem of unmeasurable regions, *and* the main objection to finite atomism, which is Weyl's tile argument. The flip side of having a rich measure theory is the admission of a vast number of unmeasurable regions. The problem is not only worse quantitatively but also "qualitatively." Under the standard view, unmeasurable regions are exotic constructions of scatter points that require the axiom of choice. Under infinitesimal atomism, it does not take the axiom of choice to construct an unmeasurable region. For instance, for any real number, the set of all atoms that are infinitesimally close to it is unmeasurable. (The construction of a hyperreal system only requires "the Boolean prime ideal theorem," which is strictly weaker than the axiom of choice.) As for Weyl's tile argument, infinitesimal atomism has no more resources than finite atomism in answering it. Although I do not think either objection is devastating—in fact, I propose a response to Weyl's tile argument in Chapter 4—they do not leave infinitesimal atomism with a clear overall advantage over the more familiar alternatives.

In Chapter 3, I advance Infinitesimal Gunk as a response to Arntzenius's (2008) and Russell's (2008) inconsistency arguments against *the finite gunky view* ("finite" as opposed to the infinitesimal version of the approach that I develop), according to which points do not exist and there are no boundary regions. As Arntzenius and Russell respectively pointed out, under some plausible assumptions, the principle that there are no boundary regions is inconsistent with countable additivity, a standard measure-theoretic principle. Facing this problem, Arntzenius suggested admitting boundary regions that have a non-zero Lebesgue measure. Russell, on the other hand, suggested rejecting countable additivity and retreating to mere finite additivity. Both approaches have significant costs. Against Arntzenius's approach, allowing a set of scattered points (that has zero dimension) to represent a region of space seems to go against our intuition behind the gunky approach, namely that we want to get rid of indivisibles and lower-dimensional parts.

Russell's suggestion also leads to an undesirable impoverishment of measure theory. For example, it entails that for countably many disjoint line segments respectively of size 1/4, 1/8, 1/16,. . . , their fusion can either have a measure of 1/2 or a measure of one depending on how you arrange them! Relatedly, many Lebesgue measurable regions would be unmeasurable under Russell's proposal.

Infinitesimal Gunk is the view that there are no ultimate components of a continuum and there are infinitesimal regions. Unlike Arntzenius's proposal, Infinitesimal Gunk does not require denying that there are no boundaries and hence better preserves our gunky intuition on this aspect. It also has a much richer measure theory than Russell's proposal and does not retreat to mere finite additivity. Like in infinitesimal atomism, we can obtain measures of any measurable region by adding up disjoint infinitesimal intervals it contains, except that in this case there are no smallest intervals representing indivisible atoms. This means that the measure theory is even richer than in infinitesimal atomism. It automatically follows that the measure of any Lebesgue measurable region can be approximated by the hyperreal measure of a corresponding region up to infinitesimal differences. Such accurate approximations do not exist under Russell's approach. Moreover, the magic of dramatically changing the measure of the fusion of countably infinitely many parts by rearrangement does not happen—although the reason is that in each case, the fusion in question is unmeasurable (but has an approximate measure). These advantages suggest that Infinitesimal Gunk is a promising gunky approach. The addition of infinitesimal regions bring rich structures that make up for the elimination of unextended points.

In Chapter 4, I engage with Weyl's tile argument against atomism. Since atomism (or its variations) is an active research project in physics, how we respond to Weyl's tile argument could be very relevant to science. Weyl's argument runs like this: if a square region is composed of atoms arranged like square tiles, then the diagonal of the region is just as long as the side. This means that Euclidean geometry is not approximately true. But our space is approximately Euclidean at least at the ordinary scale, so our space is not atomistic.

A crucial assumption in this argument is that the distance between any two atoms is equal to the number of atoms between them. This amounts to a simple path-dependent account of distance for atomistic space, initially proposed by Riemann (1866). Nevertheless, this account is not the only reasonable option. I propose an alternative account—*the mixed account*—according to which only local pairs of atoms bear primitive distances while other distances can be derived from them. The distance between any two atoms is the least sum of the primitive distances between them. As long as we allow for enough primitive distances, atomistic space can approximate Euclidean geometry as closely as we want. For example, suppose an atom is of the Planck size (10^{-35} meters). Also suppose we want space to approximate Euclidean geometry with a distortion smaller than $\pm 1.6 \times 10^{-9}$, the relative accuracy of one of the best measurements for small distances (NIST, n.d.). Then, the longest primitive distance should be 10^{-26} meter, the diameter of a neutrino. (Among all current approaches, only the mixed account allows for such a physical model where the size of an atom is at the Planck level.)

One crucial feature of this account is that not every distance is primitive. According to general relativity, the metric of spacetime is determined by the distribution of mass-energy under Einstein's field equations, which implies that the presence of a massive body would distort the paths nearby. If all distances were primitive, then the distance of two atoms, however far away, would not depend on anything between them, which is incompatible with general relativity.

The mixed account might strike one as unnatural or overcomplicated because it involves two concepts of distance. Against this, I argue that the standard account of distance from differential geometry has a similarly mixed form. We start with local metrics, which are analogous to primitive distances, and similarly obtain distances by adding up those local metrics (though technically, it's integration rather than addition).

The value of this new account of distance does not lie in its technical novelty—because there isn't much. Rather, it's mostly a philosophical defense of a (technically more or less obvious) mathematical model as a realistic theory of space that allows us to circumvent the technical difficulties that other approaches might need to overcome. In theoretical physics, causal set theory is an active research program that aims at giving an atomistic account for relativistic spacetime. Because of the close analogy between primitive distances and metric tensors, the mixed account can be extended to relativistic settings rather straightforwardly (see 4.4). The extended mixed account is very much like causal set theory except that it has the additional structure of primitive distances, which allows us to avoid some technical difficulties that have arisen in causal set theory.

The title of the dissertation *Continua* is the subject of my study, but also a pun about multiple theories of continua. I do not aim to nail down the best approach to continua. But through developing a few viable alternative theories, I hope to show that there is little reason for us to be satisfied with the standard view, and that we should continue to expand our stock of possible theories rigorously developed and closely examined. Given the important role of spacetime, this is not only central to metaphysicians' task of understanding reality but also valuable to our ongoing scientific practice.

CHAPTER 2

DO SIMPLE INFINITESIMAL PARTS SOLVE ZENO'S PARADOX OF MEASURE

In this chapter, I develop an original view of the structure of space—called *infinitesimal atomism*—as a reply to Zeno's paradox of measure. According to this view, space is composed of ultimate parts with infinitesimal size, where infinitesimals are understood within the framework of Robinson's (1966) nonstandard analysis. Notably, this view satisfies a version of additivity: for every region that has a size, its size is the sum of the sizes of its disjoint parts. In particular, the size of a finite region is the sum of the sizes of its infinitesimal parts. Although this view is a coherent approach to Zeno's paradox and is preferable to Skyrms's (1983) infinitesimal approach, it faces both the main problem for the standard view (the problem of unmeasurable regions) and the main problem for finite atomism (Weyl's tile argument), leaving it with no clear advantage over these familiar alternatives.

Keywords: continuum; Zeno's paradox of measure; infinitesimals; unmeasurable regions; Weyl's tile argument

2.1 Zeno's Paradox of Measure

A continuum, such as the region of space you occupy, is commonly taken to be indefinitely divisible. But this view runs into Zeno's famous paradox of measure. If a finite line segment is indefinitely divisible, then it can be divided into infinitely many parts with the same length. In this case, if every part has a finite length, then the whole line segment would have an infinite length. But if every part has zero length, then the whole would have zero length. Either way, the whole would not have a finite length. This contradicts the assumption. (Skyrms 1983; see also Furley 1967)

At its core, Zeno's paradox reveals the tension between two intuitive claims on the composition of a continuum: on the one hand, it is intuitive that every extended part of a continuum is further divisible, and given that, it is natural to consider an extensionless point to be an ultimate component of a continuum; on the other hand, an extended continuum cannot be exhaustively composed of extensionless points because zero sizes add up to zero. Leibniz, for example, was deeply puzzled by this tension and eventually concluded that continua are not real (Russell 1958).

We can break Zeno's paradox down into the following assumptions:

> INFINITE DIVISIBILITY. A continuum can be divided into smaller and smaller parts without limit.
>
> INFINITY CONDITIONAL. If INFINITE DIVISIBILITY, then a continuum can be divided into infinitely many parts of equal size.
>
> DICHOTOMY. Any part of a continuum has either zero size or at least a finite size.
>
> ADDITIVITY. The size of the whole is the sum of the sizes of its disjoint parts.
>
> ZEROS-SUM-TO-ZERO. Zeros, however many, always sum up to zero.

Since all these assumptions are required for the paradox to arise, each assumption provides a possible way to escape from the paradox. If we deny INFINITE DIVISIBILITY, we will arrive at the view called *finite atomism*, according to which the ultimate parts of a continuum are of finite sizes. If we deny INFINITY CONDITIONAL, then we will come to *the gunky view*, according to which a continuum does not have ultimate parts. This option can be traced back to Aristotle, who famously rejected INFINITY CONDITIONAL by distinguishing between potential infinity and actual infinity. Finally, according to *the standard view* in modern mathematics, a continuum is composed of uncountably many extensionless points, and therefore ADDITIVITY is false. Unfortunately, these approaches have their own problems (Section 2).

Yet another possible approach to Zeno's paradox, following Skrym (1983), is to deny DICHOTOMY by claiming instead that a part of a continuum can have an infinitesimal size, a size that is non-zero but smaller than any finite size. In the past, many people have found the idea of an infinitesimal size unintelligible.[1] But the situation started to change with the arrival of various systems of infinitesimals, such as Robinson's (1966) *nonstandard analysis* and *smooth infinitesimal analysis* developed by Lawvere (1980) and others. In this paper, I will explore the option of rejecting DICHOTOMY by extracting a new theory of continua, called *infinitesimal atomism*, from the framework of nonstandard analysis.[2] According to this theory, the ultimate parts of a continuum have an infinitesimal size. I will compare this approach with the standard view and finite atomism, the two more familiar approaches to Zeno's paradox that admit indivisible parts. My conclusion is that infinitesimal atomism, although having some attractive features, has no clear advantages over these familiar alternatives.

Note that infinitesimal atomism is not the only way to reject DICHOTOMY. Another possible approach is to have an infinitesimal version of the gunky view, according to which every part of a continuum is further divisible and some parts have infinitesimal sizes. I argue in Chapter 3 that this new gunky approach has distinct advantages over current gunky theories. This essay is part of the project of examining what resources nonstandard analysis can provide for rethinking traditional paradoxes.

2.2 Solutions and Their Problems

The standard solution to Zeno's paradox, classically defended by Grünbaum (1973, 158-76), is based on standard analytic geometry. According to this view, a geometric line can be algebraically represented by the set of real numbers, with each real number representing a point on the line. This means that a line is exhaustively composed of points. The length of a point under standard measure theory (the Lebesgue measure) is zero, while the length of any line segment is a finite number. This is what we call *the standard view*, according to which a line is exhaustively composed of points of zero length.

[1] Infinitesimals first appeared in Democritus's work (450 B.C.E) only to be banished by Eudoxus (350 B.C.E). They reappeared during the invention of modern calculus, but were attacked and eventually abandoned in mainstream nineteenth-century mathematics. (Bell 2013)

[2] I explore other technical approaches to infinitesimals such as smooth infinitesimal analysis elsewhere.

The standard view violates ADDITIVITY, the principle that the size of the whole is the sum of the sizes of its disjoint parts. In standard analysis, there is a general definition of an arithmetic sum that satisfies ZEROS-SUM-TO-ZERO, even in the uncountable case.[3] However, according to the standard view, a point has zero length, and yet a line segment composed of uncountably many points has a finite length. Therefore, the length of the line segment is not the sum of the lengths of its constituent points.

More straightforwardly, the standard view violates the principle that an extended whole cannot be composed of unextended parts. This principle—call it REGULARITY—is highly intuitive and remains an important consideration for evaluating a theory of continua. Thus, the violation of this principle or ADDITIVITY is a substantial cost for the standard view.

Apart from contradicting these intuitive principles, the standard view also leads to other measure-theoretical paradoxes, which are similar in spirit to Zeno's paradox. For instance, the Banach-Tarski paradox says that, assuming the axiom of choice, we can divide a sphere into finitely many disjoint parts, move them around rigidly (no stretching or squishing), and rearrange them to form *two* spheres, each of which has exactly the same size as the original one (Wagen 1985; see also Forrest 2004).[4] The measure theory of the standard view (the Lebesgue measure) satisfies finite additivity (and indeed countable additivity): for any finitely many disjoint parts of a continuum, the measure of their fusion is the sum of the measure of those parts. Moreover, the measure of a part does not change under rigid transformation. Therefore, standard measure theory cannot assign any measure to those parts: if they have any measures, it would be impossible for them to compose something

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[3]In standard analysis, even though we typically only sum up countably many numbers, summing up zeros is a notable exception. The following general definition of a sum over non-negative numbers indexed by an arbitrary set implies that zeros always sum up to zero. Let HUGE be an indexing set for some non-negative numbers. Let PAR be the set of partial sums of numbers indexed by finite subsets of HUGE. We define the sum of the numbers indexed by HUGE to be the least upper bound of PAR. It follows from this definition that zeros, even if uncountably many, sum up to zero. This definition, when applies to a countable set of positive numbers, coincides with the more familiar notion of sum in standard analysis defined as the limit of partial sums of an infinite sequence.

[4]There is also a more elementary analogous result in the one-dimensional case, which involves the construction of "Vitali sets" (see Skryms 1983, 238-9).

twice as large as before without undergoing a change in measure. Call this problem "the problem of unmeasurable regions." I will come back to this issue in Section 5.

The standard view not only faces *a priori* objections but is also questionable on empirical grounds (for example, see Geroch 1972, Arntzenius 2003). Moreover, physicists are actively designing empirical experiments to determine the structure of space (Hogan 2012). What we philosophers can contribute, then, is to steer toward a clear view of what space could *possibly* be like—stocking our warehouse with a rich store of possibilities, in the hopes that actuality might be one of them.

Finite atomism is one of the main alternatives to the standard view. This view, which says that a continuum is composed of finitely extended indivisible parts, was once considered unappealing because it violates INFINITE DIVISIBILITY, the highly intuitive principle that every extended part can be further divided. But this view has become more popular because of favorable considerations from physics. It is even considered "received wisdom" among physicists that a certain atomistic structure is required for reconciling quantum theory and general relativity (Maudlin 2015, 46). However, this view faces a troublesome argument given by Weyl (1949)—called *Weyl's tile argument*—which purports to show that there are no natural distance functions over atomistic space that approximate Euclidean geometry. Since our space is approximately Euclidean at certain scales, Weyl's conclusion implies that finite atomism cannot describe our actual space. This problem, if unsolved, would leave finite atomism quite unattractive, for considerations from actual physics play an important role in motivating this view. I will explain Weyl's argument more in Section 6.

The gunky view, the view that every part of a continuum is further divisible, is another main alternative to the standard view. This approach is attractive because it does not violate highly intuitive principles such as INFINITE DIVISIBILITY and REGULARITY. However, a natural development of this view turns out to be inconsistent (Arntzenius 2008; Russell 2008). Alternative gunky approaches are further discussed in Chapter 3.

Because of these problems, the situation is far from settled, and philosophers continue to look for alternative theories of continua or solutions to Zeno's paradox. The option of solving the paradox by appealing to infinitesimals has been contemplated from time to time but never developed into a full view of continua (for example, see Skyrms 1983). Without a clear and coherent view, it's hard to compare this option with other approaches to the paradox. So in this paper, I will first flesh out this option into a more developed view of continua—infinitesimal atomism—within the framework of nonstandard analysis. This view, I believe, is the most attractive atomistic theory of continua that we can extract from the mathematical ideas of nonstandard analysis. Notably, according to infinitesimal atomism, we can indeed sum up the lengths of infinitely many disjoint infinitesimal parts and get a finite length, although this kind of "sum" is a new technical notion introduced by nonstandard analysis.

Once we have a clear and coherent view in focus, we can start to evaluate it philosophically. Unfortunately, it turns out that the news for infinitesimal atomism is mostly bad. On the good side, the view indeed satisfies REGULARITY and other intuitive assumptions of Zeno's paradox to a certain extent (which I discuss in Section 4). On the bad side, it suffers *both* from the main problem for the standard view (the problem of unmeasurable regions, which I discuss in Section 5) *and* from the main problem for finite atomism (Weyl's tile argument, which I discuss in Section 6). Neither of these difficulties is decisive, but they don't leave infinitesimal atomism with any clear advantages over more familiar alternatives.

2.3 Infinitesimal Atomism

In this section, I will develop the view *infinitesimal atomism*, according to which a continuum is composed of ultimate parts of infinitesimal size, which I call "minims," where infinitesimals are understood in the framework of nonstandard analysis.

In nonstandard analysis (NSA), the real line $\mathbb{R}$ is extended to the *hyperreal line* $*\mathbb{R}$, which includes infinitesimals and infinite numbers along with the familiar real numbers.[5] A number is *infinitesimal* iff its absolute value is smaller than any positive real number. A number is *infinite* iff its absolute value is larger than any positive real number. On the

[5]My introduction of the hyperreal system is based on Goldblatt (1998). Note that there are multiple non-isomorphic hyperreal systems. The hyperreal system I introduce here is the smallest one. The surreal number system introduced by Conway (1976) is considered to be the largest hyperreal system.

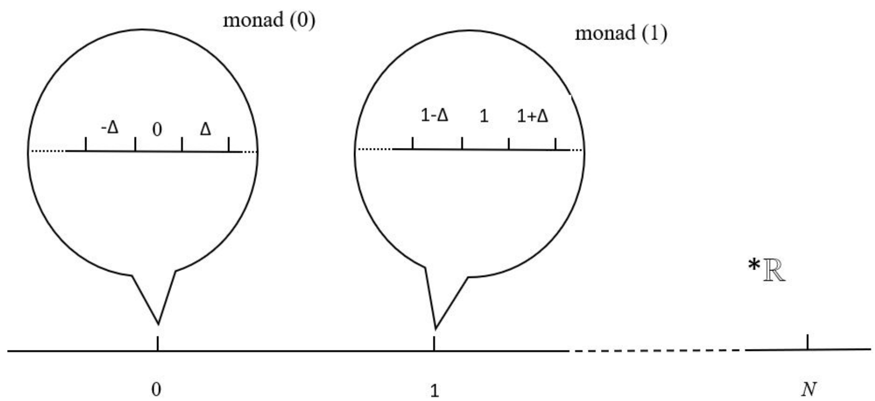

Figure 2.1. Monads on the hyperreal line

hyperreal line, each real number is surrounded by a "cloud" of hyperreal numbers that are infinitesimally close to it, called its *monad* (Figure 2.1). The monads of different real numbers do not overlap. Moreover, we can do arithmetic with the hyperreal numbers just like with the standard real numbers. For example, for any positive infinitesimal δ and any infinite number N, we have $\sqrt{\delta}$, $\frac{1}{\delta}$, δ^2, $\delta + N$, $\frac{\delta}{N}$, etc. More generally, the hyperreal numbers satisfy all the first-order truths about the real numbers in standard analysis.

The basic picture of infinitesimal atomism in the one-dimensional case is that the minims are lined up one by one with adjacent ones connected. Let Δ be the infinitesimal length of a minim. Then, it is natural to represent minims by consecutive intervals of length Δ on the hyperreal line, such as $...[0, \Delta), [\Delta, 2\Delta), ...$[6] For convenience, these intervals will be named by their left endpoints $...0, \Delta...$ Then, all minims are represented by members of the set $M = \{k \cdot \Delta : k \in *\mathbb{Z}\}$, where $*\mathbb{Z}$ is the hyperreal extension of the set of integers $\mathbb{Z}$ that satisfies all the first-order truths about $\mathbb{Z}$. Note that although minims can be lined up one by one as shown in Figure 2.1, there are actually *uncountably* many minims between, for

[6]The half-open intervals, while they make a helpful intuitive picture, should not be taken too literally. I am using certain mathematical objects to represent parts of space, but not every mathematical feature corresponds to a genuine spatial feature. In particular, the asymmetric *half-openness* does not correspond to any real feature of the minims that the intervals represent. Rather, the representation in question is a one-to-one correspondence between the half-open intervals and the minims that preserves length, order, and connectedness. Note that in the case of atomistic space, I take "connectedness" (or "adjacency") to be primitive (see Forrest 1995 and Roeper 1997). Intuitively, two regions are connected if there is no gap between them.

example, 0 and 1. In fact, for *every* real number, there are uncountably many minims in its monad.[7]

Minims are the smallest parts of the continuum, but what about others parts of the continuum? For brevity, let "space" and "region" be synonymous with "continuum" and "part of a continuum" respectively. Since space is entirely composed of minims, every region is also a fusion of minims. But I will not assume standard mereology and in particular, the principle of unrestricted composition:

UNRESTRICTED COMPOSITION. Every collection of regions has a fusion.

The question of exactly what collections of minims should compose regions is a bit delicate, and I will postpone the discussion until Section 5. But for now, we can at least assume that collections of regions of a special kind—*hyperfinite* collections of regions—have fusions.

The notion of "hyperfiniteness" is a distinct notion of cardinality in NSA. When extending $\mathbb{R}$ to $*\mathbb{R}$, the set of all natural numbers is extended to the set of *hypernatural* numbers, which preserves all the first-order truths about natural numbers in standard analysis. Just as every real number is bounded by a natural number, every hyperreal number is bounded by a hypernatural number. Since there are infinite hyperreal numbers, there are also infinite hypernatural numbers. Let N be an infinite hypernatural number. Consider the set $\{1, 2, ..., N\}$. How many elements does it have? The answer is N—according to the new notion of *hyperfinite cardinality* in NSA. Just as in standard analysis the set $\{1, 2, ..., n\}(n \in \mathbb{N})$ has a cardinality of n, in NSA the set $\{1, 2, ..., N\}$ has a hyperfinite cardinality of N. Note that the notion of hyperfinite cardinality is finer-grained than the notion of cardinality in standard set theory. According to standard set theory, if N is infinite, then $\{1, 2, ..., N\}$ and $\{1, 2, ..., N + 1\}$ have the same cardinality. But in NSA, they do not have the same hyperfinite cardinality. Two sets have the same hyperfinite cardinality just when there is a bijective *internal* function between them, where "internal" means being expressible in

[7] This can be derived from the theorem that every infinite hyperfinite set is uncountable (Goldblatt 1998, 141). In every monad, there are more than hyperfinitely many minims.

the language of standard analysis.[8] (For "hyperfiniteness," see Goldblatt 1998, 178-81; for "internal entities," see 166-73.)

Given the notion of hyperfinite cardinality, we assume the following composition principle:

> HYPERFINITE COMPOSITION. Every hyperfinite collection of regions has a fusion.

Let's assume that Δ is a multiplicative inverse of an infinite hypernatural number. Then, according to HYPERFINITE COMPOSITION, the minims between 0 and 1, which are represented by $0, \Delta, ..., \frac{1}{\Delta} \cdot \Delta$, compose a region, because $\{0, \Delta, ..., \frac{1}{\Delta} \cdot \Delta\}$ is hyperfinite. But this principle does not tell us whether, for instance, the minims in the monad of 0 compose a region, because the set of those minims is not hyperfinite—it's not hyperfinite because there is no maximal hypernatural N such that $N \cdot \Delta$ is infinitesimal.

The measure theory for infinitesimal atomism is especially simple: we can find the measure of a region just by counting the minims it contains. The trick that makes this work is that we have the notion of *hyperfinite sum* in NSA. Just as every finite sequence of numbers in standard analysis has a sum, in NSA, every hyperfinite sequence of numbers has a hyperfinite sum. As a simple example, if we add up the infinitesimal length $1/N$ N-many times, then the result is one. Note that this notion is very different from the notion of "infinite sum" in standard analysis (Footnote 3).

> HYPERFINITE MEASURE. For any hyperfinite collection of minims, the measure of their fusion is equal to the hyperfinite sum of the measures of those minims.[9]

Then, the fusion of N-many minims has the measure of $N \cdot \Delta$.

One desirable feature of the measure is that it generally gives familiar results: it approximates the Lebesgue measure (the standard measure over the real line) up to infinitesimal differences. For example, consider again the set of minims between 0 and 1:

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[8] Note that the language of standard analysis includes quantifiers over *sets* and *functions*, as in this definition of hyperfiniteness (Goldblatt 1998, 178-80). These quantifiers also receive non-standard "internal" interpretations in the hyperreal system (Goldblatt 1998, 168-70).

[9] For those who are mathematically informed, the measure is closely related to Loeb measure in NSA. The main difference is that Loeb measure is real-valued and is defined over a σ-algebra which includes the domain of the hyperreal measure as a proper subset.

$\{0, \Delta, ..., (\frac{1}{\Delta} - 1) \cdot \Delta\}$, which has a hyperfinite cardinality of $\frac{1}{\Delta}$. Thus the fusion of those minims has a measure of $\frac{1}{\Delta} \cdot \Delta = 1$, which is equal to the Lebesgue measure of the interval [0,1]. (For more general results, see Goldblatt 1998, 215-7.)

Although all measurable regions are fusions of hyperfinitely many minims, the measure theory is actually very rich. The *shadow* of a hyperreal number (or its corresponding minim), if it exists, is the real number that is infinitesimally close to it (note that an infinite hyperreal does not have a shadow). For *every* Lebesgue-measurable set X, we can find minims whose shadows are elements of X such that the hyperreal measure of the fusion of those minims is infinitesimally close to the Lebesgue measure of X (212-3, 216-7). Let each Lebesgue-measurable set be mapped to such a measurable region. Then sets with different Lebesgue measures are mapped to regions with different hyperreal measures. Meanwhile, it's clear that regions with different hyperreal measures do not correspond to sets with different Lebesgue measures in an analogous sense. For example, a region composed of a single minim and a region composed of two each correspond to a singleton of a point, which has Lebesgue measure zero. This means that we have an injective but not surjective map from Lebesgue-measurable sets to measurable regions that preserves their measures up to infinitesimal differences. In this sense, infinitesimal atomism has an even richer measure than the standard measure!

2.4 Revisiting Zeno's Paradox

Infinitesimal atomism avoids Zeno's paradox by dispensing with DICHOTOMY, the principle that implies there are no infinitesimal regions. But whether the view solves Zeno's paradox satisfactorily depends on how well it satisfies other intuitive assumptions in the paradox. In this section, I will give a preliminary examination of this question and leave some of the more complicated issues to Section 5.

First, consider INFINITE DIVISIBILITY. In the context of hyperreals, we can distinguish between the following two versions:

> FINITE DIVISIBILITY. Every finitely extended region can be divided into smaller parts.
>
> GENERAL DIVISIBILITY. Every extended region can be divided into smaller parts.

The standard view satisfies both principles. Finite atomism satisfies neither. Infinitesimal atomism satisfies FINITE DIVISIBILITY but not GENERAL DIVISIBILITY, since according to the view, any finitely extended region can be divided into infinitely many minims, but each minim is an indivisible extended region.

One might think that regardless of whether FINITE DIVISIBILITY holds, it is equally bad to deny GENERAL DIVISIBILITY. The intuition behind this principle can be spelled out through the following argument given by Zimmerman (1996). Every extended region has a size and a shape. Every region that has a size and a shape has proper parts—for instance, a unit square must have a left half and a right half, each of which is a half-unit rectangle. Thus, every extended region must have proper parts and hence be divisible.[10]

Let's grant that it is intuitive that every extended region has a proper part. One thing we could say in reply is that this intuition is reliable only when the region in question is finite. When it comes to the infinitesimals, our intuition need not be reliable: we have a technical theory about them, but we don't necessarily have an intuitive grasp of them. After all, infinite numbers are notoriously counterintuitive: Hilbert's hotel is fully occupied and yet can accommodate infinitely more! So it might well be the case that our intuition is equally unreliable in the case of infinitesimals, which are just reciprocals of infinite numbers. In this sense, the violation of GENERAL DIVISIBILITY in infinitesimal atomism might be more acceptable than the violation of FINITE DIVISIBILITY in finite atomism.

The case of ADDITIVITY is tricky. On the one hand, the measure theory satisfies the following version of additivity:[11]

> HYPERFINITE ADDITIVITY. For hyperfinitely many measurable disjoint regions, the measure of their fusion is the (hyperfinite) sum of the measures of those regions.

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[10]There is a related worry of arbitrariness that I do not discuss in this paper. The worry is that it is arbitrary that a minim has a length of a particular infinitesimal. As pointed out by an anonymous referee, Reeder (2015) discussed this worry and suggested that we should represent a minim by a monad instead. I prefer the approach in this paper because it allows for a richer measure theory with attractive features. We can add up the lengths of minims to obtain an infinitesimal or a finite length, and the measure of any measurable region is the sum of its ultimate parts. In contrast, monads are unmeasurable, and we cannot add up their lengths. In fact, the measure theory of Reeder's approach would be less rich than the standard measure: they are otherwise the same except that points of length zero are replaced by unmeasurable monads.

[11]Note that this is a distinct feature of infinitesimal atomism that is not present in, for example, Skryms's (1983) suggestion. The measure theory Skyrms suggested is only *finitely additive* (241-2).

This is true because the fusion of any hyperfinitely many measurable disjoint regions is a fusion of hyperfinitely many minims.[12] The hyperfinite number we obtain by counting the minims in the fusion is the same number as we get by counting the minims in each region and then adding them together.

On the other hand, the original principle of ADDITIVITY is not restricted to hyperfinite collections of regions. Since a non-hyperfinite collection of regions may not have a measurable fusion at all, "the measure of the fusion" in ADDITIVITY can be an empty description. How should we understand this principle then? Here's a natural option:

> WEAK ADDITIVITY. For arbitrarily many measurable disjoint regions, if they have a fusion, and if their fusion is measurable, then the measure of their fusion is the sum of the measures of those regions.

Infinitesimal atomism satisfies this principle.[13] For arbitrarily many measurable disjoint regions, either they are hyperfinitely many, or their fusion is not measurable, since every measurable region is a fusion of hyperfinitely many measurable regions. If the regions are hyperfinitely many, the principle is reduced to HYPERFINITE ADDITIVITY. If their fusion is unmeasurable, then the principle is vacuously true. Either way, the principle is satisfied. In contrast, the standard view does not satisfy this principle, for according to the standard view, any finite line segment is a fusion of uncountably many points of zero size, but zeros sum up to zero.

Having WEAK ADDITIVITY is desirable, but not fully satisfactory. A stronger version of additivity would require that the fusion of arbitrarily many measurable disjoint regions is also measurable:

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[12] This is because the union of hyperfinitely many disjoint hyperfinite sets is still hyperfinite, just like the union of finitely many disjoint finite sets is still finite.

[13] In contrast, Skyrms's (1983) approach violates WEAK ADDITIVITY. The fact that Skyrms's suggested measure is only finitely additive (see Footnote 11) means that for any region of a finite size, its measure is *not* the sum of the measures of its composing points, since every finite region contains uncountably many points. This feature, I think, makes his approach less attractive than mine. The violation of the intuitive principle of WEAK ADDITIVITY alone is a cost. Moreover, this violation also undermines the motivation for REGULARITY. If the measure of an extended region is not determined by the measures of its ultimate parts, does it matter whether the ultimate parts have measure zero? It doesn't seem so bad to have an unextended point *per se*. What's bad rather seems to be the failure of ADDITIVITY when those unextended points compose an extended region. So, it is not clear that Skyrms's approach is an improvement over the standard view.

STRONG ADDITIVITY. For arbitrarily many measurable disjoint regions, if they have a fusion, then their fusion is measurable, and the measure of their fusion is the sum of the measures of those regions.[14]

However, whether infinitesimal atomism satisfies STRONG ADDITIVITY depends on what composition principles hold. I will take up this question in the next section.

So far, infinitesimal atomism does reasonably well on meeting the intuitive assumptions of Zeno's paradox. It satisfies REGULARITY, since every region is extended. It also satisfies INFINITE DIVISIBILITY in the finite case, though not in general. Moreover, the view satisfies WEAK ADDITIVITY and ZEROS-SUM-TO-ZERO. Whether it satisfies STRONG ADDITIVITY, or at what cost, will be discussed in the next section—where the bad news begins.

2.5 The Problem of Non-Measurable Regions

Suppose UNRESTRICTED COMPOSITION is true: every collection of minims has a fusion. Then some regions are not measurable.[15] This could happen, for example, when the region is the fusion of countably infinitely many disjoint regions, each of which is measurable. Imagine that you are walking along a straight line from the minim 0. You first walked 1/2 mile, then 1/4 mile, then 1/8 mile, and so on.[16] How many miles have you walked in total? The answer is "Unmeasurable." This is because the minims that are infinitesimally close to the minim 1 are not included in your journey. The fusion of these minims does not have a measure because these minims are not hyperfinitely many. So the total distance you have traveled is not measurable. Also, a region composed of minims in a monad is not measurable, since those minims are not hyperfinitely many (Section 3).

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[14]This version of additivity is as strong as it can get without building in UNRESTRICTED COMPOSITION, the principle that any regions have a fusion. Under UNRESTRICTED COMPOSITION, STRONG ADDITIVITY entails the full-blown version of additivity: for arbitrarily many measurable disjoint regions, they have a fusion, their fusion is measurable, and the measure of their fusion is the sum of the measures of those regions.

[15]Here I assume that the fusion of regions is also a region. This follows from my stipulation that "region" simply means a part of the continuum, together with the standard mereological principle that the fusion of parts of X is still a part of X.

[16]More precisely, the sequence of distances you walked is $\langle 1/2^n \rangle$, where *n* ranges over all *natural* numbers.

Indeed, if we assume UNRESTRICTED COMPOSITION, then there are many more unmeasurable regions than those under the standard view. For every set of real numbers, let's map it to the fusion of all minims that are infinitesimally close to those real numbers. All such fusions are unmeasurable, while sets of real numbers are generally Lebesgue measurable.[17] For example, the singleton set of a real number, which has a Lebesgue measure of zero, is mapped to the fusion of the minims in its monad, which is unmeasurable. The interval [0,1] is mapped to the fusion of the minims in the hyperreal interval $*[0,1]$ together with 0's and 1's monads, which is again unmeasurable. Thus we have an injective but not surjective map from non-Lebesgue-measurable sets of real numbers to unmeasurable regions that preserves mereological relations. It is unclear—and unlikely—that there is such a map from unmeasurable regions to non-Lebesgue-measurable sets. After all, even within a single monad, there are already uncountably many unmeasurable regions![18]

Therefore, if every collection of minims has a fusion, then like the standard view, infinitesimal atomism faces the problem of unmeasurable regions. The situation seems even worse for infinitesimal atomism, since there are more unmeasurable regions. Moreover, one strategy to get rid of non-Lebesgue-measurable sets in standard analysis is to reject the axiom of choice, the principle that for any collection of nonempty sets, there is a set that has exactly one element from each of those sets (Solovay 1970). But this option is not available for infinitesimal atomism: the existence of unmeasurable regions does not require the axiom of choice.[19]

For the sake of argument, let's assume that it is problematic to have unmeasurable regions. Does infinitesimal atomism have a better response than the standard view? In particular, can we deny UNRESTRICTED COMPOSITION instead? Although those who are convinced of this principle (such as Lewis 1991 and Bricker 2015) are unlikely to be moved

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[17]Like a monad, the sets of the minims in question are not *internal sets*, sets that are characterizable in the language of standard analysis.

[18]Every region composed of countably infinitely many minims is unmeasurable, and every monad contains uncountably many countably infinite collections of minims.

[19]The ultrafilter construction of the hyperreal system requires only *the Boolean prime ideal theorem*, which ensures the existence of nonmeasurable sets and is strictly weaker than the axiom of choice. Thanks to an anonymous referee for pointing out this difference.

by considerations about measure theory alone, this option would still be attractive if something principled and intuitively relevant can be said about what collections of regions have fusions. The following principle is a natural candidate if we want to avoid unmeasurable regions:

> HYPERFINITE COMPOSITION†. A collection of regions has a fusion if and only if it's hyperfinite.[20]

If we adopt HYPERFINITE COMPOSITION†, it will follow that all regions are measurable. But is the condition of hyperfiniteness intuitively relevant for composition? Here's one line of reasoning that may encourage such an idea. In fact, the hyperfinite subsets of M are precisely those *internal* subsets of M, namely those that are expressible in the language of standard analysis.[21] Let ϕ be a schematic variable for any one-place formula in a suitable language, and let $\text{Fus}_\phi z$ mean that z is a fusion of all the things that satisfy ϕ.[22] Consider the following schematic way of capturing UNRESTRICTED COMPOSITION (Cotnoir and Varzi 2018):

$$(\Phi)\ \exists x\phi(x) \rightarrow \exists z \text{Fus}_\phi z.$$

If we take ϕ as only ranging over the formulas in the language of standard analysis (call it $\mathcal{L}$), then given that all and only hyperfinite subsets of M are $\mathcal{L}$-expressible, Φ is equivalent to HYPERFINITE COMPOSITION†.

But why should we restrict ϕ in Φ to the $\mathcal{L}$-formulas? Such a restriction is reasonable only if we think the nonstandard model only has internal entities. For example, one might think that, because the countably infinite set of minims $\{\Delta, 2\Delta, ...\}$ is not $\mathcal{L}$-expressible, there is no such set in the nonstandard model. Therefore, the minims $\Delta, 2\Delta, ...$ do not compose a region. But such a connection between existence and $\mathcal{L}$-expressibility is indefensi-

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[20]To adopt this principle, we must impose a further constraint on the set of minims M: it must be bounded by hyperreal numbers. Otherwise the minims in M would not be hyperfinitely many, and thus would not compose a region.

[21]Here, we assume that M is bounded by hyperreal numbers (Footnote 20). Also see Goldblatt (1998, 166-73) for more information about internal sets.

[22]$\text{Fus}_\phi z$ can be defined in terms of parthood in a first-order language (let "Pxy" mean x is a part of y): $\text{Fus}_\phi z$ iff $\forall x(\phi x \rightarrow Pxz) \wedge \forall y(\forall x(\phi x \rightarrow Pxy) \rightarrow Pzy)$. (Cotnoir and Varzi 2018)

ble. After all, the notion of *infinitesimal* is not $\mathcal{L}$-expressible.[23] But we still think the notion of infinitesimal intelligible and that there is, for example, a collection of regions infinitesimally close to zero—at least if we want to take infinitesimal atomism seriously. So I don't think that a restriction of composition to *internal* collections of regions is well-motivated.

Therefore, infinitesimal atomism doesn't appear to have better options than the standard view in answering the problem of unmeasurable regions. Like the standard view, infinitesimal atomism leads to the result that there are unmeasurable regions, and therefore also violates STRONG ADDITIVITY. Note that I am *not* arguing it is necessarily bad to have unmeasurable regions—the point is just that the problem of unmeasurable regions cannot be the reason for favoring infinitesimal atomism over the standard view.

Notice that *finite* atomism does not face this problem. For every region, it is either composed of finitely many atoms or else of infinitely many. If a region is composed of finitely many atoms, which have finite sizes, the region has a finite size; otherwise it has a size of positive infinity $+\infty$. So every region is measurable.

2.6 Weyl's Tile Argument

Weyl's tile argument purports to show that if finite atomism is true, then the Pythagorean theorem is not even approximately true. So, as long as we hold onto the Pythagorean theorem as an approximate law of geometry, finite atomism is false. Philosophers and physicists have found the argument challenging. In what follows, I will first explain Weyl's tile argument and its underlying assumptions and only turn to infinitesimal atomism at the end.

Weyl wrote:

> How should one understand the metric relations in space on the basis of this idea? If a square is built up of miniature tiles, then there are as many tiles along the diagonal as there are along the side; thus the diagonal should be equal in length to the side. (Weyl 1949, 42)

[23] Recall that infinitesimals are those whose absolute values are smaller than any positive real number. But in NSA, the set of positive hyperreal numbers satisfies the exact same first-order truths as the set of positive real numbers in the language of standard analysis. So the set of positive real numbers is not expressible, which means that the set of infinitesimals is not expressible either.

For example, consider the region composed of 4×4 square tiles shown in Figure 2.[24] A, B, C are corner tiles. The side of the square AC is four units long. There are four tiles along the diagonal BC, which, according to Weyl, implies that BC is four units long. But if the Pythagorean theorem is approximately true, then BC should be about 5.7 units long. Adding more tiles does not help. If the square is composed of 8×8 tiles, the diagonal is still as long as the side—both are eight units long.

The argument relies on an important assumption about length in atomistic space: the length of a “line segment" is equal to the number of tiles the “line segment" contains. This assumption can also be put in terms of distance:[25]

> DISTANCE. The distance between any two tiles is equal to the number of tiles between them.

What is the rationale for DISTANCE? As Bricker (1993) pointed out, our best theory of physics supports a *path-dependent local* account of distance: the distance between any two points is the length of the shortest path between them, and the length of a path is determined by the local metric properties of the points along the path. Although the standard definition of path does not directly apply to atomistic space, we can define “path" in atomistic space as a fusion of minims in a “chain of adjacency."[26] More precisely, a *path* between minims M_1, M_n is the fusion of minims $M_1, M_2, ..., M_n$ such that M_k and M_{k+1} are connected for every $k = 1, ..., n-1$.[27] Given this definition, Weyl's tile argument can be understood as implicitly assuming that two tiles are connected just in case they are horizontally, vertically, or diagonally adjacent. It follows that the four tiles along the diagonal in Figure 1 compose a path.[28]

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[24]My presentation of the argument draws on Salmon (1980).

[25]Although it is possible to distinguish between “length," a *measure* defined over regions of space, and “distance," a *metric* defined over pairs of points or atoms, it is standardly assumed that the argument can be put in terms of distances.

[26]In standard differential geometry, a path between two points is a continuous function from the real interval [0,1] to the space, which takes 0 to one of the two points and 1 to the other one.

[27]*Connectedness* is a primitive topological notion for finite atomism and infinitesimal atomism.

[28]Another intuitive option is to stipulate that two tiles are connected *iff* they are horizontally or vertically adjacent. Under this option, the diagonal BC is the zigzag region. This option is similarly problematic.

Furthermore, we can take Weyl's tile argument as assuming that, in atomistic space, the size of a minim (which is one unit) is the only primitive local metric property. Then, according to the path-dependent local account of distance, the length of a path in atomistic space is equal to the number of minims the path contains. Given that the distance between any two minims is equal to the length of the shortest path, DISTANCE follows.

Suppose that finite atomism is subject to Weyl's tile argument. In that case, can infinitesimal atomism do better? The answer is no. Let N be an infinite hypernatural number: Consider a region composed of $N \times N$ tiles. Let each tile be represented by a pair of hypernatural numbers. For example, the tiles contained in a side can be represented by $(1,1),(1,2),...,(1,N)$. The tiles contained in a diagonal can be represented as $(1,1),(2,2),...,(N,N)$. Both sets of tiles clearly have the hyperfinite cardinality of N. Therefore, if we assume DISTANCE applies to hyperfinite cardinality, the same departure from the Pythagorean theorem ensues.[29]

As before, I am not saying that Weyl's tile argument is devastating for infinitesimal atomism, but rather that it does not provide a reason to favor infinitesimal atomism over ordinary finite atomism. (In fact, I think finite atomism has a good response to Weyl's tile argument, which I defend in Chapter 4; see also Forrest 1995, Van Bendegem 1987, 1997 for other responses.)

2.7 Conclusion

In this paper, I have developed a theory of continua with a non-standard measure, infinitesimal atomism, in response to Zeno's paradox of measure. Among the principles Zeno's paradox relies on, DICHOTOMY is denied because the ultimate parts of continua have an infinitesimal measure, while other principles are satisfied to some degree. Most notably, the theory satisfies WEAK ADDITIVITY, the principle that the measure of every measurable region is the sum of the measures of its disjoint parts. The measure theory

[29]Skyrms's continuum might be able to escape this argument. Although Skryms did not explicitly define a distance function over the continuum, he can resort to the standard distance function over standard space because Skyrms's continuum is isomorphic to standard space. The infinitesimal measure of a "point" does not play any role in determining distances.

is very rich and approximates the familiar Lebesgue measure over standard space. REGULARITY is satisfied, in which respect infinitesimal atomism is more intuitive than the standard view.

However, it turned out that the theory does not score well against its main competitor, finite atomism, the familiar view that continua are composed of finitely extended atoms. Finite atomism has most of the benefits of infinitesimal atomism, such as REGULARITY and WEAK ADDITIVITY. Furthermore, it satisfies STRONG ADDITIVITY and is free of the problem of unmeasurable regions, which troubles both the standard view and infinitesimal atomism. Weyl's tile argument is the main drawback for finite atomism, but infinitesimal atomism has no special resources for answering it that are unavailable to finite atomism. Infinitesimal atomism does have two distinctive advantages over finite atomism: first, it satisfies FINITE DIVISIBILITY, the principle that every finite region can be further divided into smaller regions, and second, it has a richer measure than either of its competitors. But all things considered, it's hard to say that these advantages count decisively in its favor.

Therefore, if we are not satisfied with the standard view and finite atomism, infinitesimal atomism provides no better consolation, and we must continue looking for alternatives.

CHAPTER 3

INFINITESIMAL GUNK

In this chapter, I advance an original view of the structure of space called *Infinitesimal Gunk*. This view says that every region of space can be further divided and some regions have infinitesimal size, where infinitesimals are understood in the framework of Robinson's (1966) nonstandard analysis. This view, I argue, provides a novel reply to the inconsistency arguments proposed by Arntzenius (2008) and Russell (2008), which have troubled a more familiar gunky approach. Moreover, it has important advantages over the alternative views these authors suggested. Unlike Arntzenius's proposal, it does not introduce regions with no interior. It also has a much richer measure theory than Russell's proposal and does not retreat to mere finite additivity.

3.1 Is Space Pointy?

Consider the space you occupy. Does it have ultimate parts? According to *the standard view*, the answer is yes: space is composed of uncountably many unextended points.[1] Although standard, this view leads to many counterintuitive results. For example, intuitively, the size of a region should be the sum of the sizes of its disjoint parts.[2] But according to the standard view, the points have zero size. Thus they cannot add up to a finite size, because zeros always add up to zero.

For another example, according to the standard view, every region of space (except the whole space) has a boundary, and a closed region includes its boundary. Now, suppose

[1] "Space" can be understood as physical space or (mathematical) geometric space: the discussions in this paper do not turn on the differences between them. Many considerations also apply to time or spacetime.

[2] This is one of the intuitions behind Zeno's paradox of measure. See Skyrms (1983) and Butterfield (2006).

that two rigid bodies which occupy closed regions come into perfect contact: there is no gap between them. Under the standard view, we cannot put two closed regions side by side without overlapping and without leaving a gap between them. Thus, to be in perfect contact, the two closed regions must overlap on their boundaries. But the bodies are rigid and impenetrable, so they should not occupy overlapping regions. Therefore, if the standard view is true, two rigid bodies that occupy closed regions cannot come into perfect contact. But perfect contact is intuitively possible. This is called "the contact puzzle."(See Zimmerman 1996, Artnzenius 2008 and Russell 2008)

Due to these problems, a *gunky* conception of space has been proposed, according to which space cannot be broken down into ultimate parts. That is, every part of space can be further divided, and extensionless points do not exist. Such a conception can be traced back to the ancient Greeks, such as Anaxagoras.[3] Its contemporary development is often associated with A. N. Whitehead (1919, 1920, 1929). Under Whitehead's theory, all regions have at least a finite size. So, it avoids the counterintuitive result that an extended region is composed of unextended points. Moreover, the contact puzzle can be avoided by denying the existence of boundaries. Call this approach *the finite gunky view* ("finite" as opposed to the infinitesimal approach that I shall soon introduce). However, both F. Arntzenius (2008) and J. Russell (2008) pointed out that the finite gunky view, in conjunction with other plausible assumptions, is inconsistent with countable additivity, an attractive measure-theoretic principle. These authors proposed their own solutions, but at the expense of some attractive features of the original view. Arntzenius suggests readmitting boundaries with nonzero measures, even though they are scattered points with no interiors. Finding this proposal unattractive, Russell suggests rejecting countable additivity instead and having merely finite additivity. But the resulting measure theory is impoverished.

However, the *finite* gunky view is not the only approach to the gunky conception of space: a different approach is to claim that space also has parts of *infinitesimal* sizes, which can be further divided. It has been argued that such a notion of divisible infinitesimals appeared in the Chrysippean doctrine of space, time and motion (White 1992). Although

[3] "Nor of the small is there a smallest, but always a smaller..."(Curd 2007, B3)

it has an ancient origin, such a doctrine was never developed further, due to the alleged obscurity of an infinitesimal size. But the situation has changed since the development of nonstandard analysis by Abraham Robinson (1966), which gives infinitesimals a rigorous foundation.[4] In this paper, I will develop a gunky view of space, *Infinitesimal Gunk*, in the framework of nonstandard analysis. Like the finite gunky view, this view implies that every part of space can be further divided, and there are no boundaries. But unlike the finite gunky view, it implies that some parts have infinitesimal sizes. Developing such a view is not straightforward, for novel technical difficulties arise as we turn to nonstandard analysis. Thus part of my goal is to solve these difficulties and present a rigorous and most plausible gunky view in the framework of nonstandard analysis. In addition, I will advance Infinitesimal Gunk as a novel reply to the inconsistency arguments of Arntzenius and Russell. I will argue that Infinitesimal Gunk has distinctive advantages over the solutions proposed by these authors. Unlike Arntzenius's proposal, it does not need to admit regions without interiors. It also has a much richer measure theory than Russell's proposal. Infinitesimal Gunk also violates countable additivity, but it has attractive measure-theoretic compensations unavailable to Russell's proposal.

3.2 Trouble for the Finite Gunky View

Before I present *the finite gunky view*, I shall first lay out the main ideas of a gunky space without assuming that every region of space has at least a finite size. The intuitive ideas of a gunky space can be put into the following:

> *Gunky Space.*[5] (Mereology) Every region has a proper part. (Topology) There are no boundary regions. (Measure theory) Every region has a subregion of a strictly smaller size.

The mereological aspect can be considered the starting point or the core claim of any gunky view of space. Since extensionless points have no proper parts, it follows that there are no

[4]This is not the only foundation for infinitesimals. I explore alternative theories, such as *smooth infinitesimal analysis*, in other work (Chen, manuscript). For my work on atomistic space in the framework of nonstandard analysis, see Chen (2019).

[5]While the notion of gunky space is usually asssociated with only the mereological aspect, I am anticipating a more developed theory.

points in space. The topological aspect needs some explanation. A boundary of a region is generally lower-dimensional than the region itself: it's like the skin of an apple if we idealize by imagining the skin to have no thickness at all.[6] The requirement that there are no boundaries thus reflects the "gunky" intuition that space has no lower-dimensional parts. This intuition is in a similar spirit to the mereological aspect: just as there are no indivisible points, there are no lines or surfaces in a higher-dimensional space because they cannot be divided along a particular dimension.[7] Finally, the measure-theoretical aspect is also closely associated with the mereological aspect. It follows from the measure-theoretic principle and the mereological aspect (together with some plausible assumptions) that every region has a positive measure.[8] The principle that every region has a positive measure is motivated by the consideration that no extended region is entirely composed of unextended ones.

Before getting to the finite gunky view, we need to know a few topological terms. In standard topology, *openness* is the only primitive topological notion: a *topological space* is a set together with some choice of its open subsets (satisfying certain constraints). A *closed* set is the complement of an open one. The *interior* of a set is the union of its open subsets—like the flesh of an apple inside its ideally thin skin. The *closure* of a set is the intersection of the closed sets including it—like a whole apple to its flesh. Further, a set that is identical to the closure of its interior is called *regular closed*. For instance, consider the real line $\mathbb{R}$ (with its standard topology). The singleton of a point is not regular closed, because the closure of its interior is empty. Similarly, a set that includes an isolated point is not regular closed (e.g. $[0, 1] \cup \{2\}$). Now, every equivalence class of sets of real numbers that differ at most on their boundaries includes exactly one regular closed set. For the finite gunky view, the intuitive idea is that, since boundaries do not exist, every region should correspond to exactly one such equivalence class (except that of the empty set). In that case,

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[6]The precise definition of "boundary" in gunky space will be given later.

[7]This reason does not apply to boundaries in general, since boundaries such as the fusion of two points in a one-dimensional space can be divided into two points.

[8]More explicitly, we need the following assumptions: (1) WEAK SUPPLEMENTATION: if a region x has a proper part y, then x also has a proper part z that is disjoint from y; (2) FINITE ADDITIVITY: for finitely many regions, the size of their fusion is the sum of the sizes of those regions.

every region can be represented by the regular closed set in its corresponding equivalence class. For simplicity, I will henceforth pretend that our space is one-dimensional (most discussion can be carried over to higher-dimensional cases straightforwardly). In the finite gunky view, we postulate the following principle for gunky space, which will be further strengthened later.

> REAL REPRESENTATION. There is a one-to-one correspondence between all regions of space and all non-empty regular closed sets of real numbers such that a region X is a part of a region Y iff X's corresponding set is a subset of Y's corresponding set.[9]

As in standard mereology, other mereological notions can be defined in terms of parthood. For example, two regions *overlap* iff they share a common part. Two regions are *disjoint* iff they do not overlap. A region X is a *fusion* of regions Ys iff each Y is a part of X and every region that overlaps X also overlaps one of Ys. For any collection of regions, their mereological fusion corresponds to the *closure* of the union of their corresponding sets because the union of regular closed sets may not be regular closed and the closure of the union is the smallest regular closed set that includes those sets. For example, the fusion of the gunky regions represented by $[0,1/2]$, $[0,3/4]$, $[0,7/8]$... is not represented by the union of those intervals, namely $[0,1)$, which is not regular closed, but by its closure $[0,1]$. Since every nonempty regular closed set includes a non-empty regular closed set as a proper subset, every region has a proper part—the mereological aspect of gunky space is confirmed. (For brevity, I will henceforth use "set" to refer to non-empty sets unless otherwise specified.)

We can also specify the topology of gunky space. In the standard topological framework, we have "openness" as the primitive notion. However, this framework is inadequate in the case of gunky space. Since we want there to be no boundaries, there should be no distinction between "open" and "closed" regions that differ at most on their boundaries. As Roeper (1997) and other authors suggest, instead of "openness," we can use the binary

[9] Note that this principle (along with the measure specified later) implies that there are infinite regions of space. Although whether physical space is infinite or not need not to be settled by a gunky view, we postulate infinite regions for convenience. If one wishes to have a gunky view of space with only finite regions, one can modify REAL REPRESENTATION easily.

relation *connectedness* as a primitive notion. To postulate the topology of gunky space, we strengthen REAL REPRESENTATION by the following clause:

> A region X is connected to a region Y iff X's corresponding set intersects Y's corresponding set.

For example, $[0,1]$ and $[1,2]$ represent two connected regions since they intersect at 1. Following Russell (2008), other topological terms can be defined in terms of connectedness. For example, a region X is a *boundary* of a region Y iff every part of X is connected to both Y and some region disjoint from Y. Intuitively, an apple's (ideally thin) skin is the boundary of the apple because every part of the skin is in contact with both the apple and its surrounding air. It follows that no region is a boundary of any region—thus the topological aspect of gunky space is met (Russell 2008, 7). Then we can define "openness": a region is *open* iff it does not overlap any of its boundaries. It follows that every region is open. I will henceforth call the topologically strengthened REAL REPRESENTATION together with the measure-theoretic principle that every region has a strictly smaller subregion *the finite gunky view*.

How should we measure regions? In standard analysis, the Lebesgue measure, which takes values in the nonnegative extended real numbers $[0, +\infty]$, is the standard way of assigning length, area, volume, and so on to subsets of a real coordinate space. Given REAL REPRESENTATION, it is natural to assign to a region the Lebesgue measure of its corresponding set. More precisely, we strengthen REAL REPRESENTATION by the following clause:

> LEBESGUE GUNKY MEASURE. The measure of any region is equal to the Lebesgue measure of its corresponding set.

We can check that every region indeed has a subregion of a strictly smaller size. However, Arntzenius (2008) shows that this measure is not countably additive.

> COUNTABLE ADDITIVITY. For any countably many disjoint regions, their fusion has a measure, which is the sum of the measures of those regions.[10]

[10]Standard measure theory satisfies COUNTABLE ADDITIVITY. In particular, the fusion of countably infinitely many disjoint regions with positive measures has a measure of $+\infty$.

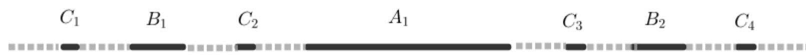

Figure 3.1. The fat cantor set

The inconsistency argument runs as follows. Consider a unit closed interval of real numbers. Take the middle closed interval of length 1/4. Then take the middle intervals of length 1/16 from both of the remaining intervals. Repeat the same process *ad infinitum*. In the i-th step, each middle interval we take is $1/4^i$-inch long. Label those intervals $A_1, B_1, B_2, C_1, ...$ such that the same letters refer to intervals of the same lengths. The intervals taken in the first three steps are illustrated below :
Call the intervals *the Cantor intervals* and their union *the Cantor union*. A point is a *limit point* of a set iff any open set including that point intersects that set. The closure of a set is the union of this set and all its limit points. We have that every point in the unit interval either belongs to the Cantor union or is a limit point of it. Thus, the closure of the Cantor union is the whole unit interval. Now, each Cantor interval represents a gunky region. Call those regions *the Cantor regions*. It follows from REAL REPRESENTATION that the fusion of the Cantor regions is represented by the closure of the Cantor union, which is the whole unit interval. Thus, the Lebesgue gunky measure of the fusion is one. However, the Cantor regions that compose the fusion respectively measure $1/4, 1/16 \cdot 2, 1/64 \cdot 4,...$, which sum up to 1/2. Suppose the measure theory is countably additive. Then the measure of the fusion is 1/2. It follows that the measure of the fusion of the Cantor regions is both 1 and 1/2. So, LEBESGUE GUNKY MEASURE is not countably additive.

What's worse, Russell (2008) pointed out that LEBESGUE GUNKY MEASURE is even inconsistent with finite additivity. Let *Big Fusion* be the fusion of the Cantor regions represented by *Big Intervals* $A_1, C_1, C_2, C_3, C_4, ...$, and let *Small Fusion* be the fusion of the Cantor regions represented by *Small Intervals* $B_1, B_2, D_1, D_2, ...$[11] It turns out that every point on the unit interval is either a point in the Cantor union or a limit point of *both* the union of

[11] The symbols are used as before: those with the same letter refer to intervals of the same length. Each of the Bs is one-fourth the length of A_1, and each of the Cs is one-fourth the length of each of the Bs, etc.

Big Intervals *and* the union of Small Intervals. Then, the Lebesgue gunky measure of Big Fusion is the Lebesgue measure of the closure of the union of Big Intervals, which is $5/6$. The measure of Small Fusion is the Lebesgue measure of the closure of the union of Small Intervals, which is $2/3$. But the fusion of Big Fusion and Small Fusion is represented by the whole unit interval which measures one. Since Big Fusion and Small Fusion are disjoint regions, and $5/6 + 2/3 \neq 1$, finite additivity fails!

Russell gave a more general argument for the inconsistency between the finite gunky view, COUNTABLE ADDITIVITY, and other plausible assumptions, which does not rely on LEBESGUE GUNKY MEASURE or any other specific measure. Instead, he appealed to an additional topological feature. In both standard topology and gunky topology, a *basis* for a topological space is a set of *basis elements* which generate all the open regions of the space by taking unions. A topological space can have many different bases. For instance, the set of all open cubes is a basis for the standard topology of three-dimensional real space $\mathbb{R}^3$. This implies that, for example, an apple-shaped open set in $\mathbb{R}^3$ is the union of many cubes. In standard topology, a topological space typically has a countable basis, that is, a basis with only countably many basis elements. For instance, one basis for the standard topology of the real line is the set of all open intervals with rational numbers as endpoints, which are countably many. Russell argued that, like a real coordinate space, our space has a countable basis.

COUNTABLE BASIS. The topology of space has a countable basis.

Russell has shown that the finite gunky view, if it satisfies COUNTABLE BASIS and some standard mereological assumptions, is inconsistent with COUNTABLE ADDITIVITY (Russell 2008, 9).[12]

In the face of the two inconsistency arguments, various solutions have been proposed. Arntzenius suggested denying that there are no boundary regions. Instead of REAL REPRESENTATION, Arntzenius suggested that a region should be represented by an equivalence class of Borel sets (formed from open sets through countable union, countable inter-

[12]The main mereological assumption is REMAINDER CLOSURE, the principle that unless a region X is proper part of a region Y, X has a part that is the remainder of Y in X (or the mereological difference $X - Y$). (Russell 2008, 4)

section and complement) that differ up to Lebesgue measure zero. Under his proposal, the fusion of the Cantor regions is distinct and smaller than the unit line segment, and there exists a boundary region that makes up the difference between the two regions. This region is represented by the equivalence class of the complement of the Cantor union and has no interior. Since this feature seems undesirable, we should check if there is a better alternative.[13] Russell suggested that we should deny COUNTABLE ADDITIVITY and retreat to *merely* finitely additive measure. Under this suggestion, the measure of any region is equal to the Jordan measure of its corresponding set (a restriction of the Lebesgue measure that is merely finitely additive). But the resulting measure theory is rather impoverished (which I discuss in Section 5). Is there another way out? Infinitesimal Gunk, the nonstandard theory of space that I shall develop in Section 3, will provide a novel reply to the inconsistency arguments. In particular, COUNTABLE BASIS fails, but without the costs that Russell assumed (Section 4). Although this theory also violates COUNTABLE ADDITIVITY, it satisfies a weaker version of it and has a much richer measure theory than Russell's proposal (Section 5). The theory is also immune to the variants of Arntzenius's argument (Section 6).

3.3 Infinitesimal Gunk

The core idea of Infinitesimal Gunk is that, instead of representing a gunky region by a set of real numbers, we represent a region by an extended set of numbers provided by nonstandard analysis (NSA). In NSA, we can extend the real line $\mathbb{R}$ to the hyperreal line $*\mathbb{R}$, which includes infinitesimals and infinite numbers, along with the familiar real numbers.[14] A number is *infinitesimal* iff its absolute value is smaller than any positive real number. A number is *infinite* iff its absolute value is larger than any positive real number. On the hyperreal line, each real number is surrounded by a "cloud" of hyperreal numbers

[13]In this paper, I do not discuss why this feature of Arntzenius's theory is undesirable. I simply assume that it is intuitively attractive for a gunky space to have no boundaries (see Russell 2008).

[14]The hyperreal system I refer to in this paper is obtained through the ultrapower construction of real number sequences (see Goldblatt 1998). It's unique up to isomorphism under the assumption of continuum hypothesis (thanks to a referee for pointing this out). There are other non-isomorphic hyperreal systems. For example, the surreal number system introduced by Conway (1976) is considered the largest hyperreal system.

that are infinitesimally close to it, called its *monad*. The monads of different real numbers do not overlap. Moreover, we can do arithmetic on the hyperreal numbers just like on the standard real numbers. For example, for any positive infinitesimal δ and any infinite number N, we have $\sqrt{\delta}$, $\frac{1}{\delta}$, δ^2, $\delta + N$, $\frac{\delta}{N}$, etc. In general, the hyperreal numbers satisfy all the first-order truths about the real numbers in standard analysis.

As in the finite gunky view, I would like to represent gunky regions by regular closed sets. But unlike in the finite gunky view, I will appeal to sets of hyperreal numbers rather than just sets of real numbers. Before presenting the view, I shall first introduce the interval topology of the hyperreal line. (For a comparison of different topologies on the hyperreal line, see Goldblatt 1998, 120-1, 143-5.) According to the interval topology, a hyperreal set is *open* iff it is the union of hyperreal intervals (a, b) (i.e., the set of hyperreal numbers strictly between hyperreal numbers a and b). For example, the set of all infinitesimals is open, because it is the union of all open intervals with infinitesimals as endpoints. Also, the set of all non-infinitesimals is also open, because it is the union of all open intervals of the form $(-N, -r)$ or (r, N), with r a positive real number and N a positive infinite number. Since the complement of an open set is closed, the set of all infinitesimals is both open and closed, and so is the set of all non-infinitesimals. Furthermore, both sets are regular, since they are identical to the closure of their interiors.

Now, for Infinitesimal Gunk, I propose that regions are represented by regular closed sets of hyperreal numbers under the interval topology.

> HYPERREAL REPRESENTATION. There is a one-to-one correspondence between all regions of space and all non-empty regular closed sets of hyperreal numbers such that a region X is a part of a region Y iff X's corresponding set is a subset of Y's corresponding set.[15]

As a basic example, the hyperreal interval $[0, 1]$ is regular closed and thus represents a region. The set of all infinitesimals—and in general each monad—is regular closed, and thus represents a region. Furthermore, the countable union of $[1/2, 1]$, $[1/4, 1]$, ..., $[1/2^n, 1]$, ... is also regular closed and represents a region.

[15] As in the finite gunky view, the set of all gunky regions forms a boolean algebra without the bottom element.

We need to define a topology for gunky space so that there are indeed no boundary regions. Like in the finite gunky view, I will follow Roeper (1997) in using *connectedness* as the primitive notion. But the difficulty here is that we cannot postulate the connectedness relation between gunky regions in Infinitesimal Gunk in the same way as in the case of the finite gunky view. Recall that, in the finite gunky view, we postulate that two regions are connected iff their representative sets have a non-empty intersection. But this will not do the trick in Infinitesimal Gunk, because every monad represents a region, and distinct monads have no elements in common. If we postulate connectedness in the same way, then every region represented by a monad would be disconnected from the region represented by its complement on the hyperreal line.[16] This feature would be bad for our theory—for we want to describe a continuous space, which is composed of regions connected with each other.

To solve this "disconnection" problem, I propose an alternative to Roeper's postulation of connectedness between gunky regions. The intuitive idea is that two regions are connected iff they each contain a part such that there is no "gap" between them. For any two sets A, B, let "$A \leq B$" mean that for any $x \in A$ and any $y \in B$, we have $x \leq y$. Similarly, for any set A and any point z, let "$A \leq z$," for example, mean that for any $x \in A$ we have $x \leq z$. Then, the relation of connectedness satisfies the following principle:

> CONNECTEDNESS. Two regions represented by sets A and B are *connected* iff there is a subset A' of A and a subset B' of B such that either (1) $A' \leq B'$, and there is no point z with $A' < z < B'$, or (2) $B' \leq A'$, and there is no point z with $B' < z < A'$.[17] (Call such a z a *separating point*.)

It immediately follows from this definition that a region represented by a monad is indeed connected with the region represented by the complement of the monad on the hyperreal line. Under this definition, connectedness is *reflexive*, *symmetric*, and *monotonic*—the three features that constitute what Russel calls "core topology" (Russell 2016, 262). In addition, connectedness is *distributive*. Together, these features are essentially what Roeper took to

[16]Indeed, the interval topology itself has this problem: under the definition of connectedness in standard topology, every monad is disconnected from the rest of the hyperreal line.

[17]In higher-dimensional space—informally speaking—two regions are connected iff they each contain a part such that there is no hyperreal hypersurface between their corresponding sets.

be the "central characteristics" of connectedness (Roeper 1997, 255).[18] Let X, Y, Z range over all regions:

> REFLEXIVITY. X is connected to itself.
> SYMMETRY. If X is connected to Y, then Y is connected to X.
> MONOTONICITY. If X is connected to Y, and Y is a part of Z, then X is connected to Z.
> DISTRIBUTIVITY. If X is connected to the fusion of Y and Z, then X is either connected to Y or to Z.

I shall explain why DISTRIBUTIVITY holds, since it's relatively less obvious. Take two arbitrary regions represented by sets B and C. First, we note that the union of any two regular closed sets is still regular closed. Thus, the fusion of the two regions is represented by the union of B and C. Suppose a region represented by set A is connected to the fusion of those two regions. According to CONNECTEDNESS, there are subsets $A' \subseteq A, B' \subseteq B$ and $C' \subseteq C$ with $A' \leq B' \cup C'$ or $B' \cup C' \leq A'$ such that there are no separating points between A' and $B' \cup C'$.[19] It follows that either there are no separating points between

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[18]A small complication is that Roeper's core axioms for connectedness assume the existence of the null region, while I assume there is no null region, which would require some small changes in the formalism.

It is worth noting that my stipulation of connectedness does not satisfy all of Roeper's axioms beyond the core axioms. In particular, with any reasonable definition of another primitive notion *limitedness* in Roeper's axioms, Infinitesimal Gunk would violate the following axiom (a region X is *well inside* a region Y iff X is not connected to Y's complement):

> A10. If A is a limited region, B is not the null region, and A is well inside B, then there is a limited region C such that A is well inside C and C is well inside B. (Roeper 1997, 256)

In whatever way we define *limitedness*, it is reasonable to assume that at least the hyperreal interval $[0, 1]$ is limited. In the following sketch of a counterexample to A10, I will use Roeper's axiom that every part of a limited region is limited. A10 can fail for some infinite fusion of infinitesimal intervals. Let ϵ be an infinitesimal. Let A be represented by the union of all intervals $[4n\epsilon, (4n+1)\epsilon]$ for all hypernatural n such that $n\epsilon$ is an infinitesimal. Let B be represented by the union of slightly larger intervals $[(4n-1)\epsilon, (4n+2)\epsilon]$ (with the same restriction on n) together with the set of all non-infinitesimal numbers. Note that A is entirely contained in the monad of zero and thus limited, and it is well inside B. But there is no region C that satisfies A10. Call a region *snuggly* iff its representing set contains arbitrarily small positive non-infinitesimal numbers. It can be straightforwardly checked that, for any region C, if A is well inside C, then C is snuggly. But if C is snuggly, then C is connected with B's complement. So A10 must be violated.

One main role of Roeper's axioms is to ensure that there is a one-to-one correspondence between gunky topologies (or "region-based topologies" in Roeper's term) and locally compact Hausdorff spaces under standard point-set topology. As a result of the violation of A10, we cannot recover a locally compact Hausdorff space from Infinitesimal Gunk through Roeper's correspondence (see Roeper 1997, 276, 278-9). This is not terribly surprising because the interval topology of the hyperreal line is not locally compact.

[19]Note that B' or C' in question could be empty, though they can't both be—every point separates the empty set from other sets.

A' and B' or there are no separating points between A' and C', for otherwise at least one of them would separate A' and $B' \cup C'$. For example, if $A' \leq B' \cup C'$, then whichever of a separating point between A' and B' and one between A' and C' is smaller, it would separate A' and $B' \cup C'$. So A is either connected to B or to C.

As in the finite gunky view, other topological terms are defined in terms of connectedness. It follows from CONNECTEDNESS that no region is a boundary of any region. Recall that a region X is a boundary of a region Y iff every part of X is connected to both Y and some region disjoint from Y. Suppose there is such a boundary region. Then it is represented by some regular closed set. The intuitive idea is that a regular closed set is "fat" enough that we can always find a regular closed set strictly inside it. This smaller regular closed set represents a region that is disconnected from any region disjoint from the boundary region. This contradicts the definition of boundary, so there are no boundaries. It follows that the condition for openness is trivially satisfied, which means that every region is open. Also, every region is closed because every region has a complement (which is open). The closure of a region is always itself.

Next, I shall postulate a measure over regions. Like in LEBESGUE GUNKY MEASURE, I will equate the measure of a region to the measure of its representing set. But first of all I shall propose a measure on the hyperreal line. The measure will be non-standard in the following senses. Instead of assigning nonnegative extended real numbers to subsets of a space, it assigns nonnegative *hyperreal* measures to certain hyperreal sets. Also, unlike standard measure theory, the measurable sets are not closed under countable union, which I will discuss more in Section 5.[20]

Similar to the construction of the Lebesgue measure on the real line, we first define the measure of a hyperreal interval:

> INTERVAL LENGTH. For a hyperreal interval with end points a, b, its measure is $|b - a|$.

Notice that the measure of an interval can be infinitesimal.

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[20]The measure I introduce here is similar to the "proto-measure" introduced in Goldblatt (1998, 207). However, Goldblatt did not consider it a measure precisely because the measurable sets are not closed under countable union. He instead used it to define an extended-real-valued measure *Loeb measure* which satisfies this requirement.

Next, we define the length of the hyperreal set that is a union of such intervals. But before that, I shall first explain the notion of *hyperfinite sum* in NSA. First, notice the following fact: for any countably infinitely many items, even if they have an infinite sum (i.e., the limit of partial sums) in standard analysis, they generally do not have an infinite sum in NSA, because the partial sums do not converge to a unique hyperreal number. For instance, the partial sums $1/2 + 1/4 + ... + 1/2^n$ do not converge to any unique hyperreal number.[21] In general, unlike the real line, the hyperreal line does not have the least upper bound property: the set of all the partial sums $1/2 + 1/4 + ... + 1/2^n$ does not have a least upper bound. For every infinitesimal ϵ, $1 - \epsilon$ is an upper bound of the set, and there is no largest infinitesimal.

However, there is a special kind of infinite "cardinality," and accordingly a special kind of infinite sum in NSA. Recall that the set of all hyperreal numbers is an extension of the set of all real numbers. In the same sense, the set of all natural numbers can be extended to the set of *hypernatural numbers*, which obey the same first-order truths of standard analysis as the natural numbers. Just as any real number is smaller than some natural number, any hyperreal number is smaller than some hypernatural number. Since there are infinite hyperreal numbers, it follows that there are also infinite hypernatural numbers. Let N be a hypernatural number. In NSA, there is a distinct notion of "cardinality"—call it *hyperfinite cardinality*—that assigns N to $\{1, 2, ..., N\}$, just as the finite set $\{1, 2, ..., n\}(n \in \mathbb{N})$ has a cardinality of n. A hyperfinite set is either finite or else continuum-sized. Furthermore, just as in standard analysis the sum of a finite sequence of real numbers is well-defined, in NSA, the sum of a hyperfinite sequence of hyperreal numbers is well-defined. This is called *the hyperfinite sum.* Note though, hyperfiniteness is different from finiteness when it comes to higher-order claims: for example, a subset of a hyperfinite set need not be

[21]There are two ways of defining "converge" here. First, we can say that the partial sums $1/2 + 1/4 + ... + 1/2^n$ *converge* to a hyperreal number h, if their difference can be made smaller than any particular real number by making n sufficiently large. Second, we can define "converge" in a non-standard way: the partial sums $1/2 + 1/4 + ... + 1/2^n$ *converge* to a hyperreal number h, if their difference can be made smaller than any particular *hyperreal* number by making n sufficiently large. Under the first definition, the partial sums in question converge to many different hyperreal numbers. Under the second definition, the partial sums in question do not converge to any hyperreal number. Either way, there is no unique hyperreal number that the partial sums converge to.

hyperfinite.[22] (For more on "hyperfinite cardinality" and "hyperfinite sum," see Goldblatt 1998, 178-81.)

For any hyperreal set, if we can list the disjoint intervals it includes in a hyperfinite sequence, then its measure is the hyperfinite sum of the measures of those intervals.[23]

> HYPERREAL MEASURE. For any hyperreal set, if it is a union of hyperfinitely many disjoint hyperreal intervals, then its measure is the sum of the measures of those intervals. Otherwise, its measure is undefined.

Such a measure is well-defined because, like in the finite case, different decompositions of a hyperreal set into hyperfinitely many disjoint intervals (if possible) lead to the same measure.[24] Since every measurable set is a union of hyperfinitely many disjoint intervals, and because hyperfinite summation is associative like in the finite case, it follows that for hyperfinitely many disjoint measurable sets, the measure of their union is the sum of the measures of those sets.

> HYPERFINITE ADDITIVITY (SET). For hyperfinitely many disjoint measurable sets, the measure of their union is the sum of the measures of those sets.

The hyperreal measure approximates the Lebesgue measure over the real line in the following sense. Recall that any finite hyperreal number is infinitely close to exactly one real number. The real number is called the *shadow* of the hyperreal number. Let the shadow of any infinite positive hyperreal number be the extended real number $+\infty$. Let the shadow of a set be the set of the shadows of its members. Then, we have the following theorem:

> LEBESGUE APPROXIMATION. For any measurable hyperreal set, the shadow of its measure is the Lebesgue measure of its shadow (Goldblatt 1998, 215-7).

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[22] A subset of a hyperfinite set can be countably infinite, but no countably infinite set is hyperfinite.

[23] A *hyperfinite sequence* is an internal bijection from $\{1, 2, ..., N\}$, for some hypernatural N. An *internal* function is a function that is expressible in the language of standard analysis. (Appendix B; see also Goldblatt 1998, 172-5 for more detail.)

[24] That is, like in the finite case, it is true in nonstandard analysis that for any hyperfinitely many disjoint hyperreal intervals $B_1, B_2, ..., B_N$ and $C_1, C_2, ..., C_M$ (N, M are hypernaturals), if the union of all B_i is the same as the union of all C_j, then the hyperfinite sum of the measures of all B_i is equal to the hyperfinite sum of the measures of all C_j. Note that the language of standard analysis quantifies over sets as well as numbers, and these quantifiers also receive nonstandard internal interpretation in the hyperreal system (Goldblatt 1998, 168-170).

In other words, the measure of a hyperreal set, if well-defined, is infinitesimally close to the Lebesgue measure of its shadow on the real line.

Notice that some hyperreal sets, including some regular closed ones, are not measurable. For example, consider the set of all infinitesimals, which is regular closed. Intuitively, what should be the measure of such a set? The measure should be smaller than any real number. But it can't be an infinitesimal number because, for any positive infinitesimal number δ, the set is larger than $(-\delta/2, \delta/2)$. Thus, the set of all infinitesimals has no measure. It's worth noting that some non-measurable hyperreal sets do have a Lebesgue-measurable shadow. For instance, the shadow of the set of all infinitesimals is {0} and therefore has Lebesgue measure zero.

Finally, we can postulate the measure over a gunky line based on the measure over the hyperreal line and HYPERREAL REPRESENTATION:

> HYPERREAL GUNKY MEASURE. The measure of a gunky region is the measure of its representing regular closed hyperreal set.

For the reason I have just discussed, it follows that some gunky regions are not measurable.

I have completed the basic picture of Infinitesimal Gunk, according to which every region is further dividable and some regions have infinitesimal sizes. Now, I shall evaluate this view in light of the inconsistency arguments and compare it with other solutions.

3.4 No Countable Basis

Recall that Russell's inconsistency argument appeals to COUNTABLE BASIS, the assumption that space has a countable basis. However, this does not hold for Infinitesimal Gunk. Informally speaking, infinitesimals are really small—in fact, for any merely *countable* set of positive sizes, there are infinitesimal sizes which are even smaller than all of those. But that means that no countable collection of regions is "fine-grained" enough to build up all the regions. (See Appendix B.3 for my proof, which draws on a standard feature of the hyperreal system called *countable saturation*.)

When defending COUNTABLE BASIS, Russell wrote,

> There are topological spaces that do not have countable bases, but generally speaking, they are exotic infinite dimensional affairs. Such space would be

> shaped nothing like Euclidean space or any other ordinary manifold. (Russell 2008, 11)

These claims are not quite justified in light of nonstandard analysis. The hyperreal line is indeed “infinite dimensional” according to the standard definition of “dimension": there is a homeomorphism from the hyperreal line to a Euclidean space of infinite dimensions.[25] But this definition is just inadequate to capture the geometric nature of the hyperreal line, namely that it's a one-dimensional line—one *hyperreal* dimension.

Denying COUNTABLE BASIS in standard topology may result in “exotic,” ill-behaved spaces because it is associated with other desirable topological features of a space. For example, it is typically associated with *metrizability*. A topological space is *metrizable* iff we can define a real-valued distance between any two points such that the set of open balls with any radius are a basis for the topology. A *metric space*, which is of special interest in physics and mathematics, is a metrizable space together with a specific distance function. Every space with a countable basis is metrizable. Although the converse is not true in general, many commonly studied metrizable spaces have a countable basis.[26] So we typically require a space to have a countable basis to ensure that it is metrizable.

What hyperreal space shows is that it's not obvious that our physical space is metrizable, for it is completely natural to have a *hyperreal-valued* distance function, rather than a real-valued one. Let's call the corresponding notion *hypermetrizability*. Unlike metrizable spaces, it is not typical for a hypermetrizable space to have a countable basis—after all, a hyperreal space with the interval topology does not have a countable basis but is nevertheless hypermetrizable. For any two points on the hyperreal line a, b, we can define the distance between them to be $|b - a|$, which is the same as the length of the interval (a, b). These open intervals constitute a basis for the interval topology of the hyperreal line.[27]

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[25] A hyperreal number can be considered as an equivalence class of infinite sequences of real numbers that agree on “almost” every position (which is defined through an ultrafilter). (Goldblatt, 1998)

[26] Some rather unusual metric spaces do not have a countable basis. For example, consider any uncountable set. Let the distance between any two distinct elements be one. Then it generates a topology that does not have a countable basis (because for each element, its singleton is open). But such a distance function is not very interesting.

[27] Higher-dimensional cases are similar. For any two points in a hyperreal coordinate space $p = (p_1, p_2, ...), q = (q_1, q_2, ...)$, we can define the Euclidean distance between them, i.e., $d(p, q) =$

3.5 More Than Finite Additivity

I will now illustrate some desirable features of the measure in Infinitesimal Gunk by comparing it with the measure in Russell's solution. To avoid the inconsistency, Russell suggested rejecting COUNTABLE ADDITIVITY and using a merely finitely additive measure, such as the Jordan measure. Recall that in Arntzenius's inconsistency argument, COUNTABLE ADDITIVITY entails that the fusion of the Cantor regions has a measure of 1/2, but this fusion is represented by the unit interval, which has measure one. By rejecting COUNTABLE ADDITIVITY, Russell was able to claim that the fusion simply has measure one. To motivate this strategy, Russell argued that while finite additivity is necessary for understanding what a measure is, COUNTABLE ADDITIVITY need not be built into the nature of a measure. However, adopting a merely finitely additive measure has many drawbacks.

To start with, it violates an attractive principle of supervenience: for countably many disjoint measurable regions, the measure of their fusion (or whether there is one) is completely determined by the measures of those regions. That is, for any countably many disjoint measurable regions, no rearrangement will change the measure of their fusion (or whether their fusion has a measure). To put it more precisely:

> COUNTABLE SUPERVENIENCE. For any two countable sets A, B of disjoint measurable regions, if there is a measure-preserving bijection from A to B, then if the fusion of A is measurable, the fusion of B is measurable and has the same measure as the fusion of A.

Russell's solution violates this principle. For instance, if you first walked 1/4 mile, and then 1/16 mile twice, and then 1/64 mile four times, and so on, in a straight line, then the total distance you walked is a half-mile. But when regions of these same sizes happen to be arranged like the Cantor intervals, the total distance becomes one mile. Thus, the measure of the fusion of countably many disjoint regions is not determined by the measures of those regions. This seems magical. Now, many people have argued that this violation is no more magical than the violation of supervenience in the uncountable case

$\sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2 + ...}$ Then, all open balls constitute a basis for the interval topology of that hyperreal space.

in the standard view, according to which the length of a line segment does not supervene on the length of its constituent points (for example, see Hawthorne and Weatherson 2004). Thus, although the violation of COUNTABLE SUPERVENIENCE may be technically inconvenient, it is philosophically no worse than the violation of uncountable supervenience, or the failure of arbitrary supervenience. But it is not obvious whether all motivations behind COUNTABLE SUPERVENIENCE will generalize to all cases.[28] In standard mathematics, the countable case is usually more well-behaved than the uncountable case, so the countable case may be of special interest. In general, I will take it as an advantage to satisfy COUNTABLE SUPERVENIENCE (which Infinitesimal Gunk does, as I shall argue soon), but I will leave it open whether this advantage is significant.

There is also the question what determines the measures of countable fusions. In many cases, no non-arbitrary answer can be given. Recall that Big Fusion, the fusion of the regions represented by $A_1, C_1, C_2, ...$, and Small Fusion, the fusion of the regions represented by $B_1, B_2, D_1, ...$, compose a region represented by the unit interval (see Section 2). But the Lebesgue gunky measures of Big Fusion and Small Fusion are respectively $5/6$ and $2/3$. Since the resulting measure is not even finitely additive, Big Fusion and Small Fusion could not have these measures. What are their measures then? Russell suggested that we either consider them to be unmeasurable, or we assume brute facts about their measures that are compatible with finite additivity. We can assign to Big Fusion any value between its inner measure of $1/3$ and its outer measure of $5/6$, and to Small fusion any value between its inner measure of $1/6$ and its outer measure of $2/3$, as long as the sum of the two values add up to one (Russell 2008, 20-1). But these suggestions have clear drawbacks. If we consider those regions to be unmeasurable (without anything more to say), then the measure theory would be very restricted. But if those regions have measures, then the measure theory would involve many brute facts that are wildly different from their equally good alternatives.

[28] In probability theory, for example, Easwaran (2013) argued that some motivations behind countable additivity do not motivate uncountable additivity. So it is in principle possible to have considerations in favor of COUNTABLE SUPERVENIENCE that do not generalize.

The situation for Infinitesimal Gunk is subtle. On the one hand, like Russell's solution, Infinitesimal Gunk also violates COUNTABLE ADDITIVITY. For example, suppose you are walking a straight line, and you first walked 1/2 mile, then 1/4 mile, then 1/8 mile, and so on for all natural numbers. How many miles have you walked in total? It's not one mile because the monad at the end of the one mile is not included in your journey. Indeed it's unmeasurable because the monad is not measurable. In fact, we can prove that for *any* countably infinitely many disjoint measurable regions, their fusion is unmeasurable. It takes two steps to prove this claim. First, for any countably infinitely many disjoint measurable hyperreal sets, their union is unmeasurable (see Appendix B.5 for my proof based on countable saturation). Second, for any countably many disjoint regular closed sets, their union is regular closed (which, as I show in Appendix B.6, also follows from countable saturation). Since measurable regions are represented by measurable regular closed sets, it follows that for any countably infinitely many disjoint measurable regions, their fusion is always unmeasurable.

On the other hand, this very result entails that Infinitesimal Gunk satisfies COUNTABLE SUPERVENIENCE, since the fusion of countably infinitely many disjoint measurable regions is always unmeasurable no matter how those regions are arranged. Thus, unlike in Russell's proposal, for countably many disjoint regions, the magic of changing the measure of their fusion (or whether there is a measure) through mere rearrangements of those regions does not occur.[29]

However, one may argue that, even though COUNTABLE SUPERVENIENCE is satisfied, the mere fact that *no* fusion of countably infinitely many disjoint measurable regions has a measure is a serious cost to the theory. This may be true. However, what makes the measure theory of Infinitesimal Gunk attractive is that it has important compensations unavailable to Russell's proposal: the measure theory satisfies HYPERFINITE ADDITIVITY, and we can define an extended-real-valued approximate measure that satisfies COUNTABLE ADDITIVITY. I will explain them in turn.

[29]Note that, like Russell's account or the standard measure theory, Infinitesimal Gunk does not satisfy arbitrary supervenience (or the restricted version of arbitrary supervenience that only involves *internal* bijections) for reasons that will become apparent in the next section.

First, unlike Russell's solution, the measure theory does not retreat to mere finite additivity. Rather, it has hyperfinite additivity as a compensation.

> HYPERFINITE ADDITIVITY. For hyperfinitely many disjoint measurable regions, their fusion has a measure, which is the (hyperfinite) sum of the measures of those regions.

This follows from HYPERFINITE ADDITIVITY (SET) in Section 3.[30] As a result, Infinitesimal Gunk has a much richer measure than Russell's solution.

To make this richness more vivid, consider **Big Fusion*, defined as the hyperreal extension of Big Fusion (that is, the fusion of the regions represented by the hyperreal extensions of $A_1, C_1, C_2, ...$).[31] What is the measure of *Big Fusion? Infinitesimal Gunk implies that it is unmeasurable. But this is not all it says. The hyperreal measure satisfies the following theorem:

> MEASURE APPROXIMATION. For any "proper" hyperreal set A, if it has a Lebesgue measurable shadow, then there are measurable sets B and C such that $B \subseteq A \subseteq C$ and the measures of B and C differ by at most an infinitesimal.[32] (Goldblatt 1998, 212-4)

Call a hyperreal set C a *Lebesgue completion* of a (proper) hyperreal set A iff A includes or is included in C and C's measure is equal to the Lebesgue measure of A's shadow. Then, it follows that every (proper) unmeasurable set whose shadow is Lebesgue measurable has a Lebesgue completion. Indeed, we can say that the hyperreal measure is richer than the Lebesgue measure in the following sense: there is an injective but not surjective function from Lebesgue-measurable sets on the real line to hyperreal-measurable sets on the hyperreal line that preserves their measures and mereological relations. In particular, if a

[30]The principle can be reduced to HYPERFINITE ADDITIVITY (SET) except for the caveat that two disjoint regions may be represented by two regular closed sets that overlap on their boundaries. But because every measurable set is the union of hyperfinitely many disjoint intervals, its boundary at most includes hyperfinitely many points. Since a point measures zero, and hyperfinitely many zeros add up to zero, the potential double counting of those boundary points in the case of overlapping boundaries does not affect the final measure.

[31]The hyperreal extension of A_1, for example, is the set of hyperreal numbers between the endpoints of A_1.

[32]The qualification of being "proper" and having a Lebesgue measurable shadow corresponds to Loeb-measurablity (see Goldblatt 1998, 215-7). Then MEASURE APPROXIMATION follows from Goldblatt (1998, 212-4). Roughly, the set of Loeb-measurable sets is like the set of hyperreal-measurable sets except that it has more members so that it is closed under countable unions. The claims to follow in the main text hold under this qualification. In particular, the set representing *Big Fusion is Loeb-measurable.

Lebesgue-measurable set of real numbers is the shadow of an unmeasurable set, then this function takes the set of the real numbers to a Lebesgue completion of that unmeasurable set. In contrast, the Jordan measure is poorer than the Lebesgue measure in the same sense.

Moreover, a natural notion of approximate measure can be defined based on MEASURE APPROXIMATION:

> APPROXIMATE MEASURE. For any region, if its corresponding set has a Lebesgue completion, then it has an *approximate measure*, which is equal to the measure of that completion.

Since *Big Fusion has a shadow that has a Lebesgue measure of 1/3, it has a Lebesgue completion that measures 1/3. This means that, even though *Big Fusion is unmeasurable, it has an approximate measure of 1/3. These approximate measures have nice properties. Notably, they satisfy COUNTABLE ADDITIVITY.

> APPROXIMATE COUNTABLE ADDITIVITY. For any countably many disjoint regions that have approximate measures, their fusion also has an approximate measure, which is equal to the sum of the approximate measures of those regions.[33] (Goldblatt 1998, 206-8, 212-4)

In comparison, in Russell's solution, no such attractive approximate measure can be systematically assigned. It is reasonable to assume that, under Russell's proposal, the approximate measure of a Jordan-measurable region is just its Jordan measure. Then, assuming APPROXIMATE COUNTABLE ADDITIVITY, the approximate measure of Big Fusion would again equal its inner measure 1/3. Similarly, Small Fusion would have an approximate measure of 1/6. But this would again violate even finite additivity, since Big Fusion and Small Fusion are disjoint regions, but their fusion has a measure of one.

We began by looking for a theory of space that escaped Russell's theorem by giving up COUNTABLE BASIS. What we have ended up with is a theory that gives up *both* COUNTABLE BASIS *and* COUNTABLE ADDITIVITY. A point worth clearing up is whether this was

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[33]The notion of approximate measure amounts to the Loeb measure, which is countably additive (Goldblatt 1998, 206-8; see also Footnote 32). Given that the Loeb measure is countably additive, it is natural to wonder why we do not use the Loeb measure instead of the hyperreal measure that I define in the paper. The main reason is that under the Loeb measure, all infinitesimal regions have zero measure. This violates the principle that every region has a strictly smaller subregion, which is one of the main intuitions behind the gunky approach to space.

inevitable. Is there a reasonable theory of space that gives up COUNTABLE BASIS without also giving up COUNTABLE ADDITIVITY? Given some reasonable assumptions, the answer is no: there is no such theory.[34] Here's a brief sketch as for why. If space does not have a countable basis, it would follow (under some reasonable assumptions) that there exist some very small regions that do not have any rational interval as a part. Moreover, assuming a measure is translation-invariant, we can find countably infinitely many disjoint such regions with the same size within a unit interval. Those regions cannot all have positive finite measures. If they have infinitesimal measures (measures smaller than any finite number), then COUNTABLE ADDITIVITY would be violated. But if they all have measure zero, then the attractive measure-theoretic principle that all regions have smaller subregions would be violated. Therefore, if we assume (among other things) that the measure is translation-invariant and every region has a smaller subregion, it is impossible to give up COUNTABLE BASIS without also giving up COUNTABLE ADDITIVITY.[35]

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[34]Without any constraint, it is possible to violate COUNTABLE BASIS without violating COUNTABLE ADDITIVITY, but this requires topological spaces that are too exotic to be a candidate for our actual physical space. A typical example is the product space $[0,1]^I$ with product topology, where I is a cardinality larger than continuumly many. It does not have a countable basis, but has a countably additive measure.

[35]I will demonstrate that it is impossible to violate COUNTABLE BASIS without violating COUNTABLE ADDITIVITY under certain reasonable assumptions including that every region has a smaller subregion and a measure is translation-invariant. As usual, I will pretend our space is "one-dimensional" in any suitable sense. The additional assumptions that the proof relies on are labeled in parentheses.

Proof. Suppose space does not have a countable basis, and furthermore, the set of all measurable regions does not have a countable basis (although this supposition is stronger than the violation of COUNTABLE BASIS, Russell's inconsistency proof effectively only involves the thesis that the set of all measurable regions has a countable basis). Also, suppose space is one-dimensional in the sense that its topology can be generated by some intervals (ASSUMPTION-1). In particular, we assume that every interval can be characterized by two endpoints, and that all endpoints are abstract entities that constitute a totally ordered Abelian group, to which rational numbers can be embedded. The set of all intervals with rational endpoints (or "rational intervals" for short) cannot be a basis since they are only countably many. As one can check, it follows that there is an interval (ϵ, δ) that do not have any rational interval as a part and therefore $\delta - \epsilon$ is smaller than any rational number. Suppose measure is translation-invariant (ASSUMPTION-2). Then we can find countably infinitely many disjoint regions with the same measure as (ϵ, δ) within the interval $(0, 1)$. For example, let $\Delta = \delta - \epsilon$, and let $I_n = (\frac{1}{n} - \Delta, \frac{1}{n})$ for all $n \in \mathbb{N}$. Assuming COUNTABLE ADDITIVITY, the measure of the fusion of I_n for all $n \in \mathbb{N}$ is well-defined—call this fusion "Big." Now, consider the fusion of I_n for all $n \geq 2$, and call this fusion "Small." Then, given COUNTABLE SUPERVENIENCE (which is entailed by COUNTABLE ADDITIVITY), Small has the same measure as Big. We further assume that for any two bounded regions, their measures have a well-defined subtraction (ASSUMPTION-3). Since Big is the fusion of the two disjoint regions I_1 and Small, it follows from finite additivity that I_1 has a measure of zero. This contradicts the principle that every region has a strictly smaller subregion (ASSUMPTION-4), which captures the measure-theoretic aspect of gunky space. Therefore, given the listed assumptions, it is impossible to violate COUNTABLE BASIS without violating COUNTABLE ADDITIVITY. QED

3.6 Variants of Arntzenius's Argument

Now let's examine how Infinitesimal Gunk avoids the specific difficulties raised by Arntzenius's Cantorian constructions as well as some variants. Arntzenius's original argument is based on REAL REPRESENTATION. Thus, it does not directly apply to Infinitesimal Gunk, since the regions are now no longer represented by regular closed sets of real numbers but of hyperreal numbers. So, let's consider the analogous argument on the hyperreal line. Take a unit hyperreal interval. Then similarly we cut out a 1/4-long hyperreal interval, two 1/16-long hyperreal intervals, four 1/64-long hyperreal intervals, and so on for all natural numbers—call them *the *Cantor intervals* and their union simply **Cantor*. These intervals each represent a gunky region—call them *the *Cantor regions*.

Recall that Arntzenius's inconsistency argument relies on the following fact:

> CANTOR CLOSURE. The closure of the Cantor union is the unit real interval.

In the case of the hyperreal line, the analogous claim would be:

> *CANTOR CLOSURE. The closure of *Cantor is the unit hyperreal interval.

But this claim is false: the closure of *Cantor is not the whole unit line segment but rather *Cantor itself. Why? Consider a point in the unit interval that is not in any of the *Cantor intervals. One such point would be one that is infinitely close to the left endpoint of the unit interval. Let δ be an infinitesimal. δ is not in *Cantor since it is smaller than all the left endpoints of the *Cantor intervals. Furthermore, δ is not a limit point of *Cantor because there is an open set that includes δ, but does not include any point in *Cantor. One example is $(0, \delta + \epsilon)$, where ϵ is any infinitesimal. Therefore, δ does not belong to the closure of *Cantor. In general, for any point x outside *Cantor, $(x - \epsilon, x + \epsilon)$ is an open set that includes x but does not overlap *Cantor. More vividly, each point outside *Cantor has an infinitesimal "cushion" that "protects" it from *Cantor. So, the closure of *Cantor on the hyperreal line is just itself.

Can we come up with a different construction from *Cantor that gives rise to similar problems as in Arntzenius's argument? One observation is that there is no analogous claim to *CANTOR CLOSURE as long as such a construction is composed of countably many intervals.

COUNTABLE UNION. The union of countably many disjoint closed intervals is regular closed.

Again, the idea is that infinitesimals are so small that for any point outside the countably many disjoint closed intervals, we can find an infinitesimally small neighborhood of that point disjoint from the union of those intervals. As a result, any point outside the union is not a limit point of it (Appendix B.6).

A perhaps cleverer revision of the argument is that, rather than cutting out countably many hyperreal intervals from the line segment, we cut out a much larger set of intervals. A countable set is in one-to-one correspondence with the set of all natural numbers. Analogously, let a *hypercountable* set be in one-to-one (internal) correspondence with the set of all hypernatural numbers. The idea then is to cut out hypercountably many hyperreal intervals from the line segment: $R_1, R_2, ...R_N, R_{N+1}, ...$, with N being some infinite hypernatural. Call the union of these intervals *Hypercantor*. In this case, every point outside Hypercantor is a limit point of it, because for any such point, its neighborhood—even if infinitesimally small—always intersects Hypercantor.[36] Consequently, the closure of Hypercantor is the whole line segment!

But this does not cause trouble for Infinitesimal Gunk, because Hypercantor is not a union of hyperfinitely many hyperreal intervals.[37] We can grant that the gunky region represented by the unit hyperreal interval is the fusion of hypercountably many gunky regions of lengths $1/4, 1/16, ..., 1/4^N, ...$ This does not result in any contradiction because we do not have "hypercountable additivity" in Infinitesimal Gunk. (Notice that this reply has the same structure as Russell's reply to Arntzenius's original argument. That is, in the "problematic" cases, both Russell and I rely on some version of additivity failing to

[36]The reason for this is analogous to the reason why every point outside the Cantor union is a limit of the Cantor union. In the case of the Cantor union, for any point x in the unit interval, for any positive real number ϵ, we can find a point y in the Cantor union such that $|y - x| < \epsilon$. Now, Hypercantor is constructed in the same way as the Cantor union, except that the cutting process does not stop with countably many intervals but continues for hypercountably many more. In particular, the Hypercantor intervals can be expressed in terms of when they are cut out in the same way as the Cantor intervals. For example, the leftmost point of the Hypercantor intervals cut out at stage N is $1/2^{N+1} + 1/2^{2N+1}$, just like the leftmost point of the Cantor intervals cut out at stage n is $1/2^{n+1} + 1/2^{2n+1}$. Thus, for instance, 0 is a limit point of Hypercantor because for any positive hyperreal δ, we can find a hypernatural N such that $1/2^{N+1} + 1/2^{2N+1} < \delta$. This reasoning can be generalized to all the points on the unit hyperreal interval.

[37]There is no internal bijection between a hypercountable set and a hyperfinite set.

hold. But as I discussed in Section 5, although hypercountable additivity fails, Infinitesimal Gunk still has several advantages over Russell's solution.)

More generally, we can prove that any union of disjoint regular closed measurable sets that is not identical to its closure must be unmeasurable.

> HYPERFINITE UNION. The union of hyperfinitely many disjoint measurable regular closed sets is regular closed.[38]

Moreover, any measurable regular closed set is the union of hyperfinitely many disjoint measurable regular closed sets (in particular, closed hyperreal intervals). It follows that any union of disjoint regular closed sets that is not regular closed is unmeasurable. As a result, Infinitesimal Gunk is safe from any Arntzenius-style trouble, since a union of regular closed sets that is not identical to their closures is unmeasurable and thus does not cause trouble, just like what we saw in the case of Hypercantor.

3.7 Conclusion

Can space be divided into ultimate parts? Does space have parts with infinitesimal sizes? These questions are more related to each other than they seem to be. In this paper, I have shown that Infinitesimal Gunk, the view that any region of space can be further divided *and* some regions are infinitesimally small, provides a novel reply to the inconsistency arguments given by Arntzenius and Russell. Moreover, this view has several important advantages over the solutions these authors suggested. It has a richer measure theory than Russell's proposal and satisfies attractive measure-theoretic principles unavailable to the latter. Unlike Arntzenius's proposal, it does not need to admit boundaries. Thus I recommend this novel theory for serious consideration.

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[38]This claim is an extension of the claim that the union of *finitely* many regular closed sets is regular. See Appendix B.7 for my proof sketch.

CHAPTER 4

INTRINSIC LOCAL DISTANCES: A MIXED SOLUTION TO WEYL'S TILE ARGUMENT

Weyl's tile argument purports to show that there are no natural distance functions in atomistic space that approximate Euclidean geometry. I advance a response to this argument that relies on a new account of distance in atomistic space, called *the mixed account*, according to which *local distances* are primitive and other distances are derived from them. Under this account, atomistic space can approximate Euclidean space very well. To motivate this account as a genuine solution to Weyl's tile argument, I argue that this account is no less natural than the standard account of distance in continuous space. I also argue that the mixed account has distinctive advantages over Forrest's (1995) account in response to Weyl's tile argument, which can be considered as a restricted version of the mixed account.

4.1 Weyl's Tile Argument

According to *the atomistic view*, space (or spacetime) is composed of extended indivisible parts—call them "atoms." This view is motivated by both conceptual and empirical puzzles for the standard view, according to which space is composed of extensionless points (for example, see Van Bendegem 1995 and Baez 2018). However, there is a famous argument given by Weyl (1949) against it:

> How should one understand the metric relations in space on the basis of this idea? If a square is built up of miniature tiles, then there are as many tiles along the diagonal as there are along the side; thus the diagonal should be equal in length to the side. (Weyl 1949, 43)

Consider the following square region composed of 4×4 atoms represented by square tiles (Figure 4.1):[1]

[1]My presentation of the argument follows Salmon (1980).

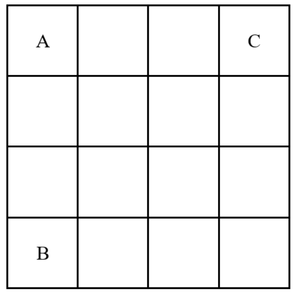

Figure 4.1. 4×4 tile space

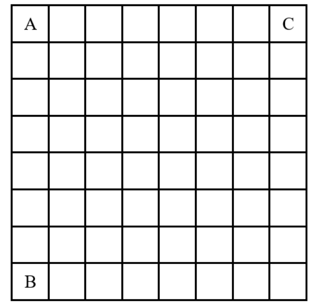

Figure 4.2. 8×8 tile space

There are four atoms on the side AC. There are also four atoms on the diagonal BC. This, according to Weyl, implies that AC and BC have the same length. But if the Pythagorean theorem is approximately true, then BC should be about $\sqrt{2}$ times as long as AC. Adding more atoms does not help. If the square is made of 8×8 atoms (Figure 2), there are still as many atoms on the diagonal as on the side. So no matter how big the square region is, the ratio between the length of its side and the length of its diagonal does not approximately satisfy the Pythagorean theorem. Weyl concluded that, since the Pythagorean theorem is approximately true, our space is not atomistic.

This conclusion is relevant to both philosophers and physicists. Whether space is atomistic is an active research question in physics. For example, experiments have been proposed to test the hypothesis that space is composed of “atoms" at the Planck scale (Hogan 2012).[2] It is a “received wisdom" that a certain sort of discrete structure is required for reconciling quantum theory and general relativity (Maudlin 2015, 46). But if Weyl's tile argument is successful, then we can conclude that space is not atomistic without doing experiments. Due to its relevance, physicists continue to be intrigued by this argument (for example, see Crouse and Skufca 2018).

Even though Weyl's tile argument has been found “devastating" (Van Bendegem 2015), the core assumptions that the argument relies on have not been explicitly motivated. For

[2] I put “atoms" in quotes because it is not entirely clear what philosophical theory of spacetime we should explicate from Hogan (2012). More technically, the tested hypothesis implies that the geometry of spacetime is not commutative below the Planck level. Among other things, this means that unextended points do not exist because the coordinates of a point are necessarily commutative (e.g., in the (x, y)-coordinate system, for any point (a, b), $ab - ba = 0$).

instance, why might we think that the length of the diagonal equals the number of the atoms on the diagonal? This, as I will explain in Section 2, amounts to a simple path-dependent account of distance in atomistic space, which fits into our best physical theory. In contrast, a perhaps equally intuitive alternative—*the intrinsic account of distance*—does not have similar merits (Section 3; see also McDaniel 2007).

Making the underlying account of distance explicit is not only useful for appreciating the force of Weyl's argument, but also for opening up new options that haven't been considered so far (for current solutions, see Van Bendegem 1987, 1995, Forrest 1995). I will propose a solution to Weyl's tile argument by appealing to a new account of distance in atomistic space, called *the mixed account*, according to which there are primitive distances at the small scale (Section 4). I will argue that this account is a successful reply to Weyl's argument by comparing it with the standard account of distance in standard space, which exemplifies a similar structure (Section 5). I will also argue that the mixed account has distinct advantages over Forrest's proposal (Section 6).

For simplicity, I will pretend that our actual space's geometry is approximately Euclidean in this paper, except that in Section 5, I will focus on the standard account of distance for continuous space in general.

4.2 Path-Dependent Distance

In this section, I will identify the account of distance implicitly assumed by Weyl's tile argument and examine the rationale behind it. I argue that there is room for rejecting this premise and propose the conditions for a successful response to Weyl's argument.

An important step in Weyl's tile argument is to claim that the lengths of the side AC and the diagonal BC are both determined by the numbers of atoms they contain. Under standard geometry, we would think that the property of length is only fundamentally instantiated by one-dimensional line segments or, more generally, a path. However, in atomistic space, there are no one-dimensional line segments or paths in the standard sense. So, how should we understand "length" in atomistic space? A natural option is to define a new notion of “path" in atomistic space to which the property of length fundamentally applies. The definition involves a primitive notion of *adjacency* that is reflexive and symmetric.

> PATH. A *path* from atom a_1 to a_n is a sequence of atoms $a_1, a_2, ..., a_k, ..., a_n$ such that for every k, a_k and a_{k+1} are adjacent ($1 \leq k \leq n-1$).

Assuming that the unit of length is the length of a path containing one atom, Weyl’s tile argument can be taken as relying on the following principle:

> LENGTH-BY-COUNTING. The *length* of a path is equal to the number of atoms it contains.

According to the standard account of distance in standard space, the distance between two points is equal to the length of a shortest path between them. This account can also apply to atomistic space:

> PATH-DEPENDENT DISTANCE. For any two atoms a and b, the distance between a and b is the length of a shortest path from a to b.[3]

(Note that a shortest path from a to b contains the same number of atoms as that from b to a, which implies that distance relation is indeed symmetric.) It follows that we can obtain the distance between two atoms by counting the atoms between them (Riemann 1866).

In order to apply these definitions to Weyl’s tile space, we need to specify which atoms count as adjacent. Weyl’s tile argument amounts to endorsing the option that two atoms are adjacent *iff* their representing tiles are horizontally, vertically, or diagonally adjacent. Under this stipulation, the diagonal BC in Figure 2 is composed of eight atoms that are diagonally adjacent.[4]

But why should we accept LENGTH-BY-COUNTING? Here’s one tempting thought. In standard measure theory, we have the principle of finite additivity:

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[3]Strictly speaking, it is more natural to think that the distance between a and b is the length of a shortest path from a to b *minus one*. For example, while the length of the side AB is four in Figure 1, it’s more natural to think that the distance between A and B is three. However, for the sake of generalization in later discussions, it’s better to use DISTANCE.

[4]Another intuitive option is to assume that two atoms are adjacent *iff* their representing tiles are horizontally or vertically adjacent. Under this option, the diagonal BC is represented by the zigzag region along the diagonal direction. But this option has the same problem: the ratio of the diagonal to the side is about 2:1 rather than $\sqrt{2}:1$.

> FINITE ADDITIVITY. For any natural number n, for any n-dimensional region X, if X is composed of finitely many disjoint regions Ys, then the measure of X is the sum of the measures of Ys.

In the case of atomistic space, if we assume that every atom has a unit size, then FINITE ADDITIVITY entails:

> SIZE-BY-COUNTING. The measure (or size) of a region is equal to the number of atoms it contains.

But SIZE-BY-COUNTING generally does not imply LENGTH-BY-COUNTING. In standard space, a path is a one-dimensional region of space and therefore only has a one-dimensional measure. But in the case of two-dimensional atomistic space, the two-dimensional measure of a path need not be numerically equal to its one-dimensional length.[5] For example, imagine that an atom has a kind of shape, which is given by primitive lengths along different directions. Say an atom has a horizontal and vertical length of 1, and a diagonal length of $\sqrt{2}$, and we still assume that each atom has a unit size. In this case, the length of the diagonal BC would be $4\sqrt{2}$, which is numerically unequal to the size of BC (which is 4).[6]

Therefore, LENGTH-BY-COUNTING is not a conceptual necessity for atomistic space. Forrest (1995), for example, considered it to be motivated by the consideration of theoretical simplicity and elegance. It's attractive that the metric property of space is founded on just one primitive dyadic relation of adjacency. However, simplicity and elegance should not be the sole factors for theory choice. (Besides, they are often hard to measure—a theory that is simpler and more elegant in one sense may be more complicated in other senses.) Moreover, granting that simplicity and elegance are important theoretical virtues, in order for Weyl's argument to be successful, there needs to be a stronger claim, namely that there are no atomistic accounts of distance that can do as well as the standard account of distance for continuous space—for otherwise it would be unfair to conclude that our space is not atomistic but continuous. This is a claim that I will challenge in this paper.

[5]Here I am using "dimension" in an informal (and hopefully intuitive) way that every region of N-dimensional atomistic space is also N-dimensional. In other words, dimensionality is an intrinsic property of an atom. But we can have alternative definitions of dimension in atomistic space, which will be briefly discussed in Section 6.

[6]Note that this example does not solve Weyl's tile argument: even though the sides and the diagonal of the square region satisfy the Pythagorean theorem, the distances along other directions don't.

More explicitly, I will argue that there is an account of distance for atomistic space that meets the following "success conditions" and therefore solves Weyl's argument: (1) it allows atomistic space to approximate Euclidean geometry; (2) it is compatible with physics as we know it; and (3) it scores reasonably well on theoretical virtues, such as intelligibility, intuitiveness, naturalness, simplicity and so on, and in particular, it scores no worse than the standard account for continuous space.

Let me briefly address another implicit assumption in Weyl's tile argument: atoms are arranged like the regular square tiling. What if atoms are arranged very differently? For example, they may be arranged like the regular hexagonal tiles. We can check that the distance relations under this arrangement approximate Euclidean geometry much better than the square tiling (e.g., the ratio between the lengths of AB, AC and BC is close to what is required by the Pythagorean theorem). Nonetheless, the deviation is still large enough to be detectable at a large scale. Indeed, so far there hasn't been any clear example of tiling arrangement that approximates Euclidean geometry sufficiently well (Van Bendegem 2017).

In this paper, I will not take on the quest of looking for such a tiling space. As shown in Fritz (2012), even if there are atomistic spaces represented by some tiling arrangements that approximate Euclidean space very well at least at the large scale, those arrangements have to be very complicated and irregular.[7] The alternative account of distance that I will propose, I believe, solves Weyl's tile argument at least as well as this approach. Thus the question about alternative tiling arrangements may only be of purely technical interest.

4.3 Against Intrinsic Global Distance

In the last section, I have been assuming the path-dependent account of distance, according to which the distance between any two atoms is equal to the length of a shortest path between them. An alternative account, *the intrinsic account of distance*, says that the

[7] In Fritz's formalism, atomistic space is modeled by an infinite graph composed of $\mathbb{Z}^d$-translates of a certain finite pattern—call each of those translates a "cell." For example, in the hexagonal tile space, each cell contains just one vertex and six edges. According to Fritz, a cell must contain a very large number of edges in order for the metric of the graph to approximate Euclidean geometry closely at the large scale. This means that, if there is an atomistic space represented by a tile space that approximates Euclidean space very well at the large scale, the repeated pattern must be very complicated.

distance between two atoms does not depend on the path between them and indeed is intrinsic to their fusion: if we duplicate the fusion without duplicating anything else, the duplicate atoms will still have the same distance.[8]

Now, if the intrinsic account is true, there would be no problem assigning distances among atoms that approximate Euclidean geometry. The trick is to assign primitive distances to all pairs of atoms that match Euclidean distances. More precisely, let each atom be represented by a pair of integers in the two-dimensional coordinate space $\mathbb{R}^2$. We assume this:

> EUCLIDEAN MODEL(I). The distance between any two atoms $(a_1, b_1), (a_2, b_2)$ is equal to $\sqrt{(a_2 - a_1)^2 + (b_2 - b_1)^2}$.

Then all distance relations trivially satisfy Euclidean geometry.

Have we solved Weyl's tile argument then? No, because the intrinsic account faces two objections, one empirical and one theoretical. Note that McDaniel (2007) proposed the intrinsic account as a "solution" to Weyl's tile argument in the sense that atomistic space that satisfies EUCLIDEAN MODEL(I) is metaphysically *possible*. But the focus here is whether such a space is a live candidate for the structure of *actual* spacetime. The answer is no, because such a space is incompatible with actual physics. According to the theory of general relativity, the metric of spacetime is determined by the distribution of mass-energy under Einstein's field equations. Roughly, the curvature of spacetime at a point is proportional to the density of mass-energy near that point, which means that the presence of a massive body would distort the paths nearby. Furthermore, our physics is *local*: there is no instantaneous action at a distance. Now, consider two spacetime points far apart. If there occurs a massive body between them, then according to general relativity, their distance will be different. But the fusion of the two points presumably does not go through any intrinsic change, especially if both points are far away from the massive body and couldn't be affected immediately. Thus the massive body changes the distance between them only by changing the length of the shortest path between them. This means that the distance between them cannot be intrinsic to them. So, the intrinsic account is false for

[8]See McDaniel (2007) for more discussion of the view. McDaniel argued that the intrinsic account is true in some possible worlds, and in such worlds, atomistic space can approximate Euclidean distance.

actual space. Insofar as we want atomistic space to be a candidate for our actual space, the intrinsic account does not help.

Apart from actual physics, there is also a theoretical consideration against the intrinsic account. Maudlin (2007), among others, has argued that if distances are all primitive, then we need to posit TRIANGLE INEQUALITY as an axiom, which says that for any "points" a, b, c, the distance between a and b plus the distance between b and c must be at least as great as the distance between a and c.[9] But if we define distance as the length of a shortest path, then TRIANGLE INEQUALITY automatically follows. Suppose TRIANGLE INEQUALITY is false: there are three points a, b, c such that the length of the shortest path from a to c is longer than the sum of the length of the shortest path from a to b and that from b to c. However, the path from a to b connected with the path from b to c *just is* a path from a to c, the length of which is equal to the sum of the two connected paths.[10] Then, this path would be shorter than the shortest path from a to c! Contradiction. So, a shortest path from a to c cannot be longer than the sum of the length of the path from a to b and the length of the path from b to c. Moreover, it seems that the path-dependent conception of distance is not only sufficient but also necessary for fully justifying TRIANGLE INEQUALITY: without thinking in terms of paths, it is mysterious why this axiom should hold for distance at all.[11] So the path-dependent account is not only simpler on this regard but also more perspicuous.

Nonetheless, the above arguments only show that not every distance is intrinsic and, in particular, that global distances are not intrinsic. They do not show that no distance can be intrinsic or that the notion of intrinsic distance is unintelligible. Indeed, I will now propose an account of distance in response to Weyl's tile argument that also relies on primitive intrinsic distances.

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[9]In a general context, I use "point" to simply refer to an ultimate part of an arbitrary space.

[10]Here, "connected" is used in the sense that a path $a_1, ..., a_k$ can be connected with a path $a_k, ..., a_n$ to form a single path $a_1, ..., a_n$ $(1 \leq k \leq n)$.

[11]In mathematics, a *semimetric* is a generalized distance function that does not satisfy TRIANGLE INEQUALITY. Under the intrinsic account, it is hard to see why a space cannot have a semimetric.

4.4 Primitive Local Distance

In this section, I will propose an alternative account of distance, which I call *the mixed account*. According to this account, we can assign primitive distances not to all pairs of atoms but to atoms in a "local neighborhood." I will argue that this allows atomistic space to be approximately Euclidean and is not subject to the previous objections to the intrinsic account.

Unlike the path-dependent account, we do not posit the primitive notion of adjacency. The only primitive notion we have is *proto-distance*, denoted by $\mathbf{d}$, which is partially defined over pairs of atoms and satisfies some standard axioms for distances (p, q range over atoms):

> NONNEGATIVITY. $\mathbf{d}(p,q) \geq 0$ if $\mathbf{d}(p,q)$ is defined.
> SYMMETRY. $\mathbf{d}(p,q) = \mathbf{d}(q,p)$ if $\mathbf{d}(q,p)$ is defined.
> NONSINGULARITY. $\mathbf{d}(p,q) = 0$ iff $p = q$.

These proto-distances determine all the metric properties of space. For any atom, an atom that bears a primitive distance to it is a *neighbor* of it, and the set of all its neighbors is its *(local) neighborhood*. For the spaces we are interested in, all primitive distances are bounded by a finite number, which means that a local neighborhood is also bounded by a finite region. (Note that because of this requirement, the model under the intrinsic account of distance discussed in the last section is ruled out here.)

Next, we define the notion of a path in terms of neighbors:

> PATH*. A *path* from a_1 to a_n is a sequence of atoms $a_1, a_2, ..., a_{n-1}$ such that for any k with $1 \leq k \leq n-1$, a_k and a_{k+1} are neighbors.

The length of a path is obtained by adding up the proto-distances along the path.

> PATH-LENGTH. If the sequence of atoms $a_1, a_2, ..., a_{n-1}$ is a path from a_1 to a_n, then the *length* of the path is equal to $\mathbf{d}(a_1,a_2) + \mathbf{d}(a_2,a_3) + ... + \mathbf{d}(a_{n-1},a_n)$.

Just like any path-dependent account, the distance (denoted by "d") between any two atoms is the length of a shortest path from one to the other. Note that when proto-distances satisfy TRIANGLE INEQUALITY, namely $\mathbf{d}(p,q) \leq \mathbf{d}(p,r) + \mathbf{d}(r,q)$, they are genuine distances under this account. In this case I'll call the proto-distance the *primitive distance*.

I claim that, under this account of distance, we can find an atomistic space that approximates Euclidean space as closely as we want at all scales (see Appendix C for the proof).

> EUCLIDEAN APPROXIMATION. Under the mixed account, for a Euclidean space of any dimension, there is an atomistic space that approximates it sufficiently well.

For instance, let each atom be represented by a pair of integers. The following model approximates Euclidean space at all scales if the number M is sufficiently large:

> EUCLIDEAN MIXED MODEL. For any two atoms $a = (x_1, y_1)$ and $b = (x_2, y_2)$, the primitive distance $\mathbf{d}(a,b) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ if $(x_2 - x_1)^2 + (y_2 - y_1)^2 \leq M^2$; otherwise $\mathbf{d}(a,b)$ is undefined.

In this model, distances within a local neighborhood are exactly Euclidean. For two atoms that are far apart, their distances will generally differ from the corresponding Euclidean distance. But the difference can get as small as we want if we choose a sufficiently large M. To illustrate, consider the following region (Figure 4.3). Consider the atoms A, B, C, D: $A = (0,0), B = (40,0), C = (40,30), D = (17,23)$. Let $M = 30$. Then A, C are path-connected through D. This means that $d(A,C) \leq \mathbf{d}(A,D) + \mathbf{d}(D,B) \approx 50.01$ (and it's clear that $d(A,C) \geq 50$). It's easy to get $d(A,B) = 40$ and $d(B,C) = 30$. Thus, the distance relations between A, B, C are very close to satisfying the Pythagorean theorem.

The mixed account does not face the difficulties that the intrinsic account of distance has. Recall that Maudlin has objected to the intrinsic account for the reason that it needs an additional axiom of TRIANGLE INEQUALITY. In the mixed account, we do not need to posit this axiom. Since the distance between any two atoms is defined to be the length of a shortest path, TRIANGLE INEQUALITY automatically follows. Note that proto-distances need not be distances. For example, if the proto-distance between a and c is longer than the sum of the proto-distances between a and b and between b and c, then the sequence of atoms a, b, c is a shorter path from a to c than the sequence of atoms a, c. Thus, the distance between a and c is not the proto-distance between them. In this case, the proto-distance between a and c does not play any role in determining other distances either. Then, insofar as physics only involves distances and insofar as the goal is to recover physics, there is

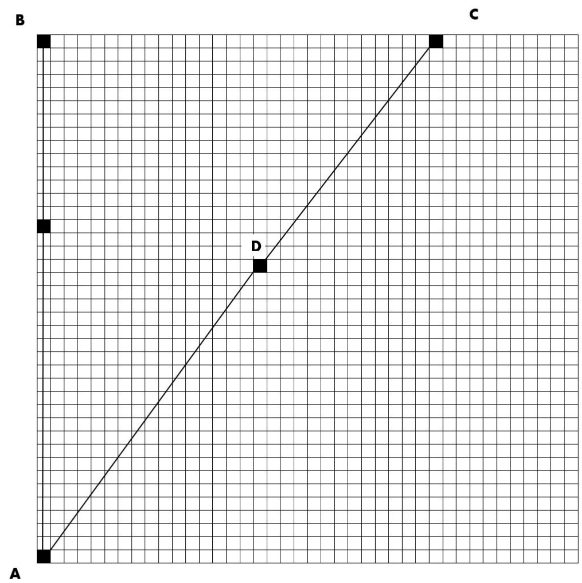

Figure 4.3. Euclidean approximation under the mixed account

no need to posit such a proto-distance. That is, we generally do not need models where proto-distances do not satisfy TRIANGLE INEQUALITY.

Since the mixed account allows large-scale distances to be path-dependent, it is compatible with our actual physics as far as we know it. For two atoms that are sufficiently far apart, their distance is not intrinsic to their fusion but depends on other atoms that compose the shortest path between them. So, when the presence of a massive body curves the shortest path between them, their distance will change accordingly. Note that the same empirical problem for large-scale primitive distances could, in principle, arise for small-scale primitive distances. So it is possible for the account to be disconfirmed by a new development in physics, supposing we can find a way to (indirectly) observe those small-scale distances and how they can be affected.

Note that the mixed account can be extended to relativistic settings relatively straightforwardly, though I won't go into much detail. Here's a sketch of how we might do so. Instead of symmetric primitive distances, we may posit directed (thus antisymmetric) *time-like primitive distances*, which will be sufficient for determining the metric structure of relativistic spacetime.[12] For any atom a, we call atom b a *future neighbor* of a if there is a directed primitive time-like distance from a to b. A *time-like path* is a sequence of atoms with each one preceding a future neighbor of it, and the length of a path is obtained by summing up the primitive time-like distances along the path. For any two atoms that are connected by a time-like path, the *time-like distance* between them is equal to the length of a *longest* path between them (a time-like distance is the *maximal* time spent on traveling from one spatiotemporal atom to another). We can derive the metric structure of spacetime from time-like paths through standard radar methods (for example, see Rosser 1992, Perlick 2007). With a suitable set of primitive time-like distances, the resulting model can be expected to approximate Minkowski spacetime as closely as we want. Let each atom be represented by a quadruple of integers $\langle t, x, y, z \rangle$, respectively representing its temporal coordinate and spatial coordinates. For any two atoms $a = \langle t_1, x_1, y_1, z_1 \rangle, b = \langle t_2, x_2, y_2, z_2 \rangle$, if $0 \leq g = (t_2 - t_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2 \leq M^2$, and $t_2 \geq t_1$, then the directed time-like primitive distance $\overrightarrow{\mathbf{d}}(a, b) = \sqrt{g}$. Like in the case of Euclidean space, if M is sufficiently large, then this atomistic model approximates Minkowski spacetime.

To take stock, the mixed account allows for an atomistic model that approximates Euclidean space (or other continuous spaces) well enough and does not face the objections the intrinsic account faces. It remains to be seen whether the mixed account scores reasonably well on other theoretic virtues. While the notion of primitive distance and of path-dependent distance are sufficiently intelligible by themselves, the mixture of the two notions may seem unnatural. To defend this account further, I will turn to a comparison between the mixed account and the standard account for continuous space.

4.5 "Local Distances" in Continuous Space

The mixed account might strike one as unnatural or overcomplicated because it involves two concepts of distance and allows the geometry of atomistic space to be deter-

[12]This "time-first" approach to the discrete analogue of relativistic spacetime bears some similarity to causal set theory (for example, see Sorkin 1990). One main difference is that my approach allows for the additional structure of primitive distances rather than just a partial ordering between atoms, which may be a useful way to circumvent certain technical difficulties that have arisen in causal set theory.

mined by a vast number of varied primitive distances. This might lead one to uphold Weyl's conclusion that our space is not atomistic after all. Against this, I will argue that the standard account of distance from differential geometry has a similarly mixed form. Under the standard account, as I will explain, we start with local metrics, which are analogous to primitive distances, and similarly obtain distances by "adding up" those "primitive distances" (though technically, it is integration rather than addition). Note that, in arguing for this, I will shift attention from Euclidean space to generally non-Euclidean continuous space—after all, our actual space is, strictly speaking, non-Euclidean.[13] In addition, I will compare the mixed account with the standard account on how they fit into a Lewisian metaphysical framework and argue for an advantage of the mixed account on this aspect.

According to the standard account, a *path* in a space is a continuous function from a unit interval to that space. We can take the unit interval in question as a unit interval of time. Then a path can be considered as the trajectory of a point-sized object in a unit interval of time. We will focus on a path that is smooth and does not intersect itself. At every point on the path, we can define a *tangent vector* to be the derivative of the path at that point, which indicates the "velocity" of the path at that point. A *metric tensor* at a point assigns a *length* to each tangent vector. Heuristically, it may be helpful to think of the length of a tangent vector at a point to be the infinitesimal distance from the point along the direction of the vector divided by an infinitesimal time—though strictly speaking there are no infinitesimals in standard analysis.[14] The length of a path is obtained by integrating the lengths of the tangent vectors along the path—or informally, by adding up those "infinitesimal distances" over the unit time interval.[15]

> RIEMANNIAN CONCEPTION. The length of a path is equal to the path integral of the lengths of the tangent vectors along the path. (Riemann 1866)

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[13]The mixed account can accommodate non-Euclidean space as well. I will not go into details here. But one can refer to Forrest (1995, 334-40), in which Forrest explained how an atomistic model can approximate curved space once we have a model that approximates Euclidean space.

[14]For instance, in two-dimensional Euclidean space (or any flat two-dimensional Riemannian manifold), the length of a tangent vector expressed by $(\frac{dx}{dt}, \frac{dy}{dt})$ is $\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}$.

[15]More formally, consider a path in two-dimensional Euclidean space. Let g be a metric tensor and T range over tangent vectors along a path. Then the length of that path is $\int \sqrt{g(T,T)}dt$.

As in any path-dependent account of distance, the distance between any two points is the length of a shortest path between them.

Thus, like the mixed account, the standard account has a mixture of two levels. The lengths of tangent vectors assigned by metric tensors are analogous to primitive local distances in the mixed account, and obtaining distances through integrating those lengths are analogous to adding up the primitive distances. Since we accept this mixed form in the standard account as unproblematic, we should not object to the mixed account on this ground—it is just as natural as in the standard account. (Note that "primitive" only means "geometrically primitive," so the primitiveness of metric tensors is compatible with their being determined by the mass-energy distribution in a dynamical theory. Analogously, we can also allow primitive distances in the mixed account to be determined in this way.)

What's more, the primitive "distances" in the standard account are no sparser than in the mixed account: a metric assigns a length to every tangent vector at each point, and there are infinitely many tangent vectors at every point. Since we do not know any simple foundation that can determine a geometry that is generally non-Euclidean, it seems unfair to charge the mixed account of unnaturalness and overcomplexity, at least not without further arguments.

Let's turn to another possible metaphysical reason to favor the standard account over the mixed account: local metrics are local properties and therefore fit better with the Humean spirit than primitive binary relations. The idea can be captured by Lewis's well-known statement of Humean supervenience: "all there is to the world is a vast mosaic of local matters of fact, just one little thing and then another." (Lewis, 1986: ix) In this picture, the fundamental properties or relations of the world are intrinsic properties of point-sized objects together with spatiotemporal relations (ix-x). Although Lewis himself did not consider fundamental metric features to be intrinsic properties of spatiotemporal points, it would be attractive to have *a Humean geometry* which says so.

The problem of invoking Humean supervenience here, however, is that it's hard to see how we could fit the standard account into the "Humean mosaic": metric tensors and tangent vectors aren't obviously anything like qualities distributed over spacetime, and there isn't an obvious way that one can reduce the former to the latter. Of course, we can still give a try. To start fitting the standard account into a Humean geometry, let's suppose a tangent vector represents a property of a point. Here's one way of putting it: a tangent vector v at a point p represents p's property of being such that there is a path f passing through p with the "velocity" v. Next, we need such a property to be intrinsic to the point. But this conflicts with the standard account under the Lewisian-Humean framework. Hume famously denied necessary connections between distinct entities, which, according to Lewis, implies that it should be possible to have a perfect duplicate of an entity regardless of how the rest of the world is. Moreover, a property is intrinsic iff it never differs between perfect duplicates. So, if a property is intrinsic to a point, it should be possible to have a duplicate of that point which has that property even if the duplicate exists all by itself. However, according to the standard account, if we simply have an isolated point, then there is only "a null vector" (a vector with length zero) at the point. So if we "delete" the surroundings of a point in a continuous space, then the originally non-null tangent vectors at the point would become a null vector. Thus it seems natural to think that a tangent vector or the property it represents is not intrinsic to a point. This picture, then, doesn't exactly depict a Humean geometry.

Nonetheless, even if the standard account is *not* local in the sense that fundamental metric features are qualities intrinsic to spatiotemporal points, there is a different important sense in which it is local. While it is not plausible that geometric features of spacetime supervene on intrinsic properties of spatiotemporal points, it *is* plausible that those features supervene on intrinsic properties of small neighborhoods. Here's a way to make this precise:

> LOCALITY. A property F of a point x is *local* to x iff for any neighborhood U of x, any point y, and any neighborhood V of y such that the pair (V, y) is a duplicate of (U, x), then y has F.[16]

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[16]It is tricky to define duplicates in this context, which I'll come back to later. Here we shall rely on our intuitive understanding of duplicates. It is important to note that the duplicates of a neighborhood preserves not only its spatial structure but also its physical content. In the "local minimum" example, the content in question includes the mass density field. Also note that a duplicate of a pair of a neighborhood and a point in it preserves among others the position of the point.

Here, when referring to spatial regions, I mean those regions together with their physical content. For example, consider a mass density field ρ in a two-dimensional Euclidean space. We say that the mass density at a point x is a *local minimum* iff there is a neighborhood of x such that for every point y in the neighborhood, $\rho(x) \leq \rho(y)$. According to the above definition, being a local mass density minimum is indeed a local property of x because for any density field ρ' that is the same as ρ within a neighborhood of x, x will still be a local density minimum. We can check that in this case, the property of being a local minimum is not intrinsic to x, since if we assign the same mass density to x as $\rho(x)$ but assign smaller density to its surrounding points, then x is no longer a local density minimum. We can also see the distinction between local properties and "global" properties: x being a local density minimum does not imply that x is a global density minimum because there may be points y such that the density at y is smaller than the density at x. It is straightforward to extend LOCALITY to apply to relations in general (which will be useful soon):

> LOCALITY*. A relation R between x, z is *local* to x iff for any neighborhood U of x, any y, and any neighborhood V of y such that the pair (V, y) is a duplicate of (U, x), V includes a duplicate z' of z and y bears R to z'.

In this case, even though a local property (or relation) is not intrinsic to a point, it is close enough: while a property intrinsic to a point is preserved by a duplicate of that point, a property (or relation) local to a point x is preserved by a duplicate of (U, x) with U being *any* neighborhood of the point, however small it is. In this sense, a local property is "almost" intrinsic to a point.

But how do we apply these notions of locality to atomistic space? Although in atomistic space the standard topological notion of "neighborhood" does not straightforwardly apply, the mixed account does provide a simple and natural notion of "neighborhood": a *neighborhood* of a point x is a set of atoms that includes all atoms that bear a primitive distance to x. It follows that, for any neighborhood U of an atom a, a duplicate of (U, a) will include the duplicates of all a's neighbors and preserve the primitive distances between them. This would mean that a primitive distance *is local to* either of its relata. Therefore, like the standard account, even though primitive distances aren't intrinsic properties of points, there is a perfectly legitimate sense in which the geometry under the mixed account is local. So there is no obvious sense in which the mixed account satisfies the spirit of a Humean geometry any less than the standard account.

On the other hand, we do have a consideration from the Humean-Lewisian framework in favor of the mixed account over the standard account. According to Lewis, duplicates are defined in terms of *perfectly natural* properties and relations.

> DUPLICATE. Two possibilia X, Y are *duplicates* iff there is a one-to-one correspondence between parts of X and parts of Y that preserves all perfectly natural properties and relations.

Meanwhile, intrinsic properties and relations are defined in terms of duplicates: they are properties or relations that do not differ between duplicates. It follows that all perfectly natural properties and relations are intrinsic. But it is hard to see how we can fit the standard account into this framework. We have already seen that, according to the standard account, the properties represented by tangent vectors are not intrinsic to the points. But these properties are presumably perfectly natural. We would then have perfectly natural and extrinsic properties, which is incompatible with the Humean-Lewisian framework. There have been attempts of resolving this tension by revising the standard account or the Lewisian framework, which I will not discuss here.[17] But it's helpful to note that the mixed account does not face this difficulty. According to the mixed account, primitive distances are perfectly natural: the fusion of atoms a, b and the fusion of atoms a', b' are geometric duplicates iff a', b' have the same primitive distance as a, b. Primitive distances are also intrinsic to their relata: if you duplicate the fusion of a and b without duplicating anything else, their primitive distance remains the same. So the Lewisian framework is directly applicable to the mixed account, and this may be considered an attractive feature of the account.

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[17]For example, Weatherson (2006) argued that we should define duplicates in terms of fundamental properties and relations in a way that weeds out neighborhood-dependent aspects. Bricker (1993) suggested that local metrics are distances in infinitesimal neighborhoods of points.

4.6 Forrest's Proposal

In this section, I will compare the mixed account with Forrest's (1995) solution to Weyl's tile argument.[18] Forrest's account, as I shall argue, is a restricted version of the mixed account. While the two accounts are not mutually exclusive, the mixed account has the advantage of allowing potentially better models for actual space that are incompatible with Forrest's account.

Like the path-dependent account in Section 2, Forrest posited exactly one fundamental dyadic relation between atoms, *adjacency*, which is symmetric and irreflexive.[19] The distance between any two atoms is equal to the least number of atoms in a "chain of adjacency" between them. What's new about Forrest's account is that, unlike what we have seen before, two adjacent atoms do not need to be represented by two square tiles that are directly next to each other but can also be represented by tiles that are far apart. In this sense, the notion of "adjacency" is analogous to the notion of "neighbors" in the mixed account. Let each atom be represented by a pair of integers. Let m be a parameter of atomistic space. To have a model that approximates Euclidean geometry, Forrest stipulated atoms to have the following adjacency relations:

> FORREST'S MODEL. Two atoms (x_1, y_1) and (x_2, y_2) are *adjacent* iff $(x_2 - x_1)^2 + (y_2 - y_1)^2 \leq m^2$.

When m is sufficiently large, this model approximates two-dimensional Euclidean space very well at the large scale.[20]

Forrest's account is a restricted version of the mixed account, because every model under Forrest's account is isomorphic to a model under the mixed account, but not *vice versa*. Since in Forrest's account, the distance between two adjacent atoms is one, FORREST'S MODEL is isomorphic to the following model under the mixed account:

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[18]Van Bendegem (1987, 1995) also proposed solutions to Weyl's tile argument. I consider his later proposal as a restricted version of Forrest's account. We can have a one-to-one correspondence between *points* (a technical notion) in Bendegem's model and atoms in Forrest's model that preserves distance. But Forrest's account allows models that are incompatible with Bendegem's account.

[19]I change some of Forrest's terminology to align with mine. He calls atomistic space "discrete space" and atoms "points."

[20]For the proof, see Forrest (1995, 344-6).

FORREST MIXED MODEL. For any two atoms $a = (x_1, y_1)$ and $b = (x_2, y_2)$, the primitive distance $\mathbf{d}(a, b) = 1$ if $(x_2 - x_1)^2 + (y_2 - y_1)^2 \leq m^2$; otherwise $\mathbf{d}(a, b)$ is undefined.

This means that Forrest's solution is also available under the mixed account.

Since FORREST'S MODEL approximates Euclidean geometry at the large scale, and since Forrest's account seems simpler than the mixed account, one may think that we do not need to go for the mixed account and should stick with Forrest's account. One problem for Forest's account may be that it is counterintuitive to assign the tiles that are "far apart" the same distance as atoms represented by tiles next to each other. But it's hard to press this issue further without a notion of "far apart" independent from distance. More importantly, the mixed account allows for models like EUCLIDEAN MIXED MODEL (Section 4) that have distinctive advantages over FORREST'S MODEL, and we should not rule out those models as candidates for our actual space.

One difference between EUCLIDEAN MIXED MODEL and FORREST'S MODEL is that the former approximates Euclidean geometry at the local level while the latter does not. In FORREST'S MODEL, when m is large, there are a large number of atoms that are equidistant from each other. There is no way to embed these atoms in Euclidean space that preserves their distances approximately. This could be a disadvantage of FORREST'S MODEL because our space may turn out to be locally Euclidean (or have a richer local geometry than what Forrest's account allows for). But one may resist this answer by arguing that if the local level is sufficiently small, that is, smaller than any observable distance, then the local geometry of the model cannot be disconfirmed by our empirical considerations. For instance, suppose the Planck length (10^{-35} meter) is the smallest physically meaningful unit. If we assume that the local level is smaller than the Planck scale, then the model will approximate Euclidean geometry above the Planck scale. Let atoms be represented by a grid of points in a scaled Euclidean plane such that the nearest points have a Euclidean distance of 10^{-65} meter. Furthermore, let two atoms be adjacent if their representative points have a Euclidean distance smaller than or equal to 10^{-35} meters. This means that the parameter

m in FORREST'S MODEL equals 10^{30}, which guarantees that, at scales larger than the Planck scale, distances are approximately Euclidean.[21]

However, even if FORREST'S MODEL can be made compatible with any empirical observations of our space, this compatibility does not come without costs. First of all, under the above configuration, the vast majority of local distances in FORREST'S MODEL would not play any role in physical theories. Every distance that is a whole number times the Planck length is determined by the length of a path consisting of $a_1, a_2, ..., a_n$ such that for any $i = 1, ..., n-1$, the distance between a_i and a_{i+1} is the Planck length. In this case, for any two atoms that are represented by points with their Euclidean distance smaller than 10^{-35} meters, the distance between them does not play any role in determining the geometry at physically meaningful scales. These distances are extraneous to physical theories, and there are a lot of them: for every distance that is physically meaningful, there are about $1/2 \cdot 10^{30}!$ distances that are not.

Apart from the problem of extraneousness, by making the local level "sub-physical," we would lose the empirical motivation for atomistic space. One motivation for atomistic space is that quantum theory and general relativity are incompatible below the Planck scale, so some physicists take the Planck length to be an indivisible unit of length.[22] Thus the models in which atoms are significantly smaller than the Planck scale seem unhelpful for such physical considerations. This would make atomistic models less motivated.

In contrast, under the mixed account, we do not have to assume the local level to be smaller than the Planck scale. For example, let the shortest primitive distance be the Planck length. What would the longest primitive distance be in order to accommodate our current observations? The relative accuracy of a diffraction measurement, one of the best measurements for small distances, is about $\pm 1.6 \times 10^{-9}$ (NIST, n.d.). Assuming we want space to approximate Euclidean geometry with a distortion smaller than this margin of error, a lo-

[21]The parameter $m = 10^{30}$ is a number given by Forrest to ensure the model to approximate Euclidean geometry at the large scale (Forrest 1995, 333).

[22]For example, see 't Hooft (2016).

cal neighborhood will have to have a diameter of 10^{-26} meter.[23] This is a scale at which physics does not dispense with geometry. Thus unlike FORREST'S MODEL, local distances in this model are not extraneous. Such a model stays relevant to our empirical interest in an atomistic theory of space.

The local geometry of EUCLIDEAN MIXED MODEL has another potentially attractive feature that FORREST'S MODEL lacks: it has a low dimension. According to Forrest, the dimension of a space is determined by the largest number of "points" that are equidistant from each other. In FORREST'S MODEL, the largest number of atoms that are equidistant from each other depends on the distance in question. As a result, Forrest proposed a scale-relative definition of dimensions for atomistic space:

> DIMENSION. A space is *N-dimensional relative to distance D* iff there are at most $N + 1$ atoms that bear distance D to each other.

According to this definition, FORREST'S MODEL has a very high dimension relative to the unit distance, since there are a vast number of atoms that are of a unit distance from each other. For example, with the parameter $m = 10^{30}$, there are about 10^{60} atoms that bear a unit distance from each other, which means that the model is about 10^{60}-dimensional at the local level. In contrast, EUCLIDEAN MIXED MODEL is exactly two-dimensional at the local level.[24] Although overall simplicity is tricky to compare, this is one aspect that EUCLIDEAN MIXED MODEL may seem simpler.

As a small bonus, under the mixed account, there is no need for defining dimensions to be relative to scales.[25] The dimension of a space can be uniformly determined by the local geometry:

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[23]As shown in Appendix C, in order for the atomistic model to approximate Euclidean space, the longest primitive distance needs to be about as large as the shortest primitive distance divided by the permitted distortion (as expressed by "$M > 3r/\delta$" in the appendix).

[24]Suppose EUCLIDEAN MIXED MODEL is more than two-dimensional locally, then there are more than three atoms in a local neighborhood equidistant from each other. But their distances just are the Euclidean distances among their representative pairs of integers. Thus there are more then three pairs of integers that are equidistant from each other on the Euclidean plane. But this is known to be impossible. Thus, EUCLIDEAN MIXED MODEL is no more than two-dimensional locally. Moreover, it is clear that EUCLIDEAN MIXED MODEL is not one-dimensional locally, so it is exactly two-dimensional.

[25]Forrest needs the definition of dimensionality to be relative to the scale because he wants to recover *some* sense in which space is three (or four) dimensional.

> DIMENSION*. A space is *N-dimensional* iff N is the least number such that every local neighborhood can be isometrically embedded in a N-dimensional continuous space.[26]

In summary, although Forrest's account does make do with economical resources in one respect, having a full-fledged local metric (instead of just primitive adjacency) has important payoffs. Thus I recommend the mixed account as a response to Weyl's tile argument: it is versatile, compatible with our best physical theory and no less natural than the standard account of distance in continuous space, and it has distinct advantages over Forrest's proposal.

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[26] This definition is analogous to the definition of the dimension of a manifold (i.e., a continuous space). One may try to translate this definition into a more intrinsic form such as this:

> DIMENSION†. A space is *N-dimensional* iff N is the least number that there are at most $N+1$ atoms that bear the same primitive distance to each other.

The problem with DIMENSION† is that it leads to counterintuitive results. For instance, if no two pairs of atoms in the same local neighborhood have the same primitive distance, then DIMENSION† would imply that the space is one-dimensional. But when such a space is not embeddable into one-dimensional continuous space, it is intuitively not one-dimensional.

APPENDIX A

THE STANDARD VIEW

In this appendix, I will introduce the detail of *the standard view* of continua, also known as *the pointy view*. The core claim of the view is that a continuum is composed of unextended points. To explain the view in details, I shall start with Euclidean n-space $\mathbb{R}^n$, as the exemplar of a pointy continuum, consisting of all n-tuples $x = (x^1, x^2...x^n)$ with each $x^i \in \mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. I will introduce three basic aspects of the standard structure of $\mathbb{R}^n$: mereology, topology, and measure theory. The alternative approaches I discuss later contrast with the standard view on these aspects. I will also introduce the more general notion of "differentiable manifold," the standard modern conception of continua.

Standard Mereology

In standard mereology, we have the primitive relation *being a part of*. For example, the handle is a part of the mug. Gin is part of the martini. The graduate years are the best part of her life. Let "be a part of" be abbreviated as "$\preceq$." Let x, y, z range over regions of space. Parthood satisfies the following core axioms:

> REFLEXIVITY. $x \preceq x$.
> TRANSITIVITY. If $x \preceq y$ and $y \preceq z$, then $x \preceq z$.
> ANTISYMMETRY. If $x \preceq y$, and $y \preceq x$, then $x = y$.

Other mereological notions are defined in terms of *parthood*.

> PROPER PART. x is *a proper part* of y (abbr. $x \prec y$) iff $x \preceq y$ and $x \neq y$.
> OVERLAP. x and y *overlap* iff there is a z such that $z \preceq x$ and $z \preceq y$.
> DISJOINT. x and y are *disjoint* iff x and y do not overlap.
> FUSION. x is a *fusion* of ys iff for every region z, z overlaps x if and only if z overlap some ys. ("ys" is a plural variable.) It follows that for any regions, there is at most one fusion.

INTERSECTION. x is a *intersection* of ys iff for any z, $z \preceq x$ if and only if for every y among ys, $z \preceq y$. It follows that for any regions, there is at most one intersection.

REMAINDER. x is a *remainder* of y in z, or the difference of z and y, iff x is the largest part of z that is disjoint from y. x is a *complement* of y iff x is the difference of the whole space and y.

Three decomposition principles below capture our intuitive notion of *proper parthood*, ordered by their increasing strongness:

WEAK SUPPLEMENTATION. If $y \prec x$, then there is a part of x that is disjoint from y.

STRONG SUPPLEMENTATION. Unless $x \preceq y$, there is a part of x that is disjoint from y.

REMAINDER CLOSURE. If $y \prec x$, then there is a z that is disjoint from y and x is the fusion of y and z.

It follows that every region that is not the whole space has a unique complement. Furthermore, we have a simple composition principle:

UNRESTRICTED COMPOSITION. For any xs, there is a y that is a fusion of xs.

The standard mereology of Euclidean space $\mathbb{R}^n$ is as follows:

THE STANDARD MEREOLOGY. There is a one-to-one correspondence between non-empty subsets of $\mathbb{R}^n$ and regions of Euclidean space such that for any regions x and y, x is a part of y iff $x \subseteq y$.[1]

This entails that the mereology is "pointy":

THE POINTY MEREOLOGY. There are regions with no proper part. All the regions represented by singletons of n-tuples are mereologically simple.

Standard Topology

In standard topology, we have the primitive notion of *openness* applied to regions of a topological space X, which satisfies the following conditions:

1. The whole space is open.
2. The fusion of any open regions is open.
3. The intersection of any finitely many open regions is open.

[1] Although "the null region," which corresponds to the empty set, is often posited as a formal element in order to form a complete Boolean algebra, it is standard not to include the null region as a region.

Basis is another important notion, which is defined as a collection $\mathcal{B}$ of regions of X (called *basis elements*) that satisfies the following conditions: (1) every point is contained in at least one basis element. (2) For every point x, if x is in the intersection of two basis elements B_1 and B_2, then there is a basis element B_3 in the intersection of B_1 and B_2 that contains x. A basis $\mathcal{B}$ of X can generate all open regions of X by taking arbitrary fusions.

The standard topology of Euclidean space $\mathbb{R}^n$ is this:

> THE STANDARD TOPOLOGY. All open intervals $\{\langle x_1, ..., x_n \rangle \mid a_1 < x_1 < b_1, ..., a_n < x_n < b_n\}$ of $\mathbb{R}^n$ form a basis, which generates all open regions of $\mathbb{R}^n$.

A topological space can have many different bases. For instance, the standard topology of $\mathbb{R}^2$ can be generated by the set of all open squares, or the set of all open disks. Note that $\mathbb{R}^n$ has a *countable basis*, namely a basis containing countably many basis elements. For example, a basis can consist of only open intervals with rational numbers as end points, which are countably many. Russell (2008) took this feature to hold for any reasonable model of our actual space, which I discuss and reject in Chapter 3.

> COUNTABLE BASIS. Space has a countable basis.

Other useful topological notions can be defined in terms of open regions. All variables ranges over regions.

> CLOSED. x is *closed* if the complement of x is open.
> INTERIOR. The *interior* of x is the fusion of all open regions that are parts of x.
> CLOSURE. The *closure* of x is the intersection of all closed sets that contains x.
> LIMIT. A point x is a *limit point* of y if every open interval containing x intersects A in some point other than x itself.
> BOUNDARY. The *boundary* of x is the difference between the interior of x and the closure of x.

It follows that a point x is a limit point of y iff x is in the closure of y. The closure of x is the fusion of x and all the limit points of x, or, the fusion of x and its boundary. A region is closed iff it includes all its limit points.

Another pair of notions that will become important in gunky topology (Chapter 3) is this:

REGULAR CLOSED. x is *regular closed* if x is identical to the closure of its interior.

REGULAR OPEN. x is *regular open* if x is identical to the interior of its closure.

Informally speaking, a region is regular if it doesn't contain isolated points or point-sized holes.

Finally, we observe the standard topology of $\mathbb{R}^n$ has the following feature:

THE POINTY TOPOLOGY. Every region that is not the whole space has a boundary. In other words, for every proper part of space, its interior is distinct from its closure.

Standard Measure Theory

In standard measure theory, we have the primitive measure function from regions of space to extended real numbers (standard real numbers plus positive and negative infinity) that satisfies the following conditions.

NONNEGATIVITY. For any region, its measure is nonnegative.

COUNTABLE ADDITIVITY. For any countably many disjoint measurable regions, their fusion is measurable and is equal to the sum of their measures.

In particular, the measure satisfies finite additivity: for any finitely many disjoint regions, the measure of their fusion is the sum of their measures. For example, suppose you are walking along a straight line. You first traversed 5 meters, and then another 5 meters. Clearly, the total distance you have traversed should be the sum of the two distances, namely 10 meters, for otherwise we wouldn't understand what distance is. COUNTABLE ADDITIVITY also entails the following weaker principle:

COUNTABLE SUBADDITIVITY. For any countably many disjoint measurable regions, their fusion is measurable and is less than or equal to the sum of their measures.

Russell (2008) rejected this principle in his proposed measure theory for gunky space (see Chapter 3). Note that one could argue that if a measure satisfies countable subadditivity, it should also satisfies countable additivity. Suppose again you are walking on a straight line. You first traversed 1 meter, then 1/2 meter, then 1/4, and so on *ad infinitum*. Assuming finite additivity, at each stage of the walk, we can add finitely many distances together, and it's clear that the total distance should be no shorter than the sum of those distances, since the infinite sum is just a limit of finite sums, and the total distance I have walked should be no shorter than the limit of all the finite distances I have traversed. This, together with COUNTABLE SUBADDITIVITY, entails COUNTABLE ADDITIVITY.

There are many different measures for $\mathbb{R}^n$. I will introduce the Lebesgue measure as the standard measure. First, we define the length of an interval:

> INTERVAL LENGTH. For any interval (a, b), $(a, b]$, $[a, b)$, or $[a, b]$ $(a, b \in \mathbb{R})$, the length of the interval is $b - a$.

The Lebesgue measure of $\mathbb{R}^n$ is formulated as follows:

> THE STANDARD MEASURE. Consider an n-rectangle $I = I_1 \times I_2 \times ... \times I_n$, where I_i is an interval in the i-th dimension of $\mathbb{R}^n$. The measure of I is the product of the measures of I_i. Let A ranges over all fusions of countably many disjoint n-rectangles. Let B be any region in $\mathbb{R}^n$. The Lebesgue measure $|B| = inf\{|A| : B \subseteq A\}$.

The Lebesgue measure of $\mathbb{R}^n$ is pointy in the following sense:

> THE POINTY MEASURE. Some regions have zero measure.

For example, any countable fusion of points has zero measure. An uncountable fusion of points like the Cantor set can also have zero measure (Henry 1876).

Manifold

Having introduced Euclidean n-space $\mathbb{R}$ as an exemplary case, I shall now proceed to the more general conception of continuum, known as "manifold," introduced in standard differential geometry. Roughly, a manifold is like $\mathbb{R}^n$ locally. To make it more precise, we need the following notions (X and Y below all refer to topological spaces):

> *Homeomorphism.* A function $f : X \to Y$ is *continuous* iff the inverse image of every open subset of Y is an open subset of X.[2] A function is a *homeomorphism* iff it is bijective and continuous, and its inverse is also continuous. In other words, a homeomorphism preserves topological structures. X and Y are *homeomorphic* iff there is a homeomorphism between them.

[2]In real analysis, we have the ϵ-δ definition of a continuous function. For a function $f : \mathbb{R} \to \mathbb{R}$, these two definitions are equivalent. But the topological definition is more general.

Then, we can define "manifold" as follows:

> MANIFOLD. A topological space M is a *manifold* iff for every $x \in M$, there is some neighborhood U of x and some $n \in \mathbb{N}^*$ such that U is homeomorphic to $\mathbb{R}^n$.[3]

More vividly, for any point in a manifold, one can find a neighborhood of it which can be deformed (through stretching, squishing or bending) into $\mathbb{R}^n$ for some $n \geq 0$. For a simple example, S^1 (1-sphere, which is a circle-like space) is a manifold, since every neighborhood in it is homeomorphic to $\mathbb{R}$, although the whole of it is not homeomorphic to $\mathbb{R}$.

Notice that the notion of *differentiable function* is not automatically well-defined on a manifold. A *differential manifold* M is a manifold equipped with an additional differential structure—called an *atlas*—consisting of a maximal set of compatible homeomorphisms from local neighborhoods of M to $\mathbb{R}^n$. Each homeomorphism in an atlas is called a *chart*.[4] We can also define other geometric structures such as a metric on a manifold. How we

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[3]"Manifold" can also be defined on basis of "metric space" instead of "topological space." Largely, the difference does not matter. The only difference is that our definition allows some pathological manifolds that are not metrizable (explanation below). For readers that are interested in the "metric space" version of manifold, I supply the definition of "metric space" below:

> *Metric.* A metric on a set X is a function $d : X \times X \to \mathbb{R}$ that satisfies:
> (1) $d(x, y) \geq 0$ for all $x, y \in X$, and equality holds only when $x = y$;
> (2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
> (3) $d(x, y) + d(y, z) \geq d(x, z)$, for all $x, y, z \in X$.
> Here, $d(x, y)$ is often called the distance between x and y.
> *Metric Topology.* Let the collection of all ϵ-ball centered at x in a set X, i.e. $\{y | d(x, y) < e\}$ $(x, y \in X)$, form a basis. The topology generated by this basis is called the *metric topology* induced by d.
> *Metric Space.* A topological space X is a *metric space* iff a metric d is specified on X, which can induce X's topology. If X does not have a such a metric, then X is not *metrizable*.

[4]Since $\mathbb{R}^n$ has a differentiable structure (namely the differentiability of functions from $\mathbb{R}^n$ to $\mathbb{R}$ is well-defined), one might expect that we can define a differentiable function on a manifold though copying that structure locally through a homeomorphism like this: let U be an $\mathbb{R}^n$-like region in a manifold M and ϕ be a homeomorphism from U to $\mathbb{R}^n$, then $f : M \to \mathbb{R}$ is differentiable on U iff $f \circ \phi^{-1} : \mathbb{R}^n \to \mathbb{R}$ is differentiable. Unfortunately, it's possible to find another homeomorphism $\psi : V \to \mathbb{R}^n$ with $U \cap V \neq \emptyset$ such that $f \circ \psi^{-1}$ is not differentiable. Then, if we stick to such definition, the property of being differentiable would be relative to a specific homeomorphism (in other words, a specific coordinate system). Fortunately, the problem can be fixed by imposing an additional requirement on homeomorphisms involved so that such a possibility does not occur. We require that two homeomorphisms in question ϕ, ψ satisfy the following condition: $\phi \circ \psi^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ and $\psi \circ \phi^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ must both be smooth (i.e. indefinitely differentiable). We call a maximal collection of so-related homeomorphisms an atlas.

might interpret those additional geometric structures realistically is controversial. In this sense, there is no comprehensive standard view about manifolds. See Chapter 4 for more discussion.

Since a manifold is locally like $\mathbb{R}^n$, many of the previous discussions about $\mathbb{R}^n$ can easily carry over to a manifold. The mereology of a manifold can be similarly defined by identifying the parthood relation with the subset relation. Since there are elements in a manifold that have no proper subset, a manifold has mereological points and hence satisfies THE POINTY MEREOLOGY.

Next, consider the topology of a manifold. While $\mathbb{R}^n$ has a standard topology, there is no such thing as the standard topology of a manifold, since a manifold is by definition equipped with a topology and there is no such thing as "the standard manifold." Nevertheless, it is clear that every manifold satisfies THE POINTY TOPOLOGY, since we can map a boundary of some region in $\mathbb{R}^n$ into a manifold through a local homeomorphism (or rather, its inverse). Since a homeomorphism preserves the topological structure, the image of the boundary of a region is still a boundary of a corresponding region in the manifold. Similar, the distinction between the closure and the interior of a region is preserved through homeomorphisms.

Finally, consider the measure of a manifold. A systematic formulation of the Lebesgue measure on a manifold is complicated. But we can just focus on regions of measure zero in order to show that a manifold satisfies THE POINTY MEASURE. A region S of a manifold M has measure zero iff for every chart $\phi : U \to \mathbb{R}^n$, $\phi(U \cap S)$ has measure zero in $\mathbb{R}^n$. Under this definition, it is clear that there are regions in M of measure zero (e.g., any point in M).

Summary

> THE POINTY VIEW. A continuum is composed of points. A point has no proper part and has a measure of zero. There are boundary regions, and the interior of a region is generally distinct from its closure.

Although the above features do not fully characterize the standard conception of continua, they capture the features that distinguish the standard view from the alternatives I will explore.

APPENDIX B

COUNTABLE SATURATION

The language of standard analysis $\mathcal{L}$ that we focus on includes constant symbols for all real numbers and set constructions from real numbers through the iterations of the powerset operation and union. More precisely, let $U_n(X) = U_{n-1}(X) \cup \mathcal{P}(U_{n-1}(X))$ and $U(X) = \bigcup_{n=0}^{\infty} U_n(X)$. The language includes constants for all members of $U(\mathbb{R})$, which have associated ranks according to the least $U_n(\mathbb{R})$ they belong. $\mathcal{L}$-terms and $\mathcal{L}$-formulas are defined in the usual way (Goldblatt 1998, 166-7). We can show that all the familiar functions and relations in standard analysis (such as addition, integration, Lebesgue measure), considered as sets, are members of $U(\mathbb{R})$ and therefore are referred to by $\mathcal{L}$-constants (Goldblatt 1998, 165-6). Note that these functions and relations are not just defined over real numbers but can have a variety of ranks. The hyperreal system under consideration (along with the set constructions) U' is an alternative model for $\mathcal{L}$ that is an expansion of $U(\mathbb{R})$. There is a unique transfer map from $U(\mathbb{R})$ to U' that preserves all the $\mathcal{L}$-truths. Members of the image of the transfer map are called *standard*. For example, the hyperreal line $*\mathbb{R}$ is the image of $\mathbb{R}$ under the transfer map, and is therefore standard. An entity is *internal* iff it is a member of a standard set (Goldblatt 1998, 172). Any hyperreal number is internal because it is a member of $*\mathbb{R}$. Any hyperreal interval is internal because it is a member of the set $\{X \mid (\exists a, b \in *\mathbb{R})(\forall x \in X)(a \leq x \leq b)\}$, which is the image of the set of all real intervals. A bijective internal function belongs to a set of functions characterized as "bijective" in $\mathcal{L}$ in the usual way. It turns out that all sets in the form of $\{x \mid \phi(x)\}$, where ϕ is a formula in the language $\mathcal{L}'$ that extends $\mathcal{L}$ with constants for internal entities, are internal sets (Goldblatt 1998, 177). For example, $\mathcal{L}'$ has constants for all hyperreal numbers. A hyperreal interval $\{x \mid a \leq x \leq b\}$ $(a, b \in *\mathbb{R})$ is therefore internal. On the other hand, we can prove that any infinite set of real numbers (e.g., $\mathbb{R}, \mathbb{N}$) is not internal (Goldblatt 1998, 176).

In the hyperreal models we are interested in, the internal sets satisfy the following property:

Theorem B.1 *(Countable Saturation) The intersection of a decreasing sequence of nonempty internal sets* $X^1 \supseteq X^2 \supseteq ...$ *is always nonempty (Goldblatt 1998, 138).*

Countable Saturation implies this principle:

Corollary B.2 *(Nested Intervals) For any countable nested sequence of intervals* $I_1 \supseteq I_2 \supseteq ...$*, their intersection is non-empty and includes an (open) interval.*

Proof. All hyperreal intervals are internal sets. Thus, according to Countable Saturation, the countable nested sequence of intervals $I_1, I_2, ...$ have non-empty intersection. Moreover, the interiors of these intervals also have non-empty intersection. Let x be a point in the intersection of their interiors. Then, the intersection of the parts of the intervals to the right of x is non-empty. Let y be a point in this intersection. Then, $[x, y]$ is included in the intersection of $I_1, I_2, ...$ (Clearly, the intersection also contains the open interval (x, y).) QED.

With Nested Intervals, we can prove that Infinitesimal Gunk violates Countable Basis:

Theorem B.3 *Under Infinitesimal Gunk, the topology of space does not have a countable basis.*

Proof. In this proof, we will use this fact: if a set of regions $\mathcal{B}$ is a basis for a gunky space, then every region in that space contains some region in $\mathcal{B}$. Let $\mathcal{C}$ be any countable set of regions. Take an arbitrary point x on the hyperreal line, and consider the set of all elements in $\mathcal{C}$ that include x in their interiors. Call this set $\mathcal{C}_x$. Since $\mathcal{C}$ is countable, $\mathcal{C}_x$ is also countable. It follows from Nested Intervals that there exists an infinitesimal neighborhood Δ of x that is included in all elements of $\mathcal{C}_x$. Take a closed infinitesimal interval that is strictly included in Δ. This interval does not contain any element of $\mathcal{C}$, so $\mathcal{C}$ is not a basis. Thus, a gunky space does not have a countable basis. QED.

In Section 3.5, I mentioned that the fusion of any countably infinitely many disjoint measurable regions is not measurable. This claim can be derived from the following corollaries of Countable Saturation:

Corollary B.4 *If an internal set X is a countable union of internal sets $X_1, X_2, ...$, then there is a natural number k such that X is the union of $X_1, ..., X_k$ (Goldblatt 1998, 139).*

Proof. The proof is adpated from Goldblatt (1998, 139-40). Suppose that for all $k \in \mathbb{N}$, $X - \bigcup_{n \leq k} X_n$ is non-empty. Since $X - \bigcup_{n \leq k} X_n = \bigcap_{n \leq k}(X - X_n)$, we have that $\bigcap_{n \leq k}(X - X_n)$ is non-empty. Call this set Y^k. Then $\langle Y^k \rangle$ is a decreasing sequence of non-empty internal sets. So, by Countable Saturation, there is a point belonging to Y^k for all k, and thus to $X - X_k$ for all k. Therefore, X is not the union of $X_1, X_2, ...$ QED.

Corollary B.5 *For any countably infinitely many disjoint measurable sets, their union is unmeasurable.*

Proof. Every measurable set is a union of hyperfinitely many disjoint intervals. This in fact guarantees that it is an internal set. Let $A_1, A_2, ...$ be countably infinitely many disjoint measurable sets and let A be their union. Suppose A is measurable. According to Corollary B.4, it follows that A is the union of finitely many A_i. But since $A_1, A_2, ...$ are infinitely many and disjoint, their union is not identical to the union of any finitely many A_i. Thus, A is not measurable. QED.

Corollary B.6 *For any countably many disjoint regular closed sets, their union is regular closed.*

Proof. Let $A_1, A_2, ...$ be countably many disjoint regular closed sets on the hyperreal line. Let A be their union. I will show that A includes all its limit points and is therefore closed. Take any point y outside A. For each regular closed set A_j, there is an open interval that includes y and is disjoint from A_j. According to Nested Intervals, the intersection of these open intervals includes an open interval which includes y and is disjoint from A. Thus y is not a limit point of A. Since y is arbitrarily chosen, no point outside A is a limit point. Therefore, A is closed. QED.

In Section 3.6, we need to show that there are no "trouble-making" boundaries when it comes to the union of hyperfinitely many regular closed sets.

Theorem B.7 *For any hyperfinitely many measurable regular closed sets, their union is regular closed.*

Proof Sketch. In standard analysis, we have the following induction principle for the natural numbers: an $\mathcal{L}$-formula ϕ with one free variable is satisfied by every natural number (taken as $1, 2, ...$) if (1) ϕ is satisfied by $n = 1$; (2) if ϕ is satisfied by any natural number n, then it is also satisfied by $n + 1$. In nonstandard analysis, we have an analogous induction principle for the hypernatural numbers: an $\mathcal{L}$-formula ψ with one free variable is satisfied by every hypernatural number if (1) ψ is satisfied by $N = 1$; (2) if ψ is satisfied by any hypernatural number N, then it is also satisfied by $N + 1$. Since any set with a hyperfinite cardinality N can be ordered under an internal bijection to $\{1, 2, ..., N\}$, we will pick any such ordering of the set of hyperfinitely many measurable regular closed sets in question. Now, we can easily confirm the following: (1) the union of the singleton set of a measurable regular closed set is (trivially) regular closed; (2) if the union of the first N measurable regular closed sets is regular closed, then the union of the first $N + 1$ measurable regular closed sets is also regular closed because the union of two regular closed sets is regular closed. Also, these expressions can indeed be put into $\mathcal{L}$-formulas. Then according to the induction principle, for any hypernatural N, the union of the first N measurable regular closed sets is regular closed, which is just what we want. QED.

APPENDIX C

EUCLIDEAN APPROXIMATION

Now I shall turn to how well space approximates Euclidean space under the mixed account. Under this account, an atomistic space can be represented by a set of points with a *shortest path metric* that assigns some pairs of points real-valued distances (bounded by a finite number) and derives other distances as their least sums.

We will understand “approximation" in terms of “almost isometry." Let $e(p,q)$ be the Euclidean distance between two points p,q in Euclidean space. Let ϵ, r be two positive numbers. A metric space X with a metric d is *ϵ-isometric* to Euclidean space E with regard to r iff there is a map f from X to E such that (1) for $x, y \in X$, we have

$$1 - \epsilon \leq \frac{e(f(x), f(y))}{d(x,y)} \leq 1 + \epsilon$$

(the smallest ϵ such that f satisfies this condition is called the *distortion* of f);[1] (2) for every $p \in E$, there is a $x \in X$ such that $e(p, f(x)) \leq r$. In other words, the embedded points cover E reasonably well so that there are no obvious “clusters" and “holes."

Theorem C.1 *For any ϵ and r, there is a set of points with a shortest path metric (with distances being bounded by a finite number) that is ϵ-isometric to Euclidean space with regard to r.*

Proof. For brevity, I will resort to the following abbreviations when applicable. Given an embedding f of a metric space into Euclidean space, for any points x, y in the space, let

[1] Here, the notion of approximation is cast in a different way from Forrest's (1995). Forrest showed that his model approximates Euclidean space in the sense that we can map Euclidean space into his model such that the distances are approximately preserved. Here, it is the other way around: a model approximates Euclidean space in the sense that we can map this model into Euclidean space that preserves distances approximately. I do not consider either interpretation of approximation to be better than the other, but I work with this one because I feel it a bit more natural.

$\|xy\|_f = e(f(x), f(y))$ (the subscript "f" is omitted if it is clear which embedding we refer to). Also, for any points p, q in Euclidean space, let $\|pq\| = e(p, q)$.

Let G be an embedding of an infinite set X to Euclidean space E such that there is an r such that for any $p \in E$, we can find an $x \in X$ with $e(p, G(x)) < r$. (For example, if G maps members of X to Euclidean points represented by pairs of integers, then r in question is at least $\sqrt{2}/2$.) We will construct a metric over X such that the resulting metric space is ϵ-isometric to Euclidean space under G, where ϵ is a small number we choose.

M is a real-number parameter that will play an important role in assigning weights and in determining the distortion of the intended embedding. For any $x, y \in X$, if $\|xy\| > M$, we can find a sequence of points $p_1, p_2, ...p_n$ in E such that $p_0 = G(x)$, $p_n = G(y)$, $\|p_0p_1\| = \|p_1p_2\| = ... = \|p_{n-2}p_{n-1}\| = M$ and $\|p_{n-1}p_n\| < M$. Let $N = \|p_{n-1}p_n\|$. Consider p_i, p_{i+1}, where $i = 1, ..., n-2$. We can find $x_i, x_{i+1} \in X$ such that $e(G(x_i), p_i) < r$ and $e(G(x_{i+1}), p_{i+1}) < r$. We know that the largest distance between points on two circles is equal to the distance between their centers plus their radii.[2] Thus, $\|x_ix_{i+1}\| < M + 2r$. Now, for any two $a, b \in X$, if $\|ab\| < M + 2r$, then let the primitive distance $d(a, b) = \|ab\|$; otherwise, $d(a, b)$ is not defined. Then, $M \leq d(x_i, x_{i+1}) < M + 2r$. Moreover, it's easy to see that $M \leq d(x, x_1) \leq M + r$ and $N \leq d(x_{n-1}, y) \leq N + r$. It follows that $d(x, y) \leq d(x, x_1) + d(x_1, x_2) + ... + d(x_{n-1}, y) < n \cdot (M + 2r) + (N + r)$. Furthermore, if $x, x_1, ...x_n, y$ is a shortest path, then $d(x, y) = d(x, x_1) + d(x_1, x_2) + ... + d(x_{n-1}, y) = \|xx_1\| + ... + \|x_{n-1}y\| \geq \|xy\|$. Thus, we have:

$$1 \leq \frac{d(x, y)}{\|xy\|} < \frac{n \cdot (M + 2r) + (N + r)}{nM + N} = 1 + \frac{(2n + 1)r}{nM + N}$$

The distortion $\delta = \dfrac{d(x, y)}{\|xy\|} - 1 < \dfrac{(2n + 1)r}{nM + N} < \dfrac{(2n + 1)r}{nM} < \dfrac{3r}{M}$. Then, for any small positive number ϵ, we can make $\delta < \epsilon$ by letting $M = 3r/\epsilon$. (Note that if we are only concerned with distances that involve a large n, we only need M to be $2r/\epsilon$.) This completes the case

[2] Here's a proof for the simple case in which two circles in question have the same radius, which is adequate for our purpose. Let two circles be $x_1 = r\cos\theta_1$, $y_1 = r\sin\theta_1$, $x_2 = r\cos\theta_2 + n$, $y_2 = r\sin\theta_2$. Then, $(x_1 - x_2)^2 + (y_1 - y_2)^2 = n^2 - 2r^2\cos(\theta_1 - \theta_2) - 2nr(\cos\theta_1 - \cos\theta_2) + 2r^2 \leq n^2 + 2r^2 + 4nr + 2r^2 = (n + 2r)^2$. That is, for two circles with the same size, the largest distance between two points on them is equal to the distance between their centers plus their radii.

for any $x, y \in X$ with $\|xy\| > M$. If $\|xy\| \leq M$, then we have $d(x, y) = \|xy\|$, in which case there is no distortion. Therefore, we have found a metric space, in which all distances are bounded by $3r/\epsilon + 2r$, that is ϵ-isometric to Euclidean space at any scale. $\square$

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