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ANTICANONICAL MODELS OF SMOOTHINGS OF CYCLIC QUOTIENT SINGULARITIES

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ANTICANONICAL MODELS OF SMOOTHINGS OF CYCLIC QUOTIENT SINGULARITIES

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by

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ABSTRACT

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In this thesis we study anticanonical models of smoothings of cyclic quotient singularities. Given a surface cyclic quotient singularity $Q \in Y$, it is an open problem to determine all smoothings of Y that admit an anticanonical model and to compute it. In [HTU], Hacking, Tevelev and Urzúa studied certain irreducible components of the versal deformation space of Y, and within these components, they found one parameter smoothings $\mathcal{Y} \to \mathbb{A}^1$ that admit an anticanonical model and proved that they have canonical singularities. Moreover, they compute explicitly the anticanonical models that have terminal singularities using Mori's division algorithm

[M02]. We study one parameter smoothings in these components that admit an anticanonical model with canonical but non-terminal singularities with the goal of classifying them completely. We identify certain class of “diagonal” smoothings where the total space is a toric threefold and we construct the anticanonical model explicitly using the toric MMP.

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CHAPTER 1

INTRODUCTION

One of the most important problems in algebraic geometry is that of classifying algebraic varieties up to birational equivalence. Any variety is birational to a projective variety (Chow's lemma) and by Hironaka's theorem every variety has a resolution of singularities. Then we can rephrase the problem as classifying smooth projective varieties up to birational equivalence. Then the canonical divisor is well defined and an interesting birational invariant is the canonical model, X_{can}, defined as the Proj of the canonical ring:

$$R(X, K_X) = \bigoplus_{m \geq 0} H^0(X, mK_X)$$

which was proven to be finitely generated in [BCHM].

In dimension one, we have that every curve C is birational to a unique smooth projective curve and they can be classified in term of their genus $g = H^0(C, \omega_C)$. Here the genus could be either 0 (and $C \simeq \mathbb{P}^1$), 1 (C is an elliptic curve) or ≥ 2 (C is of general type), the canonical divisor, K_C, is either antiample, trivial or ample respectively and the canonical model is empty, a point or isomorphic to the curve C respectively. Then for each value of g we can study the moduli problem: Find a variety whose points parametrize smooth projective curves of genus g.

For dimension two, every smooth projective surface X is birational to a minimal surface X_m. Here the elemental birational maps are the blow-up and its inverse, the blow-down. Then X_m is obtained from X by performing a finite sequence of blow-ups and blow-downs. The minimal surfaces can be classified in terms of their Kodaida dimension $\kappa = dim X_{can}$. When X_{can} is empty we say that $\kappa = -\infty$, X_m is a Fano fibration over a curve and $-K_{X_m}$ is ample. If $\kappa = 0$, X_m is a K-trivial surface. If $\kappa = 1$, X_m is a K-trivial fibration over a curve. Finally, if $\kappa = 2$, X_m is of general type, $X_m \simeq X_{can}$ and $K_{X_{can}}$ is ample. This is the minimal model program (MMP) for surfaces. The minimal model is not unique, but the canonical model is and it may be singular.

For higher dimension, one would expect to see the same three cases for the MMP: $\kappa(X) = -\infty$ and then X_m is a Fano fibration (also called Mori fiber space), $0 \leq \kappa < dim X$ then X_m is a K-trivial fibration over X_{can}, or $\kappa(X) = dim X$ and X_m is of general type and $X_m \simeq X_{can}$ and $K_{X_{can}}$ is ample. Here the elemental birational maps are divisorial contractions and a new operation called flip.The existence of these birational maps has been established, but in order to complete MMP, it is necessary ti show that flips terminate, i.e. there are no infinite sequence of flips (this has been done in dimension 3, but it has not been proved for higher dimensions).

Similar to the canonical ring, one can define the anticanonical ring for a smooth projective variety X:

$$R(X, -K_X) = \bigoplus_{m \geq 0} H^0(X, -mK_X).$$

This ring is not a birational invariant and is not necessarily finitely generated, see [S82] for a discussion in the case of ruled surfaces. When the anticanonical ring is finitely generated, then Proj $R(X, -K_X)$ is called the *anticanonical model* of X. Two varieties isomorphic in codimension one clearly have the same anticanonical model (if it exists). Anticanonical models have been proven to be useful in the study of birational properties of Fano threefolds (see [I80]), and a characterization of varieties of Fano type is given in [CG] in terms of the singularities of their anticanonical models.

In this thesis we are interested in anticanonical models of smoothings $\mathcal{Y}$ of Y over a disc. The canonical model of $\mathcal{Y}$, which is given by a deformation of a P-resolution of Y (defined in section 1.2), is very useful in the study of deformations of Y [KSB88] and moduli spaces of surfaces of general type [HTU]. We do not assume $\mathcal{Y}$ is $\mathbb{Q}$-Gorenstein, so $K_{\mathcal{Y}}$ is not $\mathbb{Q}$-Cartier in general. The variety $\mathcal{Y}$ is normal and not projective, so we have to define what we mean by an anticanonical model of $\mathcal{Y}$ in this case. Note that $K_{\mathcal{Y}}$ is not Cartier so $\mathcal{O}(-nK_{\mathcal{Y}})$ is a divisorial sheaf, not a line bundle. We define the anticanonical model of $\mathcal{Y}$ as

$$\operatorname{Proj}_{\mathcal{Y}} \bigoplus_{n \geq 0} \mathcal{O}(-nK_{\mathcal{Y}})$$

under the condition that this is a sheaf of finitely-generated algebras.

In this chapter we define the singularities we will be using throughout the thesis and describe some results from birational geometry and toric MMP that will be needed when proving the main results of this project. Moduli spaces of minimal surfaces of general type as well as the background and results on extremal neighbourhoods are discussed in Chapter 2. In Chapter 3 we compute the anticanonical model for a class of one parameter smoothings of cyclic quotient singularities. These are the main results of this thesis. Finally, in Chapter 4, we discuss some problems for future projects.

1.1 Semi log canonical singularities

The notion of semi log canonical singularities was introduced in [KSB88] in order to investigate deformations of surface singularities and compactifications of moduli spaces for surfaces of general type. They are non-normal analogs of log canonical singularities. Here we give the basic definitions in the case of surfaces.

Definition 1.1 *A surface X will be called semi-smooth if every closed point of X is either smooth, or normal crossing (isomorphic to $(xy = 0) \subset \mathbb{C}^3$) or a pinch point (isomorphic to $(x^2 = zy^2) \subset \mathbb{C}^3$). The singular locus of a semi-smooth surface is a smooth curve C, called the double curve of X. The normalization $\nu\colon X^\nu \to X$ is smooth, $C^\nu = \nu^{-1}(C)$ is again smooth and $\nu\colon C^\nu \to C$ is 2:1 and ramifies at the pinch points.*

Definition 1.2 *A map $f\colon Y \to X$ is called a semi-resolution of X if f is proper, Y is semi-smooth, if C_Y is the double curve of Y, then no component of Y is mapped to a point and there is a finite set $S \subset X$ such that f restricted to $f^{-1}(X - S)$ is an isomorphism.*

A curve $E_i \subset Y$ is called exceptional if $f(E_i)$ is a point. Let $E = \cup E_i$. We say f is a good semi-resolution if in addition $E \cup C_Y$ has smooth components and transverse intersections.

For a semi-smooth surface, let $E \subset X$ be a curve not contained in the double curve of X and let $E^\nu = \nu^* E$. Then we have the usual adjunction formula

$$2g(E^\nu) - 2 = (\omega_{X^\nu} + E^\nu)E^\nu$$

where $\omega_{X^\nu} = \nu^* w_X(-C^\nu)$.

Given a semi-smooth surface X and a semi-smooth resolution $f\colon Y \to X$, then X can be obtained from Y by contracting (-1)-curves. A semi-resolution is minimal if no (-1)-curve is contracted by f.

Definition 1.3 *Let (x, X) be a $\mathbb{Q}$-Gorenstein surface singularity such that $X - x$ is semi-smooth. Let $f\colon Y \to X$ be a good semi-resolution of X. Then we can write*

$$\omega_Y^m \simeq \omega_X^{[s]} \otimes \mathcal{O}(\sum s a_i E_i)$$

where E_i are the exceptional divisors and $a_i \in \mathbb{Q}$ are called the discrepancies. Then (x, X) is called

(i) semi-canonical if $a_i \geq 0$,

(ii) semi-log-terminal if $a_i > -1$,

(iii) semi-log-canonical if $a_i \geq -1$.

It should be noted that these definitions are independent of the chosen good semi-resolution. Clearly $(i) \subset (ii) \subset (iii)$.

Now we recall the definition of certain normal singularities. A normal Gorenstein surface singularity is called

- simple elliptic if the exceptional divisor of the minimal resolution is a smooth elliptic curve.

- a cusp if the exceptional divisor of the minimal resolution is a cycle of smooth rational curves or a rational nodal curve.

We also say that a Gorenstein surface singularity X is a degenerate cusp if X is not normal and it has a minimal semi-resolution where the exceptional divisor is a cycle of smooth rational curves or a rational nodal curve.

For surfaces, the semi-log-canonical singularities were classified in [KSB88]. Here we list only the Goresntein cases.

Theorem 1.4 *Let (x, X) be a Gorenstein surface singularity such that $X - x$ is semi-smooth. Then*

1. *X is semi-canonical if and only if $x \in X$ is either smooth, a normal crossing point, a pinch point or a DuVal singularity.*

2. *X is semi-log-canonical if and only if X is either a simple elliptic singularity, a cusp, a degenerate cusp or semi-canonical.*

The semi-log-canonical singularities which are not Gorenstein, are obtained from the ones in the Theorem by taking a quotient by a cyclic group action $\mathbb{Z}_r$. A complete list of these singularities can be found in [KSB88].

These semi-log-canonical singularities are exactly the singularities that appear on surfaces on the boundary of the moduli space of surfaces of general type, also known as stable surfaces.

Let $(P \in X)$ be an slc surface germ. The index of P is defined as the least integer N such that $\omega_X^{[N]}$ is invertible. There is a cyclic cover of degree N, $p\colon Z \to X$, called the index one cover of $(P \in X)$, that satisfies the following properties: $p^{-1}(P)$ is a point $Q \in Z$, the morphism p is étale over $X \setminus P$, and Z is Gorenstein. The surface Z can be explicitly defined as

$$Z\colon = \underline{Spec}_X(\mathcal{O}_X \oplus \omega_X \oplus \ldots \oplus w_X^{[n-1]})$$

and the multiplication on $\mathcal{O}_Z$ is defined by fixing an isomorphism $w_X^{[n]} \to \mathcal{O}_X$.

A deformation $(P \in \mathcal{X}) \to (0 \in S)$ of an slc surface germ $(P \in X)$ is called $\mathbb{Q}$-Gorenstein if it is induced by an equivariant deformation of the index one cover of $(P \in X)$.

In the next section we recall the notion of cyclic quotient singularity and define extremal P-resolutions which will be used repeteadly in the rest of the thesis.

1.2 Surface singularities and extremal P-resolutions

A surface cyclic quotient singularity $Q \in Y$ is a germ at the origin of the quotient of $\mathbb{C}^2$ by a group action $(x, y) \mapsto (\mu x, \mu^q y)$ where μ is a primitive m-th root of unity, $1 \leq q < m$ and $gcd(m, q) = 1$. We denote this singularity by $\frac{1}{m}(1, q)$. Let $\tilde{Y} \to Y$ be the minimal resolution of Y, then on $\tilde{Y}$ we have a chain of exceptional curves E_i, $1 \leq i \leq s$, such that $E_i^2 = -b_i$ where the numbers b_i appear in the Hirzebruch-Jung continued fraction

$$\frac{m}{q} = [b_1, \ldots, b_s].$$

Example 1 *Let Y be the cone over the rational normal curve of degree n. It can be described as the subvariety in $\mathbb{A}^{n+1}$ defined by the vanishing of the ideal of 2×2 minors of the matrix*

$$\begin{bmatrix} x_1 & x_2 & \ldots & x_n \\ x_2 & x_3 & \ldots & x_{n+1} \end{bmatrix}$$

It has a cyclic singularity at the origin corresponding to $\frac{1}{n}(1, 1)$. Let X^+ denote the minimal resolution of Y. The exceptional divisor, denoted by C^+, is a rational curve with $(C^+)^2 = -n$.

An important class of cyclic quotient singularities are the T-singularities. These are normal surface singularities that admits a $\mathbb{Q}$-Gorenstein smoothing. There is an explicit description of these singularities ([KSB88]):

1. The singularities [4] and $[3, 2, \ldots, 2, 3]$ are T-singularities.

2. If $[b_1, \ldots, b_s]$ is a T-singularity, then so are

$$[2, b_1, \ldots, b_s + 1] \text{ and } [b_1 + 1, b_2, \ldots, b_s, 2].$$

3. Every T-singularity which is not a RDP (rational double point), can be obtained by starting with one of the singularities mentioned in (1), and applying the steps in (2).

It can be shown that a T-singularity that is not a RDP is of the form

$$\frac{1}{dm^2}(1, dma - 1)$$

for some $1 \leq a < m$ and $gcd(a, m) = 1$. In particular, T-singularities include Wahl singularities, i.e., cyclic quotient singularities of the form

$$\frac{1}{m^2}(1, ma - 1)$$

where $1 \leq a < m$ and $gcd(m, a) = 1$.

Definition 1.5 *Let $Q \in Y$ be a cyclic quotient singularity. A P-resolution of Y is a partial resolution $f^+ \colon X^+ \to Y$, such that X^+ has only T-singularities and K_{X^+} is relatively ample. An extremal P-resolution is a P-resolution $X^+ \to Y$ with the additional properties that the exceptional set is a curve $C^+ \simeq \mathbb{P}^1$, and X^+ has at most two Wahl singularities*

$$(P \in X^+) \simeq \mathbb{A}^2 / \frac{1}{m^2}(1, ma - 1)$$

along C^+

Example 2 *Consider the cyclic quotient singularity $\frac{1}{n}(1,1)$ from Example 1. Then its minimal resolution is an extremal P-resolution. For $n \neq 4$, $\frac{1}{n}(1,1)$ does not have any other P-resolutions and for $n = 4$, then the surface $\frac{1}{4}(1,1)$ itself is a P-resolution, since this singularity is a Wahl singularity with $m = 2$ and $a = 1$.*

By [KSB88, Theorem 3.9], there is a correspondence between irreducible components in the versal deformation space of a cyclic quotient singularity Y and P-resolutions of Y, i.e., partial resolutions $f^+ \colon X^+ \to Y$, such that X^+ has only T-singularities and K_{X^+} is relatively ample. Any deformation $\mathcal{Y}$ of Y within the corresponding component is obtained by blowing down a $\mathbb{Q}$-Gorenstein deformation $\mathcal{X}^+$ of X^+, which gives the canonical model of $\mathcal{Y}$.

1.3 Toric geometry and toric MMP

Here we review the Minimal Model Program for toric varieties. The idea is as follows: let $f \colon X \to Y$ a projective toric morphism, i.e., it is induced by a map of lattices. Here X is a $\mathbb{Q}$-factorial toric variety. Let D be a $\mathbb{Q}$-divisor on X. The goal is to recursively create intermediate varieties $f_i \colon X_i \to Y$ and $\mathbb{Q}$-divisors D_i in X_i, such that X_i is $\mathbb{Q}$-factorial and f_i is projective. After finitely many steps, we get a final object $\bar{f} \colon \bar{X} \to Y$ and a $\mathbb{Q}$-divisor $\bar{D}$ such that either $\bar{D}$ is $\bar{f}$-nef or we can find an extremal ray R of $NE(\bar{X}/Y)$ with $R \cdot \bar{D} < 0$. Then we have a contraction morphism $\phi_R \colon \bar{X} \to Z$ over Y and if $dim(Z) < dim(\bar{X})$ then ϕ_R is called a Fano contraction an we stop the process. On the other hand, if ϕ_R is birational, then it is either a divisorial contraction if ϕ_R contracts a divisor or a flipping contraction if ϕ_R contracts a subvariety of codim > 1. Next we list the main theorems for the toric MMP (see [FS] for more details).

Theorem 1.6 (Cone Theorem) *Let $f\colon X \to Y$ be a proper toric morphism. Then the cone*

$$NE(X/Y) \subset N_1(X/Y)$$

is a polyhedral convex cone. Moreover, if f is projective, then the cone in strongly convex.

Theorem 1.7 (Contraction Theorem) *Let $f\colon X \to Y$ be a projective toric morphism. Let F be an extremal face of $NE(X/Y)$. Then there exist a projective surjective toric morphism*

$$\phi_F\colon X \to Z$$

over Y with the following properties:

1. *Z is a toric variety and is projective over Y.*

2. *ϕ_F has connected fibers.*

3. *Let C be a curve in a fiber of f. Then $[C] \in F$ if and only if $\phi_F(C)$ is a point.*

Furthermore, if F is an extremal ray R and X is $\mathbb{Q}$-factorial, then Z is $\mathbb{Q}$-factorial and $\rho(Z/Y) = \rho(X/Y) - 1$ if ϕ_R is not small.

We will use the description of (terminal) three-dimensional toric flips given in [**?**], so we write it here for later reference.

Theorem 1.8 *Let $\phi_R\colon X(\Delta) \to Y(\Sigma)$ be the contraction morphism of an extremal ray R with $K_X \cdot R < 0$ of flipping type from a toric threefold with only $\mathbb{Q}$-factorial terminal singularities. Assume that Y is affine. Then we have the following description of the flipping contraction: There exist two three-dimensional cones*

$$\sigma_1 = \langle v_1, v_2, v_3 \rangle \in \Delta,$$

$$\sigma_2 = \langle v_1, v_2, v_4 \rangle \in \Delta,$$

sharing the two-dimensional wall

$$\omega = \langle v_1, v_2 \rangle$$

such that $[V(\omega)] \in R$. *Therefore*

$$\Delta = \{\sigma_1, \sigma_2, \ \textit{and their faces}\}$$

and

$$\Sigma = \{\langle v_1, v_2, v_3, v_4 \rangle, \ \textit{and their faces}\}.$$

This theorem basically says that the cone of X is obtained by subdiving the cone Y and this is exactly what we will use in Chapter 3.

CHAPTER 2

MORE BACKGROUND

2.1 Moduli spaces

A moduli space is a space whose points parametrize geometric objects. A well known example is $\mathcal{M}_g$, the moduli space of smooth algebraic curves (or equivalently Riemann surfaces) of genus g. An interesting fact about moduli spaces is that they have the structure of an algebraic variety (or more general, the structure of a scheme or stack). Often we study families of curves in $\mathcal{M}_g$ and the curves in these families may degenerate to a singular curve which is not an element of $\mathcal{M}_g$. Then it becomes necessary to consider a compactification of this space which in the case of curves is due to Deligne and Mumford. For dimension two we have the Gieseker moduli space $\mathcal{M}_{K^2,\chi}$ where each point represents an isomomorphism class of a minimal surface of general type with invariants K^2, χ. Similarly to the case of curves, it is necessary to consider a compactification of this space, and the one I am interested in is the compactification due to Kollár, Shepherd-Barron and Alexeev (known as KSBA compactification) denoted by $\overline{\mathcal{M}}_{K^2,\chi}$. A surface X on the boundary has semi-log canonical (slc) singularities and nearby smooth surfaces are obtained by a $\mathbb{Q}$-Gorenstein smoothing of these, i.e. a one-parameter smoothing $\mathcal{X}$ of X such that a multiple of the canonical class of $\mathcal{X}$ is Cartier.

One of the motivations to study the KSBA compactification is that in general is really difficult to construct examples of surfaces in $\mathcal{M}_{K^2,\chi}$ and for some values of the invariants it is not even known if the moduli space is not empty. The smallest possible values for the invariants are $\chi = 1$ and $K^2 = 1$. Surfaces with these invariants are known as Godeaux surfaces. It is known that the algebraic fundamental group, π^{alg}, of a Godeaux surface is a cyclic group of order ≤ 5. Miles Reid conjectured that $\mathcal{M}_{1,1}$ is formed by five unirational components labeled according to the possible algebraic fundamental groups (and these should coincide with the topological fundamental group). Godeaux surfaces with $\pi_1 = \mathbb{Z}/3, \mathbb{Z}/4, \mathbb{Z}/5$ are completely described in [R78] but a description in the cases $\pi = 1, \mathbb{Z}/2$ is still not known. I am interested in the component corresponding to simply connected Godeaux surfaces. The first known examples of such surfaces is the family constructed by Barlow in [B85]. These were the only known examples until the work of Y. Lee and J. Park in [LP07] where the authors construct stable simply connected Godeaux and Campedelli surfaces (they have $K^2 = 2$ and $\chi = 1$) which admit a $\mathbb{Q}$-Gorenstein smoothing where the general fiber is a surface in $\mathcal{M}_{1,1}$ and $\mathcal{M}_{1,2}$ respectively. The fact that the surfaces admit a $\mathbb{Q}$-Gorenstein smoothing imposes extra conditions that together with $\chi = 1$ implies that $1 \leq K^2 \leq 4$. Following the Lee-Park method, in [PPS09] and [PPS09-2] the authors construct examples of stable simply connected surfaces with $K^2 = 3, 4$ admitting a $\mathbb{Q}$-Gorenstein smoothing thus proving that $\overline{\mathcal{M}}_{3,1}$ and $\overline{\mathcal{M}}_{4,1}$ are nonempty. In my joint work with Giancarlo Urzúa, we use the Lee-Park method to construct new examples of smoothable KSBA normal surfaces with Wahl singularities, i.e., cyclic quotient singularities of the form $\frac{1}{m^2}(1, ma - 1)$, with $gcd(a, m) = 1$. Among log-canonical surface singularities, the ones which have a rational homology disk smoothing (a smoothing with Milnor number equal to zero), are Wahl singularities and three distinguished elliptic quotient singularities. Elliptic quotient singularities are normal two dimensional singularities with discrepancy (-1) and such that its canonical cover is a simple elliptic singularity, and the three distinguished ones, have a minimal resolution where the exceptional divisor consists of 4 smooth rational curves E_1, E_2, E_3, and F. The curves E_i are disjoint, each meets the central curve F transversally at one point, and

$$[-E_1^2, -E_2^2, -E_3^2; -F^2] = [3,3,3;4], [4,2,4;3] \text{ or } [2,3,6;2].$$

In [SU], using the Lee-Park method we were able to prove

Theorem 2.1 *There exist a smoothable KSBA simply connected, normal surfaces with $p_g = 0$ and $K^2 = 1, 2$ having each of the distinguished elliptic quotient singularities.*

In the next section we discuss extremal neighborhoods, and show that they can be viewed as the total space of a flat family of surfaces. This fact is used to study families of surfaces on the moduli space $\overline{\mathcal{M}}_{K^2,\chi}$ that degenerate to a singular surface which is not stable. Then it becomes necessary to compute the so called stable limit, which amounts to run the relative MMP for the family. The main results of [HTU] and [M02] show how to compute the flip and divisorial contractions of extremal neighborhoods of type $K1A$ and $k2A$ (defined in the next section).

2.2 Overview of HTU

Let $(C \subset \mathcal{X})$ be a germ of a threefold along a proper reduced irreducible curve C. $\mathcal{X}$ will be called an extremal neighborhood if there is a germ $(Q \in \mathcal{Y})$ of a threefold and a proper birational morphism

$$f \colon (C \subset \mathcal{X}) \to (Q \in \mathcal{Y})$$

satisfying that $f_*\mathcal{O}_{\mathcal{X}} = \mathcal{O}_{\mathcal{Y}}$, $f^{-1}(Q) = C$ as sets and $K_{\mathcal{X}} \cdot C < 0$. The exceptional locus of f is either equal to C, in which case we call f a flipping contraction, or is a divisor whose image is a curve passing through Q, in which case we call f a divisorial contraction.

In the flipping case, we need to construct the flip of f, which is defined as a germ $(C^+ \subset \mathcal{X}^+)$ of a threefold along a proper reduced curve such that $\mathcal{X}^+$ has terminal singularities and there is a proper birational morphism

$$f^+ \colon (C^+ \subset \mathcal{X}^+) \to (Q \in \mathcal{Y})$$

such that $f^+_*\mathcal{O}_{\mathcal{X}^+} = \mathcal{O}_{\mathcal{Y}}$, $(f^+)^{-1}(Q) = C^+$ as sets and $K_{\mathcal{X}^+} \cdot C^+ > 0$.

Extremal neighborhoods can be classified by looking at a general member $E_X \in |-K_{\mathcal{X}}|$ and $E_Y = f(E_X) \in |-K_{\mathcal{Y}}|$. Then by [?], E_X and E_Y are normal surfaces with at worst DuVal singularities. When $Q \in E_Y$ is a singularity of type A, we say that the extremal neighborhood is semistable. A semistable extremal neighborhood will be of type k1A (k2A respectively) if E_X has one singular point (has two singular points respectively).

Consider a general element $s \in m_{\mathcal{Y},Q} \subset \mathcal{O}_{\mathcal{Y},Q}$ and the corresponding morphism $\mathcal{Y} \to \mathbb{A}^1_s$. Then we have an induced morphism $\mathcal{X} \to \mathbb{A}^1_s$. Let $X \subset \mathcal{X}$ and $Y \subset \mathcal{Y}$ be the fibers over $s = 0$. Then we can think of an extremal neighborhood as a total space of a flat family of surfaces.

Given a family $\mathcal{X}^* \to \mathbb{D}^*$ of surfaces in the moduli space $\mathcal{M}_{K^2,\chi}$ over a punctured smooth curve germ, then we want to complete this family to a flat family $\mathcal{X} \to \mathbb{D}$ where the special fiber is a stable surface in the boundary of $\overline{\mathcal{M}}_{K^2,\chi}$. By results in [KSB88], one can reduce to the case of flipping and divisorial contractions of type k1A or k2A with $b_2(X_s) = 1$. Now given an extremal neighborhood of type k1A or k2A with $b_s(X_s) = 1$, then in [HTU] it is shown that the special fiber X is normal and has one or two Wahl singularities.

Mori gave an explicit algorithm to compute flips of extremal neighborhoods of type K2A in [M02], and in [HTU] they extend this algorithm to compute flips of extremal neighborhoods of type K1A by constructing a universal family of extremal neighborhoods thus showing that K1A and K2A neighborhoods belong to the same deformation family. In the flipping case, they show that $Y \subset \mathcal{Y}$ is a germ of a cyclic quotient singularity $\frac{1}{\Delta}(1, \Omega)$ and $X^+ \subset \mathcal{X}^+$ is its extremal P-resolution.

We describe the results in [HTU], but we will start with an extremal P-resolution X^+ of Y, and from here we will see that this gives a universal family of k1A and k2A extremal neighbourhoods (antiflips) such that its flip has X^+ as central fiber.

Let X^+ be an extremal P-resolution of a cyclic quotient singularity $Q \in Y$, and let $\mathcal{X}^+$ be a $\mathbb{Q}$-Gorenstein smoothing of X^+. Let $\frac{1}{m_1'^2}(1, m_1'a_1' - 1)$, $\frac{1}{m_2'^2}(1, m_2'a_2' - 1)$ be the singularities of X^+ with Hirzebruch-Jung continued fractions $\frac{m_1'^2}{m_1'a_1'-1} = [e_1, \ldots, e_{r_1}]$ and $\frac{m_2'^2}{m_2'a_2'-1} = [f_1, \ldots, f_{r_2}]$. Then the singularity $Q \in Y$ is given by

$$\frac{\Delta}{\Omega} = [f_{r_2}, \ldots, f_1, c, e_1, \ldots, e_{r_1}]$$

where $-c$ is the self-intersection of the proper transform of C^+ in the minimal resolution of X^+. Define

$$\delta = cm_1'm_2' - m_1'a_2' - m_2'a_1'$$

and define

$$m_2 = m_1', \qquad a_2 = m_1' - a_1' \text{ if } m_1' \neq a_1', \text{ or } a_2 = 1 \text{ otherwise}$$

and

$$m_1 = \delta m_1' + m_2', \qquad a_1 = \frac{\delta + m_1 m_2 - a_2 m_1}{m_2}$$

.

The numbers (m_1, a_1, m_2, a_2) represent an "initial" extremal neighbourhood $\mathcal{X}^-$ of type $k2A$ where the special fiber has Wahl singularities $\frac{1}{m_1^2}(1, m_1 a_1 - 1)$, $\frac{1}{m_2^2}(1, m_2 a_2 - 1)$ and such that the flip of $\mathcal{X}^-$ is $\mathcal{X}^+$. In fact, as shown in [HTU], there is a toric surface M (of locally finite type if $\delta > 1$), corresponding to a fan Σ with cones $\{0\}$, $\mathbb{R}_{\geq 0} v_i$, $\langle v_i, v_{i+1} \rangle_{\mathbb{R}_{\geq 0}}$ for some primitive vectors $v_i \in \mathbb{Z}^2$ defined as follows:

$$v_1 = (1, 0),\ v_2 = (\delta, 1),$$

$$v_{i+1} + v_{i-1} = \delta v_i$$

There is a toric birational morphism $p\colon M \to \mathbb{A}^2_{u_1' u_2'}$, flat irreducible families of surfaces

$$\mathbb{X}^- \to M, \qquad \mathbb{Y} \to M, \qquad \mathbb{X}^+ \to M$$

and morphisms

$$\pi\colon \mathbb{X}^- \to \mathbb{Y} \times M, \text{ and } \pi^+\colon \mathbb{X}^+ \to \mathbb{Y}.$$

There exist a morphism $g\colon \mathbb{A}^1_t \to M$ such that $p(g(0)) = 0 \in A^2$ and such that the flip of $\mathcal{X}^- \to \mathcal{Y}$ is the pullback of $\mathbb{X}^- \to \mathbb{Y}^+$ under $p \circ g$. Each v_i has a label (m_i, a_i). If we take a smoothing $\mathcal{Y}$ of Y with axial multiplicites (α_1, α_2) (defined below) corresponding to one of the v_i's (or corresponding to a ray which is between two consecutive primitive vectors v_i, v_{i+1}), then we get an extremal neighbourhood of type $k1A$ (of type k2A, respectively). The special fiber has a Wahl singularity $\frac{1}{m_i^2}(1, m_i a_i - 1)$ (respectively, has two Wahl singularities $\frac{1}{m_i^2}(1, m_i a_i - 1)$, $\frac{1}{m_{i+1}^2}(1, m_{i+1} a_{i+1} - 1)$). All these extremal neighbourhoods have a flip with central fiber equal to X^+.

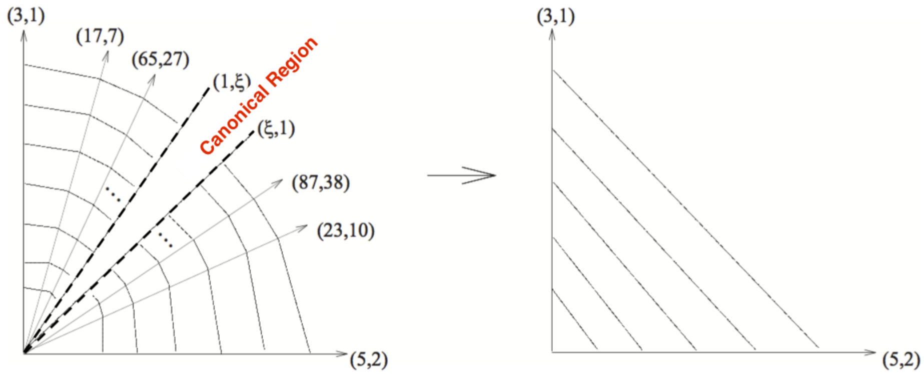

Figure 1: Example of a map of fans $p\colon M \to \mathbb{A}^2$

In addition, Corollary 3.23 in [HTU] shows that given an extremal P-resolution $C^+ \subset X^+$ of some cyclic quotient singularity $Q \in Y$, if we choose a toric structure for X^+ and let B^+ be its toric boundary, then a smoothing $\mathcal{X}^+ \to \mathbb{A}^1$ of X^+ admits a terminal antiflip (anticanonical model) if and only if the following conditions are satisfied:

1. There is a divisor $D \in |-K_{\mathcal{X}^+}|$ such that $D|_{\mathcal{X}^+} = B^+$
2. The axial multiplicities α_1, α_2 of the singularitites of $\mathcal{X}^+$ satisfy $\alpha_1^2 - \delta\alpha_1\alpha_2 + \alpha_2^2 > 0$.

Let

$$\mathcal{R} = \{(\alpha_1, \alpha_2) | \alpha_1^2 - \delta\alpha_1\alpha_2 + \alpha_2^2 \leq 0\}$$

We will call this set the *Canonical Region* (see Figure 1). If $(\alpha_1, \alpha_2) \in \mathcal{R}$, then the anticanonical model has non-terminal singularities [HTU], and no explicit construction or description is known .

Definition 2.2 *By Corollary 3.23 in [HTU], there is a divisor $D \in |-K_{\mathcal{X}^+}|$ such that the restriction $D|_{X^+}$ is equal to a chain of smooth rational curves $L_1 - C^+ - L_2 \subset$*

X^+. Let P_1, P_2 be points where L_1 and L_2 intersect C^+ respectively. The points P_1, P_2 are either smooth or a Wahl singularity.

(i) If $P = \frac{1}{m^2}(1, ma-1)$ is a singular point of X^+, there is an analytic isomorphism (over $\mathbb{C}$)

$$(P \in X^+) \simeq (\xi\eta = \zeta^m) \subset A^3_{\xi,\eta,\zeta}/\frac{1}{m}(1,-1,a)$$

and then for the deformation $(P \in \mathcal{X}^+) \to \mathbb{A}^1_t$ we get an analytic isomorphism

$$(P \in \mathcal{X}^+) \simeq (0 \in \xi\eta = \zeta^m + t^\alpha) \subset A^3_{\xi,\eta,\zeta}/\frac{1}{m}(1,-1,a) \times \mathbb{A}^1_t$$

for some $\alpha \in \mathbb{N}$ called the axial multiplicity of $P \in \mathcal{X}^+$.

(ii) If P is a smooth point of X^+ then the local deformation $(P_i \in D \subset \mathcal{X}^+) \to (0 \in \mathbb{A}^1_t)$ of $(P_i \in C^+ \subset X^+)$ is of the form

$$(0 \in (\xi\eta = t^{\alpha_i} h_i(t))) \subset \mathbb{A}^2_{\xi,\eta} \times \mathbb{A}^1_t$$

for some $\alpha_i \in \mathbb{N}$ and convergent power series $h_i(t)$ with $h_i(0) \neq 0$. The number α_i is called the axial multiplicity of $P_i \in \mathcal{X}^+$.

2.2.1 Construction of the flip

Next we describe an explicit model for $\mathcal{X}^+$ following [M02] and [HTU].

Definition 2.3 *Define a sequence*

$$d(1) = m_1, \quad d(2) = m_2$$

and

$$d(i+1) = \delta d(i) - d(i-1) \text{ for } i \geq 2.$$

By [HTU, Lemma 3.3], there exists $k \geq 3$ such that $d(k-1) > 0$ and $d(k) \leq 0$. Consider the sequence $c\colon \mathbb{Z} \to \mathbb{Z}$ defined as

$$c(1) = a_1, \quad c(2) = m_2 - a_2$$

and

$$c(i-1) + c(i+1) = \delta c(i) \text{ for } 2 \leq i \leq k-1.$$

We also define $c(k+1) = -c(k-1)$ and $c(k+2) = -c(k)$.

Define

$$W' = (x_1' y_1' = z^{m_1'} x_2'^{\delta} + u_1', x_2' y_2' = z^{m_2'} x_1'^{\delta} + u_2') \subset \mathbb{A}^5_{x_1', x_2', y_1', y_2', z} \times \mathbb{A}^2_{u_1', u_2'}$$

and

$$\Gamma' = \{\gamma' = (\gamma_1', \gamma_2') \quad | \quad \gamma_1'^{m_1'} = \gamma_2'^{m_2'}\} \subset \mathbb{G}_m^2.$$

Define an action of Γ' on W' by

$$\gamma \cdot (x_1', x_2', y_1', y_2', z, u_1', u_2') \mapsto (\gamma_1' x_1', \gamma_2' x_2', \gamma_1'^{-1} y_1', \gamma_2'^{-1}, \gamma_1'^{c(k-1)} \gamma_2'^{c(k)} z, u_1', u_2').$$

Define

$$W'^0 = W' \setminus (x_1' = x_2' = 0)$$

and

$$\mathbb{X}^+ = (W'^0)/\Gamma'.$$

Write $U_1' = (x_2' \neq 0) \subset \mathbb{X}^+$ and $U_2' = (x_1' \neq 0) \subset \mathbb{X}^+$.

Then $\mathbb{X}^+ = U_1' \cup U_2'$,

$$U_i' = (\xi_i' \eta_i' = \zeta_i'^{m_i'} + u_i') \subset \mathbb{A}^3_{\xi_i, \eta_i', \zeta_i'} / \frac{1}{m_i'}(1, -1, a_i') \times \mathbb{A}^2_{u_1', u_2'}$$

for each $i = 1, 2$, and glueing is given by

$$U_1' \supset (\xi_1' \neq 0) = (\xi_2' \neq 0) \subset U_2',$$

$$\xi_1'^{m_1'} = \xi_2'^{-m_2'}, \qquad \xi_1'^{-c(k-1)} \zeta_1' = \xi_2'^{-c(k)} \zeta_2'.$$

2.3 A results of Mori and Prokhorov

In [MP], Mori and Prokhorov classified terminal threefold extremal contractions of type (IA) and $(IA^\vee)$. These threefolds are germs of an irreducible curve C which has negative intersection with the canonical divisor, and they study a general element $H \in |\mathcal{O}_C(-K)|$. We are interested in the result they obtain in the case when every member of $H \in |\mathcal{O}_C(-K)|$ is non normal. in this context they prove the following theorem.

Theorem 2.4 *Let $\nu \colon H^\nu \to H$ be the normalization of H, and let $C^\nu = \nu^{-1}(C)$. The dual graph of (H^ν, C^ν) is of the form*

$$[a_r, \dots, a_1] - \circ - [c_1, \dots, c_l] - \circ - [b_1, \dots, b_s].$$

In particular $C^\nu = C_1 \cup C_2$ is reducible where C_1 and C_2 are represented by $\circ$ in the graph. The chain $[a_r, \dots, a_1]$ corresponds to the singularity $\frac{1}{m}(1, a)$ and the chain $[b_1, \dots, b_s]$ corresponds to its conjugate singularity $\frac{1}{m}(1, -a)$ for some integers m, a with $gcd(m, a) = 1$. The chain $[c_1, \dots, c_l]$ corresponds to the point $Q = C_1 \cap C_2$. Moreover,

$$\sum (c_i - 2) \leq 2 \text{ and } \bar{C}_1^2 + \bar{C}_2^2 + 5 - \sum (c_i - 2) \geq 0$$

where $\bar{C}_i^2$ represents the proper transform of C_i in the minimal resolution of H^ν.

In Chapter 3, we will prove that if we take a smoothing $\mathcal{X}^+ \to \mathbb{A}^1$ of an extremal P-resolution of some cyclic quotient singularity $Q \in Y$ with equal axial multiplicities $\alpha_1 = \alpha_2$, then $\mathcal{X}^+$ is actually a toric threefold and using tools from toric geometry and the toric MMP, we will compute the anticanonical model $\mathcal{X}^-$ explicitly. In particular, when we analyze the special fiber X^- of $\mathcal{X}^-$, we will get a non normal surface with semi-log-canonical singularities and for $\delta \geq 3$, the minimal resolution of the normalization of X^- will have a configuration of curves of the same form as in the previous theorem, but our threefolds will be non terminal, then the condition

$$\sum(c_i - 2) \leq 2 \text{ and } \bar{C}_1^2 + \bar{C}_2^2 + 5 - \sum(c_i - 2) \geq 0$$

is not true in our case. It would be interesting to classify canonical but non-terminal extremal neighborhoods having these configuration of exceptional curves.

CHAPTER 3

DIAGONAL CASE

3.1 Change of coordinates

Now we consider the diagonal case, corresponding to smoothings $\mathcal{X}^+$ where the axial multiplicities α_1 and α_2 are equal, which translates into $u_1 = u_2 = u$ in the equations of $\mathbb{X}^+$. The equations for $\mathcal{X}^+$ appear complicated, but we show that in the diagonal case $\mathcal{X}^+$ is a toric variety using a non-trivial change of variables.

Lemma 3.1 *$\mathcal{X}^+$ is a toric threefold.*

Proof. We will show that $\mathcal{X}^+$ is isomorphic to the variety $Z = V_1 \cup V_2$ where

$$V_i = (\xi_i' \tilde{\eta}_i = q_i) \subset \mathbb{A}^3_{\xi_i', \tilde{\eta}_i, \zeta_i} / \frac{1}{m_i'}(1, -1, a_i') \times \mathbb{A}^1_{q_i}$$

with gluing given by

$$\xi_1^{m_1'} = \xi_2^{-m_2'}, \quad \xi_1' \tilde{\eta}_1 = \xi_2' \tilde{\eta}_2$$

Consider the map $f_1 \colon \mathbb{A}^3_{\xi_1', \eta_1', \zeta_1'} \times \mathbb{A}^1_u \to \mathbb{A}^3_{\xi_1', \tilde{\eta}_1, \zeta_1'} \times \mathbb{A}^1_{q_1}$ given by

$$(\xi_1', \eta_1', \zeta_1', u) \mapsto (\xi_1', \eta_1' + \xi_1'^{\delta - 1} \zeta_1'^{m_2'}, \zeta_1', \zeta_1'^{m_1'} + \xi_1'^{\delta} \zeta_1'^{m_2'} + u).$$

The group $\mathbb{Z}_{m'_1}$ acts on $\mathbb{A}^3_{\xi'_1,\eta'_1,\zeta'_1}$ with weights $1,-1,a'_1$ and on $\mathbb{A}^3_{\xi'_1,\tilde{\eta}_1,\zeta'_1}$ with the same weights. Let γ be a primitive m'_1-th root of unity, then

$$\begin{aligned}
&f_1(\gamma\cdot(\xi'_1,\eta'_1,\zeta'_1,u))\\
&=(\gamma\xi'_1,\gamma^{-1}\eta'_1+\gamma^{\delta-1+a'_1m'_2}\xi_1'^{\delta-1}\zeta_1'^{m'_2},\gamma^{a'_1}\zeta'_1,\zeta'_1+\gamma^{\delta+a'_1m'_2}\xi_1'^{\delta}\zeta_1'^{m'_2}+u)\\
&=(\gamma\xi'_1,\gamma^{-1}\eta'_1+\gamma^{-1}\xi_1'^{\delta-1}\zeta_1'^{m'_2},\gamma^{a'_1}\zeta'_1,\zeta'_1+\xi_1'^{\delta}\zeta_1'^{m'_2}+u)\\
&=\gamma\cdot f_1(\xi'_1,\eta'_1,\zeta'_1,u)
\end{aligned}$$

where we have used that $\delta=cm'_1m'_2-m'_1a'_2-m'_2a'_1$ and then $\delta+m'_2a'_1$ is a multiple of m'_1. Therefore f_1 is equivariant and it descends to a map $f_1\colon U'_1\to V_1$. Similarly, consider the map $f_2\colon \mathbb{A}^3_{\xi'_2,\eta'_2,\zeta'_2}\times\mathbb{A}^1_u\to\mathbb{A}^3_{\xi'_2,\tilde{\eta}_2,\zeta'_2}\times\mathbb{A}^1_{q_2}$ given by

$$(\xi'_2,\eta'_2,\zeta'_2,u)\mapsto(\xi'_2,\eta'_2+\xi_2'^{\delta-1}\zeta_2'^{m'_1},\zeta'_2,\zeta_2'^{m'_2}+\xi_2'^{\delta}\zeta_2'^{m'_1}+u).$$

The group $\mathbb{Z}_{m'_2}$ acts on $\mathbb{A}^3_{\xi'_2,\eta'_2,\zeta'_2}$ with weights $1,-1,a'_2$ and on $\mathbb{A}^3_{\xi'_2,\tilde{\eta}_2,\zeta'_2}$ with the same weights. A similar argument as above shows that f_2 is equivariant with respect to this action and therefore it descends to a map $f_2\colon U'_2\to V_2$.

Note that f_1 and f_2 are isomorphisms since we can write their inverses by expressing η'_i and u in terms of $\xi'_i,\tilde{\eta}_i,\zeta'_i,q_i,\ i=1,2$. Therefore $U'_i\simeq V_i$. Now we show that the maps f_1, f_2 can be glued.

Let $(\xi'_1,\eta'_1,\zeta'_1,u)\subset(\xi'_1\neq 0)\subset U'_1$, then

$$\begin{aligned}
\xi'_1\eta'_1 &= \zeta_1'^{m'_1}+u\\
\xi'_1(\tilde{\eta}_1-\xi_1'^{-1}\zeta_2'^{m'_2}) &= \zeta_1'^{m'_1}+u\\
\xi'_1\tilde{\eta}_1 &= \zeta_1'^{m'_1}+\zeta_2'^{m'_2}+u\\
\xi'_1\tilde{\eta}_1 &= q_1.
\end{aligned}$$

Similarly, if $(\xi_2', \eta_2', \zeta_2', u) \subset (\xi_2' \neq 0) \subset U_2'$, then

$$
\begin{aligned}
\xi_2'\eta_2' &= \zeta_2'^{m_2'} + u \\
\xi_2'(\tilde{\eta}_2 - \xi_2'^{-1}\zeta_1'^{m_1'}) &= \zeta_2'^{m_2'} + u \\
\xi_2'\tilde{\eta}_2 &= \zeta_1'^{m_1'} + \zeta_2'^{m_2'} + u \\
\xi_2'\tilde{\eta}_2 &= q_2.
\end{aligned}
$$

Since $U_1' \supset (\xi_1' \neq 0) = (\xi_2' \neq 0) \subset U_2'$ we conclude that the gluing on $V_1 \supset f_1((\xi_1' \neq 0)) = f_2((\xi_2' \neq 0)) \subset V_2$ is given by

$$\xi_1'^{m_1'} = xi_2'^{-m_2'}, \quad \xi_1'\tilde{\eta}_1 = \xi_2'\tilde{\eta}_2.$$

Finally note that all the relations defining Z are monomial relations and Z is normal, so Z is a toric variety. $\diamond$

3.2 Toric computations

Next we work out the fan of this toric variety. Say $U_1 = Spec(\mathbb{C}[S_{\sigma_1}])$ for $\sigma_1 \subset N_{\mathbb{R}}$ where $\sigma_1^\vee = Cone(e_1^\vee, e_2^\vee, e_3^\vee)$ and $U_2 = Spec(\mathbb{C}[S_{\sigma_2}])$ for $\sigma_2 \in N_{\mathbb{R}}$ where $\sigma_2^\vee = Cone(f_1^\vee, f_2^\vee, f_3^\vee)$.

Notation 3.2 *Given an element $r_1e_1^\vee + r_2e_2^\vee + r_3e_3^\vee \in S_{\sigma_1}$, we denote the corresponding element of its semigroup algebra $\mathbb{C}[S_{\sigma_1}]$ as $\xi_1'^{r_1}\tilde{\eta}_1^{r_2}\zeta_1'^{r_3}$. Similarly, to an element $s_1f_1^\vee + s_2f_2^\vee + s_3f_3^\vee \in S_{\sigma_2}$ we associate the element $\xi_2'^{s_1}\tilde{\eta}_2^{s_2}\zeta_2'^{s_3} \in \mathbb{C}[S_{\sigma_2}]$.*

The lattices of U_1 and U_2 are given by

$$L_1^\vee = \{\sum b_i e_i^\vee : b_1 - b_2 + a_1' b_3 \text{ is divisible by } m_1', b_i \in \mathbb{Z}\}$$

and

$$L_2^\vee = \{\sum d_i f_i^\vee : d_1 - d_2 + a_2' d_3 \text{ is divisible by } m_2', d_i \in \mathbb{Z}\}$$

respectively and from the glueing relations we get

$$\begin{aligned} m_1' e_1^\vee &= -m_2' f_1^\vee \\ -c(k-1)e_1^\vee + e_3^\vee &= -c(k) f_1^\vee + f_3^\vee \\ e_1^\vee + e_2^\vee &= f_1^\vee + f_2^\vee. \end{aligned}$$

Let $w_1^\vee = m_1' e_1^\vee = -m_2' f_1^\vee$, $w_2^\vee = e_1^\vee + e_2^\vee = f_1^\vee + f_2^\vee$, and $w_3^\vee = e_3^\vee - c(k-1)e_1^\vee = f_3^\vee - c(k) f_1^\vee$.

Proposition 3.3 *The lattices $L_1^\vee$ and $L_2^\vee$ are both equal to the lattice $N^\vee = \langle w_1^\vee, w_2^\vee, w_3^\vee \rangle$.*

Proof. It is clear that $N^\vee \subset L_1^\vee$ and $N^\vee \subset L_2^\vee$. Let $b_1 e_1^\vee + b_2 e_2^\vee + b_3 e_3^\vee \in L_1^\vee$. We write the $e_i^\vee$'s in terms of the $w_i^\vee$'s

$$\begin{aligned} e_1^\vee &= \frac{1}{m_1'} w_1^\vee \\ e_2^\vee &= w_2^\vee - \frac{1}{m_1'} w_1^\vee \\ e_3^\vee &= w_3^\vee + \frac{c(k-1)}{m_1'} w_1^\vee \end{aligned}$$

Then

$$b_1 e_1^\vee + b_2 e_2^\vee + b_3 e_3^\vee = \left(\frac{b_1 - b_2 + c(k-1) b_3}{m_1'} \right) w_1^\vee + b_2 w_2^\vee + b_3 w_3^\vee.$$

But this element is in $L_1^\vee$ so $b_1 - b_2 + a_1' b_3 = m_1' l$ for some $l \in \mathbb{Z}$. In the construction of the flip in [HTU], a_1' is defined as a number $0 \le a_1' \le m_1'$ with $a_1 \equiv c(k-1) \bmod m_1'$. Then the coefficient of $w_1^\vee$ in the above expression in an integer and $L_1^\vee = \{\sum b_i w_i^\vee | b_i \in \mathbb{Z}\} = N^\vee$.

Similarly, if $d_1 f_1^\vee + d_2 f_2^\vee + d_3 f_3^\vee \in L_2^\vee$ then writing the $f_i^\vee$'s in terms of the $w_i^\vee$'s

$$
\begin{aligned}
f_1^\vee &= -\frac{1}{m_2'}w_1^\vee \\
f_2^\vee &= w_2^\vee + \frac{1}{m_2'}w_1^\vee \\
f_3^\vee &= w_3^\vee - \frac{c(k)}{m_2'}w_1^\vee
\end{aligned}
$$

we get

$$
d_1 f_1^\vee + d_2 f_2^\vee + d_3 f_3^\vee = -\left(\frac{d_1 - d_2 + c(k)d_3}{m_2'}\right)w_1^\vee + d_2 w_2^\vee + d_3 w_3^\vee
$$

But $d_1 - d_2 + a_2' d_3$ is divisible by m_2' and by definition $a_2' \equiv c(k) \bmod m_2'$. Therefore the coefficient of $w_1^\vee$ is an integer and $L_2^\vee = N^\vee$. $\diamond$

Theorem 3.4 *Let $\mathcal{X}^+$ be a Q-Gorenstein smoothing of an extremal P-resolution with axial multiplicities $\alpha_1 = \alpha_2$, and singularities*

$$
\frac{1}{m_1'^2}(1, m_1' a_1' - 1), \qquad \frac{1}{m_2'^2}(1, m_2' a_2' - 1)
$$

Let $\langle w_1, w_2, w_3 \rangle$ be a basis of $N = \mathbb{Z}^3$. Then there exist vectors

$$
w_4 = m_1' w_1 + w_2 + c w_3 \text{ and } w_5 = -m_2' w_1 + w_2 + d w_3
$$

for some $c, d \in \mathbb{Z}$ such that $\mathcal{Y}, \mathcal{X}^+$ and the anticanonical model $\mathcal{X}^-$ are analytically isomorphic to toric varieties given by the fans in Figure 1. In this way we get $\mathcal{X}^- = W_1 \cup W_2$ where $W_1 = \frac{1}{\delta}(-\rho - 1, \rho, 1)$ and $W_2 = \frac{1}{F}(\lambda, 1, -1)$ for some $\rho, \lambda \in \mathbb{Z}$ (given explicitly in the proof) and $F = m_1' + m_2'$.

By the classification of Ishida and Iwashita ([IsIw]), we have that the singularities of $\mathcal{X}^-$ are canonical. Note that W_1 is non-terminal and W_2 is terminal if and only if $gcd(F, \lambda) = 1$.

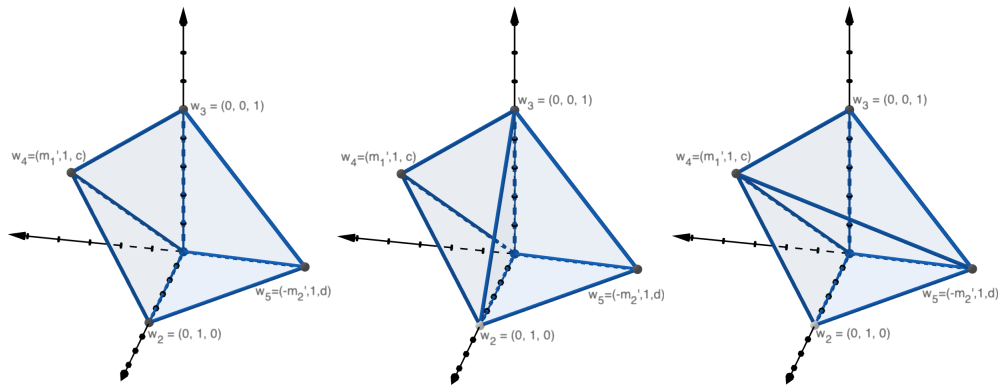

Figure 2: Fans of $\mathcal{Y}$, $\mathcal{X}^+$ and $\mathcal{X}^-$

Proof.

Note that we can rewrite the cones as

$$\sigma_1^\vee = Cone\bigg(\frac{1}{m_1'}w_1^\vee, w_2^\vee - \frac{1}{m_1'}w_1^\vee, \frac{c(k-1)}{m_1'}w_1^\vee + w_3^\vee\bigg)$$

and

$$\sigma_2^\vee = Cone\bigg(-\frac{1}{m_2'}w_1^\vee, \frac{1}{m_2'}w_1^\vee + w_2^\vee, w_3^\vee - \frac{c(k)}{m_2'}w_1^\vee\bigg).$$

Let

$$w_4 = m_1'w_1 + w_2 - c(k-1)w_3 \text{ and } w_5 = -m_2'w_1 + w_2 - c(k)w_3)$$

Then the duals of these cones are given by

$$\sigma_1 = Cone(w_2, w_3, w_4) \text{ and } \sigma_2 = Cone(w_2, w_3, w_5)$$

so we conclude that $\mathcal{X}^+$ is the toric variety corresponding to the fan with maximal cones σ_1, σ_2 and $\mathcal{Y}$ is the toric variety corresponding to the fan generated by w_2, w_3, w_4, w_5. Note that the fan of $\mathcal{X}^+$ is obtained by subdividing the fan of $\mathcal{Y}$. There is only one more possible subdivision of the fan of $\mathcal{Y}$ given by σ_3 and σ_4. We will prove that these cones correspond to the two charts of the antiflip.

We know that a toric variety given by a simplicial fan has only quotient singularities, and for each maximal cone σ, the affine open U_σ is the quotient of $\mathbb{C}^n$ by the action of the finite abelian group $G = N/N'$ where N' is the lattice obtained by the primitive generators of σ. Let M, M' be the duals of N and N' respectively. Then we have an action of G on $\mathbb{C}[M']$, determined by the canonical pairing

$$M'/M \times N/N' \to \mathbb{C}^n$$

and given by

$$v(X^{u'}) = exp(2\pi i\langle u', v\rangle)X^{u'}$$

for $v \in N$, $u' \in M'$. Then we consider the lattices

$$\begin{aligned} L_3 &= \langle w_2, w_4, w_5\rangle \\ &= \langle w_2, m_1'w_1 - c(k-1)w_3, -m_2'w_1 - c(k)w_3\rangle \subset N \end{aligned}$$

and

$$\begin{aligned} L_4 &= \langle w_3, w_4, w_5\rangle \\ &= \langle w_3, m_1'w_1 + w_2, -m_2'w_1 + w_2\rangle \subset N. \end{aligned}$$

Let $G = N/L_3$ and $H = N/L_4$, then we have that $U_{\sigma_3} = \mathbb{C}^3/G$ and $U_{\sigma_4} = \mathbb{C}^3/H$. Note that

$$\begin{bmatrix} w_2 \\ w_4 \\ w_5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & m_1' & -c(k-1) \\ 0 & -m_2' & -c(k) \end{bmatrix} \begin{bmatrix} w_2 \\ w_1 \\ w_3 \end{bmatrix}$$

To determine G we find the Smith normal form of this matrix. Recall that $\delta = -(m_1'c(k) + m_2'c(k-1))$. By definition, we have that

$$c(k-1) \equiv a_1' \pmod{m_1'} \text{ and } gcd(a_1', m_1') = 1.$$

We can find integers r, s such that $rm_1' - sc(k-1) = 1$. Let $\rho = rm_2' + sc(k)$, then we obtain the Smith normal form of the matrix by multiplying by the following invertible matrices

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \rho & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & m_1' & -c(k-1) \\ 0 & -m_2' & -c(k) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & r & c(k-1) \\ 0 & s & m_1' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \delta \end{bmatrix}$$

so $G = \mathbb{Z}_\delta$ and is generated by the element $-sw_1 + rw_3$. Now $L_3^\vee = \langle w_2^\vee, w_6^\vee, w_7^\vee \rangle$ where

$$\begin{aligned} w_6^\vee &= -\frac{c(k)}{\delta} w_1^\vee + \frac{m_2'}{\delta} w_3^\vee \\ w_7^\vee &= \frac{c(k-1)}{\delta} w_1^\vee + \frac{m_1'}{\delta} w_3^\vee \end{aligned}$$

Now $\sigma_3^\vee = Cone(w_{10}^\vee, w_6^\vee, w_7^\vee)$ where $w_{10}^\vee = w_2^\vee - w_6^\vee - w_7^\vee$. Then $U_{\sigma_3} = \mathbb{C}^3/\mathbb{Z}_\delta$, the pairing is given by

$$\langle -sw_1 + rw_3, w_{10}^\vee \rangle = \frac{-\rho - 1}{\delta}\ ,\ \langle -sw_1 + rw_3, w_6^\vee \rangle = \frac{\rho}{\delta}\ ,\ \langle -sw_1 + rw_3, w_7^\vee \rangle = \frac{1}{\delta}$$

and then the weights are $-\rho - 1, \rho, 1$.

For the other chart, note that

$$\begin{bmatrix} w_3 \\ w_4 \\ w_5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & m_1' & 1 \\ 0 & -m_2' & 1 \end{bmatrix} \begin{bmatrix} w_3 \\ w_1 \\ w_2 \end{bmatrix}$$

To determine H we find the Smith normal form of this matrix. Let $F = m_1' + m_2'$. Then we obtain the Smith normal form of the matrix by multiplying by the following invertible matrices

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & m_1' & 1 \\ 0 & -m_2' & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -m_1' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & F \end{bmatrix}$$

so $H = \mathbb{Z}_F$ and is generated by the element w_1. Now $L_4^\vee = \langle w_3^\vee, w_8^\vee, w_9^\vee \rangle$ where

$$\begin{aligned} w_8^\vee &= \frac{1}{F} w_1^\vee + \frac{m_2'}{F} w_2^\vee \\ w_9^\vee &= -\frac{1}{F} w_1^\vee + \frac{m_1'}{F} w_2^\vee \end{aligned}$$

and $\sigma_4^\vee = Cone(w_{11}^\vee, w_8^\vee, w_9^\vee)$ where

$$w_{11}^\vee = w_3^\vee + c(k-1)\left(\frac{1}{F} w_1^\vee + \frac{m_2'}{F} w_2^\vee\right) + c(k)\left(-\frac{1}{F} w_1^\vee + \frac{m_1'}{F} w_2^\vee\right)$$

Then $U_{\sigma_4} = \mathbb{C}^3/\mathbb{Z}_F$, the pairing is given by

$$\langle w_1, w_{11}^\vee \rangle = \frac{c(k-1) - c(k)}{F} \ , \ \langle w_1, w_8^\vee \rangle = \frac{1}{F} \ , \ \langle -w_1, w_9^\vee \rangle = \frac{-1}{F}$$

and then the weights are $c(k-1) - c(k), 1, -1$.

$\diamond$

Corollary 3.5 *If $\mathcal{X}^+$ is the one parameter deformation of the minimal resolution of the rational normal curve of degree $n \geq 3$ with equal axial multiplicities, then the antiflip is $\mathcal{X}^- = W_1 \cup W_2$ where $W_1 = \frac{1}{n-2}(-2, 1, 1)$ and*

$$W_2 = \begin{cases} \frac{1}{2}(0,1,1) & \textit{if } n \textit{ is even,} \\ \\ \frac{1}{2}(1,1,1) & \textit{if } n \textit{ is odd} \end{cases}$$

Proof. Follows directly from the theorem since in this case we have $m_1' = m_2' = a_1' = a_2' = 1$, $\delta = n-2$, $c(k-1) = 0$ and $c(k) = -\delta$. $\diamond$

3.3 Special Fiber

Now we analyze the special fiber of $\mathcal{X}^-$. To a linear combination $r_1A_i + r_2B_i + r_3C_i \in S_{\sigma_i}$ we associate the element $X_i^{r_1}Y_i^{r_2}Z_i^{r_3} \in \mathbb{C}[S_{\sigma_i}]$.

Lemma 3.6 *The special fiber $X^- = S_1 \cup S_2$ where*

$$S_1 = (X_1Y_1Z_1 = Y_1^\delta + Z_1^\delta) \subset \mathbb{A}^3/\frac{1}{\delta}(-\rho-1, \rho, 1),$$

$$S_2 = (Y_2Z_2 = X_2^{m_1'}Z_2^\delta + X_2^{m_2'}Y_2^\delta) \subset \mathbb{A}^3/\frac{1}{F}(c(k-1)-c(k), 1, -1)$$

and the gluing is given by

$$X_1^\delta = X_2^{-F}\ ,\ Y_1^\delta = X_2^{m_2'}Y_2^\delta\ ,\ Z_1^\delta = X_2^{m_1'}Z_2^\delta.$$

Proof. From the equations of $\mathcal{X}^+$ we have $u = \xi_1'\tilde{\eta}_1 - {\zeta_1'}^{m_1'} - {\zeta_2'}^{m_2'} \in \mathbb{C}[S_{\sigma_1}]$ which becomes

$$u = X_1Y_1Z_1 - Y_1^\delta - Z_1^\delta \in \mathbb{C}[S_{\sigma_3}]$$

or

$$u = Y_2Z_2 - X_2^{m_1'}Z_2^\delta - X_2^{m_2'}Y_2^\delta \in \mathbb{C}[S_{\sigma_4}]$$

$\diamond$

Let $(X^-)^\nu, S_1^\nu$ and S_2^ν be the normalization of X^-, S_1 and S_2. Then $(X^-)^\nu = S_1^\nu \cup S_2^\nu$. Let

$$T_1 = (X_1Y_1Z_1 = Y_1^\delta + Z_1^\delta) \subset \mathbb{A}^3$$

and

$$T_2 = (Y_2Z_2 = X_2^{m_2'}Y_2^\delta + X_2^{m_1'}Z_1^\delta) \subset \mathbb{A}^3.$$

Let T_1^ν, T_2^ν be their normalizations, then S_1^ν is obtained by taking the quotient of T_1^ν by the $\mathbb{Z}_\delta$ action and S_2^ν is obtained by taking the quotient of T_2^ν by the $\mathbb{Z}_F$ action.

Theorem 3.7 *If $\delta \geq 3$, then the special fiber X^- is a non normal surface, singular along a curve $C^- \simeq \mathbb{P}^1$. A transversal slice of $\mathcal{X}^-$ through a general point of C^- is a surface with an $A_{\delta-1}$ singularity. Let X^ν be the normalization of X^- and let $\bar{X}$ be the minimal resolution of X^ν. Then $\bar{X}$ has the following configuration of smooth rational curves:*

$$[a_r, \ldots, a_1] - \circ - [c_1, \ldots, c_l] - \circ - [b_1, \ldots, b_s]$$

where the chains $[a_1, \ldots, a_r]$ and $[b_1, \ldots, b_s]$ correspond to conjugate cyclic quotient singularities and $\circ$ denotes a (-1)-curve. X^ν is obtained from $\bar{X}$ by contracting the $[a], [b], [c]$ chains of rational curves. Finally to obtain C^- and X^- we glue the two (-1)-curves creating a orbifold normal crossing point. Locally around the non-terminal point, X^- is given by the equation

$$(XYZ = Y^\delta + Z^\delta) \subset A^3/\frac{1}{\delta}(-\rho - 1, \rho, 1).$$

Proof. Let

$$w = \frac{-Z_1^{\delta-1} + X_1Y_1}{Y_1}.$$

Note that w is an integral element in $\mathbb{C}[X_1, Y_1, Z_1]$ since it satisfies the monic equation $w^2 - wX_1 + Y_1^{\delta-2}Z_1^{\delta-2} = 0$. We will prove that T_1^ν is the spectrum of

$$\mathbb{C}[X_1, Y_1, Z_1, w]/(wZ_1 - Y_1^{\delta-1}, WY_1 + Z_1^{\delta-1} - X_1Y_1, w^2 - wX_1 + Y_1^{\delta-2}Z_1^{\delta-2}).$$

First we apply the following change of coordinates: $X_1 \mapsto \tilde{X}_1 = w - X_1$ to get

$$\mathbb{C}[\tilde{X}_1, Y_1, Z_1, w]/(wZ_1 - Y_1^{\delta-1}, \tilde{X}_1Y_1 + Z_1^{\delta-1}, \tilde{X}_1w + Y_1^{\delta-2}Z_1^{\delta-2}).$$

It is known that for a cyclic quotient singularity $\frac{1}{r}(1, a)$, where a and r are coprime, the invariant monimials are given by $u_0 = p^r$, $u_1 = p^{r-a}q$, $\ldots$, $a_k = q^r$ and they satisfy the relations

$$u_{i-1}u_{i+1} = u_i^{a_i} \text{ for } i = 1, \ldots, k$$

where the a_i come from the continued fraction of $\frac{r}{r-a} = [a_1, \ldots, a_k]$. Consider the case when $r = \delta(\delta-2)$ and $a = (\delta-2)(\delta-1)-1$. Then we get invariants $u_0 = p^{\delta(\delta-2)}$, $u_1 = p^{\delta-1}q$, $u_2 = pq^{\delta-1}$, $u_3 = q^{\delta(\delta-2)}$ and they satisfy the relations

$$u_0u_2 = u_1^{\delta-1}, u_1u_3 = u_2^{\delta-1}, u_0u_3 = u_1^{\delta-2}u_2^{\delta-2}.$$

Notice that these equations are the same equations obtained above under the identification

$$u_0 \mapsto \tilde{X}, u_1 \mapsto Z_1, u_2 \mapsto Y_1 \text{ and } u_3 \mapsto w$$

Therefore the surface given by the Spec of

$$\mathbb{C}[\tilde{X}_1, Y_1, Z_1, w]/(wZ_1 - Y_1^{\delta-1}, \tilde{X}_1Y_1 + Z_1^{\delta-1}, \tilde{X}_1w + Y_1^{\delta-2}Z_1^{\delta-2})$$

is isomorphic to $\frac{1}{\delta(\delta-2)}(1, (\delta-2)(\delta-1)-1)$. Since it is a toric variety, then in particular it is normal so we conclude that it is the normalization of T_1, i.e., T_1^ν. Now $\mathbb{Z}_\delta$ acts on $(\tilde{X}, Y_1, Z_1, w)$ with weights $(-\rho-1, \rho, 1, -\rho-1)$ and we obtain S_1^ν by taking the quotient of T_1^ν under this action. To do this, we use the previous identification

$$\tilde{X} = p^{\delta(\delta-2)}, Z_1 = p^{\delta-1}q, Y_1 = pq^{\delta-1} \text{ and } w = q^{\delta(\delta-2)}$$

Note that $\mathbb{Z}_\delta$ acts monomially on T_1^ν so the quotient will be again a toric variety and if $L_{T_1^\nu}$ is the lattice of T_1^ν , then $L_{T_1^\nu}$ is a sublattice of $\mathbb{Z}^2_{p,q}$. Then we can look for invariant monomials in $L_{T_1^\nu}$. It is enough to find two subsequent invariant monomials, as it is known that these will generate the lattice for S_1^ν. Note that $w^\delta = q^{\delta^2(\delta-2)}$ is invariant. If we find an invariant monomial of the form $w^tY_1 = pq^{t\delta(\delta-2)+(\delta-1)}$ then these two points of $L_{T_1^\nu}$ would form a basis of the lattice of S_1^ν. Now w^tY_1 is invariant if and only if

$$t(-\rho-1) + \rho \equiv 0 \pmod{\delta}.$$

If $gcd(\rho+1, \delta) = 1$ then the congruence has a unique solution and we conclude that

$$S_1^\nu \simeq \frac{1}{\delta^2(\delta-2)}(1, \delta^2(\delta-2) - t\delta(\delta-2) - \delta + 1).$$

If $gcd(\rho+1, \delta) = d > 1$, then $L_{T_1^\nu}$ is a sublattice of $\mathbb{Z}^2_{p^d,q^d}$. Note that $w^{\delta/d}$ is invariant and so we need to find an invariant monomial of the form $w^t Y_1^d = p^d q^{t\delta(\delta-2)+d(\delta-1)}$. Now this monomial is invariant if and only if

$$t(-\rho-1) + d\rho \equiv 0(\text{mod } \delta)$$

Say $\rho + 1 = dh$ and $\delta = dj$, then the previous congruence is equivalent to

$$th + \rho \equiv 0(\text{mod } \delta)$$

which has a unique solution and we conclude that

$$S_1^\nu \simeq \frac{1}{j\delta(\delta-2)}(1, j\delta(\delta-2) - t\delta(\delta-2) - d(\delta-1)).$$

For the other chart, first note that at the origin the tangent cone is given by

$$T_{(0,0,0)} = (Y_2 Z_2 = 0).$$

Then at the origin T_2 is analytically isomorphic to its tangent cone, thus its quotient by the $\mathbb{Z}_F$ action at the origin will be analytically isomorphic to the quotient

$$(Y_2 Z_2 = 0) \subset \frac{1}{F}(c(k-1) - c(k), 1, -1)$$

which is an orbifold normal crossing.

Along $C^- = (Y_2 = Z_2 = 0)$, if $X_2 \neq 0$, then the $\mathbb{Z}_F$ action is free and therefore at these points the surface has the same singularities as the corresponding points of S_2^ν, namely two transversal branches.

Away from C^-, if $X_2 \neq 0$ then the action is free and therefore these are smooth points. Away from C^-, if $X_2 = 0$ then the action is also free so the points of the form $(0, Y_2, 0)$ and $(0, 0, Z_2)$ are smooth. $\diamond$

From now on let $\delta = 2$. So

$$T_1 = (X_1 Y_1 Z_1 = Y_1^2 + Z_1^2) \subset \mathbb{A}^3$$

and

$$T_2 = (Y_2 Z_2 = X_2^{m_2'} Y_2^2 + X_2^{m_1'} Z_2^2) \subset \mathbb{A}^3.$$

Then S_1^ν and S_2^ν are obtained by taking the quotient of T_1^ν and T_2^ν by the $\mathbb{Z}_2$ and the $\mathbb{Z}_F$ actions respectively.

Proposition 3.8 *S_1 has two pinch points on $C^- = (Y_1 = Z_1 = 0)$ and it has normal crossings elsewhere along this line.*

Proof. In the first chart, we take the quotient by either $\frac{1}{2}(1,0,1)$ if ρ is even or $\frac{1}{2}(0,1,1)$ if ρ is odd. Notice that T_1 has pinch points at $(2,0,0)$ and $(-2,0,0)$. If ρ is even the action interchanges the pinch points and is free everywhere except on the points where $X_1 = 0$. Now

$$\frac{1}{2}(1,0,1) \simeq A_1 \times \mathbb{A}^1_{Y_1}$$

with $A_1 = (uw = v^2)$ where $u = X_1^2$, $v = X_1 Z_1$ and $w = Z_1^2$. Then the equation of T_1 can be written as

$$Y_1 v = Y_1^2 + w$$

and then the quotient of T_1 by Z_2 is given by the spectrum of

$$\mathbb{C}[Y_1, u, v, w]/(uw - v^2, Y_1 v - Y_1^2 - w)$$

Now we can write $w = Y_1 v + Y^2$ so the quotient becomes the spectrum of

$$\mathbb{C}[Y_1, u, v]/(uvY_1 - uY_1^2 - v^2)$$

Note that we can rewrite the equation

$$uvY_1 - uY_1^2 = v^2$$

as

$$u(Y_1 - \frac{1}{2}v)^2 = v^2(1 - \frac{1}{4}u)$$

and since $(1 - \frac{1}{4}u)$ is a unit close to the origin, we see that the quotient also has a pinch point at the origin and normal crossings elsewhere on the line $C^- = (Y_1 = Z_1 = 0) = (u = v = 0)$.

If ρ is odd the action does not interchange the pinch points and does not produce a pinch point at the origin when we quotient by the action. So in this case S_1 has two pinch points at the images of $(2, 0, 0)$ and $(-2, 0, 0)$ in the quotient $T_1/\frac{1}{2}(0, 1, 1)$ and it has normal crossings elsewhere on the line $C^- = (Y_1 = Z_1 = 0)$. $\diamond$

Recall that X^+ is an extremal P-resolution of Y and it has at most two Wahl singularities (m_1', a_1') and $(m_2', a_2,)$ along C^+. If $\delta = 2$, then it follows that

$$2 = \delta = cm_1'm_2' - m_1'a_2' - m_2'a_1'$$

where $-c$ is the self intersection of C^+ in the minimal resolution of X^+. Also recall that $F = m_1' + m_2'$.

Lemma 3.9 *If $\delta = 2$ then F is even and we have the following posibilities:*

- *X^+ is smooth so we have $m_i', a_i' = 1$, $i = 1, 2$,and $c = 4$. Therefore X^+ is the minimal resolution of the cone over the rational normal curve of degree 4.*

- *X^+ has one singularity along C^+. Then $m'_1, a'_1 = 1$ and $m'_2 = 2k+1$, $a'_2 = 2k-1$ for some $k \in \mathbb{N}$ and $c = 2$.*
- *X^+ has two singularities along C^+. Then $c = 1$.*

Proof. We will assume that $m'_2 \geq m'_1$. If X^+ is smooth, then $m'_i, a'_i = 1$, $i = 1, 2$. Replacing this values in the formula for δ we get that $c = 4$. Therefore Y is the cone over the rational normal curve of degree 4 and X^+ is its minimal resolution. In this case $F = 2$.

If X^+ has only one singularity, then $m'_1, a'_1 = 1$ and $m'_2 \geq 2$. Replacing in the formula for δ we get

$$2 = (c-1)m'_2 - a'_2$$

Note that $c = 1$ is not possible since we would get that $a'_2 < 0$. Also $c \geq 3$ is not possible since $m'_2 - a'_2 \geq 1$ then we get that

$$2 = (c-1)m'_2 - a'_2 \geq 2m'_2 - a'_2 \geq m'_2 + 1 \geq 3$$

Therefore we must have that $c = 2$ and then

$$2 = m'_2 - a'_2$$

But $gcd(m'_2, a'_2) = 1$, then we must have that they are consecutive odd numbers, i.e., $m'_2 = 2k+1$ and $a'_2 = 2k-1$ for some $k \in \mathbb{N}$. In this case $F = 2k+2$.

Finally, if X^+ has two singularities, then $1 < m'_1 \leq m'_2$. Note that $c \geq 2$ is not possible since we would get

$$\begin{aligned} 2 &= cm'_1m'_2 - m'_1a'_2 - m'_2a'_1 \\ &\geq 2m'_1m'_2 - m'_1a'_2 - m'_2a'_1 \\ &= m'_1(m'_2 - a'_2) + m'_2(m'_1 - a'_1) \end{aligned}$$

but $m'_i - a'_i \geq 1$, $i = 1, 2$ so we get that

$$2 \geq m'_1 + m'_2 \geq 4$$

Therefore we must have that $c = 1$. To show that F is even, we will show that m'_1 and m'_2 are either both even or both odd. Assume otherwise, say $m'_1 = 2l$ and $m'_2 = 2j + 1$ (the case m'_1 odd and m'_2 even follows the same argument). Then $m'_1 m'_2$ and $m'_1 a'_2$ are even and since $2 = m'_1 m'_2 - m'_1 a'_2 - m'_2 a'_1$, then $m'_2 a'_1$ has to be even. But m'_2 is odd so we conclude that a'_1 has to be even. But this contradicts the fact that $gcd(m'_1, a'_1) = 1$. Therefore m'_1 and m'_2 are either both even or both odd, and in any of these cases we get that F is even. $\diamond$

Observation 3.10 *For an initial extremal neighbourhood of type $k2A$, the number k in Definition 2.3 is always equal to 3, so $c(k-1) - c(k) = c(2) - c(3)$, which we will assume from now on.*

Now we look at the chart S_2. Note that the proof done in the case $\delta \geq 3$ also works in this case. So S_2^{ν} is given by

$$(Y_2 Z_2 = 0) \subset \frac{1}{F}(c(2) - c(3), 1, -1).$$

We will analyze the minimal resolution of S_2^{ν} in the three cases of Lemma 3.9.

Proof. [Proof of Proposition **??**] If X^+ is smooth, we have $F = 2$ and $c(2) - c(3) = 2$, therefore in this case we have

$$(Y_2 Z_2 = 0) \subset \frac{1}{2}(0, 1, -1)$$

so S_2^{ν} is smooth. The fact that $C^2 = (-4)$ follows from the fact that $(C^+)^2 = (-4)$. $\diamond$

Proposition 3.11 *If X^+ has one singularity, then S_2^ν has conjugate cyclic quotient singularities $\frac{1}{k+1}(1,1)$ and $\frac{1}{k+1}(1,-1)$ for some $k \geq 1$, its minimal resolution is equal to the minimal resolution of $Q \in Y$ and $C^2 = (-5)$.*

Proof. If X^+ has one singularity, by Lemma 3.9 we have $m_1' = a_1' = 1$, $m_2' = 2k+1$, $a_2' = 2k-1$ for some $k \geq 1$ and $F = 2k+2$. One can also check that $c(2)-c(3) = 2$, therefore in this case we have that S_2^ν is given by

$$(Y_2 Z_2 = 0) \subset \frac{1}{k+1}(1,1,-1)$$

so it has conjugate singularities

$$\frac{1}{k+1}(1,1), \text{ and } \frac{1}{k+1}(1,-1).$$

We will prove that the Hirzebruch-Jung continued fraction corresponding to the Wahl singularity with $m_2' = 2k+1$ and $a_2' = 2k-1$ for $k \geq 1$ is of the form

$$[\underbrace{2,\ldots,2}_{k-1},5,k+1]$$

We will use the following well-known facts about Wahl singularities: If $[a_1,\ldots,a_r]$ is the continued fraction of a Wahl singularity (m,a), then the conjugate cyclic quotient singularity is the Wahl singularity $(m, m-a)$ and its continued fraction is $[a_r,\ldots,a_1]$. Then it is enough to prove that the Hirzebruch-Jung continued fraction corresponding to the Wahl singularity with $m = 2k+1$ and $a = 2$ for $k \geq 1$ is of the form

$$[k+1,5,\underbrace{2,\ldots,2}_{k-1}]$$

If $m = 2k+1$ and $a = 2$ then note that

$$\frac{(2k+1)^2}{2(2k+1)-1} = \frac{4k^2+4k+1}{4k+1} = (k+1) - \frac{k}{4k+1}$$

so $k+1$ is the first number in the continued fraction. Then

$$\frac{4k+1}{k} = 5 - \frac{k-1}{k}$$

so 5 is the second number in the continued fraction and now we are left with $\frac{k}{k-1}$ which is a A_{k-1} singularity and we know that its continued fraction is $[2,\ldots,2]$ where we have $k-1$ curves. Therefore we conclude that

$$\frac{(2k+1)^2}{2(2k+1)-1} = [k+1, 5, \underbrace{2,\ldots,2}_{k-1}]$$

Now X^+ has a Wahl singularity with $m_2' = 2k+1$ and $a_2' = 2k-1$ which is represented by the continued fraction

$$\frac{m_2'^2}{m_2'a_2' - 1} = [k+1, 5, \underbrace{2,\ldots,2}_{k-1 \text{ curves}}].$$

In Lemma 3.9 we showed that in this case $c = 2$, therefore in the minimal resolution of X^+ we have a configuration of rational curves

$$[k+1, 5, \underbrace{2,\ldots,2}_{k \text{ curves}}]$$

Note that

$$\frac{1}{k+1}(1,1) = [k+1], \text{ and } \frac{1}{k+1}(1,1) = [\underbrace{2,\ldots,2}_{k \text{ curves}}]$$

so the minimal resolution of X^+ is equal to the minimal resolution of S_2^ν and we conclude then the proper transform of C^- has to be the (-5)-curve in this configuration. $\diamond$

Proposition 3.12 *If X^+ has two singularities, then S_2^ν has conjugate cyclic quotient singularities $\frac{1}{f}(p,1)$ and $\frac{1}{f}(p,-1)$ where $f = F/2$ and $p = (c(2)-c(3))/2$. Its minimal resolution is equal to the minimal resolution of $Q \in Y$ and $C^2 = (-5)$.*

Proof. If X^+ has two singularities, then F is even and $c(2) - c(3) = m_2' - a_2' + a_1'$ is also even. Therefore in this case we have

$$(Y_2 Z_2 = 0) \subset \frac{1}{f}(p, 1, -1)$$

where $f = F/2$ and $p = (c(2) - c(3))/2 = (m_2' - a_2' + a_1')/2$, so S_2^ν has conjugate singularities

$$\frac{1}{f}(p, 1), \text{ and } \frac{1}{f}(p, -1).$$

Now we need to prove that in the minimal resolution of $(X^-)^\nu$, the proper transform of C^- is a (-5)-curve.

Let $\tilde{C}$ be the proper transform of C^- in X^ν and C be the proper transform of $\tilde{C}$ in $\bar{X}$. In $\bar{X}$, let A and B be the exceptional curves that intersect C obtained by resolving the singularities of X^ν. Let $\pi \colon \bar{X} \to X^\nu$, then

$$\pi^* \tilde{C} = C + \frac{p}{f} A + \frac{f-p}{f} B$$

so using the projection formula we have

$$\tilde{C}^2 = (\pi^* \tilde{C})(\pi_* C) = C^2 + \frac{p}{f} + \frac{f-p}{f} = C^2 + 1$$

Therefore, proving that $C^2 = -5$ is equivalent to proving that $\tilde{C}^2 = -4$.
Now germs $(C^- \subset X^-)$ and $(C^+ \subset X^+)$ have $\mathbb{Q}$-Gorenstein smoothings with the same general fiber, then we must have $(K_X^+)^2 = (\omega_{X^-})^2$, see [L86] for a definition of the self-intersection in the non-compact case. Let $\pi^+ \colon X^+ \to Y$ and $\pi^- \colon X^- \to Y$. Then

$$K_{X^+} = (\pi^+)^* K_Y + aC^+$$

and

$$K_{X^\nu} = (\pi^-)^* K_Y + b\tilde{C}.$$

We abuse notation and let $\pi^-\colon X^\nu \to Y$. Then

$$\nu^*\omega_{X^-} = K_{X^\nu} + \tilde{C} = (\pi^-)^* K_Y + (b+1)\tilde{C}.$$

And since the self-intersections are equal then we must have

$$a^2(C^+)^2 = (b+1)^2(\tilde{C})^2$$

Now

$$(C^+)^2 = -1 + \frac{m_1' a_1' - 1}{m_1'^2} + \frac{m_2' a_2' - 1}{m_2'^2} = -(\frac{1}{m_1'} + \frac{1}{m_2'})^2$$

On the other hand, using the *different formula* (see Section 16 of [K92]), we have

$$(K_{X^+} + C^+)C^+ = -2 + (1 - \frac{1}{m_1'^2}) + (1 - \frac{1}{m_2'^2}) = -(\frac{1}{m_1'^2} + \frac{1}{m_2'^2})$$

but also

$$(K_{X^+} + C^+)C^+ = (a+1)(C^+)^2$$

so from this two expressions we get that

$$a = \frac{-2m_1' m_2'}{(m_1' + m_2')^2}$$

and we conclude that

$$a^2(C^+)^2 = -\frac{4}{(m_1' + m_2')^2} = -\frac{4}{F^2}$$

We can do the same calculation for X^ν and we get

$$(K_{X^\nu} + \tilde{C})\tilde{C} = -2 + 2(1 - \frac{2}{F}) = -\frac{4}{F}$$

but also

$$(K_{X^\nu} + \tilde{C})\tilde{C} = (b+1)\tilde{C}^2$$

so we get that

$$(b+1) = -\frac{4}{F(\tilde{C})^2}$$

and then

$$(b+1)^2(\tilde{C})^2 = -\frac{16}{F^2(\tilde{C})^2}.$$

Finally

$$-\frac{4}{F^2} = a^2(C^+)^2 = (b+1)^2(\tilde{C})^2 = -\frac{16}{F^2(\tilde{C})^2}$$

and we get that $\tilde{C}^2 = -4$ and then $C^2 = -5$. $\diamond$

Observation 3.13 *In the previous propositions we have assumed that $m_2' \geq m_1'$, but the same proofs work if $m_1' \geq m_2'$. In fact, as it will be shown in the next proposition, when $\delta = 2$, then $Q \in Y$ has two extremal P-resolutions, one with $m_2' > m_1'$ and one with $m_1' > m_2'$ unless $Q \in Y$ is the cone over the rational normal curve of degree 4 which has only one extremal P-resolution.*

Proposition 3.14 *Given integers f, p with $1 \leq p < f$ and $gcd(p, f) = 1$, there are two extremal P-resolutions X^+ with Wahl singularities (m_1', a_1') and (m_2', a_2') such that:*

1. *$\delta = 2$*

2. *If $\mathcal{X}^+$ is a $\mathbb{Q}$-Gorenstein smoothing of X^+ with axial multiplicities $\alpha_1 = \alpha_2$ and $\mathcal{X}^-$ is the antiflip, then the normalization of the special fiber X^- has singularities*
$$\frac{1}{f}(p,1), \text{ and } \frac{1}{f}(p,-1).$$

Proof.

If $p = 1$, then the two extremal P-resolutions have singularities given by the data

(i) $m_1' = a_1' = 1$, $m_2' = 2f - 1$ and $a_2' = 2f - 3$.

(ii) $m_1' = f+1$, $a_1' = 1$, $m_2' = f-1$ and $a_2' = f-2$.

In the first case X^+ has only one singularity and the second case X^+ has two singularities. For both surfaces we have that $\delta = 2$ and by Proposition 3.11 and Proposition 3.12 it follows that $(X^-)^\nu$ has singularities

$$\frac{1}{f}(p,1), \text{ and } \frac{1}{f}(p,-1)$$

.

If $p = f-1$, then the two extremal P-resolutions have singularities given by the data

(i) $m_1' = f-1$, $a_1' = f-2$, $m_2' = f+1$ and $a_2' = 1$.

(ii) $m_1' = 2f-1$, $a_1' = 2f-3$, and $m_2' = 1a_2' = 1$.

In the first case X^+ has two singularities and the second case X^+ has one singularity. For both surfaces we have that $\delta = 2$ and by Proposition 3.12 and Proposition 3.11 it follows that $(X^-)^\nu$ has singularities

$$\frac{1}{f}(p,1), \text{ and } \frac{1}{f}(p,-1)$$

.

If $1 < p < f$, since $gcd(p,f) = 1$, then there exist unique integers m_1', a_1' with $1 < m_1' < f$ such that

$$m_1'p - a_1'f = 1.$$

Note that $a_1' > 0$, otherwise we would get that $m_1'p - a_1'f > 1$. Also we must have that $a_1' < m_1'$, otherwise we would get that $f < p$. Define $m_2' = 2f - m_1'$ and $a_2' = m_2' + a_1' - 2p$. Then we need to check that $gcd(m_2', a_2') = 1$ and that

$\delta = 2$. Note that $1 < m_1' < m_2'$, therefore we are in the case when X^+ has two singularities. Then $c = 1$ and

$$\begin{aligned}
\delta &= m_1'm_2' - m_1'a_2' - m_2'a_1' \\
&= m_1'm_2' - m_1'(m_2' + a_1' - 2p) - (2f - m_1')a_1' \\
&= 2(pm_1' - fa_1') \\
&= 2
\end{aligned}$$

Note that we can write f and p in terms of m_1', a_1', m_2', a_2' as

$$\begin{bmatrix} 2f \\ 2p \end{bmatrix} = \begin{bmatrix} m_1' & m_2' \\ a_1' & m_2' - a_2' \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and note that

$$det \begin{bmatrix} m_1' & m_2' \\ a_1' & m_2' - a_2' \end{bmatrix} = \delta = 2$$

Then

$$det \begin{bmatrix} 2f & m_2' \\ 2p & m_2' - a_2' \end{bmatrix} = 2 \Rightarrow det \begin{bmatrix} f & m_2' \\ cp & m_2' - a_2' \end{bmatrix} = 1$$

so we conclude that $1 = gcd(m_2', m_2' - a_2') = gcd(m_2', a_2')$.

For the other extremal P-resolution, using again the fact that $gcd(f, p) = 1$, there are unique integers m_2', q with $1 < m_2' < f$ such that

$$qf - m_2'p = 1.$$

Clearly $q > 0$, and note that $q < m_2'$, otherwise we would get that

$$1 = qf - m_2'p \geq m_2'(f - p)$$

which is not possible since the expression on the right hand side is greater that one. Define $a_2' = m_2' - q$, $m_1' = 2f - m_2'$ and $a_1' = 2p - q = 2p - m_2' + a_2'$. Then we need to check that $\delta = 2$ and $gcd(m_1', a_1,) = 1$. Note that $gcd(m_2', a_2') = 1$ and $1 < m_2' < m_1'$, therefore we are in the case when X^+ has two singularities. Then $c = 1$ and

$$\begin{aligned}
\delta &= m_1'm_2' - m_1'a_2' - m_2'a_1' \\
&= m_1'm_2' - m_1'(m_2' - q) - m_2'(2p - q) \\
&= q(m_1' + m_2') - m_2'p \\
&= 2(qf - m_2'p) \\
&= 2
\end{aligned}$$

Note that we can write f and p in terms of m_1', a_1', m_2', a_2' as

$$\begin{bmatrix} 2f \\ 2p \end{bmatrix} = \begin{bmatrix} m_1' & m_2' \\ a_1' & m_2' - a_2' \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and note that

$$det \begin{bmatrix} m_1' & m_2' \\ a_1' & m_2' - a_2' \end{bmatrix} = \delta = 2$$

Then

$$det \begin{bmatrix} m_1' & 2f \\ a_1' & 2p \end{bmatrix} = 2 \Rightarrow det \begin{bmatrix} m_1' & f \\ a_1' & p \end{bmatrix} = 1$$

so we conclude that $1 = gcd(m_1', a_1')$. $\diamond$

Theorem 3.15 *There is a two-to-one correspondence between*

$$\left\{ \begin{array}{l} \mathbb{Q}\text{-}Gorenstein\ smoothings\ X^+ \subset \mathcal{X}^+\ with\ \delta = 2 \\ and\ axial\ multiplicities\ \alpha_1 = \alpha_2 \end{array} \right\}$$

and

$$\left\{ \{(p,f) \in \mathbb{Z}^2 | 1 \leq p \leq f/2, gcd(p,f) = 1\} \right\}$$

The special fiber of the antiflip $X^- \subset \mathcal{X}^-$ is a non normal surface, singular along $C^- \simeq \mathbb{P}^1$. A transversal slice of $\mathcal{X}^-$ through a general point of C^- is a surface with an A_1 singularity. Let X^ν be the normalization of X^- and $\bar{X}$ is its minimal resolution, then $\bar{X}$ has the following configuration of curves:

$$[a_r, \ldots, a_1] - C - [b_1, \ldots, b_s]$$

where the chains $[a_1, \ldots, a_r]$ and $[b_1, \ldots, b_s]$ correspond to the conjugate cyclic quotient singularities

$$\frac{1}{f}(p,1), \text{ and } \frac{1}{f}(p,-1)$$

and C is the proper transform of C^- in $\bar{X}$ and $C^2 = -5$. X^- is obtained from $\bar{X}$ by first contracting the $[a]$ and $[b]$ chains of rational curves and then folding the curve C onto itself, producing an orbifold normal crossing point, two pinch points, and the singularity with local equation

$$(XYZ = Y^2 + Z^2) \subset A^3 / \frac{1}{2}(-\rho - 1, \rho, 1).$$

Proof. Using Propositions 3.11 and 3.12, we show how to associate a pair (f,p) to a $\mathbb{Q}$-Gorenstein smoothing with equal axial multiplicities of a given extremal P-resolution X^+. Using Proposition 3.14, we see that to each pair (f,p) we can associate two different extremal P-resolutions, thus obtaining the two-to-one correspondence. $\diamond$

Remark 3.16 *Proposition 3.14 shows that, with the exception of the cone over the rational normal curve of degree 4, every cyclic quotient singularity $Q \in Y$ having an extremal P-resolution with $\delta = 2$, in fact has exactly two extremal P-resolutions* *(recall that a cyclic quotient singularity has at most two extremal P-resolutions [HTU]). This fact, together with Lemma 3.9 show that all these singularities satisfy the conditions of Theorem 1.2 and Theorem 1.3 of [UV], so in particular they satisfy the Wormhole conjecture (see Conjecture 1.1 in [UV]).*

For the case when $Q \in Y$ is the rational normal curve of degree $n \geq 5$, and X^+ its minimal resolution, we can give an algorithm that shows how to go from X^+ to the minimal resolution of the normalization of X^- by blowing up some points. We separate the cases when n is even and odd.

The numerical data associated to the rational normal curve of degree n is as follows: $m_i' = a_i' = 1$, $i = 1, 2$, $\delta = n - 2$, $c(k-1) = 0, c(k) = -\delta$, $\rho = 1$, $F = 2$ and $\lambda = \delta$.

- If $n = 2l + 1$ is odd, then $gcd(\rho + 1, \delta) = 1$, so following the proof of Theorem 3.7, the solution to the congruence $t(\rho - 1) + \rho \equiv 0 \mod \delta$ is $t \equiv l \mod \delta$ and then on the first chart, the normalization of X^- has the singularity

$$\frac{1}{\delta^2(\delta-2)}(1, \delta^2(\delta-2) - l\delta(\delta-2) - \delta + 1)$$

  and on the second chart, it has two A_1 singularities $\frac{1}{2}(1,1)$ and $\frac{1}{2}(-1,1)$.

  Example 3 *For $n = 5, 7$ and 9, we calculate the singularity of the first chart of the normalization of X^-, its continued fraction and show how to obtain the minimal resolution of the normalization by blowing up X^+. We denote the (-1)-curves by $\circ$.*

  $n = 5$. *Singulartity:* $\frac{1}{9}(1,1) = [9]$.

$$[5] \Rightarrow \circ - [7] - \circ \Rightarrow [2] - \circ - [9] - \circ - [2].$$

$n = 7$. *Singulartity:* $\frac{1}{75}(1, 26) = [3, 9, 3]$.

$$[7] \Rightarrow \circ - [9] - \circ \Rightarrow \circ - [2, 9, 2] - \circ \Rightarrow [2] - \circ - [3, 9, 3] - \circ - [2].$$

$n = 9$. *Singulartity:* $\frac{1}{245}(1, 99) = [3, 2, 11, 2, 3]$.

$$[9] \Rightarrow \circ{-}[11]{-}\circ \Rightarrow \circ{-}[2, 11, 2]{-}\circ \Rightarrow \circ{-}[2, 2, 11, 2, 2]{-}\circ \Rightarrow [2]{-}\circ{-}[3, 2, 11, 2, 3]{-}\circ{-}[2].$$

In general, for $n = 2l + 1 \geq 7$, we get the the following chain of rational curves:

$$[2] - \circ - [3, \underbrace{2, \dots, 2}_{l-3 \text{ curves}}, n + 2, \underbrace{2, \dots, 2}_{l-3 \text{ curves}}, 3] - \circ - [2].$$

To get X_{-}, as explained in Theorem 3.7, we contract the chains of rational curves and then glue the (-1)-curves.

- If $n = 2l$ is even, then $gcd(\rho + 1, \delta) = 2$, so following the proof of Theorem 3.7, the solution to the congruence $-t + \rho \equiv 0 \bmod \delta$ is $t \equiv 1 \bmod \delta$, and then on the first chart, the normalization of X^{-} has the singularity

$$\frac{1}{l\delta(\delta - 2)}(1, l\delta(\delta - 2) - \delta(\delta - 2) - 2(\delta - 1))$$

and the second chart is smooth.

Example 4 *For $n = 6, 8$ and 10, we calculate the singularity of the first chart of the normalization of X^{-}, its continued fraction and show how to obtain* *the minimal resolution of the normalization by blowing up X^+. We denote the (-1)-curves by $\circ$.*

$n = 6$. *Singularity:* $\frac{1}{8}(1,1) = [8]$.

$$[6] \Rightarrow \circ - [8] - \circ.$$

$n = 8$. *Singularity:* $\frac{1}{36}(1,19) = [2,10,2]$.

$$[8] \Rightarrow \circ - [10] - \circ \Rightarrow \circ - [2,10,2] - \circ.$$

$n = 10$. *Singularity:* $\frac{1}{96}(1,65) = [2,2,12,2,2]$.

$$[10] \Rightarrow \circ - [12] - \circ \Rightarrow \circ - [2,12,2] - \circ \Rightarrow \circ - [2,2,12,2,2] - \circ$$

In general, for $n = 2l \geq 6$, we get the the following chain of rational curves:

$$\circ - [\underbrace{2,\ldots,2}_{l-3 \text{ curves}}, n+2, \underbrace{2,\ldots,2}_{l-3 \text{ curves}}] - \circ.$$

To get X_-, as explained in Theorem 3.7, we contract the chains of rational curves and then glue the (-1)-curves.

CHAPTER 4

PARTIAL RESULTS AND FUTURE PROJECTS

4.1 Some calculations on the non-diagonal case

Here we discuss some calculations made in the case when $Q \in Y$ is the cone over the rational normal curve of degree $n \geq 5$ and X^+ is its minimal resolution. We believe these calculations would give a good approach to compute the anticanonical models for non-diagonal rays in the canonical region.

Recall that the cone over the rational normal curve of degree n can be described as the subvariety in $\mathbb{A}^{n+1}$ defined by the vanishing of the ideal of 2×2 minors of the matrix

$$\begin{bmatrix} x_1 & x_2 & \dots & x_n \\ x_2 & x_3 & \dots & x_{n+1} \end{bmatrix}$$

It has a cyclic singularity at the origin usually denoted as $\frac{1}{n}(1,1)$. Let X^+ denote the minimal resolution of Y, which can be explicitly described as the blowup of Y at the origin, i.e. $X^+ = U_1 \cup U_2$ where $U_1 = \mathbb{A}^2_{(t,x_1)}$, $U_2 = \mathbb{A}^2_{(s,x_{n+1})}$ and the gluing on the overlap is given by $s = t^{-1}$, $x_{n+1} = t^n x_1$. The exceptional divisor, denoted by C^+, is a rational curve with $(C^+)^2 = -n$ and in the coordinates of X^+ is given by $(t, 0)$ on U_1 and $(s, 0)$ on U_2. It is known that X and X^+ are toric surfaces, and we choose the toric structure on X so that its toric boundary is given by $l_1 \cup l_2 \subset X$

where $l_1 = \{x_1 = x_2 = \ldots = x_n = 0\}$ and $l_2 = \{x_2 = x_3 = \ldots = x_{n+1} = 0\}$. Then the toric boundary B^+ of X^+ is given by the nodal chain of curves l_1^+, C^+, l_2^+ where l_i^+ denotes the strict transform of l_i and then $K_X = -(l_1 + l_2)$ and $K_{X^+} = -(l_1^+ + C^+ + l_2^+)$. Note that X^+ is in particular an extremal P-resolution of Y, so we obtain deformations of Y by blowing down deformations of X^+.

By Corollary 3.23 in HTU, we know that a one parameter smoothing $\mathcal{X}^+ \to \mathbb{A}^1_u$ of X^+ admits a terminal antiflip if and only if

1. There is a divisor $D \in |-K_{\mathcal{X}^+}|$ such that $D|_{X^+}$ is the toric boundary of X^+
2. The axial multiplicities α_1, α_2 of the singularities of $\mathcal{X}^+$ satisfy $\alpha_1^2 - \delta\alpha_1\alpha_2 + \alpha_2^2 > 0$.

Even more, if the inequality in (2) is not satisfied, then a smoothing still admits an antiflip which is no longer terminal but canonical (See Remark 3.25 in HTU).

Let $\mathcal{Y}$ be the one parameter smoothing of Y defined by the vanishing of the ideal of the 2×2 minors of the matrix

$$\begin{bmatrix} x_1 & x_2 & \ldots & x_{n-1} & x_n + u^{\alpha_1} \\ x_2 + u^{\alpha_2} & x_3 & \ldots & x_n & x_{n+1} \end{bmatrix}$$

and let $\mathcal{X}^+$ be the corresponding deformation of X^+ and let $f^+ \colon \mathcal{X}^+ \to \mathcal{Y}$.

Claim 4.1 *$\mathcal{X}^+$ satisfies the first condition of Corollary 3.23 in HTU, so it admits an antiflip.*

Proof. Consider the form

$$\omega = \frac{1}{sx_{n+1} - u^{\alpha_1}} dx_{n+1} \wedge ds \wedge du = \frac{-1}{tx_1 - u^{\alpha_2}} dx_1 \wedge dt \wedge du.$$

Then $-K_{\mathcal{X}^+} = \{div(tx_1 - u^{\alpha_2}), div(sx_{n+1} - u^{\alpha_1})\}$. Then $-K_{\mathcal{X}^+}|_{u=0} = \{tx_1 = 0, sx_{n+1} = 0\}$ which is exactly $l_1^+ \cup C^+ \cup l_2^+$, i.e. the toric boundary of X^+ so $D = -K_{\mathcal{X}^+}$ satisfies condition (1) of Corollary 3.23. ◇

Note that $\mathcal{X}^+$ is smooth, so in this case, the axial multiplicities of $\mathcal{X}^+$ are defined as follows: we look at a general member D of $|-K_{\mathcal{X}^+}|$, which satisfies condition (1) as proved in the previous claim. Let P_i be the singular points of the toric boundary of X^+, then the local deformation $(P_i \in D \subset \mathcal{X}^+) \to (0 \in \mathbb{A}^1_u)$ of $(P_i \in B^+ \subset X^+)$ is of the form

$$(0 \in (xy = u^{\alpha_i} h(u)) \subset \mathbb{A}^2_{x,y} \times \mathbb{A}^1_u)$$

for some $\alpha_i \in \mathbb{N}$ and convergent power series $h(u)$ with $h(0) \neq 0$. Then α_i are the axial multiplicities.

Claim 4.2 *The α_i appearing in the matrix defining $\mathcal{Y}$ correspond to the axial multiplicities of the singularities of the toric boudary of X^+.*

Proof. The toric boundary B^+ of X^+ has equations $\{tx_1 = 0, sx_{n+1} = 0\}$. Then $P_1 = \{t = 0, x_1 = 0\}$ and $P_2 = \{s = 0, x_{n+1} = 0\}$ and the deformation of B^+ is given by $\{tx_1 = u^{\alpha_2}, sx_{n+1} = u^{\alpha_1})\}$ so the axial multiplicities of P_1 and P_2 are α_2 and α_1 respectively. ◇

The anticanonical model is given by

$$\mathcal{X}^- = R(\mathcal{X}, -K_{\mathcal{X}^+}) = Proj(\bigoplus_{m \geq 0} \Gamma(\mathcal{X}^+, \mathcal{O}(-mK_{\mathcal{X}^+})))$$

so if one wants to compute it explicitly, then one needs to find a set of generator of this anticanonical ring and relations between them.

Consider the form

$$\omega = t^{n-2} dx_1 \wedge dt \wedge du = -dx_{n+1} \wedge ds \wedge du.$$

Let $A = div(\omega) = \{div(t^{n-2}), \emptyset\}$ and $B = \{div(tx_1 - u^{\alpha_2}), div(sx_{n+1} - u^{\alpha_1})\}$, then $A \sim K_{\mathcal{X}^+}$, $B \sim -K_{\mathcal{X}^+}$ and $A + B = div(x_n)$. Let $R = \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. Since A, B are Cartier divisors, there is a short exact sequence

$$0 \to \mathcal{O}_{\mathcal{X}^+}(-A - B) \to \mathcal{O}_{\mathcal{X}^+}(-A) \to \mathcal{O}_B(-A) \to 0$$

Note that $A + B$ is principal so $\mathcal{O}_{\mathcal{X}^+}(-A - B) \simeq \mathcal{O}_{\mathcal{X}^+}$ and $\mathcal{O}_{\mathcal{X}^+}(-A) \hookrightarrow \mathcal{O}_{\mathcal{X}^+}$ so we get a map $\mathcal{O}_{\mathcal{X}^+} \to \mathcal{O}_{\mathcal{X}^+}$ which is multiplication by x_n. Taking push forward of this short exact sequence we get

$$0 \to (f^+)_*\mathcal{O}_{\mathcal{X}^+}(-A - B) \to (f^+)_*\mathcal{O}_{\mathcal{X}^+}(-A) \to (f^+)_*\mathcal{O}_B(-A) \to \cdots$$

By the adjunction formula, $K_B = (K_{\mathcal{X}^+} + B)|_B \sim 0$, then $(C^+)^2 = -2$ on B. Let $\overline{B} = f^+(B)$, then $(f^+)_*\mathcal{O}_B(-A) = \mathcal{O}_{\overline{B}}(-f^+(A))$.

Proposition 4.3 *The map $(f^+)_*\mathcal{O}_{\mathcal{X}^+}(-A) \to \mathcal{O}_{\overline{B}}(-f^+(A))$ is surjective for all axial multiplicities.*

Proof. We will prove that $R^1(f^+)_*\mathcal{O}_{\mathcal{X}^+} = 0$ which implies $R^1(f^+)_*\mathcal{O}_{\mathcal{X}^+}(-A-B) = 0$ and then the map is surjective. Consider the short exact sequence

$$0 \to \mathcal{O}_{X^+} \to \mathcal{O}_{X^+} \to \mathcal{O}_{X^+} \to 0$$

where the first map is multiplication by u. Taking the pushforward of this short exact sequence we get sequences:

$$0 \to \mathcal{O}_{\mathcal{X}} \to \mathcal{O}_{\mathcal{X}} \to \mathcal{O}_X \to 0$$

and

$$0 \to R^1(f^+)_*\mathcal{O}_{\mathcal{X}^+} \to R^1(f^+)_*\mathcal{O}_{\mathcal{X}^+} \to R^1(f^+)_*\mathcal{O}_{X^+}$$

But $R^1(f^+)_*\mathcal{O}_{X^+} = 0$ because X^+ is a rational singularity. Then $R^1(f^+)_*\mathcal{O}_{\mathcal{X}^+} \simeq R^1(f^+)_*\mathcal{O}_{\mathcal{X}^+}$ where the map is given by multiplication by u. Note that $R^1(f^+)_*\mathcal{O}_{\mathcal{X}^+}$

is a coherent sheaf supported at the singularity, then by Nakayama's lemma we conclude that $R^1(f^+)_*\mathcal{O}_{\mathcal{X}^+} = 0$. $\diamond$

Similarly, for every $m > 0$ we get a sequence

$$0 \to (f^+)_*\mathcal{O}_{\mathcal{X}^+}(-mA - B) \to (f^+)_*\mathcal{O}_{\mathcal{X}^+}(-mA) \to \mathcal{O}_{\overline{B}}(-mf^+(A)) \to \cdots$$

and then

$$0 \to \bigoplus_{m\geq 0} \Gamma(\mathcal{X}^+, -mA - B) \to \bigoplus_{m\geq 0} \Gamma(\mathcal{X}^+, -mA) \to \bigoplus_{m\geq 0} \Gamma(\overline{B}, -m(f^+)A) \to \cdots$$

Note that $\overline{B}$ is a cone with a $A_{\alpha_1+\alpha_2-1}$ singularity at the vertex with equation $x_1x_{n+1} = u^{\alpha_1+\alpha_2}$ and that $f^+(A) \cap \overline{B}$ corresponds to the multiple ruling $(n-2)(f^+(C) \cap \overline{B})$ where $C = \{t, \emptyset\}$.

Proposition 4.4 *The map*

$$\Gamma(\mathcal{X}^+, -mA) \to \bigoplus_{m\geq 0} \Gamma(\overline{B}, -m(f^+)A)$$

is surjective for every $m > 0$.

Proof. Note that $\mathcal{X}^+$ and $\mathcal{X}^-$ are isomorphic in codimension one, then their canonical and anticanonical rings are isomorphic. Let $f^- \colon \mathcal{X}^- \to \mathcal{Y}$ and let S be the proper transform of $\overline{B} \subset \mathcal{Y}$ in $\mathcal{X}^-$. Note that $S \cdot C^- > 0$ so $f^-|_S$ is a finite birational map onto $\overline{B}$, which is a normal surface. Then $f^-|_S$ must be an isomorphism and since A^- is the proper transform of $f^+(A)$, there is an isomorphism $\bigoplus_{m\geq 0} \Gamma(S, -mA^-) \simeq \bigoplus_{m\geq 0} \Gamma(\overline{B}, -mf^+(A))$. Then the proposition is equivalent to proving that the map $\bigoplus_{m\geq 0} \Gamma(\mathcal{X}^-, -mA^-) \to \bigoplus_{m\geq 0} \Gamma(S, -mA^-)$ is surjective. But $\mathcal{X}^-$ has canonical singularities and $-A^- \sim -K_{\mathcal{X}^-}$ is ample, so $R^1(f^-)_*\mathcal{O}_{\mathcal{X}^-}(-mA - S) = 0$ by Kawamata-Viehweg vanishing theorem [KMM] and then the map is surjective.

$\diamond$

Diagonal case: Let's look at the case when $\alpha_1 = \alpha_2 = 1$. Then $\overline{B}$ has an A_1 singularity at the vertex.

Claim 4.5 *If n is even then*

1. *$\mathcal{O}_{\overline{B}}(-f^+(A))$ is a locally principal sheaf of ideals and the generator of $\mathcal{O}_{\overline{B}}(-f^+(A))$ lifts to some $G \in (f^+)_*\mathcal{O}_{\mathcal{X}^+}(-A)$ so $\mathcal{O}_{\mathcal{X}^+}(-A)$ is generated by x_{n-1} and G*

2. *$\mathcal{O}_{\mathcal{X}^+}(-mA)$ is generated by $x_n - 1$ and G^m as an R-module.*

Proof. If n is even then $f^+(A) \cap \overline{B}$ is a Cartier Divisor on $\overline{B}$ so it is a locally principal sheaf of ideals. Let $\tilde{G}$ be a generator and let G be its lift. Note that $(f^+)^{-1}(\text{double ruling } t = 0) = (t^2 x_0)$ and G restricted to B is equal to $(t^2 x_0)^{\frac{n-2}{2}}$. Finally, since $f^+(A)$ is Cartier, then $f^+(mA)$ is also Cartier and G^m surjects onto $(f^+)_*\mathcal{O}_B(-mA)$ so the last claim follows by the same argument above applied to the short exact sequence

$$0 \to \mathcal{O}_{\mathcal{X}^+}(-mA - B) \to \mathcal{O}_{\mathcal{X}^+}(-mA) \to \mathcal{O}_B(-mA) \to 0$$

$\diamond$

Let's use this method to compute the anticanonical model in the case of the rational normal curve of degree 4, to check that we get the same result as in Chapter 3.

If $n = 4$, then $A = \{t^2, \emptyset\}$, $C = \{t, \emptyset\}$ and $\overline{B}$ has an A_1 singularity of equation $x_1 x_5 = u^2$ and $f^+(C) = (x_5, u)$ is a ruling. Then $\bigoplus_{m \geq 0} \Gamma(\overline{B}, -m(f^+)C)$ is generated by $f_1 = x_5$ and $g_1 = u$ in degree 1 and $f_2 = x_5$ in degree 2, so

$\bigoplus_{k\geq 0}\Gamma(\overline{B}, -2k(f^+)C)$ is generated by f_2. Its lift is $G = x_5+x_3$ so $\bigoplus_{m\geq 0}\Gamma(\mathcal{X}^+, -mA)$ is generated by $F = x_4$ and G, both of them in degree one and it is easy to check that these generators satisfy the relations $r_1 = ((x_2+x_4)F = x_3G), r_2 = ((x_1+x_3+u)F = x_2G), r_3 = ((x_0+x_2)F = x_1G)$ and in fact, $\mathcal{X}^- = Proj(R[F, G]/\langle r_1, r_2, r_3\rangle)$. To see this, note that in the chart where $F \neq 0$, x_3, x_4 and x_5 can be expressed in terms of x_1, x_2, u and $\overline{G} = \frac{G}{F}$, i.e. we get an hypersurface inside $\mathbb{A}^4$ whose equation is $x_1x_2\overline{G} - x_1^2 - x_2^2 - x_2u = 0$. Notice that this polynomial is irreducible so the three-fold $\mathcal{X}^- \setminus \{F = 0\}$ is also irreducible. Then we get a flat map $\mathcal{X}^- \setminus \{F = 0\} \to \mathbb{A}^1_{\overline{G}}$ where each fiber is a surface with an A_1 singularity, i.e. $\mathcal{X}^-$ has transversal A_1 singularities along $C^- = \{x_1 = x_2 = u = 0\}$. The special fiber has two pinch points at $x_1 = x_2 = 0$ and $G = \pm 2$ and all the other points on C^- are normal crossings. Note that this is exactly what we got in Chapter 3.

4.2 Questions for future project

There are many unanswered questions related to this thesis, and here we will describe two directions which we think would be really interesting to pursue in a future project.

1. It would be interesting to study the deformations of X^-. In particular, the chart containing the canonical but not terminal singularity. We saw that this chart is given by the local equation

$$(XYZ = Y^\delta + Z^\delta) \subset \mathbb{A}^3/\frac{1}{\delta}(-\rho - 1, \rho, 1).$$

   Then we are interested in deformations of $(XYZ = Y^\delta + Z^\delta) \subset \mathbb{A}^3$ which are equivariant under the $\mathbb{Z}_\delta$ action. Let $f(X, Y, Z) = XYZ - Y^\delta - Z^\delta$ and assume for now that $\delta > 2$. Then the versal deformation space of $(f = 0) \subset \mathbb{A}^3$ is given by

$$f(X,Y,Z) + tg(X,Y,Z) = 0$$

where

$$g(X,Y,Z) \in \mathbb{C}[X,Y,Z]/(f, \frac{\partial f}{\partial X}, \frac{\partial f}{\partial Y}, \frac{\partial f}{\partial Z}).$$

Note that in this ideal we can get rid of f since

$$f = \frac{\delta - 2}{\delta} X \frac{\partial f}{\partial X} + \frac{1}{\delta}(Y \frac{\partial f}{\partial Y} + Z \frac{\partial f}{\partial Z}).$$

We also have that

$$Y^\delta = \frac{1}{\delta}(Y \frac{\partial f}{\partial Y} - x \frac{\partial f}{\partial X}) \text{ and } Z^\delta = \frac{1}{\delta}(\frac{\partial f}{Z \partial Z} - X \frac{\partial f}{\partial X}).$$

Then, $g(X,Y,Z) = p(X) + q(Y) + r(Z)$ with $deg(q), deg(r) \leq \delta - 1$. Moreover, we want this deformation to be equivariant with respect to the $\mathbb{Z}_\delta$ action. The easiest case is when $\rho + 1$ and ρ are coprime to δ, as this implies that $g(X,Y,Z) = p(X^\delta)$. So we conclude that the versal deformation space is given by $f(X,Y,Z) + tp(X^\delta) = 0$. Note that if $p(0) \neq 0$, then for every value of t we get a smooth surface. The interesting case is when $p(0)$. In particular we are interested in two questions. The local question: identify $\mathbb{Q}$-Gorenstein deformations that stay semi-log-canonical, and the global question: deformations that can be glued with deformations of the other chart to obtain a deformation of X^-. For example, we could get

$$XYZ = X^\delta + Y^\delta + Z^\delta$$

which is an elliptic singularity denoted by $T_(\delta, \delta, \delta)$, and after taking the quotient by the $\mathbb{Z}_\delta$ action we get an isolated log-canonical singularity.

2. In the diagonal case we proved that $\mathcal{X}^-$ is a toric threefold obtained by gluing two charts, each with a canonical cyclic quotient singularity. A natural question related to this result is: given two canonical cyclic quotient singularities from the classification of Ishida and Iwashita (with at least one of them being non terminal), can we find a way to glue them such that we get a canonical extremal neighborhood? This could help to find other rays in the canonical region where the threefolds are toric.

3. In the case of the cone over the rational normal curve, we have shown that generators for the anticanonical ring of X^+ can be obtained by lifting generators of another graded ring which is much easier to compute. Using mathematical software, especifically Macaulay2, we were able to write a program to find the generators of the canonical ring explicitly and compute the ideal of relations, thus obtaining $\mathcal{X}^-$ as a subvariety in a weighted projective space. The problem is that this program works only for smaller values of the sum of the axial multiplicities $\alpha_1 + \alpha_2$, as the number of generators for the canonical ring depends on this number. This restrictive condition on the axial multiplicities, implies in most cases that we are outside of the canonical region. It would be interesting to improve the program so that we can do computations for more rays in the canonical region and in this way obtained more examples of non-terminal anticanonical models. Even more, it would be interesting to see if this program can be extended to work with other cyclic quotient singularities. Following work of Stevens and Altman, it is not hard to write deformations of cyclic quotient singularities that satisfy the first condition of Corollary 3.23 in [HTU]. In particular, for other cyclic quotient singularities, is it still true that we can fit the canonical ring into a short exact sequence? If this were

true, then following the same process as in the case of the cone over rational normal curve, we should be able to find generators for the canonical ring and hopefully compute the ideal of relations to obtain $\mathcal{X}$ inside of a weighted projective space.

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