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A Cone Conjecture for Log Calabi-Yau Surfaces

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A CONE CONJECTURE FOR LOG CALABI-YAU SURFACES

A Dissertation Presented

By

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ABSTRACT

A CONE CONJECTURE FOR LOG CALABI-YAU SURFACES

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We consider log Calabi-Yau surfaces (Y, D) with maximal boundary. We denote by (Y_e, D_e) the unique surface in each deformation type such that the mixed Hodge structure on $H_2(Y \setminus D)$ is split. The generic log Calabi-Yau surface (Y_{gen}, D_{gen}) does not contain any (-2)-curves. We prove that (1) if K is the kernel of the action of $\mathrm{Aut}(Y_e, D_e)$ on $H^2(Y_e \setminus D_e)$, then $\mathrm{Aut}(Y_e, D_e)/K$ acts on the nef effective cone of Y_e with a rational polyhedral fundamental domain; and (2) The monodromy group acts on the nef effective cone of Y_{gen} with a rational polyhedral fundamental domain. We also prove that for a log Calabi-Yau surface (Y_e, D_e) of boundary length $n \leq 6$, the cone of curves of Y_e is finitely generated, and we explicitly describe the cones. This provides infinite series of new examples of Mori Dream spaces.

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CHAPTER 1

INTRODUCTION

1.1. Organization

This paper is organized as follows. In the Introduction, we give an overview of the project with some informal definitions. We also give a brief explanation of our motivations for this project in the Introduction. Section 2 consists of the main definitions, Section 3 is where we state major theorems and lemmas used, and Section 4 contains the proof of the conjecture. In Section 5, we give explicit descriptions of certain cones of curves, which provide infinite series of new examples of Mori Dream Spaces. We end the paper with Section 6, where we explain in some more detail the motivations of our project.

1.2. Known Results

Given a smooth projective variety Y over $\mathbb{C}$, the closed cone of curves of Y is the closure of the set of all nonnegative linear combinations of classes of irreducible curves in $H_2(Y, \mathbb{R})$. The cone of curves of any Fano variety is rational polyhedral, meaning it has finitely many rational generators (see Theorem 1.24 on p.22 of [**KM98**]). But this is not true in general for Calabi-Yau varieties - if Y is Calabi-Yau, the cone of curves of Y could be round, for example (see Figure 1.2.1). The nef cone is the dual of the cone of curves.

The Morrison cone conjecture states that if Y is a Calabi-Yau variety, then there exists a rational polyhedral cone which is a fundamental domain for the action of the automorphism group of Y on the nef cone. This can be pictured in dimension two

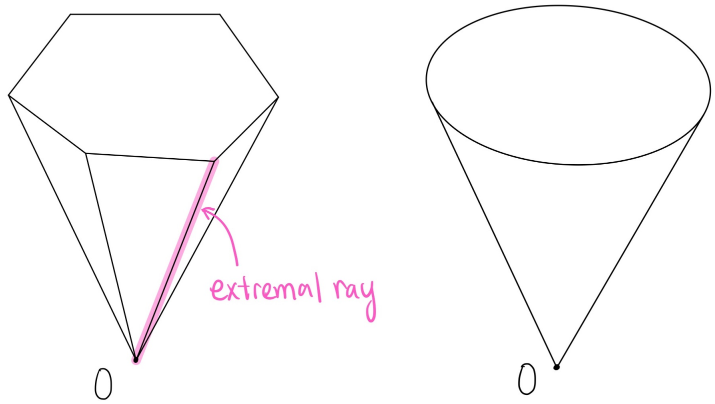

FIGURE 1.2.1. The left drawing shows a rational polyhedral cone, which has finitely many extremal rays. The cone on the right is round.

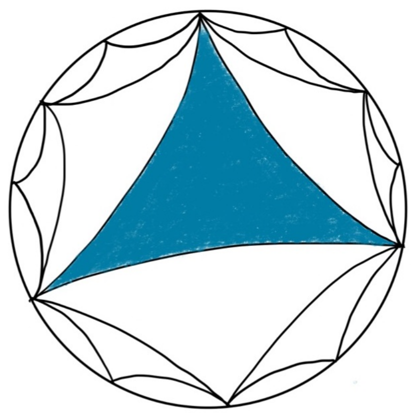

FIGURE 1.2.2. The drawing above shows a rational polyhedral fundamental domain, which is the shaded region, that tessellates the interior of a cone.

using hyperbolic geometry - in this case, there exists a rational polyhedral "piece" that tessellates the interior of the cone, as shown in Figure 1.2.2.

The conjecture is known to be true in dimension two, but for higher dimensions, it is an open question. In [**T10**], Totaro has shown that a generalization of this conjecture is true in dimension two: if (Y, Δ) is a klt Calabi-Yau pair, then the automorphism group of Y acts on the nef cone with a rational polyhedral fundamental domain.

We are studying a cone conjecture for log Calabi-Yau surfaces that is similar to, but different from, the conjecture proved by Totaro. We consider a log Calabi-Yau pair (Y, D) where Y is a smooth projective surface and D is a reduced normal crossing divisor on Y such that $K_Y + D = 0$. Additionally, we require D to be singular, and write $D = D_1 + \cdots + D_n$ for the irreducible components of D. By the Gross-Hacking-Keel Torelli theorem for log Calabi-Yau surfaces (Theorem 1.8 in [**GHK15b**]), in each deformation type of log Calabi-Yau surfaces there exists a unique pair $(Y, D) = (Y_e, D_e)$ such that the mixed Hodge structure on $Y \setminus D$ is split. The conjecture we study is stated as follows:

1.3. Conjecture statement

There are two statements:

Conjecture 1.3.1. *Let (Y_e, D_e) be a log Calabi-Yau surface such that the mixed Hodge structure on $H_2(Y_e \setminus D_e, \mathbb{Z})$ is split. Let K be the kernel of the action of the automorphism group of the pair on $H^2(Y, \mathbb{Z})$. Then $Aut(Y_e, D_e)/K$ acts on the nef effective cone $Nef^{\mathrm{e}}(Y_e)$ with a rational polyhedral fundamental domain.*

Conjecture 1.3.2. *Let (Y_{gen}, D_{gen}) be a generic log Calabi-Yau surface. Then the monodromy group Adm acts on the nef effective cone $Nef^{\mathrm{e}}(Y_{gen})$ with a rational polyhedral fundamental domain.*

1.4. Results

Conjecture 1.3.1 and Conjecture 1.3.2 hold.

1.5. Motivation

The Morrison cone conjecture, stated in 1993, is originally inspired by mirror symmetry. The log Calabi-Yau surface version of this conjecture is also related to mirror symmetry through the deformation theory of cusp singularities of surfaces.

Given a log Calabi-Yau surface (Y, D), we may contract the boundary D to a cusp singularity p, resulting in (Y', p) (see Grauert [**G62**]) and Definition 2.23). Cusp singularities come in dual pairs such that the links are diffeomorphic but have opposite orientations. If (Y', p) is obtained by contracting the boundary of a log Calabi-Yau surface (Y, D) to a cusp singularity $p \in Y'$, then, conjecturally, (Y, D) corresponds to an irreducible component of the deformation space of the dual cusp. This is expected as a consequence of mirror symmetry: $Y \backslash D$ is mirror to the Milnor fiber of the corresponding smoothing of the dual cusp. Again conjecturally, the component of the deformation space of the dual cusp can be described in terms of the action of the monodromy group Adm on Nef (Y'), by a construction of Looijenga ([**L03**], §4). However, to use this construction, the group Adm must act with a rational polyhedral fundamental domain on the effective nef cone of Y', and this is the original motivation for our conjecture, cf. [**M93**].

CHAPTER 2

BACKGROUND

Let Y be a smooth complex projective variety. A *divisor* on Y is a formal integral linear combination $\sum a_i D_i$ of codimension one subvarieties D_i of Y. A *1-cycle* on Y is a formal integral linear combination $C = \sum a_i C_i$ of curves $C_i \subset Y$. The intersection product $D \cdot C$ is an integer for D a divisor and C a 1-cycle. Two divisors D_1 and D_2 are said to be numerically equivalent if $D_1 \cdot C = D_2 \cdot C$ for all curves $C \subset Y$. Two 1-cycles C_1 and C_2 are said to be numerically equivalent if $D \cdot C_1 = D \cdot C_2$ for all divisors $D \subset Y$.

We define $N^1(Y)$ to be the space of divisors with real coefficients modulo numerical equivalence, and the space $N_1(Y)$ to be the space of 1-cycles with real coefficients modulo numerical equivalence. Then the intersection product defines a nondegenerate pairing

$$N^1(Y) \times N_1(Y) \to \mathbb{R}. \tag{2.1}$$

By the Néron-Severi theorem, the space $N^1(Y)$ is finite dimensional, and by 2.1, the dimensions of $N^1(Y)$ and $N_1(Y)$ are equal. Alternatively by Hodge theory,

$$N^1(Y) = H^{1,1}(Y) \bigcap H^2(Y, \mathbb{R}),$$

where $H^2(Y, \mathbb{R})$ is a subset of $H^2(Y, \mathbb{C}) = H^{2,0} \oplus H^{1,1} \oplus H^{0,2}$.

Remark 2.2. Although we define some terms for varieties of any dimension, we only consider the case that Y is two-dimensional. Moreover, the surface Y is rational in

our setting (see [**GHK15b**], p.1). It follows that

$$N^1(Y) = H^2(Y, \mathbb{R}) = \mathrm{Pic}(Y) \otimes \mathbb{R}, \tag{2.3}$$

$$N_1(Y) = H_2(Y, \mathbb{R}) = \mathrm{Cl}(Y) \otimes \mathbb{R}, \tag{2.4}$$

and $N^1(Y) = N_1(Y)$ because $\dim(Y) = 2$. Here is a proof of the lines 2.3 above: since Y is smooth, its class group is isomorphic to its Picard group, that is, $\mathrm{Cl}(Y) \cong \mathrm{Pic}(Y)$. We have $H^1(\mathcal{O}_Y) = H^2(\mathcal{O}_Y) = 0$ because Y is rational, so by the exponential exact sequence, the first Chern class gives an isomorphism $\mathrm{Pic}(Y) \cong H^2(Y, \mathbb{Z})$. By Poincaré Duality, it follows that $H^2(Y, \mathbb{Z}) = H_2(Y, \mathbb{Z})$, and the intersection product on $H_2(Y, \mathbb{R})$ is nondegenerate.

Properties of a smooth projective surface Y are encoded using convex geometry through cones.

Definition 2.5. Let V be a finite dimensional real vector space, so that $V \simeq \mathbb{R}^\rho$ for some $\rho \geq 0$. Then $C \subset V$ is called a *cone* if for any $v \in C$, the product $\lambda \cdot v \in C$ also, for any $\lambda \in \mathbb{R}_{>0}$.

Remark 2.6. For us, the vector space V is $\mathrm{Pic}(Y) \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{R}^\rho$.

Remark 2.7. We only consider convex cones. Recall that a set $S \subset V$ is *convex* if $x, y \in S$ implies that $\lambda_1 x + \lambda_2 y \in S$ for all $\lambda_1, \lambda_2 \in \mathbb{R}_{\geq 0}$ such that $\lambda_1 + \lambda_2 = 1$. Geometrically, for any two points x, y in S, the line segment L joining x and y also lies in S. If $S = C$ is a cone, this is equivalent to the condition that $x, y \in C$ implies $x + y \in C$.

In the following definitions, let Y be a smooth complex projective variety.

Definition 2.8. The *nef cone of* Y is defined as follows:

$$\mathrm{Nef}\,(Y) = \{L \in N^1(Y) \mid L \cdot C \geq 0 \text{ for all irreducible curves } C \subset Y\}$$

Definition 2.9. The *effective cone of* Y is defined by:

$$\mathrm{Eff}(Y) = \{\sum a_i[D_i] \in N^1(Y) \mid a_i \in \mathbb{R}_{\geq 0} \text{ and } D_i \subset Y \text{ are codimension one subvarieties.}\}$$

Definition 2.10. Following Kawamata in [**K97**], we define the *nef effective cone of* Y as follows:

$$\mathrm{Nef}^{\,\mathrm{e}}(Y) = \mathrm{Nef}\,(Y) \bigcap \mathrm{Eff}(Y).$$

$\mathrm{Conv}(S)$, where S is subset of a real vector space, is used to denote the *convex hull* of the set S.

Lemma 2.11. *For a smooth projective surface,*

$$\mathit{Nef}^{\,\mathrm{e}}(Y) = \mathit{Conv}\{[L] \in N^1(Y) \mid L \in \mathrm{Pic}(Y) \textit{ is nef and } h^0(L) \neq 0\}.$$

Proof. For any element $[L] \in \mathrm{Conv}\{[L] \in N^1(Y) \mid L \in \mathrm{Pic}(Y)$ is nef and $h^0(L) \neq 0\}$, the line bundle $L \in \mathrm{Pic}(Y)$ is nef so that $[L] \in \mathrm{Nef}\,(Y)$. It is also effective by definition, proving the inclusion $\mathrm{Conv}\{[L] \in N^1(Y) \mid L \in \mathrm{Pic}(Y)$ is nef and $h^0(L) \neq 0\} \subseteq \mathrm{Nef}^{\,\mathrm{e}}(Y)$.

If $L \in \mathrm{Nef}^{\,\mathrm{e}}(Y)$, then L is nef, and L is effective, meaning we can write $L = \sum_{i=1}^{r} a_i[C_i]$ for some curves C_i and $a_i \in \mathbb{R}_{\geq 0}$. Define a rational polyhedral cone $\sigma := \langle C_1, \ldots, C_r\rangle_{\mathbb{R}\geq 0}$, and $\tau := \sigma \bigcap \mathrm{Nef}\,(Y)$. Then $L \in \tau \subset \mathrm{Nef}^{\,\mathrm{e}}(Y)$. Moreover, since $\tau \subset \sigma$ is defined by a a finite list of inequalities $\{M \in \sigma \mid M \cdot C_i \geq 0 \text{ for all } i\}$, τ is rational polyhedral, so we may write $L \in \tau$ as a linear combination $\sum c_i M_i$ of integral classes $M_i \in \tau$ with nonnegative real coefficients. Therefore $L \in \mathrm{Conv}\{[L] \in N^1(Y) \mid L \in \mathrm{Pic}(Y)$ is nef and $h^0(L) \neq 0\}$, proving that $\mathrm{Nef}^{\,\mathrm{e}}(Y) \subseteq \mathrm{Conv}\{[L] \in N^1(Y) \mid L \in \mathrm{Pic}(Y)$ is nef and $h^0(L) \neq 0\}$. □

Corollary 2.12. *For (Y, D) a log Calabi-Yau surface (see Definition 2.16), the nef effective cone of Y is equal to the convex hull of the integral points of the nef cone of Y, that is,*

$$\mathit{Nef}^{\,\mathrm{e}}(Y) = \mathit{Conv}(\mathit{Nef}\,(Y) \bigcap \mathrm{Pic}(Y)).$$

Proof. This result follows from Lemma 2.11 and Lemma 2.22. ☐

Definition 2.13. The *cone of curves of* Y is defined as follows:

$$\mathrm{Curv}(Y) = \{\sum a_i[C_i] \in N_1(Y) \mid a_i \in \mathbb{R}_{\geq 0} \text{ and each } C_i \subset Y \text{ an irreducible curve}\}.$$

We write $\overline{\mathrm{Curv}}(Y)$ to mean the closure of the cone of curves.

Remark 2.14. The nef cone $\mathrm{Nef}\,(Y)$ and the closed cone of curves $\overline{\mathrm{Curv}}(Y)$ are dual cones. This can be understood as follows. In an arbitrary real vector space V, if $\sigma \subset V$ is a cone, then its dual cone is defined as $\sigma^* := \{\theta \in V^* \mid \theta(v) \geq 0 \text{ for all } v \in \sigma\}$.

Definition 2.15. Let L be a finitely generated free Abelian group, ie., $L \simeq \mathbb{Z}^\rho$ for some $\rho \geq 0$. A cone $C \subset L \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{R}^\rho$ is said to be *rational polyhedral* if

$$C = \langle v_1, \ldots, v_r \rangle_{\mathbb{R}\geq 0} = \{a_1 v_1 + \cdots + a_r v_r \mid a_i \in \mathbb{R}_{\geq 0}\},$$

for some $v_1, \ldots, v_r \in L$. That is, the cone C is generated by finitely many integral vectors $v_1, \ldots, v_r \in L$.

Definition 2.16. A *log Calabi-Yau surface* is a pair (Y, D) where Y is a smooth complex projective surface and $D \subset Y$ is a reduced normal crossing divisor such that $K_Y + D = 0$. We say that (Y, D) has maximal boundary if D is singular. We write $D = D_1 + \cdots + D_n$, where n is the number of components or the *length* of D.

In this thesis, we always assume that (Y, D) has maximal boundary. If (Y, D) is a log Calabi-Yau surface with maximal boundary, then Y is a rational surface, i.e., Y is birational to $\mathbb{P}^2$ (see [**GHK15b**], top of p.2).

Remark 2.17. The boundary D is either a rational curve of arithmetic genus one with a single node (i.e., a copy of $\mathbb{P}^1$ with two points identified to form a node), or it is a cycle of smooth rational curves (i.e., a cycle of n copies of $\mathbb{P}^1$). This follows from the adjunction formula.

We fix a cyclic ordering $D = D_1 + \cdots + D_n$ of the components of D and a compatible orientation (an isomorphism $H_1(D, \mathbb{Z}) \simeq \mathbb{Z}$). This orientation is uniquely determined by the cyclic ordering for n greater than two.

Definition 2.18. We say that a log Calabi-Yau surface (Y, D) is *generic* if there are no (-2)-curves C contained in $Y \setminus D$. We sometimes write (Y_{gen}, D_{gen}) to denote one such log Calabi-Yau surface.

Definition 2.19. Two log Calabi-Yau surfaces (Y^1, D^1) and (Y^2, D^2) are said to be *deformation equivalent* if there exists a flat family $(\mathcal{Y}, \mathcal{D})$ over a connected base S such that there are points $p, q \in S$ with fibers $f^{-1}(p) = (Y^1, D^1)$ and $f^{-1}(q) = (Y^2, D^2)$. Since S is connected, there is a path from p to q. In this case, we say that (Y^1, D^1) and (Y^2, D^2) are of the same *deformation type*.

By the GHK Torelli Theorem in [**GHK15b**], given a log Calabi-Yau surface (Y, D), the moduli space $\mathcal{M}$ of log Calabi-Yau surfaces that are deformation equivalent to (Y, D) can be described explicitly and the locus of generic surfaces is the complement of a countable union of divisors in $\mathcal{M}$ (see [**GHK15b**], Section 6). For any two generic surfaces of the same deformation type, the nef cones of the two surfaces are the same. This cone for Y_{gen} is described after the following definition:

Definition 2.20. For a log Calabi-Yau surface (Y, D), an *interior* (-1)*-curve* is a smooth rational curve of self-intersection -1 that is not contained in the boundary D. By the adjunction formula, such a curve must intersect the boundary transversely at a single point.

Proposition 2.21. *(See Gross-Hacking-Keel* [**GHK15b**]*, Lemma 2.15)*

$$\begin{aligned} \mathit{Nef}\,(Y_{gen}) = \{L \in \operatorname{Pic}(Y) \otimes_{\mathbb{Z}} \mathbb{R} \mid L^2 \geq 0 \ \textit{and}\ L \cdot D_i \geq 0 \ \textit{ for all }\ i \ \textit{and} \\ L \cdot C \geq 0 \ \textit{ for any interior (-1)-curve}\ C\}. \end{aligned}$$

Lemma 2.22. *Let (Y, D) be a log Calabi-Yau surface. If $L \in \operatorname{Pic}(Y)$ is nef, then L is effective.*

PROOF. Let $L \in \mathrm{Pic}(Y)$ be nef. By Riemann-Roch, we have

$$\begin{aligned}\chi(L) &= \chi(\mathcal{O}_Y) + \frac{1}{2}L(L - K_Y)\\ &= 1 + \frac{1}{2}(L^2 + L \cdot D)\\ &\geq 1,\end{aligned}$$

since L being nef and D being effective give $L \cdot D \geq 0$ and L nef gives $L^2 \geq 0$. On the other hand, we have

$$\begin{aligned}\chi(L) &= h^0(L) - h^1(L) + h^2(L)\\ &\leq h^0(L) + h^2(L)\end{aligned}$$

Next we show that $h^2(L) = 0$. By Serre Duality, we have $h^2(L) = h^0(K_Y - L) = h^0(-D - L)$. If H is ample and L is nef and D is effective, then we have $H \cdot D > 0$ and $H \cdot L \geq 0$. Then $H \cdot (-D - L) < 0$, so $h^0(-D - L) = 0$. Thus $h^0(L) \geq \chi(L) \geq 1$, and therefore L is linearly equivalent to an effective divisor. $\square$

Definition 2.23. A *cusp singularity* is a surface singularity whose minimal resolution is a cycle of smooth rational curves that meet transversally. That is, the exceptional locus of the minimal resolution of a cusp singularity is a union of copies of $\mathbb{P}^1$ with nodal singularities such that the dual graph is a cycle.

Given a log Calabi-Yau surface (Y_{gen}, D_{gen}) with D_{gen} having a negative definite intersection matrix $(D_i \cdot D_j)$, it is possible to contract D_{gen} to a cusp singularity p (by a theorem of Grauert on the contractibility of a negative definite configuration of curves on a smooth complex surface in the analytic category - see [**G62**] and Figure 2.1). Let $f : Y_{gen} \to Y'_{gen}$ be the morphism contracting D_{gen} to a point. Then we have the induced isomorphism

$$Y_{gen} \setminus D_{gen} \cong Y'_{gen} \setminus \{p\},$$

and $f^{-1}(p) = D_{gen}$. In addition, the surface Y'_{gen} is normal and compact (for the usual Euclidean topology). We note that although Y_{gen} is a projective variety, the new surface Y'_{gen} is in general no longer a projective variety, but a normal, complex analytic space. We make the following definitions.

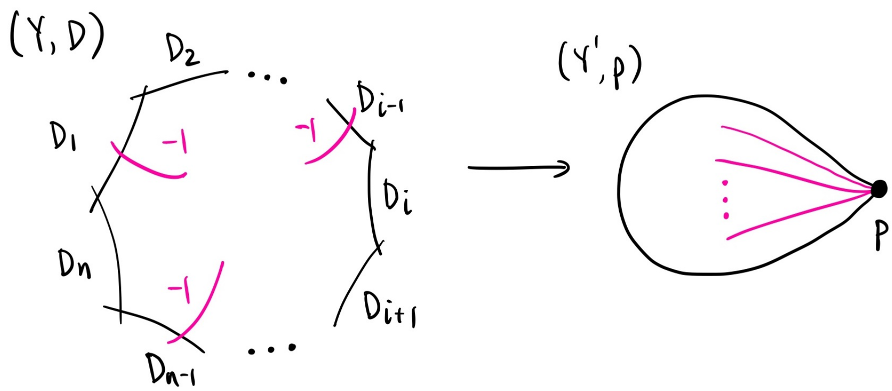

FIGURE 2.1. The drawing above shows the contraction of the boundary D of a log Calabi-Yau surface (Y, D) to another pair (Y', p), where p is a cusp singularity. The new surface is an analogue to a $K3$ surface. Here we note that the interior curves pictured in (Y, D) are (-1)-curves. When D is contracted to the cusp p, these curves all pass through the point p. It may be that in Y', such a curve is contractible to a point and so defines an extremal ray of the cone of curves of Y'. Although the surface Y' is similar to a $K3$ surface, there is no analogue of such curves in the case of a $K3$ surface. For this reason, the cone of curves of Y' is more complicated than that of a $K3$.

Definition 2.24. (Mumford's definition of intersection numbers on a normal surface, on p.17 of [**M61**]). Let X be a normal surface and let $\pi : \tilde{X} \to X$ be a resolution of singularities of X. Then for a divisor D on X, we may define $\pi^* D$ by

$$\pi^* D := D' + \sum a_i E_i,$$

where D' is the strict transform of D, the E_i's are exceptional curves of π, and the $a_i \in \mathbb{Q}$ are chosen such that

$$\left(D' + \sum a_i E_i\right) \cdot E_j = 0 \text{ for all } j.$$

We note that because the intersection matrix $(E_i \cdot E_j)$ is negative definite, it is non-degenerate and therefore the coefficients a_i are uniquely determined. Then $\pi^* D \in$

$\mathrm{Pic}(\tilde{X}) \otimes_{\mathbb{Z}} \mathbb{Q}$ satisfies $(\pi^* D) \cdot E_i = 0$ for all exceptional curves E_i. Now we may define the intersection number of two divisors D_1 and D_2 on the surface X in the following way:

$$D_1 \cdot D_2 := \pi^* D_1 \cdot \pi^* D_2 \in \mathbb{Q},$$

where the dot '$\cdot$' denotes the intersection product on $\mathrm{Pic}(\tilde{X}) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Definition 2.25. We define the *nef cone of* Y'_{gen} in the following way:

$$\mathrm{Nef}\,(Y'_{gen}) = \{L \in \mathrm{Cl}(Y'_{gen}) \otimes_{\mathbb{Z}} \mathbb{R} \mid L \cdot C \geq 0 \text{ for all curves } C \subset Y'_{gen}\},$$

where the dot '$\cdot$' in the intersection '$L \cdot C \geq 0$' represents Mumford's intersection product on $\mathrm{Cl}(Y'_{gen})$.

Definition 2.26. We define the *nef effective cone of* Y'_{gen} in the following way:

$$\mathrm{Nef}^{\,\mathrm{e}}(Y'_{gen}) = \mathrm{Nef}\,(Y'_{gen}) \,\cap\, \mathrm{Eff}(Y'_{gen}) \subset \mathrm{Cl}(Y'_{gen}) \otimes_{\mathbb{Z}} \mathbb{R}.$$

Lemma 2.27. *The cone Nef* (Y'_{gen}) *may be described as follows:*

$$Nef\,(Y'_{gen}) = Nef\,(Y_{gen}) \bigcap \langle D_1, \ldots, D_n \rangle_{\mathbb{R}}^{\perp}$$

PROOF. First we prove that

$$\mathrm{Nef}\,(Y'_{gen}) = \mathrm{Nef}\,(Y_{gen}) \bigcap \langle D_1, \ldots, D_n \rangle_{\mathbb{R}}^{\perp}.$$

Because we have the birational morphism of surfaces $f : Y_{gen} \to Y'_{gen}$ with exceptional locus $D \subset Y_{gen}$, we have the following exact sequence:

$$0 \to \langle D_1, \ldots, D_n \rangle_{\mathbb{Z}} \to \mathrm{Cl}(Y_{gen}) \xrightarrow{f_*} \mathrm{Cl}(Y'_{gen}) \to 0. \tag{2.28}$$

Then $\mathrm{Cl}(Y'_{gen}) \cong \mathrm{Cl}(Y_{gen}) / \langle D_1, \ldots, D_n \rangle_{\mathbb{Z}}$. Tensoring both sides by $\mathbb{R}$ results in

$$\begin{aligned} \theta : \mathrm{Cl}(Y'_{gen}) \otimes \mathbb{R} &\xrightarrow{\sim} \mathrm{Cl}(Y_{gen}) \otimes \mathbb{R} / \langle D_1, \ldots, D_n \rangle_{\mathbb{R}} \\ &\cong \langle D_1, \ldots, D_n \rangle_{\mathbb{R}}^{\perp}, \end{aligned}$$

where the second isomorphism holds because $\mathrm{Cl}(Y_{gen}) = \langle D_1, \ldots, D_n\rangle_{\mathbb{R}} \oplus \langle D_1, \ldots, D_n\rangle_{\mathbb{R}}^{\perp}$ (since the intersection product on $\langle D_1 \ldots, D_n\rangle_{\mathbb{R}}$ is negative definite and so nondegenerate).

Because Nef (Y'_{gen}) is contained in $\mathrm{Cl}(Y'_{gen}) \otimes \mathbb{R}$, it remains to show that θ induces an isomorphism $\tau : \text{Nef }(Y'_{gen}) \cong \text{Nef }(Y_{gen}) \bigcap \langle D_1, \ldots, D_n\rangle_{\mathbb{R}}$, as shown below.

$$\begin{array}{ccc}
\text{Nef }(Y'_{gen}) & \subset & \mathrm{Cl}(Y'_{gen}) \otimes \mathbb{R} \\
\downarrow \tau & & \downarrow \theta \\
\text{Nef }(Y_{gen}) \bigcap \langle D_1, \ldots, D_n\rangle_{\mathbb{R}}^{\perp} & \subset & \langle D_1, \ldots, D_n\rangle_{\mathbb{R}}^{\perp}
\end{array}$$

We use the definitions:

$$\text{Nef }(Y_{gen}) = \{L \in \mathrm{Pic}(Y_{gen}) \otimes \mathbb{R} \mid L \cdot C \geq 0 \text{ for all curves } C \subset Y_{gen}\},$$

and

$$\text{Nef }(Y'_{gen}) = \{L \in \mathrm{Cl}(Y'_{gen}) \otimes \mathbb{R} \mid L \cdot C \geq 0 \text{ for all curves } C \subset Y'_{gen}\},$$

where $L \cdot C$ refers to Mumford's intersection product (explained above in 2.24).

First we show that $L \in \text{Nef }(Y'_{gen})$ implies that $\theta(L) \in \text{Nef }(Y_{gen})$. Let $C \subset Y'_{gen}$ be a curve. Then

$$\theta(C) = C' + \sum a_i D_i,$$

such that $\theta(C) \cdot D_i = 0$ for all i. Here we use C' to denote the strict transform of C. Now we have $\theta(L) \cdot \theta(C) = \theta(L) \cdot C'$. Since $\theta(L)$ is perpendicular to each D_i by definition, for each i, the intersection $\theta(L) \cdot D_i$ is zero. Then $\theta(L) \cdot C' = \theta(L) \cdot \theta(C) = L \cdot C \geq 0$. The last inequality is by our assumption that $L \in \text{Nef }(Y'_{gen})$.

For any curve $\Gamma \subset Y_{gen}$, we have $\theta(L) \cdot \Gamma \geq 0$. This is because Γ is either exceptional, so that $\Gamma = D_i$ and $\theta(L) \cdot \Gamma = 0$, or Γ is not exceptional, so that $f(\Gamma) = C$ is some curve on Y'_{gen}, so that $\Gamma = C'$ and $\theta(L) \cdot \Gamma \geq 0$ by the inequality in the previous paragraph. Thus by definition of Nef (Y_{gen}), we obtain $\theta(L) \in \text{Nef }(Y_{gen})$.

Conversely, we want to show that $M \in \text{Nef }(Y_{gen}) \bigcap \langle D_1, \ldots, D_n\rangle_{\mathbb{R}}^{\perp}$ implies $M = \theta(L)$ for some $L \in \text{Nef }(Y'_{gen})$. Since $\theta : \mathrm{Cl}(Y'_{gen}) \otimes_{\mathbb{Z}} \mathbb{R} \xrightarrow{\sim} \langle D_1, \ldots, D_n\rangle_{\mathbb{R}}^{\perp}$ is an

isomorphism, we have $M = \theta(L)$ for some unique $L \in \mathrm{Cl}(Y'_{gen}) \otimes \mathbb{R}$. If $C \subset Y'_{gen}$ is a curve, then

$$\begin{aligned} L \cdot C &= \theta(L) \cdot \theta(C) \\ &= \theta(L) \cdot C' \\ &= M \cdot C' \\ &\geq 0, \end{aligned}$$

where the last line comes from the assumption that M is nef. Therefore $L \in \mathrm{Nef}\,(Y'_{gen})$. This completes the proof that $\mathrm{Nef}\,(Y'_{gen}) \cong \mathrm{Nef}\,(Y_{gen}) \bigcap \langle D_1, \ldots, D_n \rangle_{\mathbb{R}}^{\perp}$. $\square$

Lemma 2.29. *The cone $Nef^{\,\mathrm{e}}(Y'_{gen})$ may be described as follows:*

$$Nef^{\,\mathrm{e}}(Y'_{gen}) = Nef^{\,\mathrm{e}}(Y_{gen}) \bigcap \langle D_1, \ldots, D_n \rangle_{\mathbb{R}}^{\perp}.$$

Proof. Because $\mathrm{Nef}^{\,\mathrm{e}}(Y) = \mathrm{Nef}\,(Y) \bigcap \mathrm{Eff}(Y)$, it suffices to show that $\mathrm{Eff}(Y'_{gen}) \xrightarrow{\sim} \mathrm{Eff}(Y_{gen})$ under the isomorphism $\theta : \mathrm{Cl}(Y'_{gen}) \otimes \mathbb{R} \to \langle D_1, \ldots, D_n \rangle_{\mathbb{R}}^{\perp}$, defined above in the proof of Lemma 2.27. We use the following to observations:

(1) The restriction of the pushforward $f_* : \mathrm{Cl}(Y_{gen}) \otimes \mathbb{R} \to \mathrm{Cl}(Y'_{gen}) \otimes \mathbb{R}$ to $\langle D_1, \ldots, D_n \rangle_{\mathbb{R}}^{\perp}$ coincides with θ^{-1}. This follows from the exact sequence 2.28 and the definition of θ. Now f_* sends effective divisors to effective divisors:

$$f_*(\sum a_i C_i) = \sum a_i f_*(C_i),$$

where $f_*(C_i) = 0$ if $f(C_i)$ is a point and $f_*(C_i) = f(C_i)$ otherwise. So θ^{-1} sends effective divisors to effective divisors.

(2) If L is effective, then $\theta(L)$ is effective: this is because

$$\theta(L) = L' + \sum a_i D_i,$$

where L' is the strict transform of L and each $a_i \in \mathbb{Q}$ and $\theta(L) \cdot D_i = 0$ for all i. If L is effective, then its strict transform L' is also effective. By Lemma

3.41 in Kollár and Mori [**KM98**], each a_i is nonnegative. Therefore $\theta(L)$ is effective.

$\square$

Definition 2.30. An *isomorphism of log Calabi-Yau surfaces* (Y^1, D^1) and (Y^2, D^2) means an isomorphism $\theta : Y^1 \to Y^2$, with the property that $\theta(D_i^1) = \theta(D_i^2)$ for each boundary component D_i^k of D^k for $k = 1, 2$, and θ respects the orientations of D^1 and D^2 (automatic for $n \geq 3$). The *automorphism group of a log Calabi-Yau surface* (Y, D) is denoted by $\mathrm{Aut}(Y, D)$.

Definition 2.31. Given any log Calabi-Yau surface (Y, D), the *admissible group* of Y is defined as follows:

$$\begin{aligned}\mathrm{Adm} = \{\theta \in \mathrm{Aut}(\mathrm{Pic}(Y), \cdot) \mid \theta([D_i]) = [D_i] \text{ for all } i = 1, \ldots, n \text{ and} \\ \theta(\mathrm{Nef}\ (Y_{gen})) = \mathrm{Nef}\ (Y_{gen})\}.\end{aligned}$$

In the definition above, the Picard group $\mathrm{Pic}(Y)$ is considered as an Abelian group. The dot '$\cdot$' symbol that appears in $(\mathrm{Pic}(Y), \cdot)$ denotes the intersection form on $\mathrm{Pic}(Y)$. Then $\mathrm{Aut}(\mathrm{Pic}(Y), \cdot)$ is by definition the group of automorphisms of the Abelian group $\mathrm{Pic}(Y)$ which preserve the intersection form. So in other words, the group Adm is the subgroup of $\mathrm{Aut}(\mathrm{Pic}(Y), \cdot)$ consisting of all automorphisms that fix the class of each component of D and preserve the cone $\mathrm{Nef}\ (Y_{gen})$.

Remark 2.32. Adm is identified with the monodromy group for (Y, D) (see [**GHK15b**] Theorem 5.15 on p.25).

Definition 2.33. Let Γ be a group and X a topological space. Suppose that Γ acts on X by homeomorphisms. We say that a closed subset $D \subset X$ is a *fundamental domain* for the action of Γ on X if the following are true:

(1) for all $x \in X$, there exists $d \in D$ and $\gamma \in \Gamma$ such that $\gamma(d) = x$; and

(2) for all $\gamma_1, \gamma_2 \in \Gamma$ such that $\gamma_1 \neq \gamma_2$, the intersection $\gamma_1 D \cap \gamma_2 D$ has no interior. Equivalently, for all $\gamma \in \Gamma$ such that $\gamma \neq 1$, the intersection $\gamma D \cap D$ has no interior.

Remark 2.34. The two conditions in the last statement of Definition 2.33 are equivalent because

$$\gamma_1 D \cap \gamma_2 D = \gamma_2((\gamma_2^{-1}\gamma_1)D \cap D)$$

Definition 2.35. Given a log Calabi-Yau surface (Y, D), the *period point* is defined to be the homomorphism $\phi : \langle D_1, \ldots, D_n \rangle^{\perp} \to \mathbb{C}^*$, where a line bundle $L \in \langle D_1, \ldots, D_n \rangle^{\perp}$ is sent to $\theta([L|_D]) \in \mathbb{C}^*$, where $\theta : \mathrm{Pic}^0(D) \xrightarrow{\sim} \mathbb{C}^*$ is the isomorphism determined by the given orientation of D (as explained in [**GHK15b**], Lemma 2.1). Here $\mathrm{Pic}^0(D)$ is the kernel of the map $c_1 : \mathrm{Pic}(D) \to H^2(D, \mathbb{Z}) \simeq \mathbb{Z}^n$, defined by sending any $L \in \mathrm{Pic}(D)$ to $\left(\deg(L|_{D_i})\right)_{i=1}^{n} = (\deg(L|_{D_1}), \ldots, \deg(L|_{D_n}))$.

By Proposition 3.12 in [**F15**], the homomorphism ϕ is the extension class of the mixed Hodge structure on $H_2(U, \mathbb{C})$, where we take $U = Y \setminus D$. There exists an exact sequence (see [**L81**], Chapter I, Section 5.1, p. 285):

$$0 \to \mathbb{Z} \to H_2(U) \to \langle D_1, \ldots, D_n \rangle^{\perp} \to 0.$$

There exists a unique log Calabi-Yau surface in each deformation type such that $\phi(\alpha) = 1$ for all $\alpha \in \langle D_1, \ldots, D_n \rangle^{\perp}$, i.e., meaning that $\phi : \langle D_1, \ldots, D_n \rangle^{\perp} \to \mathbb{C}^*$ is the constant map sending everything to 1. This follows from the Torelli theorem and is stated at the beginning of Section 5 of [**GHK15b**]; it is also stated without proof in [**F15**], in Corollary 9.7. This unique surface corresponds to the mixed Hodge structure on $H_2(U)$ being split. In this case we denote the log Calabi-Yau surface by (Y_e, D_e). To summarize, for any log Calabi-Yau surface (Y, D), there is a unique surface denoted (Y_e, D_e) such that

(1) the mixed Hodge structure on $H_2(Y_e \setminus D_e, \mathbb{Z})$ is split, and
(2) (Y_e, D_e) is deformation equivalent to (Y, D).

Definition 2.36. Given a log Calabi-Yau surface (Y, D), the associated *root system* is the subset of $\mathrm{Pic}(Y)$ defined by:

$$\Phi = \{\alpha \in \langle D_1, \ldots, D_n\rangle^{\perp} \mid \alpha^{\perp} \cap \mathrm{Int}(\mathrm{Nef}\,(Y_{gen})) \neq \emptyset \text{ and } \alpha^2 = -2\}$$

Definition 2.37. We define the *Weyl group* of a root system $\Phi \subset \mathrm{Pic}(Y)$ as follows:

$$W = \langle s_\alpha \mid \alpha \in \Phi\rangle \subset \mathrm{Aut}(\mathrm{Pic}(Y), \cdot),$$

where the generators $s_\alpha(\beta) = \beta + (\alpha \cdot \beta)\alpha$ are the reflections in the hyperplanes $\alpha^{\perp}$ for $\alpha \in \Phi$.

An equivalent but more efficient presentation of the Weyl group involves the simple roots of a log Calabi-Yau surface.

Definition 2.38. Given (Y_e, D_e), we define the *simple roots* as the set:

$$\Delta = \{[C] \mid C \subset Y_e \setminus D_e \text{ is a } (-2)\text{-curve}\}.$$

Proposition 2.39. *The Weyl group W is generated by the reflections s_δ for $\delta \in \Delta$, i.e.,*

$$W = \langle s_\delta \mid \delta \subset \Delta\rangle.$$

We note that $\Delta \subseteq \Phi$ and $W \cdot \Delta = \Phi$ (see Definition 1.6 and Proposition 3.4 in [**GHK15b**]). By Lemma 2.15 in [**GHK15b**],

$$\mathrm{Nef}\,(Y_e) = \mathrm{Nef}\,(Y_{gen}) \bigcap (\delta \geq 0 \text{ for all } \delta \in \Delta),$$

where $(\delta \geq 0 \text{ for all } \delta \in \Delta)$ means $\{L \in \mathrm{Pic}(Y) \otimes \mathbb{R} \mid L \cdot \delta \geq 0 \text{ for all } \delta \in \Delta\}$.

Remark 2.40. The Weyl group is a normal subgroup of Adm: it follows from the definitions 2.31 and 2.36 that Adm preserves Φ. If $g \in \mathrm{Adm}$ and $\alpha \in \Phi$, then $g s_\alpha g^{-1} = s_{g(\alpha)}$, which implies that $W \lhd \mathrm{Adm}$.

By Theorem 3.2 of Gross-Hacking-Keel [**GHK15b**], the group W acts on $\mathrm{Nef}^{\,\mathrm{e}}(Y_{gen})$ with fundamental domain $\mathrm{Nef}^{\,\mathrm{e}}(Y_e)$. This is called the *fundamental chamber* in $\mathrm{Nef}\,(Y_{gen})$. By Theorem 5.1 in Gross-Hacking-Keel [**GHK15b**] on p.19, there is an exact sequence

$$1 \to K \to \mathrm{Aut}(Y_e, D_e) \to \mathrm{Adm}/W \to 1,$$

where K is the kernel of the action of $\mathrm{Aut}(Y_e, D_e)$ on $\mathrm{Pic}(Y)$.

CHAPTER 3

TOOLS

Here we include some main results that we used in the proof of our results.

Theorem 3.1. *The Global Torelli Theorem for* (Y, D) *(Gross-Hacking-Keel* [**GHK15b**], *Theorem 1.8, p.5). Suppose that* (Y^1, D^1) *and* (Y^2, D^2) *are log Calabi-Yau surfaces. Consider the following three statements:*

(1) $\theta : \mathrm{Pic}(Y^1) \to \mathrm{Pic}(Y^2)$ *is an isometry such that* $\theta([D_i^1]) = [D_i^2]$ *for* $i = 1, \ldots, n$.

(2) $\theta(L)$ *is ample for some ample* L *on* Y^1.

(3) $\phi_{Y^2} \circ \theta = \phi_{Y^1}$, *where* $\phi_Y : \langle D_1, \ldots, D_n\rangle^{\perp} \to \mathbb{C}^*$ *is the period point of* Y.

$(1), (2)$ *and* (3) *hold if and only if* $\theta = f^*$ *for some isomorphism* $f : (Y^2, D^2) \to (Y^1, D^1)$.

Theorem 3.2. *(Engel-Friedman* [**EF16**], *Proposition 1.5, p.13). Let* (Y_{gen}, D_{gen}) *be a generic log Calabi-Yau surface, where* D_{gen} *has at least three boundary components. If* E *is a divisor on* Y_{gen} *with nonnegative integer coefficients, then* E *is linearly equivalent to a divisor of the form*

$$\sum a_i D_i + \sum b_j C_j,$$

where the C_j*'s are disjoint interior* (-1)*-curves and* a_i, b_j *are nonnegative integers.*

Remark 3.3. Although the Engel-Friedman Theorem 3.2 is stated for E with nonnegative *integer* coefficients, the statement also holds for E with nonnegative *real* coefficients. There is a sketch of the proof in the paper ([**EF16**], p.55), which uses a continuity argument and the assertion that the collection of subsets

$$\left\{\sum a_j D_j + \sum b_i E_i \mid a_j, b_i \in \mathbb{R}_{\geq 0}\right\},$$

where the E_i's are disjoint interior (-1)-curves, is locally finite in $\mathrm{Nef}^{\,\mathrm{e}}(Y_{gen})$ in a sense that is made precise below. Since this is important for our results, we give a complete proof (see Corollary 3.10).

Friedman showed in [**F15**] that Adm acts with finitely many orbits on the set of faces of $\mathrm{Nef}^{\,\mathrm{e}}(Y_{gen})$ corresponding to interior (-1)-curves. This is stated in Theorem 3.4.

Theorem 3.4. *(Friedman* [**F15**]*, Theorem 9.8, p.74) Let (Y, D) be a generic log Calabi-Yau surface. Let $\mathcal{E}(Y, D)$ be the set of all interior (-1)-curves of Y. Then the admissible group Adm acts on $\mathcal{E}(Y, D)$ and there are finitely many Adm-orbits for this action.*

The following Corollary 3.5 by Friedman is similar to the statement above. Specifically, it is a statement about the action of Adm on the set of collections of disjoint interior (-1)-curves.

Corollary 3.5. *(Friedman* [**F15**]*, Corollary 9.10, p.75) Given a generic log Calabi-Yau surface (Y, D), let $\mathcal{E}_k(Y, D)$ be the set of collections $\{E_1, \ldots, E_k\}$, where the curves E_i are disjoint, interior (-1)-curves. Then the admissible group Adm acts on $\mathcal{E}_k(Y, D)$ and the number of Adm orbits for this action is finite.*

Theorem 3.6. *(Looijenga* [**L14**]*, Proposition-Definition 4.1; and Application 4.14; and Proposition 4.7) Let Γ be a group and L be a lattice, i.e., a finitely generated free abelian group, and let $C \subset L \otimes_{\mathbb{Z}} \mathbb{R}$ be an open nondegenerate convex cone. Define*

$$C_+ := Conv(\bar{C} \cap L).$$

Assume that Γ acts on L faithfully, preserving the cone C. If there exists a polyhedral cone $\Pi \subset C_+$ such that $\Gamma \cdot \Pi = C_+$, then there exists a rational polyhedral fundamental domain for the action of Γ on C_+. Moreover, in this case, the group $N_\Gamma F/Z_\Gamma F$ acts on any face F of C_+ with a rational polyhedral fundamental domain.

Remark 3.7. In Proposition 3.6, following the notation of Looijenga, we use $N_\Gamma F$ to mean the normalizer of F in Γ and $Z_\Gamma F$ to mean the centralizer (i.e., elements of Γ that fix F pointwise). The last statement is a special case of Proposition 4.7 in [**L14**].

Lemma 3.8. *Let (Y_{gen}, D_{gen}) be a generic log Calabi-Yau surface. For a collection $\{E_1, \ldots, E_k\}$ of disjoint (-1)-curves, define*

$$C'(E_1, \ldots, E_k) := \langle D_1, \ldots, D_n, E_1, \ldots, E_k \rangle_{\mathbb{R}_{\geq 0}} \cap Nef^{\mathrm{e}}(Y_{gen}).$$

Then

(1) *$C'(E_1, \ldots, E_k)$ is a rational polyhedral cone; and*

(2) *If D_{gen} consists of at least three components, the set of cones $C'(E_1, \ldots, E_k)$ covers $Nef^{\mathrm{e}}(Y'_{gen})$.*

Proof. Let C be the cone defined by

$$C := \langle D_1, \ldots, D_n, E_1, \ldots, E_k \rangle_{\mathbb{R}_{\geq 0}},$$

so that $C'(E_1, \ldots, E_k)$ can be expressed as $C \cap \mathrm{Nef}^{\mathrm{e}}(Y_{gen})$. The cone $C'(E_1, \ldots, E_k)$ is rational polyhedral because C is rational polyhedral by definition, and the intersection with the nef cone is given by finitely many inequalities $L \cdot D_i \geq 0$ and $L \cdot E_j \geq 0$ for all $1 \leq i \leq n$ and all $1 \leq j \leq k$. This shows that $C'(E_1, \ldots, E_k)$ is rational polyhedral. Now assume that D_{gen} has at least three components. To see why,

$$\mathrm{Nef}^{\mathrm{e}}(Y_{gen}) = \bigcup C'(E_1, \ldots, E_k),$$

where the union is over the set $\bigcup\limits_k \mathcal{E}_k(Y, D)$ of collections $\{E_1, \ldots, E_k\}$ of disjoint interior (-1)-curves. We apply Corollary 3.10 below. $\square$

Theorem 3.9. *(The Siegel Property, as stated in Looijenga's paper* [**L14**]*, Theorem 3.8.) Let L be a lattice and $V = L \otimes \mathbb{R}$. Let $C \subset V$ be an open convex nondegenerate cone. We denote the convex hull $Conv(\bar{C} \cap L)$ by C_+. Let Γ be a subgroup of $GL(V)$ such that Γ leaves the cone C and the lattice L invariant. Then Γ has the Siegel*

Property in C_+, that is, if Π_1 and Π_2 are polyhedral cones in C_+, then the collection $\{\gamma\Pi_1 \bigcap \Pi_2\}_{\gamma\in\Gamma}$ is finite.

Next, we give the precise statement of Engel-Friedman's Proposition 3.2 for real coefficients, followed by a careful proof, which uses the Siegel Property 3.9 and the result 3.5 of Friedman.

Corollary 3.10. *(Engel-Friedman 3.2 for E with real coefficients) Let (Y_{gen}, D_{gen}) be a generic log Calabi-Yau surface, where D_{gen} has at least three boundary components. If E is a divisor on Y_{gen} with nonnegative real coefficients, then E is linearly equivalent to a curve of the form*

$$\sum a_i D_i + \sum b_j C_j,$$

where the C_j's are disjoint interior (-1)-curves and a_i, b_j are nonnegative real numbers.

Proof. We use the same notation introduced in Lemma 3.8 above, that is, for a log Calabi-Yau surface (Y_{gen}, D_{gen}) where D_{gen} is of length at least three, we let

$$C := \langle D_1, \ldots, D_n, E_1, \ldots, E_k \rangle_{\mathbb{R}\geq 0}.$$

We want to show that

$$\operatorname{Curv}(Y_{gen}) = \bigcup C(E_1, \ldots, E_k), \tag{3.11}$$

where $\operatorname{Curv}(Y) := \{\sum a_i[C_j] \mid a_i \in \mathbb{R}_{\geq 0} \text{ and } C_i \subset Y \text{ are irreducible curves}\}$.

Remark 3.12. Because $\dim(Y_{gen}) = 2$, the cones $\operatorname{Eff}(Y_{gen})$ and $\operatorname{Curv}(Y_{gen})$ coincide.

By definition, for any collection $\{E_1, \ldots, E_k\}$ of disjoint interior (-1)-curves,

$$\begin{aligned} C(E_1, \ldots, E_k) &:= \langle D_1, \ldots, D_n, E_1, \ldots, E_k \rangle_{\mathbb{R}\geq 0} \\ &= \{a_1 D_1 + \cdots + a_n D_n + b_1 E_1 + \cdots + b_k E_k \mid a_i, b_j \in \mathbb{R}_{\geq 0}\}. \end{aligned}$$

Thus the following inclusion holds for any cone $C(E_1, \ldots, E_k)$:

$$C(E_1, \ldots, E_k) \subseteq \mathrm{Curv}(Y_{gen})$$

Therefore we also have

$$\bigcup C(E_1, \ldots, E_k) \subseteq \mathrm{Curv}(Y_{gen}),$$

where the union is taken over all $C(E_1, \ldots, E_k)$ where $\{E_1, \ldots, E_k\}$ are collections of disjoint interior (-1)-curves. Therefore, in order to prove the equality in Equation 3.11, we need to prove the following inclusion:

$$\mathrm{Curv}(Y_{gen}) \subseteq \bigcup C(E_1, \ldots, E_k), \tag{3.13}$$

Let $x \in \mathrm{Curv}(Y_{gen})$ be an arbitrary point. A convex cone is the disjoint union of the relative interiors of its faces. This follows from the supporting hyperplane theorem (see [**S11**], Proposition 8.5 on p.122). There are two cases we need to consider.

Case 1. Suppose that $x \in \mathrm{Int}(\mathrm{Curv}(Y_{gen}))$. Then we may construct a small rational polyhedral cone around x such that this is contained in $\mathrm{Curv}(Y_{gen})$. This can be done by choosing rational points that lie inside of $\mathrm{Curv}(Y_{gen})$ and are close to x. Taking the convex hull of these rational points gives a rational polyhedral cone that is contained in $\mathrm{Curv}(Y_{gen})$ and also contains x.

Case 2. Suppose that $x \notin \mathrm{Int}(\mathrm{Curv}(Y_{gen}))$. Then $x \in \mathrm{relInt}(F)$, where F is some face of $\overline{\mathrm{Curv}}(Y_{gen})$. Since $\mathrm{Curv}(Y_{gen})$ is generated by rational points, the same is true for any face of $\mathrm{Curv}(Y_{gen})$. In particular, the face F is the convex hull of its rational points, so the rational points are dense in F. Thus we may choose a sequence of points $x_n \in F \bigcap (\mathrm{Pic}(Y) \otimes \mathbb{Q})$ that converge to x as n approaches infinity. The original Engel-Friedman statement (see Proposition 3.2) was stated for integer coefficients, but this implies that the statement for rational coefficeints is also true. So for every n, the point x_n belongs to some cone $C(E_1, \ldots, E_k)$, as defined above.

Since $x \in \mathrm{relInt}(F)$, i.e., the interior of F regarded as a subset of $\langle F \rangle_{\mathbb{R}}$, there exists a rational polyhedral cone $\Pi \subset F$ such that $x \in \mathrm{relInt}(\Pi)$ and $\dim(\Pi) = \dim(F)$. Then $\mathrm{relInt}(\Pi)$ is an open subset of F. Since the points $\{x_n\}$ converge to x in

the face F, there exists some number $N \in \mathbb{N}$ such that $x_n \in \Pi$ for all $n \geq N$. Friedman's results tells us that Adm acts on the cones $C(E_1, \ldots, E_k)$ with finitely many orbits. Say we choose a representative C_i from each orbit, so we have finitely many representatives $C_1, \ldots, C_r$ (note that we drop the $\{E_1, \ldots, E_k\}$ part here to keep the notation simpler). By the Siegel property 3.9, there exists finitely many elements $g \in \mathrm{Adm}$ such that $g(C_i) \bigcap \Pi \neq \emptyset$. Suppose these elements are $g_{i,1}, \ldots, g_{i,m_i}$ for $i = 1, \ldots, r$. Then the following cones intersect Π:

$$\begin{aligned}
&g_{1,1}C_1, \ldots, g_{1,m_1}C_1 \\
&g_{2,1}C_2, \ldots, g_{2,m_2}C_2 \\
&\vdots \\
&g_{r,1}C_r, \ldots, g_{r,m_r}C_r.
\end{aligned}$$

As a result, we have a (finite) total of $m = m_1 + \cdots + m_r$ cones σ_l of the form $C(E_1, \ldots, E_k)$ intersecting the cone Π. Since each cone $C(E_1, \ldots, E_k)$ is closed, the finite union

$$\bigcup_{l=1}^{m} \sigma_l$$

is also closed. Recall that each x_n is contained in some cone in the union above, so their limit point x must also lie in the union, i.e.,

$$x \in \bigcup_{l=1}^{m} \sigma_l \subset \bigcup C(E_1, \ldots, E_k).$$

Now we have shown that $\mathrm{Curv}(Y_{gen}) \subseteq \bigcup C(E_1, \ldots, E_k)$. Therefore,

$$\mathrm{Curv}(Y_{gen}) = \bigcup C(E_1, \ldots, E_k) = \bigcup \langle D_1, \ldots, D_n, E_1, \ldots, E_k \rangle_{\mathbb{R}\geq 0},$$

proving Corollary 3.10. □

Theorem 3.14. *If L is a nef divisor on $Y = Y_e$, then L is semiample.*

Proof. Let L be nef on $Y = Y_e$. Then $L^2 \geq 0$. If $L^2 > 0$, then by Friedman's results (see Theorem 4.8 on p.35 in Friedman [**F15**]), the divisor L is semiample. For

the remainder of this proof, we suppose that $L^2 = 0$ and $L \neq 0$. Using Riemann-Roch, we obtain

$$\begin{aligned}\chi(L) &= \chi(\mathcal{O}) + \frac{1}{2}L \cdot (L - K_Y) \\ &= 1 + \frac{1}{2}L^2 + \frac{1}{2}(L \cdot D) \quad \text{since } Y \text{ is rational and } K_Y + D = 0 \\ &= 1 + \frac{1}{2}(L \cdot D) \quad \text{using } L^2 = 0.\end{aligned}$$

Here we note that $\chi(L) \geq 1$, since L nef and D effective imply that $L \cdot D \geq 0$. On the other hand, the Euler characteristic of L may also be expressed as

$$\chi(L) = h^0(L) - h^1(L) + h^2(L).$$

By the last paragraph of the proof of Lemma 2.22, since L is nef, we have $h^2(L) = 0$. Now $\chi(L) = h^0(L) - h^1(L) = 1 + \frac{1}{2}(L \cdot D)$. Recall that $L \cdot D \geq 0$ (because L is nef). Next we split this last inequality into two subcases, and in each situation we prove that $h^0(L) \geq 2$.

Subcase (i). Suppose that $L \cdot D > 0$, or $L \cdot D \geq 1$. Then

$$\begin{aligned}\chi(L) &= h^0(L) - h^1(L) \\ &= 1 + \frac{1}{2}(L \cdot D) \\ &\geq 1 + \frac{1}{2} \cdot 1 = \frac{3}{2}.\end{aligned}$$

Since $\chi(L) \in \mathbb{Z}$, we must have $\chi(L) = h^0(L) - h^1(L) \geq 2$. Since $h^1(L)$ is by definition the dimension of a vector space, we have $h^1(L) \geq 0$. Then $h^0(L) - h^1(L) \leq h^0(L)$. Combining these inequalities, we have $2 \leq h^0(L) - h^1(L) \leq h^0(L)$, or $h^0(L) \geq 2$.

Subcase (ii). Suppose that $L \cdot D = 0$. We still have $\chi(L) = h^0(L) - h^1(L) = 1$; the point here is to show that $h^1(L) \geq 1$, which would prove that $h^0(L) \geq 2$. In other words, we eliminate the possibility that $h^0(L) = 1$. Since $L \cdot D = 0$ and L is nef, we have $L \cdot D_i = 0$ for all i. Then because $Y = Y_e$, by Friedman's result in [**F15**] (see

Prop 3.12 on p.22 and Def 3.7 on p.30), it follows that $\mathcal{O}_D(L|_D) \simeq \mathcal{O}_D$. From the exact sequence

$$0 \longrightarrow \mathcal{O}_Y(L-D) \longrightarrow \mathcal{O}_Y(L) \longrightarrow \mathcal{O}_D \longrightarrow 0,$$

we obtain

$$\begin{array}{ccccc} H^1(\mathcal{O}_Y(L)) & \xrightarrow{\delta} & H^1(\mathcal{O}_D) & \longrightarrow & H^2(\mathcal{O}_Y(L-D)) \\ & & \wr\| & & \\ & & \mathbb{C} & & \end{array}$$

By Serre Duality, we have

$$\begin{aligned} h^2(\mathcal{O}_Y(L-D)) &= h^0(\mathcal{O}_Y(K_Y-(L-D))) \\ &= h^0(\mathcal{O}_Y(-L)) \quad \text{since } K_Y+D=0 \\ &= 0. \end{aligned}$$

Then the map δ in the exact sequence above is surjective, so $H^1(\mathcal{O}_Y(L)) \neq 0$, i.e.,

$$\dim H^1(\mathcal{O}_Y(L)) = h^1(L) > 0,$$

so $h^1(L) \geq 1$. Now $h^0(L) = 1 + h^1(L) \geq 1 + 1 = 2$.

Therefore $h^0(L) \geq 2$. This means that in the linear system $|L|$, there is a moving part. Writing $L = M + F$, where M is the moving part and F is the fixed part, we have

$$\begin{aligned} L^2 &= L \cdot (M+F) \\ &= L \cdot M + L \cdot F, \end{aligned}$$

and L nef gives $L \cdot M \geq 0$ and $L \cdot F \geq 0$. Since $L^2 = 0$ by assumption, we obtain $L \cdot M = 0 = L \cdot F$. Now we have

$$\begin{aligned} L \cdot M &= (M + F) \cdot M \\ &= M^2 + M \cdot F \\ &= 0, \end{aligned}$$

and M is nef (since it is moving) so $M^2 \geq 0$ and $M \cdot F \geq 0$, so $M^2 = M \cdot F = 0$. Also,

$$\begin{aligned} L^2 &= (M + F)^2 \\ &= M^2 + 2M \cdot F + F^2 \\ &= F^2, \quad \text{since } M^2 = 0 = M \cdot F, \end{aligned}$$

so that $F^2 = L^2 = 0$. We make two conclusions from the computations above.

(a) The linear system $|M|$ has no fixed part, so $|M|$ is basepoint free: there exists $M' \sim M$ such that M and M' have no common components (since M is moving). Then $M \cdot M' = M^2 = 0$, so $\mathrm{Supp}(M) \cap \mathrm{Supp}(M') = \emptyset$, and therefore $|M|$ is basepoint free. It follows that there exists a map $\phi_{|M|} : Y \to C$, where $C \subset \mathbb{P}^N$ is a curve. By Stein factorization (see Hartshorne [**H77**], Chapter III (11.5) on p.276), replacing $L = M + F$ by $kL = kM + kF$ for sufficiently large k, we may assume that C is a smooth curve and ϕ has connected fibers.

Remark 3.15. The point of the last statement above is to avoid the possibilities of the map $Y \to C$ having disconnected fibers and of the curve C being singular.

(b) Secondly we conclude that L is semiample, using the results that $F^2 = 0$ and $F \cdot M = 0$. Since $F \cdot M = 0$, the divisor F is contained in a union of fibers of the map $\phi : Y \to C$. A fiber has negative semidefinite intersection matrix with kernel generated over $\mathbb{Q}$ by the class of the fiber. Therefore kF is a sum of fibers

for some $k > 0$. Then $k'F$ is basepoint free for some $k' > 0$ such that $k|k'$ by Riemann-Roch on the curve C. Now $k' \cdot L = k' \cdot M + k' \cdot F$ is basepoint free, so L is semiample.

$\square$

CHAPTER 4

PROOF OF THE CONJECTURE

Theorem 4.1. *The cone conjecture for Y_{gen} holds. That is, the group Adm acts on $Nef^{\mathrm{e}}(Y_{gen})$ with a rational polyhedral fundamental domain.*

Proof. First, assume $n \geq 3$. By Friedman's result (Corollary 3.5), the group Adm acts on the set of finite collections of disjoint interior (-1)-curves with finitely many orbits. Since each cone C' described above (in Lemma 3.8) is determined by a finite set of disjoint interior (-1)-curves, we conclude that Adm acts on the set of all such cones C' with fnitely many orbits. This result, together with Lemma 3.8 and Looijenga's results (see Theorem 3.6) allow us to conclude that Adm acts on the nef effective cone $\mathrm{Nef}^{\mathrm{e}}(Y_{gen})$ with a rational polyhedral fundamental domain. To see why, let $C'_1, \ldots, C'_r$ be representatives for the finitely many orbits of Adm on the set of cones C'. Let $\Pi = \mathrm{Conv}(C'_1, \ldots, C'_r)$. Then Π is rational polyhedral because the cones C'_i are, by Lemma 3.8 (1). Moreover $\mathrm{Adm} \cdot \Pi = \mathrm{Nef}^{\mathrm{e}}(Y_{gen})$ by Lemma 3.8 (2). Therefore Adm acts on $\mathrm{Nef}^{\mathrm{e}}(Y_{gen})$ with a rational polyhedral fundamental domain by Theorem 3.6 of Looijenga: in our setting, the lattice L is $\mathrm{Pic}(Y_{gen})$ and C is the ample cone of Y_{gen} (which is the interior of $\mathrm{Nef}(Y_{gen})$). Its closure $\bar{C}$ is $\mathrm{Nef}(Y_{gen})$. The group Γ acting on L is Adm. By Corollary 2.12, $C_+ = \mathrm{Nef}^{\mathrm{e}}(Y_{gen})$. This proves the cone conjecture for Y_{gen} in the case when D_{gen} has at least three components.

If the number of components n of D_{gen} is one or two, then we show below in Section 5 that the nef cone is rational polyhedral for Y_e. Moreover in these cases, the groups Adm and the Weyl group W are equal (see Looijenga [**L81**], Chapter I, Proposition 4.7 on p.284, or see Friedman [**F15**], Theorem 9.13, p.76). Because the action of W on $\mathrm{Nef}^{\mathrm{e}}(Y_{gen})$ has fundamental domain $\mathrm{Nef}^{\mathrm{e}}(Y_e)$ (see Gross-Hacking-Keel [**GHK15b**],

Theorem 3.2, p.15), we conclude that Adm $= W$ acts on Nef $^{\mathrm{e}}(Y_{gen})$ with the *rational polyhedral* fundamental domain Nef (Y_e), proving the cone conjecture.

□

Theorem 4.2. *The cone conjecture for Y'_{gen} holds. That is, the group Adm acts on Nef $^{\mathrm{e}}(Y'_{gen})$ with a rational polyhedral fundamental domain.*

Remark 4.3. We use the definition of Nef $^{\mathrm{e}}(Y'_{gen})$ as given in 2.25.

PROOF. By Theorem 4.1, we know that the cone conjecture holds for Y_{gen}. Since Nef $^{\mathrm{e}}(Y'_{gen})$ is a face F of Nef $^{\mathrm{e}}(Y_{gen})$, by Looijenga's result (see the last statement of Theorem 3.6), the cone conjecture also holds for Y'_{gen}. In our setting, the normalizer $N_\Gamma F =$ Adm and the centralizer $Z_\Gamma F = \{e\}$. □

Theorem 4.4. *$Aut(Y_e, D_e)/K$ acts on Nef $^{\mathrm{e}}(Y_e)$ with a rational polyhedral fundamental domain.*

Remark 4.5. If the action of $\mathrm{Aut}(Y_e, D_e)$ on $\mathrm{Pic}(Y)$ is not faithful, then there exists a nontrivial kernel K and the action of the group quotiented by K is faithful.

Remark 4.6. The proof of Theorem 4.4 is similar to the argument of Sterk for $K3$ surfaces (see [**S85**]).

PROOF. (Theorem 4.4) By Theorem 4.1, the group Adm acts on Nef $^{\mathrm{e}}(Y_{gen})$ with a rational polyhedral fundamental domain. Moreover, by Looijenga's Application 4.14 in 3.6, we can choose $y \in \mathrm{Int}(\mathrm{Nef}^{\mathrm{e}}(Y_{gen}))$ such that y has trivial stabilizer in Adm, then we obtain a rational polyhedral fundamental domain $\sigma(y)$ defined as follows:

$$\sigma(y) = \sigma := \{x \in \mathrm{Nef}^{\mathrm{e}}(Y_{gen}) \mid \gamma x \cdot y \geq x \cdot y \text{ for all } \gamma \in \mathrm{Adm}\}.$$

Let $\gamma = s_\alpha$, the reflection associated to a simple root $\alpha = [C]$ where $C \subset Y_e \setminus D_e$ is a (-2)-curve. Because $s_\alpha(x) = x + (x \cdot \alpha)\alpha$, the condition

$$\gamma x \cdot y \geq x \cdot y \tag{4.7}$$

is equivalent to

$$\begin{aligned} s_\alpha(x)\cdot y \geq x\cdot y &\iff (x+(x\cdot\alpha)\alpha)\cdot y \geq x\cdot y \\ &\iff x\cdot y + (x\cdot\alpha)(\alpha\cdot y) \geq x\cdot y \\ &\iff (x\cdot\alpha)(\alpha\cdot y)\geq 0. \end{aligned}$$

Because α is effective and y is ample (since $y \in \mathrm{Int}(\mathrm{Nef}\,(Y_e))$, which is the ample cone), the intersection $(\alpha\cdot y)$ is positive. Then $(x\cdot\alpha)(\alpha\cdot y)\geq 0$ if and only if $(x\cdot\alpha)\geq 0$. In particular, this shows the following:

$$\sigma \subset \mathrm{Nef}^{\,\mathrm{e}}(Y_{gen}) \cap (\alpha \geq 0 \;\forall\, \alpha\in\Delta) = \mathrm{Nef}^{\,\mathrm{e}}(Y_e),$$

where Δ above denotes the simple roots (see Definition 2.38) and the equality follows from the description of the nef cone in Gross-Hacking-Keel [**GHK15b**] (see Lemma 2.15).

The following statements are true:

(1) σ is rational polyhedral (this is from Looijenga's construction, Application 4.14 [**L14**]) and $\sigma \subset \mathrm{Nef}^{\,\mathrm{e}}(Y_e)$, as shown above;

(2) $\mathrm{Adm} = W \rtimes \mathrm{Aut}(Y_e, D_e)/K$ (from Theorem 5.1 of Gross-Hacking-Keel [**GHK15b**] or Theorem 9.6 of Friedman [**F15**]);

(3) $\mathrm{Nef}^{\,\mathrm{e}}(Y_e)$ is a fundamental domain for the action of W on $\mathrm{Nef}^{\,\mathrm{e}}(Y_{gen})$ (this follows from Gross-Hacking-Keel [**GHK15b**], Theorem 3.2), and by the Torelli Theorem (see Gross-Hacking-Keel [**GHK15b**], Theorem 1.8), $\mathrm{Aut}(Y_e, D_e)/K \leq \mathrm{Adm}$ is the normalizer of $\mathrm{Nef}^{\,\mathrm{e}}(Y_e)$.

Next, we show how the three statements above imply that σ is a rational polyhedral fundamental domain for the action of $\mathrm{Aut}(Y_e, D_e)/K$ on $\mathrm{Nef}^{\,\mathrm{e}}(Y_e)$. Let g be an element of Adm. By (2) above, there exist unique $w \in W$ and $\theta \in \mathrm{Aut}(Y_e, D_e)/K$ such that $g = w\theta$. We claim that the following inclusion holds:

$$(g\sigma) \bigcap \mathrm{Nef}^{\,\mathrm{e}}(Y_e) \subset \theta\sigma \tag{4.8}$$

To see why, let $\mathcal{C}$ be the cone

$$\mathcal{C} = (\alpha \geq 0 \text{ for } \alpha \in \Delta),$$

which is the fundamental chamber for the action of W on $\operatorname{Pic}(Y) \otimes_{\mathbb{Z}} \mathbb{R}$, and we have $\sigma \subset \operatorname{Nef}^{\mathrm{e}}(Y_e) \subset \mathcal{C}$. From above, $g = w\theta$. Let $x \in (g\sigma) \bigcap \operatorname{Nef}^{\mathrm{e}}(Y_e)$. The group $\operatorname{Aut}(Y_e, D_e)/K$ acts on $\operatorname{Nef}^{\mathrm{e}}(Y_e)$. Then $x \in \mathcal{C}$ and $x = gu = w\theta u$, where $u \in \sigma \subset \operatorname{Nef}^{\mathrm{e}}(Y_e)$ and so $\theta u \in \operatorname{Nef}^{\mathrm{e}}(Y_e) \subset \mathcal{C}$. Thus θu and $w\theta u$ are in $\mathcal{C}$. So $\theta u \in w\mathcal{C} \bigcap \mathcal{C} \subset \operatorname{Fix}(w)$ by Sterk (see Lemma 1.2 in [**S85**]), which means that $w\theta u = \theta u$, i.e., $x = \theta u$. Then $x = \theta u \in \theta\sigma$, proving the inclusion 4.8.

Finally, we want to show that σ is a rational polyhedral fundamental domain for the action of $\operatorname{Aut}(Y_e, D_e)/K$ on $\operatorname{Nef}^{\mathrm{e}}(Y_e)$. By (1), σ is rational polyhedral, so it remains to show that it is a fundamental domain. Because σ is a fundamental domain for the action of Adm on $\operatorname{Nef}^{\mathrm{e}}(Y_{gen})$,

$$\operatorname{Nef}^{\mathrm{e}}(Y_{gen}) = \bigcup_{g \in \mathrm{Adm}} g\sigma.$$

We can write

$$\begin{aligned}
\operatorname{Nef}^{\mathrm{e}}(Y_{gen}) \bigcap \operatorname{Nef}^{\mathrm{e}}(Y_e) &= \Big(\bigcup_{g \in \mathrm{Adm}} g\sigma \Big) \cap \operatorname{Nef}^{\mathrm{e}}(Y_e) \\
&= \bigcup_{g \in \mathrm{Adm}} (g\sigma \bigcap \operatorname{Nef}^{\mathrm{e}}(Y_e)) \\
&= \bigcup_{\theta \in \operatorname{Aut}(Y_e, D_e)/K} \theta\sigma.
\end{aligned}$$

Here is why the last equality holds: if $g = w\theta$, then we showed above that $(g\sigma) \bigcap \operatorname{Nef}^{\mathrm{e}}(Y_e) \subset \theta\sigma$. If $w = 1$, then $g = \theta \in \operatorname{Aut}(Y_e, D_e)/K$ and

$$(g\sigma) \bigcap \operatorname{Nef}^{\mathrm{e}}(Y_e) = (\theta\sigma) \bigcap \operatorname{Nef}^{\mathrm{e}}(Y_e) = \theta\sigma,$$

because θ preserves $\operatorname{Nef}^{\mathrm{e}}(Y_e)$ and $\sigma \subset \operatorname{Nef}^{\mathrm{e}}(Y_e)$. Moreover, because

$$\operatorname{Int}\big(g_1\sigma \bigcap g_2\sigma\big) = \emptyset \ \forall\, g_1, g_2 \in \mathrm{Adm},$$

it follows that the same statement holds for g_1, g_2 in the smaller group $\mathrm{Aut}(Y_e, D_e)/K$. That is,

$$\mathrm{Int}\big(g_1\sigma \bigcap g_2\sigma\big) = \emptyset \; \forall \; g_1, g_2 \in \mathrm{Aut}(Y_e, D_e)/K,$$

which is the second property in the definition of a fundamental domain. Therefore we have shown, using conditions (1), (2), and (3), that σ is a rational polyhedral fundamental domain for the action of $\mathrm{Aut}(Y_e, D_e)/K$ on $\mathrm{Nef}^{\,\mathrm{e}}(Y_e)$.

$\square$

CHAPTER 5

NEW EXAMPLES OF MORI DREAM SPACES

Definition 5.1. (cf. Hu and Keel [**HK00**], Definition 1.10 on p.4) Let Y be a smooth projective surface. Then Y is a Mori Dream space if the following are true:

(1) $\mathrm{Pic}(Y) \otimes \mathbb{R} = N^1(Y)$;

(2) Nef (Y) is rational polyhedral; and

(3) If L is a nef divisor on Y, then L is semiample.

Theorem 5.2. *A log Calabi-Yau surface (Y_e, D_e) in which boundary D_e consists of no more than six components has a rational polyhedral cone of curves.*

In addition, for each such surface, we give an explicit description of the cone of curves. We note that some similar calculations for $n \leq 5$ appear in Looijenga's paper [**L81**].

Remark 5.3. When the cone of curves of Y has finitely many generators, it is automatically closed. Because the cones we describe below are all rational polyhedral, we have $\overline{\mathrm{Curv}}(Y) = \mathrm{Curv}(Y)$.

In this next part, we only consider log Calabi-Yau surfaces with the split mixed Hodge structure. We will show that each surface Y described for each $n \leq 6$ is the surface Y_e in the given deformation type with the split mixed Hodge structure (that is, the period point ϕ given by $\phi(x) = 1$ for all $x \in \langle D_1, \ldots, D_n\rangle^{\perp}_{\mathbb{Z}}$).

Lemma 5.4. *Let (Y, D) be a log Calabi-Yau surface and suppose that $\langle D_1, \ldots, D_n\rangle^{\perp}_{\mathbb{Z}}$ is generated by classes of curves $C \subset Y \setminus D$. Then $\phi(x) = 1$ for all $x \in \langle D_1, \ldots, D_n\rangle^{\perp}_{\mathbb{Z}}$.*

Proof. Using the notation $\theta : \mathrm{Pic}^0(D) \xrightarrow{\sim} \mathbb{C}^*$ from Definition 2.35, we recall that $\phi([C]) = \theta(\mathcal{O}_Y(C)|_D)$. Because $C \bigcap D = \emptyset$, the restriction $\mathcal{O}_Y(C)|_D = \mathcal{O}_D$ is the

trivial bundle on D. Then $\phi([C]) = 1$. From our assumption it follows that $\phi(x) = 1$ for all $x \in \langle D_1, \dots, D_n\rangle^{\perp}_{\mathbb{Z}}$. $\square$

The lemma applies in our situation, because for the surfaces we describe in cases $n \leq 6$, the lattice $\langle D_1, \dots, D_n\rangle^{\perp}_{\mathbb{Z}}$ is generated by the classes of (-2)-curves C in $Y \setminus D$.

Remark 5.5. We cover every deformation type for each $n \leq 6$ of log Calabi-Yau surfaces such that the intersection matrix $(D_i \cdot D_j)$ is negative definite or negative semidefinite. This follows from two theorems below: Looijenga for $n \leq 5$ (see Theorem 5.7) and Simonetti for $n = 6$ (see Theorem 5.9).

Remark 5.6. If the intersection matrix $(D_i \cdot D_j)$ is not negative definite or negative semidefinite, then Nef (Y) is rational polyhedral by the Cone Theorem (see [**GHK15a**], Lemma 6.9).

To keep the notation simple, we will use (Y, D) to mean (Y_e, D_e) in this section, unless otherwise specified. The proof of Theorem 5.2 is split into the six cases $n = 1, \dots, 6$. We use the following theorem and lemma. We note that in each of the cases considered, the number of boundary components remains the same.

Theorem 5.7. *(Looijenga, Theorem 1.1,* [**L81**]*) Let Y be a smooth rational surface endowed with an anti-canonical cycle $D = D_1 + \cdots + D_n$ of length $n \leq 5$. Suppose that if $n \geq 2$, then $D_i^2 \leq 4 - n$ for all $i \in \mathbb{Z} \mod n$. Then there exists a sequence of blowdowns of interior (-1)-curves which gives a smooth rational surface $\bar{Y}$ endowed with an anticanonical cycle $\bar{D} = \bar{D}_1 + \dots, +\bar{D}_n$, where $\bar{D}_i$ is the image of D_i for each i, such that:*

(1) If $n = 1$, then $\bar{Y} \cong \mathbb{P}^2$ and $\bar{D}$ is a cubic curve with a node;

(2) If $n = 2$, then $\bar{Y} \cong \mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_2$ and $\bar{D}_1$ is linearly equivalent to $\bar{D}_2$.

(3) If $n = 3$, then $\bar{Y} \cong \mathbb{P}^2$ and $\bar{D}$ is a triangle of lines (i.e., $\bar{D}$ is a choice of toric boundary);

(4) If $n = 4$, then $\bar{Y} \cong \mathbb{P}^1 \times \mathbb{P}^1$ and $\bar{D}$ is a square consisting of two fibers of each of the two projections $\bar{Y} \to \mathbb{P}^1$ (i.e., $\bar{D}$ is a choice of toric boundary); and

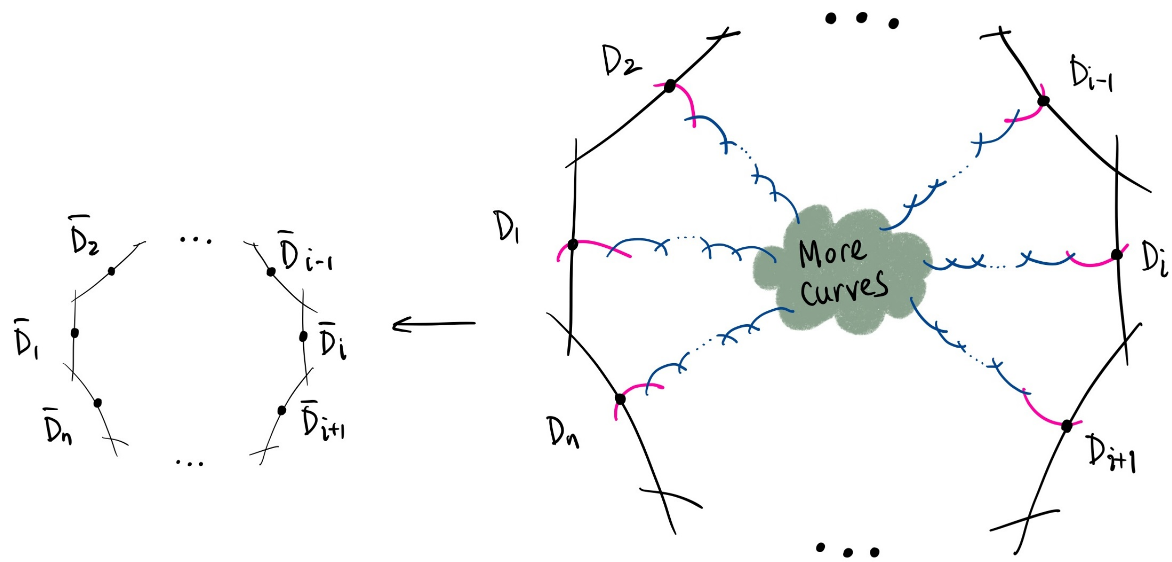

FIGURE 5.1. This drawing shows arbitrarily many blowups of the surface $(\bar{Y}, \bar{D})$. The blowup at each point creates a chain of one (-1)-curve, intersecting the boundary component at one point, followed by some number (-2)-curves that lead to a common "central region". The center consists of additional curves.

(5) If $n = 5$, then $\bar{Y}$ is a del Pezzo surface of degree five (i.e., the blowup of four points on $\mathbb{P}^2$) and each component $\bar{D}_i$ of $\bar{D}$ is a (-1)-curve.

The general blowup picture is shown in Figure 5.1: each boundary component is linked to a "chain" of a single (-1)-curve, followed by arbitrarily many (-2)-curves. A general chain is shown in Figure 5.2.

Lemma 5.8. *Let Y be a smooth projective complex surface. Let $\mathcal{B}$ be a basis for $N_1(Y)$ consisting of irreducible curves. Suppose the dual basis may be expressed as effective combinations of a set $\mathcal{C}$ of curves. Then Curv$(Y) = \langle \mathcal{B} \cup \mathcal{C} \rangle_{\mathbb{R}\geq 0}$.*

PROOF. Let $C \subset Y$ be a curve and suppose that $C \notin \mathcal{B}$. Then $C \cdot B_i \geq 0$ for all $B_i \in \mathcal{B}$. Since B_i^* is an effective linear combination of elements in $\mathcal{C}$ and $C \cdot B_i \geq 0$, it follows that $C = \sum (C \cdot B_i) B_i^*$ belongs to $\langle\ \mathcal{C}\ \rangle_{\mathbb{R}_{\geq 0}}$. $\square$

Number of boundary components $n = 1$. Let $\bar{Y} = \mathbb{P}^2$ with a rational nodal curve $\bar{D}_1$ and a flex point q. In coordinates, we may take

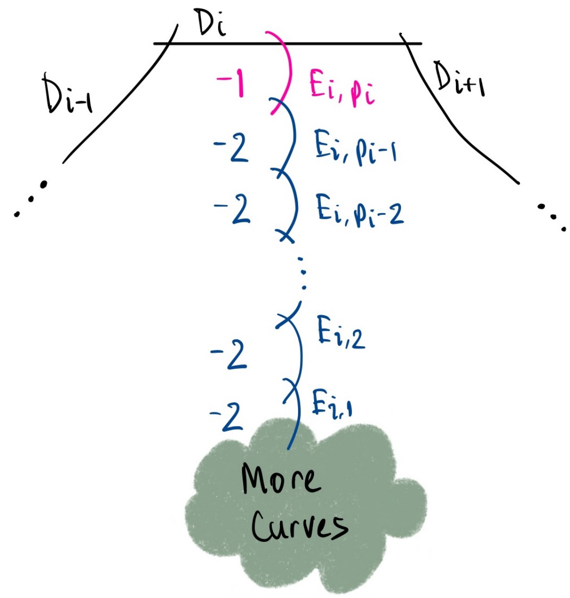

FIGURE 5.2. This drawing depicts a general "chain" that forms after arbitrarily many blowups. For a boundary of length n, the value of i ranges from 1 to n.

$$\bar{D}_1 : (X_0 X_2^2 = X_1^2(X_1 + X_0)) \subseteq \mathbb{P}^2_{(X_0:X_1:X_2)} \text{ and } q = (0:0:1).$$

We denote the tangent line at point q by $\bar{F}$, and we blow up the point q some number p_1 of times.

A basis for $\mathrm{Pic}(Y)$ is

$$\mathcal{B}_1 = \{E_{1,j}, F \mid 1 \leq j \leq p_1\}.$$

and its dual basis $\mathcal{B}_1^*$ consists of the following elements:

$$E^*_{1,p_1} = D_1$$

$$E^*_{1,p_1-1} = D_1 + E_{1,p_1}$$

$$E^*_{1,p_1-2} = D_1 + 2E_{1,p_1} + E_{1,p_1-1}$$

$$E^*_{1,p_1-3} = 2D_1 + 8E_{1,p_1} + 6E_{1,p_1-1} + 4E_{1,p_1-2} + 2E_{1,p_1-3} + E_{1,p_1-4}$$

$$\vdots$$

$$E^*_{1,j} = D_1 + (p_1 - j)E_{1,p_1} + (p_1 - j - 1)E_{1,p_1-1} + \cdots + 2E_{1,j+2} + E_{1,j+1} \quad \text{for } 3 \leq j \leq p_1;$$

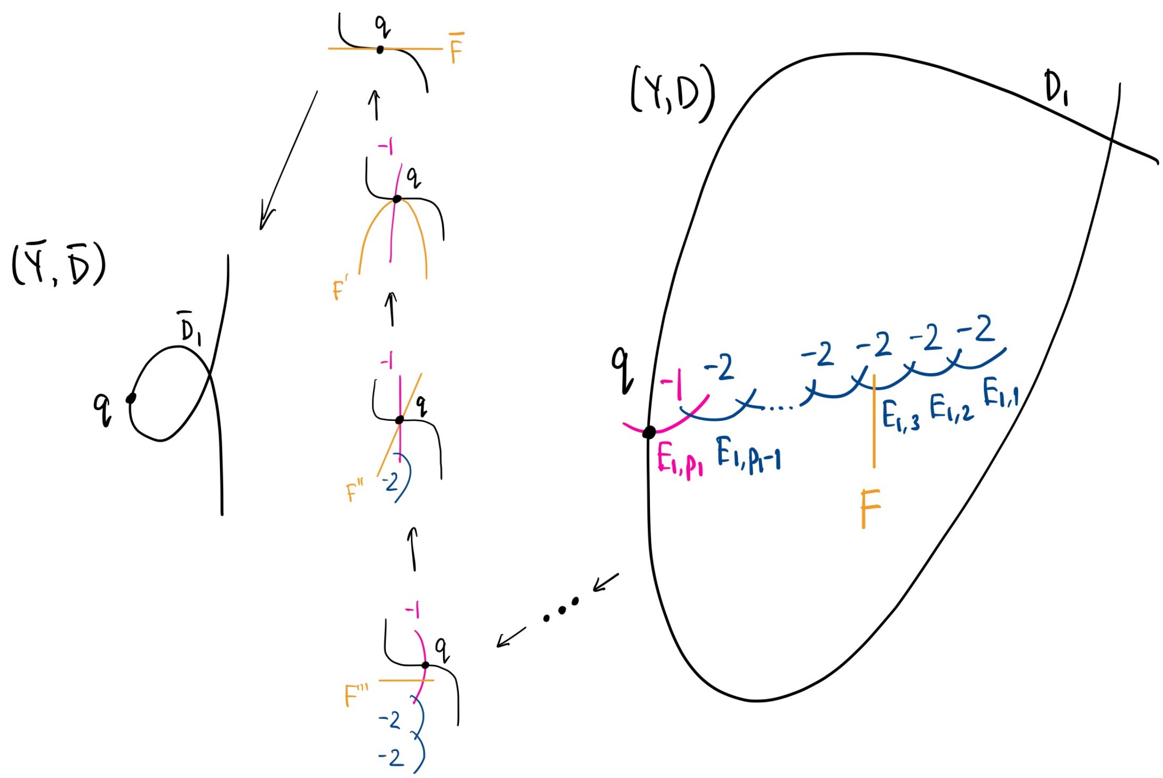

FIGURE 5.3. The drawing on the far left shows $(\bar{Y}, \bar{D})$ before any blowups. The middle drawing shows the first three blowups at the point q, and the figure on the right depicts (Y, D) after arbitrarily many, say p_1, blowups at q.

and

$$E_{1,2}^* = 4E_{1,p_1} + 4E_{1,p_1-1} + \cdots + 4E_{1,4} + 4E_{1,3} + 2E_{1,2} + E_{1,1} + 2F$$

$$E_{1,1}^* = 2E_{1,p_1} + 2E_{1,p_1-1} + \cdots + 2E_{1,4} + 2E_{1,3} + E_{1,2} + F$$

$$F^* = 3E_{1,p_1} + 3E_{1,p_1-1} + \cdots + 3E_{1,4} + 3E_{1,3} + 2E_{1,2} + E_{1,1} + F$$

By Lemma 5.8, we can describe the cone of curves as follows:

$$\mathrm{Curv}(Y) = \langle D_1, E_{1,j}, F \mid 1 \leq j \leq p_1 \rangle_{\mathbb{R}_{\geq 0}}$$

Number of boundary components $n = 2$**.** Let $\bar{Y}$ be the Hirzebruch surface $\mathbb{F}_2$ with two smooth curves $\bar{D}_1$ and $\bar{D}_2$ in the linear system $|B + 2A|$. Here B denotes the negative section of the $\mathbb{P}^1$ fibration $\mathbb{F}_2 \to \mathbb{P}^1$ and A denotes the fiber. We may assume that the curves $\bar{D}_1$ and $\bar{D}_2$ intersect transversely. We fix two points $q_i \in \bar{D}_i$

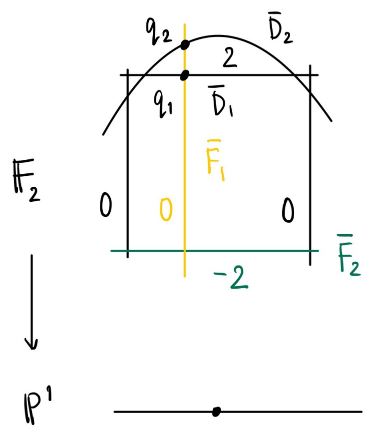

FIGURE 5.4. This drawing shows the curves $\bar{F}_1$ and $\bar{F}_2$ in the case $n = 2$.

where $i = 1, 2$, such that the points lie on a common fiber $\bar{F}_1$, and let $\bar{F}_2$ be the (-2)-curve (see Figure 5.4). Then blow up at the points q_i some number of times, which is shown in Figure 5.5 (our notation is that we blow up a total of p_i times at points q_i for $i = 1, 2$). The curves F_i are the strict transforms of $\bar{F}_i$ for $i = 1, 2$.

A basis $\mathcal{B}_2$ for $\mathrm{Pic}(Y)$ is given by

$$\mathcal{B}_2 = \{E_{i,j}, F_i \mid i = 1, 2 \text{ and } 1 \leq j \leq p_i\}.$$

The dual basis $\mathcal{B}_2^*$ consists of the following elements:

$$\mathcal{B}_2^* = \{E_{i,j}^*, F_1^*, F_2^* \mid i = 1, 2 \text{ and } 1 \leq j \leq p_i\},$$

where for $i = 1, 2$, the dual $E_{i,j}$ elements are defined as

$$\begin{aligned}
E_{i,p_i}^* &= D_i \\
E_{i,p_i-1}^* &= D_i + E_{i,p_i} \\
E_{i,p_i-2}^* &= D_i + 2E_{i,p_i} + E_{i,p_i-1} \\
&\vdots \\
E_{i,1}^* &= D_i + (p_i - 1)E_{i,p_i} + (p_i - 2)E_{i,p_i-1} + \cdots + 2E_{i,3} + E_{i,2}
\end{aligned}$$

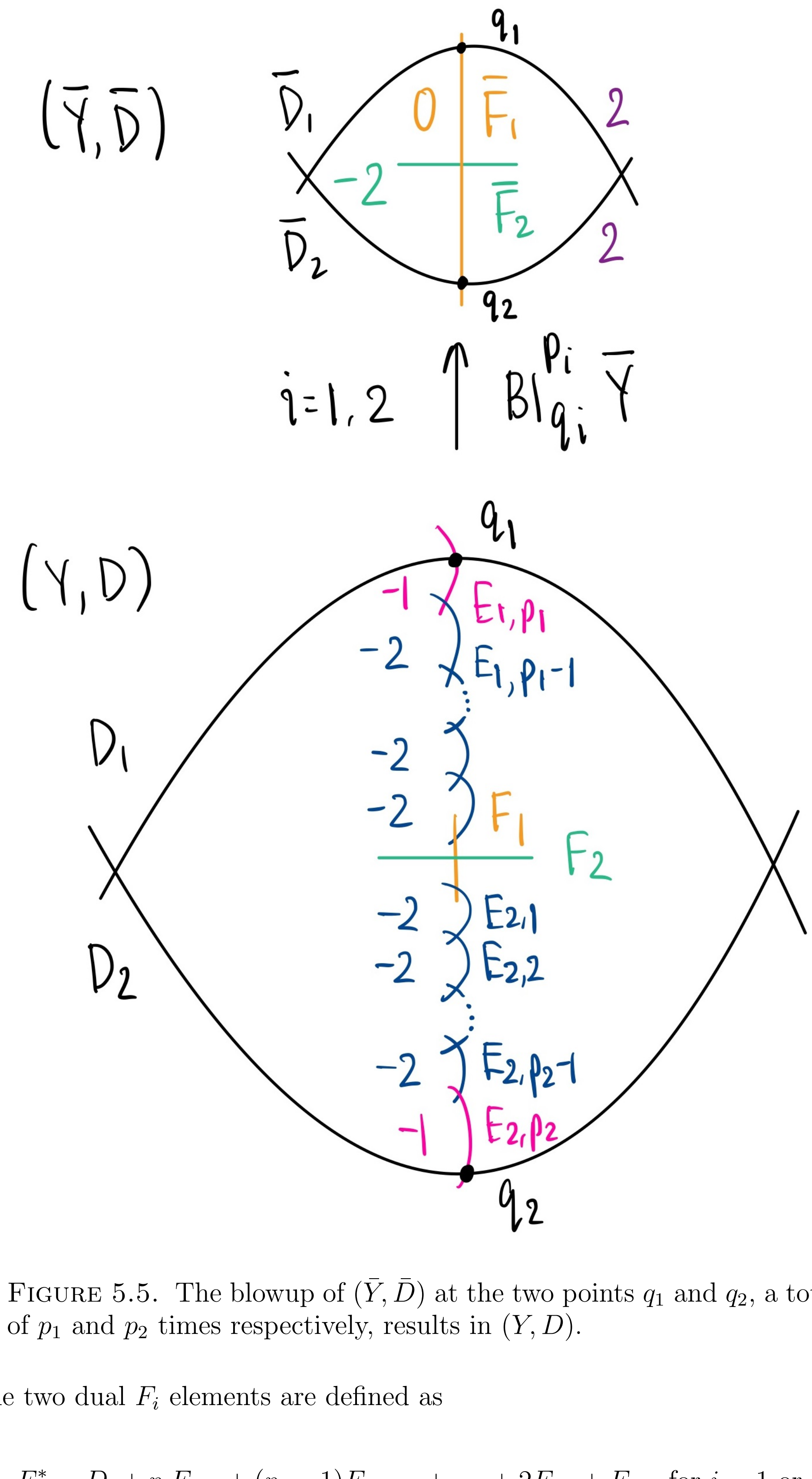

FIGURE 5.5. The blowup of $(\bar{Y}, \bar{D})$ at the two points q_1 and q_2, a total of p_1 and p_2 times respectively, results in (Y, D).

and the two dual F_i elements are defined as

$$F_1^* = D_i + p_i E_{i,p_i} + (p_i - 1)E_{i,p_i-1} + \cdots + 2E_{i,2} + E_{i,1} \text{ for } i = 1 \text{ or } 2,$$

and

$$F_2^* = D_i + (p_i + 1)E_{i,p_i} + p_i E_{i,p_i - 1} + \cdots + 2E_{i,1} + F_1 \text{ for } i = 1 \text{ or } 2.$$

By Lemma 5.8, we can describe the cone of curves as follows:

$$\mathrm{Curv}(Y) = \langle D_i, E_{i,j}, F \mid i = 1,2 \text{ and } 1 \leq j \leq p_i \rangle_{\mathbb{R}_{\geq 0}}.$$

Number of boundary components $n = 3$. Let $\bar{Y} = \mathbb{P}^2$ with $\bar{D} = \bar{D}_1 + \bar{D}_2 + \bar{D}_3$ its toric boundary, which is the union of three lines. Fix three collinear points $q_i \in \bar{D}_i$ where $i = 1,2,3$ and blow them up some number of times. Let F be the strict transform of the line $\bar{F}$ passing through the three points (see Figure 5.6).

A basis $\mathcal{B}_3$ for $\mathrm{Pic}(Y)$ is given by

$$\mathcal{B}_3 = \{E_{i,j}, F \mid 1 \leq i \leq 3 \text{ and } 1 \leq j \leq p_i\}.$$

A dual basis $\mathcal{B}_3^*$ consists of the elements, for $i = 1,2,3$:

$$\begin{aligned}
E_{i,p_i}^* &= D_i \\
E_{i,p_i-1}^* &= D_i + E_{i,p_i} \\
E_{i,p_i-2}^* &= D_i + 2E_{i,p_i} + E_{i,p_i-1} \\
&\vdots \\
E_{i,1}^* &= D_i + (p_i - 1)E_{i,p_i} + (p_i - 2)E_{i,p_i-1} + \cdots + 2E_{i,3} + E_{i,2}
\end{aligned}$$

and

$$F_i^* = D_i + p_i E_{i,p_i} + (p_i - 1)E_{i,p_i-1} + \cdots + 2E_{i,2} + E_{i,1}.$$

By Lemma 5.8, we can describe the cone of curves as follows:

$$\mathrm{Curv}(Y) = \langle D_i, E_{i,j}, F \mid 1 \leq i \leq 3 \text{ and } 1 \leq j \leq p_i \rangle_{\mathbb{R}_{\geq 0}}.$$

Number of boundary components $n = 4$. Let $\bar{Y} = \mathbb{P}^1 \times \mathbb{P}^1$ with its toric boundary, which is the union of two fibers of each of the two projections $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$. Fix four points $q_i \in \bar{D}_i$ where $i = 1, \ldots, 4$ such that q_1 and q_3 lie on a fiber $\bar{F}_1$ of the first

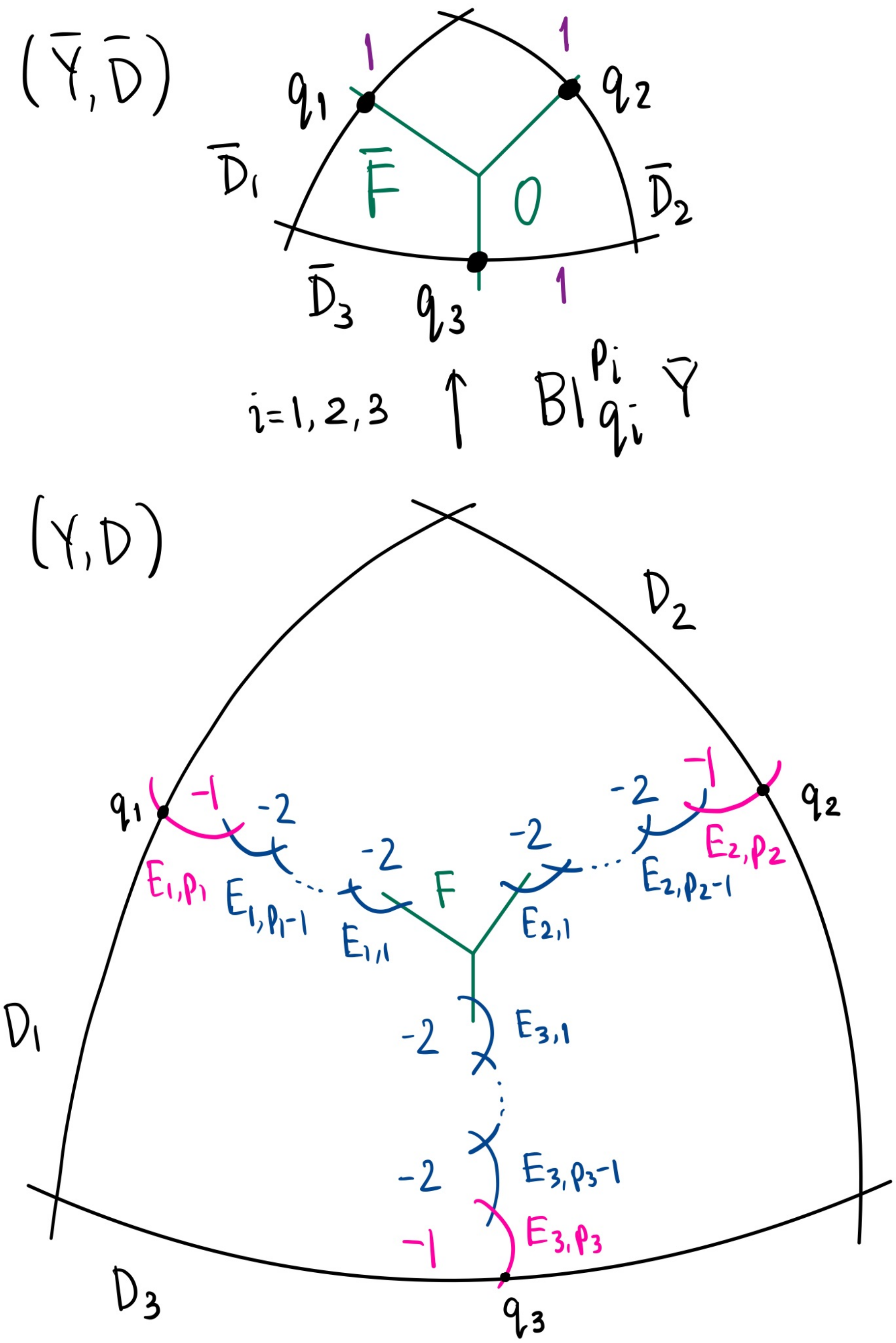

FIGURE 5.6. After $(\bar{Y}, \bar{D})$ is blown up arbitrarily many (p_i) times at each point q_i, the resulting pair is (Y, D).

projection and q_2 and q_4 lie on a fiber $\bar{F}_2$ of the second projection. Then blow them up some number of times (see Figure 5.7).

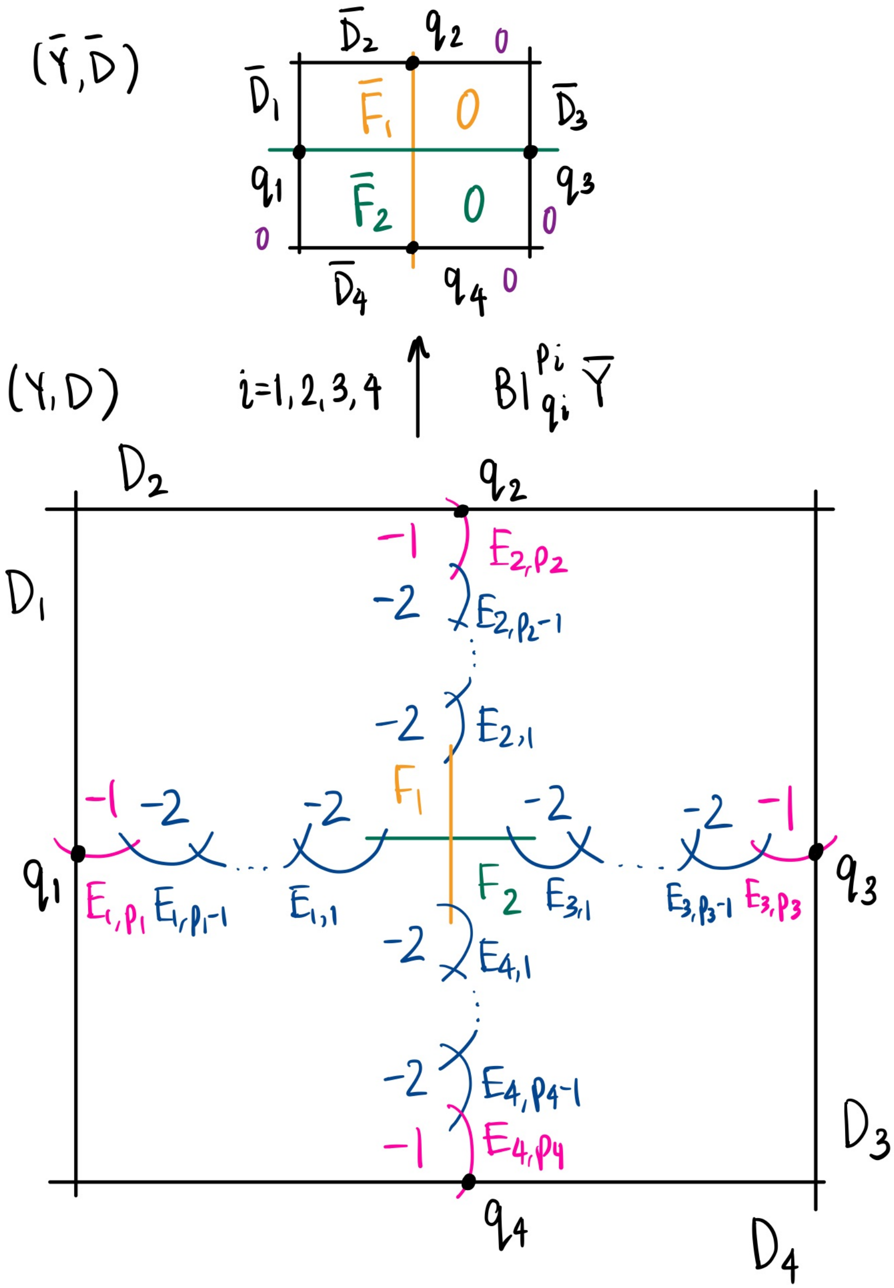

FIGURE 5.7. This shows the blowup of the toric pair $(\bar{Y}, \bar{D})$ at the points q_i. At each point there are arbitrarily many (p_i) blowups, resulting in (Y, D).

A basis for $\mathrm{Pic}(Y)$ is $\mathcal{B}_4$:

$$\mathcal{B}_4 = \{E_{i,j}F_1, F_2 \mid 1 \leq i \leq 4 \text{ and } 1 \leq j \leq p_i\}.$$

The dual basis $\mathcal{B}_4^*$ consists of the following elements:

$$\begin{aligned}
E_{i,p_i}^* &= D_i \\
E_{i,p_i-1}^* &= D_i + E_{i,p_i} \\
E_{i,p_i-2}^* &= D_i + 2E_{i,p_i} + E_{i,p_i-1} \\
&\vdots \\
E_{i,1}^* &= D_i + (p_i - 1)E_{i,p_i} + (p_i - 2)E_{i,p_i-1} + \cdots + 2E_{i,3} + E_{i,2}
\end{aligned}$$

and for each of F_j where $j = 1, 2$, there are two possibilities:

$$F_j^* = D_i + p_i E_{i,p_i} + (p_i - 1)E_{i,p_i-1} + \cdots + 2E_{i,2} + E_{i,1}$$

with $i = 2$ or 4 for $j = 1$ and $i = 1$ or 3 for $j = 2$. By Lemma 5.8, we can describe the cone of curves as follows:

$$\mathrm{Curv}(Y) = \langle D_i, E_{i,j}, F_1, F_2 \mid 1 \leq i \leq 4 \text{ and } 1 \leq j \leq p_i \rangle_{\mathbb{R}_{\geq 0}}.$$

Number of boundary components $n = 5$. Let $\bar{Y}$ be the blowup of four points in general position in $\mathbb{P}^2$ and let $\bar{D}$ be a cycle of five (-1)-curves. The surface $\bar{Y}$ contains ten (-1)-curves:

(1) Four are exceptional curves E_i from blowing up the points p_i, for $i = 1, 2, 3, 4$.
(2) Six (obtained by $6 = \binom{4}{2}$) are strict transforms l'_{ij} of lines l_{ij} defined by points p_i and p_j.

The process of blowing up points p_i for $i = 1, \ldots, 4$ on $\mathbb{P}^2$ to obtain a surface with ten curves is shown in Figure 5.8. Taking the dual of this figure (see Figure 5.9), we choose a pentagon inside and rearrange vertices so that this pentagon encloses all other vertices. Then the interior vertices can be rearranged to form a star, resulting in the Petersen graph. The dual of the interior five-pointed star is a pentagon, and the dual of the outside pentagonal boundary is again a pentagon. Together, these

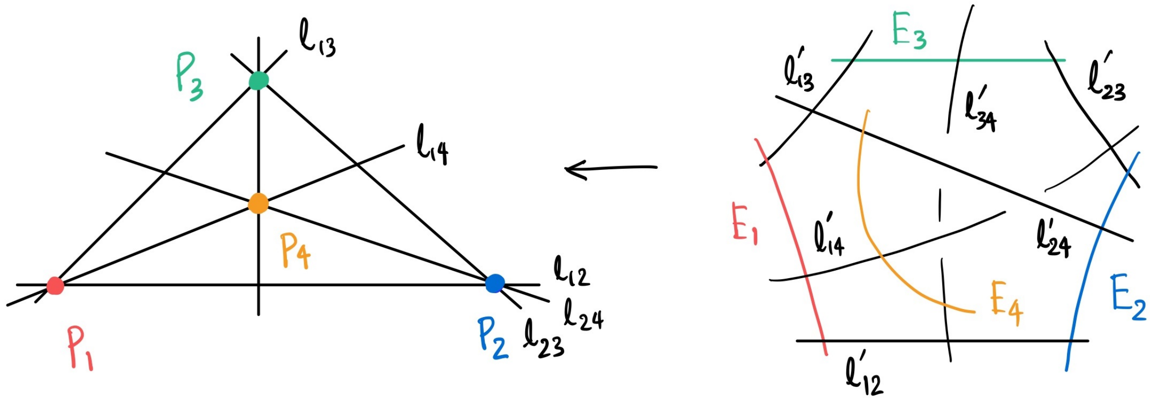

FIGURE 5.8. Blowing up once at each point $p_1, \ldots, p_4$ results in the diagram above.

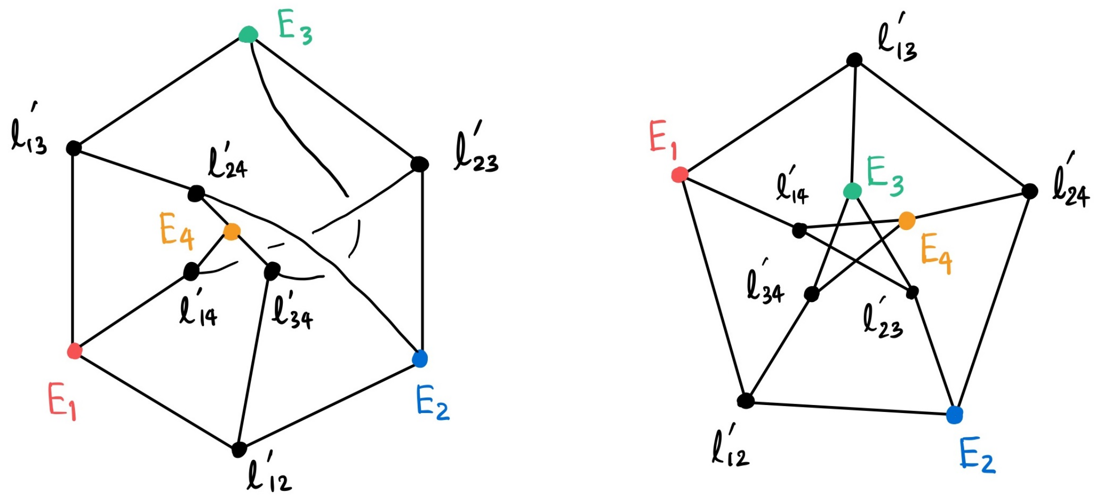

FIGURE 5.9. The dual graph of the blowup shown in Figure 5.8 is drawn on the left, and it is equivalent to the Petersen graph shown on the right.

two parts form the configuration of (-1)-curves on the surface $\bar{Y}$. The $\bar{F}_i$'s are (-1)-curves not contained in the boundary in $\bar{Y}$; the $\bar{F}_i$'s correspond to a pentagonal star. Each $\bar{F}_i$ intersects the boundary component $\bar{D}_i$, and we denote their strict transforms by F_i.

The remaining (-1)-curves on $\bar{Y}$ intersect $\bar{D}$ transversely in five points q_i where $i = 1, \ldots, 5$. Blow up these points some number of times to obtain Figure 5.11. A basis $\mathcal{B}_5$ for $\mathrm{Pic}(Y)$ is the collection:

$$\mathcal{B}_5 = \{E_{i,j}, F_i \mid 1 \leq i \leq 5 \text{ and } 1 \leq j \leq p_i\}.$$

The dual elements $E^*_{i,j}$ and F_i, where $1 \leq i \leq 5$ and $1 \leq j \leq p_i$, are defined as follows:

$$\begin{aligned}
E^*_{i,p_i} &= D_i \\
E^*_{i,p_i-1} &= D_i + E_{i,p_i} \\
E^*_{i,p_i-2} &= D_i + 2E_{i,p_i} + E_{i,p_i-1} \\
&\vdots \\
E^*_{i,1} &= D_i + (p_i - 1)E_{i,p_i} + (p_i - 2)E_{i,p_i-1} + \cdots + 2E_{i,3} + E_{i,2} \\
F^*_i &= D_i + p_iE_{i,p_i} + (p_i - 1)E_{i,p_i-1} + \cdots + 2E_{i,2} + E_{i,1}
\end{aligned}$$

By Lemma 5.8, we can describe the cone of curves as follows:

$$\mathrm{Curv}(Y) = \langle D_i, E_{i,j}, F_i \mid 1 \leq i \leq 5 \text{ and } 1 \leq j \leq p_i\rangle_{\mathbb{R}_{\geq 0}}.$$

Number of boundary components $n = 6$.

Theorem 5.9. *(A. Simonetti, Ph.D. Thesis, 2021) Let (Y, D) be a log Calabi-Yau surface with negative definite or negative semidefinite boundary of length $n = 6$, such that D does not contain any (-1)-curves. Then (Y, D) is obtained as a blowup of the toric surface $(\bar{Y}, \bar{D})$ with boundary a cycle of six (-1)-curves.*

We have the toric surface $\bar{Y}$ with toric boundary $\bar{D} = \bar{D}_1 + \cdots + \bar{D}_6$, a hexagon of (-1)-curves. Take q_i where $i = 1 \ldots, 6$ to be the points $(-1) \in \mathbb{C}^* \subset \mathbb{P}^1 = \bar{D}_i$ for some choice of toric coordinates on $\bar{Y}$, and Y is the blowup of $\bar{Y}$ some number of times at each q_i. We consider the following five curves $\bar{F}_k$ on $\bar{Y}$, which are strict transforms of curves F_k on Y:

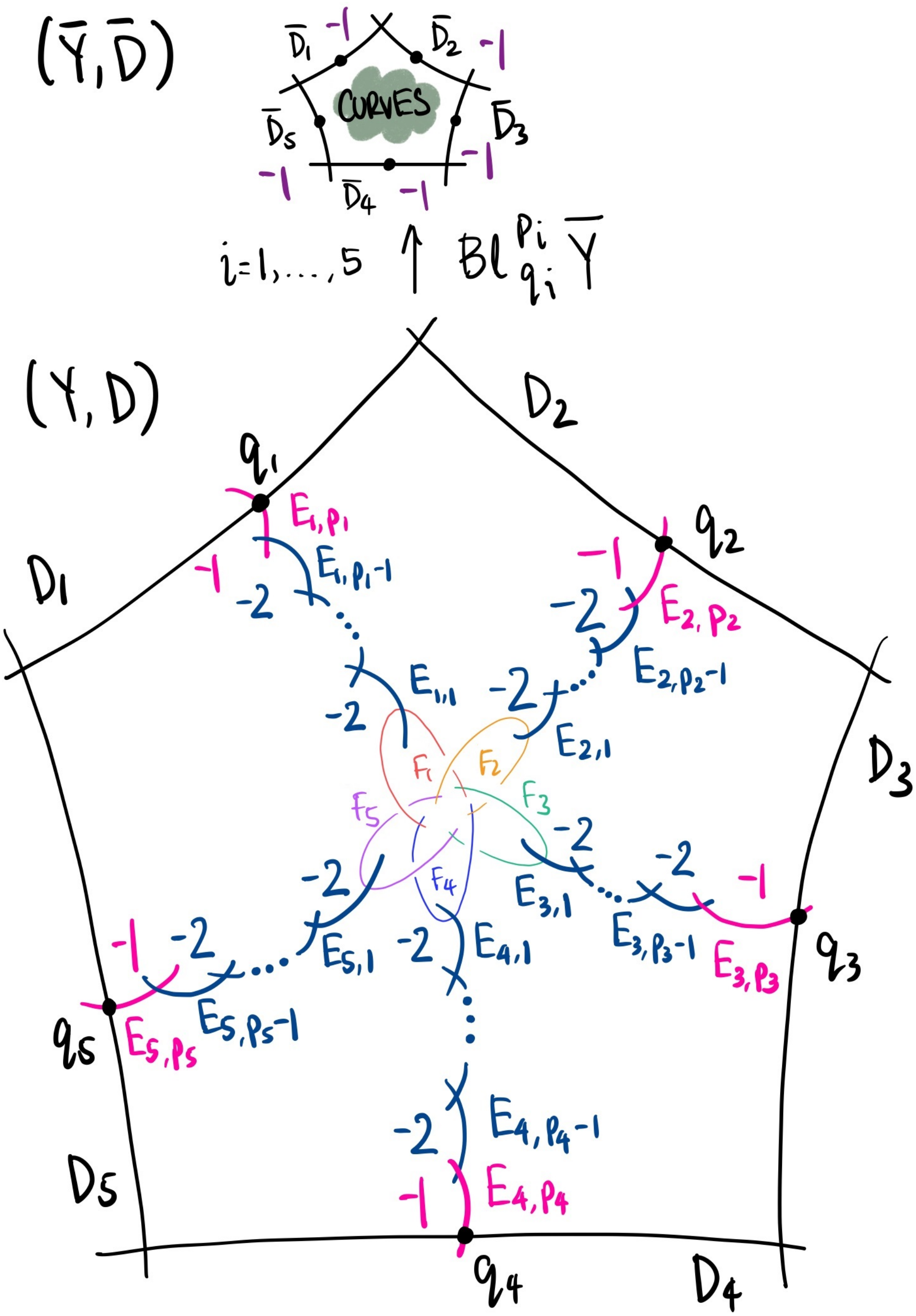

FIGURE 5.10. Blowing up once at each point $q_1, \dots, q_5$ results in the diagram above.

$\bar{F}_{1,4}$ and $\bar{F}_{2,5}$ and $\bar{F}_{3,6}$ and $\bar{F}_{1,3,5}$ and $\bar{F}_{2,4,6}$.

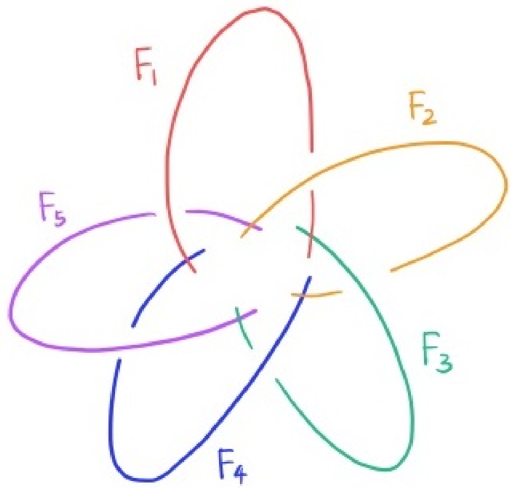

FIGURE 5.11. This drawing shows the curves in the center of case $n = 5$.

The subscript numbers for the curves $\bar{F}$ above are used to denote the boundary components that it intersects. For example, the curve $\bar{F}_{1,3,5}$ intersects components $\bar{D}_1$ and $\bar{D}_3$ and $\bar{D}_5$ of boundary $\bar{D}$. In particular:

(1) $\bar{F}_{1,3,5}$ and $\bar{F}_{2,4,6}$ are pullbacks of a line in $\mathbb{P}^2$ for two different birational morphisms $\bar{Y} \to \mathbb{P}^2$, as shown in Figure 5.13; and
(2) $\bar{F}_{1,4}$ and $\bar{F}_{2,5}$ and $\bar{F}_{3,6}$ are fibers of three different morphisms $\bar{Y} \to \mathbb{P}^1$, as shown in Figure 5.14.

The classes of the curves $\{\bar{F}_k\}$ span $\mathrm{Pic}(\bar{Y})$ with one relation:

$$\bar{F}_{1,4} + \bar{F}_{2,5} + \bar{F}_{3,6} = \bar{F}_{1,3,5} + \bar{F}_{2,4,6}.$$

Define an index set as follows:

$$K = \{\{1,4\},\{2,5\},\{3,6\},\{1,3,5\},\{2,4,6\}\}, \tag{5.10}$$

so that $\{\bar{F}_k\}$ where $k \in K$ refers to the set of five curves described above.

For $i = 1, \ldots, 6$ and $j = 0, \ldots, p_i$, define a divisor

$$A_{i,j} = D_i + (p_i - j)E_{i,p_i} + (p_i - j - 1)E_{i,p_i-1} + \ldots, +E_{i,j+1}.$$

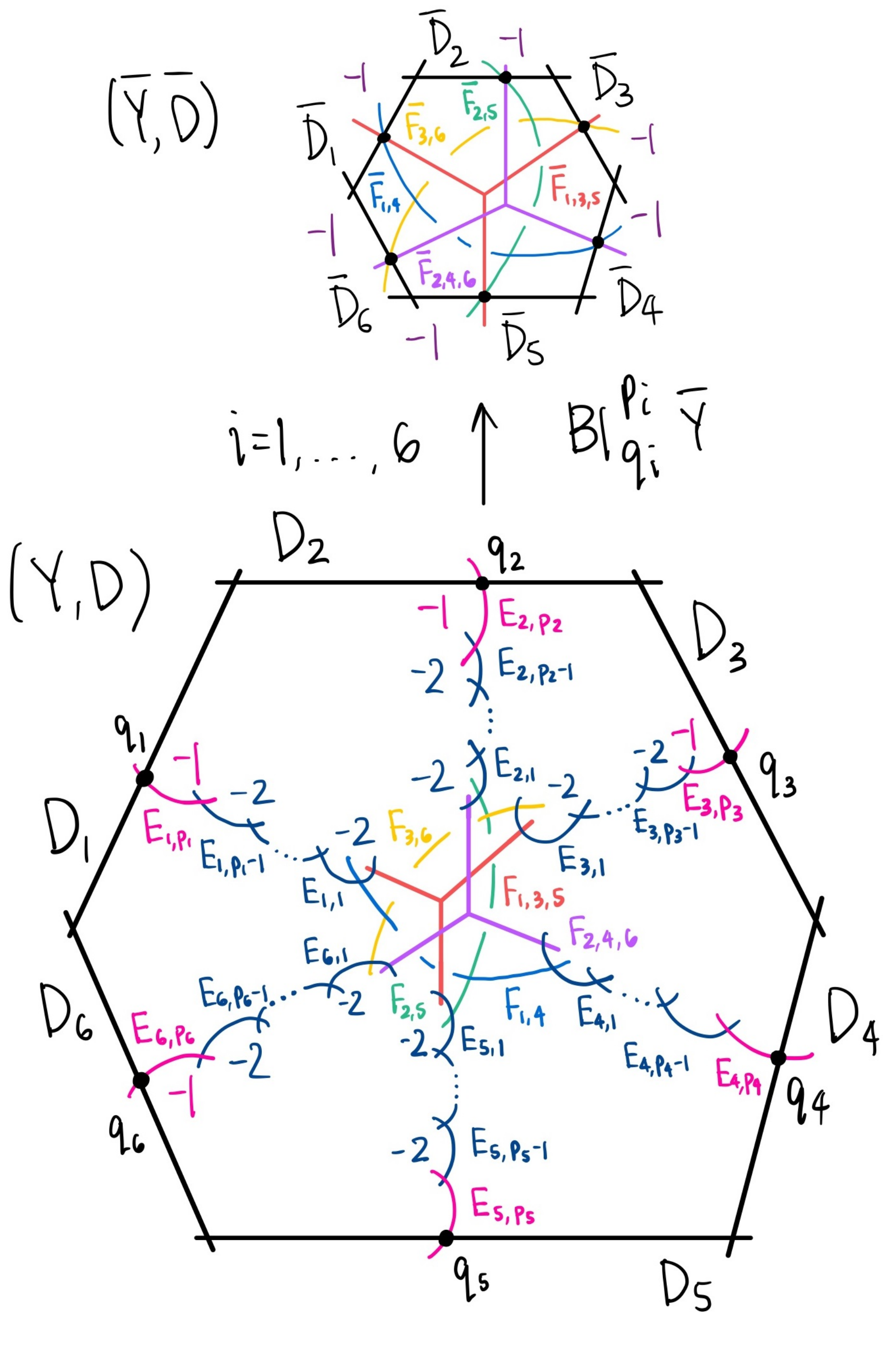

FIGURE 5.12. This diagram shows the surface $(\bar{Y}, \bar{D})$ blown up at points q_i, each a total of p_i times, for when $n = 6$.

Then for $j > 0$ we have

$$A_{i,j} \cdot E_{s,t} = \begin{cases} 1 & \text{if } i = s \text{ and } j = t; \\ 0 & \text{otherwise.} \end{cases} \tag{5.11}$$

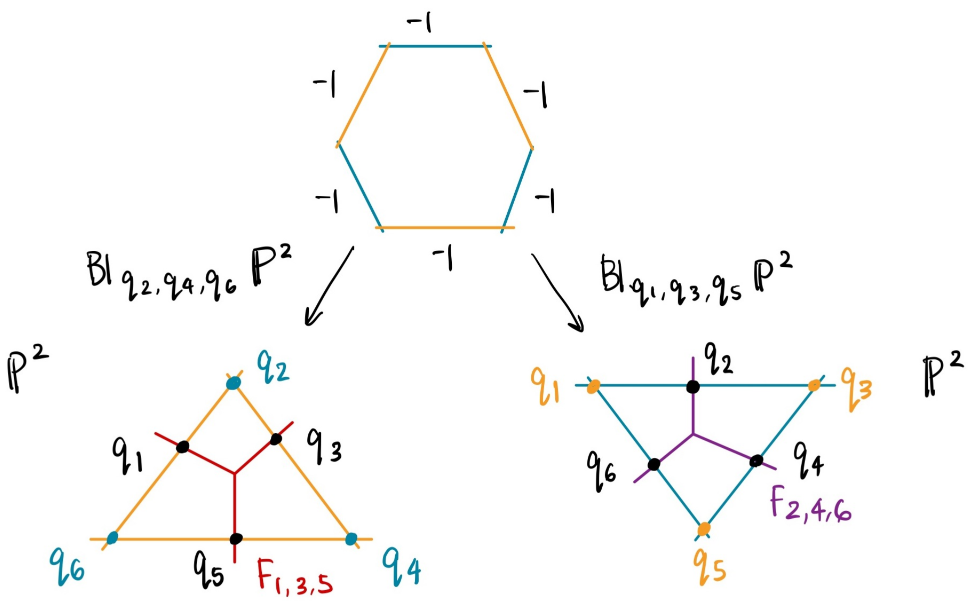

FIGURE 5.13. Two different ways to blow down the hexagon to $\mathbb{P}^2$.

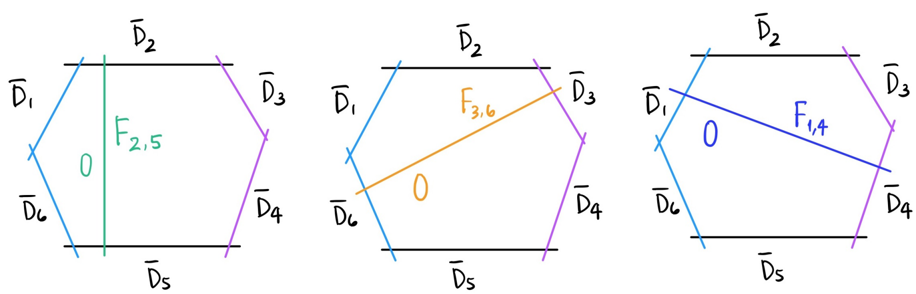

FIGURE 5.14. The top figure shows one of the three fibers of the morphism $f : \bar{Y} \to \mathbb{P}^2$. The bottom figures show all three different fibers, which are obtained by rotations.

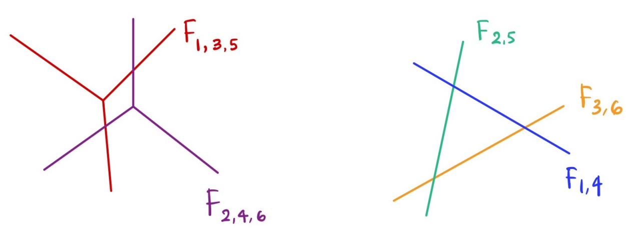

FIGURE 5.15. Two sets of curves that lie in the interior of the surface of $n = 6$.

The set $S = \{E_{i,j} \mid i = 1, \ldots, 6\} \cup \{F_k \mid k \in K\}$ spans $\mathrm{Pic}(Y)$. This follows from two facts: (1) The set $\{\bar{F}_k \mid k \in K\}$ spans $\mathrm{Pic}(\bar{Y})$ and (2) if S is a smooth projective surface and $\mathcal{B}$ is a basis of $\mathrm{Pic}(S)$ and $\tilde{S} \to S$ is the blowup of a point $p \in S$ with exceptional curve E, then $\tilde{\mathcal{B}}$, consisting of the strict transforms of divisors in $\mathcal{B}$ with E, is a basis of $\mathrm{Pic}(\tilde{S})$.

Let $C \subset Y$ be an irreducible curves. Suppose that $C \neq D_i, E_{i,j}$ for all i, j. Then $C \cdot A_{i,j} \geq 0$ for all i, j. We can write

$$C = \sum a_{i,j} E_{i,j} + \sum b_k F_k \in \mathrm{Pic}(Y)$$

Computing the intersection numbers $A_{i,j} \cdot E_{s,t}$ and $A_{i,j} \cdot F_k$ results in the following inequalities:

$$\begin{aligned}
&a_{i,j} \geq 0 \text{ for all } i, j;\\
&b_{1,4} + b_{1,3,5} \geq 0;\\
&b_{1,4} + b_{2,4,6} \geq 0;\\
&b_{2,5} + b_{1,3,5} \geq 0;\\
&b_{2,5} + b_{2,4,6} \geq 0;\\
&b_{3,6} + b_{1,3,5} \geq 0; \text{ and}\\
&b_{3,6} + b_{2,4,6} \geq 0;
\end{aligned}$$

The last six inequalities define the cone

$$\sigma := \langle [F_k] \mid k \in K \rangle_{\mathbb{R}\geq 0} \subset V,$$

where $V := \langle [F_k] \mid k \in K \rangle_{\mathbb{R}}$. Using the spanning set

$$\{[F_k]\} \text{ where } k \in K = \{\{1,4\}, \{2,5\}, \{3,6\}, \{1,3,5\}, \{2,4,6\}\}$$

of V, we can identify σ with the cone

$$\langle \bar{e}_1, \ldots, \bar{e}_5 \rangle_{\mathbb{R}\geq 0} \subset \mathbb{R}^5 / \langle (1,1,1,-1,-1) \rangle_{\mathbb{R}}.$$

Then, we may assume that $b_k \geq 0$ for all $k \in K$, so that C lies in the cone generated by the $E_{i,j}$ and the F_k. Therefore $\overline{\mathrm{Curv}}(Y) = \langle D_i, E_{i,j}, F_k \mid i = 1, \ldots, 6 \text{ and } j = 0, \ldots, p_i \text{ and } k \in K \rangle_{\mathbb{R}_{\geq 0}}$.

Corollary 5.12. *A log Calabi-Yau surface (Y_e, D_e) which has boundary D_e consisting of no more than six components is an example of a Mori Dream space.*

Proof. This follows from Theorem 5.2, Theorem 3.14, Remark 2.2, and Definition 5.1. □

CHAPTER 6

MOTIVATIONS

Let (Y, D) be a log Calabi-Yau surface, and let (Y', p) be obtained by contracting D to a cusp singularity p. Then there is the following conjecture:

(1) The smoothing components of the deformation space of Y, up to isomorphism, are in bijective correspondence with deformation types of log Calabi-Yau pairs (Y, D) such that D (which does not contain any (-1)-curves) contracts to the dual cusp p, i.e.,

$$\pi : (Y, D) \to (Y', p),$$

where π is the minimal resolution. A cusp might not have a smoothing, or it could have more than one smoothing component.

(2) The smoothing component of $p \in Y$ associated to (Y, D) is the *Looijenga space* determined by the action of Adm on the nef effective cone $\mathrm{Nef}^{\,\mathrm{e}}(Y'_{gen})$, which is contained in $\langle D_1, \ldots, D_n \rangle^{\perp} \otimes_{\mathbb{Z}} \mathbb{R}$ - this construction is described in [**L03**], Section 4.

Looijenga's construction requires that Adm acts on $\mathrm{Nef}^{\,\mathrm{e}}(Y'_{gen})$ with a rational polyhedral fundamental domain, and this is a motivation for the cone conjecture for log Calabi-Yau surfaces. This was also the motivation for the Morrison cone conjecture for Calabi-Yau threefolds.

The log Calabi-Yau cone conjecture can provide insight into the original Morrison cone conjecture because it includes more accessible cases. For instance, in every dimension there are many log Calabi-Yau pairs (Y, D) such that the variety Y is rational. In addition, the cone conjecture is related to the abundance conjecture, which is a long-standing open question of the minimal model program.

Remark 6.1. The cone conjecture for log Calabi-Yau surfaces suggests that the Morrison cone conjecture is false in general, because it is the monodromy group Adm that acts with a rational polyhedral fundamental domain on $\mathrm{Nef}^{\,\mathrm{e}}(Y_{gen})$, and not the automorphism group.

Remark 6.2. The explicit description of $\mathrm{Nef}\,(Y_e)$ can be used to verify the conjecture (1) stated above. For $n \leq 5$, this follows from work of Looijenga [**L81**], and for $n = 6$, we expect that it can be verified using work of Brohme (see [**B95**]). The deformation theory for $n > 6$ is not known.

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