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Equivariant smoothings of cusp singularities

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EQUIVARIANT SMOOTHINGS OF CUSP SINGULARITIES

A Dissertation Presented

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ABSTRACT

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Let $(p \in X)$ be the germ of a cusp singularity and let ι be an antisymplectic involution, that is an involution free on $X \setminus \{p\}$ and such that there exists a nowhere vanishing holomorphic 2-form Ω on $X \setminus \{p\}$ for which $\iota^*(\Omega) = -\Omega$. We prove that a sufficient condiition for such a singularity equipped with an antisymplectic involution to be equivariantly smoothable is the existence of a Looijenga (or anti-canonical) pair (Y, D) that admits an involution free on $Y \setminus D$ and that reverses the orientation of D.

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CHAPTER 1

INTRODUCTION

Cusp singularities are a specific type of surface singularities: more precisely a point p on a complex algebraic surface X is said to be a cusp singularity if the exceptional locus $E = \pi^{-1}(p)$ of its minimal resolution $\pi : \tilde{X} \to X$ is either an irreducible nodal curve or a cycle of smooth rational curves meeting transversally. Every cusp singularity $(p \in X)$ has an associated dual cusp: we will refer to the exceptional locus D of this dual cusp singularity as the cycle dual to $(p \in X)$. If the germ of a cusp singularity $(p \in X)$ admits an involution ι which is free away from p and such that there exists a nowhere vanishing holomorphic 2-form Ω on $X \setminus \{p\}$ for which $\iota^*(\Omega) = -\Omega$, then we say ι is *antisymplectic* and one can consider the associated quotient. This gives a new singularity which is rational and log canonical. Cusp singularities and their quotients by the action of $\mathbb{Z}/2\mathbb{Z}$ are among the surface singularities which appear at the boundary of the compactification of the moduli space of surfaces of general type due to Kollár and Shepherd Barron (cfr. [11]). Since only those singularities that admit a smoothing family occur at the boundary of this moduli space, it is useful to find nice conditions under which they happen to be smoothable. This question has been answered completely when it comes to cusp singularities. Indeed in 1981 Loijenga proposed the following conjecture, also proving the necessity of the condition.

Theorem 1.0.1 ([14]). *A cusp singularity* $(p \in X)$ *is smoothable if and only if the dual cycle* D *sits as an anticanonical divisor on a smooth rational surface.*

The proof of the sufficiency of this conjecture came later in a broader paper on the mirror symmetry of log Calabi-Yau surfaces by Gross, Hacking and Keel [6]. This result is interesting because it connects the deformation theory of this type of singularities to the existence of certain surfaces, the Looijenga pairs which can be checked algorithmically.

Inspired by these results, the main goal of this thesis is to address the same problem for quotient cusp singularities, or, in other words, to investigate under which conditions a smoothable cusp singularity is equivariantly smoothable with respect to the action of $\mathbb{Z}/2\mathbb{Z}$. We have been able give a sufficient condition for cusp singularities to be $\mathbb{Z}/2\mathbb{Z}$-equivariantly smoothable, which can be stated as follows:

Theorem 1.0.2. *Let* $(p \in X)$ *be the germ of a cusp singularity equipped with an antisymplectic involution* ι *and let* D *be its dual cycle. If there exists a Looijenga pair* (Y, D) *endowed with an antisymplectic involution that extends the one induced on* D *by* ι*, then the cusp singularity* $(p \in X)$ *is equivariantly smoothable.*

Here an involution $j : (Y, D) \to (Y, D)$ is said to be antisymplectic if it is free on the complement of D and it reverses the orientation of D. The proof of this result is based on the work of Gross, Hacking and Keel ([6]): we use the involution defined on the surface Y to get an equivariant version of the GHK construction. From this family we then obtained the required equivariant smoothing for the cusp singularity. Theorem 1.0.2 is already very useful. Indeed it allows to prove the following interesting fact.

Corollary 1.0.3. *All cusp singularities of multiplicity* $n \leq 10$ *admitting an antisymplectic involution are equivariantly smoothable.*

In order to prove these results, a great importance had the study of the other main character of theorem 1.0.2, that is Looijenga pairs. A Looijenga pair is a smooth projective surface Y, together with an anticanonical divisor D which is either an irreducible rational curve with a single node or a cycle of smooth rational curves. Examples of Looijenga pairs are provided for instance by the toric surfaces $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ with their respective toric boundaries. In his paper Looijenga showed that if the number n of irreducible components of D (the *length* of D) is less or equal to 5 then for each n and for each fixed D with self intersections $D_1^2, \ldots, D_n^2$, there esists one deformation type of negative Looijenga pairs (Y, D). We were able to extend this result and study the deformation types of Looijenga pairs of length $6 \leq n \leq 9$. Note that in the following theorem we identify D with its *cycle of integers*, $(-D_1^2, \ldots, -D_n^2)$.

Theorem 1.0.4. *If $n = 6, 7$ or $n = 8$ and D has associated cycle of integers different from $(a, 2, b, 2, c, 2, d, 2)$ then there is one deformation type of negative definite Looijenga pairs (Y, D) of length n with fixed D. If $n = 8$ and D is of type $(a, 2, b, 2, c, 2, d, 2)$ there are two deformation types, distinguished by $\pi_1(U)$, where $U = Y \setminus D$. Finally, if $n = 9$, then there are at most three deformation types of negative definite Looijenga pairs (Y, D) of length 9 with fixed D.*

Having information about the deformation types of negative definite Looijenga pairs is interesting also for the implications it can have on the deformation theory of cusp singularities.

Conjecture 1.0.5. *Let $(p \in X)$ be a cusp singularity. Then the set of smoothing components of its deformation space modulo the action of automorphisms of $(p \in X)$ is in bijective correspondence with the set of deformation types of negative definite Looijenga pairs (Y, D) such that D contracts to the dual cusp.*

Going back to the main goal of this work, we studied Looijenga pairs equipped with an *antisymplectic involution*, that is an involution that is fixed point free on $Y \setminus D$ and reverses the orientation of D. We proved the following result

Theorem 1.0.6. *Let $(\mathbb{P}^1 \times \mathbb{P}^1, \Delta)$ be the toric Looijenga pair given by $\mathbb{P}^1 \times \mathbb{P}^1$ together with its toric boundary Δ. Given a negative definite Looijenga pair (Y, D) with $n \geq 4$ equipped with an antisymplectic involution j, there always exists a sequence of contractions of disjoint pairs of (-1) curves*

$$(Y, D) \xrightarrow{\psi_1} (Y_1, D_1) \xrightarrow{\psi_2} \dots \xrightarrow{\psi_{m-1}} (Y_{m-1}, D_{m-1}) \xrightarrow{\psi_m} (\mathbb{P}^1 \times \mathbb{P}^1, \Delta) \tag{1.1}$$

that respects the $\mathbb{Z}/2\mathbb{Z}$-action defined on (Y, D) and induces on $(\mathbb{P}^1 \times \mathbb{P}^1, \Delta)$ the action given by the map $j : (z, w) \mapsto (1/z, -w)$

Conversely, if the length n of D is such that $4 \leq n \leq 10$ and D is the dual cycle to a symmetric cusp singularity which admits an antisymplectic involution, then there always exists a smooth projective surface Y containing D as an anticanonical divisor and a $\mathbb{Z}/2\mathbb{Z}$-action defined on it. This, together with theorem 1.0.2, implies the result stated in corollary 1.0.3. Moreover, already among cusps of multiplicity equal to twelve, using theorem 1.0.6, there can be found examples of cusp singularities that are equipped with an antisymplectic involution and are smoothable, but for which there does not exist a Looijenga pair (Y, D) that admits an antisymplectic involution extending the action defined on D.

Finally we would like to observe that theorem 1.0.2 is in fact part (the sufficient condition) of a more comprehensive conjecture, modeled on Looijenga's theorem for which we would like to find a complete proof in the coming years.

Conjecture 1.0.7 (Main conjecture)**.** *Let $(p \in X)$ be a cusp singularity equipped with an antisymplectic involution ι. Then $p \in X$ admits an equivariant smoothing* *if and only if the dual cycle D sits as an anticanonical divisor on a smooth rational surface which admits an antisymplectic involution extending the one induced on D by ι on $(p \in X)$.*

The thesis is structured as follows. The second chapter (the first one after this introduction) contains the results on cusp singularities admitting an antisymplectic involution, the third chapter deals with Looijenga pairs and contains the proof of theorems 1.0.4 and 1.0.6, among the others. The final chapter contains the main theorem, 1.0.2, and the results on smoothability of symmetric cusps of multiplicity $n \leq 12$.

CHAPTER 2

CUSP SINGULARITIES

2.1 Definitions and general results

Let $(p \in X)$ be the germ of an isolated normal surface singularity. Let $\pi : \tilde{X} \longrightarrow X$ be its minimal resolution and $E = \pi^{-1}(p)$ the exceptional locus. We summarize some well known facts about this type of singularities, see for instance Looijenga [14] and Friedman [3].

Definition 2.1.1. We say that $(p \in X)$ is a *cusp singularity* if the exceptional locus E is a union of smooth rational curves meeting transversally, $E = \bigcup_{i=1}^{n} E_i$, with dual graph a cycle and $n \geq 2$ or a rational curve with one node.

Note that the negative definiteness of the intersection matrix for E and the fact that π is a minimal resolution, imply the following three conditions:

i. Each self intersection $-e_i = {E_i}^2$ is such that $e_i \geq 2$

ii. There exists at least one j such that $e_j \geq 3$

iii. If E only has one irreducible component, then $-E^2 \geq 1$

Moreover, the cycle of integers $(e_1, \ldots, e_n)$ determines the analytic type of the cusp singularity; in other words cusp singularities are taut ([13]).

The *multiplicity* of $(p \in X)$ is equal to 2 if $E^2 = -1$, otherwise it is equal to $-E^2$; n is called the *length* of the cycle. Following Friedman, we will occasionally abuse notation and use E to indicate the cycle of curves, the cycle of integers and the cusp singularity itself.

Remark 2.1.2. Every cusp singularity comes with an associated *dual cusp*: one way to describe it is in terms of its cycle of integers. If the cusp $(p \in X)$ is given by the cycle

$$(a_1, \underbrace{2, \ldots, 2}_{b_1}, a_2, \underbrace{2, \ldots, 2}_{b_2}, \ldots, a_l, \underbrace{2, \ldots, 2}_{b_l})$$

then the dual cusp D is obtained as:

$$(b_1 + 3, \underbrace{2, \ldots, 2}_{a_2 - 3}, b_2 + 3, \underbrace{2, \ldots, 2}_{a_3 - 3}, \ldots, b_l + 3, \underbrace{2, \ldots, 2}_{a_1 - 3})$$

unless the length of the cycle is 1 or $E^2 = -1$. In these cases we have:

- If $E = (1)$, then $D = (1)$
- If $E = (e)$ with $e \geq 2$ then $D = (3, \underbrace{2, \ldots, 2}_{e-1})$
- If $E = (3, \underbrace{2, \ldots, 2}_{e})$ with $e \geq 1$ then $D = (e + 1)$

The duality of cusp singularities D, E can be described from various points of view, some of which will appear later in this section. To give an idea of how E and its dual D are related to each other we include the following result.

Proposition 2.1.3 (Lemma 1.4, [3]). *Let E represent a cusp singularity and D represent its dual, then*

i. the dual to D is E

ii. the length of D is equal to $-E^2$

iii. the embedding dimension of E is equal to $\max(3, -E^2)$

The proposition above implies that, since we can have cusp singularities with an exceptional cycle of arbitrary length, then we can have cusp singularities of arbitrary embedding dimension. For cusps of multiplicity $m \leq 5$ we actually know more about the geometry of these embeddings: if $m \leq 3$ then $(p \in X)$ embeds in $\mathbb{C}^3$ as a hypersurface, if $m = 4$ it embeds as a complete intersection in $\mathbb{C}^4$, finally if $m = 5$ it embeds in $\mathbb{C}^5$ as the zero locus of the 4×4 pfaffians of a 5×5 skew matrix.

Example 2.1.4. Consider $0 \in X = (x^p + y^q + z^r + xyz = 0) \subset \mathbb{A}^3$ with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$ and $p, q, r > 4$ (in this case the cusp singularity is a hypersurface). Here is a rough description of the minimal resolution: First we perform a blowup at 0, obtaining $\hat{\pi} : \hat{X} \to X$, with exceptional locus $\hat{E} = \hat{\pi}^{-1}(0) \subset \hat{X}$ (figure 1: note that this picture is meant only as a sketch of the exceptional divisor: the three components of $\hat{E}$ meet transversally at points p_1, p_2, p_3). The points $p_1, p_2, p_3 \in \widehat{E}$

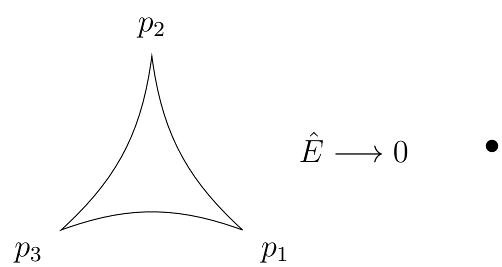


Figure 1. First blowup

are respectively A_{p-4}, A_{q-4}, A_{r-4} singularities. Indeed, let $\hat{X} \subset \mathrm{Bl}_0\mathbb{A}^3$. Then, in the chart given by $(u, y', z') \mapsto (u, uy', uz')$, $\hat{X}$ is given by the equation $u^{p-3} + u^{q-3}y' + u^{r-3}z' + y'z' = 0$ which, after an analytic change of coordinates becomes $u^{p-3} + y''z'' = 0$. The situation is analogous in the other two charts. Resolving

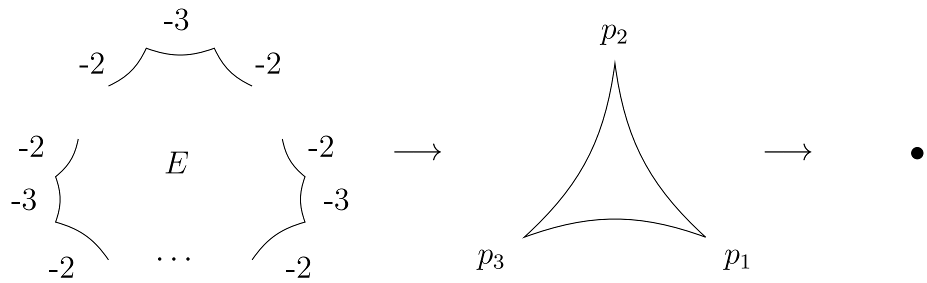


Figure 2. Resolution of the three singular points

each one of the three singularities of type A we get the map $\widetilde{\pi} : \widetilde{X} \to \widehat{X}$, with exceptional locus $\widetilde{E} \subset \widetilde{X}$ (figure 2) where $\widetilde{X}$ is smooth, therefore $\pi : \widetilde{X} \to X$ is the minimal resolution of $(0 \in X)$ and the cusp singularity is associated to the cycle $(3, \underbrace{2, \ldots, 2}_{p-4}, 3, \underbrace{2, \ldots, 2}_{q-4}, 3, \underbrace{2, \ldots, 2}_{r-4})$. One way to see that the strict transforms of the three exceptional divisors of the first blow up $\hat{\pi} : \hat{X} \to X$ are (-3)-curves in $\tilde{X}$ is the following. Let C be one of these curves: since $\tilde{X}$ is smooth and C is rational, then C has self intersection (-3) if and only if $K_{\tilde{X}} \cdot C = 1$. In our case $\tilde{\pi} : \tilde{X} \to \hat{X}$ is the minimal resolution of some Du Val singularities, therefore it satisfies $K_{\tilde{X}} = \tilde{\pi}^* K_{\hat{X}}$ and using the projection formula we get $K_{\tilde{X}} \cdot C = K_{\hat{X}} \cdot \tilde{\pi}_* C$. Thus

$$K_{\tilde{X}} \cdot C = K_{\hat{X}} \cdot \tilde{\pi}_* C = (\hat{\pi}^* K_X - \hat{E}) \cdot \tilde{\pi}_* C = -\hat{E} \cdot \tilde{\pi}_* C = -(-1) = 1$$

. Using remark 2.1.2, we see that the dual cusp is represented by the cycle $(p - 1, q - 1, r - 1)$.

Cusp singularities have an interesting quotient construction which is due to Hirzebruch [9]. Let us describe the idea of this construction as it appears in [6]. Let $N \cong \mathbb{Z}^2$ and let $A \in \mathrm{SL}(N)$ be a hyperbolic transformation, that is A has a real eigenvalue $\lambda > 1$. A determines a pair of dual cusps as follows.

Choose two linearly independent eigenvectors $w_1, w_2 \in N_{\mathbb{R}} \cong N \otimes \mathbb{R}$ for A with eigenvalues respectively $1/\lambda$ and λ so that $w_1 \wedge w_2 > 0$ (with the standard counter-clockwise orientation of $\mathbb{R}^2$). Let C, C' be the interiors of the strictly convex cones spanned by $\{w_1, w_2\}$ and $\{w_2, -w_1\}$. We observe that C, C' are invariant for A. Now let $U_C, U_{C'}$ be the corresponding tube domains

$$U_C = \{z \in N_{\mathbb{C}} \text{ such that } \Im z \in C\}/N \subset N_{\mathbb{C}}/N \cong (\mathbb{C}^*)^2$$

A acts freely and properly discontinuously on $U_C, U_{C'}$. Write $Y_C, Y_{C'}$ for the holomorphic hulls of $U_C/\Gamma, U_C'/\Gamma$ where Γ is the subgroup of $\mathrm{SL}(N)$ generated by A. At the level of sets Y_C and $Y_{C'}$ are obtained from $U_C/\Gamma, U_C'/\Gamma$ by adding one point to each of them, respectively $p \in Y_C, p' \in Y_{C'}$.

Proposition 2.1.5 (cfr. Chapter III, Section 2 of [14])**.** *$(p \in Y_C)$ and $(p' \in Y_{C'})$ are germs of two cusp singularities which are dual to each other. Moreover all cusp singularities arise in this way.*

Idea of the proof. Let E represent a cusp singularity with associated cycle $(e_1, \ldots, e_n)$. Then the matrix

$$A := \begin{pmatrix} 0 & -1 \\ 1 & e_1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & e_2 \end{pmatrix} \cdots \begin{pmatrix} 0 & -1 \\ 1 & e_n \end{pmatrix}$$

will produce, through the process we described, the cusp E and its dual D. □

Remark 2.1.6. Suppose we are given an hyperbolic transformation $A \in \mathrm{SL}(N)$. Consider the cone C as before and call Ξ the convex hull of lattice points in C: since w_1, w_2 are irrational, there are infinitely many points in the boundary of Ξ. Label them v_i with $i \in \mathbb{Z}$. Now consider the fan Σ with two-dimensional cones generated by v_i, v_{i+1}. The set Ξ is invariant for A, more precisely A acts on the vectors $\{v_i\}$ by translation $A(v_i) = v_{i+n}$ for some n, which can be assumed to be positive. Σ defines a toric variety X_Σ with toric boundary given by an infinite chain of $\mathbb{P}^1$'s: the action of A on Σ induces an action on X_Σ which is free and properly discontinuous on a tubular neighbourhood V of the toric boundary. The quotient $V/\Gamma =: \widetilde{V}$ is the minimal resolution of $(p \in Y_C)$ and its exceptional locus is the cycle of divisors E_i, where $i = 1, \ldots, n$, corresponding to the rays of Σ generated by the vectors v_i. Note that each pair v_i, v_{i+1} is a basis for the lattice N and that, if we let $e_i = -E_i^2$, or $e_1 = -E_1^2 + 2$ if $n = 1$, and define $e_0 = e_n$ then we get relations (cfr. section 2.5, first exercise in [5])

$$e_{i \bmod n} v_i = v_{i-1} + v_{i+1} \qquad \text{for } i \in \mathbb{Z},\ n > 1 \tag{2.1}$$

Remark 2.1.7. The tube domain U_C is diffeomorphic to $(S^1 \times S^1) \times \mathbb{R}^2$ through the map $(\tau, \varphi) : N_\mathbb{R}/N \times C \to (S^1 \times S^1) \times \mathbb{R}^2$ where τ is the diffeomorphism between $N_\mathbb{R}/N \cong \mathbb{R}^2/\mathbb{Z}$ and $S^1 \times S^1$ determined by a choice of basis of N while φ is the composition of the diffeomorphisms:

i. $\varphi_1 : C \to (\mathbb{R}^2_{>0})_{x_1,x_2}$ defined by $w_1, w_2 \mapsto e_1, e_2$,

ii. $\varphi_2 : (\mathbb{R}^2_{>0})_{x_1,x_2} \to (\mathbb{R}^2_{>0})_{u_1,u_2}$ given by $(x_1, x_2) \mapsto (x_1 * x_2, x_1/x_2)$,

iii. $\varphi_3 : (\mathbb{R}^2_{>0})_{u_1,u_2} \to (\mathbb{R}^2)_{s_1,s_2}$ where $(u_1, u_2) \mapsto (\log u_1, \log u_2)$.

Similarly, $U_{C'}$ is diffeomorphic to $(S^1 \times S^1) \times \mathbb{R}^2$ via an analogous map. As a consequence, the action of A on C and on C' corresponds to an action of $\mathbb{Z}$ by translation on $\mathbb{R}^2$: on U_C, if λ, λ^{-1} are the eigenvalues of A, then $\varphi_1 \circ A \circ \varphi_1^{-1}$ is the diagonalization of A, while $\varphi_2\varphi_1 \circ A \circ (\varphi_2\varphi_1)^{-1}$ is the map acting as $(u_1, u_2) \mapsto (u_1, \lambda^2 u_2)$ and finally $\varphi \circ A \circ \varphi^{-1}$ is given by $(s_1, s_2) = (\log u_1, \log u_2) \mapsto (\log u_1, \log \lambda^2 u_2) = (s_1, s_2 + 2\log\lambda)$. Therefore U_C/Γ and $U_{C'}/\Gamma$ are diffeomorphic to torus bundles over a cylinder:

$$U_C/\Gamma \cong \frac{N_\mathbb{R}/N \times \mathbb{R}^2}{\sim_A} \quad \text{and} \quad U_{C'}/\Gamma \cong \frac{N_\mathbb{R}/N \times \mathbb{R}^2}{\sim_{A^{-1}}}$$

where the equivalence relation $\sim_A$ (respectively $\sim_{A^{-1}}$) is given by $(v, s_1, s_2) \sim_A (w, t_1, t_2)$ if and only if $(w, s_1, s_2) = (A^k v, s_1, s_2+k)$ (resp. $(w, s_1, s_2) = (A^{-k} v, s_1, s_2+k)$) for some $k \in \mathbb{Z}$, once we have rescaled the translation. As a consequence, the links $L_p, L_{p'}$ of the singularities $p \in Y_C$ and $p' \in Y_{C'}$ are diffeomorphic to torus bundles over S^1 and thus they are diffeomorphic to each other via the orientation-reversing map ψ:

$$\psi : L_p \cong \frac{N_{\mathbb{R}}/N \times \mathbb{R}}{\sim_A} \longrightarrow \frac{N_{\mathbb{R}}/N \times \mathbb{R}}{\sim_{A'}} \cong L_{p'}$$

$$(v, s) \mapsto (v, -s)$$

Here $s = s_2$ and s_1 is held fixed, moreover the cusp appears at $s_1 \to \infty$. Note that the map ψ is well defined on equivalence classes. Let $[(v, s)] = [(A^k v, s + k)]$ be a point in L_p, then $\psi(v, s) = (v, -s)$ while $\psi(A^k v, s + k) = (A^k v, -s - k)$. On the other side, $[(v, -s)] = [(A^k v, -s - k)] = [\psi(A^k v, s + k)]$.

We conclude this section with the following lemma, which appears in [6] and relates a cusp singularity and its dual from another perspective.

Lemma 2.1.8. *[Lemma 7.3, [6]] Let A, C, Ξ and the v_i be as above giving a cusp singularity $p \in Y_C$. Let Z be the toric variety associated to the polytope Ξ with character lattice $M_Z := N$. Let $H \subset Z$ be the toric boundary of Z, an infinite chain of smooth rational curves corresponding to the boundary of Ξ. Then there exists a tubular neighborhood $H \subset N \subset Z$ such that the Γ action on Ξ induces a properly discontinuous Γ action on N. Let $F \subset \widetilde{X}$ denote the quotient of $H \subset N$ by Γ. So F is a cycle of smooth rational curves. Then $F \subset \widetilde{X}$ can be contracted to a singularity $p' \in X$, which is a copy of the dual cusp $p' \in Y_{C'}$. Moreover, $\widetilde{X}$ is obtained form the minimal resolution of $p' \in X$ by contracting all the (-2)-curves.*

2.2 On the action of $\mathbb{Z}/2\mathbb{Z}$ on a cusp singularity

In this work we are interested in studying germs of cusp singularities that admit a $\mathbb{Z}/2\mathbb{Z}$-action and their equivariant smoothings. Let us begin this section with the following definition.

Definition 2.2.1. Let $(p \in X)$ be the germ of a cusp singularity. An involution ι is *antisymplectic* if it is fixed point free on $X \setminus \{p\}$ and there exists a nowhere vanishing holomorphic 2-form Ω on $X \setminus \{p\}$ for which $\iota^*(\Omega) = -\Omega$.

Equivalently, the induced involution on the minimal resolution $\pi : \tilde{X} \longrightarrow X$ reverses the orientation of the exceptional cycle E and it is fixed point free away from E. We observe that not all cusp singularities admit such an involution, as shown in the following proposition.

Proposition 2.2.2. *Let $(p \in X)$ be a cusp singularity and let ι be an antisymplectic involution. Then E has the following properties:*

i. If $E = \bigcup_{i=1}^{n} E_i$, then each irreducible component E_i is sent to some other irreducible component $E_{\sigma(i)}$ where σ is a reflection in the dihedral group D_n.

ii. None of the nodes in E is fixed by ι. Instead ι fixes setwise two of the irreducible components of E. In particular n is even: without loss of generality we can always label the fixed components E_n and $E_{n/2}$.

iii. Let e_i be equal to ${-E_i}^2$. Then e_n and $e_{n/2}$ are even and $e_i = e_{\sigma(i)}$ for all $i = 1, \ldots, n$.

Proof. Suppose there exists an involution ι as stated above. It is clear that each E_i has to be sent to some other E_j, where j might be equal to i. This implies that the corresponding action on the dual graph has to be given by an element σ of order 2 in the dihedral group of order $2n$. Moreover we claim that ι cannot fix a node of the exceptional cycle E. Indeed suppose it does, then we can choose local coordinates on a neighbourhood U of the fixed node p so that locally $E = (xy = 0)$ and the action is given by the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

It follows that ι fixes a line, which gives a contradiction to the original assumption. As a consequence the number of irreducible components of E has to be even and the induced action on the dual graph has to be given by a reflection fixing 2 vertices, because we want the orientation of the cycle to be reversed. This proves (i) and (ii). Finally, it is easy to check that if E_i is sent to $E_{\sigma(i)}$ then e_i has to be equal to $e_{\sigma(i)}$. Let us focus now on $E_{n/2}, E_n$, which according to our labeling are the components of E fixed by the action: each of these curves is a rational curve with an involution defined on it. Since the automorphism we are considering does not fix these curves pointwise, then it has to fix 2 distinct points on each one of them. Consider the quotient $\tilde{\rho} : \tilde{X} \to \hat{Z}$ given by the involution. Here $E_{n/2}, E_n$ are mapped to the rational curves $\hat{F}_{n/2}, \hat{F}_n$ and each curve contains two A_1 singularities corresponding to the fixed points: the minimal resolution of these singularities, $\tilde{\pi} : \tilde{Z} \to \hat{Z}$, is the composition of four blowups at the four distinct singular points on $\hat{F}_{n/2}, \hat{F}_n$. For $i = n/2, n$, let $F_i \subset \tilde{Z}$ be the strict transform of $\hat{F}_i$ and $G_1, \dots, G_4$ be the exceptional divisors. We have

$$F_{n/2} = \tilde{\pi}^* \hat{F}_{n/2} - \frac{1}{2} G_1 - \frac{1}{2} G_2 \quad \text{and} \quad F_n = \tilde{\pi}^* \hat{F}_n - \frac{1}{2} G_3 - \frac{1}{2} G_4$$

which gives

$$F_{n/2}^2 = \hat{F}_{n/2}^2 + \left(\frac{1}{2} G_1\right)^2 + \left(\frac{1}{2} G_2\right)^2 = \hat{F}_i^2 - \frac{1}{2} - \frac{1}{2} = \hat{F}_i^2 - 1$$

and similarly, $F_n^2 = \hat{F}_n^2 - 1$. Therefore $\hat{F}_{n/2}^2 = 1/2E_{n/2}^2$ and $\hat{F}_n^2 = 1/2E_n^2$ have to be integers, which implies that $e_{n/2}, e_n$ have to be even. Thus (iii) is verified. □

The proposition above and its proof give necessary conditions for a cusp singularity to admit an involution with the required properties, which can be summarized in the following definition.

Definition 2.2.3. We say a cusp E is *symmetric* if there exists a labeling of E and a reflection σ in the dihedral group of order $2n$ fixing n and $n/2$ such that $E_n^2, E_{n/2}^2$ are even and $E_i^2 = E_{\sigma(i)}^2$ for every $i = 1, \ldots, n$. In particular, the length of a symmetric cusp E has to be even.

Remark 2.2.4. If a cusp E is symmetric then the dual cusp D is symmetric as well. This follows immediately from the way the cycle of integers for the dual cusp is constructed starting from the one of E. Indeed, consider E_n, one of the two curves fixed by σ according to our convention on labels, and suppose that $e_n = -E_n^2 > 2$. Then, since e_n is even, it produces an odd number of (-2) curves in the dual cycle: call D_m the central curve among them. Viceversa, if $e_n = 2$, then E_n is the central curve in a chain of $2l+1$ (-2)-curves for some integer l and therefore it corresponds in D to a curve of self intersection $-2l+2$: again, label this curve D_m. The same reasoning applied to $E_{n/2}$ gives a curve labeled $D_{m/2}$. Finally the symmetry of the remaining self intersections carries on to D, thus giving a $\mathbb{Z}/2\mathbb{Z}$ action on the dual cycle which fixes exactly $D_{m/2}$ and D_m.

The conditions given in definition 2.2.3 are also sufficient to construct an antisymplectic involution on a cusp singularity.

Proposition 2.2.5. *Given a cusp singularity* $(p \in X)$*, an antisymplectic involution exists if and only if the associated exceptional cycle* E *is symmetric.*

Proof. For this proof we refer to the quotient construction of a cusp singularity described in section 2.1: let A be the matrix associated to $(p \in X)$, let w_1, w_2 be a pair of eigenvectors in $N_{\mathbb{R}}$ for A chosen appropriately so that we may assume $Bw_1 = w_2$. Recall that C is the open cone generated by w_1, w_2 and v_i for $i \in \mathbb{Z}$ are the lattice points on the boundary of Ξ, where Ξ is the convex hull of the lattice points contained in C.

The forward direction of the statement follows from (i) and (iii) of the above proposition. Now suppose E is symmetric and $n/2, n$ are the indices fixed by the reflection σ. It follows from the definition of a symmetric cusp that e_n and $e_{n/2}$ are even and $e_i = e_{n-i}$ for $i \neq n$, $i \neq n/2$. Moreover, recall that we have the relation $e_0 v_0 = v_{-1} + v_1$ (where $e_0 = e_n = E_n^2$). We can thus define an action on N fixing v_0 as follows: choose $\{v_0, v_1\}$ as a basis (we can do that since the toric chart is smooth) for the lattice and let B be the linear map given by the matrix

$$B = \begin{pmatrix} 1 & e_0 \\ 0 & -1 \end{pmatrix}$$

Then B satisfies:

$$B^2 = I, \quad Bv_0 = v_0 \quad \text{and} \quad Bv_1 = e_0 v_0 - v_1 = v_{-1}$$

Similarly, the involution B maps each v_i to v_{-i}: by induction, suppose $Bv_{i-1} = v_{1-i}$ and $Bv_{i-2} = v_{2-i}$, then $Bv_i = B(e_{i-1 \bmod n} v_{i-1} - v_{i-2}) = e_{i-1 \bmod n} v_{1-i} - v_{2-i}$ and $e_{i-1 \bmod n} = e_{1-i \bmod n}$, thus $Bv_i = v_{-i}$. It follows that the cone C is invariant under B, considered as a map on $N_{\mathbb{R}}$. Thus B induces an involution on $U_C = N_{\mathbb{R}} + iC/N$.

Observe that v_0 is an eigenvector for B which belongs to the lattice N and it is primitive. The second eigenspace for B relative to the eigenvalue -1 is given by the equation $2x + e_0 y = 0$, therefore it is generated by the eigenvector $u_0 = -e_0/2 v_0 + v_1$ which again belongs to N because e_0 is an even integer and it is a primitive vector for this lattice. In fact the pair of eigenvectors v_0, u_0 forms a basis for the lattice N because $\{v_0, v_1\}$ is a basis of N. To see this, it suffices to consider the change of basis given by the matrix P described below:

$$P = \begin{pmatrix} 1 & \frac{e_0}{2} \\ 0 & -1 \end{pmatrix} \qquad \Rightarrow \qquad P^{-1}BP = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Moreover, u_0 is contained in C', the cone dual to C. In order to show this let us first write down w_1, w_2 in terms of v_0, v_1. The fact that $Bw_1 = w_2$ and viceversa, the condition that $w_1 \wedge w_2 > 0$ and the convention we use to label the vectors v_i give us the following:

$$w_1 = (\alpha + e_0\beta)v_0 - \beta v_1 \qquad w_2 = \alpha v_0 + \beta v_1$$

where $\alpha < 0 < \beta$ and α/β is irrational. Given that $u_0 = -e_0/2v_0 + v_1$, we can then write u_0 in terms of w_1, w_2, obtaining that $u_0 = -(2\beta)^{-1}w_1 + (2\beta)^{-1}w_2$. Since β is positive, then we can conclude that $u_0 \in C'$. Now, if we defined our involution on the cusp singularity using only the linear involution B, we would obtain a map that is not fixed point free on $Y_C \setminus \{p\}$. Therefore we compose B with the translation by an element of the torus $t \in N_{\mathbb{C}}/N$. More precisely, since we still want to construct an involution, we have to choose a two torsion element contained in $1/2N/N$. To describe t more in detail, let us analyze B under the isomorphism between $N_{\mathbb{C}}/N$ and $(\mathbb{C}^*)^2$ given by the exponential map. Fix v_0, u_0 as a basis for N. Then B is associated to the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Thus, under the isomorphism between $N_{\mathbb{C}}/N$ and $(\mathbb{C}^*)^2$, the involution B corresponds to the map $(x, y) \mapsto (x, y^{-1})$. In order for the final involution to be fixed point free we need the translation by t to correspond to the map $(x, y) \mapsto (-x, \pm y)$, so that the pair (B,t) acts on $(\mathbb{C}^*)^2$ via the map

$$\vartheta_{B,t} : (x,y) \mapsto (-x, \pm y^{-1})$$

Indeed the variable x corresponds to the character v_0^* and, thinking in terms of the minimal resolution of $p \in Y_C$, it corresponds to the direction which is normal to the divisor E_n. With respect to the basis chosen above, we want $t = av_0 + bu_0$ such that $(a,b) + i(0,0) \mapsto (e^{a\cdot 2\pi i}, e^{b\cdot 2\pi i}) = (-1, \pm 1)$ in $(\mathbb{C}^*)^2$, thus we need $a = 1/2$. In particular t cannot be equal to $0, 1/2u_0$, that is it cannot be a pure multiple of u_0. The same reasoning has to be made for the other fixed component of E, $E_{n/2}$, thus t cannot be a multiple of $u_{n/2} := e_{n/2}/2v_{n/2} + v_{n/2+1}$. Therefore $t \in 1/2N/N \setminus \{0, 1/2u_0, 1/2u_{n/2}\}$. Since $1/2N/N$ has order four, there is always an element t that works.

We can now define our involution given by the pair (B,t) on $N_{\mathbb{C}}/N$ as

$$\vartheta_{B,t} : \; v + iw \mapsto B(v+iw) + t = (Bv + t) + iw$$

Observe that if $[v+iw] \in U_C$, meaning that $w \in C$, then $[Bv + iBw]$ still belongs to U_C because the cone C is invariant under B and $[Bv+t+iw] \in U_C$ as well, since the translation by t does not affect the imaginary part w. Now let us consider the transformation $A \in \mathrm{SL}(N)$ and the quotient U_C/Γ, where Γ is the infinite cyclic group generated by A. For all $v_i \in N$ we have that $B(Av_i) = Bv_{i+n} = v_{-(i+n)}$ and $v_{-(i+n)} = A^{-1}(v_{-i}) = A^{-1}(Bv_i)$. This holds true in particular for v_0, v_1, therefore we get the relation

$$BA(v+iw) = A^{-1}B(v+iw) \text{ for all } v+iw \in N_{\mathbb{C}} \tag{2.2}$$

The involution $\vartheta_{B,t}$ is constant on the equivalence classes relative to the action of A, that is $[\vartheta_{B,t}(v+iw)]_A = [\vartheta_{B,t}([A(v+iw)])_A$, thus giving a well defined map on U_C/Γ. Indeed $\vartheta_{B,t}([A(v+iw)]) = BAv + t + iBAw$ and to show that $[Bv+t+iBw]_A = [BAv+t+iBAw]_A$ it suffices to prove that $A^{-1}Bv + A^{-1}t + iA^{-1}Bw = BAv + t + iBAw \bmod N$. Given (2.2) we only need to show that $At = t \bmod N$, or equivalently that $A(2t) = 2t \bmod 2N$. Observe that since $t \in 1/2N/N$, then $2t$ lies in N, therefore the fact that A is congruent to the identity matrix mod 2, as proved in lemma 2.2.6, immediately implies that $A(2t) = 2t \bmod 2N$.

Given what we discussed above, the pair (B,t) defines a fixed point free map on the algebraic torus and therefore on U_C that descends to a well defined and still fixed point free analytic map on the quotient U_C/Γ which extends to its partial compactification Y_C as an analytic map fixing the cusp singularity $p \in Y_C$. □

Lemma 2.2.6. *Let N be a two dimensional lattice and let A be a transformation in* $\mathrm{SL}(N)$ *associated to a symmetric cusp. Then A is congruent to the identity matrix mod 2N.*

Proof. A matrix A associated to a symmetric cusp can always be written as a product of n matrices. More precisely, if we fix v_0, v_1 as a basis for N then A is given by:

$$A = \begin{pmatrix} 0 & -1 \\ 1 & e_1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & e_2 \end{pmatrix} \cdots \begin{pmatrix} 0 & -1 \\ 1 & e_{n/2} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & e_{n/2-1} \end{pmatrix} \cdots \begin{pmatrix} 0 & -1 \\ 1 & e_1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & e_n \end{pmatrix} \tag{2.3}$$

since $e_i = e_{n-i}$ for $i = 1, \ldots, n$. Besides $e_{n/2}, e_n$ are even integers, therefore the two corresponding matrices are congruent to the matrix

$$J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

mod 2. Now observe that for a fixed (positive) integer e, we have that

$$\begin{pmatrix} 0 & -1 \\ 1 & e \end{pmatrix} J \begin{pmatrix} 0 & -1 \\ 1 & e \end{pmatrix} = J$$

Therefore, mod 2, we get

$$\begin{pmatrix} 0 & -1 \\ 1 & e_{n/2-1} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & e_{n/2} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & e_{n/2-1} \end{pmatrix} \equiv \begin{pmatrix} 0 & -1 \\ 1 & e_{n/2-1} \end{pmatrix} J \begin{pmatrix} 0 & -1 \\ 1 & e_{n/2-1} \end{pmatrix} = J$$

and, using this recursively in (2.3) for each i until $i = 1$ we obtain

$$A \equiv J \cdot \begin{pmatrix} 0 & -1 \\ 1 & e_n \end{pmatrix} \equiv J \cdot J = I \qquad \text{mod } 2$$

Therefore A is congruent to the identity matrix mod 2, as required.

□

Remark 2.2.7. Note that the generators of the subgroups $\langle A \rangle \cong \mathbb{Z}$ and $\langle B \rangle \cong \mathbb{Z}/2\mathbb{Z}$ of $\mathrm{GL}(N)$ described in proposition 2.2.5 satisfy $BAB = A^{-1}$ thus defining an infinite dihedral group $\mathcal{D}_{A,B}$ isomorphic to the semidirect product $\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$. One may ask whether this group gives a complete description of the normalizer of A in $\mathrm{GL}(N)$. More precisely if $\mathrm{N}_{\mathrm{GL}(N)}(A)$ is generated by $A, B, -Id$. To answer this question, consider for instance the cusp singularity given by $E = (-E_1^2, -E_2^2, -E_3^2, -E_4^2) = (4, 6, 4, 6)$. Then two distinct reflections can be defined on E: σ_1, fixing E_1, E_3 or σ_2, fixing E_2, E_4. Following proposition 2.2.5, σ_1 and σ_2 induce two different matrices B_1, B_2 acting on the cone C relative to the cusp E and satisfying $B_i A B_i = A^{-1}$, thus we get $\mathcal{D}_{A,B_i} < \langle A, B_1, B_2 \rangle \leq \mathrm{N}_{\mathrm{GL}(N)}(A)$. This is, however, a special case since this cusp is the double cover of the cusp associated to the cycle $(4, 6)$, therefore a better posed question might be if the normalizer of *primitive* cusps (meaning those that are not covers of other cusps) can be described via the dihedral group mentioned above.

An involution on a cusp singularity $(p \in X)$ is determined by the $\mathbb{Z}/2\mathbb{Z}$-action on the exceptional cycle E, in the following sense. Let $(p \in X)$ be a symmetric cusp singularity, let E be its exceptional cycle and σ a reflection acting on the components of E, as in definition 2.2.3. Suppose we are given an antisymplectic involution ι which acts as the reflection σ on E and let $\rho : (p \in X) \to (q \in Z)$ be the quotient map associated with the action induced by an antisymplectic involution ι acting freely away from p. Then $(q \in Z)$ is the germ of a rational isolated singularity, usually referred to as the cusp quotient singularity [19] and the map ρ gives an étale covering of $Z \setminus \{q\}$. Let $\tilde{\rho} : \tilde{X} \to \hat{Z}$ be the quotient of $\tilde{X}$ by the action corresponding to the involution $\tilde{\iota}$ on the minimal resolution of $(p \in X)$. We get the commutative diagram:

$$\begin{array}{ccc} E \subset \tilde{X} & \xrightarrow{\pi} & (p \in X) \\ \downarrow{\scriptstyle \tilde{\rho}} & & \downarrow{\scriptstyle \rho} \\ \hat{F} \subset \hat{Z} & \xrightarrow[\hat{\pi}]{} & (q \in Z) \end{array}$$

where $\hat{F} = \bigcup_{i=1}^{n/2-1} \hat{F}_i \cup \hat{F}_{n/2} \cup \hat{F}_n$ is the image of E under $\tilde{\rho}$, more precisely $\tilde{\rho}^{-1}(\hat{F}_i) = E_i \cup E_{n-i}$ for $i \neq n/2, n$ while $\tilde{\rho}^{-1}(\hat{F}_i) = E_i$ for $i = n/2, n$. The proof of proposition 2.2.5 shows that $\hat{Z}$ contains four isolated A_1 singularities which lie in pairs on $\hat{F}_{n/2}, \hat{F}_n$. The minimal resolution $\pi_Z : \tilde{Z} \to Z$ of the singularity $(q \in Z)$ is obtained by composing the minimal resolution $\tilde{\pi} : \tilde{Z} \to \hat{Z}$ of the A_1 singularities on $\hat{Z}$ with the map $\hat{\pi}$. In particular the exceptional locus $F = \pi_Z^{-1}(q)$ is the union of the rational curves $\bigcup_{i=1}^{n/2-1} F_i \cup F_{n/2} \cup F_n \cup G_1 \cup \cdots \cup G_4$ where $G_1, \ldots, G_4$ are the exceptional divisors of the resolution of the A_1 singularities, $F_i = \tilde{\pi}^{-1}(\hat{F}_i)$ for $i \neq n/2, n$ and finally $F_{n/2}, F_n$ are the strict transforms of $\hat{F}_{n/2}, \hat{F}_n$ under $\tilde{\pi}$. The dual graph for F is described in figure 3.

Note that the arrangement of the irreducible components of F and their self intersections only depends on σ. Moreover, since quotient cusp singularities are

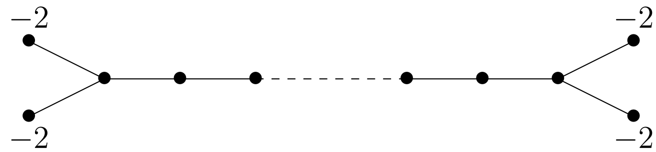


Figure 3. Dual graph of the exceptional cycle F

taut (see for instance [19]), then their isomorphism type is determined by their exceptional cycle. As a consequence, the quotient cusp $(q \in Z)$ only depends on the $\mathbb{Z}/2\mathbb{Z}$-action defined by σ on E. A priori though, the covering map $\rho : (p \in X) \to (q \in Z)$ could depend on the specific involution we consider for the germ of the singularity $(p \in X)$. The following proposition shows that in this case the map ρ is the same up to isomorphism, for any choice of involution ι as long as the action of ι on E is given by the same reflection, σ and we only consider antisymplectic involutions which are free away from the cusp singularity.

Proposition 2.2.8. *Let ι be any antisymplectic involution defined on a cusp singularity $(p \in X)$ which acts as the reflection σ on the associated exceptional cycle E. Then the quotient map $\rho : (p \in X) \to (q \in Z)$ associated to ι coincides, up to isomorphism, with the index one covering map (as described for example in [10]) of the quotient singularity $(q \in Z)$. As a consequence, given two involutions with the properties described above, there exists an isomorphism $\vartheta : (p \in X) \to (p \in X)$ that makes the following diagram commute:*

$$\begin{array}{ccc} (p \in X) & \xrightarrow{\vartheta} & (p \in X) \\ {\scriptstyle \iota_1}\downarrow & & \downarrow{\scriptstyle \iota_2} \\ (p \in X) & \xrightarrow[\vartheta]{} & (p \in X) \end{array}$$

Proof. Let us begin this proof with the observation that the index one covering map for Z described in [10], definition 5.19 has domain the cusp $(p \in X)$ and is induced by a covering $\tilde{X} \to \hat{Z}$ which is étale on the smooth locus of $\hat{Z}$. Now, the étale coverings of the smooth locus of $\hat{Z}$ are in bijective correspondence with the 2-torsion elements of its class group, $\mathrm{Cl}(\hat{Z})$. Indeed, if $D \in \mathrm{Cl}(\hat{Z})$ and $2D \sim 0$, then $\hat{X} := \underline{\mathrm{Spec}}_{\hat{Z}}(\mathcal{O}_{\hat{Z}} \oplus \mathcal{O}_{\hat{Z}}(D))$ with multiplication defined by an isomorphism $\theta : \mathcal{O}_{\hat{Z}}(D)^{\otimes 2} \to \mathcal{O}_{\hat{Z}}$ is a double cover of $\hat{Z}$ étale over the smooth locus and every such coveer arises this way (cf. [21], Cor 2.6). Note that the isomorphism type of the cover does not depend on the choice of the isomorphism θ because $\pi_1(\hat{Z}) = 1$. The map θ is determined up to a unit $u \in \mathrm{H}^0(\mathcal{O}^*_{\hat{Z}})$: since $\pi_1(\hat{Z})$ is trivial, u amits a square root $v \in \mathrm{H}^0(\mathcal{O}^*_{\hat{Z}})$. Then the map $\mathcal{O}_{\hat{Z}} \oplus \mathcal{O}_{\hat{Z}}(D) \to \mathcal{O}_{\hat{Z}} \oplus \mathcal{O}_{\hat{Z}}(D)$ defined by $(a, b) \mapsto (a, v \cdot b)$ induces an isomorphism of the double covers defined by $u \cdot \theta$ and θ. Thus in order to study the map $\tilde{\rho}$ and therefore the quotient map ρ it is useful to give a description of $\mathrm{Cl}(\hat{Z})$. If $G_1, \ldots, G_4$ are the exceptional divisors associated to the resolution of the four A_1 singularities of $\hat{Z}$, then the exact sequence

$$0 \to \langle G_1, \ldots, G_4 \rangle \to \mathrm{Cl}(\tilde{Z}) \to \mathrm{Cl}(\hat{Z}) \to 0$$

and the fact that $\tilde{Z}$ is smooth give the isomorphism

$$\mathrm{Cl}(\hat{Z}) \cong \frac{\mathrm{Pic}(\tilde{Z})}{\langle G_1, \ldots, G_4 \rangle}$$

Since $(q \in Z)$ is a rational singularity, then the Picard group of its minimal resolution is the free abelian group $H^2(\tilde{Z}, \mathbb{Z})$ generated by the dual basis to the basis of $H_2(\tilde{Z}, \mathbb{Z})$ given by classes of the $k + 4$ irreducible components of the exceptional locus (note that here $k = n/2 + 1$). Thus

$$\mathrm{Cl}(\hat{Z}) \cong \mathbb{Z}^{k+4}/Q\mathbb{Z}^4$$

where Q is the $(k + 4 \times 4)$ intersection matrix relative to $G_1, \ldots, G_4$. This matrix

Q can always be put in the form

$$Q = \begin{pmatrix} -2 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & -2 & 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & -2 & 0 & 0 & \cdots & 1 \end{pmatrix}^T$$

Note that Q gives the relations $-2G_1^* + F_{n/2}^* = 0$, $-2G_2^* + F_{n/2}^* = 0$, $-2G_3^* + F_n^* = 0$, $-2G_4^* + F_n^* = 0$, therefore $G_1^* - G_2^*$ and $G_3^* - G_4^*$ are elements of order two in $\mathrm{Cl}(\hat{Z})$. The Smith normal form of Q is the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 2 & \cdots & 0 \end{pmatrix}^T$$

which implies that $\mathrm{Cl}(\hat{Z}) \cong \mathbb{Z}^k \times (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$, thus the class group of $\hat{Z}$ contains three non trivial 2-torsion elements corresponding to three possible covers of $\hat{Z}$ of degree two which are étale on the smooth locus, one of them giving the index one covering of $(q \in Z)$.

Now, proposition 2.2.5 allows us to view the singularity $(q \in Z)$ as the partial compactification of the quotient of the tube domain U_C by the action of the hyperbolic matrix A and a pair (B, t) described explicitly in the proof of the proposition. Thus the singularity $(q \in Z)$ is obtained by taking the quotient of U_C by the action of the infinite dihedral group $\mathcal{D}_\infty$ generated by $a = A, b = (B, t)$.

$$U_C \xrightarrow{/\langle a\rangle} (p \in X) \xrightarrow{/\langle b\rangle} (q \in Z) \tag{2.4}$$

The maps above can be understood completely by looking at the corresponding maps on neighborhoods $N, \tilde{N}$ of the exceptional locus E and its universal covering

$\tilde{E}$, which is a chain of rational curves indexed by $\mathbb{Z}$. In particular the matrix A acts on $\tilde{E}$ by translation giving the cycle of curves E while the pair (B,t) acts as a reflection, giving $\hat{F}$.

Each subgroup of index 2 of $\mathcal{D}_\infty$ corresponds to a covering of $(q \in Z)$ of degree 2, and therefore to a covering of $\hat{Z}$ which is étale on the smooth locus, and it is easy to see that there are three such subgroups: $H_1 = \langle a^2, b\rangle$, $H_2 = \langle a^2, ab\rangle$, $H_3 = \langle a\rangle$. In particular, by the description in terms of the covering map above any such covering of $\hat{Z}$ arises in this way. In order to understand these coverings, as already stated above, it suffices to understand how the quotient maps induced by each subgroup H_i act on $\tilde{E} \subset \tilde{N}$ and $E \subset N$. Clearly H_3 corresponds to the quotient maps given in (2.4). As for H_1, we get the following diagram

$$\tilde{E} \to E_1' \to E_1 \to \hat{F}$$

where E_1' is a cycle of length $2n$, given that n is the length of E, and E_1 is a chain of $n+1$ rational curves finally mapping to $\hat{F}$ through the action of $\overline{a} \in \mathcal{D}_\infty/H_1$. A very similar description holds true for H_2. Thus we see that the only subgroup giving us a covering of Z by the cusp X, or alternatively of $\hat{Z}$ by $\tilde{X}$ is H_3. Since H_3 is the only subgroup giving a covering map from X to Z we deduce that this covering map is the one associated to the index one cover.

□

Given an action of $\mathbb{Z}/2\mathbb{Z}$ on a cusp singularity one may ask if it induces in a natural way an action on the dual cusp as well as an action on the exceptional dual cycle D. The answer is affirmative.

Theorem 2.2.9. *Let $(p \in X)$ be a cusp singularity and let ι be an antisymplectic involution. Then ι induces an antisymplectic involution on the dual cusp $p' \in X'$.*

Proof. Let E be the exceptional divisor associated with $(p \in X)$, as usual. Observe that E is symmetric, since it admits an involution. Then the theorem follows from remark 2.2.4 and proposition 2.2.5. $\square$

Finally, we can describe the relation between the involution on a cusp $(p \in X)$ and the one on its dual in light of lemma 2.1.8 from [6].

Remark 2.2.10. As usual, let $(p \in Y_C)$ be a cusp singularity, E its exceptional cycle. Let ι be the involution defined on it and σ the reflection induced by ι on E. On the other side, let $(p' \in Y_{C'})$ be the dual cusp and D its associated exceptional cycle. Finally let G be the cycle of curves obtained from E contracting all the (-2) curves and let F be the one obtained from D through the same process. Note that this F is exacly the divisor defined in lemma 2.1.8. Observe that σ gives directly (by restriction to the indices that are left after the contraction) an involution σ' on G that respects the self intersections of the irreducible components of G. Now, because of the way F, G are related to each other (in terms of their dual graphs, the vertices of Γ_G are the edges of Γ_F and viceversa), σ' can be seen as a reflection on F that respects self intersections, once we choose the appropriate labeling for it. Moreover σ' extends from F to D thus giving a reflection σ'' on D. Using proposition 2.2.5, σ'' induces an involution on $(p' \in Y_{C'})$ which is the one given by theorem 2.2.9.

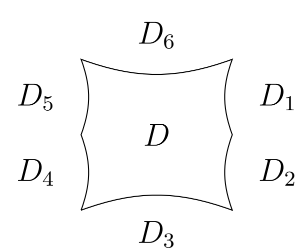
CHAPTER 3

LOOIJENGA PAIRS

An important role in the deformation theory of cusp singularities is played by Looijenga pairs. We present some preliminary facts on them following closely Friedman [4] and Gross, Hacking, Keel [7].

Definition 3.0.1. A *Looijenga pair* (Y, D) is a smooth projective surface Y together with a connected singular nodal divisor $D \in |-K_Y|$ which is either an irreducible rational curve with a single node or a cycle of smooth rational curves, $D = \sum_{i=1}^{n} D_i$, where each D_i meets D_{i+1} transversally, with i understood mod n.

We will also refer to pairs (Y, D) as *anticanonical pairs*. The integer n is called the length of D, if the components of D are indexed as above, we refer to (Y, D) as a *labeled Looijenga pair* and to the sequence of self intersections $(-D_1^2, -D_2^2, \ldots, -D_n^2)$ as the *cycle of integers* associated to it. To fix the notation, we will always label the components of D starting from the top-right one, for instance, for $n = 6$ we have:



Note, as always, that all the pictures that will apear in this work are merely sketches: all components of D should be understood as meeting transversally. An orientation of D is an orientation of its dual graph, or equivalently the choice of a generator of $H_1(D, \mathbb{Z}) \cong \mathbb{Z}$. Observe that for $n \geq 3$ an orientation determines a natural labeling of the components of D up to cyclic permutation and viceversa a labeling induces an orientation on D.

Lemma 3.0.2 (Lemma 2.1, [7]). *Let D be a cycle of n rational curves, with a choice of orientation. This orientation induces an identification* $\mathrm{Pic}^0(D) \cong \mathbb{G}_m$, *where* $\mathrm{Pic}^0(D)$ *is the group of numerically trivial line bundles.*

Construction of the isomorphism. Let us describe how the isomorphism is obtained if $n \geq 3$. For $L \in \mathrm{Pic}^0(D)$ choose a nowhere-vanishing section $s_i \in \Gamma(L|_{D_i})$ for all i. Then define the map λ as

$$\lambda : \mathrm{Pic}^0(D) \to \mathbb{G}_m \qquad L \mapsto \prod_i \frac{s_{i+1}(p_{i,i+1})}{s_i(p_{i,i+1})}$$

where $p_{i,i+1} = D_i \cap D_{i+1}$. We have that λ does not depend on the choice of the sections s_i and it is an isomorphism. □

Definition 3.0.3. An *isomorphism* of labeled Looijenga pairs (Y, D) and (Y', D') is an isomorphism $f : Y \to Y'$ such that $f(D_i) = D'_i$ for each $i = 1, \ldots, n$ which is compatible with the orientation of D and D'. Let $\mathrm{Aut}(Y, D)$ be the group of automorphisms of a labeled Looijenga pair mapping each component of D to itself and preserving the orientation of D.

If the intersection matrix $(D_i \cdot D_j)$ is negative definite, we call (Y, D) a *negative definite Looijenga pair* and say that D is negative definite. A useful invariant of anticanonical pairs is their charge.

Definition 3.0.4 ([4], Definition 1.1)**.** The *charge* $Q(Y,D)$ of a Looijenga pair is defined as

$$Q(Y,D) = 12 - D^2 - n$$

To give a glimpse of the cohomology theory of anticanonical pairs, let $\Lambda(Y,D) \subset H^2(Y,\mathbb{Z})$ be the orthogonal complement of the lattice spanned by the classes of the D_i. Then $\Lambda(Y,D)$ is free ([4], Lemma 1.5) and, if D is negative definite (which implies that the classes D_i are independent in cohomology), its rank is equal to the charge minus two ([4], Lemma 1.5). We also note that in the case D is negative definite, then $Q(Y,D) \geq 3$ ([4], Corollary 1.3).
Always with the aim of fixing our notation let us give the following definitions

Definition 3.0.5. Let (Y,D) be a Looijenga pair. A curve C in Y is an *interior curve* if none of its irreducible components is contained in D. An *internal (-2)-curve* instead is a smooth rational curve of self intersection -2 that is disjoint from D. We say that (Y,D) is *generic* if it has no internal (-2)-curves.

Define a *simple toric blowup* to be the blowup of a Looijenga pair (Y,D) at a node of D and an *interior blowup* to be a blowup of Y at a smooth point on D. For a toric blowup $\widetilde{Y} \to Y$, set $\widetilde{D} = \sum_i \widetilde{D}_i$, where $\widetilde{D}_i$ is the strict transform of D_i, while for an interior blowup define $\widetilde{D} = \sum_i \widetilde{D}_i + E$, where $\widetilde{D}_i$ is the strict transform of D_i and E is the exceptional divisor. Then in both cases $(\widetilde{Y}, \widetilde{D})$ is still a Looijenga pair. Interior blowups increase the charge $Q(Y,D)$ by one, while corner blowups do not change it ([3], Lemmas 3.3 and 3.4). Finally we observe that the charge of a Looijenga pair (Y,D) has a topological interpretation: let $U = Y \setminus D$, then $e(U) = Q(Y,D)$ where $e(U)$ is the Euler number of U ([4], Lemma 1.2).

3.1 Toric models for Looijenga pairs of length $n \leq 9$

Among Looijenga pairs there are some special ones which can be used to classify and analyze all the others, namely toric models and minimal pairs.

Definition 3.1.1. A Looijenga pair $(\bar{Y}, \bar{D})$ is a *toric pair* if $\bar{Y}$ is a smooth projective toric surface and $\bar{D} = \bar{Y} \setminus (\mathbb{C}^*)^2$ is the toric boundary. Now let $\pi : Y \to \bar{Y}$ be a sequence of interior blowups and let D be the strict transform of $\bar{D}$; we call π a *toric model* for the Looijenga pair (Y, D).

In other words, we say that (Y, D) admits a toric model if there exists a sequence of interior blow-downs $\pi : (Y, D) \to (\bar{Y}, \bar{D})$ where $(\bar{Y}, \bar{D})$ is toric. We explicitly note that the charge of a toric pair is equal to zero.

Remark 3.1.2 ([7]). The general theory of smooth projective toric varieties implies that the isomorphism type of a toric Looijenga pair is determined by its cycle of integers.

Changing perspective, given any Looijenga pair we can always contract a sequence of (-1)-curves on it until we get to a pair (Y', D'), that we will call *minimal*, where Y' is a minimal rational surface. We have the following result by Miranda [15]. Note that from now until the end of the section we will assume that the divisor D does not contain any (-1)-curves.

Theorem 3.1.3. *Let (Y, D) be a negative definite Looijenga pair with $n \geq 4$. Then Y can be blown down to $\mathbb{P}^1 \times \mathbb{P}^1$ so that D is mapped to the standard square $D' = (\mathbb{P}^1 \times \{0, \infty\}) \cup (\{0, \infty\} \times \mathbb{P}^1)$*

We observe that we can always arrange the sequence of blowups from $\mathbb{P}^1 \times \mathbb{P}^1$ to (Y, D) so that we first perform all the toric blowups and then all the interior blowups, thus every negative definite anticanonical pair admits a map to a toric pair (with an exceptional cycle of the same length) which consists of a sequence of interior blowups, or, equivalently, every negative definite Looijenga pair admits a toric model. Let's focus on toric pairs for which $n \leq 9$.

Proposition 3.1.4. *Every negative definite Looijenga pair* (Y, D) *with cycle* D *of length* $n = 6, 7$ *or* 9 *can be blown down, for each* n*, to one common toric pair* $(T_{(n)}, G_{(n)})$ *with length*$(D) =$ *length*$(G_{(n)})$*. Looijenga pairs* (Y, D) *with length* $n = 8$ *can always be blown down to one of the two toric pairs* (T_i, G_i) *or* (T_{ii}, G_{ii}) *whose toric boundaries are described in figure 4, along with the ones for* $n = 6, 7, 9$*.*

Proof. We begin with the case where $n = 6$. From Miranda's theorem we know that there always exists a ruling on (Y, D) such that two disjoint components of D are sections of it. Thus, up to symmetry, either D_6 and D_3 are sections (a), or D_6 and D_2 are sections (b). Consider fibres of this ruling which do not contain any component of D: they are always chains of interior (-2)-curves with two (-1)-curves at the ends of the chain intersecting D. Indeed every negative definite Looijenga pair (Y, D) is obtained from a Looijenga pair $(\bar{Y}, \bar{D})$ such that $\bar{Y}$ is a $\mathbb{P}^1$-bundle, by a sequence of blowups. Therefore the fibres of the ruling on $\bar{Y}$ are smooth irreducible curves of self intersection 0. Since (Y, D) is obtained from $(\bar{Y}, \bar{D})$ through a series of (either toric or interior) blowups, then the fibres of the ruling on Y not containing any component of D have to be chains of the type we described above. Fibres containing components of D have a similar configuration: let $f = \cup F_i$ be such a fibre. Then some of the curves F_i are components of D and have no restrictions on their self intersections (other than the negative definiteness condition) while the others will be arranged in chains of (-2)-curves with a (-1)-curve at the end that intersects D.

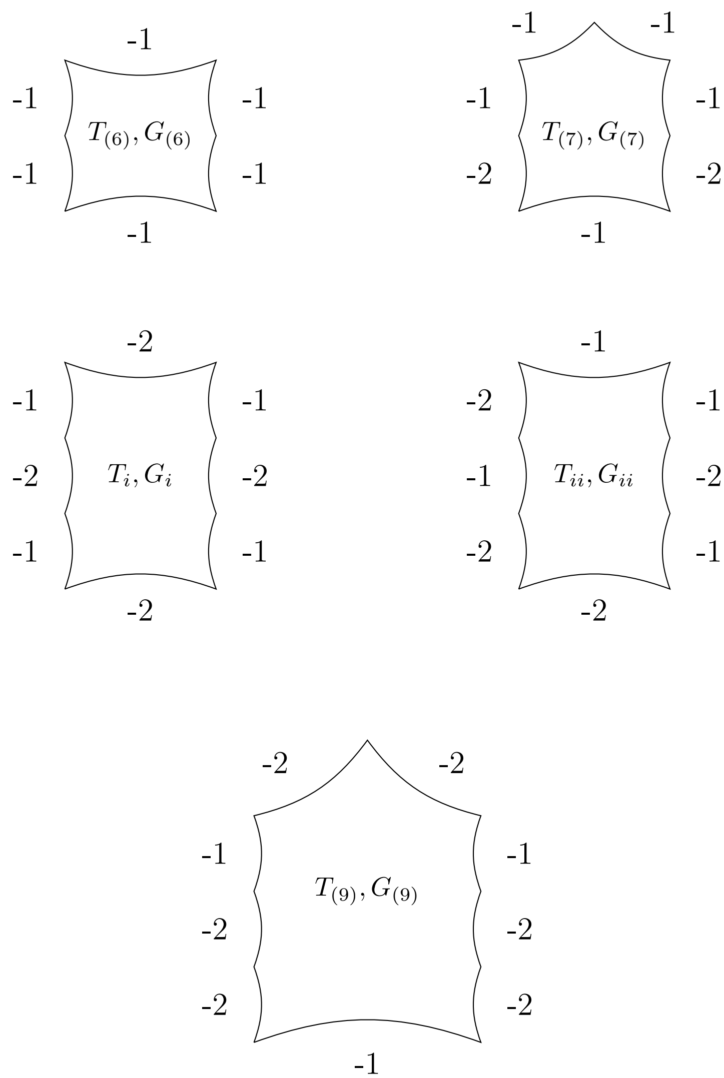


Figure 4. Boundary cycles of the common toric pairs

Now we can contract all singular fibres in the ruling which do not contain any components of D (always on the most negative section) until they are irreducible fibres and blow down singular fibres containing components of D to a chain contained in D. We get a map to a new pair $\pi : (Y, D) \rightarrow (\bar{Y}, \bar{D})$ where $\bar{Y}$ is a toric surface and $\bar{D} = \pi(D)$ its toric boundary. Let $\bar{D}_i$, for $i = 1, \ldots, 6$, be the irreducible components of $\bar{D}$.

Suppose we are in case (a). Then $\bar{D}_1 + \bar{D}_2$ and $\bar{D}_4 + \bar{D}_5$ are fibers of the ruling on $(\bar{Y}, \bar{D})$ induced by the one on (Y, D), therefore they must be pairs of (-1)-curves. Moreover, the fact that $(\bar{Y}, \bar{D})$ is a toric pair implies that

$$-\sum \bar{D}_i^2 = 3 \cdot n - 12 \tag{3.1}$$

hence, in our case, $-\sum \bar{D}_i^2 = 6$, so that $-\bar{D}_6^2 - \bar{D}_3^2 = 2$. Furthermore, we know that $-D_3^2 \geq 2$ and $-D_6^2 \geq 2$, and by the algorithm we used we must have $|\bar{D}_3^2 - \bar{D}_6^2| \leq 1$, therefore $\bar{D}_6^2 = -1 = \bar{D}_3^2$. In this case remark 3.1.2 implies that $(\bar{Y}, \bar{D})$ is isomorphic to the pair $(T_{(6)}, G_{(6)})$, where $T_{(6)}$ is a Del Pezzo surface of degree 6 and $G_{(6)}$ is the cycle of (-1)-curves contained in it.

Next, suppose we are in case (b). Then $\bar{D}_1$ has to be a simple fibre with self intersection equal to zero. Similarly, $\bar{D}_3 + \bar{D}_4 + \bar{D}_5$ is a singular fibre and there are only two possible arrangements of self intersections for this triple of curves:

i. $\bar{D}_3^2 = -1$, $\bar{D}_4^2 = -2$ and $\bar{D}_5^2 = -1$

ii. $\bar{D}_3^2 = -2$, $\bar{D}_4^2 = -1$ and $\bar{D}_5^2 = -2$

Let us start with (i). Using (3.1) we get that $-\bar{D}_6^2 - \bar{D}_3^2 = 2$, thus, using the argument given for case (a), we can assume that $-\bar{D}_6^2 = -1 = -\bar{D}_3^2$ and $(\bar{Y}, \bar{D})$ has associated cycle of integers $(0, 1, 1, 2, 1, 1)$.

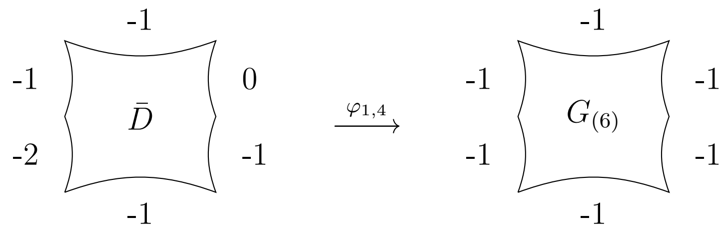
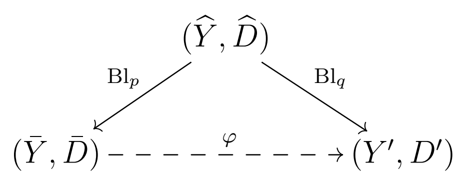


Figure 5. Case (b)-i

Now we observe (figure 5) that there exists a ruling on $(\bar{Y}, \bar{D})$ with sections given by the curves $\bar{D}_1$ and $\bar{D}_4$ with self intersections respectively 0 and -2. Since the cycle D we started with is negative definite (with no (-1)-curves on it), then it follows that D_1^2 and D_4^2 are both less or equal to -2. This means that when blowing down on singular fibres meeting D_1 and D_4 we contracted (at least) two (-1)-curves intersecting D_1 and none intersecting D_4, and we can always change that and contract one curve on D_4 and one on D_1. In other words, there exist an *elementary transformation* φ



given by the composition of the blowup of a smooth point p on $\bar{D}_1$ and the blowdown of a (-1)-curve intersecting $\widehat{D}_4$, the strict transform of $\bar{D}_4$, to a smooth point q on D'_4, and a map $\psi : (Y, D) \to (\widehat{Y}, \widehat{D})$ such that $\mathrm{Bl}_p \circ \psi = \pi$. This gives us a new toric model for the anticanonical pair we started with, $(Y, D) \to (Y', D')$ and the diagram:

$$\begin{array}{ccccc} (Y,D) & & & & \\ & \overset{\psi}{\searrow} & & & \\ \Big\downarrow \pi & & (\widehat{Y},\widehat{D}) & & \\ & \swarrow^{\mathrm{Bl}_p} & & \searrow^{\mathrm{Bl}_q} & \\ (\bar{Y},\bar{D}) & & \overset{\varphi}{\dashrightarrow} & & (Y',D') \end{array}$$

where D' is a cycle of six (-1)-curves (figure 5). As a consequence, the toric pair (Y', D') is again isomorphic to the anticanonical toric pair $(T_{(6)}, G_{(6)})$.

Remark 3.1.5. From now on let us denote by $\varphi_{i,j}$ the elementary transformation from $(\bar{Y}, \bar{D})$ to (Y', D') that is given by the composition of the blowup of a smooth point p on $\bar{D}_i$ and the blowdown of a (-1)-curve intersecting $\widehat{D}_j$, the strict transform of $\bar{D}_j$, to a smooth point q on D'_j (when it exists, i.e., when there is a ruling of $(\bar{Y}, \bar{D})$ s.t. $\bar{D}_i$ and $\bar{D}_j$ are sections). For the birational map φ described above, we would have $\varphi = \varphi_{1,4}$.

Finally, in case (ii), we have $-\bar{D}_0^2 - \bar{D}_3^2 = 1$ thus $\bar{D}$ is associated to the cycle of integers $(0, 0, 2, 1, 2, 1)$ (see figure 6). First we use the existence of the ruling with sections given by $\bar{D}_1, \bar{D}_3$ to get to the pair (Y', D') with D' given by the cycle of integers $(1, 0, 1, 1, 2, 1)$ through the elementary transformation $\varphi_{1,3}$. Then we use the ruling with sections given by $\bar{D}_2, \bar{D}_5$ to get to the pair (Y'', D'') with D'' given by the cycle of integers $(1, 1, 1, 1, 1, 1)$. We have obtained again a map made of of interior blow-downs from (Y, D) to $(T_{(6)}, G_{(6)})$, thus proving the proposition for $n = 6$.

For $n = 7, 8, 9$ we proceed similarly, proving the required statement case by case. First we distinguish different cases, based on the list of possible arrangement of the two sections contained in D, up to symmetry. Then for each case we suppose to contract the singular fibres of the given ruling for (Y, D) following the proce-

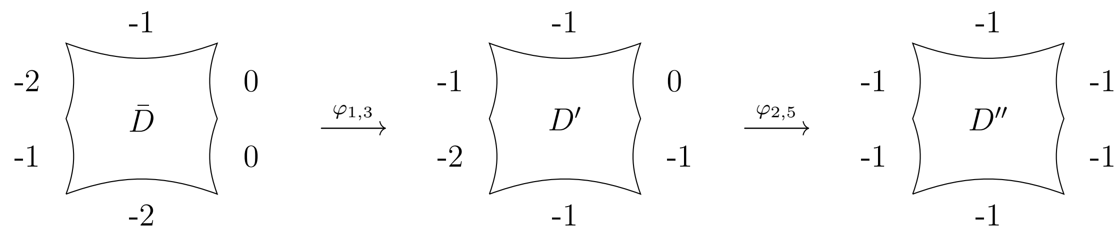


Figure 6. Case (b)-ii

dure discussed above until we end up with a toric pair $(\bar{Y}, \bar{D})$ of the same length. We make a list of all such toric pairs, depending on the possible arrangements of fibers and using (3.1) to determine the cycles of self intersections. Finally, for all toric pairs that are different from the ones described in figure 4 we use a sequence of elementary transformations to obtain the toric model we are looking for. For $n = 7$ the common toric pair $(T_{(7)}, G_{(7)})$ is obtained from $(T_{(6)}, G_{(6)})$ through a toric blowup. There are two possible choices for the positions of the two sections in D and each of these admits different toric pairs, depending on the self intersection of the fibres: these toric pairs and the corresponding elementary transformations are described in table 1. For $n = 8$ we fix two toric pairs which are, again, obtained from $(T_{(7)}, G_{(7)})$ via one appropriate toric blowup: (T_i, G_i) has cycle of integers $(1, 2, 1, 2, 1, 2, 1, 2)$ and (T_{ii}, G_{ii}) is associated to $(1, 2, 1, 2, 2, 1, 2, 1)$. In this case the possible configurations of the sections in D are three: for each of them table 2 lists all the toric pairs we could get and the relative elementary transformations used to obtain either (T_i, G_i) or (T_{ii}, G_{ii}). Finally, for $n = 9$, $(T_{(9)}, G_{(9)})$ is constructed from (T_{ii}, G_{ii}) through a toric blowup in such a way that the correspondent cycle is given by $(2, 2, 1, 2, 2, 1, 2, 2, 1)$. Here there are again three possible configurations of sections contained in D. The list of toric pairs and relative elementary transformations can be found in table 3 at the end of the chapter. $\square$

Sections: D_3 and D_7	
Toric pair, cycle of integers	Elementary transformations
(1,1,1,1,2,1,2)	Not needed
Sections: D_2 and D_7	
Toric pair, cycle of integers	Elementary transformations
(0,1,1,2,2,1,2)	$\varphi_{4,1}$
(0,1,1,3,1,2,1)	$\varphi_{4,1}$
(0,0,2,2,1,3,1)	$\varphi_{3,1}, \varphi_{6,2}$

Table 1. Elementary transformations for $n = 7$

Remark 3.1.6. Notice that the pairs a, b in table 2 are associated to the same ruling and share the same arrangement of fibers: $\bar{D}_1, \bar{D}_2, \bar{D}_3$ with self intersections $-1, -2, -1$ and $\bar{D}_5, \bar{D}_6\bar{D}_7$ again with self intersections $-1, -2, -1$. The only differences between them are the self intersections of the two sections. This is to take into account the negative definite Looijenga pairs (Y, D) obtained from a if none of the interior blowups are performed on either $\bar{D}_4$ or $\bar{D}_8$. Indeed in this case there will be no reducible fibres intersecting D_4 and D_8, who will still have self intersections equal to -2. As a consequence such pairs can only be contracted to the toric pair a and not to the toric pair b (in other words there does not exists an elementary transformation from pair a to pair b). The same reasoning applies to pairs c, d. We observe explicitly that a and d are the only two pairs who admit a sequence of elementary transformations taking them to (T_i, G_i) but not one to (T_{ii}, G_{ii}), while for all the other toric pairs listed in table 2 there exists a sequence of maps $\varphi_{i,j}$ bringing them to (T_i, G_i), (T_{ii}, G_{ii}) or even both of them.

Remark 3.1.7. A direct consequence of the result above is that every non negative Looijenga pair with cycle D of length $n = 6, 7, 9$ can be contracted to a pair $(Y_{(n)}, D_{(n)})$ that is obtained from the toric pair $(T_{(n)}, G_{(n)})$ of the appropriate length performing an interior blowup on every component of self intersection -1. Similarly,

Sections: D_4 and D_8		
Toric pair, cycle of integers	Elementary transformations	Toric model
a. (1,2,1,2,1,2,1,2)	Not needed	(T_i, G_i)
b. (1,2,1,1,1,2,1,3)	$\varphi_{6,3}, \varphi_{8,5}$	(T_{ii}, G_{ii})
(2,1,2,1,1,2,1,2)	Not needed	(T_{ii}, G_{ii})
Sections: D_3 and D_8		
Toric pair, cycle of integers	Elementary transformations	Toric model
(1,1,2,1,2,2,1,2)	Not needed	(T_{ii}, G_{ii})
(1,1,1,1,2,2,1,3)	$\varphi_{8,3}$	(T_{ii}, G_{ii})
Sections: D_2 and D_8		
Toric pair, cycle of integers	Elementary transformations	Toric model
c. (0,1,1,2,2,2,1,3)	$\varphi_{4,1}, \varphi_{8,3}$	(T_{ii}, G_{ii})
d. (0,2,1,2,2,2,1,2)	$\varphi_{5,1}$	(T_i, G_i)
(0,1,1,2,3,1,2,2)	$\varphi_{8,2}, \varphi_{5,1}$	(T_{ii}, G_{ii})
(0,1,1,3,1,3,1,2)	$\varphi_{4,1}, \varphi_{4,1}, \varphi_{6,3}$	(T_{ii}, G_{ii})
(0,1,1,3,2,1,3,1)	$\varphi_{4,1}, \varphi_{7,2}$	(T_{ii}, G_{ii})
(0,1,1,4,1,2,2,1)	$\varphi_{4,1}, \varphi_{4,1}$	(T_{ii}, G_{ii})
(0,1,2,1,4,1,2,1)	$\varphi_{5,1}, \varphi_{3,8}, \varphi_{5,2}$	(T_{ii}, G_{ii})
(0,1,2,2,2,1,4,0)	$\varphi_{7,1}, \varphi_{3,8}, \varphi_{7,2}$	(T_{ii}, G_{ii})
(0,1,2,3,1,2,3,0)	$\varphi_{7,1}, \varphi_{4,1}$	(T_{ii}, G_{ii})
(0,1,3,1,3,1,3,0)	$\varphi_{7,1}, \varphi_{7,1}, \varphi_{4,8}$	(T_{ii}, G_{ii})

Table 2. Elementary transformations for $n = 8$

Looijenga pairs with cycle of length $n = 8$ can always be contracted to either one of the anticanonical pairs (Y_i, D_i) or (Y_{ii}, D_{ii}), which again are obtained respectively from (T_i, G_i) and (T_{ii}, G_{ii}) through four interior blowups on the (-1)-curves. For ease of notation let us assume, without loss of generality, that these blowups are performed on components $2, 3, 4, 5, 7$ for $n = 7$, $1, 3, 5, 7$ for D_i, $1, 3, 6, 8$ for D_{ii} and $1, 4, 7$ for $n = 9$. Note that, since all the irreducible components of the toric boundaries described in figure 4 have self intersections less than or equal to 2 in absolute value, then such a pair $(Y_{(n)}, D_{(n)})$ can always be constructed: from now on we will refer to these pairs as *elliptic* pairs, since they are deformation euivalent to elliptic surfaces and their anticanonical cycles are singular fibres of the corresponding elliptic fibration.

Let us focus on the two elliptic pairs with cycles of length 8. Clearly the cycles of integers for these two pairs coincide, since both D_i and D_{ii} are made of eight curves of self intersection (-2), thus we may ask ourselves if these anticanonical pairs are isomorphic or not. Observe that they can be distinguished by the fundamental group of the complement of their anticanonical divisors, $U_i := Y_i \setminus D_i$ and $U_{ii} := Y_{ii} \setminus D_{ii}$. More precisely, $\pi_1(U_i) = \mathbb{Z}/2\mathbb{Z}$ while $\pi_1(U_{ii}) = 0$. This can be seen recalling that if (Y, D) is obtained from a toric surface through a sequence of interior blowups, then $\pi_1(Y \setminus D) = N/\langle v_1, \ldots, v_p \rangle$, where N is the lattice containing the fan of the toric variety and $v_1, \ldots, v_p \in N$ are the primitive vectors corresponding to the curves of the toric boundary where the blowups are performed. Now, for Y_{ii}, the set of vectors $\{w_1, \ldots, w_4\}$ corresponding to the four (-1)-curves contains a basis for the lattice N, thus the fundamental group of U_{ii} is trivial, while the set of vectors $\{w_1, \ldots, w_4\}$ in the fan associated to T_i corresponding to the four (-1)-curves share the linear relations $v_1 = -v_3, v_2 = -v_4, v_1 + v_2 = 2e_1, v_1 + v_4 = 2e_2$ where e_1, e_2 are generators for the lattice N, thus $\pi_1(U_i) = \mathbb{Z}/2\mathbb{Z}$.

This remark, together with the proof of proposition 3.1.4 allows us to refine the result of that proposition for Looijenga pairs of length eight.

Corollary 3.1.8. *Let (Y, D) be a negative definite Looijenga pair with anticanonical cycle of length $n = 8$. Then, if $\pi_1(Y \setminus D) = \mathbb{Z}/2\mathbb{Z}$ it can only be contracted to the toric pair (T_i, G_i), otherwise if $\pi_1(Y \setminus D)$ is trivial it can be contracted to both (T_{ii}, G_{ii}) and (T_i, G_i).*

Proof. Let (Y, D) be such that $\pi_1(Y \setminus D) = \mathbb{Z}/2\mathbb{Z}$. Then (Y, D) must be obtained from the toric model (T_i, G_i), since, as we saw in remark 3.1.7, this is the only way to keep the fundamental group non trivial. More precisely, the Looijenga pair (Y, D) which is obtained from (Y_i, D_i) blowing up points only on the odd components of

D_i. Observe now that the fact that all the interior blowups happen on the odd components of D_i implies that the fundamental group of $Y \setminus D$ is equal to the fundamental group of $(Y_i \setminus D_i) = \mathbb{Z}/2\mathbb{Z}$. Thus, $\pi_1(Y \setminus D) = \mathbb{Z}/2\mathbb{Z}$ if and only if (Y, D) is of the kind described above, in which case it only admits a birational map to (T_i, G_i) On the other hand, if (Y, D) is a negative definite Looijenga pair of length eight with trivial fundamental group, then it can be contracted to either of the toric models listed in proposition 3.1.4 (see table 2). □

Proposition 3.1.4 allows us to give a complete description of the number of deformation of Looijenga pairs (Y, D) with fixed D of length $6 \leq n \leq 9$. In order to do that we need to recall briefly the description of the Mordell-Weil group of the surfaces $(Y_{(n)}, D_{(n)})$ and $(Y_i, D_i), (Y_{ii}, D_{ii})$ constructed in remark 3.1.7,and more generally on their automoprhism groups. Given a rational elliptic surface Y with section, the Mordell-Weil group can be thought of as a subgroup of the automorphism group of the surface itself and it acts transitively on sections of $Y \to \mathbb{P}^1$: each element of the Mordell-Weil group gives an automorphism of Y that acts as a translation on (smooth) fibres. Thus if (Y, D) is an elliptic Looijenga pair, this group can also be identified with a subgroup of the generalized automoprhism group $\overline{\mathrm{Aut}}(Y, D)$ containing all automorphisms of (Y, D) fixing D set-wise but not component wise. Note that the automorphism group of (Y, D) as defined in 3.0.3 is also contained in $\overline{\mathrm{Aut}}(Y, D)$ as a subgroup, more precisely it can be thought of as the kernel of the map $\gamma : \overline{\mathrm{Aut}}(Y, D) \to \mathcal{D}_n$, with n equal to the length of D, that sends each automorphism ϕ in $\overline{\mathrm{Aut}}(Y, D)$ to the element of the dihedral group which corresponds to the action of ϕ to the dual graph for D. Hence we get the sequence of maps:

$$0 \to \mathrm{Aut}(Y, D) \to \overline{\mathrm{Aut}}(Y, D) \xrightarrow{\gamma} \mathcal{D}_n \tag{3.2}$$

Now for the elliptic Looijenga pairs of remark 3.1.7, we can prove the following result.

Proposition 3.1.9. *Let* $(Y_{(n)}, D_{(n)})$ *be the elliptic Looijenga pair of length* $n = 6, 7$*. Then the automophism group* $\overline{\mathrm{Aut}}(Y_{(n)}, D_{(n)})$ *projects onto the dihedral group* $\mathcal{D}_n$ *of order* $2n$*. Similarly the automorphism group* $\overline{\mathrm{Aut}}(Y_{ii}, D_{ii})$ *admits a sujective map onto* $\mathcal{D}_8$*. The automorphism group* $\overline{\mathrm{Aut}}(Y_i, D_i)$ *instead admits a surjective map to* $\mathcal{D}_4$*, but not onto* $\mathcal{D}_8$*. Similarly, the automorphism group* $\overline{\mathrm{Aut}}(Y_{(9)}, D_{(9)})$ *admits a surjective map to* $\mathcal{D}_3$*, but not onto* $\mathcal{D}_9$*.*

Proof. The proof will proceed as follows: for each n listed above, first we will show that the Mordel Weil group admits a surjective map onto $\mathbb{Z}/n\mathbb{Z}$, then we will construct explicitely an involution of the elliptic anticanonical pair thus showing that the map γ in sequence 3.2 is surjective.

For $n = 6$, then there is a (-1)-curve intersecting every component of $D_{(6)}$. Since each of these (-1)-curves is a section for the elliptic surface, and $\mathrm{MW}(Y_{(6)})$ acts transitively on sections, then there must esists an automorphism of $Y_{(6)}$ mapping $D_{(6),i} \mapsto D_{(6),i+1}$ and thus acting as a rotation of order 6 on the dual graph. We therefore get a surjective map $\mathrm{MW}(Y_{(6)}) \to \mathbb{Z}/6\mathbb{Z}$. As for the involution, let us describe the toric pair of length 6 through its fan. Let N be the lattice isomorphic to $\mathbb{Z}^2$ and let e_1, e_2 be the vectors corresponding to $(1, 0), (0, 1)$. Then the cones of the fan for $(T_{(6)}, G_{(6)})$ are generated by the vectors $\{e_1, -e_2\}$,$\{-e_2, -e_1-e_2\}$,$\{-e_1-e_2, -e_1\}$,$\{-e_1, e_2\}$,$\{e_2, e_1+e_2\}$ and $\{e_1+e_2, e_1\}$. The lattice isomorphism mapping $e_1 \mapsto e_2$ induces an involution of the toric Looijenga pair which lifts to an involution of $(Y_{(6)}, D_{(6)})$ if the 6 interior blowups are made at the appropriate points. Since this involution acts on the dual graph of $D_{(6)}$ as a reflection our claim is proven.

Similarly, if $n = 7$, since there is a curve intersecting the second and third components of $D_{(7)}$ (see remark 3.1.7) then again there exists an automorphism of the elliptic pair of length 7 mapping $D_{(7),i} \mapsto D_{(7),i+1}$ and thus acting as a rotation of order 7 on the dual graph. The fan for the toric pair of length 7 is obtained from the one above adding the ray generated by $-e_1 + e_2$ and the required involution is obtained as a lift of the one induced on $(T_7, G_{(7)})$ by the linear map sending e_2 to $-e_1$ (and fixing the new ray). The case of (Y_{ii}, D_{ii}) is treated similarly noticing that components $1, 8$ are adjacent and both intersect a (-1)-curve given by an interior blowup. Moreover the linear map used for the previous case still gives us an involution of (Y_{ii}, D_{ii}) following the same procedure as above once we observe that the toric pair (T_{ii}, G_{ii}) can be constructed adding the rays $e_1 + 2e_2, -2e_1 - e_2$ to the fan we gave for the case $n = 6$.

Now consider (Y_i, D_i). Given the arrangement of the interior blowups, the rotation of the dual graph with greatest order is the one induced by the automorphism mapping $D_{i,j}$ to $D_{i,j+2}$ which has order 4. Therefore we get a surjective map $\mathrm{MW}(Y_i) \to \mathbb{Z}/4\mathbb{Z}$, but not one onto $\mathbb{Z}/8\mathbb{Z}$ because $\mathrm{MW}(Y_i) \cong \mathbb{Z}/4\mathbb{Z}$ (see [17], page 82). An involution can be constructed as usual, once we notice that the fan for the toric pair of length eight (T_i, G_i) can be obtained from the one of the toric pair of length seven adding the ray $e_1 - e_2$ and using the same linear map.

Finally, consider the pair $(Y_{(9)}, D_{(9)})$. Again, because of the arrangement of the interior blowups, the rotation of the dual graph with greatest order is the one induced by the automorphism mapping $D_{i,j}$ to $D_{i,j+3}$ which has order 4. Therefore we get a surjective map $\mathrm{MW}(Y_{(9)}) \to \mathbb{Z}/3\mathbb{Z}$, but not one onto $\mathbb{Z}/9\mathbb{Z}$, again because $\mathrm{MW}(Y_{(9)}) \cong \mathbb{Z}/3\mathbb{Z}$ (see [17], page 82). Moreover, an involution can be constructed as usual, once we notice that the fan for the toric pair of length nine $(T_{(9)}, G_{(9)})$ can be obtained from the one of the toric pair (T_{ii}, G_{ii}) adding the ray $e_1 - e_2$ and using the same linear map. □

Theorem 3.1.10. *If $n = 6, 7$ or $n = 8$ and D has associated cycle of integers different from $(a, 2, b, 2, c, 2, d, 2)$ then there is one deformation type of negative definite Looijenga pairs (Y, D) of length n with fixed D. If $n = 8$ and D is of type $(a, 2, b, 2, c, 2, d, 2)$ there are two deformation types, distinguished by $\pi_1(U)$, where $U = Y \setminus D$. Finally, if $n = 9$, then there are at most three deformation types of negative definite Looijenga pairs (Y, D) of length 9 with fixed D.*

Proof. Let (Y, D), (Y', D') be negative definite Looijenga pairs of length $n = 6, 7$. Thanks to remark 3.1.7, we know that they can always be obtained from the elliptic pair of the same length pair through a sequence of m interior blowups, $\pi : (Y, D) \to (Y_{(n)}, D_{(n)})$ and $\pi' : (Y', D') \to (Y_{(n)}, D_{(n)})$. Let us fix labelings on $D, D', D_{(n)}$ so that $\pi(D_i) = D_{(n),i}$ and similarly $\pi'(D'_i) = D_{(n),i}$. Finally let us assume that they share the same cycle of integers $(a_1, \dots, a_n)$. Then there must be an element σ of the dihedral group of order $2n$ such that $(-D_1^2 - 2, \dots, -D_n^2 - 2) = (-D'^2_{\sigma(1)} - 2, \dots, -D'^2_{\sigma(n)} - 2)$. If σ is the identity map, then all the blowups happen on the same components of $D_{(n)}$ for both (Y, D) and (Y', D'), therefore they are deformation equivalent. Otherwise, proposition 3.1.9 guarantees that there exists an automorphism ϕ of the elliptic Looijenga pair of length n which maps $D_{(n),i} \mapsto D_{(n),\sigma(i)}$, therefore (Y, D) and (Y', D') are obtained from isomorphic pairs by performing the blowups on components of the anticanonical cycle which are identified via the isomorphism ϕ and again they are deformation equivalent.

Now let us assume that (Y, D), (Y', D') are negative definite Looijenga pairs of length eight with fixed cycle of integers $(a_1, \dots, a_8)$. If $\pi_1(Y \setminus D) = \pi_1(Y' \setminus D') = 0$, then both pairs can be contracted to (Y_{ii}, D_{ii}) and thanks to proposition 3.1.9 the same argument used above will show that the two Looijenga pairs are deformation equivalent. If $\pi_1(Y \setminus D) = \pi_1(Y' \setminus D') = \mathbb{Z}/2\mathbb{Z}$, then it follows from corollary 3.1.8

that (Y, D) and (Y', D') can only be contracted to the elliptic pair (Y_i, D_i) and the blowups happen only on the odd components of D_i. Since $\mathrm{Aut}(Y_i, D_i)$ admits a surjective map to $\mathcal{D}_4$ (proposition 3.1.9), then again the argument used in the previous cases allows us to show that (Y, D) and (Y', D') are deformation equivalent. Finally let us suppose $\pi_1(Y \setminus D) = \mathbb{Z}/2\mathbb{Z}$ and $\pi_1(Y' \setminus D') = 0$ and suppose that they are deformation equivalent. Then there would exists a diffeomorphism mapping (Y, D) to (Y', D') and sending D to D': this would imply in particular that the fundamental groups of $Y \setminus D$ and $Y' \setminus D'$ are isomorphic, thus giving a contradiction. Thus they cannot be deformation equivalent and the statement is proved.

If $n = 9$, then given we only have a surjective map from the elliptic pair $\mathrm{Aut}(Y_{(9)}, D_{(9)})$ to $\mathcal{D}_3$, there are at most three choices for the arrangement of the interior blowups which do not need to give an automorphism of the elliptic pair. Therefore there are at most three deformation types. □

Having information about the deformation types of negative definite Looijenga pairs is interesting also for the implications it can have on the deformation theory of cusp singularities.

Conjecture 3.1.11. *Let $(p \in X)$ be a cusp singularity. Then the set of smoothing components of its deformation space modulo the action of automorphisms of $(p \in X)$ is in bijective correspondence with the set of deformation types of negative definite Looijenga pairs (Y, D) such that D contracts to the dual cusp.*

3.2 On the action of $\mathbb{Z}/2\mathbb{Z}$ on a Looijenga pair

In this section and the following one, similarly to what we did for cusp singularities, we will characterize Looijenga pairs who admit a $\mathbb{Z}/2\mathbb{Z}$-action. More precisely we will give sufficient conditions for an involution on the Picard group of a Looijenga pair (Y, D) to lift to an involution of the pair itself. We start with a precise definition of involution of a given Looijenga pair.

Definition 3.2.1. An *involution* of a labeled Looijenga pair (Y, D) is an involution $j : Y \to Y$ such that $j(D_i) = D_{\sigma(i)}$ for each $i = 1, \ldots, n$, with σ an element of order two in the dihedral group of order $2n$. We say that j is *antisymplectic* if it reverses the orientation of D and it is free on $Y \setminus D$.

Remark 3.2.2. We will be mainly interested in involutions of Looijenga pairs which are are antisymplectic. Observe that if j is such an involution then $j(D_i) = D_{\sigma(i)}$ where σ is a reflection in the dihedral group of order $2n$.

Now, recall that Λ is defined to be the orthogonal complement of $D_1, \ldots, D_n$ in $\mathrm{Pic}(Y)$.

Definition 3.2.3. The canonical map

$$\phi_Y : \Lambda \longrightarrow \mathrm{Pic}^0(D) \cong \mathbb{G}_m \qquad L \mapsto L|_D$$

determined by restriction of line bundles is called the *period point* of Y

Note that the period map ϕ_Y depends on the orientation of D, since it is used in lemma 3.0.2 for the construction of the isomorphism $\mathrm{Pic}^{\circ} D \cong \mathbb{G}_m$. We have the following:

Theorem 3.2.4 (see [7] or [4]). *Let (Y, D), (Y', D') be labeled Looijenga pairs with D, D' of the same length and let*

$$\mu : \mathrm{H}^2(Y, \mathbb{Z}) \to \mathrm{H}^2(Y', \mathbb{Z})$$

be an isomorphism of lattices. Then $\mu = f^*$ for an isomorphism f of labeled Looijenga pairs compatible with the orientations if and only if the following hold:

a. $\mu([D_i]) = [D_i']$ *for all* i.

b. $\mu(\mathrm{Nef}\ Y) = \mathrm{Nef}\ Y'$.

c. $\phi_{Y'}(\mu(q)) = \phi_Y(q)$ *for all* $q \in \Lambda$.

Moreover if f and f' are two such isomoprhisms, then there exists a $\phi \in K(Y, D) := \ker(\mathrm{Aut}(Y, D) \to \mathrm{Aut}(\mathrm{Pic}\ Y))$ such that $f' = \phi \circ f$. Conversely, if $\phi \in K(Y, D)$, then $f' = \phi \circ f$ is an isomorphism from Y to Y' such that $(f')^* = \mu$.

A direct consequence of this result is the next theorem.

Theorem 3.2.5. *Let (Y, D) be a labeled Looijenga pair and let*

$$\theta : \mathrm{H}^2(Y, \mathbb{Z}) \to \mathrm{H}^2(Y, \mathbb{Z})$$

be an involution of lattices. Then $\theta = f^*$ for an automorphism f of the labeled Looijenga pair (Y, D) if and only if the following hold:

i. $\theta([D_i]) = [D_{\sigma(i)}]$ *for all* i.

ii. $\theta(\mathrm{Nef}\ Y) = \mathrm{Nef}\ Y$.

iii. $\phi_Y(\theta(q)) = \phi_Y(q)^{-1}$ *for all* $q \in \Lambda$.

Moreover f is an involution if $K(Y, D)$ is trivial.

Proof. We want to reduce ourselves to theorem 3.2.4. Let Y' be equal to Y and define D' by $D'_i := D_{\sigma(i)}$. Then (Y', D') is a labeled Looijenga pair. The map $\mu : \mathrm{H}^2(Y, \mathbb{Z}) \to \mathrm{H}^2(Y', \mathbb{Z})$ given by $\mu(L) := \theta(L)$ gives an isomorphism of lattices such that $\mu([D_i]) = \theta([D_i]) = D_{\sigma(i)} = D'_i$, therefore condition (a) is equivalent to (i). Moreover the condition $\theta(\mathrm{Nef}\, Y) = \mathrm{Nef}\, Y$ directly implies that $\mu(\mathrm{Nef}\, Y) = \mathrm{Nef}\, Y'$, giving that (b) is equivalent to (ii). Finally consider $\phi_{Y'}$. Using lemma 3.0.2, we get that

$$\phi_Y : \Lambda \to \mathrm{Pic}^0(D) \to \mathbb{G}_m, \qquad L \mapsto \prod_i \frac{s_{i+1}(p_{i,i+1})}{s_i(p_{i,i+1})}$$

where the s_i's are sections of $L|_{D_i}$ for each i. We can use the same sections to define $\phi_{Y'}$: indeed, define s'_i to be $s_{\sigma(i)}$. Then

$$\phi_{Y'} : \Lambda' \to \mathrm{Pic}^0(D') \to \mathbb{G}_m, \qquad L \mapsto \prod_i \frac{s'_{i+1}(p_{i,i+1})}{s'_i(p_{i,i+1})}$$

Now, if $\sigma(i) = j$, for some j, then $\sigma(i+1)$ has to be equal to $j-1$, because we assumed that σ is a reflection, thus it changes the orientation of D. This gives us

$$\frac{s'_{i+1}(p_{i,i+1})}{s'_i(p_{i,i+1})} = \left(\frac{s_j(p_{j-1,j})}{s_{j-1}(p_{j-1,j})}\right)^{-1}$$

therefore $\phi_{Y'}(q) = \phi_Y(q)^{-1}$ for all $q \in \Lambda = \Lambda'$. As a consequence, $\phi_{Y'}(\mu(q)) = \phi_Y(\theta(q))^{-1} = \phi(q)$ and condition (c) is equivalent to (iii). Hence we can apply theorem 3.2.4 and we get that $\theta = f^*$ for an isomorphim $f : (Y', D') \to (Y, D)$ of Looijenga pairs if and only if conditions (i), (ii) and (iii) hold. By construction of (Y', D') this isomorphism f can in fact be viewed as an automorphism of the labeled Looijenga pair (Y, D) which reverses the orientation of D. Finally consider $f^2 : (Y, D) \to (Y, D)$. It is an automorphism of (Y, D) mapping each D_i to itself. Therefore, again by theorem 3.2.4, if $K(Y, D)$ is trivial then f^2 has to be equal to the identity map and f is an involution of (Y, D).

□

Proposition 3.2.6. *Suppose θ is an isometry of* $\operatorname{Pic} Y$ *and σ is a reflection in the dihedral group of order $2n$ such that $\theta([D_i]) = [D_{\sigma(i)})]$ for all i and $\theta(\operatorname{Nef} Y_{\mathrm{gen}}) = \operatorname{Nef} Y_{\mathrm{gen}}$. Let S be the locus in the moduli space*

$$T = \operatorname{Hom}(\Lambda, \mathbb{C}^*)$$

of (marked) pairs (Y', D') deformation equivalent to (Y, D) for which there exists an isomorphim j with $\theta = j^*$ and $j(D_i') = D_{\sigma(i)}$. Then we have

$$S \cap T^{\mathrm{gen}} = \{\phi \in T^{\mathrm{gen}} \mid \phi \circ \theta(q) = \phi(q)^{-1} \text{ for all } q \in \Lambda\}$$

where

$$T^{\mathrm{gen}} = T \setminus \bigcup_{\alpha \in \Phi} \{\chi \in T \mid \chi(\alpha) = 1\}$$

and Φ is the set of roots in $\operatorname{Pic} Y$

Proof. It follows from theorem 3.2.5 and the structure of the moduli space of marked Looijenga pairs described in [7]. □

3.3 Equivariant minimal model program for pairs (Y, D)

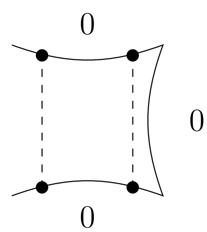
For this section we assume that the length n of D is greater than or equal to 4 and that D does not contain any curves with self intersection -1. Consider a negative definite Looijenga pair and assume it admits an antisymplectic involution (as a labeled Looijenga pair) j. Let (Z, F) be the quotient induced by the action: we know that Z contains four singularities of type A_1 lying in pairs on two of the irreducible components of F. Moreover note that we have $K_Z + F = 0$ in $\operatorname{Cl}(Z) \otimes \mathbb{Q}$. Let us study minimal models for (Z, F).

Theorem 3.3.1. *Let (Z, F) be as above. Then there exists a sequence of contractions $(Z, F) \to (Z_1, F_1) \to \cdots \to (Z_m, F_m) = (Z', F')$ such that (Z', F') satisfies:*

i. *(Z', F') is a minimally ruled surface with four singularities of type A_1.*

ii. *these singularities lie in pairs on two distinct fibres and on 2 distinct sections of self intersection equal to 0.*

iii. *F' consists of 3 rational curves of self intersection 0, one of which is a fibre of the ruling, while the other two are sections and contain the four A_1 singularities.*



Before proving this theorem let us state a more general result on the minimal model program for projective surfaces with A_1 singularities:

Theorem 3.3.2. *Let Z be a projective surface containing isolated singularities of type A_1. Then there exists a sequence of contractions $Z \longrightarrow Z'$ such that Z' satisfies one of the following*

i. *Z' has at worst A_1 singularities and $K_{Z'}$ is nef.*

ii. *Z' has at worst A_1 singularities and it admits a map $Z' \xrightarrow{\varphi} C$ where C is a curve and the fibres of φ are smooth rational curves.*

iii. *Z' is a Del Pezzo surface with at worst A_1 singularities and the Picard number is $\rho(Z') = 1$.*

Proof. First, using the cone theorem in its generalized version (see for example [10], p.76, Theorem 3.7), we know that the contraction map $c_R : Z \to Z'$ exists for every extremal ray R contained in the cone of curves of Z such that $R \cdot K_Z < 0$. Moreover if C is a rational curve such that $[C] \in R$ then we get:

1. If $C^2 < 0$, then Z' has dimension 2 and $\rho(Z') = \rho(Z) - 1$. Here every curve whose class is contained in R is contracted to one point p and in fact the fiber over this point p, $c_R^{-1}(p)$, consist of one irreducible curve

2. If $C^2 = 0$ then Z' has dimension 1 and $\rho(Z) = 2$. In this case an argument analogous to the one used for the smooth case shows that the fibres are connected and irreducible. Moreover they are still smooth and rational: let F be a fibre. Then by assumption $F \cdot K_Z < 0$ and, using the adjunction formula for the singular case we get $K_Z \cdot F + F^2 = 2p_a(F) - 2 + \mathrm{Diff}$, where Diff is always a non negative quantity and $F^2 = 0$. Therefore we must have $2p_a(F) - 2 < 0$, which implies that $p_a(F) = 0$ so that F is smooth and rational, as expected.

3. If $C^2 > 0$ then Z' is a point and $\rho(Z) = 1$

Let us focus on case 1. There are only 2 types of curves C satisfying $C^2 < 0$ and $C \cdot K_Z < 0$: either (-1)-curves (as for smooth surfaces) or rational smooth curves passing through one surface singularity. To see this suppose that C goes through two or more singularities. Resolve these singularities, and consider the correspondent map $\pi : \widetilde{Z} \to Z$. Let E_1 and E_2 be the exceptional divisors and $\widetilde{C}$ the strict transform of C, then

$$\widetilde{C} = \pi^*(C) - \mu_1 E_1 - \mu_2 E_2$$

where $\mu_1, \mu_2 \in \frac{1}{2}\mathbb{Z}$ and they are non negative. We have, on one side

$$\widetilde{C}^2 = C^2 - 2\mu_1^2 - 2\mu_2^2 < 0 \tag{3.3}$$

and on the other side

$$\widetilde{C} \cdot K_Z = C \cdot K_Z < 0 \tag{3.4}$$

Therefore, given that $\widetilde{Z}$ is smooth, 3.3 and 3.4 imply that C is a (-1)-curve. Thus, by 3.3

$$-1 = C^2 - 2\mu_1^2 - 2\mu_2^2 \implies -C^2 = 1 - 2\mu_1^2 - 2\mu_2^2$$

so that

$$0 < 1 - 2\mu_1^2 - 2\mu_2^2 \implies 2\mu_1^2 + 2\mu_2^2 < 1$$

The latter is impossible unless there is in fact only one exceptional curve E and the corresponding $\mu = \frac{1}{2}$, in which case $\widetilde{C} \cdot E = \pi^*(C) - \frac{1}{2}E = -\frac{1}{2}(-2) = 1$ hence $\widetilde{C}$ meets E trasversally and there can be only one singularity on C. Locally $p \in C \subset Z$ is analytically isomorphic to $0 \in (u = 0) \subset \mathbb{C}^2/\frac{1}{2}(1,1)$.

The contraction of a (-1)-curve, exactly as for the smooth case, corresponds to a standard blow up. As for the second type of curve, the map $c_R : Z \to Z'$ is such that $c_R(C) = p$, where p is a smooth point on Z. Indeed, suppose we resolve the singularity through C, we get a map $\pi : \widetilde{Z} \to Z$. As we did above, let E be the exceptional divisor and $\widetilde{C}$ the strict transform of C: we can now first contract E and then contract the image of $\widetilde{C}$ which has become a (-1)-curve, thus the composition of these two contractions, ϕ, gives a new smooth surface and in particular $E, \widetilde{C}$ are mapped to a smooth point.

$$\begin{array}{ccccccc}
E \cup \widetilde{C} & \subset & \widetilde{Z} & \xrightarrow{\pi} & Z & \supset & C \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
p & \in & \overline{Z} & \xrightarrow{\cong} & Z' & \ni & p'
\end{array}$$

Now the proof continues as in the usual (smooth) case:

- We start with the surface Z. If K_Z is a nef divisor then we stop: we have obtained the result stated in (i).
- Otherwise there exists an extremal ray in the cone of curves of Z whose intersection with the canonical divisor is negative: the contraction will produce one of the outcomes described at the beginning of this proof. If we are in case 2 or 3 then we stop and we get the result stated in (ii) or (iii).
- If we are in case 1 then we go back to the first step and keep iterating the algorithm.

Note that given what we said about the possible types of curves that get contracted, we end up with a surface with at most the same number of singularities Z had. □

Remark 3.3.3. Let us analyze cases (ii) and (iii) more in detail. Firstly, there is only one singular Del Pezzo surface with Picard number equal to 1 only containing A_1 singularities: it is the weighted projective space $\mathbb{P}(1,1,2)$ (cfr [1], chapter 8). Now let us consider the second possible outcome of the minimal model program applied to Z. So far we know that we get a $\mathbb{P}^1$-fibration $Z' \xrightarrow{\varphi} C$ where C is a curve, but we can actually say more about the arrangement of singularities along the fibers of this ruling: thanks to lemma 3.4 in [12], given a fibre F then Z' is smooth over F or F contains exactly two A_1 singularities or there's a unique singularity, which is a D_n singularity, along F. Since the last case cannot happen with our initial assumptions, then if there are singularities along F they must be exactly two and of type A_1.

We can now go back to the theorem stated at the beginning of this section

Proof of theorem 3.3.1. Given a pair (Z, F) obtained as a quotient of a negative definite anticanonical pair (Y, D), run the minimal model program as described in the result we just proved: we know there are three possible outcomes for Z'. Since the surface we start with is rational with four singularities of type A_1, then it clearly has to be either as in (ii) or as in (iii). Moreover, there are no curves C such that $C^2 = -1/2$ and $K_Z \cdot C = -1$ at the same time. Indeed, suppose C is such a curve: then we must have $(C \cdot F)_p = 1/2(2k+1)$ where p is the unique A_1 singularity on C and k is a non negative integer. But this contradicts the hypothesis that $C \cdot F = C \cdot (-K_Z) = 1 \in \mathbb{Z}$.

Therefore, running the minimal model program preserves all four singularities and, at the end, we get a new surface (Z', F') still containing 4 A_1 singularities. Thanks to remark 3.3.3 we can thus conclude that (Z', F') is a surface admitting a ruling $\phi : Z' \to \mathbb{P}^1$ such that each fiber is a smooth rational curve and the A_1 singularities lie in pairs on two distinct fibres. It remains to prove that the two components of F on which they lie are sections for the ruling and there is at most one intermediate component. This follows from $F' \cdot A = (-K_{Z'}) \cdot A = 2$, where A is a fiber of the ruling. Also we need to rule out the case where the two sections F'_1, F'_2 in F' meet the third component A which is a fiber at the same point q. In this case we can always blowup at q thus obtaining a new pair $(\hat{Z}, \hat{F})$ where $\hat{F}$ is made of the strict transforms $\hat{F}_1, \hat{F}_2$ of F'_1, F'_2 plus the exceptional divisor E. Note that now the strict transform of A has self intersection (-1) and meets E transversallty but it is disjoing from $\hat{F}_1, \hat{F}_2$. Therefore we can contract the strict transform of A and we get a pair (Z'', F'') where F'' consists of the images of $\hat{F}_1, \hat{F}_2$ and the image of E which has now self intersection 0 and is a fiber for the ruling on (Z'', F''). This implies that there exists a sequence of contractions giving a map $(Z, F) \to (Z'', F'')$, where the latter pair has the right properties. Finally, we may always assume that the sections have self intersection 0, by contracting as usual always on the most negative section. □

Remark 3.3.4. Consider $\mathbb{P}^1 \times \mathbb{P}^1$ along with its toric boundary

$$\Delta = (\mathbb{P}^1 \times \{0, \infty\}) \cup (\{0, \infty\} \times \mathbb{P}^1)$$

Let (z, w) be complex coordinates on $\mathbb{P}^1 \times \mathbb{P}^1$ and set $\Delta_1 = \{\infty\} \times \mathbb{P}^1_w$, $\Delta_2 = \mathbb{P}^1_z \times \{\infty\}$, $\Delta_3 = \{0\} \times \mathbb{P}^1_w$, $\Delta_4 = \mathbb{P}^1_z \times \{0\}$. Define the map $j_0 : (z, w) \mapsto (1/z, -w)$. Then j_0 is an involution with 4 fixed points $(1, 0), (-1, 0), (1, \infty), (-1, \infty)$ that interchanges Δ_1 and Δ_3 and preserves Δ_2, Δ_4. Thus j_0 is an involution of the labeled anticanonical pair $(\mathbb{P}^1 \times \mathbb{P}^1, \Delta)$ and it defines a $\mathbb{Z}/2\mathbb{Z}$-action on it which is free on $U = \mathbb{P}^1 \times \mathbb{P}^1 \setminus \Delta$.

Theorem 3.3.5. *Given a negative definite Looijenga pair (Y, D) with $n \geq 4$ equipped with an antisymplectic involution j, there always exists a sequence of contractions of pairs of disjoint (-1) curves.*

$$(Y, D) \xrightarrow{\psi_1} (Y_1, D_1) \xrightarrow{\psi_2} \dots \xrightarrow{\psi_{m-1}} (Y_{m-1}, D_{m-1}) \xrightarrow{\psi_m} (\mathbb{P}^1 \times \mathbb{P}^1, \Delta) \qquad (3.5)$$

that respects the $\mathbb{Z}/2\mathbb{Z}$-action defined on (Y, D) and induces on $(\mathbb{P}^1 \times \mathbb{P}^1, \Delta)$ the action defined in remark 3.3.4.

Proof. Observe that (Z', F') can be obtained from $\mathbb{P}^1 \times \mathbb{P}^1$ as a quotient by the action of $\mathbb{Z}/2\mathbb{Z}$. To see this, recall that (Z', F') is a ruled surface and let $p : Z' \to \mathbb{P}^1$ be the associated map to $\mathbb{P}^1$. Let p_1, p_2 be the two points in $\mathbb{P}^1$ such that the two corresponding fibres $p^{-1}(p_1), p^{-1}(p_2)$ contain the two pairs of A_1 singularities. Now consider the quotient $\pi : \mathbb{P}^1 \to \mathbb{P}^1$ by the $\mathbb{Z}/2\mathbb{Z}$-action that fixes p_1, p_2 and let (Z'', F'') be the pullback of (Z', F') along π. The map $Z'' \to \mathbb{P}^1$ is ramified at $\pi^{-1}(p_1), \pi^{-1}(p_2)$. Normalizing (Z'', F''), we obtain a smooth ruled surface $(\hat{Z}, \hat{F})$

over $\mathbb{P}^1$ that is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ (because the two sections in $\hat{F}$ have self intersections equal to 0) and the map $(\mathbb{P}^1 \times \mathbb{P}^1, \Delta) \to (Z', F')$ we get is the required quotient map. More precisely, (Z', F') is the quotient by the action defined in remark 3.3.4. Indeed, let f be the involution on $\mathbb{P}^1 \times \mathbb{P}^1$ such that the associated quotient space is given by (Z', F'). Then f is induced by an automorphism of the algebraic torus $(\mathbb{C}^*)^2$: since Aut $(\mathbb{C}^*)^2 \cong \mathrm{GL}(2, \mathbb{Z}) \rtimes (\mathbb{C}^*)^2$, the involution on the torus has to be of the form (B, t), where B is a linear involution. Explicitely B is associated to the matrix

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

because of how it has to act on the toric boundary of $\mathbb{P}^1 \times \mathbb{P}^1$. Therefore the map has to have the form $(x_1, x_2) \mapsto (t_1 x_1^{-1}, t_2 x_2)$. Finally for this map to be an involution we must have $(t_1 t_1^{-1} x_1, t_2^2 x_2) = (x_1, x_2)$. It follows that on the one hand, we may always assume $t_1 = 1$ and on the other hand $t_2^2 = 1$ gives $t_2 = \pm 1$. Since we want the involution to have only isolated fixed points, $t_2 = -1$ and the claim is proved. Using theorem 3.3.1 we get the diagram:

$$\begin{array}{ccccccccc} (Z, F) & \xrightarrow{\phi_1} & (Z_1, F_1) & \xrightarrow{\phi_2} & \dots & \longrightarrow & (Z_{m-1}, F_{m-1}) & \xrightarrow{\phi_m} & (Z', F') \\ {\scriptstyle p}\uparrow & & & & & & & & \uparrow{\scriptstyle q} \\ (Y, D) & & & & & & & & (\mathbb{P}^1 \times \mathbb{P}^1, \Delta) \end{array}$$

The map ϕ_1 is the blow up of a single point p in (Z_1, F_1) lying on one of the irreducible components of F_1: let E be the exceptional curve $E = \phi_1^{-1}(p)$ and consider the preimage of this (-1)-curve via the quotient map p, $\{E_1, E_2\} = p^{-1}(E)$. These two rational curves are in fact (-1)-curves which do not intersect: if they did, they would share a fixed point, hence E would contain a singularity and this leads to a contradiction. Therefore we can subsequently contact E_1 and E_2. Let (Y_1, D_1)

be the composition of the contractions of E_1, E_2 in (Y, D). The diagram becomes

$$\begin{array}{ccccccc}
(Z,F) & \xrightarrow{\phi_1} & (Z_1,F_1) & \xrightarrow{\phi_2} \dots \longrightarrow & (Z_{m-1},F_{m-1}) & \xrightarrow{\phi_m} & (Z',F') \\
{\scriptstyle p}\uparrow & & {\scriptstyle q_1}\uparrow & & & & {\scriptstyle q}\uparrow \\
(Y,D) & \xrightarrow{\psi_1} & (Y_1,D_1) & & & & (\mathbb{P}^1\times\mathbb{P}^1,\Delta)
\end{array}$$

where the square on the left commutes. Repeating the same process $m-1$ times we obtain the sequence of maps:

$$\begin{array}{ccccccc}
(Z,F) & \xrightarrow{\phi_1} & (Z_1,F_1) & \xrightarrow{\phi_2} \dots \longrightarrow & (Z_{m-1},F_{m-1}) & \xrightarrow{\phi_m} & (Z',F') \\
{\scriptstyle p}\uparrow & & {\scriptstyle q_1}\uparrow & & {\scriptstyle q_{m-1}}\uparrow & & {\scriptstyle q}\uparrow \\
(Y,D) & \xrightarrow{\psi_1} & (Y_1,D_1) & \xrightarrow{\psi_2} \dots \longrightarrow & (Y_{m-1},D_{m-1}) & \longrightarrow & (\mathbb{P}^1\times\mathbb{P}^1,\Delta)
\end{array}$$

thus concluding the proof. $\square$

Remark 3.3.6. Theorem 3.3.5 implies that a symmetric Looijenga pair (Y, D) admits an antisymplectic involution j if and only if there exists a toric model $(Y, D) \to (\bar{Y}, \bar{D})$ such that $(\bar{Y}, \bar{D})$ admits an antisymplectic involution $\bar{j}$.

Theorem 3.3.5 gives us a useful criterion to decide, given a cusp D, whether it is possible or not to find a smooth rational surface Y where D sits as an anticanonical divisor (in other words if there exists a negative definite Looijenga pair (Y, D) equipped with an involution j which is free on $Y \setminus D$). Indeed, suppose you can. If there exists (Y, D) admitting the said action, then there must exist a map consisting of blowups going from $\mathbb{P}^1 \times \mathbb{P}^1$ to (Y, D) which respects the action at each step. Now, the existence of the latter map can be checked algorithmically only using the information coming from the cycle of self intersections of D. Moreover, if D is a cusp of length $4 \leq n \leq 10$ we have a result going in the other direction.

Proposition 3.3.7. *Let D be a symmetric cusp of length $4 \leq n \leq 10$. Then there always exists a surface Y and a sequence of blowups giving a map*

$$\psi : (Y, D) \longrightarrow (\mathbb{P}^1 \times \mathbb{P}^1, \Delta)$$

such that (Y, D) is a Looijenga pair and the involution j_0 defined in 3.3.4 lifts to an antisymplectic involution j on (Y, D) along ψ.

Remark 3.3.8. Let (Y, D) be a Looijenga pair and $j : (Y, D) \to (Y, D)$ an antisymplectic involution such that $j(D_i) = D_{\sigma(i)}$ with σ a reflection in the dihedral group. Let the pair $\{p, j(p)\}$ be made of a point sitting on one of the irreducible components of D and its image through j. Let $\pi : (\widetilde{Y}, \widetilde{D}) \to (Y, D)$ be the composite map obtained by blowing up at p and $j(p)$. Then the map j lifts to a unique map $\widetilde{j} : (\widetilde{Y}, \widetilde{D}) \to (\widetilde{Y}, \widetilde{D})$ (cfr. [8], chapter 7) which is still an involution.

We explicitly note that if p is a smooth point on D, so is $j(p)$, thus in this case $\widetilde{D} = \sum_i \widetilde{D}_i$ where $\widetilde{D}_i$ is the strict transform of D_i and $\widetilde{j}(\widetilde{D}_i) = \widetilde{D}_{\sigma(i)}$. Similarly if p is a node, then $j(p)$ is a node as well, therefore $\widetilde{D} = \sum_i \widetilde{D}_i + E_p + E_{j(p)}$ where E_p and $E_{j(p)}$ are the exceptional divisors of π, $\widetilde{j}(\widetilde{D}_i) = \widetilde{D}_{\sigma(i)}$ and $\widetilde{j}(E_p) = E_{j(p)}$. As a consequence $\widetilde{j}$ is still an involution of labeled Looijenga pairs.

proof of 3.3.7. We begin with $(\mathbb{P}^1 \times \mathbb{P}^1, \Delta)$ with the action defined in 3.3.4 associated to the involution j_0: note that j_0 is such that $j_0(D_i) = D_{\sigma(i)}$, where σ is the reflection given by $1 \mapsto 3$, 2,4 are fixed.

If $n = 4$, then the required (Y, D) is obtained from $(\mathbb{P}^1 \times \mathbb{P}^1, \Delta)$ via a sequence of interior blowups. Moreover given the symmetries of the self intersections of the D_i's, every time we blow up at a point p on D_1 we need to blow up at a point on the correspondent $D_{\sigma(1)} = D_3$ (and viceversa) and we can always choose this point to be $j(p)$; similarly the number of blowups needed at points lying on D_2, D_4 is even, thus we can always perform them in pairs at points $p, j(p)$, respectively on D_2 or D_4 avoiding the points fixed by the action. We are in the situation described in remark 3.3.8, hence the involution j lifts to the negative definite Looijenga pair (Y, D).

If $n > 4$ then we first perform $n-4$ corner blowups to get a toric pair of the right length among those whose cycles are described in figure 7. We observe that:

i. The self intersections for these cycles are minimal, in the following sense: all of the $-D_i^2$ are either 1 or 2, except possibly for a pair of curves with self intersection -3 or a single curve with self intersection -4, so that any other negative definite Looijenga pair with symmetric D can be obtained by a sequence of non toric blowups from one of these pairs.

ii. The number of nodes we need to blow up is always even and they come in pairs $\{p, j(p)\}$.

Therefore using remark 3.3.8 we can extend the action to each one of these toric pairs and thanks to the properties of the cycle of integers $(d_1, \ldots, d_n)$ every cusp D sits on a smooth rational surface Y that can be obtained by at least one of the toric pairs in our list through a sequence of interior blowups. We can thus repeat the argument used for $n = 4$ to conclude the proof. $\square$

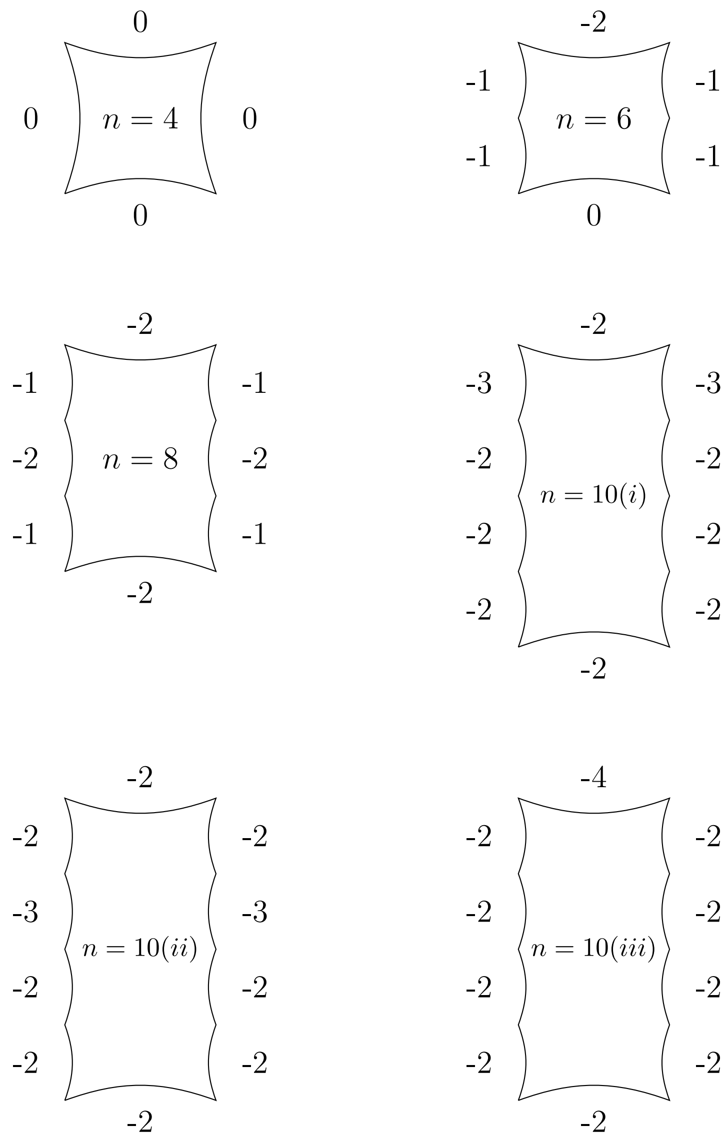


Figure 7. Cycles of the toric pairs

Sections: D_1 and $D5$	
Toric pair, cycle of integers	Elementary transformations
(i) (2,1,2,1,3,1,2,2,1)	$\varphi_{7,2},\varphi_{1,6},\varphi_{9,5}$
(3,1,2,1,2,1,2,2,1)	$\varphi_{8,4}$, back to (i)
(2,1,2,1,2,1,3,1,2)	$\varphi_{9,4}$, back to (i)
(ii) (1,1,2,1,2,2,2,1,3)	$\varphi_{1,6},\varphi_{9,5}$
(2,1,2,1,1,2,2,1,3)	$\varphi_{1,5}$ back to (ii)
(2,2,1,2,2,1,2,2,1)	Not needed
Sections: D_1 and D_4	
Toric pair, cycle of integers	Elementary transformations
(2,1,1,2,1,2,3,1,2)	$\varphi_{6,2}$, back to (ii)
(2,1,1,2,1,3,1,3,1)	$\varphi_{8,3}$, back to (i)
(2,1,1,1,1,3,2,1,3)	$\varphi_{6,3},\varphi_{9,4}$
(iii) (1,1,1,2,1,3,2,1,3)	$\varphi_{9,3},\varphi_{6,1}$
(iv) (2,1,1,1,1,4,1,2,2)	$\varphi_{6,3},\varphi_{1,5},\varphi_{6,2}$
(v)(1,1,1,2,1,4,1,2,2)	$\varphi_{6,2}$, back to (i)
(vi) (1,1,1,1,3,1,3,1,3)	$\varphi_{9,3},\varphi_{5,1}$, back to (i)
Sections: D_1 and D_3	
Toric pair, cycle of integers	Elementary transformations
(2,0,3,1,2,2,2,2,1)	$\varphi_{5,9},\varphi_{8,4},\varphi_{3,7}$
(0,0,1,2,2,2,2,1,5)	$\varphi_{9,2},\varphi_{4,1},\varphi_{9,3},\varphi_{5,1},\varphi_{9,4}$
(1,0,1,1,3,2,2,1,4)	$\varphi_{5,2}$, back to (iv)
(0,0,1,3,1,3,2,1,4)	$\varphi_{9,2},\varphi_{4,1}$, back to (iii)
(1,0,1,2,1,4,2,1,3)	$\varphi_{6,2},\varphi_{6,2}$, back to (i)
(2,0,1,1,2,3,2,1,3)	$\varphi_{5,2},\varphi_{6,3},\varphi_{9,4}$
(0,0,1,2,3,1,3,1,4)	$\varphi_{9,2},\varphi_{4,1}$back to (vi)
(1,0,1,1,4,1,3,1,3)	$\varphi_{5,2}$, back to (vi)
(1,0,0,3,2,1,4,1,3)	$\varphi_{4,2},\varphi_{9,3}$, back to (v)
(1,0,1,2,2,1,5,1,2)	$\varphi_{7,2}$, back to (v)
(1,0,2,1,3,1,4,1,2)	$\varphi_{7,2},\varphi_{7,2}$, back to (i)
(1,0,2,2,1,3,3,1,2)	$\varphi_{7,2},\varphi_{5,2}$
(2,0,2,1,2,2,3,1,2)	$\varphi_{6,2}$, back to (i)
(1,0,0,2,2,3,1,2,4)	$\varphi_{4,2},\varphi_{9,3},\varphi_{6,2}$
(1,0,1,1,3,3,3,2,3)	$\varphi_{5,2},\varphi_{9,3},\varphi_{5,2}$
(1,0,2,1,2,4,1,2,2)	$\varphi_{6,2}\varphi_{6,2}$
(1,0,0,2,3,2,1,3,3)	$\varphi_{4,2},\varphi_{9,3},\varphi_{5,2},\varphi_{8,3}$
(1,0,1,1,4,2,1,3,2)	$\varphi_{5,2},\varphi_{8,3}$
(1,0,0,3,2,2,1,2,4)	$\varphi_{4,2},\varphi_{9,3},\varphi_{8,2},\varphi_{4,9},\varphi_{8,3}$
(1,0,1,2,2,2,1,5,1)	$\varphi_{8,2},\varphi_{4,9},\varphi_{8,3}$
(1,0,2,1,3,2,1,4,1)	$\varphi_{8,2},\varphi_{5,9}$
(2,0,2,1,2,3,1,3,1)	$\varphi_{6,2},\varphi_{9,3}$, back to (i)

Table 3. Elementary transformations for $n = 9$

CHAPTER 4

DEFORMATION THEORY

Let $(p \in X)$ be a cusp singularity and $p' \in X'$ be its dual. Let $\pi_X : \tilde{X} \to X$ and $\pi_{X'} : \widetilde{X}' \to X$ be their respective minimal resolutions with $E = \pi_X^{-1}(p)$ and $D = \pi_{X'}^{-1}(p')$. Looijenga proposed a conjecture (that now has been completely proved, see [14] and [6]) which gives a sufficient and necessary condition for smoothability of cusp singularities.

Theorem 4.0.1 (Looijenga's theorem)**.** *A cusp singularity $(p \in X)$ is smoothable if and only if the dual cycle D sits as an anticanonical divisor on a smooth rational surface.*

The aim of this section is to provide a similar result for cusp singularities that admit an antisymplectic involution and therefore a $\mathbb{Z}/2\mathbb{Z}$-action. Here we emphasize that we always assume that the involutions we consider on $(p \in X)$ (respectively on (Y, D)) are fixed point free on $X \setminus \{p\}$ (respectively on $Y \setminus D$). The following conjecture, modeled on theorem 4.0.1, would give necessary and sufficient conditions for any of these cusp singularities to be equivariantly smoothable.

Conjecture 4.0.2 (Main conjecture)**.** *Let $(p \in X)$ be a cusp singularity equipped with an antisymplectic involution ι. Then $p \in X$ admits an equivariant smoothing with respect to the $\mathbb{Z}/2\mathbb{Z}$-action induced by ι if and only if the dual cycle D sits as an* *anticanonical divisor on a smooth rational surface Y which admits an antisymplectic involution j that extends the one induced on D by ι.*

Even though we strongly believe that this conjecture is true, as for now, we do not have a full proof of this result, but we have a proof of the sufficiency of the conjecture

4.1 Proof of the sufficiency of Conjecture 4.0.2

We will use some of the results contained in [6] to prove the following. Let $(p \in X)$ be a smoothable cusp singularity, ι an antisymplectic involution defined on it which is free on $X \backslash p$ and σ the reflection induced by it on E. By theorem 2.2.9 the $\mathbb{Z}/2\mathbb{Z}$-action on $(p \in X)$ gives an action on the dual cusp and an associated reflection σ' on the cycle D. Theorem 4.0.1 states that there exists a smooth rational surface Y containing D as an anticanonical divisor: suppose this Looijenga pair (Y, D) admits an antisymplectic involution j which agrees with σ' on D. Then the deformation space described in the paper mentioned above gives an equivariant smoothing of the cusp singularity $(p \in X)$.

We start by fixing some notation. Let (B, Σ) be the tropicalization of (Y, D), where B is an integral affine manifold with singularities and Σ is a decomposition of B into two dimensional cones $\{\sigma_{i,i+1}\}$ generated by rays ρ_i, ρ_{i+1}. Let $f : Y \to Y'$ be the contraction of D to the cusp singularity $q \in Y'$. The involution j on (Y, D) induces an involution, namely j^*, on Pic Y as well as one on $A_1(Y, \mathbb{R})$ and an involution θ on (B, Σ).

Now, let L be a nef divisor such that

$$\mathrm{NE}(Y)_{\mathbb{R}_{\geq 0}} \cap L^{\perp} = \langle D_1, \ldots, D_n \rangle_{\mathbb{R}_{\geq 0}}$$

Note that, possibly replacing L with $L+j^*L$, we can always choose such a divisor so that it is nef and invariant under the action of α. Indeed, given that j^* permutes the classes $[D_1], \dots, [D_n]$, then $\langle D_1, \dots, D_n\rangle \subset (j^*L)^\perp$ and $\mathrm{NE}(Y)_{\mathbb{R}_{\geq 0}} \cap (L + j^*L)^\perp = \langle D_1, \dots, D_n\rangle_{\mathbb{R}_{\geq 0}}$. Let $\sigma \subset A_1(Y, \mathbb{R})$ be a strictly convex rational polyhedral cone containing $\mathrm{NE}(Y)$. Set $\sigma_P = \sigma \cap j^*(\sigma)$. Then σ_P is invariant under j^*, it still contains $\mathrm{NE}(Y)$ and, possibly intersecting σ_P with the halfspace of curves β such that $\beta \cdot L > 0$, we can always assume that $\sigma_{bdy} := \sigma_P \cap L^\perp$ is a face of σ_P and we have that $j^*(\sigma_{bdy}) = \sigma_{bdy}$. Let $P = \sigma_P \cap A_1(Y, \mathbb{Z})$ be the toric monoid associated to σ_P, $\mathfrak{m} = P \setminus 0$ and $J = P \setminus P \cap L^\perp$. We will call $S = \mathrm{Spec}\ \mathbb{C}[P]$ and $S_I = \mathrm{Spec}\ \mathbb{C}[P]/I$ for any monomial ideal I.

The map j^* gives an involution on S and S_I for any j^*-invariant monomial ideal I defined by

$$\alpha : z^\beta \mapsto (-1)^{\beta \cdot (D_n + D_{n/2})} z^{j^*(\beta)} \tag{4.1}$$

where $D_n, D_{n/2}$ are the two components of D fixed by j, in particular we get

$$\alpha(z^{[D_i]}) = \begin{cases} (-1)^{D_i \cdot D_i} z^{j^*([D_i])} = z^{[D_i]} & \text{if } i = n/2, n \\ -z^{j^*([D_i])} = -z^{[D_{n-i}]} & \text{if } i = 1, n/2-1, n/2+1, n-1 \\ z^{j^*([D_i])} = z^{[D_{n-i}]} & \text{otherwise} \end{cases}$$

From [6] we get the following theorem.

Theorem 4.1.1. *Fix $R > 1$. There exists an analytic open neighborhood S'_J of $0 \in S_J$ and an analytic flat family $f_J : X_J \to S'_J$ together with a section $s : S'_J \to X_J$ satisfying the following properties:*

i. The general fibre $X_{J,t}$ of f_J is a Stein analytic surface with a unique singularity $s(t) \in X_{J,t}$ isomorphic to the cusp singularity $(p \in X)$.

ii. *For each ray* $\rho_i \in \Sigma$ *there is an open analytic subset* $V_{\rho_i,J} \subset X_J$ *and open analytic embeddings*

$$V_{\rho_i,J} \subset \{(x_{i-1}, x_i, x_{i+1}) \in U_{\rho_i,J} |\ |x_{i-1}| < R|x_i|,\ |x_{i+1}| < R|X_i|\} \subset U_{\rho_i,J}$$

where

$$U_{\rho_i,J} := V(x_{i-1}x_{i+1} - z^{[D_i]}x_i^{-D_i^2}) \subset \mathbb{A}^2_{x_{i-1},x_{i+1}} \times (\mathbb{G}_m)_{x_i} \times S_J$$

such that

a. $X_J^\circ := X_J \setminus s(S'_J) = \bigcup_{\rho\in\Sigma} V_{\rho,J}$

b. $V_{\rho,J} \cap V_{\rho',J} = \emptyset$ *unless* $\rho = \rho'$ *or* ρ *and* ρ' *are the edges of a maximal cone* $\sigma \in \Sigma$

iii. *The restriction of* X_J/S'_J *to* $S_{J+\mathfrak{m}^{N+1}}$ *is identified with an analytic neighborhood of the vertex in the restriction of the family* $X_{\mathfrak{m}^{N+1}}/S_{\mathfrak{m}^{N+1}}$ *given by theorem 2.28 in [6] with* $\mathfrak{D} = \mathfrak{D}^{\text{can}}$ *for each* $N \geq 0$.

The first step of the proof consists of showing that the family $f_J : X_J \to S'_J$ admits a $\mathbb{Z}/2\mathbb{Z}$ action. In order to do so, let us define maps $\iota_{i,J}$, for $i = 1, \dots, n$ as follows:

$$\iota_{i,J} : \mathbb{A}^2_{x_{i-1},x_{i+1}} \times (\mathbb{G}_m)_{x_i} \times S_J \to \mathbb{A}^2_{x_{n-i-1},x_{n-i+1}} \times (\mathbb{G}_m)_{x_{n-i}} \times S_J$$
$$(x_{i-1}, x_{i+1}, x_i, z_J) \mapsto ((-1)^{\varepsilon(i+1)}x_{i+1}, (-1)^{\varepsilon(i-1)}x_{i-1}, (-1)^{\varepsilon(i)}x_i, \alpha(z_J))$$

where $\varepsilon(i) = 1$ if $i \equiv 0 \mod (n/2)$ and 0 otherwise and indices are meant mod n within the range $1, \dots, n$ when needed. Then the image of each $U_{\rho_i,J}$ under the corresponding involution $\iota_{i,J}$ is $U_{\rho_{n-i},J}$: let $p = (x_{i-1}, x_{i+1}, x_i, z_J)$ be a point in $U_{\rho_i,J}$, then we have

$$\iota_{i,J}(p) = ((-1)^{\varepsilon(i+1)}x_{i+1}, (-1)^{\varepsilon(i-1)}x_{i-1}, (-1)^{\varepsilon(i)}x_i, \alpha(z_J))$$

which satisfies

$$(-1)^{\varepsilon(i-1)}(-1)^{\varepsilon(i+1)}x_{i-1}x_{i+1} - \alpha(z^{[D_i]})[(-1)^{\varepsilon(i)}x_i]^{-D_i^2} = 0$$

Indeed, if $\varepsilon(i) = 1$, then i is either $n/2$ or n, but $(-x_i)^{-D_i^2} = x_i^{-D_i^2}$ since $-D_i^2$ is even, while if either $\varepsilon(i-1) = 1$ or $\varepsilon(i+1) = 1$, then $i \in \{1, n/2-1, n/2+1, n-1\}$ therefore $\alpha(z^{[D_i]}) = -z^{[D_i]}$. If all the exponents are equal to 0, then the equality holds trivially, thus $\iota_{i,J}(p) \in U_{\rho_{n-i},J}$. The other containment also follows, since $\iota_{i,J}$ is an involution. As a consequence, the open analytic subsets $V_{\rho_i,J}$ are permuted accordingly, with $\iota_J(V_{\rho_i,J}) = V_{\rho_i,J}$ if $i = n/2, n$ and $\iota_J(V_{\rho_i,J}) = V_{\rho_{n-i},J}$ otherwise. Moreover the maps $\iota_{i,J}$ agree on the intersections, thus giving an involution ι_J on $\bigcup_i V_{\rho_i,J}$. Indeed, for $V_{\rho_i,J} \cap V_{\rho_{i+1},J} = (\mathbb{G}_m^2)_{x_i,x_{i+1}} \times S_J$ we have

$$\iota_{i,J}(x_i, x_{i+1}, z_J) = ((-1)^{\varepsilon(i)}x_i, (-1)^{\varepsilon(i+1)}x_{i+1}, \alpha(z_J)) = \iota_{i+1,J}(x_i, x_{i+1}, z_J)$$

This gives an analytic involution ι_J on $X_J^\circ = \cup_{\rho\in\Sigma}V_{\rho,J}$ that can be extended to X_J/S'_J, since its fibres satisfy Serre's condition S_2. The analytic involution obtained through the extension to the singular locus, that we will still denote by ι_J, is compatible with the one given in (4.1) on the base space S'_J. Moreover this involution is fixed point free and thanks to remark 2.2.10 and proposition 2.2.8 the action induced by ι_J on the cusp singularities $s(t) \in X_{J,t}$ of the general fibres of f_J is exactly the $\mathbb{Z}/2\mathbb{Z}$ action given by hypothesis on $(p \in X)$.

The next step is to consider the thickening of the cusp family as it is presented in [6].

Theorem 4.1.2. *Let $f_J : X_J \to S'_J$ be the analytic family of theorem 4.1.1. Possibly after replacing S'_J by a smaller neighborhood of $0 \in S'_J$ and X_J by a smaller neighborhood of $s(S'_J) \subset X_J$, independent of the choice of I below, the following holds. Let $I \subset P$ be a monomial ideal such that $\sqrt{I} = J$ and let $S'_I \subset S_I$ denote the* *induced thickening of* $S'_J \subset S_J$. *There is an infinitesimal deformation* $f_I : X_I \to S'_I$ *of* $f_J : X_J \to S'_J$ *such that for each* $N > 0$ *the restriction to* $\mathrm{Spec}\ \mathbb{C}[P]/(I + \mathfrak{m}^{N+1})$ *is identified with an analytic neighborhood of the vertex in the restriction of the family* $X_{\mathfrak{m}^{N+1}}/S_{\mathfrak{m}^{N+1}}$ *given by theorem 2.28 in [6] with* $\mathfrak{D} = \mathfrak{D}^{\mathrm{can}}$ *for each* $N \geq 0$.

The involution defined on the family $f_J : X_J \to S'_J$ extends to its thickening $f_I : X_I \to S'_I$. Indeed the scattering diagram $\mathfrak{D}$ is θ-invariant, in the sense that if $(\mathfrak{d}, f_{\mathfrak{d}}) \in \mathfrak{D}$ then $(\theta(\mathfrak{d}), f_{\theta(\mathfrak{d})})$ is still contained in $\mathfrak{D}$: in order to show this let us first analyze the rays defining the scattering diagram and then the associated functions. Now consider the functions associated to the rays in Σ: each f_{ρ_i} is a truncated version of

$$\exp\left[\sum_{\beta} (\beta \cdot D_i) N_\beta z^\beta x_i^{-\beta \cdot D_i}\right]$$

where the sum runs over all classes $\beta \in \mathbb{A}_1(Y, \mathbb{Z})$ satisfying the property

$$\beta \cdot D_i \neq 0 \text{ and } \beta \cdot D_j = 0 \text{ for all } j \neq i \tag{4.2}$$

and N_β is defined in [6], definition 3.1. Thus, when applying $\iota_{i,J}$, for each admissible β we get

$$(\beta \cdot D_i) N_\beta (-1)^{\beta \cdot (D_{n/2} + D_n)} z^{j^*(\beta)} (-1)^{\varepsilon(i) \beta \cdot D_i} x_{n-i}^{-\beta \cdot D_i} \tag{4.3}$$

If $i = n/2, n$, then $(-1)^{\beta \cdot (D_{n/2} + D_n)}$ becomes $(-1)^{\beta \cdot D_i}$ and $\varepsilon(i) = 1$, giving $(-1)^{\varepsilon(i)\beta \cdot D_i} = (-1)^{\beta \cdot D_i}$. Otherwise, $\beta \cdot (D_{n/2} + D_n) = 0$ and $\varepsilon(i)$, thus we obtain

$$(4.3) = \begin{cases} (\beta \cdot D_i) N_\beta z^{j^*(\beta)} x_i^{-\beta \cdot D_i} & \text{if } i = n/2, n \\ (\beta \cdot D_i) N_\beta z^{j^*(\beta)} x_{n-i}^{-\beta \cdot D_i} & \text{otherwise} \end{cases}$$

Since j is an involution, if β is a class in $A_1(Y, \mathbb{Z})$, then $j^*\beta \cdot j^*C = \beta \cdot C$ for any curve C in Y intersecting β properly. Therefore if β satisfies (4.2) for some i, then $j^*\beta$ is such that (4.2) is true with i replaced by $n - i$, and indices considered mod

n as usual. Moreover, $N_{j^*\beta} = N_\beta$ because the definition of N_β is determined by a moduli space that in turns only depends on the isomorphism class of $((Y, D), \beta)$. Besides, f γ is a class in $A^1(Y, \mathbb{Z})$ satisfying (4.2) for $n - i$, then $\gamma = j^*(\beta)$ for some β satisfying (4.2) as well, hence

$$\sum_\beta (\beta \cdot D_i) N_\beta z^{j^*(\beta)} x_{n-i}^{-\beta \cdot D_i} = \sum_\gamma (\gamma \cdot D_i) N_\gamma z^\gamma x_{n-i}^{-\gamma \cdot D_i}$$

where the sum runs over the appropriate classes β and γ respectively. Therefore $f_{\rho_i} \circ \iota_{i,J} = f_{\rho_i}$ for $n/2, n$ and $f_{\rho_i} \circ \iota_{i,J} = f_{\rho_{n-i}}$ otherwise. On the other hand, suppose $f_\mathfrak{d}$ is the function relative to any other ray $\mathfrak{d}$ of rational slope, with $\mathfrak{d}$ contained in the cone of Σ generated by ρ_i, ρ_{i+1}. Then $f_\mathfrak{d}$ is the truncated version of

$$\exp \left[\sum_\beta \kappa_\beta N_\beta z^\beta x_i^{-a\kappa_\beta} x_{i+1}^{-b\kappa_\beta} \right]$$

with β defined as in [6], definition 3.1, and a, b chosen to satisfy $\mathfrak{d} = \mathbb{R}_{\geq 0}(a\kappa v_i + b\kappa v_{i+1})$, with $\rho_i = \mathbb{R}_{\geq 0} v_i, \rho_{i+1} = \mathbb{R}_{\geq 0} v_{i+1}$ and κ_β the positive integer such that $\beta \cdot D_i = a\kappa_\beta$ and $\beta \cdot D_{i+1} = b\kappa_\beta$. Therefore, applying $\iota_{i,J}$, for each admissible β we obtain

$$\kappa_\beta N_\beta (-1)^{\beta \cdot (D_{n/2} + D_n)} z^{j^*(\beta)} g_i \tag{4.4}$$

with

$$g_i = \begin{cases} (-1)^{\varepsilon(i) a\kappa_\beta + \varepsilon(i+1) b\kappa_\beta} x_{n-i}^{-a\kappa_\beta} x_{n-i-1}^{-b\kappa_\beta} & \text{if } i \neq n/2, n \textit{ and } i+1 \neq n/2, n \\ (-1)^{\varepsilon(i) a\kappa_\beta + \varepsilon(i+1) b\kappa_\beta} x_i^{-a\kappa_\beta} x_{i-1}^{-b\kappa_\beta} & \text{if } i = n/2, n \\ (-1)^{\varepsilon(i) a\kappa_\beta + \varepsilon(i+1) b\kappa_\beta} x_{n-i}^{-a\kappa_\beta} x_{i+1}^{-b\kappa_\beta} & \text{if } i+1 = n/2, n \end{cases}$$

If $i = n/2$ or $i = n$, then $\varepsilon(i) = 1$ while $\varepsilon(i+1) = 0$, therefore reasoning as we did above, we get $(-1)^{\beta \cdot D_i} (-1)^{a\kappa_\beta} = 1$, since $\beta \cdot D_i = a\kappa_\beta$. Similarly, all negative signs cancel if $i+1=n/2,n$. Thus we get

$$(4.4) = \begin{cases} \kappa_\beta N_\beta z^{j^*(\beta)} x_{n-i}^{-a\kappa_\beta} x_{n-i-1}^{-b\kappa_\beta} & \text{if } i \neq n/2, n \textit{ and } i+1 \neq n/2, n \\ \kappa_\beta N_\beta z^{j^*(\beta)} x_{i}^{-a\kappa_\beta} x_{i-1}^{-b\kappa_\beta} & \text{if } i = n/2, n \\ \kappa_\beta N_\beta z^{j^*(\beta)} x_{n-i}^{-a\kappa_\beta} x_{i+1}^{-b\kappa_\beta} & \text{if } i+1 = n/2, n \end{cases}$$

Arguing as before we can now show that $f_{\mathfrak{d}} \circ \iota_{i,J} = f_{\theta(\mathfrak{d})}$ as needed: this concludes the proof that the scattering diagram $\mathfrak{D}$ is θ-invariant. The hypersurfaces $U_{\rho_i,I}$ are defined by the equations

$$x_{i-1}x_{i+1} - z^{[D_i]} x_i^{-D_i^2} f_{\rho_i} = 0$$

thus the analysis above implies that the maps $\iota_{i,J}$ extend to them and respect all the gluing isomorphisms, giving a new involution ι_I on X_I/S'_I.

Finally, we need to describe the subspace $S' \subset S$ of points fixed by the involution α. In order to do this let us recap our notation. Let T be the algebraic torus contained in the affine toric variety S and recall that $S = \text{Spec } \mathbb{C}[\nu^* \cap M]$, where $\nu = \sigma_P^*$ $M = \text{A}_1(Y)$ and the dual lattice $N = \text{Pic } Y$. Since by assumption the involution j defined on the pair (Y,D) is fixed point free away from D, then we can use theorem 3.3.5 to get a precise description of the pullback map $j*$ on the Picard group of Y. Indeed, the theorem states that there exists a sequence of maps

$$(Y,D) \xrightarrow{\psi_1} (Y_1,D_1) \xrightarrow{\psi_2} \cdots \xrightarrow{\psi_m} (\mathbb{P}^1 \times \mathbb{P}^1, \Delta)$$

where each ψ_i corresponds to the blowup of two points on the anticanonical divisor of the pair (Y_i,D_i) which belong to the same orbit with respect to the action induced on (Y_i,D_i) by the original involution defined on (Y,D). We may always assume that the maps $\psi_1,\ldots,\psi_t$ are pairs of interior blowups, while the remaining ones are pairs of toric blowups, thus implying that (Y_t,D_t) is an equivariant toric model for (Y, D) (see remark 3.3.6). This sequence of maps induces on $(\mathbb{P}^1 \times \mathbb{P}^1, \Delta)$ the involution j_0 defined as $(z, w) \mapsto (z^{-1}, -w)$ (crf. remark 3.3.4). Observe that the pullback of j_0 acts trivially on the Picard group of $\mathbb{P}^1 \times \mathbb{P}^1$ and that the generators for $\mathrm{Pic}(Y)$ are given (possibly with some redundancy) by the classes of the divisors $D_1, \ldots D_n$ and by those of the exceptional divisors $\mathcal{E} = \{E_{i,j}, E'_{i,\sigma(j)}\}_{i=1,\ldots,t}$, of the maps $\psi_1, \ldots, \psi_t$ or, to be more precise, of their strict transforms in Y. Here the index j refers to the divisor D_j intersected by $E_{i,j}$. Since the involution j maps each $E_{i,j}$ to $E'_{i,\sigma(j)}$ (and viceversa), the involution induced by it on the Picard group of Y corresponds to the block diagonal matrix

$$\begin{pmatrix} 1 & 0 & & & & & 0 \\ 0 & 1 & & & & & \\ & & 0 & 1 & & & \\ & & 1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & 1 \\ 0 & & & & & 1 & 0 \end{pmatrix}$$

Furthermore, let $(z_1, z_2, w_1, \ldots, w_{2k}, u_1, \ldots, u_{2l})$ be the coordinates of the torus T, arranged so that z_1, z_2 correspond to the classes of the generators of $\mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1)$, the pairs $(w_1, w_2), \ldots, (w_{2k-1}, w_{2k})$ are of the form $z^{[E_{i,j}]}, z^{[E'_{i,\sigma(j)}]}$ with $j = n/2, n$ or $z^{[D_j]}, z^{[D_{\sigma(j)}]}$ where D_j does not meet $D_{n/2}, D_n$ and the pairs $(u_1, u_2), \ldots, (u_{2l-1}, u_{2l})$ correspond to the divisors in $E_{i,j}, E'_{i,\sigma(j)}$ with $j \neq n/2, n$ or to divisors $D_j, D_{\sigma(j)}$ which do not intersect $D_{n/2}$, D_n. If we call still α (as in 4.1) the involution defined on T by the formula

$$\alpha : z^\beta \mapsto (-1)^{\beta \cdot ([D_n] + [D_{n/2}])} z^{j^*(\beta)}$$

then in coordinates we get that this involution maps $(z_1, z_2, w_1, \ldots, w_{2k}, u_1, \ldots, u_{2l})$

to $(z_1, z_2, -w_2, -w_1, \ldots, -w_{2k}, -w_{2k-1}, u_2, u_1, \ldots, u_{2l}, u_{2l-1})$. Therefore the fixed locus for α in T is described by the equations $w_{2i} = -w_{2i-1}$ for $i = 1, \ldots, k$ and $u_{2i} = u_{2i-1}$ for $i = 1, \ldots, l$ and it is the translate of a subtorus $T' \subset T$ of dimension $2 + k + l$ by a point of type $(\pm 1, \ldots, \pm 1)$, where $T' = N' \otimes \mathbb{C}^*$ and $N' \subset N$ is the $\mathbb{Z}/2\mathbb{Z}$-invariant sublattice in N. Finally let $\nu' \subset \nu$ be defined as $\nu \cap N'_{\mathbb{R}}$ and similarly $\tau' = \tau \cap N'_{\mathbb{R}}$, where $\tau = \sigma^*_{bdy}$. Then the closure of T' in S is the toric variety S' which corresponds to ν'. S' meets the interior of the stratum Z corresponding to τ when $N'_{\mathbb{R}}$ meets the relative interior of τ and in this case $\overline{S' \cap \mathrm{Int}(Z)}$ automatically contains $0 \in Z$. Observe that τ and $N'_{\mathbb{R}}$ are invariant under the $\mathbb{Z}/2\mathbb{Z}$-action induced by j^*. We claim that $N'_{\mathbb{R}}$ intersects the interior of τ non trivially. Indeed, let $V := \mathrm{span}_{\mathbb{R}}\langle \tau \rangle$, then τ is full dimensional in V and this vector space decomposes in eigenspaces for j^* as $V^+ \oplus V^-$, with corresponding eigenvalues $1, -1$. Let $p : V \to V^-$ be the projection onto the second eigenspace and consider the restriction of the involution to V^-: here j^* acts as $-\mathrm{Id}$ and $p(N'_{\mathbb{R}}) = 0$. The image $\tau^- := p(\tau)$ under p of τ is full dimensional and invariant under the action of $-\mathrm{Id}$. It follows that 0 must be contained in the interior of τ^-, and therefore $N'_{\mathbb{R}}$ must intersect the interior of τ non trivially as well.

Finally in order to be sure that S' provides a smoothing of the cusp singularity dual to D, we need to consider the Gross-Siebert $\hat{S}$ locus contained in S (cfr. definition 3.14 in [6]). The map π determines a face of Nef Y, namely $\pi^*(\mathrm{Nef}\ \bar{Y})$, or equivalently a face of NE Y. We may always assume that there is a corresponding face F of ν that is $\mathbb{Z}/2\mathbb{Z}$-invariant. On $\hat{S}$ we have coordinates $\{z^{[E_{i,j}]}, z^{[E'_{i,\sigma(j)}]}\}_{i=1,\ldots,t}$ and $j \in \{1, \ldots, n\}$, while the fixed locus in $\hat{S}$ is described by the equations

$$\begin{cases} z^{[E_{i,j}]} = -z^{[E'_{i,\sigma(j)}]} & \text{if } j = n/2, n \\ z^{[E_{i,j}]} = z^{[E'_{i,\sigma(j)}]} & \text{otherwise} \end{cases} \tag{4.5}$$

therefore it has coordinates $z^{[E_{i,j}]}$ with $i = 1, \dots, t$ and $j \in \{1, \dots, n\}$. In general if $\{z_{i,j}\}$ are the coordinates of the Gross-Siebert locus corresponding to the exceptional divisors meeting D_j, for the smoothness argument we must have $z_{i,j} \neq z_{k,j}$ for all j and for all $i \neq k$. In our case, because of (4.5) and the way σ is defined, this reduces to check that $z^{[E_{i,j}]} \neq z^{[E'_{k,j}]}$ for $j = n/2, n$ and for all $i \neq k$. This is always true, because of the first equation in (4.5). Now if I is an α-invariant monomial ideal, then we can consider $S'_I \subset S_I$ and, by restriction, we get an equivariant family $f' : X'_I \to S'_I$. From this family, using the techniques of theorem 7.13 in [6] we finally obtain an equivariant smoothing of the cusp singularity $(p \in X)$.

4.2 Cusps with embedding dimension $n \leq 12$

We describe what is known about the conjecture 4.0.2 for cusp singularities of embedding dimension $n \leq 12$. It can be proved that for $n \leq 10$ it is always possible to find an equivariant smoothing of the cusp $(p \in X)$.

Proposition 4.2.1. *Every germ of a symmetric cusp singularity of embedding dimension $n \leq 10$ is equivariantly smoothable.*

Proof. It can be checked that symmetric cusp singularities of multiplicity 2 are always equivariantly smoothable, since they embed in $\mathbb{A}^3$ as hypersurfaces and have an explicit description of their smoothings. Indeed, the equation of a cusp $(p \in X)$ of multiplicity 2 is given by $(z^2 + x^p + y^q + xyz = 0) \subset \mathbb{A}^3$; we can change coordinates so that it is given by $(z^2 + x^p + y^q - 1/4x^2y^2 = 0) \subset \mathbb{A}^3$. Note that since the cusp is symmetric, then p, q have to be even. Now, the involution on $(p \in X)$ is given by $(x, y, z) \mapsto (-x, -y, -z)$: this is the right involution since it preserves the cusp and it is antisymplectic, as it can be checked on the minimal resolution.

Finally, a smoothing of $(p \in X)$ can be described as the one parameter family $(z^2+x^p+y^q-1/4x^2y^2+t=0) \subset \mathbb{A}^3 \times \mathbb{A}^1_t$. This family gives an equivariant smoothing with respect to the extended involution given by $(x,y,z,t) \mapsto (-x,-y,-z,t)$. Now suppose $(p \in X)$ is a cusp singularity of embedding dimension $4 \leq n \leq 10$. Theorem 2.2.9 gives an involution j on the dual cusp $(p' \in X')$ acting freely on $X' \setminus \{p'\}$ that induces a $\mathbb{Z}/2\mathbb{Z}$-action on its exceptional cycle D. Thus the dual cusp D is symmetric and by proposition 3.3.7 there exists a rational surface Y on which D sits as an anticanonical divisor and a map $\psi : (Y, D) \to (\mathbb{P}^1 \times \mathbb{P}^1, \Delta)$ lifting to (Y, D) the involution j_0 defined on $(\mathbb{P}^1 \times \mathbb{P}^1, \Delta)$ in remark 3.3.4. Therefore there exists a Loojenga pair (Y, D) together with an antisymplectic involution j and we can use the GHK construction to find an equivariant smoothing of $(p \in X)$. □

To find an example of a cusp that is equipped with an antisymplectic involution ι but does not admit an equivariant smoothing, we have to look among symmetric cusps with embedding dimension at least equal to 12. In fact we conjecture that $n = 12$ is big enough. More precisely, the following result can be proved.

Proposition 4.2.2. *All symmetric cusp singularities $(p \in X)$ of embedding dimension $n = 12$ are equivariantly smoothable, except possibly for the ones listed below, which are divided according to their length:*

- $(3, 10, 3, 4), (4, 4, 8, 4), (6, 4, 6, 4), (12, 3, 2, 3), (10, 4, 2, 4), (4, 7, 2, 7)$
- $(3, 10, 3, 3, 2, 3), (3, 4, 3, 6, 2, 6)$.

These eight cusps correspond to Looijenga pairs that do not admit an antisymplectic involution which is free away from the anticanonical cycle and acts as a reflection on it.

Proof. Similarly to the proof of proposition 3.3.7, an antisymplectic involution can be constructed for a list of minimal Looijanga pairs of length equal to 12, given in table 4.

Looijenga pair	Correspondent dual cusp singularity
(3,2,2,3,2,2,2,3,2,2,3,2)	(5,6,5,4)
(2,2,2,2,2,4,2,2,2,2,2,4)	(8,2,8,2)
(2,2,2,2,2,2,2,2,2,2,2,6)	(14,2,2,2)
(4,2,2,2,2,2,2,2,2,2,4,2)	(12,2,4,2)
(2,4,2,2,2,2,2,2,2,4,2,2)	(10,2,6,2)
(2,2,3,2,2,2,2,2,3,2,2,4)	(8,5,2,5)
(2,2,2,3,2,2,2,3,2,2,2,4)	(6,6,2,6)

Table 4. *Minimal* symmetric Looijega pairs of length 12 and associated cusp singularities

Therefore the cusp singularities corresponding to these pairs are equivariantly smoothable thanks to the sufficient condition of conjecture 4.0.2 proved in section 3.1. Now, all symmetric cusps corresponding to anticanonical pairs that can be obtained from the pairs mentioned above through interior blowups will also be equivariantly smoothable, as we observed in proposition 4.2.1.

The cusps listed in the statement of this proposition are those for which it is not possible to construct a symmetric Looijenga pair starting from $\mathbb{P}^1 \times \mathbb{P}^1$ and subsequently performing pairs of toric and interior blowups in a symmetric way. This has been proved by contradiction: for each pair we assume it admits an antisymplectic involution. Then there should exists an equivariant sequence of pairs of contractions of (-1)-curves that gives $\mathbb{P}^1 \times \mathbb{P}^1$. We checked by hand exausting all possible cases that this is not possible for the given pairs. This implies that, for each of these cusps, there does not exist a pair (Y, D) equipped with an antisymplectic involution. Indeed, suppose it does: then, using proposition 3.3.5, we would be able to find an equivariant map to $\mathbb{P}^1 \times \mathbb{P}^1$ thus giving us a contradiction. □

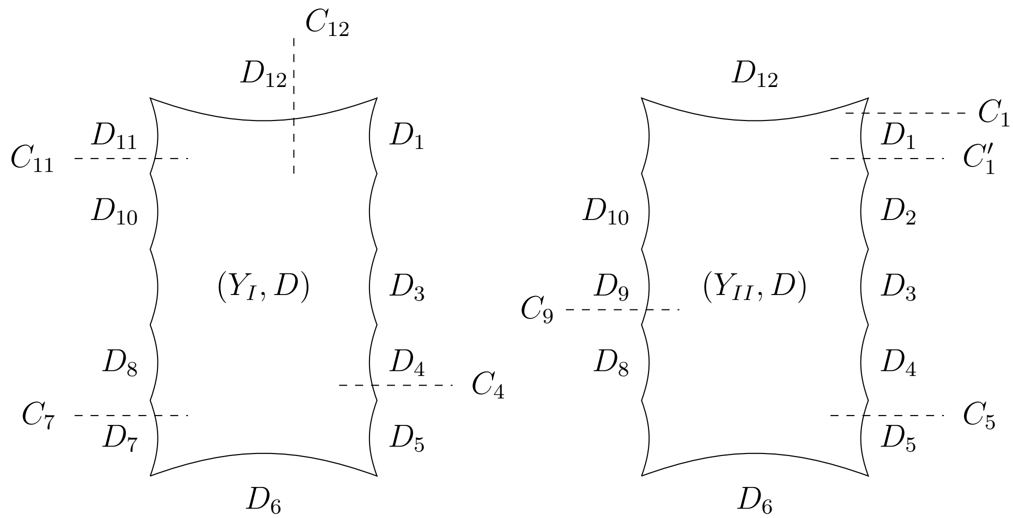


Figure 8. Schematic description of the two possible Looijenga pairs for D

Example 4.2.3. Among the cusps listed in proposition 4.2.2 there is the one with associated cycle of integers $(3, 10, 3, 4)$: its dual cusp D corresponds to the exceptional cycle with self intersections $(3, 3, 2, 2, 2, 2, 2, 2, 2, 3, 3, 2)$. By inspection it can be proved that, up to isomorphim, there are two anticanonical pairs with cycle D (see figure 8) and that for both of them there does not exist a map to $\mathbb{P}^1 \times \mathbb{P}^1$ which allows us to lift the involution j defined on $\mathbb{P}^1 \times \mathbb{P}^1$ as we did for cycles of length $n \leq 10$.

Conjecture 4.2.4. *The cusp singularities listed in proposition 4.2.2 do not admit an equivariant smoothing.*

4.3 Equivariant smoothings of simple elliptic singularities

Let us describe what happens if, instead of cusp singularities, we consider the case of simple elliptic singularities, namely cones over elliptic curves. This is an interesting case, since these singularities and their deformation space have been studied extensively (see for example [20] and [16]) and can be described quite explicitly.

For simple elliptic singularities $p \in C(E)$, where $C(E)$ is the cone over a smooth elliptic curve E of degree d smaller than eight, there exists essentially one smoothing component with associated Milnor fiber given by $M = S \setminus E$, where S is the del Pezzo surface of corresponding degree d. To be more precise in these cases, a smoothing family can be obtained as follows. Let S be a del Pezzo surface of degree d and consider the projective closure of the affine cone over this surface $\overline{C(S)} \subset \mathbb{P}^{d+1}$. Let $\mathcal{H}_t$, with $t \in \mathbb{A}^1_t$ be a family of hyperplanes in $\mathbb{P}^n$ such that $p \in \mathcal{H}_t$ if and only if $t = 0$ and let us consider $\mathcal{X}_t := \mathcal{H}_t \cap \overline{C(S)}$. Then $\mathcal{X} \to \mathbb{A}^1_t$ is a smoothing family for $p \in C(E)$ with $\mathcal{X}_0 \cong \overline{C(E)}$ and $\mathcal{X}_t \cong S$ for $t \neq 0$. On the other hand, given a cone over an elliptic curve E of degree eight, its deformation space is isomorphic to $(\bigcup_{i=1}^4 \mathbb{A}^1 \times \mathbb{A}^2) \cup C(\mathcal{E}))$, where $C(\mathcal{E})$ is the cone over the universal elliptic curve $\mathcal{E} \to \mathbb{A}^1$. Each plane $\mathbb{A}^2$ and the cone $C(E)$ give a smoothing component for the singularity, thus implying that every simple elliptic singularity of degree 8 is smoothable, and they are distinguished by the associated Milnor fibre. The Milnor fibre M_i corresponding to $\bigcup_{i=1}^4 \mathbb{A}^1 \times \mathbb{A}^2$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1 \setminus E$, where E again is the elliptic curve we start with. While the other smoothing component has Milnor fiber given by $M_{ii} = \mathbb{F}_1 \setminus E$.

Moreover, let M be the Milnor fibre of a smoothing of a simple elliptic singularity $p \in C(E)$ of degree d. Then M is mirror to the surface $U_d = Y_{(d)} \setminus D_{(d)}$,where $(Y_{(d)}, D_{(d)})$ is the negative semidefinite Looijenga pair that appears already in section 3.1. The two Milnor fibres M_i, M_{ii}, which correspond to the two different smoothing components of a simple elliptic singularity of degree eight are mirror to the surfaces $U_i = Y_i \setminus D$ and $U_{ii} = Y_{ii} \setminus D$, where (Y_i, D) and (Y_{ii}, D) are semidefinite Looijenga pairs and D is a cycle of eight rational curves of self intersection -2. They can be obtained explicitly from the toric pairs (T_i, G_i) and (T_{ii}, G_{ii}) respectively through four interior blowups on the (-1)-curves contained in the toric boundaries. We observe that M_i and M_{ii} are not diffeomorphic, since they are diffeomorphic to the open manifolds U_i, U_{ii} and these manifolds have have different fundamental groups (see remark 3.1.7).

Let now ι be an involution on $X = C(E)$ for a given smooth elliptic curve E of even degree $d \leq 8$, inducing the hyperelliptic involution on E and acting by (-1) on the fixed fibres. More precisely, we can construct the said involution as follows. Recall that $C(E) = C(E, L) = \operatorname{Spec}(\bigoplus_{k\geq 0} \mathrm{H}^0(E, L^{\otimes k}))$, for some line bundle L of degree n equal to the degree of the elliptic curve E. Thus, given the hyperelliptic involution h on E with ramification points $p_0, \ldots, p_3$, we need to define the line bundle L and a lift of h to it, i.e. a $\mathbb{Z}/2\mathbb{Z}$-linearization of L. Now, the line bundle L is of the form $\mathcal{O}_E(D)$ for some divisor D od degree n and we require that $h^*L = L$. Consider the isomorphism $E \to \operatorname{Pic}^\circ E$ given by $p \mapsto (p - p_0)$ then the involution h induces an involution of $\operatorname{Pic}^\circ E$ which is the multiplication by (-1), therefore the condition $h^*L = L$ gives:

$$h^*(D) \sim D \Leftrightarrow -(D - np_0) \sim D - np_0 \Leftrightarrow$$

$$\Leftrightarrow 2(D - np_0) \sim 0 \Leftrightarrow D - np_0 \sim (p_i - p_0) \Leftrightarrow D \sim (n-1)p_0 + p_i$$

for some $i = 0, \ldots, 3$. However, since we want to construct an involution that acts as multiplication by (-1) on the fibres above the ramification points $p_0, \ldots, p_3$, we must have $i = 0$ and n even. With these conditions on D we get an isomorphism $\theta : h * L \to L$ that is an involution (possibly up to scaling) and we can compose it with multiplication by (-1) on the fibers of L to get the desired involution on the minimal resolution of the elliptic cone. We can ask whether the existence of an equivariant smoothing for $p \in X$ corresponds to the existence of an involution on the correspondent Looijenga pair $(Y_{(d)}, D_{(d)})$ or, in the case $d = 8$, on either (Y_i, D) or (Y_{ii}, D), which is free on the complement of D and acts as a reflection on D, in the spirit of the main conjecture 4.0.2 for cusp singularities stated previously in this chapter. Let us start by stating a result on the existence of $\mathbb{Z}/2\mathbb{Z}$-equivariant smoothings.

Theorem 4.3.1. *Let E be a smooth elliptic curve of even degree $d \leq 8$ and ι an involution defined as above. Then there always exists a $\mathbb{Z}/2\mathbb{Z}$-equivariant smoothing of the singularity $p \in C(E)$. Furthermore, if $d = 8$, then the Milnor fibre of any $\mathbb{Z}/2\mathbb{Z}$-equivariant smoothing is isomorphic to M_i.*

Proof. First, let us assume E is a smooth elliptic curve of degree eight. As described above, a smoothing of $p \in C(E)$ can be obtained by considering one of the two associated del Pezzo surfaces: let us choose $\mathbb{P}^1 \times \mathbb{P}^1$. The description of how to obtain a smoothing of a simple elliptic singularity given at the beginning of this section, for this specific case, gives us the following set up: $\mathbb{P}^1 \times \mathbb{P}^1$ is embedded in $\mathbb{P}^8$ via the anticanonical divisor, $\mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{i} \mathbb{P}(H^0(-K_{\mathbb{P}^1 \times \mathbb{P}^1}))$, hence its projective cone is embedded in $\mathbb{P}^9$. Now, let us consider the involution j defined on this surface by $(z, w) \mapsto (z^{-1}, -w)$ which, in homogeneous coordinates, becomes

$$(x_0 : x_1), (y_0 : y_1) \mapsto (x_1, x_0), (y_0 : -y_1)$$

or, after a change of coordinates

$$(x_0 : x_1), (y_0 : y_1) \mapsto (x_0 : -x_1), (y_0 : -y_1)$$

Now we can use this involution to define a $\mathbb{Z}/2\mathbb{Z}$-action on the line bundle $-K_{\mathbb{P}^1 \times \mathbb{P}^1} = \mathcal{O}(2, 2)$. More precisely we define an involution ϑ on its global sections $z_0, \ldots, z_8$ as the map directly induced by j_0 on the bi-homogeneous monomials of degree $(2, 2)$ correspondent to each z_i composed with the map $z_i \mapsto -z_i$. Note that his involution fixes four of the nine monomials. Moreover we use the composition with the latter map to obtain a $\mathbb{Z}/2\mathbb{Z}$-action that is non trivial on the fibres above the four points in $\mathbb{P}^1 \times \mathbb{P}^1$ fixed by j. The map ϑ gives an involution on $\mathbb{P}^8_{z_0,\dots,z_8}$ therefore one on $\mathbb{P}^9_{x_0,\dots,x_9}$, extending the former involution so that it act trivially on the last coordinate. In order to show that there exists an equivariant smoothing of $p \in C(E)$ we just need to show that we can construct a family of planes $\mathcal{H}_t$ in $\mathbb{P}^9$ which is equivariant with respect to the action just defined on $\mathbb{P}^9$ and when intersected with $X := i(\mathbb{P}^1 \times \mathbb{P}^1) \subset \mathbb{P}^8$ gives the smooth elliptic curve E. To make the family $\mathcal{H}_t$ equivariant, we define it setting the coefficients $a_i(t)$ of the five coordinates which are not fixed by the involution to be equal to 0 for all t. Then $\hat{\mathcal{H}}_t = \mathcal{H}_t \cap \mathbb{P}^8$ is of the form $a_1(t)z_1 + a_3(t)z_3 + a_5(t)z_5 + a_7(t)z_7 = 0$, and its generic intersection with $i(\mathbb{P}^1 \times \mathbb{P}^1)$ is smooth. Indeed the linear system associated to $\hat{\mathcal{H}}_t$ in $H^0(-K_{\mathbb{P}^1\times\mathbb{P}^1}, \mathbb{C})$ is generated by the sections $x_0^2y_0y_1, x_1^2y_0y_1, x_0x_1y_0^2, x_0x_1y_1^2$. The base points are the fixed points on $\mathbb{P}^1 \times \mathbb{P}^1$ and it can be checked directly that $\hat{\mathcal{H}}_t$ is smooth at these points for generic coefficients $a_i(t)$. Thus we can conclude it is smooth everywhere by Bertini's theorem and it follows that $\mathcal{X}_t := \mathcal{H}_t \cap \overline{C(X)} \subset \mathbb{P}^9$ is an equivariant smoothing of the simple elliptic singularity of degree 8.

Every del Pezzo surface of degree $8 - i$ can be obtained from $\mathbb{P}^1 \times \mathbb{P}^1$ through the blowup of i points satisfying some suitable conditions. More precisely the del Pezzo surface of degree 6 can be obtained from $\mathbb{P}^1 \times \mathbb{P}^1$ blowing up two points on E which lie in the same j-orbit: this allows us to lift the involution defined on $\mathbb{P}^1 \times \mathbb{P}^1$ to a new involution which has the same properties as j. Similarly for the del Pezzo surfaces of degree 2 and 4. Then the same construction used above will provide the required equivariant smoothing.

On the other side, let us consider a simple elliptic singularity $p \in C(E)$ of degree eight, let ι be the involution defined on it and let us assume that there exists a $\mathbb{Z}/2\mathbb{Z}$-equivariant smoothing of this singularity whose Milnor fibre is isomorphic to M_{ii}. Then ι induces an involution on M_{ii} which can be extended to an involution of $\mathbb{F}_1$. Using Pinkham's work we see that a $\mathbb{Z}/2\mathbb{Z}$-equivariant smoothing can be globalized to a $\mathbb{Z}/2\mathbb{Z}$-equivariant smoothing of the projective cone over the elliptic curve, with general fiber $\mathbb{F}_1$. Now since there is a unique (-1)-curve on $\mathbb{F}_1$, it is necessarily fixed by the involution and can be blown down to obtain an involution on $\mathbb{P}^2$. The involution on $\mathbb{F}_1$ has only isolated fixed points (by upper semicontinuity of fiber dimension applied to the fixed locus in the family, since the involution has isolated fixed points on the special fiber). It follows that the involution on $\mathbb{P}^2$ has isolated fixed points, which is a contradiction (by the classification of involutions of $\mathbb{P}^2$). □

On the other hand if $(Y_{(d)}, D_{(d)})$, for $d \leq 6$ and even, and (Y_i, D), (Y_{ii}, D), where D has length 8, are the Looijenga pairs corresponding to the Milnor fibres of the smoothings of the appropriate simple elliptic singularity, then we can prove what follows.

Proposition 4.3.2. *The Looijenga pairs $(Y_{(d)}, D_{(d)})$ with $d \leq 6$ and even, and (Y_i, D) admit an antisymplectic involution which is free away from the anticanonical divisor and acts on it as a reflection, while the pair (Y_{ii}, D) does not admit an involution with this properties.*

Proof. To see that (Y_i, D) admits an involution with the properties described in the statement of the proposition, we follow the steps of proposition 3.3.7 to lift the involution given on $(\mathbb{P}^1 \times \mathbb{P}^1, \Delta)$ to an involution on $(T_{(d)}, D_{(d)})$ for $d = 4, 6$ and (T_i, G_i) for $d = 8$. Now, since the interior blowups are performed in a symmetric way in each of these cases, this involution lifts to one on $(Y_{(d)}, Y_{(d)})$ and (Y_i, D) respectively, again following proposition 3.3.7. In order to construct an involution on the Looijenga pair $(Y_{(2)}, D_{(2)})$ where $D_{(2)}$ is a cycle of two smooth rational curves of self intersection -2 we can proceed as follows. Let (Y, D) be the Looijenga pair obtained from $(\mathbb{P}^1 \times \mathbb{P}^1, \Delta)$ performing four interior blowups Δ_2, and Δ_4 each and one interior blowup on Δ_1 and δ_3. Then, again following proposition 3.3.7 we can lift the involution on $(\mathbb{P}^1 \times \mathbb{P}^1, \Delta)$ to a new involution on this (Y, D). This map is such that the two (-1) curves contained in D are in the same orbit, therefore we can contract them both and obtain map of Looijenga pairs $(Y, D) \to (Y_{(2)}, D_{(2)})$ and an induced involution on $(Y_{(2)}, D_{(2)})$, as desired.

Now consider (Y_{ii}, D), suppose it admits an involution ι. First note that, by the condition $\phi(i(q)) = (\phi(q))^{-1}$ for all $q \in \langle D_1, \ldots, D_8\rangle$ of proposition **??**, we have $\phi(D) = \pm 1$. It follows that there is an elliptic fibration $f : Y \to \mathbb{P}^1$ such that D is a fibre of f of multiplicity 1 or 2 (for $\phi(D)$=1 or -1 respectively). First suppose that D is a fibre of multiplicity 1. The involution defined on (Y_{ii}, D) gives an involution of the elliptic fibration, which on the base space $\mathbb{P}^1$ is given by $z \mapsto -z$. Note that the action on $\mathbb{P}^1$ cannot be the trivial one, since we are assuming that the involution ι has no fix points on $Y_{ii} \setminus D$. The $\mathbb{Z}/2\mathbb{Z}$-action we obtain on the fibration must have two fixed fibers: by assumption one of them has to be the anticanonical divisor D, we claim that the other fixed fiber has to be a smooth elliptic curve. Indeed, it cannot be a node or a cusp, since an involution on these singular curves would have to fix the singularity necessarily, contradicting the assumption; on the other hand this fiber cannot be reducible, namely a union of curves of self intersection -2, since the only curves with self intersection equal to -2 are the ones contained in D. Therefore the second fixed fiber F is a smooth elliptic curve where the involution acts as a translation. Now let us consider the quotient of $f : Y \to \mathbb{P}^1$ by the

$\mathbb{Z}/2\mathbb{Z}$-action. We get a new elliptic fibration $\hat{f} : Z \to S$, where S has two points corresponding to the fixed points in $\mathbb{P}^1$. The associated fibers are a smooth elliptic curve, which is given by the quotient of F and a new reducible fibre B whose dual graph looks as in figure 3: define $V := Z \setminus B$. Consider the exact sequence in relative cohomology:

$$\cdots \to H^2(Z, V) \to H^2(Z) \to H^2(V) \to H^3(Z, V) \to \cdots$$

We have $H^2(Z, V) \cong H^2(B) \cong \mathbb{Z}^9$ and $H^3(Z, V) \cong H^3(B) \cong 0$, therefore $H^2(V) \cong \mathrm{Pic}(Z)/\langle B_1, \ldots, B_5, E_1, \ldots, E_4 \rangle$ and on the other hand Tors $H_1(V) \cong$ Tors $H^2(V)$. Finally, if $G = -K_Z$ then the smooth fiber F has multiplicity 2 with $F = 2G$ and, as sublattices of $\mathrm{Pic}(Z)$, $G \cong \widetilde{D}_8$ and $G^\perp \cong \widetilde{E}_8$, hence we get inclusions $\widetilde{D}_8 \subset \widetilde{D}_8 + \mathbb{Z}G \subset \widetilde{E}_8$. Thus from the short exact sequence

$$0 \to G^\perp/\widetilde{D}_8 \to \mathrm{Pic}Z/\widetilde{D}_8 \to \mathbb{Z}$$

we obtain Tors $H_1(V) \cong$ Tors $\mathrm{Pic}(Z)/\widetilde{D}_8 \cong$ Tors $G^\perp/\widetilde{D}_8$ and the latter has order 4. The space $U := Y_{ii} \setminus D$ however has trivial fundamental group, therefore the map $U \to V$ is actually the universal cover map for the space V. Since it is a degree two normal cover, then the index of $\pi_1(U)$ in $\pi_1(V)$ has to be equal to two, thus giving $|\pi_1(V)| = 2$. This contradicts our previous conclusion that $|\mathrm{Tors}\ (H_1(V))| = 4$. Finally consider the case that D is a fibre of f of multiplicity 2. Then, considering the quotient as above, we obtain a rational elliptic surface with two multiple fibres, which is impossible.

□

These two propositions combined show that if a simple elliptic singularity admits an equivariant smoothing then there exists a correspondent negative semidefinite Loojienga pair equipped with an antisymplectic involution and viceversa.

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