GENERAL COVARIANCE WITH STACKS AND THE BATALIN-VILKOVISKY FORMALISM

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GENERAL COVARIANCE WITH STACKS
AND THE BATALIN-VILKOVISKY FORMALISM

A Dissertation Presented

by

FILIP DUL

Submitted to the Graduate School of the
University of Massachusetts Amherst in partial fulfillment
of the requirements for the degree of

DOCTOR OF PHILOSOPHY

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Department of Mathematics and Statistics
Dedicated to Kazimiera Szwaja,
the only grandparent I did not meet:
May her memory be a blessing.
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ABSTRACT

GENERAL COVARIANCE WITH STACKS
AND THE BATALIN-VILKOVISKY FORMALISM

MAY 2022

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In this thesis we develop a formulation of general covariance, an essential property for many field theories on curved spacetimes, using the language of stacks and the Batalin-Vilkovisky formalism. We survey the theory of stacks, both from a global and formal perspective, and consider the key example in our work: the moduli stack of metrics modulo diffeomorphism. This is then coupled to the Batalin-Vilkovisky formalism—a formulation of field theory motivated by developments in derived geometry—to describe the associated equivariant observables of a theory and to recover and generalize results regarding current conservation.
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CHAPTER 1

INTRODUCTION

The focus of the following dissertation is to understand classical and quantum field theories on curved spaces and spacetimes using modern technology, including stacks, derived geometry, and the Batalin-Vilkovisky formalism for field theories—a formalism which is partially motivated by the preceding ideas. Higher categorical methods forged in recent decades are beginning to gain traction as powerful tools within theoretical physics, and this thesis is an attempt to implement those tools in the special case of field theories coupled to gravitational backgrounds.

1.0.1 Historical remarks

About a hundred years ago, when Albert Einstein and a group of others were laying the foundations of general relativity, general covariance became an essential ingredient in formulating physics in curved spacetimes. Roughly, a field theory coupled to a background semi-Riemannian metric on a space (or spacetime) $X$ is said to be generally covariant if the diffeomorphism group of $X$ is a symmetry of the theory. Physicists usually interpret diffeomorphisms as coordinate changes, so they may say...
that a theory exhibits general covariance if it is coordinate-invariant: i.e. a theory
may superficially change to a distinct one if the coordinates are changed, but if it
is generally covariant, then those “two” theories are isomorphic in a way which we
will make rigorous. Although general covariance can be understood in the context
of all field theories, it is often considered in the context of field theories coupled to
semi-Riemannian metrics: this particular case is what we hone in on in the following
work. Broader ideas of general covariance are discussed, for example, in [FT21].

To make the above use of the word “symmetry” more precise, let’s introduce
a bit of rigor. A field theory coupled to a background metric in fact constitutes
a family of field theories parameterized by the space of semi-Riemannian metrics,
usually denoted Met(X), although we will often denote it M for brevity whenever the
space(time) X is implicit. This family defines a vector bundle π : \mathcal{F} \to \mathcal{M}, where each
fiber π⁻¹(g) = \mathcal{F}_g is a field theory: in particular, each fiber is a Batalin-Vilkovisky
(BV) classical field theory, which we will define precisely in Chapter 4. The group
of diffeomorphisms of X, traditionally denoted Diff(X) and here called \mathcal{D}, acts on
this vector bundle. If this vector bundle (along with its differential grading or L∞
structure which makes it a bundle of BV field theories) is equivariant with respect
to \mathcal{D}, we call the theory generally covariant.

A first thought given the above might be that a generally covariant family of the-
ories should descend to a bundle on the quotient space \mathcal{M}/\mathcal{D}; however, many impor-
tant manifolds naturally considered in physics have metrics with myriad isometries,
so that the quotient \mathcal{M}/\mathcal{D} would be singular at those metrics. For example, in the
simple case of $X = \mathbb{R}^n$ the go-to metric is usually the flat metric $\eta = dx_1^2 + \ldots + dx_n^2$, which has isometry group $O(n) \ltimes \mathbb{R}^n$: this is a subgroup of $\text{Diff}(\mathbb{R}^n)$ which fixes $\eta \in \text{Met}(\mathbb{R}^n)$. Therefore, $\eta$ is a singular point in the quotient $\text{Met}(\mathbb{R}^n)/\text{Diff}(\mathbb{R}^n)$. Since the quotient space is not differentiable, we cannot assign to it any vector bundles near those singular points, e.g. the bundle of Batalin-Vilkovisky field theories. Moreover, we cannot invoke the equivalence of categories between $G$-equivariant vector bundles on a $G$-space $M$ and vector bundles on the quotient space $M/G$ which holds if the $G$-action on $M$ is free:

$$\text{VectBun}(M/G) \cong \text{VectBun}_G(M).$$

(1.1)

However, if we expand our usual notion of a manifold and understand the quotient stack $[M/G]$, we get the following equivalence regardless of whether or not the $G$-action on $M$ is free:

$$\text{VectBun}([M/G]) \cong \text{VectBun}_G(M).$$

(1.2)

We can therefore make sense of a generally covariant family of theories descending to a quotient as long as it is a stack and not an ordinary manifold.

Stacks arise in many modern geometric contexts, although historically they were motivated by issues in algebraic geometry. Chronologically, the foundational notions behind the definition were introduced in Alexander Grothendieck’s work [Gro59] on descent in 1959. The term “stack” formally entered the vernacular with Deligne and Mumford’s 1969 paper [DM69] in which they describe the moduli stack of curves of a fixed genus $g \geq 2$. Eventually, quotient stacks were introduced for an affine group scheme $G$ acting on a scheme $M$, and that theory is easily altered to adapt to the case
above, in which a Lie group $G$ acts on a smooth manifold $M$. Obstructions to forming “good” moduli spaces in these cases are any automorphism/stabilizer subgroups of $G$ of points in $M$. If $M$ is in the smooth category (in which we could define on it various structures which depend on $M$’s smoothness, like bundles), but has points stabilized by a subgroup of $G$, then $M/G$ is no longer in the smooth category. A large portion of this thesis is thus dedicated to figuring out why the category of stacks is the appropriate one for dealing with these poorly behaved moduli spaces.

We then couple the theory of stacks to the Batalin-Vilkovisky (BV) formalism to introduce general covariance within that framework. The BV formalism is rooted—at least in a modern sense—in a theory of derived geometry which arose somewhat concurrently with the theory of stacks.

1.0.2 Dissertation Outline

The goal of Chapter 2 will be to review the theory of stacks with a particular aim at expounding on the usefulness of quotient stacks in our context. Much of this will follow what is written in [Hei04] and [Car11]. Many of the concrete computations in our work are for perturbative field theories, so we must consider formal neighborhoods in quotient stacks, which in turn lead us to derived deformation theory. In brief, we can associate to a generally covariant perturbative field theory a formal moduli problem, in the style of [Lur11]: Theorem 2.0.2 in that work describes an equivalence of $(\infty, 1)$-categories between pointed formal moduli problems and differential graded Lie algebras. We take time to understand what that means and along the way, we show how certain Chevalley-Eilenberg cochains arise as derived rings
of functions on formal neighborhoods in a quotient stack. Chapter 3 describes how we might take the results of the preceding chapter and port them over to the case of infinite dimensional stacks, and in particular the moduli stack of metrics modulo diffeomorphism. We review some topological constructions concerning equivariant cohomology and describe its significance in this work.

In Chapter 4, we apply the general theory of quotient stacks which we developed in the preceding chapter to field theories. We introduce the Batalin-Vilkovisky formalism, as defined in [Cos11] and [CG21], to discuss field theories. Generally covariant families of such theories descend to vector bundles over the quotient stack $[\mathcal{M}/\mathcal{D}]$. We then make perturbative considerations with constructions from the preceding chapter in mind to generalize the usual conservation laws associated to Noether’s theorems, both computationally and interpretively: for example, there is a “perturbative equivalence” of observables when the theory is deformed by vector fields (which generate certain classes of diffeomorphisms), and we describe how this equivalence describes the data of “higher” stress-energy tensors. Chapter 4 subsumes [Dul21]: in that paper, I only considered the case of free BV classical field theories, but in this chapter we describe the interacting and perturbative cases, which requires the use of $L_\infty$ algebras. I combine techniques from the preceding chapters to state the main theorem, Theorem 4.2.21:

**Theorem 1.0.1.** For a generally covariant family $([\mathcal{L}/\mathcal{D}], \ell) \to [\mathcal{M}/\mathcal{D}]$ of BV classical field theories and for a fixed $[g] \in [\mathcal{M}/\mathcal{D}]$, we have

$$O(\text{Tot}(g^*[\mathcal{L}/\mathcal{D}]))) \cong C^\bullet(G_g, \text{Obs}^{cl}(X, \mathcal{L}_g),$$

(1.3)
where
\[ \mathfrak{g}_g = \text{Vect}(X) \xrightarrow{L^g} \Gamma(X, \text{Sym}^2(T_X^\vee))[1] \]

is the dg Lie algebra associated to the formal neighborhood of \([g] \in [\mathcal{M}/\mathcal{D}]\), and the space \(\text{Tot}(g^*[\mathcal{L}/\mathcal{D}])\) is the pullback of the family of generally covariant theories over that formal neighborhood.

Following this, we make connections between the above theorem and both the work on equivariant observables in [CG21] and the “usual” perspectives on current conservation found in physics literature.

Finally, Chapter 5 is an expository summary of what quantization entails in the Batalin-Vilkovisky formalism and how in the case of free theories, the determinant line bundle (in the style of Quillen’s work [Qui85]) takes center stage. I briefly outline future research directions in this context and how the work of this dissertation might be useful to extend results in [Rab20].
CHAPTER 2
STACKS AND FORMAL MODULI SPACES

2.1 Stacks

We primarily use stacks when dealing with moduli spaces which may be poorly behaved in some way, for example if certain points in the original space have non-trivial automorphism groups. In our main example, we focus on the space of semi-Riemannian metrics on a smooth manifold $X$, denoted $\text{Met}(X)$ or $\mathcal{M}$ if $X$ is implicit. It has a natural action of the diffeomorphism group of $X$, similarly denoted $\text{Diff}(X)$ or $\mathcal{D}$, by pullback: given $g \in \mathcal{M}$ and $f \in \mathcal{D}$, $f \cdot g = f^*g$. Many if not most smooth manifolds have naturally defined metrics which have isometry groups. For example, the usual metric on $S^2$, namely $g_{S^2} = d\theta^2 + \sin^2 \theta d\phi^2$, has isometry group $O(3) \subset \text{Diff}(S^2)$. $O(3)$ is thus the automorphism group for $g_{S^2} \in \text{Met}(S^2)$, and constitutes a singular point in $\text{Met}(S^2)/\text{Diff}(S^2)$.

In this chapter we set out to formally introduce and understand stacks. We will cover the theory broadly, but eventually hone in on the case of quotient/moduli stacks: the type of stack which is of primary interest to us. Once we cover the
“global” theory of such stacks, we will introduce notions associated to them which deal with highly “local”, in particular perturbative, data. To begin, we must first understand groupoids: the cornerstone of the theory of stacks and the objects which allow us to think about stacks concretely. In this chapter we primarily follow the constructions presented in [Car11] and [Hei04], citing others as needed.

### 2.1.1 Groupoids

In this section, we use the words “arrow” and “morphism” interchangeably.

**Definition 2.1.1.** A groupoid $\mathcal{G}$ is a small category in which all arrows are invertible. Common notation is $\mathcal{G} = \mathcal{G}_1 \xrightarrow{s} \mathcal{G}_0$, where $\mathcal{G}_1$ is the set of arrows and $\mathcal{G}_0$ the set of objects; $s$ sends an arrow to its source object, and $t$ sends it to its target. Every such $\mathcal{G}$ has an identity map $e : \mathcal{G}_0 \to \mathcal{G}_1$ sending an object to its identity arrow, an inverse map $i : \mathcal{G}_1 \to \mathcal{G}_1$ sending an arrow to its inverse, and a multiplication map $m : \mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1 \to \mathcal{G}_1$ that concatenates arrows. $s, t, e, i,$ and $m$ are called the structure maps of $\mathcal{G}$.

**Example 2.1.2.** A premier example of a groupoid is the **action groupoid** which can be associated to any smooth $G$-space $M$. Its set of objects is $\mathcal{G}_0 = M$ and its set of arrows is $\mathcal{G}_1 = M \times G$, so that we can write it as

$$M \times G \xrightarrow{\cdot} M =: M//G$$

In this case, $s(p,g) = p$ and $t(p,g) = g \cdot p$. The action groupoid is defined as a first step toward understanding quotient spaces which are not necessarily smooth, in
the sense that the action of $G$ could fix certain points in $M$ and so $M/G$ could be singular.

**Example 2.1.3.** In the spirit of introducing examples that fall outside of the usual scope of our work, let’s consider the **fundamental groupoid** $\Pi_1(X)$. The objects of $\Pi_1(X)$ are simply the points of $X$, and the set of morphisms between two points $x, y \in X$ are all continuous paths $\gamma : [0, 1] \to X$ up to homotopy such that $\gamma(0) = x$ and $\gamma(1) = y$.

A key fact to observe from the above definition is that the automorphism group of a fixed point $x \in X$ is easily recognizable:

$$\text{Aut}_{\Pi_1(X)}(x) = \pi_1(X, x).$$

**Remark 2.1.4.** An advantage to considering this example is that it quickly generalizes to give an example of an **$\infty$-groupoid**: the 1-morphisms are the morphisms above, 2-morphisms are homotopies between paths, 3-morphisms are homotopies between those homotopies, and so on. The fundamental $\infty$-groupoid is denoted $\Pi_\infty(X)$.

In the action groupoid example, we could also define morphisms between morphisms: the canonical choice is $M \times G \times G$. The higher morphisms are thus $M \times G \times \ldots \times G$, etc., and thus we can associate an $\infty$-groupoid to a $G$-manifold $M$, as well.

Although it is interesting to think about $\infty$-groupoids given that they arise naturally from what we’ve seen, let us return to thinking about ordinary groupoids. We can restrict what kinds of groupoids we’re considering by noting that for the
action groupoid we’ve defined above, the objects and morphisms constitute smooth manifolds. We get the following definition.

**Definition 2.1.5.** A Lie groupoid \( \mathcal{G} = \mathcal{G}_1 \xrightarrow{s,t} \mathcal{G}_0 \) is a groupoid such that both the space of arrows \( \mathcal{G}_1 \) and space of objects \( \mathcal{G}_0 \) are smooth manifolds, all structure maps are smooth, and the source and target maps \( s, t : \mathcal{G}_1 \to \mathcal{G}_0 \) are surjective submersions.

In other words, a Lie groupoid is a groupoid internal to the category of smooth manifolds.

**Remark 2.1.6.** If \( \pi : V \to M \) is a smooth \( G \)-equivariant vector bundle, then we could also define its action groupoid \( V \times G \xrightarrow{\rho} V =: V//G \). Both \( V//G \) and \( M//G \) are in fact Lie groupoids, and \( V//G \) is a vector space object over \( M//G \) in the category \( \text{LieGrpd} \) of Lie groupoids. By some abuse of notation, we get a vector bundle in \( \text{LieGrpd} \):

\[
\pi : V//G \to M//G.
\]

**Example 2.1.7.** An important, if unexciting, example of a Lie groupoid given a smooth manifold \( M \) is \( M \) itself: the objects are the points of \( M \) and the only morphisms are the identity morphisms. We denote this as \( M \xrightarrow{\text{id}} M \) or simply \( M \), if it’s clear in context. With this, we obtain a fully faithful functor \( \text{Mfd} \to \text{LieGrpd} \).

**Example 2.1.8.** A groupoid which is well known to physicists is that of gauge fields and the gauge transformations between them. The set of objects is the set \( \mathcal{A} \) of gauge fields—or connection one-forms to the mathematically inclined—and the set of arrows is \( \mathcal{A} \times \mathcal{G} \), where \( \mathcal{G} \) is the group of gauge transformations. Gauge-invariant
theories are those that produce equivalent physics for two gauge fields $A$ and $A'$ as long as $A \xrightarrow{g} A'$ for some $g \in \mathcal{G}$. Therefore, it is essential to understand the quotient $\mathcal{A}/\mathcal{G}$. Issues may arise when particular $A \in \mathcal{A}$ are fixed points of a subgroup of $\mathcal{G}$; thus, groupoids and in particular stacks become useful tools here.

In the context of $\text{Met}(X) \circ \text{Diff}(X)$, the gauge fields get replaced by metrics (and perhaps additional “matter fields”), and so diffeomorphisms play the role of gauge transformations. Due to this, many analogies can be drawn between the two examples.

As the preceding narrative implies, groupoids are perfectly designed to keep track of moduli data associated to individual points in some space. Indeed, they are the essential ingredient required in a definition of stacks, as we will now see.

### 2.1.2 Stack: Definitions and an Example

Notice that if we slightly alter the definition of $M//G$ by replacing points with maps $\ast \to M$, the result is the same category. However, this version of the definition is amenable to a generalization via the functor of points perspective: the key is that instead of defining a functor $F : \text{Mfd}^{\text{op}} \to \text{Set}$ (or out of $\text{CommRing}^{\text{op}}$ if one is an algebraic geometer), we replace $\text{Set}$ by $\text{Grpd}$, the category of groupoids. This allows us to retain any desired “equivalence data” specific to the model at hand.

**Remark 2.1.9.** A reasonable conjecture for $F$ is to let $F(N)$ be the groupoid whose objects are maps $N \xrightarrow{f} M$ and arrows are $f_1 \xrightarrow{A} f_2$, defined as functions $A : N \to G$ such that $A(x)f_1(x) = f_2(x)$ for $x \in N$; this is a natural generalization of the action
groupoid $M//G$. However, unlike in the case of $N = \ast$ (where $F(\ast) = M//G$), great care must be taken when considering the topological non-triviality of an arbitrary test manifold $N$, and how we might glue together the data provided by the arrows as we “move around” in $N$. This is where descent (sheaf-like) conditions take the stage. Indeed, $F$ as defined here is a prestack, inasmuch as it encodes what we’d like our quotient to do: it must be stackified (analogous to sheafified) to be sufficiently sensitive to the topology of the test spaces. A physicist might notice that the objects and arrows of $F(N)$ look like fields and gauge transformations: this suggests that this is what the data looks like locally, and that we must be careful globally.

Given this motivation, we present the definition and provide a key example.

**Definition 2.1.10 ([Hei04]).** A prestack, is a (pseudo-)functor $\mathcal{X} : \text{Mfd}^{\text{op}} \to \text{Grpd}$. I.e. (1) For any $N \in \text{Mfd}$, $\mathcal{X}(N)$ is a category where all arrows are invertible.

(2) For any arrow $f : N_2 \to N_1$, we have a functor $f^* : \mathcal{X}(N_1) \to \mathcal{X}(N_2)$.

(3) For any concatenation $N_3 \xrightarrow{g} N_2 \xrightarrow{f} N_1$, there is a natural transformation $\Phi_{f,g} : g^*f^* \cong (g \circ f)^*$ which is associative for 3 composable arrows.

**Definition 2.1.11.** For $X \in \text{Mfd}^{\text{op}}$, $\mathcal{X}(N)$ is called the $N$-points of $\mathcal{X}$.

**Remark 2.1.12.** Technically, the above is a differentiable stack: its source is the opposite category of smooth manifolds. One can take the source to be $\text{Top}^{\text{op}}$ and define a topological stack. We stick with the shortened terminology because all of our stacks will be differentiable.

As it stands, a prestack doesn’t have to satisfy any gluing conditions. Indeed, $F$ as conjectured above is a prestack, but not a stack, because it doesn’t satisfy descent.
We will use the following shorthand whenever it appears: for $\iota : U \hookrightarrow X$, we write $|U$ in place of $\iota^*$.

**Definition 2.1.13 ([Hei04]).** A prestack $\mathfrak{X}$ over $\text{Mfd}^{\text{op}}$ is a stack if for any $N \in \text{Mfd}^{\text{op}}$ and open cover $\mathcal{U} = \{U_i\}$ of $N$, it satisfies descent, in other words:

1. Given objects $P_i \in \mathfrak{X}(U_i)$ and isomorphisms $\varphi_{ij} : P_i|_{U_i \cap U_j} \to P_j|_{U_i \cap U_j}$ such that $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}|_{U_i \cap U_j \cap U_k}$, there is an object $P \in \mathfrak{X}(N)$ and isomorphisms $\varphi_i : P|_{U_i} \to P_i$ such that $\varphi_{ij} = \varphi_j \circ \varphi_i^{-1}$. This is called effective descent data.

2. Given $P, P' \in \mathfrak{X}(N)$ and isomorphisms $\varphi_i : P|_{U_i} \to P'|_{U_i}$ such that $\varphi_i|_{U_i \cap U_j} = \varphi_j|_{U_i \cap U_j}$, there is a unique map $\varphi : P \to P'$ such that $\varphi_i = \varphi|_{U_i}$.

Someone familiar with the definition of a sheaf may recognize the above as strongly analogous to sheaves, with appropriate additional data. However, we would like to keep things simple, so we will dive into our key example.

**Definition 2.1.14.** Given a smooth $G$-manifold $M$, the associated quotient stack is the functor $[M/G] : \text{Mfd}^{\text{op}} \to \text{Grpd}$ such that the objects of $[M/G](N)$ are pairs of maps

$$
P \xrightarrow{\alpha} M, \quad \quad \quad \quad \quad \pi \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad N$$

where $\pi : P \to N$ is a principal $G$-bundle and $\alpha : P \to M$ is a $G$-equivariant map, and the morphisms are isomorphisms $P \xrightarrow{\sim} P'$ of principal $G$-bundles over $N$ commuting with the $G$-equivariant maps to $M$. 

13
Remark 2.1.15. An essential qualification is that \([M/G]\) evaluated on a point recovers \(M//G\). Therefore, \([M/G]\) rightly gives a natural generalization of \(M//G\), as desired. Notice as well that this definition “generalizes” our conjecture \(F\) from earlier: it maps \(N\) into \(M\) as the base manifold of a \(G\)-bundle \(P\) over \(N\). In this sense, \(N\) is being mapped into \(M\) while encoding any interesting \(G\)-actions that may be associated to it.

Example 2.1.16. A significant example derived from the above definition is that of \([pt/G]\), for \(pt\) a point. Applying the definition shows that \([pt/G](X)\) is precisely \(\text{Bun}_G(X)\), the category of principal \(G\)-bundles over \(X\) (any morphism of \(G\)-bundles over the same base space is necessarily an isomorphism). Because of this, it is common to identify \([pt/G]\) with \(BG\), since \([X, BG]\) is equivalent to \(\text{Bun}_G(X)\) modulo bundle isomorphisms.

In addition, defining a vector bundle \(V \to [pt/G]\) amounts to fixing the vector space \(V\) (a vector bundle over the point) as well as a representation \(\rho : G \to \text{End}(V)\). In other words, we have an equivalence of categories:

\[
\text{VectBun}([pt/G]) \cong \text{Rep}(G).
\] (2.1)

This equivalence becomes useful when we consider deformation theoretic computations in the following section. In that case we can make sense of vector bundles over \([pt/g]\), alternatively named \(Bg\), and see how rings of functions on vector bundles over such a formal stack are naturally modules over \(O(Bg)\), which is equal to the Chevalley-Eilenberg cochains \(C^*(g)\). This will be introduced later, beginning with Section 2.2.
**Remark 2.1.17.** The preceding example is a simple but beautiful illustration of how specifying a vector bundle $V$ over a quotient stack $[M/G]$ is equivalent to specifying a $G$-equivariant vector bundle over a $G$-manifold $M$. We therefore have the following.

**Theorem 2.1.18.** For $M$ a smooth $G$-space, we have the following equivalence of categories:

$$\text{VectBun}([M/G]) \cong \text{VectBun}_G(M).$$

The perspective outlined here shows us how a vector bundle over a $G$-space generalizes the notion of a $G$-representation. For example, say we have an $H$-representation $V$. The above implies that we might have a $G$-equivariant bundle $V \to M$ where $H$ is a stabilizer subgroup of $G$ for a point $p \in M$: in this case, the fiber $V_p \cong V$ is a representation of $H$, and so the $G$-equivariant bundle $V \to M$ provides a “broadening” of the usual notion of symmetry that representation theory buys us.

A totally analogous statement holds for sheaves\(^1\) on $[M/G]$ and $G$-equivariant sheaves on $M$, as can be found in section 4 of [Hei04]:

**Theorem 2.1.19.** For $M$ a smooth $G$-space, we have the following equivalence of categories:

$$\text{Shv}([M/G]) \cong \text{Shv}_G(M).$$

**Remark 2.1.20.** Theorem 2.1.18 is in fact a special case of Theorem 2.1.19, but one must be careful to represent a vector bundle by its sheaf of sections: this will

\(^1\)Actually, the sheaves here must be *Cartesian*, but we won’t dwell on this point.
become important for us in Chapter 4 when we consider perturbative or interacting
theories. Theorem 2.1.18 is sufficient when thinking about free (and to a degree
even free nonperturbative) theories, but Theorem 2.1.19 is essential in the case of
perturbative and interacting theories, in which case the relevant object is a family
(a sheaf) of $L_{\infty}$ algebras, not vector spaces.

**Example 2.1.21.** At this point, we can understand why the conjectural functor $F$
at the beginning of this section does not define a stack. We’ll consider the case of
$\mathbb{Z}/2\mathbb{Z}$ acting on a point. Then for $F : \text{Mfd}^{\text{op}} \to \text{Grpd}$ as above, the objects of
$F(N)$ are maps $f : N \to \text{pt}$ and arrows are $A : f_1 \to f_2$ such that $A : N \to \mathbb{Z}/2\mathbb{Z}$
and $A(x)f_1(x) = f_2(x)$. Let’s look at the case of $F(S^1)$, since $S^1$ has an interesting
topology. We can cover $S^1$ with two copies of $\mathbb{R}$—call them $U_1$ and $U_2$—each covering
slightly over half of two opposite hemispheres, so that their intersection $U_{12}$ is a
disjoint union of two open sets situated antipodally on $S^1$. This means gluing two
objects, $a_1 \in F(U_1)$ and $a_2 \in F(U_2)$, to get an object of $F(S^1)$.

Unpacking the definition shows that $a_1$ and $a_2$ must be points and equivalent
on the intersection: $a_1|_{U_{12}} = * = a_2|_{U_{12}}$. Next, choose an arrow $\varphi_{12} : * \to *$
between them in $F(U_{12})$ that is $+1$ on one open in the disjoint union and $-1$ on the
other. There’s only one object $s \in F(S^1)$ (the constant map), and to have effective
descent data means there must be $\varphi_1 : s|_{U_1} \to a_1$ and $\varphi_2 : s|_{U_2} \to a_2$ such that
$\varphi_{12} \circ \varphi_1|_{U_{12}} = \varphi_2|_{U_{12}}$. Since the $\varphi_i$ must be the constant maps $\pm 1$, either $\varphi_1 = \varphi_2$
or $\varphi_1 = -\varphi_2$ on $U_{12}$, which clearly violates the requirement for descent data to be

---

\[\text{Credit goes to Jesse Selover for helping me understand this example.}\]
effective, from the definition!

**Example 2.1.22.** A fundamental example of a stack over $\text{Mfd}^{op}$ is an ordinary manifold. For such a manifold $M$, we can define the stack $\underline{M}$ as $\underline{M}(N) := \text{Map}(N, M) = C^\infty(N, M)$ for $N \in \text{Mfd}^{op}$. This embeds $\text{Mfd}^{op}$ into the category $\text{Stk}$ of (differentiable) stacks, and moreover we get the following version of an essential lemma.

**Lemma 2.1.23** (Yoneda Lemma for Stacks). Let $\mathcal{X}$ be a stack and let $M$ be a manifold. We have the following equivalence of categories:

$$\mathcal{X}(M) \cong \text{Mor}_{\text{Stk}}(\underline{M}, \mathcal{X}).$$

One might expect that since (differentiable) stacks are designed to generalize the notion of an ordinary manifold, there should be an analogous notion of an atlas for stacks. Indeed, one could use this route to define a differentiable stack in the first place, similarly to how an atlas is used to define an ordinary manifold. Although we chose a different route, we’ll present the definition. We may sometimes denote $\underline{M}$ simply as $M$, when it’s implicit in context.

**Definition 2.1.24.** An atlas (or covering) for a stack $\mathcal{X}$ is a manifold $X$ and map $p : X \to \mathcal{X}$ such that (1) for any manifold $Y$ and $Y \to \mathcal{X}$, the stack $X \times_{\mathcal{X}} Y$ is a manifold, and (2) $p$ is a submersion, i.e. for all $Y \to \mathcal{X}$, the projection $Y \times_{\mathcal{X}} X \to Y$ is a submersion.

**Example 2.1.25.** We have already seen an example of an atlas in a previous example: the quotient map $M \to [M/G]$ is an atlas (and also a principal $G$-bundle).
Much like how we use atlases to define principal and vector bundles over an ordinary manifold, we use atlases to define such bundles over stacks, as follows.

**Definition 2.1.26.** A principal $G$-bundle $\mathcal{P} \to \mathfrak{X}$ is given by a $G$-bundle $\mathcal{P}_X$ over an atlas $X \to \mathcal{P}$ with an isomorphism of the two pullbacks $p_1^*\mathcal{P}_X \simeq p_2^*\mathcal{P}_X$ from $X \times_X X \to X$ satisfying the cocycle condition on $X \times_X X \times_X X$.

**Remark 2.1.27.** The definition of a vector bundle over a stack $\mathfrak{X}$ is completely analogous to this. Of course, one could instead invoke that a vector bundle $V \to \mathfrak{X}$ of rank $n$ is equivalent to a principal $GL(n, \mathbb{R})$-bundle and then use the preceding definition. Either way, this is the definition required to make sense of the equivalence in Theorem 2.1.18.

Atlases also provide a concrete manifestation of the relationship between groupoids and stacks. For an atlas $X \to \mathfrak{X}$, the two projections $X \times_X X \xrightarrow{\cong} X$ define source and target maps which make the preceding object into a groupoid. A simple example of this comes from the atlas $M \to [M/G]$; in that case we have

$$M \times_{[M/G]} M \cong M \times G,$$

and the projections are the familiar action and identity maps.

On the other hand, any groupoid $\mathcal{G} = \mathcal{G}_1 \xrightarrow{\sim} \mathcal{G}_0$ defines a stack as follows.\(^3\) Let

\(^3\)This can be found in more detail in [Hei04].
$[\mathcal{G}_0/\mathcal{G}_1] : \text{Mfd}^\text{op} \to \text{Grpd}$ be a (pseudo-)functor such that $[\mathcal{G}_0/\mathcal{G}_1](N)$ is

$$
\begin{array}{ccc}
P & \xrightarrow{\alpha} & \mathcal{G}_0 \\
\downarrow \pi & & \downarrow \\
N & & 
\end{array}
$$

Together with an action $\mathcal{G}_1 \times_{\mathcal{G}_0} P \to \mathcal{G}_0$ which is equivariant with respect to composition of morphisms in $\mathcal{G}_1$, and such that there exists a covering $U \to N$ and maps $\alpha_i : U \to \mathcal{G}_0$ such that $P|_U \cong \alpha_i^* \mathcal{G}_i$. Notice that this slightly generalizes Definition 2.1.14.

**Remark 2.1.28.** We say in this context that the groupoid $\mathcal{G}$ is a *presentation* of the stack $[\mathcal{G}_0/\mathcal{G}_1]$. It is important to note that a stack $\mathfrak{X}$ can have many different groupoids which present it. We won’t delve into that here: a good reference to learn more is [Car11].

### 2.2 Formal Stacks/Moduli Spaces

In the preceding section we outlined the theory of “global” stacks, with a focus on quotient/moduli stacks, which are of primary interest in mathematical physics. Much of our work is concerned with perturbative field theory, so a key step will be to associate to a (differential graded) equivariant vector bundle\(^4\) an appropriate vector bundle over a formal moduli space: in our case, this formal moduli space is

\(^4\)Or a bundle of $L_\infty$ algebras in the case of an interacting BV theory.
a formal neighborhood in a quotient stack. Along the way, we’ll discover the right Chevalley-Eilenberg cochains necessary to perform concrete computations.

Understanding formal neighborhoods in moduli stacks may seem like a daunting task, but many techniques have been developed in recent decades which make associated computations rather concrete. Much of this is thanks to a central guiding statement which serves as the cornerstone of modern deformation theory:

**Theorem 2.2.1.** There is an equivalence of $(\infty,1)$-categories between the category $\text{Lie}_k$ of differential graded Lie algebras over a characteristic zero field $k$ and the category $\text{Moduli}_k$ of formal pointed moduli problems over $k$.

This theorem has a long history, going back at least to Dan Quillen’s work on rational homotopy theory in the late 1960s and slowly growing out to cover more ground, as can be seen in [Dri88], V. Drinfeld’s famous letter to V. Schechtman. We find the most satisfying treatment of it to be Theorem 2.0.2 in [Lur11], Jacob Lurie’s Derived Algebraic Geometry X.

**Remark 2.2.2.** This statement will manifest in at least two distinct guises for us: one will be in understanding the natural analogue of tangent spaces to quotient stacks—which we will cover shortly—and the other will be in resolving intersection singularities of critical loci associated to action functionals for a classical field theory. It should be noted that especially in the latter case, we will be more concerned with $L_\infty$ algebras; however, we are safe because the homotopy category of $L_\infty$ algebras is equivalent to the homotopy category of dg Lie algebras.

Although we won’t explicitly refer to the definition of a formal moduli problem
very often, we present it here for completion and for the geometric intuition we believe it provides: it shows the analogy between (algebraic) geometry, where the fundamental objects are commutative algebras, to derived geometry, where the key objects are differential graded commutative algebras. In the formal case, these dg commutative algebras are also nilpotent. We state the definition for any characteristic zero field $k$, although for us it will usually be $\mathbb{R}$ or $\mathbb{C}$. Much of what follows can be found in greater detail in Chapter 3 of [CG21]. To begin, we must formally introduce differential graded (dg) Lie algebras and $L_\infty$ algebras.

**Definition 2.2.3.** A **differential graded Lie algebra** over a commutative ring $R$ is a $\mathbb{Z}$-graded $R$-module $\mathfrak{g}$ such that (1) it has a differential $d$ making $(\mathfrak{g}, d)$ into a dg $R$-module, and (2) it has a bilinear bracket $[-, -] : \mathfrak{g} \otimes R \mathfrak{g} \to \mathfrak{g}$ such that for $x, y \in \mathfrak{g}$,

(i) $[x, y] = -(-1)^{|x||y|}[y, x]$ (graded antisymmetry),

(ii) $d[x, y] = [dx, y] + (-1)^{|x||y|}[x, dy]$ (graded Leibniz rule), and

(iii) $[x, [y, z]] = [[x, y], z] + (-1)^{|x||y|}[y, [x, z]]$ (graded Jacobi rule), where $|x|$ denotes the cohomological degree of $x$.

**Example 2.2.4.** Rather trivially, an ordinary Lie algebra $\mathfrak{g}$ with its bracket and the zero differential is a dg Lie algebra. To make things more interesting, we can fix a smooth manifold $X$ and tensor $\mathfrak{g}$ with the de Rham forms on $X$ to get a dg Lie algebra. In this case, the differential on $\Omega^\bullet(X) \otimes \mathfrak{g}$ is the usual exterior derivative $d$ and the bracket is

$$[\alpha \otimes x, \beta \otimes y] = \alpha \wedge \beta \otimes [x, y].$$

This is an essential dg Lie algebra in the study of gauge theories.
Although dg Lie algebras are homotopic to $L_\infty$ algebras, we will often use the latter, since they computationally encompass a broader range of phenomena we’d like to describe.

**Definition 2.2.5.** An $L_\infty$ algebra over $R$ is a $\mathbb{Z}$-graded, projective $R$-module $\mathfrak{g}$ with a sequence of multilinear maps of cohomological degree $2 - n$:

$$\ell_n : \mathfrak{g} \otimes_R \cdots \otimes_R \mathfrak{g} \rightarrow \mathfrak{g},$$

where $n \in \mathbb{N}$, such that all $\ell_n$ are (1) graded antisymmetric and (2) satisfy the $n$-Jacobi rule.\(^5\)

**Remark 2.2.6.** Low values of $n$ recover familiar rules, which don’t take as much space to detail. For example, the 1-Jacobi rule says that $\ell_1 \circ \ell_1 = 0$: i.e. $\ell_1$ defines a differential on $\mathfrak{g}$. Denoting $\ell_1$ by $d$ and $\ell_2$ by $[-,-]$, the 2-Jacobi rule says $-[dx_1, x_2] + [dx_2, x_1] + d[x_1, x_2] = 0$, encoding the graded Leibniz rule. For a less trivial example, the 3-Jacobi rule is:

$$[[x_1, x_2], x_3] + [[x_2, x_3], x_1] + [[x_3, x_2], x_1]$$

$$= d\ell_3(x_1, x_2, x_3) + \ell_3(dx_1, x_2, x_3) + \ell_3(dx_2, x_3, x_1) + \ell_3(dx_3, x_1, x_2).$$

In other words, $\mathfrak{g}$ doesn’t satisfy the usual Jacobi rule precisely: only up to a relation dependent on the higher brackets. Indeed, $L_\infty$ algebras are defined as a generalization of dg Lie algebras where the Jacobi rule is only satisfied up to a hierarchy of higher homotopies. Note that at the level of cohomology, the usual Jacobi rule is satisfied.

\(^5\)We are partially sweeping a long definition under the rug here: Definition in A.1.2 in [CG21] is the whole megillah.
Example 2.2.7. The most natural example of an $L_{\infty}$ algebra for us comes from encoding nonlinear partial differential equations: i.e. those associated to an interacting field theory, with a degree three or higher action functional.

For example, say we want to encode $\Delta_g \varphi + \frac{1}{3!} \varphi^3 = 0$, the Euler Lagrange equation associated to the action functional

$$S_g(\varphi) = -\frac{1}{2} \int_X \varphi \Delta_g \varphi \text{vol}_g + \frac{1}{4!} \int_X \varphi^4 \text{vol}_g,$$

The pertinent $L_{\infty}$ algebra has underlying cochain complex

$$\mathcal{L} = C^\infty(X)[-1] \to \text{Dens}(X)[-2],$$

where the differential is $Q_g \varphi = \Delta_g \varphi \text{vol}_g$ and the only higher bracket is $\ell_3 : C^\infty(X)^{\otimes 3} \to \text{Dens}(X)$, defined as $\ell_3 : \varphi_1 \otimes \varphi_2 \otimes \varphi_3 \mapsto \varphi_1 \varphi_2 \varphi_3 \text{vol}_g$. Letting $(R, \mathfrak{m}_R)$ be a nilpotent Artinian ring in degree 0, we get that $\varphi \in C^\infty(X) \otimes \mathfrak{m}_R$ satisfies the Maurer-Cartan equation $\mathcal{L}$ if and only if

$$Q_g \varphi + \frac{1}{3!} \varphi^3 \text{vol}_g = 0,$$

which recovers the desired partial differential equation (with values in densities).

Thus we see how an $L_{\infty}$ algebra quantifies how a given equation fails to be linear: a free theory has only nontrivial $\ell_1$, and so only requires a dg Lie algebra to be described. We will cover this in much higher detail in Chapter 4.

Definition 2.2.8. A formal (pointed) moduli problem$^6$ over $k$ is a functor of simplicially enriched categories

$$F : \text{dgArt}_k \to \text{sSets},$$

$^6$We sometimes refer to this as a formal moduli space, but for now use the more common name.
where \( \text{dgArt}_k \) is the category of (local) Artinian dg algebras over \( k \) and \( \text{sSets} \) the category of simplicial sets, which satisfies:

1. \( F(k) \) is contractible.
2. \( F \) takes surjective maps in \( \text{dgArt}_k \) to fibrations in \( \text{sSets} \).
3. For \( A, B, C \in \text{dgArt}_k \) and surjections \( B \to A \) and \( C \to A \) (meaning we can define the fiber product \( B \times_A C \)), we require that the following natural map is a weak equivalence:

\[
F(B \times_A C) \to F(B) \times_{F(A)} F(C).
\]

**Remark 2.2.9.** The word “local” in the definition of \( \text{dgArt}_k \) is in parentheses because it is sometimes omitted in the literature, as it is often implicit. A classical example of a local Artinian algebra is \( k[[\varepsilon]]/(\varepsilon^2) \), the *dual numbers* associated to a field \( k \). Thus, in the correspondence between commutative rings and their spectra, local Artinian algebras correspond to infinitesimal pointed spaces. \( \text{Spec}(k[[\varepsilon]]/(\varepsilon^2)) \) is the base space for one dimensional first order deformations; \( \text{Spec}(k[[\varepsilon]]/(\varepsilon^{n+1})) \) is the same for one dimensional \( n \)th order deformations, and so on. The above definition provides the analogous formulation in the derived geometric setting.

**Construction 2.2.10.** We will take Theorem 2.2.1 for granted, but examining the functor which provides the equivalence is essential for us. For an \( L_\infty \) algebra \( g \) and a dg Artinian algebra \( R \) with its unique maximal differential ideal \( m_R \) (i.e. \( R/m_R = k \) and \( m_R^N = 0 \) for \( N \gg 0 \)), we denote the simplicial set of solutions to the Maurer-Cartan equation in \( g \otimes m_R \) as

\[
\text{MC}(g \otimes m_R).
\]

An \( n \)-simplex in this simplicial set is an element \( \alpha \in g \otimes m_R \otimes \Omega^* (\Delta^n) \) of cohomological
degree 1 which satisfies the Maurer-Cartan (MC) equation:

\[ d\alpha + \sum_{n \geq 2} \frac{1}{n!} \ell_n(\alpha, \ldots, \alpha) = 0. \] (2.4)

The above process defines a functor \( B\mathfrak{g} : \text{dgArt}_k \to \text{sSet} \) taking \((R, \mathfrak{m}_R) \) to \( \text{MC}(\mathfrak{g} \otimes \mathfrak{m}_R) \), which shows us clearly how it is a functor of points in the setting of formal derived spaces. A standard result in derived deformation theory is that \( B\mathfrak{g} \) defines a formal moduli problem [Get09].

**Remark 2.2.11.** Although we won’t prove the preceding theorem, we think it’s helpful to note that passing from \( \mathfrak{g} \) to \( \mathfrak{g} \otimes \mathfrak{m}_R \)–which is also an \( L_\infty \) algebra but inherits nilpotence properties from \( \mathfrak{m}_R \)–is what makes \( \text{MC}(\mathfrak{g} \otimes \mathfrak{m}_R) \) well defined.\(^7\)

In some sense, passing to this tensor product is what makes the data supplied by \( \mathfrak{g} \) *perturbative* in a bona fide way.

**Remark 2.2.12 (The Dictionary).** All of the above is meant to bring into a focus a sort of *dictionary* between formal moduli problems and \( L_\infty \) algebras. For example, if \( \mathfrak{g} \) is finite dimensional, then a Maurer-Cartan element in \( \mathfrak{g} \otimes \mathfrak{m}_R \) is equivalent to a map of dg commutative algebras

\[ C^*(\mathfrak{g}) \to R \]

taking the maximal ideal of \( C^*(\mathfrak{g}) \) to \( \mathfrak{m}_R \). We can thus identify the dg ring \( C^*(\mathfrak{g}) \) of Chevalley-Eilenberg (CE) cochains as \( \mathcal{O}(B\mathfrak{g}) \), the algebra of functions on \( B\mathfrak{g} \). As a quick reminder, we take a moment to define CE cochains:

\(^7\)In this context especially, the author must impart gratitude toward Chapter 2 of [Lur11].
Definition 2.2.13 (Definition A.4.2 in [CG16]). For a (dg)\(^8\) Lie algebra \(g\) with underlying field \(k\), the Chevalley-Eilenberg complex for Lie algebra cohomology of a \(g\)-module \(M\) is
\[
C^\bullet(g, M) := (\text{Sym}_k(g^\vee[-1]) \otimes_k M, d).
\] (2.5)

For a linear basis \(\{e^k\}\) for \(g^\vee\) and \(m \in M\), the differential \(d\) is defined to be
\[
d(e^k \otimes m) = - \sum_{i<j} e^k([e_i, e_j])e^i \wedge e^j \otimes m + \sum_i e^k \wedge e^i \otimes [e_i, m],
\] (2.6)
where the lower indices denote elements of \(g\). This is then extended to all of \(C^\bullet(g, M)\) as a derivation of cohomological degree 1.

Remark 2.2.14. This definition skips ahead from the usual definition of Lie algebra cohomology. There, using the fact that the ground field \(k\) is a quotient of the universal enveloping algebra \(Ug\) by the ideal \((g)\), we get
\[
H^\bullet(g, M) := \text{Ext}^\bullet_{Ug}(k, M).
\]
This form of \(k\) is used to provide a resolution
\[
(\cdots \rightarrow \Lambda^n g \otimes_k Ug \rightarrow \cdots \rightarrow g \otimes_k Ug \rightarrow Ug) \xrightarrow{\sim} k,
\]
which in turn provides the more standard expression of CE cochains:
\[
\text{Hom}_g(\Lambda^\bullet g \otimes_k Ug, M) \cong \text{Hom}_k(\Lambda^\bullet g, M).
\]

\(^8\)This definition is for non-dg \(g\) and \(M\): the dg definition is analogous, but for a few complications due to internal differentials and signs from the grading.
Computations using $C^\bullet(g, M)$ can tell us a lot: for example, it’s quick to show that $H^0(g, M) = M^g$, the $g$-invariants of $M$, and still higher cohomology groups for various $M$ tell us a lot about the representation theory of $g$. Computations with this dg ring expounding on its usefulness for us can be found in Example 2.2.26.

Resuming the dictionary, we can identify a dg vector bundle over $Bg$ with a dg module over $g$, denoted $M$, so that the sections of that bundle are thus $C^\bullet(g, M)$. This allows us to recover familiar expressions for one-form fields, as

$$\Omega^1(Bg) = C^\bullet(g, g^\vee[-1])$$

and vector fields, as

$$\text{Vect}(Bg) = C^\bullet(g, g[1]).$$

We interpret the latter in the usual sense as derivations of $\mathcal{O}(Bg) = C^\bullet(g)$, and eventually (up to a shift up by one) as the Lie algebra of deformations of the $L_\infty$ structure on $g$: this is of central importance for us in the context of perturbative field theory later on.

### 2.2.1 Vector Bundles over Formal Moduli Spaces

Given what we have broadly introduced above, we will now focus on providing the specific machinery from formal derived geometry which is relevant in perturbative field theory. The key step will be to associate to a (differential graded) equivariant vector bundle a vector bundle over a formal moduli problem: in our case, this formal moduli space is a formal neighborhood in a quotient stack.
In Chapter 4, the dg vector bundle (eventually the bundle of $L_\infty$ algebras) in question will be the family of free Batalin-Vilkovisky field theories, and the quotient stack which is the base space of the bundle will be that of metrics modulo diffeomorphism. To begin, we will introduce tangent complexes: the analogue of tangent spaces for differentiable stacks.

**Construction 2.2.15.** Let $M$ be a smooth $G$-space and let $\text{Stab}(p) \subseteq G$ be the stabilizer subgroup of $p \in M$. The $G$-orbit of $p$ thus looks like a copy of $G/\text{Stab}(p)$ lying in $M$. If we consider the map $t_p : G \to M$ defined as $t_p(g) = g \cdot p$, then its differential $dt_p$ can be used to define a 2-term cochain complex of vector spaces:

$$0 \to \mathfrak{g}[1] \xrightarrow{dt_p} T_p M \to 0 =: T_p[M/G], \quad (2.7)$$

where $\mathfrak{g}$ is in cohomological degree $-1$ and $T_p M$ is in degree 0. Alternative notation is $T_p[M/G] = (\mathfrak{g}[1] \oplus T_p M, dt_p)$. Note that $\text{Stab}(p)$ could be discrete here, although that isn’t seen in $T_p[M/G]$. We can also compute

$$\ker(dt_p) = H^{-1}(T_p[M/G]) = \text{Lie}(\text{Stab}(p)).$$

Thus, if $H^{-1}(T_p[M/G]) = 0$, then the coarse quotient $M/G$ is an ordinary manifold in a small enough neighborhood of $p$, since the action is free nearby it. $H^0(T_p[M/G])$ is the quotient of $T_p M$ by $\text{im}(dt_p)$: it’s the usual tangent space of the coarse quotient at points $p \in M$ where the action is free. As it turns out, $T_p[M/G]$ is exactly the tangent object we are looking for, as the notation suggests: further details are wonderfully detailed in [An21].

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$^9$ $t_p$ is in fact the target map for the Lie groupoid $M \times G \rightrightarrows M$ with $p \in M$ fixed in $M \times G$. 

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**Proposition 2.2.16.** The **tangent complex** to the quotient stack \( [M/G] \) at a point \( p \) is \( T_p[M/G] \), as defined in equation (2.7).

**Remark 2.2.17.** This inspired the saying that “smooth stacks are geometric spaces whose tangent spaces are complexes concentrated in nonpositive cohomological degree”. The above is concentrated in degrees \(-1\) and \(0\), but if we were to consider higher stacks, there would be data in lower degrees, representing “higher” infinitesimal symmetries. In the case of quotient stacks, we’re lucky to have a concrete way of realizing their associated tangent complexes.

**Example 2.2.18.** Although we will dive much more deeply into the topic of the moduli space of metrics modulo diffeomorphism in the upcoming section, the current context is appropriate for introducing a a tangent complex that will be useful for us later. Consider the natural action of the group of diffeomorphisms \( \mathcal{D} \) of a manifold \( X \) on the space of Riemannian metrics \( \mathcal{M} \): \( t_g(f) = f^*g \). According to [KM96], the Lie algebra of \( \mathcal{D} \) at the identity diffeomorphism is \( \text{Vect}(X) = \Gamma(X, T_X) \), the set of vector fields on \( X \). Thus, we must use this in our definition of the tangent complex.

Given that \( T_g\mathcal{M} \cong \Gamma(X, \text{Sym}^2(T_X^\vee)) \), we can compute

\[
T_g[M/\mathcal{D}] = (\Gamma(X, T_X)[1] \oplus \Gamma(X, \text{Sym}^2(T_X^\vee)), dt_g).
\]

Then, given \( V \in \Gamma(X, T_X) \), \( dt_g(V) = L_V g \), where \( L_V g \) is the Lie derivative of \( g \) along \( V \): one can see this by considering the one-parameter family of diffeomorphisms \( f = \exp(tV) \)–i.e. letting \( V \) be the infinitesimal generator of \( f \)–and computing the derivative at \( t = 0 \) of the action of \( f \) on \( g \). Not all diffeomorphisms can be written
this way: after all, \( \mathcal{D} \) isn’t even a simply connected Lie group. Even worse, there are
diffeomorphisms which are infinitesimally close to the identity diffeomorphism which
cannot be written as \( \exp(tV) \) for some \( V \) \cite{KM96}; however, we choose to consider
only those of this form in what’s to come.

**Remark 2.2.19.** If we take the union of all the complexes \( T_p[M/G] \) over all \( p \in M \),
we get a complex of vector bundles over \( M \):

\[
0 \to \mathfrak{g} \xrightarrow{dt} TM \to 0,
\]

where \( \mathfrak{g} = M \times \mathfrak{g} \), considering that the base space \( M \) is implicit. \( \mathfrak{g} \) is called the *Lie algebroid* associated to the action Lie groupoid \( M//G \), and \( dt \) is called the *anchor map* of the Lie algebroid. This is a primordial example of a Lie algebroid.

To introduce vector bundles over a formal neighborhood in a quotient stack, we’ll
consider an “alternative route” to the importance of \( \text{dg Lie} \) (and \( L_\infty \)) algebras in the
theory. Along this route, we will pick up important computation tools by means of
working out an example. We start with an action of a finite dimensional Lie group
\( G \) on a finite dimensional manifold \( M \), and then specialize to the case of \( M = \mathbb{R}^n \)
to consider some concrete computations. In the example of diffeomorphisms of a
manifold \( X \) acting on the space of Riemannian metrics on \( X \), \( \text{Met}(X) = \mathcal{M} \) is a
convex cone in \( \Gamma(X, \text{Sym}^2(T^\vee X)) \), so that we will be eventually specializing these
constructions to vector spaces or “nice” subsets thereof anyway.

**Construction 2.2.20.** Let \( \tilde{M}_p \) denote the formal neighborhood of \( p \in M \), defined so
that its ring of functions \( \mathcal{O}(\tilde{M}_p) \) is the jets of \( \mathcal{O}(M) := C^\infty(M) \) at \( p \), and denote the
inclusion map \( \hat{\rho} : \hat{M}_p \to M \): this is equivalent to the restriction map \( \mathcal{O}(M) \to \mathcal{O}(\hat{M}_p) \).

It is known that \( \mathcal{O}(\hat{M}_p) \cong \text{Sym}(T_p^\vee M) \), the Taylor series ring around \( p \in M \), although this isomorphism is not canonical. We will use the latter, and call the Taylor series ring \( \hat{\mathcal{O}}_p \) when unambiguous.

**Remark 2.2.21.** In the case that the tangent bundle of \( M \) is trivial, i.e. \( TM = M \times T_pM \) for any \( p \in M \), the preceding non-canonical isomorphism is in fact canonical. Because the space of metrics \( \mathcal{M} \) is contractible, its tangent bundle is trivial, so that we can rest easy in what follows.

The action of \( G \) on \( M \) is defined by a map \( P : G \to \text{Diff}(M) \). Taking its total derivative gives us a map \( \rho : \mathfrak{g} \to \text{Vect}(M) \) of Lie algebras, where we can then choose to view \( \text{Vect}(M) \) as derivations of \( \mathcal{O}(M) \). We then restrict the action of \( \text{Vect}(M) \) on \( C^\infty(M) \) to get an action of \( \text{Vect}(\hat{M}_p) \) on \( C^\infty(\hat{M}_p) \cong \hat{\mathcal{O}}_p \). The differential on \( T_p[M/G] \) encodes \( \rho : \mathfrak{g} \to \text{Vect}(M) \) at the point \( p \) and thus on the formal neighborhood \( \hat{M}_p \) of \( p \) since \( \rho \) is a map of Lie algebras: this gives us \( \mathfrak{g} \to \text{Vect}(\hat{M}_p) \). Noting that \( \text{Vect}(\hat{M}_p) \cong \text{Der}(\hat{\mathcal{O}}_p) \) recovers the action of \( \mathfrak{g} \) on \( \hat{\mathcal{O}}_p \) via derivations, this allows us to define the Chevalley-Eilenberg (CE) cochains \( C^\bullet(\mathfrak{g}, \hat{\mathcal{O}}_p) \) in the traditional way.

**Lemma 2.2.22.** Chevalley-Eilenberg cochains of the differential graded Lie algebra defined by shifting \( T_p[M/G] \) up one degree, denoted \( C^\bullet(\mathfrak{g} \xrightarrow{dt_p} T_pM[−1]) \), and \( C^\bullet(\mathfrak{g}, \hat{\mathcal{O}}_p) \) are isomorphic as differential graded commutative algebras. Moreover, \( C^\bullet(\mathfrak{g}, \hat{\mathcal{O}}_p) \) is the ring of functions on the formal neighborhood of \( [p] \in [M/G] \).

**Proof.** The underlying graded commutative algebras of \( C^\bullet(\mathfrak{g} \xrightarrow{dt_p} T_pM[−1]) \) and \( C^\bullet(\mathfrak{g}, \hat{\mathcal{O}}_p) \) are identical. As long as one is careful to employ the noncanonical iso-
morphism \( \mathcal{O}(\hat{M}_p) \cong \widehat{\text{Sym}}(T^\vee_p M) \), it’s quick to show that both graded commutative algebras are

\[
\text{Sym}(\mathfrak{g}^\vee[-1]) \otimes \widehat{\text{Sym}}(T^\vee_p M).
\]

From there, it is sufficient to show that the Chevalley-Eilenberg differentials are equivalent, which is left as a brief exercise.

\[\square\]

**Remark 2.2.23.** This lemma shows clearly that the dg Lie algebra

\[
\mathfrak{g}_p := (\mathfrak{g} \oplus T_p M[-1], dt_p, [\cdot, \cdot]_g)
\]

(2.9)
is of central importance. Indeed, by the equivalence in Theorem 2.2.1 between dg Lie algebras and pointed formal moduli spaces, the dg Lie algebra \( \mathfrak{g}_p \) completely defines the data of the formal neighborhood of \([p]\) in \([M/G]\).

**Example 2.2.24.** In light of this lemma and Example 2.2.18, the dg Lie algebra we must consider in the context of general covariance is thus

\[
\mathfrak{g}_g = \Gamma(X, T_X) \xrightarrow{L_{\mathfrak{g}}} \Gamma(X, \text{Sym}^2(T^\vee_X))[\cdot][\cdot][\cdot][\cdot]_g
\]

(2.10)

By applying Lemma 2.2.22, we see that the ring of functions on the formal neighborhood of \([g] \in [M/\mathcal{D}]\) is \( C^\bullet(\mathfrak{g}_g) = C^\bullet(\text{Vect}(X), \mathcal{O}(\hat{M}_g)) \), which we interpret as the derived invariants of \( \mathcal{O}(\hat{M}_g) \) with respect to the \( \mathcal{D} \)-action. Our definition of general covariance in Chapter 4 when properly “localized” will imply that the observables of a BV field theory \( \mathcal{F}_g \) over \( g \in \mathcal{M} \) form a module over \( C^\bullet(\mathfrak{g}_g) \).

**Remark 2.2.25.** It should be noted that because \( \text{Vect}(X) \) and \( \mathcal{O}(\hat{M}_g) \) are infinite dimensional, the definition of \( C^\bullet(\mathfrak{g}_g) \) is not precisely the one from the finite dimensional case. In a rigorous way, \( C^\bullet(\mathfrak{g}_g) \) represents the same data as it would if its
inputs were finite dimensional, but we must be careful with functional analytic issues to ensure that all of the rings are well defined. The precise definition will be presented in Chapter 4.

**Example 2.2.26.** Fix coordinates \((x_1, \ldots, x_n)\) on \(\mathbb{R}^n\) and consider an action \(P : G \to \text{Diff}(\mathbb{R}^n)\) for a finite-dimensional Lie group \(G\). The total derivative of this map is \(\rho : \mathfrak{g} \to \text{Vect}(\mathbb{R}^n) \cong C^\infty(\mathbb{R}^n) \otimes \mathbb{R}^n\), which for \(\alpha \in \mathfrak{g}\) has a coordinate expression like

\[
\alpha \mapsto \sum_{i=1}^n f(x_i, \alpha) \partial_i,
\]

where we use the shorthand \(\partial / \partial x_i = \partial_i\). If we restrict to a formal neighborhood of the origin, \(\hat{\mathbb{R}}_0^n\), and compute its space of functions, we get the usual Taylor series of functions about the origin, \(C^\infty(\hat{\mathbb{R}}_0^n) \cong \hat{\text{Sym}}(T_0^\vee \mathbb{R}^n) \cong \mathbb{R}[x_1, \ldots, x_n]\), which we’ll denote \(\mathbb{R}[x]\) when convenient. Thus, restricting the preceding derivative to the formal neighborhood of 0 gives us \(\rho_0 : \mathfrak{g} \to \text{Vect}(\hat{\mathbb{R}}_0^n) \cong \mathbb{R}[x] \otimes \hat{\mathbb{R}}_0^n\), which looks like:

\[
\alpha \mapsto \sum_{i=1}^n \hat{f}_0(x_i, \alpha) \partial_i,
\]

where \(\hat{f}_0\) denotes the Taylor expansion of \(f\) at 0. This defines an action of \(\mathfrak{g}\) on \(\mathbb{R}[x]\) by derivations, and so we can thus define \(C^\bullet(\mathfrak{g}, \mathbb{R}[x])\).

Fixing a basis \(\{\alpha_1, \ldots, \alpha_m\}\) for \(\mathfrak{g}\) (assuming finite dimension \(m\)), denote the dual basis for \(\mathfrak{g}^\vee\) as \(\{\alpha^1, \ldots, \alpha^m\}\). With these coordinates, we can write \(C^\bullet(\mathfrak{g}, \mathbb{R}[x])\) as \(\mathbb{R}[[\alpha^1, \ldots, \alpha^m, x_1, \ldots, x_n]]\), where the \(\alpha^k\) are in degree 1 and the \(x_k\) in degree 0. Thus, it is sufficient to see what the differential \(d_{CE}\) does on an element of the form \(\alpha^k \otimes x_l\), for \(\alpha^k \in \mathfrak{g}^\vee[-1]\) and \(x_l \in \mathbb{R}[x]\), to classify its behavior. Momentarily
viewing $\alpha^k$ as a degree 1 element of just $C^\bullet(g) = \text{Sym}(g^\vee[-1])$, and noting that $d_{CE} : g^\vee[-1] \to \text{Sym}^2(g^\vee[-1])$ is dual to the bracket $[-,-] : \text{Sym}^2(g^\vee[-1]) \to g^\vee[-1]$, we have:

$$d_{CE}\alpha^k = -\frac{1}{2} \sum_{i,j=1}^{m} c_{ij}^k \alpha^i \wedge \alpha^j,$$

where $c_{ij}^k$ are the structure constants for $g$. Concurrently, for $x_l \in R[x]$, we have

$$d_{CE}x_l = \sum_{i=1}^{m} \alpha^i \otimes \alpha_i \cdot x_l.$$

Therefore, by requiring the usual derivation rules, we get,

$$d_{CE}(\alpha^k \otimes x_l) = -\frac{1}{2} \sum_{i,j=1}^{m} c_{ij}^k \alpha^i \wedge \alpha^j \otimes x_l + \sum_{i=1}^{m} \alpha^k \wedge \alpha^i \otimes \alpha_i \cdot x_l,$$

which we extend to the rest of $C^\bullet(g, R[x])$ with the Leibniz rule. A coordinateless way of writing this is $d_{CE} = [-,-]^\vee + \rho_0^\vee$, where $\rho_0^\vee$ encodes a dual to the action map $\rho_0 : \text{Vect}(\tilde{R}_0^g) \to g^\vee \otimes \text{Vect}(\tilde{R}_0^g)$ as described implicitly above. In this example, $d$ is in fact a vector field, specifically

$$d_{CE} = -\frac{1}{2} c_{ij}^k \alpha^i \wedge \alpha^j \frac{\partial}{\partial \alpha^k} + \alpha^i \otimes (\alpha_i \cdot x_l) \frac{\partial}{\partial x_l}, \tag{2.11}$$

on the formal neighborhood of 0 in the stack $[R^n/G]$, where we have used the Einstein summation convention over repeated indices in the last step.

Considering the case of $SO(2)$ acting on $R^2$ via rotations makes things easier to grasp. So we have that $g = \mathfrak{so}(2)$ and the Taylor series ring about the origin is $R[x,y]$. The representation map $\rho_0 : \mathfrak{so}(2) \to \text{Der}(R[x,y]) \cong R[x,y] \otimes R^2$ is

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mapsto y \partial_x - x \partial_y,$$

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from which we can define $C^*(\mathfrak{so}(2), \mathbb{R}[x, y])$. We will leave it as an exercise to the reader to show that $H^0(\mathfrak{so}(2), \mathbb{R}[x, y])$ is the set of rotation-invariant Taylor series around 0. This isn’t surprising; more generally, in the case of $SO(n)$ acting on $\mathbb{R}^n$, $H^0(\mathfrak{so}(n), \mathbb{R}[x_1, \ldots, x_n])$ is the set of $SO(n)$-invariant Taylor series around the origin in $\mathbb{R}^n$.

Another enlightening exercise is to consider the appropriate Chevalley-Eilenberg cochains coming from formal neighborhoods of points away from the origin; e.g. the zeroth cohomology group of the cochains around $(x_0, 0)$ is isomorphic to $\mathbb{R}[x - x_0]$. Notice there that the vector fields coming from the action at these non-fixed points have constant coefficient terms.

**Remark 2.2.27.** Much like how a quotient stack “builds in” group action data into its definition, functions on a formal neighborhood $[\hat{M}/\hat{G}]_p$ in the stack, namely $C^*(\mathfrak{g}_p) = C^*(\mathfrak{g}, \hat{\mathcal{O}}_p)$, have “built into” them all of the $\mathfrak{g}$-invariant data. Concretely, $C^*(\mathfrak{g}, \hat{\mathcal{O}}_p)$ has the usual ring of functions $\hat{\mathcal{O}}_p$ as a subset: tensoring with $\text{Sym}(\mathfrak{g}^*[1])$ and imposing the Chevalley-Eilenberg differential remembers the data of $\mathfrak{g}$ acting on $\hat{M}_p$, and therefore on $\mathcal{O}(\hat{M}_p) \simeq \hat{\mathcal{O}}_p$ as well.

Now that we’ve made things concrete with an example, we’d like to understand vector bundles in this context. We’re primarily concerned with perturbative computations (those in the style of Construction 2.2.20); however, we’ll present the global picture first, since general covariance is first presented in such a context.

**Construction 2.2.28.** Let $V$ be a $G$-equivariant vector bundle over $M$, for which the action $\tau_M : G \to \text{Diff}(M)$ is not necessarily free. Denote the action on the total
space of $V \to M$ as $\tau_V: G \to \text{Diff}(V)$. Recall from Example 2.1.2 that we get
the pair of Lie groupoids $\mathcal{V}_G$ and $\mathcal{M}_G$ with a map $\pi: \mathcal{V}_G \to \mathcal{M}_G$ between them.
This information in turn presents a pair of stacks, and the projection gives us a map
$\pi: [V/G] \to [M/G]$ between those stacks. Here, $[V/G]$ is a vector space object in the
category of stacks over the stack $[M/G]$, much like how $\mathcal{V}_G$ is a vector space object
in the category of Lie groupoids over the Lie groupoid $\mathcal{M}_G$.

The action of a finite dimensional Lie group $G$ on a finite dimensional $M$ restricts
to an action of the formal group $\hat{G} \cong \mathfrak{g}$ (defined as the formal neighborhood of the
identity in $G$) on $\hat{M}_p$, the formal neighborhood of $p \in M$. This defines a formal
Lie groupoid which then presents the formal stack $[\hat{M}_p/\hat{G}] \cong [\hat{M}/\hat{G}]_p$, whose rings of
functions we computed earlier to be $C^\bullet(\mathfrak{g}_p)$, so that $\mathfrak{g}_p$ is the dg Lie algebra associated
to the formal moduli problem $[\hat{M}/\hat{G}]_p$.

We can pull back the $G$-equivariant vector bundle $V \to M$ along $\hat{p}: \hat{M}_p \to M$
to get a $\mathfrak{g}$-equivariant vector bundle $\hat{p}^*V \to \hat{M}_p$. Topologically, the total space of
$\hat{p}^*V$ is the formal neighborhood of the entire fiber $\pi^{-1}(p) = V_p$, which we can think
of heuristically as $V_p \times \hat{M}_p$. Both parts of this product have an action of $\hat{G}$, even
though one of the directions is a formal space and the other a vector space which is
not viewed as formal (i.e. its ring of functions is polynomials, not power series).
Thus, we can consider the associated formal Lie groupoid here as well, and it presents
the stack $[(\hat{p}^*V)/\hat{G}]$.

The vector bundle which plays the local role of the global stack $[V/G] \to [M/G]$ is therefore

$$[(\hat{p}^*V)/\hat{G}] \to [\hat{M}_p/\hat{G}]$$.
On account of $C^\bullet(g_p)$ being the space of functions on $[\hat{M}_p/\hat{G}]$, we see that a section
$\sigma : [\hat{M}_p/\hat{G}] \to [(\hat{p}^*V)/\hat{G}]$ is an element of $C^\bullet(g_p) \otimes V_p$. This is the stackified and local
version of a section of $V \to M$ being an element of $C^\infty(M) \otimes V_p$ in local coordinates
near $p$. Moreover, this reasoning results in the following lemma.

**Lemma 2.2.29.** The ring of functions on $[(\hat{p}^*V)/\hat{G}] \cong [(\hat{p}^*V)/g]$ is $C^\bullet(g, \hat{\mathcal{O}}_p \otimes \text{Sym}(V^\vee_p)) \cong C^\bullet(g_p, \text{Sym}(V^\vee_p))$, which is isomorphic as a graded ring to $C^\bullet(g_p) \otimes \text{Sym}(V^\vee_p)$.

**Proof.** The definition of $[(\hat{p}^*V)/g]$ implies that $\mathcal{O}([(\hat{p}^*V)/g])$ must be the derived $g$-invariant functions on the space $\hat{M}_p \times V_p$. Given that $\mathcal{O}(\hat{M}_p \times V_p) = \hat{\mathcal{O}}_p \otimes \text{Sym}(V^\vee_p)$
and that both parts of this tensor product are $g$-modules, we can define the CE
cochains $C^\bullet(g, \hat{\mathcal{O}}_p \otimes \text{Sym}(V^\vee_p))$. In conjunction with Lemma 2.2.22, these are the
derived $g$-invariant functions we are looking for. To see that the differential graded
rings are isomorphic, we simply note that the CE differential on both is

$$d_{CE} = [-,-]_g + \tau^V_{M_p} + \tau^V_{V_p},$$

where $\tau^V_{M_p}$ and $\tau^V_{V_p}$ are the “duals” (as in Example 2.2.26) to the induced actions $\tau_{M_p}$
and $\tau_{V_p}$ on $\hat{\mathcal{O}}_p$ and $\text{Sym}(V^\vee_p)$, respectively. \hfill \Box

**Remark 2.2.30.** In finite dimensions, the isomorphism $[(\hat{p}^*V)/\hat{G}] \cong [(\hat{p}^*V)/g]$ holds
true, since $\hat{G} \cong^\exp g$. However, we mentioned in Example 2.2.18 that it was no longer
the case that there is a diffeomorphism between the formal neighborhood of the
identity diffeomorphism in $\text{Diff}(X)$ and its Lie algebra $\text{Vect}(X)$ of vector fields with
compact support. Therefore from here onwards, we will stick to $[(\hat{p}^*V)/g]$, as it
is well-defined in the infinite-dimensional Fréchet manifold case and is the relevant object when we consider $\mathfrak{g} = \text{Vect}(X)$.
A generally covariant field theory is only sensitive to the \textit{diffeomorphism class} of a metric defined on the underlying manifold or spacetime. We will define what a field theory is in Chapter 4 and give a more precise definition for general covariance there; but we’d like to consider the moduli space of interest for its own sake here: namely, the space of Riemannian\footnote{Much of what we do works just as well for Lorentzian or other metrics, but we stick with this for simplicity.} metrics modulo diffeomorphism. We’d like to use what we’ve previously introduced to define the associated moduli stack; however, \( \text{Met}(X) =: \mathcal{M} \) and \( \text{Diff}(X) =: \mathcal{D} \) are infinite dimensional! We will thus briefly describe when they happen to be manifolds before diving into some examples which describe the geometry and topology of \( \mathcal{M}/\mathcal{D} \) and introduce some associated moduli spaces which are easier to handle.
3.1.1 Fréchet Manifolds

As one might expect, infinite dimensional manifolds are more difficult to define than finite dimensional ones. In particular, one must choose what kind of linear space should locally model an infinite dimensional manifold in the way that $\mathbb{R}^n$ locally models an $n$-manifold $X$. Doing this will allow us to do, for example, homological algebra and compute differentials in the infinite dimensional setting. We use a variety of references here: primarily [Ham82], [KM96], [TW15]. and Appendix B of [CG16].

**Definition 3.1.1.** A Fréchet space is a complete, Hausdorff, metrizable locally convex topological vector space (abbreviated LCTVS).

**Example 3.1.2.** Let $X$ be a closed, smooth, finite dimensional manifold, and let $F \to X$ be a vector bundle with space of sections $\Gamma(X, F) =: \mathcal{F}$. Choose Riemannian metrics and connections on $TX$ and $F$, let $\nabla^i \phi$ denote the $i^{th}$ covariant derivative of $\phi \in \mathcal{F}$, and set

$$||f||_n := \sum_{i=0}^{n} \sup |\nabla^i \phi(x)|.$$  

By means of the topology defined by the sequence of norms $\{|| - ||_n\}$, $\mathcal{F}$ is a Fréchet space.

The preceding example is of course essential, since fields in a field theory are often such a space of sections $\mathcal{F}$. The space of Riemannian metrics $\text{Met}(X)$ on the other hand is not a vector space, but in the case of a compact $X$ can be locally modeled by one.
**Definition 3.1.3.** A Fréchet manifold is a Hausdorff topological space with an atlas of coordinate charts taking values in Fréchet spaces such that the transition functions are smooth maps between Fréchet spaces. Fréchet Lie groups (resp. groupoids) are groups (resp. groupoids) internal to the category of Fréchet manifolds.

**Example 3.1.4.** For a manifold $N$ and a compact manifold $M$, $C^\infty(M, N)$ is a Fréchet manifold.

**Example 3.1.5.** For a compact finite dimensional manifold $X$, the vector space of smooth, symmetric $(0,2)$ tensor fields on $X$, $\Gamma(X, \text{Sym}^2(T_X^\vee))$, is a Fréchet space, by Example 3.1.2. $\text{Met}(X)$ is a convex, open cone in $\Gamma(X, \text{Sym}^2(T_X^\vee))$, and if $X$ is compact it is a Fréchet manifold with the smooth topology of uniform convergence on compact subsets. Moreover, the preceding relation allows us to compute the tangent space to $\text{Met}(X) = \mathcal{M}$:

$$T_p\mathcal{M} \cong \Gamma(X, \text{Sym}^2(T_X^\vee)).$$

Similarly, if $X$ is compact, then the group $\text{Diff}(X)$ of diffeomorphisms of $X$ is a Fréchet Lie group. Thus, we will usually assume that $X$ is compact or even closed in much of what follows, even though in the Lorentzian case, $X$ is usually not compact. However, it should be noted that many physically relevant Lorentzian manifolds are assumed to have the topological form $\Sigma \times \mathbb{R}$, for $\Sigma$ a spacelike compact submanifold and $\mathbb{R}$ the “time” dimension. This is the path through which many of the Riemannian results are translated into the Lorentzian regime.
3.1.2 The Topology of the Moduli Space

Since the action of $\mathcal{D} = \text{Diff}(X)$ on $\mathcal{M}$ is rarely free, many techniques are employed to simplify things: one could consider the subspace of metrics in $\mathcal{M}$ with no isometries or restrict to certain subgroups of $\mathcal{D}$ which act freely on $\mathcal{M}$ or otherwise simplify the problem. Now that we have established that $\mathcal{M}$ is an infinite dimensional manifold in a workable way, we’ll introduce some basic aspects of its topology by reviewing some of the aforementioned techniques. We assume $X$ is compact and connected unless otherwise stated.

Construction 3.1.6. Let’s warm up by considering the instance in which $\mathcal{M} \circlearrowleft \mathcal{D}$ is in fact free. We thus get the usual fibration $q : \mathcal{M} \to \mathcal{M}/\mathcal{D}$ with fiber $q^{-1}([g]) \cong \mathcal{D}$. We can then compute the associated homotopy exact sequence and invoke that $\mathcal{M}$ is a contractible space to conclude that

\[
\pi_i(\mathcal{D}) \cong \pi_{i+1}(\mathcal{M}/\mathcal{D}).
\]

Although $\mathcal{M} \circlearrowleft \mathcal{D}$ is rarely free in the cases we’re interested in, we can apply thinking similar to the above to suss out some information.\[^2\] For example, let $\text{Met}_0(X) =: \mathcal{M}_0$ be the subspace of metrics with no isometries, so that $\mathcal{D}$ acts freely on it. Then $\mathcal{M}/\mathcal{M}_0$ should be “relatively low dimensional” in $\mathcal{M}$, so that we can expect that for $i >> 0$,

\[
\pi_{i+1}(\mathcal{M}/\mathcal{D}) \cong \pi_i(\mathcal{M}_0/\mathcal{D}).
\]

\[^2\]Thanks go to Prof. Steven Rosenberg for the following heuristic but motivational commentary.
Moreover, because $\mathcal{M} \supset D$ is “almost free” in the preceding sense and given the contractibility of $\mathcal{M}$, $\mathcal{M}/D$ should simply be the classifying space $B\mathcal{D}$. This would in turn imply that characteristic classes for $\mathcal{D}$ should be elements of $H^\bullet(\mathcal{M}/D)$. Of course, in our case we should instead consider $H^\bullet([\mathcal{M}/D])$ for the quotient stack $[\mathcal{M}/D]$, which is equivalent to $H^\bullet_P(\mathcal{M})$ the $\mathcal{D}$-equivariant cohomology of $\mathcal{M}$ (the theory of which is described for example in [Beh02]).

**Example 3.1.7.** A standard technique which simplifies our current object of study is to consider the observer moduli space of metrics modulo diffeomorphism. The key maneuver is to introduce the following subgroup of $\mathcal{D}$.

**Definition 3.1.8.** Given $x_0 \in X$, the **diffeomorphism group with observer**, denoted $\text{Diff}_{x_0}(X) =: \mathcal{D}_{x_0}$, is the subgroup of $\mathcal{D}$ consisting of diffeomorphisms $f : X \to X$ such that $f(x_0) = x_0$ and $df_{x_0} = \text{Id}_{r_{x_0}X}$.

**Lemma 3.1.9** (Lemma 7.1.2 in [TW15]). $\mathcal{D}_{x_0}$ acts freely on $\mathcal{M}$.

**Proof.** Let $g \in \mathcal{M}$ and suppose that $f \in \mathcal{D}_{x_0}$ is an isometry of $g$: i.e. it fixes $g$ under the action of $\mathcal{D}_{x_0}$. By assumption $f$ also fixes $x_0$ and $T_{x_0}X$, so that it must also fix any geodesics passing through $x_0$. Since any two points are connected by a geodesic, $f$ must therefore fix every point in $X$. \qed

We can define the appropriate moduli spaces, and since the action above is free, the spaces inherit a smooth manifold structure.

**Definition 3.1.10.** The **observer moduli space** is $\mathcal{M}/\mathcal{D}_{x_0}$. The **observer moduli space of positive scalar curvature metrics** is $\mathcal{M}^+_{\text{scal}}/\mathcal{D}_{x_0}$, where $\mathcal{M}^+_{\text{scal}}$ is the space
of all positive curvature Riemannian metrics on $X$.

**Remark 3.1.11.** This allows us to define (without the need for stacks) the principal $\mathcal{D}_{x_0}$-bundle $\pi : \mathcal{M} \to \mathcal{M}/\mathcal{D}_{x_0}$. Since $\mathcal{M}$ is contractible, $\mathcal{M}/\mathcal{D}_{x_0}$ is homotopic to $B\mathcal{D}_{x_0}$ and so $\mathcal{M}$ is homotopic to $E\mathcal{D}_{x_0}$. Associated to this bundle is the universal bundle

$$X \hookrightarrow \mathcal{M} \times_{\mathcal{D}_{x_0}} X \to \mathcal{M}/\mathcal{D}_{x_0} = B\mathcal{D}_{x_0}.$$  

The fibers of this universal $X$-bundle can be equipped with Riemannian metrics in a canonical way, and this construction plays an essential role in proving results like the following, due to Botvinnik, Hanke, Schick, and Walsh:

**Theorem 3.1.12** (Theorem 7.2.1 in [TW15]). Given $k \in \mathbb{N}$, there’s an integer $N(k)$ such that for all odd $n > N(k)$ and $n$-manifolds $X$ admitting a positive scalar curvature metric $g$, $\pi_i(\mathcal{M}_{\text{scal}}^{+}\mathcal{D}_{x_0}, [g])$ is non-trivial when $i \leq 4k$, for $i \equiv 0 \mod 4$.

A central fact which is exploited in this context is that the homotopy theory of $\mathcal{M}/\mathcal{D}_{x_0}$ is the same as that of $B\mathcal{D}_{x_0}$, and progress has been made in computing homotopy groups of diffeomorphism groups (and their classifying spaces) for certain manifolds. For example, for odd $n$ and given $k$ and $i \leq 4k$ as above, we have

$$\pi_i(B\text{Diff}_{x_0}(S^n)) \otimes \mathbb{Q} = \begin{cases} 
\mathbb{Q} & i \equiv 0 \mod 4 \\
0 & \text{else.}
\end{cases}$$

**Remark 3.1.13.** In the long run, for example when we discuss anomalies in Chapter 5, we will be more concerned with $H^\ast([\mathcal{M}/\mathcal{D}])$; however, we find it appropriate regardless to understand constructions and examples in the associated homotopy theory as a way of “gearing up” toward our eventual aims.
3.2 A Note on Equivariant Cohomology

Since we have invoked its existence a few times, and since we have been able to make sense of certain infinite dimensional spaces as (Fréchet) manifolds (in which we can port over finite dimensional definitions), we will properly introduce *equivariant cohomology*. This serves as as kind of appendix to the preceding pages, primarily taken from [Tu20].

**Construction 3.2.1.** Consider a finite dimensional manifold $M$ equipped with an action of a Lie group $G$.\(^3\) We know from prior constructions that we can quickly define the associated action groupoid $M \times G \rightrightarrows M$ and thus the quotient stack $[M/G]$. Then, since one can define cohomology for a stack, as in [Beh02], it’s reasonable to simply define the equivariant cohomology as

$$H^*_G(M) := H^*([M/G]).$$

Taking the historical arc of the object on the left hand side for granted, this wouldn’t be such a bad definition; however, some critical and useful details could be missed. Thus, we will provide a brief construction of the usual definition.

To any Lie group $G$ we can associate its classifying space $BG$ and universal bundle $EG \to BG$, such that any $G$-bundle over a manifold $M$ is a pullback of the universal bundle by means of a map $M \to BG$. As stacky thinkers, we should view $BG$ as $[pt/G]$. By design, the action of $G$ on $EG$ is free. Thus, we can consider the action of $G$ on the space $EG \times M$, defined as $g \cdot (e, m) = (eg^{-1}, gm)$. Since $EG$ is contractible,

\(^3\)This works for infinite dimensional Fréchet manifolds, we we keep things simple to begin with.
$EG \times M$ has the same homotopy type as $M$, with the upside that the action of $G$ we’ve described is free on $EG \times M$, since it is free on $EG$. Hence, we can define $X_G := EG \times_G M = EG \times M/\sim$, where $\sim$ is the equivalence relation imposed by $g \cdot (e, m) = (eg^{-1}, gm)$.

**Definition 3.2.2** (Section 4.2 in [Tu20]). The **equivariant cohomology of** $M$ **with respect to** (its action by) **$G$** is

$$H_G^*(M; R) := H^*(X_G; R),$$

where $H^*$ denotes ordinary singular cohomology, and $R$ is the coefficient ring. If not specified, we assume the coefficient ring to be $\mathbb{R}$.

**Example 3.2.3.** An immediate example one can define from the above is if we fix the trivial action of the group $G$ on $M$. In this case, $X_G = EG \times_G M = BG \times M$, and so

$$H_G^*(M) = H^*(BG) \otimes H^*(M).$$

*In fact*, if both $G$ and $M$ are compact—even if the action is not free—this statement is true, and is particularly helpful in understanding how $H_G^*(M)$ is a $H^*(BG)$-module.⁴

**Remark 3.2.4.** All of this is very much analogous to the fact that the underlying graded ring of the algebra of functions on a formal neighborhood about a point $[p]$ in the quotient stack $[M/G]$ (i.e. the appropriate Chevalley-Eilenberg cochains) takes the form

$$\widehat{\text{Sym}}(g^\vee[-1]) \otimes \widehat{\text{Sym}}(T_p^\vee M).$$

⁴Although it is not always a free module.
Since we know that $H^\bullet([pt/G]) \cong H^\bullet_G(pt) \cong H^\bullet(BG)$, and since $H^\bullet(BG)$ is important given the above equation, it is a worthwhile task to compute these cohomology rings. To give a quick example, consider the case of $G = T = U(1) \times \ldots \times U(1)$ ($k$ times). Then $BT = \mathbb{C}P^\infty \times \ldots \times \mathbb{C}P^\infty$ and $ET = S^\infty \times \ldots \times S^\infty$. From here, we get

$$H^\bullet([pt/T]) = H^\bullet(BT) = H^\bullet(\mathbb{C}P^\infty) \otimes \ldots \otimes H^\bullet(\mathbb{C}P^\infty)$$

$$= \mathbb{R}[x_1] \otimes \ldots \otimes \mathbb{R}[x_k]$$

$$= \mathbb{R}[x_1, \ldots, x_k],$$

the polynomial ring in $k$ variables. Since $T$ is a compact group, if it acts on a compact manifold $M$, we get $H^\bullet_T(M)$ immediately, from the preceding information. Moreover, we can for example deduce that

$$H^\bullet_{U(1)}(S^2) \cong H^\bullet(BU(1)) \otimes H^\bullet(S^2)$$

$$\cong \mathbb{R}[u] \otimes (\mathbb{R}[\omega]/(\omega^2))$$

$$\cong \mathbb{R}[u] \oplus \mathbb{R}[u]_\omega,$$

where $\omega$ in degree two is the volume form on $S^2$ and $u$ in degree zero generates $H^\bullet(BU(1))$. Further computations can be found in [Tu20], and can be used to conclude that the cohomology ring of the quotient stack $[S^2/U(1)]$ is equivalent to $\mathbb{R}[u, \beta]/(\beta^2 - u^2)$, for an appropriate $\beta$ in degree two.

**Example 3.2.5.** Another nice example is when $G$ acts on $M$ freely. In that case, the ordinary quotient $M/G$ remains a smooth manifold and the fibers of the $G$-bundle
$M_G \to M/G$ can be contracted, and so $M_G \to M/G$ is a homotopy equivalence. Thus:

$$H^*_G(M) \cong H^*(M/G).$$

In addition, if $M$ is a contractible $G$-space, then its equivariant cohomology $H^*_G(M)$ is equivalent to $H^*_G(\text{pt}) = H^*(BG)$. This is particularly useful within the context of physics, because the space of semi-Riemannian metrics $\mathcal{M}$ is contractible, so that the cohomology of the moduli stack $[\mathcal{M}/\mathcal{D}]$ is simply $H^*(B\mathcal{D})$. An analogous statement holds for gauge theory, since the space of gauge fields $\mathcal{A}$ is an affine space—and so it is sufficient to consider the cohomology ring $H^*(B\mathcal{G})$ of the classifying space of the gauge group when trying to understand anomalies.
4.1 A Primer on the Classical Batalin-Vilkovisky Formalism

The basic ingredients required from the outset are a space of fields, which define the \textit{kinematics} of a physical model, and an action functional, which fixes the \textit{dynamics} of that model. The fields on a space (or spacetime) $X$ are sections of some bundle $F \to X$, denoted $\mathcal{F} := \Gamma(X, F)$. The action functional is a function $S : \mathcal{F} \to \mathbb{R}$ whose critical locus $\text{Crit}(S)$\footnote{This is computed via variational calculus, and described for example in Appendix E of [Wald84].} is the set of $\phi \in \mathcal{F}$ that satisfy the Euler Lagrange equations associated to $S$ via functional differentiation. The following construction is based on the one given in Chapter 2 of [Gwi12].

**Construction 4.1.1.** We can think of this situation geometrically, albeit $\mathcal{F}$ is usually infinite dimensional. In particular, $\text{Crit}(S)$ is the intersection of the graph

\[ \Gamma(dS) \subset T^*\mathcal{F} \]
with the zero section $\mathcal{F} \subset T^\vee \mathcal{F}$. We thus get its commutative algebra of functions to be

$$\mathcal{O}(\text{Crit}(S)) = \mathcal{O}(\Gamma(dS)) \otimes_{\mathcal{O}(T^\vee \mathcal{F})} \mathcal{O}(\mathcal{F}).$$

Note that we have not yet defined what $\mathcal{O}(\mathcal{F})$ is: we will do so in the upcoming section, and for now implore the reader to think of it “naïvely” as $\text{Sym}(\mathcal{F}^\vee)$ for the vector space $\mathcal{F}$.

The problem with the above is that $\text{Crit}(S)$ could very well be singular: for example, the intersection does not necessarily need to be transverse. To get around this issue, we follow the philosophy of derived (algebraic) geometry and replace the above critical locus with the derived critical locus $\text{Crit}^h(S)$, which has ring of functions

$$\mathcal{O}(\text{Crit}^h(S)) = \mathcal{O}(\Gamma(dS)) \otimes_{\mathcal{O}(T^\vee \mathcal{F})}^L \mathcal{O}(\mathcal{F}).$$

This is now a commutative dg algebra instead of an ordinary commutative algebra, and more importantly can be realized as the complex

$$\mathcal{O}(T^\vee [-1] \mathcal{F}) = \Gamma(\mathcal{F}, \Lambda^* T \mathcal{F}),$$

with differential $\vee dS$, contraction with the one form $dS \in \Omega^1(\mathcal{F})$.

$T^\vee [-1] \mathcal{F} = \mathcal{F} \oplus \mathcal{F}^\vee [-1] \subset \mathcal{O}(T^\vee [-1] \mathcal{F})$ provides a cochain complex with the original fields $\mathcal{F}$ in degree zero. An essential maneuver in the Batalin-Vilkovisky formalism for classical field theories is to take $T^\vee [-1] \mathcal{F}$ to be the upgraded space of fields: a differential graded enhancement found from the original by computing the $-1$-shifted cotangent bundle and finding the appropriate differential from the Euler-Lagrange equations.

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Remark 4.1.2. From now on we will refer to $\mathcal{F} \oplus \mathcal{F}^\vee[-1]$ with its differential $Q := \wedge dS$ as the space of fields and denote it $\mathcal{F}$, where the original fields will now be $\mathcal{F}^0$, the degree zero part. By definition, these fields are the space of sections of a now differential graded vector bundle $F \to X$. We should consider an example to make all of this concrete.

Example 4.1.3. Our running example through much of this text will be scalar field theory. We’ll consider the free case first. Fix a semi-Riemannian manifold $(X, g)$ and consider its space of smooth functions $\Gamma(X, \mathbb{R}) = C^\infty(X)$: these are the a priori fields. The action functional is

$$S_g(\varphi) = \frac{-1}{2} \int_X \varphi \Delta_g \varphi \text{vol}_g,$$

(4.2)

where $\varphi \in C^\infty(X)$, $\text{vol}_g$ is the volume form associated to the metric $g$, written in coordinates as $\sqrt{\det g} dx_1 \wedge \ldots \wedge dx_n$, and the Laplace-Beltrami operator $\Delta_g$ associated to $g$ should not be mistaken for the BV Laplacian discussed later in the text. The Euler-Lagrange equation here is Laplace’s equation, $\Delta_g \varphi = 0$, so that $\text{Crit}(S)$ is the set of harmonic functions.

The derived critical locus is then

$$\mathcal{F}_g = C^\infty(X) \xrightarrow{Q} \text{Dens}(X)[-1],$$

(4.3)

where $\text{Dens}(X)$ is the appropriate dual to $C^\infty(X)$ and $Q_g \varphi = \Delta_g \varphi \text{vol}_g$ is the differential, which imposes the Euler-Lagrange equations: it is written so as to land in $\text{Dens}(X)$ but also to capture all of the dependence on $g \in \text{Met}(X)$ in the action functional.
Remark 4.1.4. As written, it is implied here that \( g \) is a Riemannian metric, because the associated partial differential operator is the elliptic Laplace-Beltrami operator. If \( g \) were Lorentzian, then we would instead have the d’Alembertian \( \Box_g \), which is hyperbolic.

Remark 4.1.5. An advantage to shifting from the nonderived fields \( C^\infty(X) \) to the derived critical locus \( \mathcal{F}_g \) is that there now is an explicit dependence in the fields on the metric \( g \in \text{Met}(X) =: M. \) Moreover, this allows us to define a differential graded vector bundle \( \pi : \mathcal{F} \to M \): the base space is the space of all (semi-)Riemannian metrics on \( X \) and the fibers \( \pi^{-1}(g) = \mathcal{F}_g \) are field theories depending on the fixed \( g \). This opens up the possibility of seeing how varying the “background metric” effects the field theory, which is the major focus of this work. Note however that we have such a dg vector bundle only when the theory is free (i.e. \( S \) is quadratic in \( \varphi \)): for an interacting theory, we will require the notion of an \( L_\infty \) algebra, and the underlying geometry of the model will be very different. To make this and the above more precise, we embark toward the next section.

4.1.1 The Batalin-Vilkovisky Formalism

We shall first introduce some preliminary requirements. The infinite dimension-ality of the BV fields \( \mathcal{F} \) means we cannot take the naïve algebraic symmetric powers to define their space of functions. We therefore have the following definition which plays an identical role, but for the infinite dimensional case. Much of what follows

\(^2\)We will denote \( \text{Met}(X) \) as \( M \) when \( X \) is implicit.
is from Chapter 5 of [Cos11] and Chapters 3 through 5 of [CG21].

Let $F \to X$ be a differential graded vector bundle whose sheaf of sections $\mathcal{F} := \Gamma(X, F)$ is the fields for a field theory.

**Definition 4.1.6.** The space of **functionals** on $\mathcal{F}$ is

$$\mathcal{O}(\mathcal{F}) := \prod_{k \geq 0} \text{Hom}(\mathcal{F}^{\otimes k}, R)_{S_k}.$$  

**Remark 4.1.7.** To be fully precise, $\mathcal{F} = \Gamma(X, F)$ is a nuclear Fréchet space and $\otimes$ denotes the completed projective tensor product, so that

$$\mathcal{F}^{\otimes k} := \Gamma(X \times \cdots \times X, F \boxtimes \cdots \boxtimes F),$$

meaning each $\text{Hom}(\mathcal{F}^{\otimes k}, R)_{S_k}$ is a space of continuous functionals endowed with the strong dual topology: i.e. a space of distributions. When unambiguous, we may sometimes denote this ring as $\text{Sym}(\mathcal{F}^\vee)$. We now introduce an important related set of functionals.

**Definition 4.1.8.** The **space of local functionals**, denoted $\mathcal{O}_{\text{loc}}(\mathcal{F})$, is the linear subspace of $\mathcal{O}(\mathcal{F}_c)$ spanned by elements of the form

$$F_k(\phi) = \int_X (D_1 \phi)(D_2 \phi) \cdots (D_k \phi) \text{vol},$$

for fields $\phi \in \mathcal{F}$ and differential operators $D_i \in D_X$, where $D_X$ denotes the differential operators on $X$.

**Lemma 4.1.9.** ([Cos11], Ch. 5, Lemma 6.6.1) There is an isomorphism of cochain complexes

$$\mathcal{O}_{\text{loc}}(\mathcal{F}) \cong \text{Dens}_X \otimes_{D_X} \mathcal{O}_{\text{red}}(\mathcal{J}(F)), $$

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where \( \mathcal{J}(F) \) denotes sections of the \( \infty \)-jet bundle \( \text{Jet}(F) \to X \), and \( \mathcal{O}_{\text{red}}(\mathcal{J}(F)) \) is the quotient of \( \mathcal{O}(\mathcal{J}(F)) = \text{Sym}(\mathcal{J}(F)^\vee) \) by the constant polynomial functions.

**Remark 4.1.10.** Sections of \( \mathcal{O}_{\text{loc}}(\mathcal{F}) \) are exactly elements of the preceding form, and integration defines a natural inclusion:

\[
\iota : \mathcal{O}_{\text{loc}}(\mathcal{F}) \to \mathcal{O}_{\text{red}}(\mathcal{F}_c).
\]

This lemma shows that \( \mathcal{O}_{\text{loc}}(\mathcal{F}) \) is the space of Lagrangian densities modulo total derivatives: this is desirable because adding a total derivative to a Lagrangian density does not affect the dynamics described in the equations of motion. Local functionals are also more manageable in terms of functional analysis; for example, the action functional \( S \) is always an element of \( \mathcal{O}_{\text{loc}}(\mathcal{F}) \), and local functionals are key in defining the Poisson bracket, as we will see below.

**Definition 4.1.11.** For \( F \to X \) a graded vector bundle, a constant coefficient \( k \)-shifted symplectic structure is an isomorphism

\[
F \cong_\omega F^s[k] := (\text{Dens}_X \otimes F^\vee)[k]
\]

of graded vector spaces that is graded antisymmetric.

**Example 4.1.12.** For BV field theories, \( k = -1 \). A good example is the one we’ve had all along: the symplectic structure \( \omega \) on \( \mathcal{F}_g = C^\infty(X) \xrightarrow{Q_g} \text{Dens}(X)[-1] \) is

\[
\omega(\varphi, \mu) = \int_X \varphi \mu,
\]

for \( \varphi \) and \( \mu \) in degrees 0 and 1, respectively. We can thus write \( S_g(\varphi) \) as \( \omega(\varphi, Q_g \varphi) \).
Remark 4.1.13. It stands to reason that a symplectic structure on a space defines a Poisson bracket on its space of functions: this is indeed the case for \( \mathcal{O}_{\text{loc}}(\mathcal{F}) \subset \mathcal{O}(\mathcal{F}) \). This is not the case however for all of \( \mathcal{O}(\mathcal{F}) \), for functional analytic reasons which are outside the scope of this paper [Cos11]. Let’s denote the Poisson (anti-)bracket induced by \( \omega \) as \( \{-, -\} \).

Definition 4.1.14. A Batalin-Vilkovisky classical field theory \((\mathcal{F}, \omega, S)\) on a smooth manifold \( X \) is a differential \( \mathbb{Z} \)-graded vector bundle \( F \to X \) equipped with a \(-1\)-shifted symplectic structure \( \omega \) and an action functional \( S \in \mathcal{O}_{\text{loc}}(\mathcal{F}) \) such that:

1. \( S \) satisfies the classical master equation (CME): \( \{S, S\} = 0 \).
2. \( S \) is at least quadratic, so that it can be written uniquely as \( S(\varphi) = \omega(\varphi, Q \varphi) + I(\varphi) \), where \( Q \) is a linear differential operator and \( I \in \mathcal{O}_{\text{loc}}(\mathcal{F}) \) is at least cubic.

*(3) The complex \((\mathcal{F}, Q)\) is elliptic.

A free theory is one in which \( I = 0 \): i.e. the action functional \( S \) is purely quadratic.

Remark 4.1.15. Although \( \{-, -\} \) is not a Poisson bracket on \( \mathcal{O}(\mathcal{F}) \), bracketing with a local functional like \( S \in \mathcal{O}_{\text{loc}}(\mathcal{F}) \) defines a derivation

\[
\{S, -\} : \mathcal{O}(\mathcal{F}) \to \mathcal{O}(\mathcal{F})[1]
\]

regardless of whether or not the BV theory is free. But for a free theory, it can be shown that \( \{S, -\} = Q \) on \( \mathcal{O}(\mathcal{F}) \), where the differential \( Q \) on \( \mathcal{F} \) is extended to \( \mathcal{O}(\mathcal{F}) \) as a derivation.

In addition, Condition (3) is singled out because the assumption of the ellipticity of \( (\mathcal{F}, Q) \)–being in correspondence with the assumption of the ellipticity of \( Q \) as a
differential operator— is not always made. In particular, the classical physics which
is the content of this thesis works just as well for the \textit{hyperbolic} operators used in
the Lorentzian regime; it is when we consider renormalization for the quantized field
theory that we may require ellipticity.

\textbf{Definition 4.1.16.} Let $F \to X$ be a differential graded vector bundle constituting a
BV classical field theory $(\mathcal{F}, \omega, S)$. Then the dg commutative ring of \textbf{global classical
observables} for this theory is

$$\text{Obs}^{\text{cl}}(X, \mathcal{F}) := (\mathcal{O}(\mathcal{F}), \{S, -\}).$$

\textbf{Remark 4.1.17.} The adjective “global” is used here because a priori one could
consider $\text{Obs}^{\text{cl}}(U, \mathcal{F})$ for any open set $U \subset X$. Here we are primarily concerned with
the observables evaluated on the entirety of $X$, but considering different open sets
(and so covers) of $X$ allows one to view the observables as a \textit{factorization algebra} on
$X$: this is the main driver behind works like [CG21] and [Gwi12].

\textbf{Example 4.1.18.} For $\mathcal{F}_g = C^\infty(X) \xrightarrow{Q_g} \text{Dens}(X)[-1]$, the underlying graded ring of
$\text{Obs}^{\text{cl}}(X, \mathcal{F}_g)$ is $\mathcal{O}(\mathcal{F}_g)$, so that it is concentrated in nonpositive degrees, as Definition
4.1.6 implies. The action functional $S_g(\varphi)$ defined in Equation (4.2) is a degree
0 element of $\mathcal{O}(\mathcal{F}_g)$, but also defines a degree 1 differential on $\mathcal{O}(\mathcal{F}_g)$ as $\{S_g, -\}$:
thus, $\{S_g, S_g\}$ must be a degree 1 element of $\mathcal{O}(\mathcal{F}_g)$. Since in this example $\mathcal{O}(\mathcal{F}_g)$ is
concentrated in nonpositive degrees, the classical master equation holds vacuously.
Thus, our ongoing example of the free massless scalar field with metric background
$g$ defines a free BV classical field theory, since the other requirements are easily
satisfied.
Remark 4.1.19. The free scalar field example we worked out above is very “tame”, in the sense that it is a free theory (i.e. $I = 0$) and we didn’t view it perturbatively, so that its observables are polynomial functions of the fields as opposed to Taylor expansions. Recall that in Example 2.2.7, we used an $L_\infty$ algebra to package the data of $\varphi^4$ theory, which we viewed as both interacting and perturbative: in what follows, we will make rigorous and transparent the necessity of such an $L_\infty$ algebra in this context. In particular, we must first introduce the notion of a local $L_\infty$ algebra.

Definition 4.1.20. A local $L_\infty$ algebra on a manifold $X$ is:

1. A graded vector bundle $L \to X$, where we denote the sections as $\mathcal{L}$,
2. a differential operator $d : \mathcal{L} \to \mathcal{L}$ of cohomological degree 0 such that $d^2 = 0$, and
3. a collection of polydifferential operators $\ell_n : \mathcal{L}^\otimes \to \mathcal{L}$ for $n \geq 2$ which are alternating, of cohomological degree $2 - n$, and which make $\mathcal{L}$ an $L_\infty$ algebra.

If the local $L_\infty$ algebra $(\mathcal{L}, d)$ is an elliptic complex, we call it an elliptic $L_\infty$ algebra.

Remark 4.1.21. A local $L_\infty$ algebra $\mathcal{L}$ on $X$ defines a presheaf $B\mathcal{L}$ of formal moduli problems on $X$. For an open set $U \subset X$, $B\mathcal{L}(U)$ is the functor which sends a dg Artinian algebra $(R, m)$ to the simplicial set

$$B\mathcal{L}(U)(R) = \text{MC}(\mathcal{L}(U) \otimes m),$$

as one might expect. We thus get the following.

Definition 4.1.22. A formal pointed elliptic moduli problem is a sheaf of formal moduli problems on $X$ that is represented by a elliptic $L_\infty$ algebra.
Remark 4.1.23. As we will soon show, the observables associated to a field theory defined in the preceding context (in other words, a perturbative field theory) are roughly $\mathcal{O}(B\mathcal{L}) = C^\bullet(\mathcal{L}(U))$. It is key to note that these observables are functions on a formal space, so that they are Taylor expansions: this is in opposition to the observables we computed in Example 4.1.18, which were polynomial functions of the fields. In addition, the above definition has us at least implicitly fixing a base point of the formal moduli problem, which in physics corresponds to choosing a solution to expand around.

Example 4.1.24. The $L_\infty$ algebra we described in Example 2.2.7 in fact defines an elliptic $L_\infty$ algebra. In the case of interacting field theories, it is often advantageous to consider perturbations of a fixed solution because solutions to nonlinear PDE (the Euler-Lagrange equations associated to an interacting theory) are usually much more difficult to compute. However, one could just as well do perturbation theory around a fixed solution in a free theory. For example, if we shifted the differential graded scalar fields in Equation (4.3) up by one, then the new space of fields $\mathcal{F}_g[-1] = C^\infty(X)[-1] \xrightarrow{Q_g} \text{Dens}(X)[-2]$ is a dg Lie algebra and thus an $L_\infty$ algebra, which we could employ to analyze perturbations of a fixed harmonic function (a solution to $Q_g\phi = 0$).

Remark 4.1.25. The above shift is due to the fact that for a pointed formal moduli problem $\mathcal{M}$, the associated $L_\infty$ algebra $\mathfrak{g}_\mathcal{M}$ is such that

$$
\mathfrak{g}_\mathcal{M} = T_p\mathcal{M}[-1].
$$

\footnote{This can be found in more detail in Chapter 4.2 of [CG21].}
Colloquially, the grading choice in Equation (4.3) is often called the “physicist’s grading” and the \(-1\)-shifted version of that is called the “dg Lie algebra grading”.

In addition, although these formal derived spaces may be harder to wrap our heads around than ordinary manifolds, some analogies can be made. For example, the corresponding notion of a coordinate patch for a formal moduli space \(\mathcal{M}\) is the \(L_\infty\) algebra \(\mathfrak{g}_\mathcal{M}\) with its bracket, which encodes any potential nonlinearity of \(\mathcal{M}\). In the case of field theory, we desire out derived space of fields to be symplectic: the following “formal” Darboux lemma extends the preceding thinking to exactly that case.

**Lemma 4.1.26** (Lemma 4.2.1 in [CG21]). For a finite dimensional \(L_\infty\) algebra \(\mathfrak{g}\), \(k\)-shifted symplectic structures on \(B\mathfrak{g}\) are equivalent to symmetric invariant nondegenerate pairings on \(\mathfrak{g}\) of cohomological degree \(k - 2\).

This analogue of the Darboux lemma for formal derived symplectic spaces motivates the following definition we’ll make use of in perturbative field theory.

**Definition 4.1.27.** Let \(L \to X\) be an elliptic \(L_\infty\) algebra on a manifold \(X\). An **invariant pairing on \(L\) of cohomological degree** \(k\) is a symmetric vector bundle map

\[
\langle -, - \rangle_L : L \otimes L \to \text{Dens}(X)[k]
\]

satisfying (1) nondegeneracy, i.e. the above pairing induces an isomorphism

\[
L \to L^\vee \otimes \text{Dens}(X)[-3]
\]
of vector bundles and (2) invariance, i.e. the inner product on \( \mathcal{L}_c \) (compactly supported sections of \( L \)) induced by the pairing on \( L \), as

\[
\langle -, - \rangle : \mathcal{L}_c \otimes \mathcal{L}_c \rightarrow \mathbb{R},
\]

\[
\alpha \otimes \beta \mapsto \int_X \langle \alpha, \beta \rangle,
\]

is an invariant pairing on the \( L_\infty \) algebra \( \mathcal{L}_c \).

More exposition on the above can be found in [CG21], but the following fact found in [Wil18] nicely summarizes the correspondence we will be using in our study of perturbative field theory.

**Proposition 4.1.28** (Proposition 2.1.11 in [Wil18]). *The following structures are equivalent: (1) a (perturbative) classical BV field theory \( (\mathcal{F}, \omega, S) \) and (2) an elliptic \( L_\infty \) algebra structure on \( L = F[1] \) equipped with a \(-3\)-shifted symplectic pairing.*

Given the above, we can now introduce a slightly different definition of observables: it is tailored from the \( L_\infty \) (and thus perturbative) perspective, and moreover does not demand that we evaluate it on all of \( X \), but only on open subsets \( U \subset X \).

**Definition 4.1.29** (Definition 5.1.1 in [CG21]). *The observables with support in the open subset \( U \) are the commutative dg algebra*

\[
\text{Obs}^{\text{cl}}(U) := C^\bullet(\mathcal{L}(U)). \tag{4.4}
\]

The **factorization algebra of observables** for this classical field theory, denoted \( \text{Obs}^{\text{cl}} \), assigns the cochain complex \( \text{Obs}^{\text{cl}}(U) \) to each open \( U \subset X \).
Note that when $\mathcal{F}$ is implicit, we sometimes omit it in our notation.

**Remark 4.1.30.** If we took Definition 4.1.16 and changed it slightly to restrict on opens—i.e., if we chose the underlying commutative algebra to be $\mathcal{O}(\mathcal{F}(U))$ instead of all of $\mathcal{O}(\mathcal{F}) = \mathcal{O}(\mathcal{F}(X))$—and thus restrict the differential $\{S, -\}$ to be over those opens, we get agreement with the preceding definition. A little detail which we must be wary of is that the bundle $L$ whose sections are the elliptic $L_\infty$ algebra is $F[-1]$.

Here we’d like to remind the reader that the only reason $\{S, -\}$ is defined on $\text{Obs}^{\text{cl}}(U)$ is because $S$ is a local functional. $\text{Obs}^{\text{cl}}(U)$ as it’s been defined does not have any kind of Poisson structure in general, though we desire to use one as follows:

**Definition 4.1.31** (Definition 5.2.1 in [CG21]). A $P_0$ algebra (in the category of cochain complexes) $\mathcal{A}$ is a commutative dg algebra with a degree 1 Poisson bracket $\{-, -\} : \mathcal{A}[-1] \otimes \mathcal{A}[-1] \to \mathcal{A}[-1]$ satisfying the Jacobi identity and Leibniz rule.

**Remark 4.1.32.** This is the $k = 0$ case of a general object called a $P_k$ algebra. The usual notion of a Poisson dg algebra is a $P_1$ algebra, in which the bracket is not shifted.

When we discuss (infinitesimal) symmetries soon, it will be advantageous to have an object similar to $\text{Obs}^{\text{cl}}$, but which is also a bona fide $P_0$ algebra. Although we won’t provide the full details,\(^4\) we will at least state some useful facts.

\(^4\)These can be found in Sections 5.2-5.4 of [CG21].
Theorem 4.1.33. For any classical field theory on $X$, there exists a $P_0$ factorization algebra $\widetilde{\text{Obs}}^{\text{cl}}$ together with a weak equivalence of commutative factorization algebras:

$$\widetilde{\text{Obs}}^{\text{cl}} \simeq \text{Obs}^{\text{cl}}.$$

Remark 4.1.34. $\widetilde{\text{Obs}}^{\text{cl}}(U)$ is built from functionals on the space of solutions to the equations of motion which have higher regularity than those in $\text{Obs}^{\text{cl}}(U)$: we don’t lose too much data in defining $\widetilde{\text{Obs}}^{\text{cl}}(U) \subset \text{Obs}^{\text{cl}}(U)$, and moreover get the desired Poisson structure at least up to quasi-isomorphism.

4.2 General Covariance

“He walks along and weeps, the bearer of the seed bag.
He will surely come in with glad song bearing his sheaves.”

– Psalms 126:6 (Translation by Robert Alter)

With all of the above in our toolkit, we can now begin our thorough investigation of general covariance. We begin by revisiting Example 4.1.3, the free scalar field. Recall that in Equation (4.3), we stated that the BV space of fields over a fixed metric $g \in \mathcal{M}$ for this example was

$$\mathcal{F}_g = C^\infty(X) \xrightarrow{Q_g} \text{Dens}(X)[-1],$$

where $Q_g \varphi = \Delta_g \varphi \text{vol}_g$ is the Laplacian modified to land in densities. This in turn allowed us to define a dg vector bundle $\pi : \mathcal{F} \to \mathcal{M}$, with base space the space of
all (semi-)Riemannian metrics on $X$ and with fibers $\pi^{-1}(g) = \mathcal{F}_g$, the field theories depending on the fixed $g$.

Both components of the dg space of fields have an action by the diffeomorphism group of $X$—denoted $\mathcal{D}$ when unambiguous—via pullback: for $f \in \mathcal{D}$, $\varphi \in C^\infty(X)$, and $\mu \in \text{Dens}(X)$, $f \cdot \varphi = f^*\varphi = \varphi \circ f$ and $f \cdot \mu = f^*\mu$. Additionally, we have the usual action of $\mathcal{D}$ on $\mathcal{M}$ via pullback: $f \cdot g = f^*g$. What’s special about this example is that the differential $Q_g$ commutes with diffeomorphisms in the following sense: $f^*(Q_g\varphi) = f^*(\Delta_g\varphi\text{vol}_g) = \Delta_{f^*g}(f^*\varphi)\text{vol}_{f^*g} = Q_{f^*g}(f^*\varphi)$ [Can13]. In plain speech: the Laplacian as a function of the metrics $\mathcal{M}$ is equivariant with respect to the diffeomorphism group $\mathcal{D}$. Thus:

**Lemma 4.2.1.** Any $f \in \mathcal{D}$ defines a cochain map between fibers of $(\mathcal{F}, Q) \to \mathcal{M}$:

\[
\begin{array}{ccc}
\mathcal{F}_g & \xrightarrow{Q_g} & \text{Dens}(X)[-1] \\
\downarrow f^* & & \downarrow f^* \\
\mathcal{F}_{f^*g} & \xrightarrow{Q_{f^*g}} & \text{Dens}(X)[-1].
\end{array}
\]

In other words, $(\mathcal{F}, Q) \to \mathcal{M}$ is a $\mathcal{D}$-equivariant differential graded vector bundle.

This result also implies a significant and useful corollary:

**Corollary 4.2.2.** If $g \in \mathcal{M}$ is a fixed point of $f \in \mathcal{D}$ (in other words if $f$ is an isometry of $g$) and if $Q_g\varphi = 0$, then $Q_g(f^*\varphi) = 0$. In other words, isometries of the metric $g$ act on the space of solutions to $\Delta_g\varphi = 0$ (Laplace’s equation).

**Remark 4.2.3.** As a topological space, the bundle above is trivial, as it is $(C^\infty(X) \oplus \text{Dens}(X)) \times \text{Met}(X)$: the differential $Q_g$ is what defines any nontriviality as a differential graded vector bundle. Lemma 4.2.1 makes use of definitions laid out in
previous chapters: in particular, we are stating that \((\mathcal{F}, Q) \to M\) is a \(\mathcal{D}\)-equivariant
differential graded vector bundle in the category of Fréchet manifolds.

The above lemma thus shows us how useful it can be to express geometric and
analytic information by a fusion of the BV and dg equivariant bundle languages.
Moreover, free theories which are diffeomorphism-equivariant in this way have a
special name.

**Definition 4.2.4.** Let the differential graded vector bundle \((\mathcal{F}, Q) \to M\) define a
family of free Batalin-Vilkovisky field theories on \(X\). If it is a \(\mathcal{D}\)-equivariant as a
differential graded vector bundle, we call the theory **generally covariant**.

**Remark 4.2.5.** Field theories which satisfy general covariance are therefore not
sensitive to *all* of \(M\), but only to the moduli space \(M/\mathcal{D}\) of metrics modulo diffeo-
morphism, as Theorem 2.1.18 (applied to our Fréchet manifolds) suggests. Because
most physically relevant (and interesting) manifolds or spacetimes have metrics with
many isometries, the naïve quotient \(M/\mathcal{D}\) is often a singular space. This is why we
use stacks.

**Example 4.2.6.** A tangible example of \(M/\mathcal{D}\) being singular is the one in which
the underlying Riemannian manifold is \(X = \mathbb{R}^n\) along with the flat metric \(\eta\). It’s
well known that the isometry group of \((\mathbb{R}^n, \eta)\) is \(O(n) \ltimes \mathbb{R}^n\), where the \(\mathbb{R}^n\) in the
semidirect product is the additive group of translations of \(X = \mathbb{R}^n\). In particular,
\(O(n) \ltimes \mathbb{R}^n\) is a subgroup of \(\text{Diff}(\mathbb{R}^n)\) which stabilizes \(\eta \in \mathcal{M}(\mathbb{R}^n)\), meaning that
the corresponding point in the quotient is singular. Thus, \(O(n) \ltimes \mathbb{R}^n\) acts on the
space of solutions to any generally covariant theory defined on \(\mathbb{R}^n\) with the metric \(\eta\).
The preceding definition therefore “enlarges” our usual idea of equivalence beyond isometries.

Further unpacking Theorem 2.1.18 in this context allows us to state the following lemma, which can effectively be used as the definition.

**Lemma 4.2.7.** A family $(F, Q) \rightarrow M$ of free BV field theories is generally covariant if and only if it descends to a dg vector bundle $([F/D], Q) \rightarrow [M/D]$ of stacks.

**Construction 4.2.8.** The above definition is specified in the particular case in which the BV theory is both free and non-perturbative: i.e. the Euler-Lagrange equations are linear in the fields $\phi \in F_g$ and we are not choosing a fixed solution to perturb around, so that the observables are polynomial functions of the fields as opposed to Taylor series.

However, one could just as well formulate a definition of general covariance for an interacting or perturbative field theory. The caveat is that the bundle $(L, \{S, -\}) \rightarrow \mathcal{M}$ representing the family of theories is no longer just a dg vector bundle, but a bundle of elliptic $L_\infty$ algebras over $\mathcal{M}$. Heuristically speaking, we will no longer view the family as a collection of vector spaces varying over $\mathcal{M}$, but rather as a collection of formal neighborhoods varying over $\mathcal{M}$: although the underlying graded structure is still a vector bundle, the geometry encoded in the $L_\infty$ structures on distinct fibers implies this shift in perspective. Although this might feel like a pedantic point, we find it to be an essential difference in interpretation worthy of mindful consideration.

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5Note that the notation has changed since the perturbative space of fields is $L = F[-1]$. 

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To approach a definition of general covariance for such a perturbative or interacting theory, let’s reconsider Example 2.2.7. Recall that the equation of motion (valued in densities) in that instance is:

\[ Q_g \varphi + \frac{1}{3!} \varphi^3 \text{vol}_g = 0. \]

If we fix a diffeomorphism \( f \in \mathcal{D} \), we see that the Euler-Lagrange form on the left hand side satisfies:

\[ f^*(Q_g \varphi + \frac{1}{3!} \varphi^3 \text{vol}_g) = Q_{f^*g}(f^* \varphi) + \frac{1}{3!} (f^* \varphi)^3 \text{vol}_{f^*g}. \quad (4.5) \]

The equivariance property for the first summand is precisely what’s shown in Lemma 4.2.1, and the second summand (the interaction term) is equivariant because polynomial functions of the fields are patently equivariant in this way and we’ve already seen that the Riemannian volume form is \( \mathcal{D} \)-equivariant.

Equation (4.5) can then be reformulated in terms of the brackets on the elliptic \( L_\infty \) algebra of Example 2.2.7 as:

\[ f^*(\ell_1^g(\varphi) + \frac{1}{3!} \ell_3^g(\varphi, \varphi, \varphi)) = \ell_1^{f^*g}(f^* \varphi) + \frac{1}{3!} \ell_3^{f^*g}(f^* \varphi, f^* \varphi, f^* \varphi), \quad (4.6) \]

where we have included the dependence of the brackets \( \ell_k \) on the underlying metric \( g \in \mathcal{M} \) as a superscript. The above equation is the \( \mathcal{D} \)-equivariance property we desire in the Euler-Lagrange term which implies that the family of theories defined by \( \varphi^4 \) theory as in Example 2.2.7 is generally covariant.

Moreover, this generalizes naturally to the case in which the interaction term is any polynomial in \( \varphi \) times \( \text{vol}_g \). In that case, \( \ell_1 = Q_g \) and \( \ell_n : C^\infty(X)[-1] \otimes^n \rightarrow \ldots \)
Dens(X)[−2] for n ≥ 2 is:
\[ \ell_n : \varphi_1 \otimes \ldots \otimes \varphi_n \mapsto \lambda_n \varphi_1 \ldots \varphi_n \operatorname{vol}_g, \]
where the \( \lambda_n \) are constants. Similarly to Equation (4.5), it’s quick to show that:
\[ f^*(\ell_1^g(\varphi) + \sum_{n \geq 2} \frac{\lambda_n}{n!} \ell_n^g(\varphi, \ldots, \varphi)) = \ell_1^{f^*g}(f^* \varphi) + \sum_{n \geq 2} \frac{\lambda_n}{n!} \ell_n^{f^*g}(f^* \varphi, \ldots, f^* \varphi). \]
(4.7)
Thus, any scalar field theory with action functional
\[ S_g(\varphi) = \int_X (\frac{-1}{2} \varphi \Delta_g \varphi + V(\varphi)) \operatorname{vol}_g, \]
where \( V(\varphi) \) is a polynomial “potential” in \( \varphi \), is generally covariant.

Construction 4.2.9. We would like to summarize the above with a lemma analogous to Lemma 4.2.1. First we must collect some information on \( L_\infty \) algebras. To begin, let \( \mathcal{L}_g \) be an elliptic \( L_\infty \) algebra defining a perturbative BV field theory with background metric \( g \in M \). Without loss of generality, suppose the underlying graded space \( \mathcal{L}^g = \mathcal{L}^1 \oplus \mathcal{L}^2 \)\(^6\) is concentrated in degrees 1 and 2, and denote its \( L_\infty \) structure as \( \ell^g \); thus, for an element \( \phi \in \mathcal{L}^1_g \) in degree 1,
\[ \ell^g(\phi) = \sum_{n \geq 1} \frac{1}{n!} \ell^g_n(\phi) \]
(4.8)
is its associated Maurer-Cartan element in degree 2. This assignment of an \( L_\infty \) structure varies smoothly in \( g \in M \). To be even more specific, an \( L_\infty \) structure is an element
\[ \ell^g \in \operatorname{Hom}_{\text{Alg}_{L_\infty}}(\Lambda^{\geq 1} \mathcal{L}_g, \mathcal{L}_g[2 - n]) \]
(4.9)
such that \([\ell^g, \ell^g] = 0\). We can thus define the following.

\(^6\)Note that we do not always include the the subscript \( g \) in the graded components of \( \mathcal{L}_g \) because they do not depend on the metric-only the “dynamical” fields do.
Definition 4.2.10. A bundle of (elliptic) $L_\infty$ algebras is a $\mathbb{Z}$-graded vector bundle $\pi: (V, \ell) \to M^7$ whose fibers $(V_p, \ell^p) := \pi^{-1}(p)$ are (elliptic) $L_\infty$ algebras, such that the $L_\infty$ structure varies smoothly over $M$. In other words,

$$\ell \in \text{Hom}_{\text{Alg}_{L_\infty}(C^\infty(M))}(\Lambda^{\geq 1}V, V[2-n]).$$

Remark 4.2.11. In our infinite dimensional case, we require that the underlying graded pieces $\mathcal{L}^k[-k]$ of $\mathcal{L}_g$ are Fréchet vector spaces: i.e. if we forget about the $L_\infty$ structure, $\pi: \mathcal{L} \to \mathcal{M}$ is just a Fréchet vector bundle $(\mathcal{L}^1[-1] \oplus \mathcal{L}^2[-2]) \to \mathcal{M}$, and it is moreover trivial because $\mathcal{M}$ is a contractible Fréchet manifold. Just as in the free case, where the differential provided nontriviality as a dg vector bundle, the $L_\infty$ structure is what provides nontriviality.

Given the above and the computations from Construction 4.2.8, we can now state a lemma generalizing what we stated in Lemma 4.2.1:

Lemma 4.2.12. Let $\pi: (\mathcal{L}, \ell) \to \mathcal{M}$ be a family over $\mathcal{M}$ of perturbative Batalin-Vilkovisky classical scalar field theories with polynomial potential. Any $f \in \mathcal{D}$ defines an $L_\infty$ map between fibers of $\pi: (\mathcal{L}, \ell) \to \mathcal{M}$:

$$\mathcal{L}_g = C^\infty(X)[-1] \xrightarrow{\ell^g} \text{Dens}(X)[-2]$$

$$\downarrow f^*$$

$$\mathcal{L}_{f^*g} = C^\infty(X)[-1] \xrightarrow{\ell_{f^*g}} \text{Dens}(X)[-2].$$

In other words, $\pi: (\mathcal{L}, \ell) \to \mathcal{M}$ is a $\mathcal{D}$-equivariant bundle of $L_\infty$ algebras.

With this lemma in mind, we posit the following definition and lemma which serve as generalizations of statements from the earlier, free case.

---

7We may sometimes omit this notation as a pair if the $L_\infty$ structure is implicit.
**Definition 4.2.13.** Let the bundle $\pi : (\mathcal{L}, \ell) \to \mathcal{M}$ of $L_\infty$ algebras define a family of perturbative (or interacting) Batalin-Vilkovisky field theories on $X$. If it is a $\mathcal{D}$-equivariant as a bundle of $L_\infty$ algebras, we call the theory **generally covariant**.

**Lemma 4.2.14.** A family $(\mathcal{L}, \ell) \to \mathcal{M}$ of perturbative/interacting BV field theories is generally covariant if and only if it descends to a bundle of $L_\infty$ algebras $([\mathcal{L}/\mathcal{D}], \ell) \to [\mathcal{M}/\mathcal{D}]$ of stacks.

**Example 4.2.15.** A few interesting directions can be taken to generate examples of generally covariant theories. One could ask: what form can the potential $V(\varphi)$ take in a scalar field theory such that the theory is generally covariant? Or what other dependence(s) can it have on the “background” metric variable? For example, the free scalar field theory defined by the functional

$$S_g(\varphi) = -\frac{1}{2} \int_X \varphi(\Delta_g + R)\varphi\text{vol}_g,$$

where $R$ is the scalar curvature associated to the metric $g$, is generally covariant: this is because $R$ as a function of the metric variable is $\mathcal{D}$-equivariant. Indeed, any polynomial $f(R)$ of the scalar curvature can replace $R$ in the above, resulting in a generally covariant theory with Euler-Lagrange equation (valued in densities)

$$(\Delta_g + f(R))\varphi\text{vol}_g = 0.$$

**Remark 4.2.16 (A Long Remark on Sheaves).** We can also reformulate all that has been provided thus far in terms of sheaves, which are more advantageous with respect to certain (co)homological maneuvers. Recall that to every vector bundle we
can associate its sheaf of sections, so that we get a functor

$$\text{VectBun}(M) \to \text{Shv}(M),$$

defined by

$$(\pi : V \to M) \mapsto \Gamma(-, V).$$

In this case, $\Gamma(-, V)$ is a functor from $\text{Open}(M)^{\text{op}}$ to $\text{Vect}$; however, if $V$ has additional structure, we could narrow down the target category. An important example of this starts with $\Lambda^\bullet(T^\vee X) \to X$, the exterior algebra of the cotangent bundle of a manifold $X$. The space of sections for this bundle is the de Rham complex

$$\Omega^\bullet : \text{Open}(X)^{\text{op}} \to \text{dgAlg},$$

which for every open set $U \subset X$ gives the differential graded algebra $\Omega^\bullet(U)$ of differential forms on $U$. The grading on $\Omega^\bullet(U)$ is inherited from the grading of the exterior algebra $\Lambda^\bullet(T^\vee X)$ at any point $p \in X$ and the differential is induced from the differential of smooth functions at a point: this makes the de Rham complex a sheaf of dg algebras over the manifold $X$.

A totally analogous construction—where a structure at a point helps to define a structure on open sets—defines a sheaf of $L_\infty$ algebras over the space of metrics $\mathcal{M}$. We must define the appropriate sheaf

$$\Gamma(-, \mathcal{L}) : \text{Open}(\mathcal{M})^{\text{op}} \to \text{Alg}_{L_\infty}$$

which captures the data of a family of perturbative BV theories over the space $\mathcal{M}$ of metrics. Note that this sheaf will actually be a pair of objects: we need a graded
vector space $\Gamma(U, \mathcal{L})$ and an $L_\infty$ structure which makes $\Gamma(U, \mathcal{L})$ into an $L_\infty$ algebra. For the latter, we simply use the smooth assignment

$$\ell^(-): U \subset M \rightarrow \text{Hom}_{\text{Alg}_{L_\infty}}(\Lambda^{\geq 1} \mathcal{L}(-), \mathcal{L}(-)[2-n])$$

(4.11)

of an $L_\infty$ structure from before. Moreover, to every open set $U \subset M$ we assign the following:

$$\Gamma(U, \mathcal{L}) := C^\infty(U, \mathcal{L}^1[-1]) \oplus C^\infty(U, \mathcal{L}^2[-2]).$$

(4.12)

Note that here $C^\infty$ denotes smooth functions between Fréchet manifolds. Given the above, the pair $(\Gamma(-, \mathcal{L}), \ell)$, which we will usually just denote as $\Gamma(-, \mathcal{L})$, is a sheaf of $L_\infty$ algebras on the space of metrics $M$.

**Lemma 4.2.17.** Let the sheaf $(\Gamma(-, \mathcal{L}), \ell)$ represent a family over $M$ of perturbative Batalin-Vilkovisky classical scalar field theories with polynomial potential as in Construction 4.2.8. Then $(\Gamma(-, \mathcal{L}), \ell)$ is a $\mathcal{D}$-equivariant sheaf of $L_\infty$ algebras over $M$.

From this example, we can identify a definition for general covariance in the case that our theory is perturbative or interacting.

**Definition 4.2.18.** Let the sheaf $\Gamma(-, \mathcal{L})$ of $L_\infty$ algebras over $M$ define a family of perturbative Batalin-Vilkovisky field theories on $X$. If it is a $\mathcal{D}$-equivariant as a sheaf of $L_\infty$ algebras, we call the theory **generally covariant**.

We thus recover a result analogous to Lemma 4.2.7:
Lemma 4.2.19. A family \( \Gamma(\cdot, \mathcal{L}) \) over \( \mathcal{M} \) of perturbative BV field theories is generally covariant if and only if it descends to a sheaf of \( L_\infty \) algebras \( \Gamma(\cdot, \mathcal{L})/\mathcal{D} \) over \( [\mathcal{M}/\mathcal{D}] \) of stacks.

4.2.1 Equivariant Observables and a connection to Noether’s Theorem

Now that we have thoroughly made sense of families of field theories parameterized by the \emph{global} stack of metrics modulo diffeomorphism, we will consider what the \emph{local}, or perturbative, picture buys us.

The idea is to pull back generally covariant families of theories over fixed formal neighborhoods in the moduli stack, which define formal moduli problems in the style of Section 2.2. We then perform associated computations to recover \emph{equivariant} classical observables and find novel perspectives on the stress-energy tensor.

Construction 4.2.20. A family \( (\mathcal{L}, \ell) \to \mathcal{M} \) of BV classical field theories defined as a bundle of \( L_\infty \) algebras pulls back to a bundle of \( L_\infty \) algebras over the formal neighborhood of \( g \in \mathcal{M} \), denoted \( \hat{\mathcal{M}}_g \), where \( \mathcal{O}(\hat{\mathcal{M}}_g) = \hat{\mathcal{O}}_g \cong \text{Sym}(T^\vee_g \mathcal{M}) \). Heuristically, the total space of this pullback looks like \( \hat{\mathcal{M}}_g \times \mathcal{F}_g \).

We get an analogous pullback of stacks when the theory is generally covariant. In this case, the \( \mathcal{D} \)-equivariant bundle of \( L_\infty \) algebras \( (\mathcal{L}, \ell) \to \mathcal{M} \) is equivalent to a dg bundle of stacks \( ([\mathcal{L}/\mathcal{D}], \ell) \to [\mathcal{M}/\mathcal{D}] \). If we consider an equivalence class of metrics \( [g] \in [\mathcal{M}/\mathcal{D}] \) and fix its formal neighborhood, we can pull back \( ([\mathcal{L}/\mathcal{D}], \ell) \) over this formal neighborhood. We denote the total space of this pullback as \( \text{Tot}(g^*[\mathcal{L}/\mathcal{D}]) \). Once more, \emph{at a purely heuristic level}, this pullback can be roughly
viewed as \([L_g/Vect(X)] \times [\hat{M}_g/Vect(X)]\). We thus get the following.

**Theorem 4.2.21.** For a generally covariant family \(([L/\mathcal{D}], \ell) \to [M/\mathcal{D}]\) of \(BV\) classical field theories and for a fixed \([g] \in [M/\mathcal{D}]\), we have

\[
\mathcal{O}(\text{Tot}(g^*[L/\mathcal{D}])) \cong C^\bullet(\mathfrak{g}_g, \text{Obs}^{cl}(X, \mathcal{L}_g)), 
\]

(4.13)

where

\[
\mathfrak{g}_g = \text{Vect}(X) \xrightarrow{L^\bullet g} \Gamma(X, \text{Sym}^2(T_X^\vee))[-1]
\]

is the dg Lie algebra associated to the formal neighborhood of \([g] \in [M/\mathcal{D}]\).

**Proof.** For our purposes, we only consider the part of the formal neighborhood of \([g] \in [M/\mathcal{D}]\) which has the form \([\hat{M}_g/Vect(X)]\) (we may be losing a bit of data here, but it is a cost worth paying for simplicity and physical relevance of the infinitesimal diffeomorphisms). By the equivalence of categories from [Lur11] which we are taking for granted, this is equivalent to the dg Lie algebra

\[
\mathfrak{g}_g = \Gamma(X, T_X) \xrightarrow{L^\bullet g} \Gamma(X, \text{Sym}^2(T_X^\vee))[-1],
\]

so that the dg ring of functions on this formal neighborhood is

\[
C^\bullet(\mathfrak{g}_g) \cong C^\bullet(\text{Vect}(X), \text{Sym}(T^\vee g \mathcal{M})).
\]

The ring of functions on the fiber of the pullback is simply \(C^\bullet(\text{Vect}(X), \text{Obs}^{cl}(X, \mathcal{L}_g))\), since it is \(\mathcal{O}(\mathcal{L}_g)\) with the differential \(\{S_g, -\}\) and the implicit action of \(\text{Vect}(X)\) on the theory and thus on its observables. Hence, the underlying dg ring of functions on
Tot($g^*[\mathcal{L}/\mathcal{D}]$) is the underlying dg ring of $C^\bullet(\mathfrak{g}_g, \text{Obs}^\text{cl}(X, \mathcal{L}_g))$. Both dg rings have Chevalley-Eilenberg differential

$$d_{CE} = [-,-]_{\text{Vect}(X)} + \tau^\nu_{\mathcal{M}_g} + \tau^\nu_{\mathcal{L}_g} + \{S_g, -\}. \quad (4.14)$$

Here, the first three terms are the usual Chevalley-Eilenberg differential concerned with the dual of the bracket on $\text{Vect}(X)$ and the actions of $\text{Vect}(X)$ on $\mathcal{S}ym(T_g^\nu \mathcal{M})$ and $\mathcal{O}(\mathcal{L}_g)$, and the fourth term is the differential on the observables over $[\widehat{\mathcal{M}}_g/\text{Vect}(X)]$.

Since the underlying rings agree and the CE differentials do, too, this gives the result. \hfill \Box

**Remark 4.2.22.** This theorem is a generalization of Theorem 4.16 in [Dul21], which was shown only for the case of free BV theories. Computations forming a link with physics literature in the following section will still largely follow the free case, but we found it appropriate to state the key results in full generality.

**Remark 4.2.23.** Since $C^\bullet(\mathfrak{g}_g, \text{Obs}^\text{cl}(X, \mathcal{L}_g)) \cong C^\bullet(\mathfrak{g}_g) \otimes \text{Obs}^\text{cl}(X, \mathcal{L}_g)$ as dg commutative algebras, this theorem confirms what we suspected in 2.2.24: that an appropriate ring of classical observables computed for a generally covariant theory will be a $C^\bullet(\mathfrak{g}_g)$-module. This is a derived-geometric manifestation of the fact that the ring of functions on a bundle is a module over the ring of functions of its base space.

### 4.2.2 The Stress-Energy Tensor

A central expectation rooted in general covariance is that if we perturb the metric by an infinitesimal diffeomorphism—\textit{in other words, if we vary $g$ to $g + \varepsilon L_V g$ for a
vector field $V$—then any induced variation on $Q_g$ (or $\ell^g$) would be trivial. This is precisely the \textit{perturbative} manifestation of general covariance, and we would like to package it in the language we have thus far described. Moreover, it would imply that \{\(S_g, -\)\} is only sensitive to diffeomorphism classes of $g$ at the perturbative level: this is a \textit{necessary} feature for it as a term in a Chevalley-Eilenberg differential.

\textbf{Remark 4.2.24.} Now that we’ve outlined the mathematical picture in the previous section, let’s get our hands dirty with some concrete computations to reinforce the bridge we are trying to build with the more “physical” perspectives common in the literature. In what follows, we will stick with the case of the \textit{free} massless scalar field to reduce computational complexity: i.e. we will return to the perspective of a dg vector bundle representing our family of theories. However, much of what follows remains true in the case of perturbative and interacting families of theories, represented by a bundle of $L_\infty$ algebras.

To begin, let’s consider an arbitrary first order deformation of the Laplacian $\Delta_g$ on a Riemannian manifold $(X, g)$: in other words, let $g_t$ be a one-parameter family of metrics such that $g_0 = g$ and let’s compute

$$\left. \frac{d}{dt} \Delta_{g_t} \varphi \right|_{t=0}.$$  

Writing $\Delta_{g_t}$ in coordinates and not evaluating at $t = 0$ for now, we have:

$$\frac{d}{dt} \left( \frac{1}{\sqrt{\det g_t}} \partial_\mu (\sqrt{\det g_t} g_t^{\mu \nu} \partial_\nu \varphi) \right).$$  \hspace{1cm} (4.15)$$

Recall that for a one-parameter family of invertible matrices $A(t)$, we have

$$\frac{d}{dt} \det A(t) = \text{Tr}(A(t)^{-1} A'(t))\det A(t).$$
Using this and a few other manipulations, expression (4.15) reduces to
\[ \frac{-1}{2} \text{Tr}(g^{-1}_t \partial_t g_t) \Delta_{g_t} \varphi + \frac{1}{\sqrt{\text{det} g_t}} \partial \mu \left( \frac{\sqrt{\text{det} g_t}}{2} \text{Tr}(g^{-1}_t \partial_t g_t) g^{\mu \nu}_t \partial \nu \varphi + \sqrt{\text{det} g_t} \partial \mu g^{\mu \nu}_t \partial \nu \varphi \right). \]

(4.16)

Denote the derivative of \( g_t \) at \( g_0 = g \) as \( \delta g := \partial_t g_t \vert_{t=0} \) (this is the traditional notation in physics). Evaluating at \( t = 0 \) gives the following lemma.

**Lemma 4.2.25.** The first order deformation of the Laplacian \( \Delta_g \) with respect to the metric \( g \) is
\[ \frac{d}{dt} \Delta_{g_t} \varphi \biggvert_{t=0} = \frac{-1}{2} \text{Tr}(g^{-1} \delta g) \Delta_g \varphi + \frac{1}{\sqrt{\text{det} g}} \partial \mu \left( \frac{1}{2} \text{Tr}(g^{-1} \delta g) g^{\mu \nu} \partial \nu \varphi + \delta g^{\mu \nu} \partial \nu \varphi \right). \]

(4.17)

Moreover, if we assume the deformation \( \delta g \in T_g \mathcal{M} \) is induced by an isometry of \( g \), then the first order deformation of the Laplacian is identically zero.

**Proof.** The first claim follows immediately from the preceding computations and evaluating expression (4.16) at \( t = 0 \). Next, saying that the deformation \( \delta g \) is induced by a diffeomorphism means that \( \delta g = L_V g \), where \( V \) is the vector field which generates that diffeomorphism. If \( V \) is a Killing field for \( g \), i.e. it generates an isometry of \( g \), then it satisfies \( L_V g = 0 \). The first two terms in the sum are thus zero, and the third is zero since \( L_V g^{\mu \nu} = 0 \) when \( L_V g_{\mu \nu} = 0 \), which is \( L_V g = 0 \) with indices.

\[ \Box \]

**Remark 4.2.26.** A few remarks are in order here. First of all, the computations above involve a combination of index-dependent and independent notation. For
example, often in the physics literature, we write

$$\text{Tr}(g^{-1}\delta g) = g^{\mu\nu}\delta g_{\mu\nu}. $$

I avoid doing so because the indices are fully repeated, and so not related to the indices left in the expression (which are necessary to retain). Moreover, the above computation is done with the action functional (4.2) in mind. Thus, any difference with the stress-energy tensor computations using the functional

$$\int_X g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi \text{vol}_g = \int_X g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi \sqrt{\text{det} gd^nx},$$

which is more common in the physics literature, differs only by boundary terms, as we’ll see.

Indeed, our next step is to understand the stress-energy (alternatively, energy-momentum) tensor for the free massless scalar; however, we give the definition for any such field theory.

**Construction 4.2.27.** Let $S_g \in \mathcal{O}_{\text{loc}}(\mathcal{F})$ be an action functional for a space of fields $\mathcal{F}$ which depends on a fixed background metric $g \in \mathcal{M}$. It can thus be written as

$$S_g(\phi) = \int_X L_g(\phi),$$

where $\phi \in \mathcal{F}$ and $L_g(\phi)$ is a Lagrangian density. If we let $g_t$ be a one-parameter family of metrics such that $g_0 = g$, we can perform computations similar to those in Lemma 4.2.25 to compute:

$$\frac{\delta}{\delta g} S_g(\phi) := \left. \frac{d}{dt} S_{g_t}(\phi) \right|_{t=0} = \left. \int_X \frac{d}{dt} L_{g_t}(\phi) \right|_{t=0}. $$
The notation invoked on the left hand side is common in physics literature, and defined this way in [Wald84]. Up to boundary terms which we can safely ignore, (4.19) can be written as

$$
\int_X \delta g^{\mu\nu} T_{\mu\nu}(g, \phi) \text{vol}_g,
$$

(4.20)

for some $T_{\mu\nu}(g, \phi)$ (or simply $T_{\mu\nu}$) which depends on the fields $\phi$ and on the metrics $g$.

**Definition 4.2.28.** $T_{\mu\nu}$ is the **stress-energy** (or **energy-momentum**) tensor of a field theory on $X$ with fields $\phi \in \mathcal{F}$ and action functional $S_g$ depending on $g \in \text{Met}(X)$.

**Example 4.2.29.** To compute the stress-energy tensor of free massless scalar field, we’ll begin by noting that according to the definition, we must compute

$$
\int_X \varphi \frac{d}{dt} Q_{g_t} \varphi \bigg|_{t=0},
$$

where $Q_{g_t} \varphi = \Delta_g \varphi \text{vol}_g$. Lemma 4.2.25 is useful, since we have already done the necessary work on the first piece. However, note that $Q_{g_t} \varphi$ is written in coordinates as

$$
\partial_\mu (\sqrt{\det g} g^{\mu\nu} \partial_\nu \varphi) d^n x,
$$

so that we have in fact stripped away some of the complexity of the computation by pairing with the Riemannian volume form. Hence, we can use Lemma 4.2.25 and toss away the first term to get

$$
\int_X \varphi \frac{d}{dt} Q_{g_t} \varphi \bigg|_{t=0} = \int_X \varphi \partial_\mu \left( \sqrt{\det g} \left( \frac{1}{2} \text{Tr}(g^{-1}\delta g) g^{\mu\nu} \partial_\nu \varphi + \delta g^{\mu\nu} \partial_\nu \varphi \right) \right) d^n x.
$$

(4.21)
This is not yet in the preferred form in (4.20), but if we integrate by parts and invoke that

\[ \text{Tr}(g^{-1} \delta g) = g^{\mu\nu} \delta g_{\mu\nu} = g_{\mu\nu} \delta g^{\mu\nu}, \]

the above becomes

\[ \int_X \delta g^{\mu\nu} \left( - \partial_\mu \varphi \partial_\nu \varphi - \frac{1}{2} g_{\mu\nu} (g^{\rho\sigma} \partial_\rho \varphi \partial_\sigma \varphi) \right) \text{vol}_g, \tag{4.22} \]

where we changed the labelling of indices in the second term to omit confusion. Thus, the stress-energy tensor for our example is \( T_{\mu\nu} = -\partial_\mu \varphi \partial_\nu \varphi - \frac{1}{2} g_{\mu\nu} (g^{\rho\sigma} \partial_\rho \varphi \partial_\sigma \varphi) \). We would have computed this without any by-parts maneuvers had we started with the action functional (4.18) more common in physics literature, but it’s a good exercise to see how these agree.

**Remark 4.2.30.** The reader may be concerned that these constructions won’t work for the case of a non-closed manifold. Physicists often consider fields which “rapidly decrease”\(^8\) (or simply fields with compact support) if the field theory is on, say, \( \mathbb{R}^n \), so the above is still true then. It would be interesting, however, to see how the boundary data for a manifold with boundary would make a difference: that is left for future work.

The above is the traditional trajectory one takes to finding the stress-energy tensor; however, since our theory is generally covariant and so we can use facts about equivariant vector bundles to simplify things, let’s consider what that buys us. To begin, let \( f_t \) be a one-parameter subgroup of diffeomorphisms. General covariance

\(^8\)Mathematically, “Schwartz functions”.
implies that
\[
\frac{d}{dt} \int_X (f_t^* \varphi) \Delta_{f_t^* g} (f_t^* \varphi) \text{vol}_{f_t^* g} \big|_{t=0} = 0. \tag{4.23}
\]

Unfolding the left hand side, this equation says that
\[
\int_X (L_V \varphi) \Delta_g \varphi \text{vol}_g + \int_X \varphi \Delta_g (L_V \varphi) \text{vol}_g + \int_X \varphi \left( \frac{d}{dt} \Delta_{f_t^* g} \big|_{t=0} \right) \varphi \text{vol}_g + \int_X \varphi \Delta_g \varphi \left( \frac{d}{dt} \text{vol}_{f_t^* g} \big|_{t=0} \right)
\]
must equal zero. Here, we assumed that \( V \) generates the flow \( f_t \), and used the fact that \( \frac{d}{dt} (f_t^* \varphi) \big|_{t=0} = L_V \varphi \). This equation is an integrated linear approximation to the equivariance property computed in Lemma 4.2.1: it states concretely that a simultaneous first order perturbation along the \( \mathcal{D} \)-orbit in \( \mathcal{M} \) and in \( \mathcal{F}_g \) is trivial.

**Remark 4.2.31.** The third and fourth terms on the left hand side are exactly those that comprise the integral of the stress-energy tensor in the special case that the derivative is computed in the direction of the \( \mathcal{D} \)-orbit. This grants us two key insights:

1. Computationally, the above amounts to the metric perturbation (an element of \( T_g \mathcal{M} \)) coming from an infinitesimal diffeomorphism (i.e. a vector field). But we’ve seen this before! Explicitly, this is saying that \( \delta g \in T_g \mathcal{M} \) is in the image of the differential in the dg Lie algebra \( \mathfrak{g}_g \) in Example 2.2.24. Hence, \( \delta g^{\mu\nu} = L_V g^{\mu\nu} \) (the computation works fine even though \( g^{\mu\nu} \) is technically the inverse). With this, Equation (4.22) becomes:
\[
\int_X L_V g^{\mu\nu} T_{\mu\nu} \text{vol}_g.
\]
A standard result from Riemannian geometry is that \( L_V g^{\mu\nu} = \nabla^{\mu} V^{\nu} + \nabla^{\nu} V^{\mu} \), and
since $T_{\mu\nu}$ is symmetric by definition, the above must be

$$
\int_X (\nabla^\mu V^\nu) T_{\mu\nu} \text{vol}_g = - \int_X V^\nu (\nabla^\mu T_{\mu\nu}) \text{vol}_g,
$$

where we invoked integration by parts and the fact that $\nabla^\mu \text{vol}_g = 0$ in the equality. Then, standard computations for generally covariant theories (which can be found in Appendix E of [Wald84]) show that for on-shell fields (here meaning $\varphi$ such that $\Delta_g \varphi = 0$), the above integral is identically zero. For this to be true, it must be the case that

$$
\nabla^\mu T_{\mu\nu} = 0. \tag{4.24}
$$

In the language of Noether’s Theorem, the stress-energy tensor $T_{\mu\nu}$ is the conserved current associated to general covariance, a symmetry of a field theory coupled to a metric.

(2) Additionally, since the third and fourth terms are (up to a sign) the same as the first two, this means considering the first two alone should give us all the relevant data of the stress-energy tensor for a generally covariant field theory. This is what we explore next: we will start to think more homologically, but remember that it is all rooted in the preceding geometry.

### 4.2.3 A Stress-Energy Theorem

We will recapitulate a few key ideas formulated thus far from a slightly different angle in anticipation of the central statement.

**Construction 4.2.32.** Let’s consider the “infinitesimal general covariance” property
more formally. Insight (1) suggests that the action functionals \( S_g(\phi) \) and
\[
S_{g+\varepsilon L V g}(\phi) = \frac{1}{2} \int_X \phi \Delta_g \phi \text{vol}_g - \frac{\varepsilon}{2} \int_X L V g^{\mu \nu} T_{\mu \nu} \text{vol}_g =: S_g(\phi) + \varepsilon I_g(L V g, \phi),
\]
where this equality holds modulo \( \varepsilon^2 \), should produce the same dynamics: this is true because for on-shell fields, the second term is zero. In other words, if we were to make sense of the differential \( Q_{g+\varepsilon L V g} \) for the BV space of fields, it should be appropriately equivalent to \( Q_g \). Moreover, \( Q_g \) induces the differential \( \{ S_g, - \} \) on \( \text{Obs}^c(X, \mathcal{F}_g) \), so that we would like \( \{ S_{g+\varepsilon L V g}, - \} = \{ S_g, - \} + \varepsilon \{ I_g(L V g), - \} \), the induced differential on \( \text{Obs}^c(X, \mathcal{F}_{g+\varepsilon L V g}) \) from \( Q_{g+\varepsilon L V g} \), to be similarly equivalent. To give all of the above hands and legs, we must rigorously define \( \mathcal{F}_{g+\varepsilon L V g} \) and its observables in the first place.

Let \( \mathbb{D}_2 = \mathbb{R}[\varepsilon]/(\varepsilon^2) \) denote the (real) dual numbers. We can tensor \( \mathcal{F}_g = C^\infty(X) \xrightarrow{Q_g} \text{Dens}(X)[-1] \) with \( \mathbb{D}_2 \) to get \( \mathcal{F}_g \otimes \mathbb{D}_2 \), whose elements can be written as \( \phi_0 + \varepsilon \phi_1 \) in degree 0 and similarly for degree 1. The differential \( Q_{g+\varepsilon L V g} \) looks like
\[
Q_g + \varepsilon D.
\]
It remains only to find \( D \), which will depend on \( g \) and \( V \) and must be so that
\[
\begin{array}{ccc}
\mathcal{F}_g \otimes \mathbb{D}_2 = C^\infty(X) \otimes \mathbb{D}_2 & \xrightarrow{Q_g + \varepsilon 0} & \text{Dens}(X)[-1] \otimes \mathbb{D}_2 \\
\downarrow \text{Id} + \varepsilon L V & & \downarrow \text{Id} + \varepsilon L V \\
\mathcal{F}_g \otimes \mathbb{D}_2 = C^\infty(X) \otimes \mathbb{D}_2 & \xrightarrow{Q_g + \varepsilon D} & \text{Dens}(X)[-1] \otimes \mathbb{D}_2
\end{array}
\]
commutes. The downward-pointing arrows are \( \text{Id} + \varepsilon L V \) since we are still assuming the diffeomorphism \( f \) is generated by the vector field \( V \): concretely, this is the first order approximation to the commuting square in Lemma 4.2.1. Thus, we are trying
to suss out a neat form of the first-order perturbation of $Q_g$ with respect to the metric when the perturbation is along a diffeomorphism orbit. Our computations from Equation (4.23) suggest that we try $D = [L_V, Q_g]$.

**Lemma 4.2.33.** Let $\tilde{\mathcal{F}}_g := (\mathcal{F}_g \otimes \mathbb{D}_2, Q_g)$ and $\tilde{\mathcal{F}}_{g+\varepsilon L_V g} := (\mathcal{F}_g \otimes \mathbb{D}_2, Q_g + \varepsilon [L_V, Q_g])$. Then the map $\text{Id} + \varepsilon L_V : \tilde{\mathcal{F}}_g \to \tilde{\mathcal{F}}_{g+\varepsilon L_V g}$ is a cochain isomorphism (i.e. it is an equivalence of free BV field theories).

**Proof.** Let’s begin by checking that $\text{Id} + \varepsilon L_V$ is indeed a cochain map: in other words, we must show that the preceding square commutes for $D = [L_V, Q_g]$. Let $\varphi_0 + \varepsilon \varphi_1 \in \tilde{\mathcal{F}}_g^0$. Then

$$(Q_g + \varepsilon [L_V, Q_g])(\text{Id} + \varepsilon L_V)(\varphi_0 + \varepsilon \varphi_1) = (Q_g + \varepsilon [L_V, Q_g])(\varphi_0 + \varepsilon (L_V \varphi_0 + \varphi))$$

(4.25)

$$= Q_g \varphi_0 + \varepsilon (Q_g L_V \varphi_0 + Q_g \varphi_1 + [L_V, Q_g] \varphi_0)$$

(4.26)

$$= Q_g \varphi_0 + \varepsilon (L_V Q_g \varphi_0 + Q_g \varphi_1).$$

(4.27)

On the other hand, we have

$$(\text{Id} + \varepsilon L_V)(Q_g + \varepsilon 0)(\varphi_0 + \varepsilon \varphi_1) = (\text{Id} + \varepsilon L_V)(Q_g \varphi_0 + \varepsilon Q_g \varphi_1)$$

(4.28)

$$= Q_g \varphi_0 + \varepsilon (L_V Q_g \varphi_0 + Q_g \varphi_1).$$

(4.29)

Thus, $\text{Id} + \varepsilon L_V$ defines a cochain map from $\tilde{\mathcal{F}}_g$ to $\tilde{\mathcal{F}}_{g+\varepsilon L_V g}$. Now we must check that
Id − εLV is an inverse. Let ϕ₀ + εϕ₁ ∈ F₀ and let’s compute:

\[(Id − εLV)(Id + εLV)(ϕ₀ + εϕ₁) = (Id − εLV)(ϕ₀ + ε(LVϕ₀ + ϕ₁)) \tag{4.30} \]

\[= ϕ₀ + εϕ₁. \tag{4.31} \]

Similarly, for ϕ₀' + εϕ₁' ∈ F₀, we have (Id + εLV)(Id − εLV)(ϕ₀' + εϕ₁') = ϕ₀' + εϕ₁'.

This computation works for elements in F₁ and F₁ just as well, and so we see that the inverse is what we may have expected. F and F are thus isomorphic as cochain complexes.

The above is the perturbative realization of general covariance: intuitively, the free BV scalar field coupled to a metric is equivalent to the free BV scalar coupled to an infinitesimally close metric in the same diffeomorphism orbit. This lemma also states that for the free scalar field with differential Q on its BV space of fields, the first order deformation of Q along the D-orbit starting at g ∈ M is exactly

\[D = [L_V, Q_g]. \]

This provides a nice coordinate-free form of the stress-energy tensor.

**Remark 4.2.34.** For any generally covariant free BV theory with dg space of fields (F, Q), a similar formula holds. The caveat is that the Lie derivative L_V manifests differently on different choices of fields, so one must be careful. Additionally, certain BV fields have more than two terms in their cochain complexes: the computations in that case are more cumbersome, but only in the sense of needing to check multiple squares commute.
Our goal was not only to show that these two “infinitesimally close” spaces of fields were equivalent, but to show that their associated observables were similarly equivalent. This is what we do next. We need the following lemma:

**Lemma 4.2.35.** If \( \alpha : (V, d_V) \to (W, d_W) \) is an isomorphism of cochain complexes, then there is an induced isomorphism \( \alpha : (\text{Sym}(V), d_V) \to (\text{Sym}(W), d_W) \) of cochain complexes, where the differentials \( d_V \) and \( d_W \) are extended to the respective symmetric algebras as derivations.

**Remark 4.2.36.** It is similarly true that \( \text{Sym}(V^\vee) \) and \( \text{Sym}(W^\vee) \) are isomorphic cochain complexes: the differentials on \( V^\vee \) and \( W^\vee \) are induced by those on \( V \) and \( W \), and using this lemma once more gives \( (\text{Sym}(V^\vee), d_V) \cong (\text{Sym}(W^\vee), d_W) \). (We abuse notation so that \( d_V \) and \( d_W \) are the differentials induced from those on \( V \) and \( W \), respectively.)

One might expect that because the naïve algebraic symmetric powers of \( \mathcal{F}_g \) aren’t what we use to define observables, we should be wary; however, the completed projective tensor product we used to define functionals is the necessary one in the case of infinite-dimensional vector spaces for these constructions to carry over. We can now state a key theorem

**Theorem 4.2.37.** We have the following isomorphism of classical observables:

\[
\text{Obs}^{cl}(X, \widetilde{\mathcal{F}}_g) \cong \text{Obs}^{cl}(X, \widetilde{\mathcal{F}}_{g+\varepsilon L_V}),
\]

where the isomorphism is induced by the isomorphism \( \text{Id} + \varepsilon L_V \) from Lemma 4.2.33.
Proof. Since Lemma 4.2.35 holds for infinite dimensional cochain complexes with the
definition of \( \text{Sym}(\mathcal{F}_g) \) as in Definition 4.1.6 (i.e. with the completed projective tensor
product), we indeed have that the isomorphism \( \text{Id} + \varepsilon L_V \) from Lemma 4.2.33 induces
an isomorphism of \((\mathcal{O}(\mathcal{F}_g), \{S_g, -\})\) and \((\mathcal{O}(\mathcal{F}_{g+\varepsilon L_V}), \{S_g, -\} + \varepsilon \{I_g(L_V g), -\})\). Here,
the two differentials are those induced by \( Q_g \) and \( Q_g + \varepsilon [L_V, Q_g] \), as computed in
Equation (4.23) and what followed. This is the result. \( \square \)

Recall that although we’ve done the precise computations in the case of the
massless free scalar field, the same statement holds in the case of any free BV theory
with differential \( Q_g \).

Remark 4.2.38. This result follows almost directly from a theory exhibiting general
covariance; however, having isomorphisms written down explicitly and recognizing
their naturality when compared to the non-perturbative definition of general covari-
ce provides a sanity check, not to mention an enhanced perspective on quantities
like the stress-energy tensor.

The preceding theorem gives a concrete description of \( C^*(g, \text{Obs}^\text{cl}(X, \mathcal{F}_g)) \) with
differential
\[
d_{CE} = [-, -]_{\text{Vect}(X)} + \tau^V_{M_g} + \tau^V_{\mathcal{F}_g} + \{S_g, -\}
\]
as a bona fide Chevalley-Eilenberg cochain complex: in particular, checking this
theorem over a fixed \( g \in M \) and invoking \( \mathcal{D} \)-equivariance implies that \( \{S_g, -\} \) is
the differential over the entire diffeomorphism orbit \( \mathcal{D} \cdot g \subset M \). Similarly, seeing
how \( \{S_g, -\} \) varies over a formal neighborhood of \( g \) (i.e. expanding \( \{S_{g+\varepsilon h}, -\} \) in
consecutive orders of \( \varepsilon h \)) really grants us a view of the formal neighborhood of all of
\( \mathcal{D} \cdot g \): this is precisely equivalent to considering the formal neighborhood of \( g \) as an element of the quotient stack \([\mathcal{M}/\mathcal{D}]\).

### 4.2.4 A note on higher orders

We would like to make sense of the main theorem in the case that we do not “cut off” the orders of \( \varepsilon \) after only a linear perturbation. The linear perturbation gives the necessary data to understand the stress-energy tensor in the usual way; however retaining higher orders of \( \varepsilon \) to compute “higher” stress-energy tensors may be relevant, and the BV formalism gives an ideal way of interpreting and packaging that data. To put it plainly, we’d like to expand \( \{ S_{g+\varepsilon h}, - \} \) in more powers of \( \varepsilon h \).

**Construction 4.2.39.** A concrete jumping-off point here would be to consider that for a generally covariant theory, we have

\[
\frac{d^k}{dt^k} \int_X (f_t^* \varphi) \Delta_{f_t^* g}(f_t^* \varphi) \text{vol}_{f_t^* g} \bigg|_{t=0} = 0 \tag{4.33}
\]

for any \( k > 0 \). This is the general form of Equation (4.23). For now, let’s stick with \( k = 2 \). The above should have an analogous unpacking to the one following Equation (4.23); however, we then need to make sense of \( \frac{d^2}{dt^2} (f_t^* \varphi) \big|_{t=0} \). A short exercise in differential geometry gives us that

\[
\frac{d^2}{dt^2} (f_t^* \varphi) \big|_{t=0} = \frac{1}{2} L_V(L_V \varphi), \tag{4.34}
\]

as one might expect; the same equation holds for any \( k > 0 \), and the right side is in fact equal to \( \frac{1}{k} L_V^k \varphi \), where \( L_V^k \) denotes taking the Lie derivative with respect to the vector field \( V k \) times.

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We can now consider a similar computation to the one in Lemma 4.2.33, replacing $\mathbb{D}_2$ with $\mathbb{D}_3 := \mathbb{R}[\varepsilon]/(\varepsilon^3)$ and using the above identity, to figure out what operator $D_2$ makes the following square commute:

\[
\begin{array}{ccc}
\mathcal{F}_g \otimes \mathbb{D}_3 = C^\infty(X) \otimes \mathbb{D}_3 & \xrightarrow{Q_g+\varepsilon^0+\varepsilon^2} & \text{Dens}(X)[-1] \otimes \mathbb{D}_3 \\
\downarrow & & \downarrow \\
\mathcal{F}_g \otimes \mathbb{D}_3 = C^\infty(X) \otimes \mathbb{D}_3 & \xrightarrow{Q_g+\varepsilon D_1+\varepsilon^2 D_2} & \text{Dens}(X)[-1] \otimes \mathbb{D}_3,
\end{array}
\]

where we have renamed $D = D_1$ from above to emphasize the order of $\varepsilon$ it is associated to. Although a proof won’t be presented here (it is more arduous, though not more difficult than that of Lemma 4.2.33), we state the following lemma.

**Lemma 4.2.40.** The above square commutes if we choose

\[
D_2 = \frac{1}{2}[L_V^2, Q_g] - [L_V, Q_g]L_V.
\]

Moreover, $\tilde{\mathcal{F}}_g := (\mathcal{F}_g \otimes \mathbb{D}_3, Q_g)$ and $\tilde{\mathcal{F}}_{g+\varepsilon L_V} := (\mathcal{F}_g \otimes \mathbb{D}_3, Q_g+\varepsilon[L_V, Q_g] + \varepsilon^2(\frac{1}{2}[L_V^2, Q_g] - [L_V, Q_g]L_V))$, are cochain isomorphic via the map $\text{Id} + \varepsilon L_V + \frac{\varepsilon^2}{2} L_V^2$.

**Remark 4.2.41.** The operator $D_2 = \frac{1}{2}[L_V^2, Q_g] - [L_V, Q_g]L_V$ thus represents a sort of “higher” stress-energy tensor for a generally covariant theory, in the same way that $D_1 = [L_V, Q_g]$ did so in the first order case. It also satisfies some conservation property (analogous to $\nabla^\mu T_{\mu\nu} = 0$): otherwise, we wouldn’t have this cochain isomorphism of field theories. However, it would be harder to pin down a physical interpretation of the associated conservation law.

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**Remark 4.2.42.** Additionally, we could now update Theorem 4.2.37 so that it holds up to second order in $\varepsilon$: the isomorphism of observables is induced from the isomorphism $\text{Id} + \varepsilon L_V + \frac{\varepsilon^2}{2} L_V^2$ of the field theories. The proof is otherwise the same.

Moreover, it might be clear to the reader by now that these results can be generalized to arbitrarily high orders of $\varepsilon$. In that case, we can expand the differential as

$$Q_{g+\varepsilon L_V g} = Q_g + \varepsilon [L_V, Q_g] + \varepsilon^2 \left( \frac{1}{2} [L_V^2, Q_g] - [L_V, Q_g] L_V \right) + \varepsilon^3 D_3 + \ldots$$

on the fields–as long as the metric perturbation is induced by a vector field–and pick out $D_k$ for all $k > 0$ so that we get an analogous commutative square, with the isomorphism

$$\text{Id} + \sum_{k=1}^{\infty} \frac{\varepsilon^k}{k!} L_V^k.$$

Thus, the isomorphism of observables

$$\text{Obs}^{\text{cl}}(X, \tilde{\mathcal{F}}_g) \cong \text{Obs}^{\text{cl}}(X, \tilde{\mathcal{F}}_{g+\varepsilon L_V g})$$

remains true to all orders of $\varepsilon$, since we see from the above that all of the appropriate $D_k$ exist, regardless of how difficult they are to compute or interpret.

An interesting course of study would be to consider general expansions of $\{S_{g+\varepsilon h}, -\}$ in $\varepsilon h$: a first step in this case would be to generalize Lemma 4.2.25 to higher derivatives with respect to $t$. The expansion of $\{S_{g+\varepsilon h}, -\}$ would concretely define the $L_\infty$ action of $\mathfrak{g}_g$ on $\text{Obs}^{\text{cl}}(X, \mathcal{F}_g)$.

**Remark 4.2.43.** As well as checking for agreement with a generally covariant theory when $h = L_V g$ as we did above, we may want to consider perturbations in the
direction of various geometric flows. For example, it would be fruitful to consider field theories over metrics related by the Ricci flow, and a first step here would be to perturb a fixed metric $g$ in the direction of that flow.
CHAPTER 5
DETERMINANT LINES AND QUANTIZATION

5.1 An Introduction to Batalin-Vilkovisky Quantization

“It is not your obligation to finish the work –
but neither are you free to desist from it.”

– R. Tarfon (Chapters of the Fathers)

The path to understanding quantization has been a long and winding one. Physicists and mathematicians have put forward many techniques over the years; here, we focus primarily on the path integral method for quantum field theory forged by Richard P. Feynman. Note that this chapter is intended as a summary of what I’ve learned about this material: it is the next natural step with respect to what I’ve outlined in the rest of this thesis, and I make some comments at the end which chart a distinct path forward.

Quantum field theory, as opposed to classical field theory, tries to quantify the non-deterministic aspects of physics at small scales and high energies: it therefore hinges on the computation of expectation values. Given a space of fields $\mathcal{F} :=$
\[ \Gamma(X, F) \ni \phi \text{ equipped with a local action functional } S : \mathcal{F} \to \mathbb{R} \text{ and an observable } O : \mathcal{F} \to \mathbb{R}, \text{ its expectation value is heuristically defined as the “path integral”:} \]
\[
\langle O \rangle := \frac{1}{Z_S} \int_{\mathcal{F}} O(\phi) \exp\left(-\frac{S(\phi)}{\hbar}\right) D\phi. \tag{5.1}
\]

The partition function which specifies the probability measure is
\[
Z_S := \int_{\mathcal{F}} \exp\left(-\frac{S(\phi)}{\hbar}\right) D\phi. \tag{5.2}
\]

This definition of an expectation value follows the usual setup given in probability theory; however, a substantial obstacle in our way is that the purported Gaussian probability measure \( \exp\left(-\frac{S(\phi)}{\hbar}\right) D\phi \) is not quite a measure in the usual sense.

**Remark 5.1.1.** Note that we are really doing statistical/Euclidean field theory in this case, since the coefficient on \( S \) is \(-1\) and not \( i \): this corresponds to \((X, g)\) being Riemannian as opposed to Lorentzian. Some results are analogous, although effort is required to have a complete correspondence between the two.

It is a well known fact that a Lebesgue measure does not exist on an infinite-dimensional Banach space. Certain efforts have been successful in defining Gaussian (or Gaussian-like) measures on particular infinite-dimensional spaces: for example, heat kernel measures on infinite dimensional orthogonal groups [Gor00] or Wiener measures on spaces of piece-wise geodesics for Riemannian manifolds with certain curvature constraints [Lae13]. Nevertheless, these examples are carefully constructed only on a case-by-case basis and a unified definition of probability measures on infinite-dimensional spaces of fields evades us.
The techniques we’ll use to approach path integral quantization are those outlined in K. Costello’s book [Cos11] and in his books with O. Gwilliam, [CG16] and [CG21]. The strategy is to treat the well-known constant $\hbar$ as a *formal parameter*: this was done in the infancy of quantum field theory and has been amazingly successful in making physical predictions. Richard Feynman introduced the path integral in 1948 to deal with foundational interpretations of quantum mechanics [Fey48]. At around the same time he introduced his famous diagrams, which are tools invented to *perturbatively approximate* the path integrals in the parameter $\hbar$, since the non-perturbative computations cannot be be carried out without a well-defined measure. The path we’ll choose to approach this technique of perturbative approximation is to find a *homological alternative* to the usual approach to integration. Much of what follows is rooted in Chapter 2 of [Gwi12] and Chapter 6 of [CG21].

5.1.1 BV Quantization: Homological Integration

Recall that the key object in our earlier discussion of the classical BV formalism was the ring of functions on the derived critical locus of a function $S : M \to \mathbb{R}$ in Equation (4.1); namely, the dg commutative algebra

$$\mathcal{O}(\text{Crit}^\hbar(S)) \cong (\Gamma(M, \Lambda^\bullet TM), \triangledown dS),$$

where $\triangledown dS$ denotes the differential defined by interior multiplication with $dS \in \Omega^1(M)$ and $\Gamma(M, \Lambda^\bullet TM)$ is the *polyvector fields* on $M$, sometimes denoted $PV^\bullet(M)$. BV quantization produces a deformation of this complex which homologically encodes integration: in our case the focus is Gaussian and oscillating integrals.
For a finite dimensional $M$, the de Rham complex already provides a homological notion of integration. On a closed, oriented $n$-manifold, we have the following commutative diagram:

$$
\begin{array}{ccc}
\Omega^n(M) & \xrightarrow{\int_M} & \mathbb{R} \\
\downarrow{[-]} & & \downarrow{\langle[M],-\rangle} \\
H^n(M) & & \\
\end{array}
$$

If $\mu$ is a smooth probability measure on $M$, i.e. $\mu \geq 0$ and $\int_M \mu = 1$, then we can think of the cohomology class $[f \mu] \in H^n(M)$ for some $f \in C^\infty(M)$–where we think of $C^\infty(M)$ as the observables associated to the space of fields $M$–as the “expectation value” of the observable $f$.

It would be nice if we could employ this tactic in the case that $M$ is infinite dimensional; however, top forms don’t exist on such an infinite dimensional manifold and we’ve already seen how integration on such spaces is problematic anyway. The maneuver is to find a complex that is isomorphic to the de Rham complex for finite dimensional $M$, but is well defined for an infinite dimensional $M$, as well.

**Construction 5.1.2.** To build this complex, we will stick to a finite dimensional manifold $M$ for now. Consider again the polyvector fields $PV^\bullet(M)$, but only the underlying graded vector space–without the differential. For a fixed probability measure $\mu$ on $M$, contracting $\mu$ with a $k$-polyvector field results in an $(n-k)$-form, giving us a map of graded vector spaces:
\[
\begin{array}{cccc}
\Gamma(M, \Lambda^n TM) & \ldots & \Gamma(M, \Lambda^2 TM) & \Gamma(M, TM) \\
\downarrow \vee \mu & & \downarrow \vee \mu & \downarrow \vee \mu & \downarrow \vee \mu \\
C^\infty(M) & \ldots & \Omega^{n-2}(M) & \Omega^{n-1}(M) & \Omega^n(M).
\end{array}
\]

On the bottom row we of course have the usual exterior differential \(d\) on the de Rham complex. However, if \(\mu\) is everywhere nonvanishing, the above map is an isomorphism: i.e. we can in fact define a differential on the top row \(PV^\bullet(M)\) as \((\vee \mu)^{-1} \circ d \circ (\vee \mu)\), which will henceforth be called the divergence operator for \(\mu\) and denoted \(\text{div}_\mu\).

**Definition 5.1.3.** The **divergence complex** associated to a nonvanishing probability measure \(\mu\) on a manifold \(M\) is \(PV^\bullet(M)\) with differential \(\text{div}_\mu\).

**Remark 5.1.4.** For a finite dimensional \(M\), by construction we get that the cohomology of \(PV^\bullet(M)\) with differential \(\text{div}_\mu\) is isomorphic to \(H^\bullet_{dR}(M)[n]\), the usual de Rham cohomology shifted down by \(n\). Given \(f \in PV^0(M) = C^\infty(M)\), its cohomology class \([f]\), i.e. the function “up to divergence”, represents its expectation value.

For an infinite dimensional manifold \(M\), we don’t have this correspondence with de Rham cohomology; but luckily we can make sense of polyvector fields, with the only caveat coming from some functional analytic choices. To begin seeing what the divergence complex provides us with, let’s consider some finite dimensional examples.

**Example 5.1.5.** Although the Lebesgue measure \(dx\) on \(\mathbb{R}\) is not a probability mea-
sure, it is nonvanishing, so that we can use it to define a divergence complex on \( PV^\bullet(\mathbb{R}) \). In this case, it has only two nontrivial terms, and looks like

\[
\Gamma(\mathbb{R}, \Lambda^\bullet \mathbb{T} \mathbb{R}) = C^\infty(\mathbb{R})[\xi],
\]

where \( \xi = \partial_x \) is in degree \(-1\). Applying the definition from above shows that \( \text{div}_{dx} : f \partial_x \rightarrow \partial f / \partial x \). We can therefore write the divergence operator as

\[
\text{div}_{dx} = \frac{\partial^2}{\partial x \partial \xi}.
\]

We will often call this the “standard BV Laplacian”, and denote it \( \Delta \). The cohomology of this complex can be computed to be \( H^{-1} \cong \mathbb{R} \cong H^0 \).

**Example 5.1.6.** Because Gaussian integrals are the ones that are of primary interest to us, let’s consider on \( \mathbb{R} \) the Gaussian probability measure

\[
\mu_b := \frac{1}{\sqrt{2\pi b}} e^{-x^2/2b} dx.
\]

The underlying graded vector space is the same as in the previous example, but the divergence operator will of course be different. This is how:

\[
\text{div}_b(\xi) = (\vee\mu_b)^{-1} d(\vee\mu_b)(\xi)
= (\vee\mu_b)^{-1} d\left( \frac{1}{\sqrt{2\pi b}} e^{-x^2/2b} \right)
= (\vee\mu_b)^{-1} \left( \frac{1}{\sqrt{2\pi b}} \left( \frac{-x}{b} e^{-x^2/2b} \right) dx \right) = \frac{-x}{b}.
\]

Notice that the divergence operator thus “picks up on” the critical set of \(-x^2/2b = S(x)\), namely wherever \(-x/b = 0\). One can similarly compute its behavior on a general vector field:

\[
\text{div}_b : f \partial_x \rightarrow \frac{\partial f}{\partial x} - \frac{x}{b} f.
\]
We can write this in a coordinateless manner as

$$\text{div}_b = \Delta + \forall dS,$$

where $S$ is the quadratic functional in the exponent of the measure.

**Remark 5.1.7.** We thus see that the divergence operator in this case provides a perturbation of the differential $\forall dS$ on the classical observables (which encode the equations of motion) by the BV Laplacian $\Delta$, so that the full differential encodes our desired expectation values.

The above for the vector field $x^n \partial_x$ gives us

$$\text{div}_b(x^n \partial_x) = nx^{n-1} - \frac{1}{b} x^{n+1},$$

from which follows Wick’s Lemma (as in [CG21]):

**Lemma 5.1.8.** The expected value of $x^n$ with respect to the above Gaussian measure is zero when $n$ is odd and $b^k(2k-1)(2k-3) \cdots \cdot 3 =: b^k(2k)!!$ when $n = 2k$ is even.

The techniques here can be generalized to a Gaussian measure on $\mathbb{R}^d$ of the form

$$\mu_A := \frac{\sqrt{\det A}}{(2\pi)^d/2} e^{-A(x)/2} dx,$$

where $A(x)$ is a positive symmetric bilinear form on $\mathbb{R}^d$, to compute similar expectation values. It may be clear at this point that the motivation behind our focus on these particular types of integrals is because the path integrals for free theories (i.e. those with quadratic action functionals) as in equations (5.1) and (5.2) should be approached in the preceding way.
Remark 5.1.9. For any function $S : \mathbb{R} \to \mathbb{R}$, the volume form $e^S dx$ defines

$$\operatorname{div}_S = \Delta + \frac{\partial S}{\partial x} \frac{\partial}{\partial x}.$$ 

By recalling the Schouten bracket $\{-,-\}$ on polyvector fields, this can be written as

$$\operatorname{div}_S = \Delta + \{S,-\}. \quad (5.3)$$

The bracket $\{-,-\}$ is a Poisson bracket, showing us more generally how the above procedure reproduces the bracket $\{S,-\}$ on the classical BV graded ring of observables as well as the BV Laplacian $\Delta$ arising from quantization.

A similar formula for $\operatorname{div}$ is true for $S : \mathbb{R}^d \to \mathbb{R}^d$ and the associated volume form $e^S dx$, where $dx$ is the Lebesgue measure on $\mathbb{R}^d$, but for the additional bookkeeping of $S$ and $dx$ depending on more coordinates. For physics, this means that the formula holds for interacting theories (when $S$ has degree greater than or equal to three) as well as free ones (when $S$ is of quadratic degree). We are primarily concerned with free theories, so the infinite dimensional analogue of what we outlined in the preceding example will be our focus; however, one can find the appropriate finite dimensional formulas for interacting theories and how they relate to Feynman diagrams in the wonderful paper [GJF12].

Note moreover that if we introduce a (formal) parameter $\hbar$ into the volume form so that it has the form $e^{S/\hbar} dx$, then $\operatorname{div}_S = \Delta + \frac{1}{\hbar} \{S,-\}$, and so $\hbar \operatorname{div}_S = \hbar \Delta + \{S,-\}$. In this sense, the divergence operator more transparently provides a deformation of the classical differential $\{S,-\}$: the BV Laplacian term depending on $\hbar$ in the “classical limit”, i.e. when $\hbar \to 0$, becomes the usual differential on the dg ring of classical
observables.

An essential quality of the divergence operator for the measure $e^{S/h}dx$ is that it satisfies

$$\text{div}(\alpha \beta) = (\text{div}\alpha)\beta + (-1)^{\ell(\beta)} \alpha(\text{div}\beta) + (-1)^{|\alpha|} \hbar \{\alpha, \beta\}.$$ 

In other words, the divergence operator may be a differential, but it is not a derivation. When we take $\hbar \rightarrow 0$, the divergence operator becomes a derivation: the classical BV differential.

**Definition 5.1.10.** A Beilinson-Drinfeld (BD) algebra $(A^q, d, \{-, -\})$ is a graded commutative algebra $A^q$ which is flat as an $\mathbb{R}[[\hbar]]$-module and is equipped with a degree 1 Poisson bracket such that

$$d(\alpha \beta) = (d\alpha)\beta + (-1)^{\ell(\beta)} \alpha(d\beta) + (-1)^{|\alpha|} \hbar \{\alpha, \beta\}.$$  \hspace{1cm} (5.4) 

Here we have simply axiomatized the structure we found earlier by performing computations with the divergence complex. Notice that if we have a BD algebra $A^q$, we can find the “classical limit” $\hbar \rightarrow 0$ by computing

$$A_{\hbar=0} := A^q \otimes_{\mathbb{R}[[\hbar]]} \mathbb{R}[\hbar]/(\hbar).$$

In this case the above equation reduces to $d_0(\alpha \beta) = (d_0\alpha)\beta + (-1)^{|\alpha|} \alpha(d_0\beta)$, so that $d$ becomes a derivation in the classical limit. Given the rest of the structure intact, we see that $A_{\hbar=0}$ is a $P_0$ algebra, and so $A^q$ provides a “quantization” of that classical $P_0$ algebra.
**Definition 5.1.11.** A BV quantization of a $P_0$ algebra $A$ is a BD algebra $A^q$ such that $A_{h=0} = A$.

**Remark 5.1.12.** The schematic for the quantization of a classical system will therefore be to consider its classical observables $\text{Obs}^{cl}(X, \mathcal{F})$, a $P_0$ algebra, and find a BD algebra $\text{Obs}^q(X, \mathcal{F})$ which is its quantum observables, in the sense that $\text{Obs}^q(X, \mathcal{F})$ is a BV quantization of $\text{Obs}^{cl}(X, \mathcal{F})$. To be totally transparent, $\text{Obs}^{cl}(X, \mathcal{F})$ here is really $\widetilde{\text{Obs}}^{cl}$ from Theorem 4.1.33 (so that we actually have a $P_0$ algebra), but since they are weakly equivalent, we slightly abuse notation.

Note moreover that if we don’t evaluate the above on the entirety of $X$, but simply leave them as functors

$$\text{Obs}^{cl}(-, \mathcal{F}) : \text{Open}(X)^\text{op} \to \text{Alg}_{P_0},$$

and

$$\text{Obs}^q(-, \mathcal{F}) : \text{Open}(X)^\text{op} \to \text{Alg}_{BD},$$

then they define factorization algebras, and $\text{Obs}^q(-, \mathcal{F})$ is a factorization algebra serving as a BV quantization of the factorization algebra $\text{Obs}^{cl}(-, \mathcal{F})$. This is the central focus of the two volumes [CG16] and [CG21], by Costello and Gwilliam.

**Remark 5.1.13.** Actually defining such a perturbative BV quantization of a classical field theory requires a significant amount of work, in which the process of renormalization plays a starring role: this is the goal of the book [Cos11]. However, if we restrict our attention to the case of free field theories, then there sometimes exist different approaches to quantization which are interesting in their own right. This will be the content of the following section.
5.1.2 The Determinant Line

What follows will be a summary of some results on the determinant line which gets defined via the quantization of free field theories (for us, in the setting of elliptic theories), with a few comments at the end signifying where the work of this dissertation could be useful.

Notice that if $S$ is a quadratic function in the variable $\phi$ in the partition function of Equation (5.2), and if the fields $\mathcal{F}$ constitute a finite dimensional vector space $\mathbb{R}^d$, then the partition function is a finite dimensional Gaussian integral of the form

$$\int_{\mathbb{R}^d} \exp\left(-\frac{1}{2} (x, Ax)\right) \frac{dx^1 \cdots dx^d}{(2\pi)^{d/2}} = (\det A)^{-1/2}. \quad (5.5)$$

The integral is really only concerned with the determinant of the relevant positive symmetric bilinear form: in other words, if we’d like to make sense of the infinite dimensional “path integral” for a free field theory, a good first step would be to understand infinite dimensional determinants of elliptic operators defining associated symmetric bilinear forms. Moreover, in the case of a fermionic (or Berezin) integral, in which the variables are odd in a $\mathbb{Z}/2$-grading of a space of fields $V = V^0 \oplus V^1$, the partition function is even simpler:

$$\int_{V^1} \exp\left(-\frac{1}{2} (\eta, A\eta)\right) d\eta d\eta^\dagger = \det A. \quad (5.6)$$

**Construction 5.1.14.** Hence, it behooves us to consider infinite dimensional integrals where the integrand looks like $\exp\left(-\frac{1}{2} \langle \psi, D_g \psi \rangle\right) D\psi D\psi^\dagger$, where $D_g$ is the Dirac

---

1As this is purely heuristic, we will omit any explanation of the conjugate fermion variables.
operator defining the theory, with a potential dependence on the background metric $g \in \mathcal{M}$, and where $\langle - , - \rangle$ is the integration pairing on the space of fields.

For an operator $D : V \to W$ between finite dimensional $n$ vector spaces, we can define a determinant map $\text{det}D : \text{det}V \to \text{det}W$ (where $\text{det}V := \Lambda^n V$) as

$$\text{det}D(v_1 \wedge \cdots \wedge v_n) = Dv_1 \wedge \cdots \wedge Dv_n.$$ 

Identifying $\text{Hom}(V, W) \cong V^\vee \otimes W$, we have that $\text{det}D \in (\text{det}V)^\vee \otimes \text{det}W$. If $V = W$, then $(\text{det}V)^\vee \otimes \text{det}V \cong \mathbb{C}$ and $\text{det}D$ is the usual determinant. If $\ker D \neq 0$, then $\text{det}D = 0$. Moreover, we have an exact sequence

$$0 \to \ker D \to V \xrightarrow{D} W \to \text{coker} D \to 0,$$

which in turn shows us that

$$(\text{det}V)^\vee \otimes \text{det}W \cong (\text{det} \ker D)^\vee \otimes \text{det} \text{coker} D.$$ 

If $V, W$ are infinite dimensional, top forms on them don’t make sense; but if $\dim \ker D$ and $\dim \text{coker} D$ are finite dimensional, then we can take the right side as a definition. This is precisely what happens when $V$ and $W$ are infinite dimensional, say, Banach or Hilbert spaces, but the operator $D$ between them is Fredholm! Indeed, any elliptic operator on a compact manifold is Fredholm, so that this is very often the case for the classical field theories we consider.

We can then define a family of operators $D_b$ parameterized by a space $B \ni b$, where $D_b$ acts on a space of sections of a vector bundle over a (compact) manifold $X_b$. One would hope that the spaces

$$(\text{Det}D)_b := (\text{det} \ker D_b)^\vee \otimes \text{det} \text{coker} D_b$$

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fit together to form a smooth line bundle over $B$; but since the dimensions of $\ker D_b$ and $\coker D_b$ jump as $b \in B$ varies, this need not be true. However, Quillen showed in his seminal paper [Qui85] that for a special class of holomorphic connections on a Riemann surface and a careful construction, one can define the determinant line and smooth sections thereof.

Further restricting attention to the massless free fermion theory (highly detailed in [Rab19]), the Dirac operator $D_b$ has two components: $D_b^+$ acts on spinors of positive chirality and $D_b^-$ on those of negative chirality. Focusing on the case of elliptic Dirac operators on compact manifolds $X_b$, we get that $\ker D_b^+$ and $\coker D_b^+$ (which is equal to $\ker D_b^-$) are finite dimensional, the preceding construction can be invoked to compute smooth sections of $\Det D \to B$. Given our discussion of the fermionic partition function above, the idea now is that the expectation value of the observable $1$ in the theory defined by the operator $D_b$ should be equal to $(\det D^+)_b$, i.e. the section $\det D^+$ of $\Det D \to B$ evaluated at $b$.

**Remark 5.1.15.** The term “partition function” is in actuality misleading, since $\det D^+$ can only be viewed as a function on $B$ if we can trivialize $\Det D$ over it.

In [Rab20], Rabinovich employs the quantum BV formalism and Quillen’s work on the determinant line (along with the necessary spectral cut-offs for the operators $D_b$) to state the following interpretation of the fermionic partition function. Let $\text{Obs}^q$ be the BD algebra quantizing the $P_b$ algebra of classical observables for the massless free fermion, viewed as an infinite rank vector bundle over $B$, so that the fiber $\text{Obs}^q_b$ is the space of global section of a factorization algebra on $X_b$ in the usual sense.
**Theorem 5.1.16** (Theorem 1.1. in [Rab20]). There is a quasi-isomorphism of complexes of sheaves of $C^\infty_B$-modules

$$
\Phi : \Gamma(-, \text{Obs}^0) \to \Gamma(-, \text{Det}D),
$$

(5.7)

where $\Gamma(-, \text{Det}D)$ is the sheaf of smooth sections of the line bundle $\text{Det}D \to B$. Moreover, $C^\infty_B \subset \Gamma(-, \text{Obs}^0)$ and $\Phi(1) = \text{det}D^+.$

**Remark 5.1.17.** If $\text{Det}D \to B$ is trivialized, then $\Phi$ takes a family of observables to an honest function on $B$. In [Fre86], Freed explicitly computes key quantities associated to the determinant line bundle: e.g. it can be equipped with a metric and connection. In fact, Freed computes the curvature of this connection, which via the Atiyah-Singer Families Index Theorem can be directly linked with understanding anomalies (i.e. failures of classical symmetries to lift to quantum symmetries) of the quantized field theory.

**Remark 5.1.18.** The special case of interest given the context of this thesis is $B = M$, the space of Riemannian metrics on the manifold $X$. Moreover, a key question is: can $\text{Det}D \to M$ be trivialized *equivariantly* with respect to the diffeomorphism group $\mathcal{D}$ of $X$? Can the metric, connection, and its curvature also be defined in a $\mathcal{D}$-equivariant way? What are the obstructions to such definitions? I suspect that trying this out for special matter orientations on particular manifolds or spacetimes would reproduce interesting topological or observable data, as in [RW05], where anomalies for a matter field in a Schwarzschild-like spacetime are associated to Hawking radiation.
BIBLIOGRAPHY


