RESERVOIR ENGINEERING OF MULTI-PHOTON STATES IN CIRCUIT QUANTUM ELECTRODYNAMICS

Jeffrey M. Gertler

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RESERVOIR ENGINEERING OF MULTI-PHOTON STATES IN CIRCUIT QUANTUM ELECTRODYNAMICS

A Dissertation Presented
by
JEFFREY M. GERTLER

Submitted to the Graduate School of the University of Massachusetts Amherst in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

September 2022

Physics
ACKNOWLEDGMENTS

Chen, thank you for pushing me to do more than I thought I could. Every time we talk about physics I walk away with not only a better understanding but with a renewed passion for the work. The way you teach your students has given me an appreciation for the care and thought required to communicate hard ideas. Thank you for being patient with me when I drop factors of $2\pi$.

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Phil to name a few. You keep me focused on what matters in life. But in reality, the only reason I’m still dating you is to get access to...

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ABSTRACT

RESERVOIR ENGINEERING OF MULTI-PHOTON STATES IN CIRCUIT QUANTUM ELECTRODYNAMICS

SEPTEMBER 2022

JEFFREY M. GERTLER

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Ph.D., UNIVERSITY OF MASSACHUSETTS AMHERST

Directed by: Professor Chen Wang

The field of experimental quantum information has made significant progress towards useful computation but has been handicapped by the dissipative nature of physical qubits. Except for unwieldy and unrealized topological qubits, all quantum information systems experience natural dissipation, which limits the time scale for useful computation. However, this same dissipation, which induces errors requiring quantum error correction (QEC), can be used as a resource to perform a variety of important and unrealized tasks. In this thesis I discuss research into three uses of dissipation: manifold stabilization, state transfer, and QEC. With reservoir engineering, these tasks can be addressed in an autonomous, non-reciprocal, and hardware-efficient manner, improving on existing implementations in important ways. I discuss experimental results from a novel method of QEC and the use manifold stabilization to make a previously unrealized quantum state. These three lines of inquiry have potential for additional research, such as improving experimental designs, incorporating existing schemes in parallel with the experiments described here, and highlighting the need for additional theoretical work.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>ACKNOWLEDGMENTS</td>
<td>iv</td>
</tr>
<tr>
<td>ABSTRACT</td>
<td>vii</td>
</tr>
<tr>
<td>LIST OF TABLES</td>
<td>xi</td>
</tr>
<tr>
<td>LIST OF FIGURES</td>
<td>xiii</td>
</tr>
<tr>
<td>CHAPTER</td>
<td></td>
</tr>
<tr>
<td>1. INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>1.1 Outline of Thesis</td>
<td>1</td>
</tr>
<tr>
<td>1.2 Quantum Information</td>
<td>2</td>
</tr>
<tr>
<td>1.3 cQED Theory</td>
<td>3</td>
</tr>
<tr>
<td>1.3.1 Cavities</td>
<td>4</td>
</tr>
<tr>
<td>1.3.2 Types of Cavities</td>
<td>6</td>
</tr>
<tr>
<td>1.3.3 Transmon in the Dispersive Regime</td>
<td>7</td>
</tr>
<tr>
<td>1.3.4 Jump Operators and Natural Dissipation</td>
<td>9</td>
</tr>
<tr>
<td>1.4 Basic Experiments</td>
<td>10</td>
</tr>
<tr>
<td>1.4.1 Qubit Readout</td>
<td>10</td>
</tr>
<tr>
<td>1.4.2 Coherent Qubit Control</td>
<td>11</td>
</tr>
<tr>
<td>1.4.3 Qubit Characterization</td>
<td>12</td>
</tr>
<tr>
<td>1.4.4 Cavity Characterization: Number Splitting</td>
<td>13</td>
</tr>
<tr>
<td>1.4.5 Cavity Tomography</td>
<td>14</td>
</tr>
<tr>
<td>1.5 cQED Experimental Design</td>
<td>15</td>
</tr>
<tr>
<td>1.5.1 Chips: Design and Fabrication</td>
<td>18</td>
</tr>
<tr>
<td>1.5.2 Storage Cavities</td>
<td>18</td>
</tr>
<tr>
<td>1.5.3 Tabletop Microwave Setup</td>
<td>18</td>
</tr>
<tr>
<td>1.5.4 Cryogenic Microwave Setup</td>
<td>19</td>
</tr>
</tbody>
</table>
# 2. DRIVEN DISSIPATIVE SYSTEMS AND RESERVOIR ENGINEERING

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>Introduction and Four Wave Mixing</td>
<td>22</td>
</tr>
<tr>
<td>2.2</td>
<td>Driven Hamiltonian</td>
<td>23</td>
</tr>
<tr>
<td>2.3</td>
<td>Fourth-Order Terms</td>
<td>25</td>
</tr>
<tr>
<td>2.4</td>
<td>Adiabatic Elimination</td>
<td>27</td>
</tr>
<tr>
<td>2.5</td>
<td>Manifold Stabilization</td>
<td>29</td>
</tr>
</tbody>
</table>

# 3. GENERATION AND STABILIZATION OF PAIR COHERENT STATES

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.1</td>
<td>Pair Coherent States in cQED</td>
<td>33</td>
</tr>
<tr>
<td>3.2</td>
<td>Device Architecture and Transmon Fabrication</td>
<td>35</td>
</tr>
<tr>
<td>3.3</td>
<td>Measurement Setup</td>
<td>37</td>
</tr>
<tr>
<td>3.4</td>
<td>Measurements of Pair Photon Population Dynamics</td>
<td>37</td>
</tr>
<tr>
<td>3.5</td>
<td>Stabilized Manifold with Free $\delta$</td>
<td>39</td>
</tr>
<tr>
<td>3.6</td>
<td>Measurement of Photon Number Difference</td>
<td>40</td>
</tr>
<tr>
<td>3.7</td>
<td>Using PND for State Tomography</td>
<td>41</td>
</tr>
<tr>
<td>3.8</td>
<td>Pump Tuneup Procedure</td>
<td>44</td>
</tr>
<tr>
<td>3.9</td>
<td>Simulation for Fitting Rates</td>
<td>46</td>
</tr>
<tr>
<td>3.10</td>
<td>Effective $\gamma$ and Effective Kerr</td>
<td>46</td>
</tr>
<tr>
<td>3.11</td>
<td>Prediction of Distribution of $\delta$ States from Loss</td>
<td>49</td>
</tr>
<tr>
<td>3.12</td>
<td>Density Matrix Reconstruction</td>
<td>50</td>
</tr>
<tr>
<td>3.13</td>
<td>Summary and Outlook</td>
<td>51</td>
</tr>
</tbody>
</table>

# 4. AUTONOMOUS QUANTUM STATE TRANSFER

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.1</td>
<td>Introduction</td>
<td>52</td>
</tr>
<tr>
<td>4.2</td>
<td>Reciprocal? Hermitian?</td>
<td>53</td>
</tr>
<tr>
<td>4.3</td>
<td>Existing Directional Channels</td>
<td>53</td>
</tr>
<tr>
<td>4.4</td>
<td>Directional Pumping Channel</td>
<td>54</td>
</tr>
<tr>
<td>4.5</td>
<td>Autonomous Quantum State Transfer: Objective</td>
<td>55</td>
</tr>
<tr>
<td>4.6</td>
<td>Minimum System Size, 2x2 or 3x2?</td>
<td>55</td>
</tr>
<tr>
<td>4.7</td>
<td>Quantum State Transfer: cQED Scheme</td>
<td>56</td>
</tr>
<tr>
<td>4.7.1</td>
<td>Driven Josephson Circuit Hamilton</td>
<td>56</td>
</tr>
<tr>
<td>4.8</td>
<td>Simulation Parameters</td>
<td>62</td>
</tr>
</tbody>
</table>

# 5. AUTONOMOUS QUANTUM ERROR CORRECTION

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.1</td>
<td>Introduction</td>
<td>66</td>
</tr>
<tr>
<td>5.2</td>
<td>Error Correction</td>
<td>67</td>
</tr>
<tr>
<td>5.3</td>
<td>Code and PReSPA</td>
<td>67</td>
</tr>
<tr>
<td>5.4</td>
<td>PReSPA Spectroscopy and Rates</td>
<td>70</td>
</tr>
<tr>
<td>5.5</td>
<td>Hamiltonian and Driven System</td>
<td>72</td>
</tr>
<tr>
<td>5.6</td>
<td>Characterization of PReSPA: State Populations</td>
<td>76</td>
</tr>
<tr>
<td>5.7</td>
<td>Characterization of PReSPA: Coherence</td>
<td>79</td>
</tr>
</tbody>
</table>
LIST OF TABLES

<table>
<thead>
<tr>
<th>Table</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td></td>
</tr>
<tr>
<td><strong>Dissipative vs Hamiltonian stabilization.</strong> Comparison of multi-photon steady states under driven dissipative processes constructed with different operators, $\hat{O} = \hat{a}, \hat{a}^2$ and $\hat{a}\hat{b}$. For each of the three cases, we consider system dynamics following the master equation EQ 2.25 written in the rotating frame of the drives, where $\mathcal{D}$ is the Lindblad superoperator with $\hat{O}$ as its jump operator. In two-photon dynamics, the complex amplitude of the steady-states are analogously determined by the one-mode or two-mode squeezing drives countered by the corresponding two-photon loss rates and Kerr Hamiltonian confinement [1]. Notably, two steady states exist (with even or odd photon number parity $\Pi$) for the case of cat-state stabilization while there are infinitely many steady states (with different photon number difference $\delta$) for the case of stabilizing pair coherent states.</td>
<td>30</td>
</tr>
</tbody>
</table>

| 3.1   |      |
| **System parameters.** | |
| †The transmon $T_2^*$ reflects any $1/e$ decay time of Ramsey oscillations. | |
| ‡During pumping the equilibrium excited state population rises to $\sim 7\%$ largely due to the strength of the mixing tone. | 45 |

| 4.1   |      |
| **Simulation parameters.** Josephson energy of Junctions I and II, and the zero point fluctuations (ZPF) across each of them due to excitations in modes A, B, and R. ZPF of $B$ mode across junction I is a variable parameter in the simulation. | 64 |

| 4.2   |      |
| **Experimental proposal.** Frequencies, nonlinear couplings ($\chi$ matrix $/2\pi$), and relaxation times of modes $A$, $B$ and $R$ used in our simulation. For the $\chi$ matrix, diagonal terms are the mode anharmonicities and off-diagonal terms are dispersive frequency shifts. $\chi_{BR}$ and the reservoir lifetime ($= 2\pi/\kappa$) are variable parameters in the simulation. The loaded $T_1$ time of $A$ and $B$ (including the Purcell effect) are varied accordingly. The $T_1$ time of $|f\rangle_A$ is assumed to be half of that of $|e\rangle_A$ (which is listed here). *The frequency of cavity B can be chosen arbitrarily with no effect to the simulation. | 65 |
5.1 System parameters.
†The transmon $T_2^*$ reflects any $1/e$ decay time of Ramsey oscillations. The transmon displays a random switching behavior between two values of $\omega_q/2\pi$ 40 kHz apart, with a dwell-time split of approximately 85%:15%. The switching time scale is on the order of sub-seconds to seconds. All our experimental data reflects the averaged result from sampling the two ancilla frequencies.
‡ The cross-Kerr $\chi_{Ar}$ is derived from other measured parameters.
#Cavity A has a distinctive switching behavior between a regular state with stable $T_{2A}$ ($380 \pm 25\, \mu s$) and occasional “bad periods” lasting for 2-8 hours where $T_{2A}$ fluctuates wildly in the range of 200-340 $\mu s$. The reduced cavity coherence during these periods is not accompanied by any other changes of system parameters, and can be recovered by a Hahn echo pulse (with echo, cavity $T_{2A} \approx 390\, \mu s$ at all times).

5.2 Comparison of calculated and measured PReSPA transition rates. Using calibrated amplitudes and phases of the transmon comb and mixing comb in the experiment, the complex-valued PReSPA transition rates ($\lambda_n$ and $\Omega_n$) can be calculated based on Eq. (5.11, 5.12). Here the transmon comb amplitude $\Lambda_n$ is calibrated from ancilla Rabi oscillations, and the amplitudes of mixing tones are approximately converted to the dimensionless displacement parameter $\xi_n$ by measuring the Stark shift $\Delta_{\text{Stark}}$ induced by that single tone: $\xi \approx \sqrt{\Delta_{\text{Stark}}/2\alpha_q}$. The experimentally measured PReSPA transition rates result from fitting the time-domain dynamics of the photon-addition processes that do not contain phases. Note that $\Omega_0$ is intentionally set opposite to others in phase to suppress multi-tone mixing effects by destructive interference.
## LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1 <strong>Qubit readout.</strong> (a) Cartoon of readout resonator response to qubit state. Frequency of the readout resonator $\omega_r$ is shifted by the dispersive coupling $\chi_{qr}$ when the qubit is in $</td>
<td>e\rangle$ and remains unchanged when the qubit is in $</td>
</tr>
<tr>
<td>1.2 <strong>Number splitting.</strong> Example qubit spectroscopy result for state $</td>
<td>\psi\rangle = \sum_n \sqrt{P_n}</td>
</tr>
<tr>
<td>1.3 <strong>Example experimental simulation.</strong> Ansys HFSS simulation of the cavity in Fig 1.5. (a) Volume shown is the interior of the cavity where (b) sapphire chips are in dark blue and (c) Al thin film is in orange. (d) Light orange cylinders are coupling pins attached to the inside of the SMA ports.</td>
<td>15</td>
</tr>
<tr>
<td>1.4 <strong>Example experimental circuit.</strong> Circuit diagram of the HFSS design from Fig 1.5. Harmonic oscillators are each represented as a capacitor (parallel lines) and inductor (curly segments) in parallel, while transmon and transmon-like objects are represented by a capacitor and Josephson junction (crossed box). Squiggly arrows indicate the gain and loss of excitations through a coupling pin and transmission line. All of these elements are capacitively coupled. BBQ and EPR translate the HFSS model into circuit elements to make parameter predictions possible.</td>
<td>16</td>
</tr>
</tbody>
</table>
1.5 **Example experimental package.** An example experimental package and clamped sapphire chip. The body is machined from Al and contains a hollow cavity. SMA ports on the side allow for transmission lines to be connected to send and receive microwave signals. The chip on the right is diced from a wafer of sapphire and has thin film Al patterned on to it. The upper dark feature is the transmon while the longer lower dark strip is the strip-line readout resonator. The clamp both holds the chip mechanically and thermalizes it. The chip is inserted into the larger open hole visible on the left of the cavity. ................................................. 17

1.6 **Fridge diagram.** The experimental packages are installed in an Oxford Triton-500 dilution refrigerator. Our device is mounted at the mixing chamber (MXC) stage of the refrigerator with a nominal temperature of about 10 mK. Coherent control signals for both cavity $A$ and the transmon are generated by IQ modulation. We use a first-stage Traveling Wave Parametric Amplifier [2] (TWPA) from MIT Lincoln Laboratory, as well as a High-Electron-Mobility Transistor (HEMT) amplifier at the 4K stage. The experiment is housed inside a MuMETAL shield to protect it from external magnetic fields. All lines connected to the experiment are filtered with a K&L 10 GHz low-pass filter and an Eccosorb low-pass filter. ................................. 21

3.1 **Pair coherent state generation: system and protocol.** (a) Cartoon of the 3D cQED system containing two high quality factor (Q) cylindrical post cavities $a$ and $b$ (blue and orange), the transmon ancilla $a$ (pink), and the stripline low-Q reservoir $r$ (green). (b) mixing drive $p$, with frequency $\omega_p = \omega_a + \omega_b - \omega_r$, coherently converts reservoir excitations to pairs of $a$ and $b$ excitations mediated by the ancilla junction. The reciprocal process is also mediated by the mixing tone. With adiabatic elimination the reservoir dissipation (dotted green arrow) is mixed into an effective pair photon dissipation (dotted blue/orange arrows). Eliminating the reservoir, the storage cavity dynamics are shown in the dotted box. All junctions shown are the same ancilla junction. (c) Cartoon of the mode frequencies and relative line-widths. Strong off-resonance CW mixing drive $p$ and weak on-resonance reservoir drive $d$ are shown as vertical arrows. (d) Cartoon of pair coherent state distribution over $\delta = 0$ Fock states. 33
3.2 **Pair coherent state characterization: population and time dynamics.** (a) Populations of entangled cavity states $|0,0\rangle$ through $|4,4\rangle$ after 15 $\mu$s of pumping sweeping the frequency of the cavity drive. We vary the difference between the mixing and reservoir drives while holding the sum constant to ensure that excess reservoir photons do not spoil the pumping condition. Cavity populations are measured by a number splitting [3], selective, ancilla transmon spectroscopy measurement after playing cw mixing and drive tones. (b) Transmon spectroscopy after 15 $\mu$s of pumping at $\omega_d = 0$. Dark vertical bars correspond to $\delta = 0$ states while light bars correspond to error states caused by single photon loss. (c) Population measurements performed starting in vacuum and pumping with both cw tones for a variable time. 1QuTiP fits of this data are used to extract the pair photon pumping rate $\epsilon_{ab} = 88 \text{kHz}$. (d) Population measurements performed starting with the $t = 15 \mu$s pair coherent state and playing only the mixing tone for variable time. This measurement demonstrates the pair dissipations ability to remove pairs of photons significantly faster then the single photon loss rate. QuTiP fits of this data are used to extract the pair photon dissipation rate $\kappa_{ab} = (11.85 \mu s)^{-1}$ which is significantly faster than either cavities single-photon loss rate.

3.3 **Pair coherent state with $\Delta \neq 0$.** (a) Level diagram of states with $\delta = -1, 0, 1$ connected by single photon loss operators. Each branch experiences the pair photon drive and dissipation stabilizing pair coherent states of equal complex amplitude $\gamma$. (b),(c) Number splitting experiment of the $\delta = 1$ ($\delta = -1$) pair coherent state. This measurement is performed by preparing a $|10\rangle$ ($|01\rangle$) Fock state with SNAP gates [4] and then performing the exact same procedure as shown in Fig. 3.2b creating the pair coherent distribution of $|\gamma, \delta = 1\rangle$ ($|\gamma, \delta = -1\rangle$). For each plot, dark vertical lines indicate the desired states while light vertical lines mark error states due to single photon loss or imperfect initial state prep.
3.4 **Pair coherent state tomography.** (a) Joint Wigner tomography of the $\delta = 1, 0, -1$ states after 15 $\mu$s of pumping. Below are corresponding theoretical plots for ideal pair coherent states with $\delta = 1, 0, -1$ after 1 $\mu$s of Kerr evolution. (b) Tomography of the $\delta = 0$ state with $|\alpha| = |\beta| = .3$ sweeping over the angle of both displacements. Position of the peaks and valleys along the diagonal cut gives the phase of $\gamma$. (c) Joint Wigner tomography measurement protocol. After preparing a pair coherent state with two tone pumping, we displace the cavities and then simultaneously and selectively rotate either all even PND states or all odd PND states. (d) Phase and amplitude of gamma over time as the state stabilizes. Phase of $\gamma$ is inferred from cuts of (b) at different times. $|\gamma|$ is calculated from the value that best fits the population data in Fig. 3.2c. 

3.5 **$\delta$ state hopping from single photon loss.** a) Plot of the $C_\delta$ coefficients for different $\delta$ values assuming a pair coherent state amplitude of $\gamma = 1.25$ and loss rates of $(530\mu s)^{-1}$ and $(216\mu s)^{-1}$ for cavity a and b respectively. b) Diagrammatic representation of single photon loss causing hopping to different $\delta$ states where the right (left) pointing red arrows represent a photon loss event from cavity b (a) with the respective rates as indicated on the figure.

3.6 **Density matrix reconstruction** (a)-(f) Comparison of the 2-D Wigner cuts from experiment (left) against the reconstructed density matrix (right) for a) $Re(\alpha) vs. Re(\beta)$, b) $Re(\alpha) vs. Im(\alpha)$, c) $Re(\alpha) vs. Im(\beta)$, d) $Im(\alpha) vs. Im(\beta)$, e) $Im(\alpha) vs. Re(\beta)$, and f) $Re(\beta) vs. Im(\beta)$ respectively. (g) Color plot of the reconstructed density matrix with only the elements corresponding to the $\delta = -1, 0, 1$ states.

4.1 **A schematic diagram of AQST.** Encoded quantum state $|\psi\rangle$ is spontaneously emitted from a subsystem $A$ and fully absorbed by another subsystem $B$. This is realized via a directional coupling channel which is blind to $|\psi\rangle$. 

xvi
4.2 Implementation of AQST in circuit QED. (a) Effective circuit diagram including a transmon qutrit $A$, a storage cavity $B$, a reservoir transmon $R$, and auxiliary elements for state preparation and readout. (b) Energy-level diagram that shows the state transfer paths. The quantum state initially encoded in $A$ is driven to a pair of virtual states by slightly-detuned Rabi drives with equal rate $\Omega$ (straight arrows), and subsequently decay to the final states in $B$ by reservoir dissipation (twisted arrows). (c) Numerical results of transferring an equator state, including decoherence, showing fidelity of instantaneous state $\rho$ against target state $\sigma$, $F = \left[ \text{Tr}(\sqrt{\rho\sigma}\sqrt{\rho})^{1/2} \right]^2$ as a function of time during the transfer. Simulation parameters correspond to $A$ and $R$ having frequencies of 5.6 GHz and 8 GHz, anharmonicities of 78 MHz and 210 MHz, and a dispersive $(ZZ)$ coupling of $\chi_a = 4.2$ MHz between them. Different color curves are simulated for different $\chi_b$ and their corresponding optimal $\kappa$. Inset shows the ideal-case infidelity due to rotation of the virtual states in the drive frame, which scales as $(\kappa/\chi_b)^{-2}$.

5.1 Autonomous quantum error correction: concept and protocol. (a), A classical bit is stored in isolated points in configuration space, which are local energy minima stabilized by dissipation in all directions. In comparison, a logical qubit $|\psi\rangle$ is encoded in a continuous two-dimensional code space $\mathcal{C}$ designed in a way that both natural errors (which map the states to the error space $\mathcal{E}$) and engineered dissipation are only allowed perpendicular to $\mathcal{C}$. (b), Schematic of the circuit QED device composed of storage cavity $A$, transmon ancilla $q$ and reservoir resonator $R$. (c), AQEC scheme against single-photon loss illustrated in a level diagram (not to scale). The level indices refer to $A$, $q$, $R$ sequentially. A continuous wave (cw) “transmon comb” is applied to resonantly excite the transmon ancilla with a Rabi rate $\lambda$ (magenta arrows), selectively targeting the four even-parity states (red levels) when $\chi_q \gg \lambda$. Similarly, a cw “mixing comb” targets the $|2n, e, 0\rangle \leftrightarrow |2n + 1, g, 1\rangle$ transitions with an equal Rabi rate $\Omega$ (black arrows). Both combs are composed of four tones equally spaced by $2\chi_q$ as indicated by the varying slopes of the magenta and black transitions. Spontaneous decay of the reservoir $R$ converts the quantum state back to the code space (blue levels) without leakage of which-path information; note the identical slopes of the light-green arrows. The numbered labels show the error and recovery sequence $\text{(1)}-\text{(4)}$ for one of the four parallel paths (using the initial state $|1g0\rangle$ as an example).
5.2 Characterization of the PReSPA operator: photon population conversion. (a) Control pulse sequence of a transmon spectroscopy measurement to infer the cavity photon distribution after PReSPA: We initialize cavity $A$ to a specific initial state using an Optimal Control Theory (OCT) pulse \[5\], apply PReSPA for a variable time $t$, apply a spectrally-selective $\pi$-pulse to transmon $q$ at a variable detuning $\Delta \omega_q$, and measure the transmon excitation probability $P(\Delta \omega_q, t)$. (b) Transmon spectroscopy data $P(\Delta \omega_q, t)$ for cavity $A$ initialized in vacuum. The bright feature is shifted from $\Delta \omega_q = 0$ to $-\chi_q$ over time, showing the $|0\rangle_A \rightarrow |1\rangle_A$ conversion. (For $0 < t < 20 \mu s$ an additional delay time of $20 \mu s$ is inserted between PReSPA pumps and the transmon $\pi$-pulse to improve clarity of the spectroscopy data by allowing the partially-excited transmon to relax.) (c) Cuts of (b) at $t = 0$ and $25 \mu s$ (grey dashed line). (d), (e) $P(\Delta \omega_q)$ for $A$ initialized in an even-parity cat state at (d) $t = 0$ and (e) $t = 25 \mu s$. All four spectroscopy peaks corresponding to even photon numbers (red) are shifted by $-\chi_q$ after $t = 25 \mu s$ indicating odd photon numbers (blue). (f) Probability of achieving the target cavity state $|2n + 1\rangle_A$, as measured by $P(t)$ for fixed $\Delta \omega_q = -(2n + 1)\chi$, for cavity $A$ initialized in $|2n\rangle_A$. Error bars reflect standard error of the mean. These four time-domain curves are fitted using a numerical model of the cascaded pumping process, resulting in $\Omega = 92, 88, 87, 85$ kHz; and $\lambda = 28, 27, 27, 26$ kHz, respectively. The inset shows a block of the cavity process $\chi$ matrix for $25 \mu s$ of PReSPA. The matrix elements $\chi_{nn,n'n'}$ are calculated from transmon spectroscopy measurements from all pairs of initial Fock states $|n\rangle_A$ and final Fock states $|n'\rangle_A$. . . . . . . . . 71
5.3 PReSPA spectroscopy. (a), (b) Control pulse sequence for two-dimensional (2d) spectroscopy to find the resonance conditions for the PReSPA mixing comb and transmon comb. We prepare an even-parity Fock state (|0⟩, |2⟩, |4⟩, or |6⟩), apply PReSPA for a fixed time (12 µs) with varying detunings of the transmon comb (Δq) and the mixing comb (Δm) in an attempt to activate dissipative photon addition. After a 1 µs wait time for the reservoir to relax, we either (a) selectively π-pulse the transmon conditioned on Cavity A being in the targeted final state (|1⟩, |3⟩, |5⟩, or |7⟩) or (b) skip this pulse (for a background measurement), and proceed to read out the transmon state. The difference between the two measurements informs the likelihood of successful photon addition. (c), 2d PReSPA spectroscopy data: likelihood of photon addition as a function of the comb detunings (Δq and Δm) for the |0⟩ to |1⟩ transition. Note that the linewidth of the four-wave-mixing transition is an order of magnitude greater than that of the transmon excitation due to the short reservoir T1r. We can repeat this procedure to find all four sets of transition frequencies. (d), Cartoon spectrum of PReSPA drive frequencies. Four transmon drives, left, and four mixing drives, right, compose PReSPA. The colored ticks indicate the actual transition frequencies while the vertical black bars show the microwave drive frequencies in PReSPA. The transmon drive for the |0⟩ to |1⟩ conversion process is approximately at the Stark-shifted transmon frequency, ωq − ΔStark, and the |0⟩ to |1⟩ mixing drive is near ωA + ωR − ωq + ΔStark. Because of the equal frequency spacing η in each comb and the unequal frequency spacing between the transitions with different photon numbers (due to the 6th-order non-linearity, χ′q), not all drives can be placed exactly on resonance. Experimentally, we settle for η slightly greater than 2χq, and Δq = Δm slightly smaller than ΔStark to compensate for the effect of χ′q.
5.4 Characterization of PReSPA operator: preservation of coherence. (a), Cavity Wigner tomography of six even-parity superposition states, \((|n\rangle_A + |m\rangle_A)/\sqrt{2}\), prepared by OCT pulses, as input states for PReSPA. (b), Wigner tomography of the six corresponding output states after 25 \(\mu s\) of PReSPA, which are converted approximately to odd-parity superpositions \((|n'\rangle_A + |m'\rangle_A)/\sqrt{2}\), with \(n' = n + 1\), \(m' = m + 1\). The Wigner function, a quasi-probability distribution in the oscillator phase space, is directly measured via photon-number-parity measurements after variable cavity displacements [6]. From each Wigner function we reconstruct the density matrix, and the most significant off-diagonal element is \(\rho_{n,m}\) (or \(\rho_{n',m'}\)) that reflects the coherence between \(|n\rangle\) and \(|m\rangle\) (or \(|n'\rangle\) and \(|m'\rangle\)). We also perform similar measurements with permutations of odd-parity superpositions as input states (not shown). The \(\chi\) matrix block characterizing PReSPA coherence, \(\chi_{nm,n'm'}\), are shown next to the vertical arrows and to a good approximation are equal to \(\rho_{n',m'}/\rho_{n,m}\). The deviations of \(\chi_{nm,n'm'}\) from unity reflects the infidelity of the PReSPA process.

5.5 Cavity Wigner and PReSPA Ramsey measurements. (a) Experimental Wigner function \(W(\alpha)\) of \(|0_L\rangle\), acquired by applying a cavity displacement operation \(\hat{D}_\alpha = \exp(\alpha \hat{a}^\dagger - \alpha^* \hat{a})\) with variable complex amplitude \(\alpha\) followed by an ancilla-assisted photon-number-parity measurement (which is composed of two \(\pi/2\) pulses of the ancilla and a delay time of \(\pi/\chi_q\) and an ancilla readout [6, 7]). The Wigner function rotates around the origin over time at a rate proportional to the frequency difference between \(|1\rangle\) and \(|5\rangle\) in the rotating frame of the experiment. (b), Measured Wigner function values at a fixed phase-space position (as indicated by the cross in (a), at \(\alpha = 0.75\)) as a function of time under PReSPA. Analogous to a qubit Ramsey measurement, this cavity PReSPA Ramsey experiment can be used to efficiently track the phase evolution of any two-component superposition states using the interference effect enabled by the coherent cavity displacement \(\hat{D}_\alpha\) before readout. The exponential envelope of the sinusoidal fit indicates the rate of decay for the coherence between \(|1\rangle\) and \(|5\rangle\) under the correction of PReSPA. Similar measurements are applied to various superposition states to provide direct calibration of the frequencies and phases of these states under PReSPA. PReSPA enhances the ability to use such Ramsey measurements at high photon numbers since it approximately preserves photon number distributions in the cavity.
5.6 Process $\chi$ matrix block for 25 $\mu$s of PReSPA. The matrix converts elements of input density matrices, top axis, to output density matrix elements, left axis, expressed in the Fock state basis. (a), Amplitude values of the $\chi$ matrix elements. The upper left block ($\chi_{nn,mm}$) describes the conversion of diagonal elements of the input and output density matrices, which is associated with transfer of photon occupation probabilities calculated from transmon spectroscopy experiments. The lower right block ($\chi_{nm,kl}$) describes the conversion of relevant off-diagonal elements of density matrices, which is calculated from Wigner tomography and density matrix reconstruction [8]. The greyed blocks are assumed to be zero due to the absence of interference between the four conversion paths in PReSPA. (b), Phase values of the $\chi$ matrix elements for the lower right block in (a). For best illustration of the PReSPA process, the phases are reported in the full rotating frame where all Fock states have zero energy. Values in grey are measured but not statistically significant as the corresponding amplitude value is not large enough. In this frame, as prescribed by EQ (5.5), PReSPA requires zero phase for the six $\chi_{nm,(n+1)(m+1)}$ elements representing the coherence of the even-to-odd conversion process, which is accomplished by our PReSPA calibration. The diagonal elements $\chi_{nm,nn}$ representing the preservation of odd-state superpositions should have zero relative phase by definition. Their systematic deviation from zero was caused by parameter drift in the experiment as that block of data was acquired at a later time than the earlier rotating-frame calibration.

5.7 AQEC performance. (a)-(c), Cavity Wigner tomography for the six cardinal-point states of our Truncated 4-component Cat (T4C) code. We measure the states at (a) $t = 0$, (b) after 143 $\mu$s of free evolution, and (c) after 143 $\mu$s of AQEC using PReSPA. Center points of the Wigner function show a measurement of state parity, which is well-preserved under PReSPA. The Wigner functions have been manually rotated into the correct rotating frame for the ease of comparison. (d), Quantum process fidelity of information storage with the AQEC protocol (green) compared with other reference methods performed with the same physical system. We use each method to store quantum states corresponding to the six cardinal points of the Bloch sphere, wait for a variable time $t$ (or apply PReSPA for the AQEC curve), retrieve the state via the transmon using a decoding unitary, and perform quantum state tomography of the transmon. Process fidelity is calculated from the quantum state fidelity for the corresponding six measurements. All curves are fitted to the model $F(t) = .25 + Ae^{-t/\tau}$ to extract characteristic times and error bars are one standard error.
CHAPTER 1
INTRODUCTION

1.1 Outline of Thesis

This document consists of an introduction, four main body chapters representing publications from my work, and a concluding section:

1. Introduction. I discuss the cQED theory and experimental architecture as used in my research. Different methods of coherent qubit control are introduced, as well as experimental tools used in the laboratory.

2. Driven Dissipative Systems and Reservoir Engineering. I explain the tool of reservoir engineering and some of its potential applications. I show that using the transmon junction, we can engineer complex Hamiltonian terms, which can be seen experimentally as complex dissipative jump operators. These high order Hamiltonian and dissipative terms can be used to stabilize manifolds of interesting quantum states.

3. Generation and Stabilization of Pair Coherent States. I discuss experimental results for the generation and stabilization of pair coherent states as well as the development of a novel coherence measurement. These results help further the development of new error-protected bosonic codes.

4. Autonomous Quantum State Transfer. State transfer, an important operation in scaling to large quantum computers, can be performed with reservoir engineering. I introduce the concept of a directional pumping channel and discuss its utility in constructing dissipative quantum maps. The scheme for using this tool to perform state transfer is discussed and numerically simulated. This work has been published [9]
5. **Autonomous Quantum Error Correction.** One of the key challenges for quantum computing is quantum error correction (QEC). While experimental implementations of QEC exist, they rely largely on a schedule of error syndrome measurement and active correction, incurring an error overhead and requiring complex hardware. Using reservoir engineering, I implemented a dissipative error correction scheme that autonomously corrects single-photon loss in a bosonic qubit. This work has been published [10].

6. **Conclusion.** I summarize my body of work and its relevance to the field. Future experimental goals for these projects, and other interests beyond the scope of my dissertation are discussed.

1.2 **Quantum Information**

In 1981 Massachusetts Institute of Technology convened the first Physics of Computation Conference, at which Richard Feynman stated some of the first serious considerations about the practicality of classical computers in a quantum universe. “Nature isn’t classical, dammit, and if you want to make a simulation of Nature, you’d better make it quantum mechanical, and by golly it’s a wonderful problem, because it doesn’t look so easy” [11]. His first point was the, now clear, observation that arbitrarily accurate simulations of nature are difficult to compute on a classical computer. The complexity of quantum systems scales exponentially with system size, meaning that there will always be problems too difficult for even the largest linearly scaling classical computer. Even today the impact of Feynman is widely acknowledged by leaders in the field [12].

Feynman’s insight sparked many ideas on the building of a quantum simulator that remain relevant to this day [13], but an even greater consequence is the concept that quantum systems can be used to solve a wide range of difficult problems. With Shor’s discovery of a quantum algorithm for integer factorization [14], followed by Grover’s search algorithm [15], an entirely new use for quantum machines was born. These
algorithms and the growing interest in digital quantum computation (in contrast to quantum simulation) revealed a major obstacle for quantum computers: dissipation.

A fundamental part of the complexity Feynman was concerned about is that, like their classical counterparts, quantum systems undergo energy relaxation and experience stochastic errors. Storing and processing information in a finite quantum system inevitably involves physical processes that spoil that quantum information. These errors must be corrected in order for us to perform quantum algorithms of useful complexity. Although quantum supremacy has been declared [16], the need for better Quantum Error Correction (QEC) is more clear than ever. The theoretical framework for QEC was solved early in the development of quantum information processing [17, 18] with discussions of fault tolerance following quickly [19], but experimental solutions remain elusive [20].

Robust quantum computation requires fault tolerance, the ability to correct for inevitable errors in information storage and processing. For information storage the greatest challenge is the natural dissipation of the physical system components. For many qubits, the naturally occurring dissipation and decoherence terms push the system into either a ground state or a mixed state, destroying encoded quantum information. While these dissipation channels usually act as errors in our systems, introducing new engineered dissipation terms could serve a function for quantum information processing [21]. Proposals for, and experiments on, the use of reservoir engineering include qubit cooling [22], single-qubit state stabilization [23], and multi-qubit entanglement [24]. The common theme of these is reliance on dissipation to introduce some non-reciprocality into the system. While most qubit gates are unitary (and therefore bi-directional), the ability to construct purely directional channels allows for a move away from external measurement-based feedback loops [25] toward autonomous processes.

1.3 cQED Theory

The rate of progress towards universal quantum computation has dramatically increased through the development of circuit quantum electrodynamics (cQED). In
this new field of study, fabricated superconducting circuitry is used to construct quantum resonators and anharmonic artificial atoms [26]. Using Josephson junctions, non-linear couplings are introduced between these modes, building a platform for quantum computation. The control over nearly every system parameter has led to significant academic and industrial investments and a promise of tailored scalability.

CQED at its core is an extremely useful framework for implementing novel quantum circuits with flexibility and precision. The field is remarkably well documented from the seminal work of Devoret [27] to more modern treatments [28]. I have benefited profoundly from the theses of many students of Devoret and Schoelkopf including Jacob Blumoff [29], Matthew Reagor [30], Brian Vlastakis [31], Matthew Reed [32], and David Schuster [33]. Other formal treatments also covered the subject [34, 26, 35, 36]. Recently a number of resources that summarize the body of work on bosonic quantum information processing have emerged, diving deeper into the specifics of my sub-field [37, 38]. I will introduce the field and discuss important points for better understanding of this work and will show how my work builds on the existing literature.

1.3.1 Cavities

Harmonic oscillators are fundamental to classical physics and their quantum analogue is one of the first examples taught to students. Moving from the classical to the quantum picture with oscillators is one of the clearest examples of quantization. In this work I am primarily concerned about cavities where the quantization of electromagnetic fields in mostly confined regions of space support oscillations of specific frequencies. With quantization we understand that the energy of these systems will be proportional to the number of excitations, or photons, in each mode and to the frequency of the modes:

$$\hat{H}/\hbar = \sum_m \omega_m (\hat{a}_m^\dagger \hat{a}_m + \frac{1}{2}),$$

(1.1)

where $\hat{a}_m$ is the annihilation operator for the mode with frequency $\omega_m$. The zero point offset of $\frac{1}{2}$ can be ignored when looking at the dynamics of the system. The geometry
of these cavities can be as simple as a box, and they can be made from a range of materials. If these cavities are coupled to the external world, through a coupling pin and transmission line, for example, we can think about how external fields can affect the system. We can apply an on-resonance drive to a single mode with annihilation operator \( \hat{a} \), frequency \( \omega \), and time-dependent complex drive strength \( \epsilon(t) \)

\[
\hat{H}/\hbar = \omega \hat{a}^{\dagger} \hat{a} + \epsilon(t)(\hat{a} + \hat{a}^{\dagger}),
\]

which seems complicated, but its effects can be easily understood. First we need to move to a rotating frame that matches the cavity frequency with the unitary operator \( \hat{U} = \exp(-i\omega \hat{a}^{\dagger} \hat{a} t) \). Unitary transformations change states and Hamiltonians in the following way

\[
|\psi\rangle \rightarrow \hat{U} |\psi\rangle \\
\hat{H} \rightarrow \hat{H}' = \hat{U} \hat{H} \hat{U}^{\dagger} - i\hbar \dot{\hat{U}} \hat{U}^{\dagger}
\]

Our Hamiltonian can be rewritten as

\[
\hat{H}' = \epsilon(t)(\hat{a} e^{i\omega t} + \hat{a}^{\dagger} e^{-i\omega t}).
\]

We can use the Magnus expansion (a exponential solution to a first-order homogeneous linear differential equation for a linear operator) \cite{39} to write the unitary that evolves this system in time

\[
\dot{\hat{U}}_t = \exp(\alpha \hat{a}^{\dagger} + \alpha^{*} \hat{a}) \equiv \hat{D}_\alpha,
\]

where \( \alpha = \int_{t=0}^{t} d\tau \epsilon(\tau) e^{i\omega \tau} \). The displacement operator \( \hat{D}_\alpha \) is fundamental to understanding the behavior of quantum bosonic modes. If we apply a displacement to a vacuum state, we get a coherent state \cite{40}, a “classical” state of the quantum system.
\[ |\alpha\rangle = \hat{D}_\alpha |0\rangle = e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle \]  \hspace{1cm} (1.6)

Here \(|n\rangle\) are the photon number Fock states of the oscillator. We can think of the amplitude \(\alpha\) as a position and momentum \((Re[\alpha]\) and \(Im[\alpha]\)) of the oscillator or the average photon number, \(\langle \hat{n} \rangle = |\alpha|^2\), and phase \(\theta(\alpha) = \tan^{-1}(Im[\alpha]/Re[\alpha])\) of the photons. Displacements are additive so if our only tool is a displacement, all we can produce are coherent states. As we will see, coherent states are superb building blocks for complicated quantum states, but a single coherent state is largely useless for quantum information applications. We need non-linearities in the system for interesting quantum physics.

### 1.3.2 Types of Cavities

In this work I experimentally use two types of cavities: strip-line resonators and cylindrical post cavities. Strip-line resonators are built from a thin film of aluminum \((\sim 50 \text{ nm})\) on a substrate (sapphire) patterned in a long, thin line \((\sim 100 \mu\text{m} \text{ by } \sim 1 \text{ cm})\). When placed inside a cylindrical cavity, this strip acts like a section of transmission line with cuts to the center pin on either side. This object supports a series of modes proportional to the length of the strip and integer multiples. This type of resonator typically does not have exceptional coherence but is extremely convenient to couple to both the qubit and external transmission lines. In this thesis, strip-line resonators are used exclusively for readout and as Markovian reservoirs, never for storing interesting states.

Cylindrical post resonators also are analogous to the altered transmission line, but the center is open on one side and shorted to ground on the other. They are milled from a solid piece of Al and the mode frequency is proportional to the length of the extending cylinder. This cavity will also support multiple modes but the first excited mode will be 3 times the bare frequency. This allows for decreased frequency conflicts between different system modes. When designed correctly, a cylindrical post cavity can have long coherence times \((\sim 1 \text{ ms})\) and is convenient as a storage mode for interesting quantum states.
1.3.3 Transmon in the Dispersive Regime

The transmon qubit [41] has emerged as a leading contender for the building block of a useful quantum computer. Many research and commercial efforts rely on it for its simplicity and flexibility. The transmon is a derivative of the Cooper-Pair Box (CPB) [42, 43] in which a Josephson junction is connected to an external voltage $V_g$ by a capacitor $C_g$.

$$\hat{H}_{\text{CPB}} = \frac{4e^2}{2(C_g + C_J)} (\hat{N} - N_g)^2 - E_J \cos \hat{\phi},$$

(1.7)

where $C_J$ and $E_J$ are the capacitance and Josephson energy of the junction respectively, $\hat{\phi}$ is the phase across the junction and $\hat{N}$ is the number of Cooper pairs on the superconducting island of the CPB. $N_g = V_g C_g / 2e$ is proportional to the gate voltage source. The transmon is a CPB shunted by a large capacitance, significantly reducing the capacitive energy of the junction itself:

$$\hat{H}_{\text{trans}} = 4E_C \hat{N}^2 - E_J \cos \hat{\phi}$$

(1.8)

Here, $E_C = \frac{e^2}{2(C_J + C_e)} \ll E_J$. The first consequence of this change is a Hamiltonian that strongly resembles a harmonic oscillator (from the $\hat{\phi}^2$ term) for its lowest energy while still retaining non-linearities ($\hat{\phi}^4$ and higher). This makes for a quantum object that is easy to manipulate and design experiments around. The other consequence is that the charge offset $N_g$ no longer plays a significant role in the lower transition frequencies, allowing for long coherence times.

Transmons have an additional convenience in that they are inherently dipole antennas by virtue of their construction. By adjusting the dipole moment of the antenna and positioning it within a cavity electric field, we can capacitively couple harmonic modes to our qubit. When coupling a transmon to a cavity we work in the dispersive regime where the detuning between cavity and transmon is significantly larger than the strength of the coupling $|\omega_{ge} - \omega_c| \gg g$. Here $\omega_{ge}$ is the frequency associated with the energy difference between the ground, $|g\rangle$, and first excited state, $|e\rangle$, of the transmon. In this limit, each of the modes (including the qubit mode itself) contributes to the phase across the junction.
\[ \dot{\phi} = \sum_{m} \phi_m (\hat{a}_{m}^{\dagger} + \hat{a}_{m}), \]  

(1.9)

where \( \phi_m \) are the Zero Point Fluctuations (ZPFs) of each mode across the junction. We can consider the junction energy to have a linear component (an inductance and capacitance) and a fundamentally non-linear component which gives rise to its interesting properties. It is important to think of these modes as the normal modes of the system before considering the non-linearities of the junction. This dimensionless number \( \phi_m \) will be largest for the qubit mode and is proportional to each of the capacitive couplings to the junction.

Here we treat the transmon like any other harmonic mode, with creation and annihilation operators, that only picks up non-linearity from the higher order terms \( (\mathcal{O}(\phi_m)) \). We now inspect the cosine expansion and observe the terms we keep. We will perform the same rotating frame transformation as for a single cavity, where the linear energy contribution of each mode is eliminated. We also take a Rotating Wave Approximation (RWA), which informs us that any term of our Hamiltonian that is rotating in the stationary frame will be eliminated. The physical interpretation of this approximation is that only energy conserving terms will contribute to the Hamiltonian until we later apply a drive to the system, as shown in Chapter 2. Modes are chosen to have distinct frequencies, so energy conserving means number conserving. The retained 4th order terms are divided into Kerr and cross-Kerr terms. The Kerr terms are mode anharmonicities while the cross-Kerr terms are couplings between modes.

\[ \hat{H}_{\text{Kerr}}/\hbar = -\sum_{m} \frac{K_{mm}}{2} \hat{a}_{m}^{\dagger} \hat{a}_{m} \hat{a}_{m} \hat{a}_{m} \]  

(1.10)

\[ K_{mm} = \frac{E_{J}}{\hbar} \phi_{m}^{4} \]  

(1.11)

\[ \hat{H}_{\text{c-Kerr}}/\hbar = -\sum_{m \neq n} K_{mn} \hat{a}_{m}^{\dagger} \hat{a}_{n} \hat{a}_{n} \hat{a}_{n} \]  

(1.12)

\[ K_{mn} = \frac{E_{J}}{\hbar} \phi_{m}^{2} \phi_{n}^{2} \]  

(1.13)
Here $K_{mm}$ is the difference in energy between the first transition $\omega_{01} \equiv \langle 1 | \hat{H} | 1 \rangle - \langle 0 | \hat{H} | 0 \rangle$ and the second transition $\omega_{12} \equiv \langle 2 | \hat{H} | 2 \rangle - \langle 1 | \hat{H} | 1 \rangle$. The difference of a factor of two between these Kerr values is due to the combinatorics of expanding the $\hat{\phi}^4$ and normal ordering. A standard transmon will have a $\omega_{01} \approx 5 \text{GHz}$ and $K_{mm} \approx 200 \text{MHz}$, allowing us to selectively drive Rabi oscillations between the first two energy levels. For cavities, this anharmonicity should be small enough ($K_{cc} \approx 5 \text{kHz}$) for us to treat them as purely harmonic oscillators. In contrast, $K_{mn}$ is a dispersive coupling between modes. We can view this as a shift to the frequency of one mode being dependent on the occupation of the other mode. This term opens many possibilities for control of quantum states, which will be discussed in greater detail in the Basic Experiments section. This term will be largest coupling cavities to the qubit mode but there will also be cross-Kerr terms between two different cavities.

The cosine will also contain all other higher even powers, but these terms are bounded as $\phi_m < 1$. The 6th order terms will have small affects on specific experiments and contexts but can generally be ignored.

### 1.3.4 Jump Operators and Natural Dissipation

Non-unitary evolution in quantum systems are represented mathematically by Lindblad master equations [44, 45]. This framework is entirely a consequence of making a distinction between a “system” and an “environment.” For example, if an excitation is lost to a transmission line, we no longer consider it a part of our “system,” and we observe a non-unitary dynamic even though if we included the transmission line, the dynamics would be unitary. The Lindblad master equation for the time evolution of a density matrix, with jump operators $\hat{L}_k$, can be written as

\begin{equation}
\partial_t \hat{\rho} = -\frac{i}{\hbar} [\hat{H}, \hat{\rho}] + \sum_k \kappa_k \mathcal{D}[\hat{L}_k] \hat{\rho}
\end{equation}

\begin{equation}
\mathcal{D}[\hat{L}_k] \hat{\rho} = \hat{L}_k \hat{\rho} \hat{L}_k^\dagger - \frac{1}{2} \{ \hat{L}_k^\dagger \hat{L}_k, \hat{\rho} \}
\end{equation}
Here $\kappa_k$ are the rates of the corresponding jumps. With this in mind, we can explore some examples.

For cavity modes, a dominant error is single photon loss. This often occurs either from coupling to some non-superconducting defect or to other modes. Single photon loss is represented by a jump operator $\hat{L} = \hat{a}$ and occurs at a rate $\kappa = 1/T_{1c}$. $T_1$ is the characteristic time of the energy relaxation and is frequently used to describe quantum modes. The only types of non-vacuum states that remain unchanged by single-photon loss are coherent states. This is clear from the fact that photon loss is the eigenoperator of a coherent state $\hat{a} |\alpha\rangle = \alpha |\alpha\rangle$. This means that coherent states are insensitive to single photon loss [34], a key factor for many experimental proposals [25]. We will explore this error and discuss how to correct it in Chapter 3.

Cavities also undergo errors that affect the phase of the different Fock states. The easiest way to think about these errors is that if the frequency of the mode fluctuates (relative to what we think the frequency is) due to occupations in other coupled modes or low-precision microwave electronics, we will lose information about the phase of the state. We can model these dephasing errors with a number operator, $\hat{L} = \hat{a}^\dagger \hat{a}$, with a rate $\kappa = 1/T_{\phi c}$. $T_\phi$ is the characteristic time of dephasing of a quantum state. Chapter 3 discusses a method that stabilizes this phase error.

1.4 Basic Experiments

This section discusses basic measurements that will be used throughout this document. There are many excellent resources that provide greater detail on these topics, such as the thesis by Brian Vlastakis [46] and the Quantum Engineers Guide [35].

1.4.1 Qubit Readout

The first objective in characterizing our experimental system and gaining coherent control is the measurement of the transmon state. This measurement is the backbone of all other measurements we perform [3]. The procedure relies on the cross-Kerr term between the transmon and a readout cavity referred to as $\chi$ or $\chi_{qr}$.
Figure 1.1. Qubit readout. (a) Cartoon of readout resonator response to qubit state. Frequency of the readout resonator $\omega_r$ is shifted by the dispersive coupling $\chi_{qr}$ when the qubit is in $|e\rangle$ and remains unchanged when the qubit is in $|g\rangle$. By measuring on-resonance transmission of the readout resonator we can infer the qubit state through the amplitude. (b) Bloch sphere representation of possible qubit states in the Hilbert space spanned by $|g\rangle$ and $|e\rangle$.

This term causes the frequency of the readout cavity to shift dependent on the transmon state. The readout cavity should be coupled to an input port and strongly coupled to an output port so that $Q_{in} \approx 10Q_{out}$. By performing a on-resonance transmission measurement of the readout cavity, we can infer its frequency and therefore the state of the transmon. The ratio of coupling $Q_s$ ensures that the majority of photons with which we drive our readout cavity will be emitted into the output transmission line. Our systems are designed such that this dispersive shift is greater than the linewidth of the readout cavity $\chi > \kappa_r$. This means that if the transmon is in its first excited state, $|e\rangle$, the readout cavity will have its frequency altered by the dispersive shift, $\chi$. If we perform a transmission measurement on the readout cavity, the amplitude of the transmission signal will be dramatically reduced when the qubit is in $|e\rangle$, due to this dispersive shift.

1.4.2 Coherent Qubit Control

With readout, we can probe the state of the qubit, which allows us to explore complicated manipulations. Driving the qubit on-resonance will put energy into it but will result in a variety of possible effects based on the spectral width. The on-
resonance pulse will perform a rotation in the Bloch sphere between the $|g\rangle$ and $|e\rangle$ states. The phase of the pulse will dictate the axis of rotation. Pulsing the qubit with an extremely spectrally broad pulse (relative to the transmon anharmonicity) has the potential to excite not only the $ge$ transition but the $ef$ transition as well (between the first and second excited states, $|e\rangle$ and $|f\rangle$). Simultaneous drives of the $ge$ and $ef$ transitions are usually undesirable, placing a practical limit on the shortest possible pulse we can apply to the system. This effect can be compensated for with DRAG pulses [47], but only for small amounts of leakage to the $|f\rangle$ state. At the other extreme are selective qubit pulses where the line-width of the pulse is comparable to the line-width of the qubit mode itself. This type of pulse will only excite the $ge$ transition and is conditional on no other modes with dispersive couplings to the qubit being occupied. This idea will be explored further when discussing number splitting. We can also produce non-selective pulses with lengths between the two extremes that excite only the $ge$ transition, irrespective of other mode occupations.

1.4.3 Qubit Characterization

Measuring the exact parameters of the experimental system is vital to engineering new Hamiltonian terms. We find the readout cavity frequency with a transmission spectroscopy measurement. Because of the dispersive coupling between transmon and readout [41] we can probe the state of the qubit by measuring the on-resonance transmission of the readout. We can measure the qubit frequency by performing spectroscopy with a qubit drive while measuring the readout transmission. With the qubit frequency we can calibrate the exact pulse energy required to excite the transmon to the first excited state, giving us coherent control of single-qubit gates. With these gates we can measure the energy relaxation time, $T_1$, and the decoherence time, $T_2$, of the first transition. By starting with a prepared excited state, we can perform similar steps to characterize the second excited state of the transmon.
1.4.4 Cavity Characterization: Number Splitting

In addition to being able to read out the state of the transmon qubit, we also can probe the states of harmonic oscillators that are capacitively coupled to the Josephson junction. From the same dispersive shift that allows for readout of the qubit state, the transmon frequency is shifted proportional to the number of photons in the coupled cavity (see Fig. 1.2). This number splitting effect [3] allows us to probe the different Fock state populations of our coupled cavities. We can use this measurement to observe how states evolve over time, which is important for extracting rates. We often use this type of measurement to optimize reservoir engineering pump conditions by sweeping pumping parameters and measuring Fock state populations.

For a single cavity each Fock state has a unique dispersive shift to the transmon frequency, but when we have multiple modes we can explore interesting degeneracies. If the dispersive shifts to the transmon from two cavities are identical, it will be impossible to distinguish between occupation in the two modes. This could be advantageous or problematic depending on the desired application. We discuss multiple cavities and dispersive couplings in Chapter 3.
1.4.5 Cavity Tomography

Number splitting experiments are sufficient for measuring populations but give no information about phase and coherence. Another probe is the Wigner parity measurement, which measures the cavity photon number parity after a coherent displacement [46]. Cavity parity is measured by mapping the parity of the cavity state onto the transmon before reading out the transmon state [6, 48]. This parity map consists of a non-selective $\pi/2$ transmon rotation, a $\pi/\chi$ delay time, and a second non-selective $\pi/2$ transmon rotation. During the delay time, the transmon will experience a $\pi$ rotation about the $\sigma_z$ axis if the cavity has an odd number of photons and a $2\pi$ rotation if the cavity has an even number of photons.

By varying the amplitude and phase of the coherent displacement, we can perform a Wigner tomography of the cavity, which measures both phase and coherence. Wigner tomography measures the Wigner function [40], a quasi-probability distribution defined as

$$W(\alpha) = \frac{2}{\pi} Tr[\rho \hat{D}(\alpha) \hat{P} \hat{D}^\dagger(\alpha)]$$  \hspace{1cm} (1.16)

where $\hat{P} = e^{i\pi\hat{a}^\dagger\hat{a}}$ is the photon-number parity operator. We can express the density matrix of a given state in terms of this function

$$\rho = \frac{2}{\pi} \int W(\alpha) D(\alpha) \hat{P} D^\dagger(\alpha) d\alpha^2$$  \hspace{1cm} (1.17)

This data set is sufficient for performing density matrix reconstruction [49, 50]. The procedure numerically optimizes the physical density matrix that most closely matches the data. Density matrix reconstruction gives us an exact measurement of the state although it is an extremely time intensive experiment. This procedure is discussed and used in Chapter 5.

Like number splitting, this measurement can also be performed for two-cavity systems. Joint photon number parity, which measures the parity of the sum of the photon numbers in both cavities, is the analogue of photon number parity for a two-cavity system. Proposals for joint tomography have been published [51] and
are straightforward in a carefully designed system. We explore a novel method for measuring joint photon number parity in Chapter 3.

1.5 cQED Experimental Design

The ability to custom design novel quantum objects is a primary benefit of cQED, but this comes with a challenge. How can we best simulate the quantum behavior of a proposed system in a classical computer? At the beginning of my doctoral work the leading method was Black Box Quantization (BBQ) [52], which uses a classical, driven modal simulation to estimate quantum behavior with Ansys HFSS, a 3D electromagnetic equation solver. HFSS is capable of simulating a variety of 3D and 2D structures of different materials, including transmission lines, an example of the field building on existing commercial microwave electronics infrastructure (see Fig. 1.3). This method probes the impedance of classical lumped circuit elements in the system, including cavities, on-chip resonators, and the classical inductance and capacitance of any junctions, and places it in parallel with the purely non-linear
elements of the junction. This method allows for the calculation of the Zero Point Fluctuations (ZPFs) of each harmonic mode across the junction, including the “qubit” mode. These ZPFs along with the junction energy, $E_J$, are sufficient to predict the Hamiltonian of the system (see Fig. 1.4), inspect couplings and energy levels, and check the design for experimental viability.

While this method is functional, it presents difficulty for reliably measuring the effects on the junction of modes with small couplings and narrow line-widths. To address this issue, the Energy Participation Ratio (EPR) method [53] probes the same types of simulations using a eigenmodal simulation instead of a driven modal simulation. It calculates the participation of every mode in the lumped element junction to estimate the ZPFs, making it easier to work with a large number of modes and with modes that may have small couplings to the qubit. All experiments in this thesis were designed with BBQ, although EPR was used to analyze potential errors in the original designs.
Figure 1.5. Example experimental package. An example experimental package and clamped sapphire chip. The body is machined from Al and contains a hollow cavity. SMA ports on the side allow for transmission lines to be connected to send and receive microwave signals. The chip on the right is diced from a wafer of sapphire and has thin film Al patterned on to it. The upper dark feature is the transmon while the longer lower dark strip is the strip-line readout resonator. The clamp both holds the chip mechanically and thermalizes it. The chip is inserted into the larger open hole visible on the left of the cavity.
1.5.1 Chips: Design and Fabrication

Our device uses a 3D-planar-hybrid cQED architecture [54], and the design specifics are similar to those used by Wang et. al. in Ref. [51]. The cavity is machined from 5N5 Al (99.9995% pure) and houses a large waveguide section containing two cylindrical re-entrant quarter-wave resonators [55] and a small waveguide tunnel to fit a sapphire chip. The transmon and the low-Q readout resonator are made from thin-film Al deposited on the sapphire chip. The transmon contains a single Al-AlO$_x$-Al Josephson junction and is fabricated using a Dolan bridge technique. Electron-beam lithography is carried out with a 30keV JEOL JSM-7001F SEM, and the evaporation/oxidation is performed with a Plassys MEB550S evaporator. As an example, the corresponding cavity and chip to the design of Fig. 1.3, and circuit of Fig. 1.4, is shown in Fig. 1.5.

1.5.2 Storage Cavities

The experiments in this work have a common interest in long-lived bosonic states. The longer the lifetime of these “storage” modes, the more time we have to perform interesting operations. For this task we use coaxial $\lambda/4$ resonators [55], which have a cylindrical post at the bottom of a long cylindrical waveguide section. If the frequency of the cavity is sufficiently below the cutoff frequency of the waveguide and the waveguide is long enough, effectively none of the mode-field reaches out of the cavity. Seam losses that plague other designs are negligible. Because this cavity design is a $\lambda/4$ resonator, the first excited mode of the system will be positioned at three times the base frequency, while many other resonator designs have only a multiple of two. This reduces frequency crowding of the modes in the system, which is especially important when implementing strong off-resonant drives.

1.5.3 Tabletop Microwave Setup

While quantum computers hope to surpass their classical counterparts, it is difficult to conceive of a quantum computer working without a classical computer as a key component. Classical computers are necessary not only for programming the
operations of a quantum computer, but to coordinate all of the hardware that performs control and measurements, including signal generators, amplifiers, and switches, for example.

The readout of the qubit is a heterodyne measurement that involves generating a signal, copying it, sending one copy through the experiment, and comparing the two signals. To do this we rely on two key pieces of equipment: an Arbitrary Waveform Generator (AWG) and an Analogue to Digital Converter (ADC), or digitizer. These two instruments are responsible for the generation of pulses and the measurement of experimental results. We use a Keysight chassis to host a M3102A PXIe Digitizer and a M3202A PXIe AWG. For our measurements we calculate pulse sequences on a coordinating measurement computer before loading them onto the AWG. The AWG then triggers other devices, including the digitizer, and sends out complex measurement pulses. The frequency of the pulses from the AWG is capped by its sample rate so we use mixers to convert the \( \sim 100 \) MHz signals into the GHz range of our experiments. This is done by using either 3 port mixers or IQ mixers along with GHz signal generators (Lab bricks or signal cores). When triggered, the digitizer records the analogue signals passed through the fridge and streams the data back to the measurement computer.

A large portion of the work in building a tabletop setup for a complex experiment is adjusting powers and filtering unwanted signals. Attenuators and amplifiers can be used to decrease or increase signal power, while a mix of lowpass and bandpass filters can alleviate many of the unwanted signals produced by signal generation, amplification, and mixing. A key tool for this is a spectrum analyzer (Signal Hound), which measures the spectral composition of a signal. This can identify unwanted signals and signal powers so that they can be debugged.

1.5.4 Cryogenic Microwave Setup

With modern commercial dilution refrigerators, the vast majority of the work of cooling samples to incredibly low temperatures is already done. While still in development, cryogenic microwave components are also widely available, although
expensive. Cryogenic attenuators, filters, and isolators help protect the system from unwanted signals on the transmission lines that connect to the experiment. Cold amplifiers that act both on quantum [2] and classical principles help amplify small signals to allow for room temperature measurement. Although many components are commercially available, their performance for extremely sensitive experiments is often questionable. Fridge setups are developing as scientists better understand specific needs and best practices [56]. An example fridge diagram is shown in Fig. 1.6.
Figure 1.6. **Fridge diagram.** The experimental packages are installed in an Oxford Triton-500 dilution refrigerator. Our device is mounted at the mixing chamber (MXC) stage of the refrigerator with a nominal temperature of about 10 mK. Coherent control signals for both cavity $A$ and the transmon are generated by IQ modulation. We use a first-stage Traveling Wave Parametric Amplifier [2] (TWPA) from MIT Lincoln Laboratory, as well as a High-Electron-Mobility Transistor (HEMT) amplifier at the 4K stage. The experiment is housed inside a MuMETAL shield to protect it from external magnetic fields. All lines connected to the experiment are filtered with a K&L 10 GHz low-pass filter and an Eccosorb low-pass filter.
2.1 Introduction and Four Wave Mixing

Probably the biggest advantage of working with superconducting transmons for driven dissipation is the ability to generate nearly arbitrary even-powered mixing terms, specifically fourth-order terms. The junction Hamiltonian, $H_{JJ} = -E_J \cos(\hat{\phi})$, when expanded into its Taylor series, contains all possible even-count combinations of creation and annihilation operators for all modes that couple to the junction. Many of these terms are either arbitrarily small due to small couplings or are higher-order terms that vanish with a small parameter. A great number of these terms are not energy conserving, meaning that they are eliminated when taking the rotating wave approximation (RWA), a common method for simplifying cQED theory. However, by adding an external drive to the system, we can compensate for this energy difference and obtain stationary terms in our Hamiltonian [9]. We are primarily concerned with the fourth-order terms, called Four-Wave Mixing (FWM) terms, which can be produced with relatively high amplitudes and experimental ease.

With FWM we can introduce new types of coherent terms to the system Hamiltonian. One interesting example consists of terms including creation and annihilation operators from different modes with dramatically different energy relaxation rates. In the cQED architecture we can engineer cavity modes, both coupled to a transmon, that can vary in $T_1$ by more than three orders of magnitude. In this regime, with the right drive strengths, the dynamics of the reservoir mode can be adiabatically eliminated in the Markovian limit [57]. An interesting consequence of this is the ability to construct not only coherent Hamiltonian terms of up to third order in mode creation/annihilation operators but also the ability to
construct dissipative jump operators of up to second order as the FWM converts a reservoir loss operator into a more complex jump operator on other modes.

2.2 Driven Hamiltonian

To understand the implementation of reservoir engineering in cQED, we first need to understand the response of our Hamiltonian to external drive terms. By tuning the frequency of these drives we can construct stationary Hamiltonain terms that mediate coherent exchange of excitations between multiple modes. To demonstrate this effect let us consider a system containing one transmon qubit (labeled Q with annihilation operator $\hat{q}$) and two coupled harmonic oscillators. One of these harmonic oscillators will function as the Markovian reservoir (labeled R with annihilation operator $\hat{r}$) in the system (accomplished by strongly coupling it to a transmission line), while the other will function as a storage mode (labeled A with annihilation operator $\hat{a}$) to hold some quantum information for relatively long time scales. The system Hamiltonain is:

$$\hat{H} = \hbar \tilde{\omega}_Q \hat{q}^\dagger \hat{q} + \hbar \tilde{\omega}_A \hat{a}^\dagger \hat{a} + \hbar \tilde{\omega}_R \hat{r}^\dagger \hat{r} - E_J \cos(\hat{\varphi}),$$

(2.1)

where $\tilde{\omega}_Q, \tilde{\omega}_A,$ and $\tilde{\omega}_R$ are the bare mode frequencies and $E_J$ is the junction energy. The phase across the junction $\varphi$ can be written as:

$$\hat{\varphi} = \phi_Q (\hat{q}^\dagger + \hat{q}) + \phi_A (\hat{a}^\dagger + \hat{a}) + \phi_R (\hat{r}^\dagger + \hat{r}),$$

(2.2)

where $\phi_Q, \phi_A, \phi_R$ are the zero-point fluctuations of each mode across the junction. For this derivation, it is useful to treat the dissipation of the reservoir mode R, with rate $\kappa_R$, as an imaginary Hamiltonian term. Separating the reservoir frequency term and adding this imaginary term we get:

$$\hat{H}_{\text{eff}} = \sum_{m \in \{Q,A\}} \hbar \tilde{\omega}_m \hat{a}_m^\dagger \hat{a}_m + \hbar (\tilde{\omega}_R + i\kappa_R) \hat{r}^\dagger \hat{r} - E_J \cos(\hat{\varphi})$$

(2.3)

Now with the bare system Hamiltonian we can consider an external drive of amplitude $\epsilon_d$ and frequency $\omega_d$, which can be written as:
\[ \hat{H}_d = \hbar \epsilon_d \cos(2\omega_d t)(\hat{r}^\dagger + \hat{r}) \]  

(2.4)

We want to limit this drive to being off resonant from any of the modes such that \( |\omega_d - \tilde{\omega}_m| \gg \kappa_m \). The effect of on-resonant drives is already discussed in Chapter 1.

To simplify this complex Hamiltonian we take the unitary transformation:

\[ \hat{U} = \hat{D}_r(\alpha = -\xi_d e^{-i\omega_d t}) \]  

(2.5)

with complex displacement amplitude \( \xi_d \), to be determined later. Our Hamiltonian now reads:

\[
\hat{H}_{\text{eff}} = \sum_{m \in \{Q,A\}} \hbar \tilde{\omega}_m \hat{a}_m^\dagger \hat{a}_m - E_J \cos \left( \sum_{m \in \{R,Q,A\}} \phi_m(\hat{a}_m^\dagger + \hat{a}_m) + \phi_r(\xi_d e^{-i\omega_d t} + \xi_d^* e^{i\omega_d t}) \right) \\
+ \hbar(\tilde{\omega}_R + i\kappa_R)(\hat{r}^\dagger + \xi_d^* e^{i\omega_d t})(\hat{r} + \xi_d e^{-i\omega_d t}) \\
+ \hbar \epsilon_d 2Re \left[ e^{-i\omega_d t} \right] (\hat{r} + \xi_d e^{-i\omega_d t} + \hat{r}^\dagger + \xi_d^* e^{i\omega_d t}) \\
+ i\hbar(\partial_t(\xi_d e^{-i\omega_d t})\hat{r}^\dagger + \partial_t(\xi_d^* e^{i\omega_d t})\hat{r}) \\
+ \lambda^* \hat{r}^\dagger + \lambda \hat{r} + (\tilde{\omega}_R + \delta \omega_R + i\kappa_R)\hat{r}^\dagger \hat{r} + \text{constants} \]  

(2.6)

We can rewrite this, collecting terms we gained from the unitary transformation into the form of

\[
\hat{H}_{\text{eff}} = \sum_{m \in \{Q,A\}} \hbar \tilde{\omega}_m \hat{a}_m^\dagger \hat{a}_m + \hbar \tilde{\omega}_R \hat{r}^\dagger \hat{r} \\
- E_J \cos \left( \sum_{m} \phi_m(\hat{a}_m^\dagger + \hat{a}_m) + \phi_r(\xi_d e^{-i\omega_d t} + \xi_d^* e^{i\omega_d t}) \right) \\
+ \lambda^* \hat{r}^\dagger + \lambda \hat{r} + (\tilde{\omega}_R + \delta \omega_R + i\kappa_R)\hat{r}^\dagger \hat{r} + \text{constants} \]  

(2.7)

Here, \( \delta \omega_R \) is a shift in the bare frequency of the mode and \( \lambda \) can be written as:

\[
\lambda^* = 2\hbar Re \{\epsilon_d e^{-i\omega_d t}\} + i\hbar \partial_t(\xi_d e^{-i\omega_d t}) + \hbar(\tilde{\omega}_R + i\kappa_R)\xi_d e^{-i\omega_d t} \]  

(2.8)
We want to choose $\xi_d$ and $\omega_d$ such that $\lambda = \lambda^* = 0$ so that terms containing $\hat{r}$ or $\hat{r}^\dagger$ disappear. This condition gives us a differential equation for $\xi_d$,

$$\partial_t \tilde{\xi}_d = (i\tilde{\omega}_R - \kappa_R)\tilde{\xi}_d + 2iRe\{\epsilon_d e^{-i\omega_d t}\},$$

(2.9)

where $\tilde{\xi}_d = \xi_d e^{-i\omega_d t}$. This equation describes the behavior of a damped oscillator under a detuned drive. On a time scale of $1/\kappa_r$ the $\xi_d$ term reaches a steady state while the $e^{-i\omega_d t}$ term continues to oscillate. We can approximate a solution as a Lorentzian response:

$$\xi_d \approx -i\epsilon_d/\left(\kappa_R + i(\omega_R - \omega_d)\right)$$

(2.10)

This all means that by applying an off resonant drive we can take that term into the cosine expansion and where the amplitude of that term $\xi_d$ is proportional to the strength of the drive and its detuning. For convenience we also choose to work in a rotating frame where the excitation energies of all three modes are zero when not considering the cosine term. Our system Hamiltonian can now be written as $\hat{H} = -E_J \cos(\hat{\varphi})$ with

$$\hat{\varphi} = \phi_Q(\hat{q}^\dagger + \hat{q} + \xi^* + \xi) + \phi_A(\hat{a}^\dagger + \hat{a}) + \phi_R(\hat{r}^\dagger + \hat{r})$$

(2.11)

The 4th order expansion contains a large number of terms, but if we apply rotating-wave approximations (RWA), the only stationary terms are diagonal terms (preserving excitation numbers in all modes) and off-diagonal terms that convert excitations between specific modes, if certain frequency-matching conditions are satisfied.

### 2.3 Fourth-Order Terms

There are a number of interesting consequences of introducing a new term, $\xi$, to the phase across the junction. It will affect terms of every order from the cosine expansion but we are particularly interested in the 4th-order terms. 0th and 2nd order terms including $\xi$ will only be constant terms in the Hamiltonian, and thus
easily ignored. 6th order terms are potentially important, but for the experiments in this document they are relatively small and considered error terms. The most straightforward terms arising from a driven Josephson junction are Stark-shift terms. A strong, off-resonant tone will shift the frequency of all the modes in our system:

$$\hat{H}_{ss}/\hbar = \sum_m \phi_m^2 |\xi|^2 \hat{a}_m^\dagger \hat{a}_m$$ (2.12)

This term will not be particularly useful for reservoir engineering but is important to consider when frequency-matching specific transitions. The fact that this term is non-rotating is expected due to the \(e^{-i\omega_d t}\) rotation from \(\xi\) and the \(e^{i\omega_d t}\) rotation from \(\xi^*\) canceling each other out. We can think of this term as effectively creating and annihilating both a drive photon and an excitation from the corresponding mode \(m\).

We can imagine terms that only include one drive photon, which will be first order in \(\xi\). By including \(\xi\) we will have a rotation of \(e^{i\omega_d t}\), which we will need to cancel out with the other three terms we combine it with. For any three other terms from the junction phase, EQ 2.11, we can pick a frequency \(\omega_d\) for which the term is non-rotating. For instance, we can construct a test term that annihilates a reservoir excitation and adds two photons to mode \(A\). In the non-rotating frame this term (and its Hermitian conjugate) can be written

$$\hat{H}_{FWM} = e^{-i(2\omega_A - \omega_R - \omega_d)t/2}(-E_J\phi_R\phi_A^2/2)(\hat{a}_R^\dagger)^2\hat{\rho} + h.c.$$ (2.13)

This term will be non-rotating if \(2\omega_A - \omega_R - \omega_d = 0\), a frequency matching condition for the conservation of energy between different modes. These terms are called Four-Wave Mixing (FWM) terms. In the rotating frame this will reduce to

$$\hat{H}_{FWM} = g^* (\hat{a}_R^\dagger)^2\hat{\rho} + h.c.$$ where \(g = -E_J\phi_R\phi_A^2/2\).

There are two exciting consequences of FWM in reservoir engineering. First, we can now create arbitrary Hamiltonian terms that are either third or second order in the creation and annihilation operators of the modes in our system. This allows us to perform operations such as a drive that simultaneously and coherently adds multiple
photons to a single mode. Second, we can use four wave mixing terms to convert a single-photon dissipation into an effective multi-photon dissipation.

2.4 Adiabatic Elimination

To convert the single photon dissipation of the reservoir mode $R$ into an effective dissipation acting on the storage mode $A$, we need to eliminate the dynamics of the reservoir from our system. The most important assumption of this derivation is that the average occupation of the reservoir mode is significantly less than one $\langle \hat{r}^\dagger \hat{r} \rangle \ll 1$. We achieve this when the single photon dissipation of the reservoir mode is the strongest term in the entire system. The master equation dictating the evolution of reservoir $R$ and storage $A$ can be written as

$$\partial_t \hat{\rho}_{AR} = -\frac{i}{\hbar} [\hat{H}_{AR}, \hat{\rho}_{AR}] + \kappa_A \mathcal{D}[\hat{a}] \hat{\rho}_{AR} + \kappa_R \mathcal{D}[\hat{r}] \hat{\rho}_{AR}$$

(2.14)

Here we only consider $H_{AR} = H_{FWM}$ from EQ 2.13 but this can also include Kerr, cross-Kerr and stark shift terms. We make an ansatz about the form of this density matrix, that it is an expansion in terms of some small, dimensionless parameter $\delta$.

$$\hat{\rho}_{AR} = \hat{\rho}_{00} |0\rangle_R \langle 0|_R + \delta (\hat{\rho}_{10} |1\rangle_R \langle 0|_R + \hat{\rho}_{01} |0\rangle_R \langle 1|_R) + \delta^2 (\hat{\rho}_{20} |2\rangle_R \langle 0|_R + \hat{\rho}_{02} |0\rangle_R \langle 2|_R + \hat{\rho}_{11} |1\rangle_R \langle 1|_R) + \mathcal{O}(\delta^3)$$

(2.15)

Here $\hat{\rho}_{nm}$ are subsections of the total $\hat{\rho}_{AR}$ density matrix corresponding to different states of the reservoir. Using this ansatz we can write out the time derivatives of these subsections $\hat{\rho}_{nm}$ by multiplying EQ 2.14 with $\langle n|$ on the left and $|m\rangle$ on the right. We are primarily interested in the dynamics of $\hat{\rho}_{00}$, so for $\hat{\rho}_{10}$, $\hat{\rho}_{01}$, and $\hat{\rho}_{11}$ we neglect terms of $\mathcal{O}(\delta)$ and higher.
\[
\frac{\partial \hat{\rho}_{00}/\kappa_R}{\partial t} = -i\delta^2(\hat{A}^\dagger \hat{\rho}_{10} - \hat{\rho}_{01}\hat{A}) - \delta^2 \hat{\rho}_{11} + \frac{\kappa_A}{\kappa_R} \mathcal{D}[\hat{a}]\hat{\rho}_{00} + \mathcal{O}(\delta^3) \quad (2.16)
\]

\[
\frac{\partial \hat{\rho}_{10}/\kappa_R}{\partial t} = -i\hat{A}\hat{\rho}_{00} - \frac{1}{2}\hat{\rho}_{10} + \mathcal{O}(\delta) \quad (2.17)
\]

\[
\frac{\partial \hat{\rho}_{11}/\kappa_R}{\partial t} = -i(\hat{A}\hat{\rho}_{01} - \hat{\rho}_{10}\hat{A}^\dagger) - \hat{\rho}_{11} + \mathcal{O}(\delta), \quad (2.18)
\]

where \(\hat{A} = (\delta\kappa_R)^{-1}g\hat{a}^2\). This is the step where the adiabatic elimination can be applied. The approximation is that \(\hat{\rho}_{10}, \hat{\rho}_{01},\) and \(\hat{\rho}_{11}\) are all continuously in steady state. We can use this information to obtain equations for \(\hat{\rho}_{10}, \hat{\rho}_{01},\) and \(\hat{\rho}_{11}\) in terms of \(\hat{\rho}_{00}\).

\[
\hat{\rho}_{10} = -2i\hat{A}\hat{\rho}_{00} + \mathcal{O}(\delta) \quad (2.19)
\]

\[
\hat{\rho}_{11} = 4\hat{A}\hat{\rho}_{00}\hat{A}^\dagger + \mathcal{O}(\delta) \quad (2.20)
\]

Plugging these back into EQ 2.16, we can write this system of equations only in terms of \(\hat{\rho}_{00}\).

\[
\frac{\partial \hat{\rho}_{00}}{\partial t} = 4\delta^2\kappa_R(\hat{A}\hat{\rho}_{00}\hat{A}^\dagger - \frac{1}{2}\hat{A}^\dagger\hat{A}\hat{\rho}_{00} - \frac{1}{2}\hat{\rho}_{00}\hat{A}^\dagger\hat{A}) \quad (2.21)
\]

Reducing this, and plugging in the value of \(\hat{A}\), we obtain a new master equation describing the dynamics of the storage mode with the reservoir eliminated.

\[
\frac{\partial \hat{\rho}_A}{\partial t} = \kappa_2 \mathcal{D}[\hat{a}^2]\hat{\rho}_A + \kappa_A \mathcal{D}[\hat{a}]\hat{\rho}_A \quad (2.22)
\]

where \(\kappa_2 = 4|g|^2/\kappa_R\). This result is not specific to only this new effective two photon loss, and is not specific to just one storage mode. We can construct arbitrary 2nd-order dissipation terms so long as the reservoir dissipation remains the fastest process in the system.

An important variation to consider is the addition of a weak on-resonance drive to the reservoir mode.

\[
\hat{H}_{int} = (g\hat{a}^2 + \epsilon_d)\hat{r}^\dagger + h.c. \quad (2.23)
\]
where $\epsilon_d$ is the strength of the reservoir drive. This tone will be converted by the FWM drive into an effective two-photon drive. EQ 2.23 is written to suggest the effect this drive has of changing $\hat{A}$ to $\hat{A} = (\delta \kappa R)^{-1}(g\hat{a}^2 + \epsilon_d)$. This has the effect of creating a new Hamiltonian term in the master equation for mode A.

$$\partial_t \rho_A = -i[\epsilon_2 \hat{a}^2 + \epsilon_2^* (\hat{a}^\dagger)^2] + \kappa_2 D[\hat{a}^2] \rho_A + \kappa_A D[\hat{a}] \rho_A,$$

where $\epsilon_2 = -2ig\epsilon_d/\kappa R$.

### 2.5 Manifold Stabilization

One of the most interesting results of being able to engineer new drive and dissipation terms is the stabilization of interesting quantum manifolds. A quantum manifold refers to a set of states inside the system Hilbert space that are continuous. We can stabilize a manifold of dark states by engineering the drive and dissipation. We consider a generalized system with arbitrary Hamiltonian terms, $\hat{H}_{eng}$, and arbitrary dissipative term $\hat{O}_{eng}$,

$$\partial_t \rho = -i\hbar [\hat{H}_{eng}, \rho] + \kappa D[\hat{O}_{eng}] \rho = 0.$$  (2.25)

Here we are ignoring the natural dissipation channels of the system. These will be important sources of error but are not the dynamics of the system we will focus on. This is made easy when the Hamiltonian drive term and the engineered dissipation have the same operator form, $\hat{H}_{eng} = \epsilon \hat{O} + \epsilon^* \hat{O}^\dagger$, $\hat{O}_{eng} = \hat{O}$. It may not be immediately obvious what the dark states of this condition are but rewriting this condition gives

$$\partial_t \rho = i\hbar [\epsilon \hat{O} + \epsilon^* \hat{O}^\dagger, \rho] + \kappa D[\hat{O}] \rho = \kappa L[(\hat{O} + 2i\epsilon^*/\kappa)] \rho$$  (2.26)

This form implies some displaced loss operator where the magnitude of the displacement is proportional to the ratio of the drive strength to the rate of dissipation.

When cast from a drive into an effective loss term, the steady states of the system become much more clear. For the most simple example with $\hat{O} = \hat{a}$, the system
Table 2.1. Dissipative vs Hamiltonian stabilization. Comparison of multi-photon steady states under driven dissipative processes constructed with different operators, $\hat{O} = \hat{a}, \hat{a}^2$ and $\hat{a}\hat{b}$. For each of the three cases, we consider system dynamics following the master equation EQ 2.25 written in the rotating frame of the drives, where $\mathcal{D}$ is the Lindblad superoperator with $\hat{O}$ as its jump operator. In two-photon dynamics, the complex amplitude of the steady-states are analogously determined by the one-mode or two-mode squeezing drives countered by the corresponding two-photon loss rates and Kerr Hamiltonian confinement [1]. Notably, two steady states exist (with even or odd photon number parity $\Pi$) for the case of cat-state stabilization while there are infinitely many steady states (with different photon number difference $\delta$) for the case of stabilizing pair coherent states.

<table>
<thead>
<tr>
<th>Process category</th>
<th>Drive Hamiltonian $\hat{H}_d/\hbar$</th>
<th>Dissipator $\mathcal{D}$</th>
<th>Hamiltonian $\hat{H}_0/\hbar$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-photon</td>
<td>$\epsilon^*\hat{a} + \epsilon\hat{a}^\dagger$</td>
<td>$\kappa\mathcal{D}[\hat{a}]$</td>
<td>$\Delta\hat{a}^\dagger\hat{a}$</td>
</tr>
<tr>
<td>1-mode 2-photon</td>
<td>$\epsilon_1^2\hat{a}^2 + \epsilon_2\hat{a}^2\hat{a}^\dagger$ (ref. [58])</td>
<td>$\kappa_2\mathcal{D}[\hat{a}^2]$ (ref. [57])</td>
<td>$\kappa_\alpha\hat{a}^2\hat{a}^\dagger$ (ref. [59])</td>
</tr>
<tr>
<td>2-mode pair-photon</td>
<td>$\epsilon_{ab}\hat{a}\hat{b} + \epsilon_{ab}\hat{a}^\dagger\hat{b}^\dagger$ (ref. [60])</td>
<td>$\kappa_{ab}\mathcal{D}[\hat{a}\hat{b}]$</td>
<td>$K_{\text{eff}}\hat{a}\hat{b}\hat{a}^\dagger\hat{b}^\dagger$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Process category</th>
<th>Steady State</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-photon</td>
<td>Coherent state $</td>
</tr>
<tr>
<td>1-mode 2-photon</td>
<td>Cat states $</td>
</tr>
<tr>
<td>2-mode pair-photon</td>
<td>Pair coherent states $</td>
</tr>
</tbody>
</table>

will be in equilibrium when the value inside the dissipation super-operator goes to 0: $\hat{a} + 2i(\epsilon/\kappa)^* = 0$. As long as our state is a coherent state, this constraint simplifies to $\alpha = -2i(\epsilon/\kappa)^*$. We recognize this as a classically driven dissipative state, as expected.

For more complicated loss terms, the dark state manifold becomes more interesting. For example, a two photon drive $\hat{O} = \hat{a}^2$ gives a constraint of $\hat{a}^2 = -2i(\epsilon/\kappa)^*$. The solution to this system is a cat state $|\psi\rangle = x|\alpha\rangle + y|-\alpha\rangle$, with amplitude $\alpha = \sqrt{-2i(\epsilon/\kappa)^*} = (1 - i)\sqrt{(\epsilon/\kappa)^*}$. $x$ and $y$ are arbitrary complex coefficients satisfying the normalization condition $|x|^2 + |y|^2 = 1$. Assigning the logical values of a qubit to $|0\rangle_L = |\alpha\rangle$ and $|1\rangle_L = |-\alpha\rangle$, we see that the manifold of dark states can be mapped onto a single logical qubit.
CHAPTER 3  
GENERATION AND STABILIZATION OF PAIR COHERENT STATES

The use of continuous-variable states of bosonic modes as a platform for quantum information processing, which originated in quantum optics [61, 62], is rapidly advancing in superconducting circuit quantum electrodynamics (cQED) [37, 38]. While many of the exotic states envisioned decades ago remain challenging to implement in the optical domain, they have become practical and valuable resources in the microwave domain thanks to the ability to engineer a wide range of mode couplings and nonlinearities in Josephson circuits [28]. For example, the Schrödinger cat states [6, 51, 59] and the Gottesman-Kitaev-Preskill grid states [63] have not only been realized but also actively pursued for encoding logical qubits with error suppression or correction capabilities.

One interesting class of bosonic states yet to be studied experimentally is the pair coherent state, an example of a Barut-Girardello generalized coherent state [64]. This state gained theoretical interest as an example of a highly entangled two-mode state [65, 66] and was initially proposed as a hypothesis to explain a suppression of amplified spontaneous emission in an atomic system [67]. A pair coherent state can be written in the Fock state basis of two harmonic oscillators as

$$|\gamma, \delta\rangle = \mathcal{N} \sum_{n=0}^{\infty} \frac{\gamma^{n+\delta/2}}{\sqrt{n!(n+\delta)!}} |n + \delta, n\rangle$$

(3.1)

where $\delta$ is an integer describing the photon number difference (PND) between the two modes, $\gamma$ a complex number describing the amplitude and phase of the state, and
\( \mathcal{N} \) a normalization factor. This state is both an eigenstate of the pair photon loss operator \( \hat{a} \hat{b} \) and the PND operator \( \hat{\delta} = \hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b} \):

\[
\hat{a} \hat{b} |\gamma, \delta \rangle = \gamma |\gamma, \delta \rangle, \hat{\delta} |\gamma, \delta \rangle = \delta |\gamma, \delta \rangle.
\] (3.2)

A pair coherent state is inseparable and already in the form of a Schmidt decomposition \([66]\). It resembles a two-mode squeezed state \([68]\) but with a Poisson-like photon number distribution approximately centered around \( \gamma \).

Pair coherent states form the basis of a recently-proposed quantum error correction (QEC) code called the pair cat code \([69]\). This code promises autonomous QEC of all types of first-order physical errors by encoding a logical qubit in the superposition of pair coherent states \( |\pm \gamma, \delta \rangle \) of two oscillators coupled to the same nonlinear ancilla. The scheme involves stabilizing PND to correct the quantum jumps of photon loss events while simultaneously stabilizing the pair coherent state manifold with a two-mode four-photon dissipation process.

Despite recent advances of unitary control \([5, 70, 71]\) and measurement feedback \([25, 63, 72, 73]\) in bosonic cavity systems, there is a clear need for novel tools for multi-mode quantum operations. For example, calculating a numerical pulse to generate a pair coherent state with optimal control theory \([5]\) becomes prohibitively difficult for larger states. Reservoir engineering \([74]\) has been of particular interest for its ability to stabilize non-classical oscillator states in a resource-efficient manner \([75]\). In cQED, realization of nonlinear dissipation operators have led to stabilization of the cat-state manifolds \([57, 76, 77]\) and autonomous QEC of photon losses \([10]\) in a single cavity. However, engineered nonlinear dissipation across two cavities remains to be explored.

In this work, we present an experimental realization of the pair coherent state and its characterization using photon number analysis and two-mode Wigner tomography. We expand the toolbox of quantum reservoir engineering by realizing a two-mode dissipation operator that removes photon pairs from two superconducting cavities and stabilizes the complex amplitude of the pair coherent state. We further demonstrate...
Figure 3.1. Pair coherent state generation: system and protocol. (a) Cartoon of the 3D cQED system containing two high quality factor (Q) cylindrical post cavities $a$ and $b$ (blue and orange), the transmon ancilla $a$ (pink), and the stripline low-Q reservoir $r$ (green). (b) Mixing drive $p$, with frequency $\omega_p = \omega_a + \omega_b - \omega_r$, coherently converts reservoir excitations to pairs of $a$ and $b$ excitations mediated by the ancilla junction. The reciprocal process is also mediated by the mixing tone. With adiabatic elimination the reservoir dissipation (dotted green arrow) is mixed into an effective pair photon dissipation (dotted blue/orange arrows). Eliminating the reservoir, the storage cavity dynamics are shown in the dotted box. All junctions shown are the same ancilla junction. (c) Cartoon of the mode frequencies and relative line-widths. Strong off-resonance CW mixing drive $p$ and weak on-resonance reservoir drive $d$ are shown as vertical arrows. (d) Cartoon of pair coherent state distribution over $\delta = 0$ Fock states.

an effective method for measuring PND without fine matching of system parameters, which may be used for discrete or continuous tracking of error syndromes in future implementations of the pair cat code.

3.1 Pair Coherent States in cQED

Our approach to generate and stabilize a pair coherent state is based on the application of a pair photon drive counter-balanced by engineered pair photon dissipation and a cross-Kerr interaction. To understand how the pair coherent state naturally emerges as the steady state under their combined effect, the two-mode system dynamics can be compared or mapped to the textbook example of the stabilized Glauber coherent state of a driven damped oscillator, as well as to previous demonstrations of stabilized single-mode cat state manifolds [57, 59]. The correspondence of relevant Hamiltonian and dissipator terms are listed in Table 2.1.
While the pair-photon drive $\hat{H}_d = \epsilon_{ab} \hat{a}^\dagger \hat{b}^\dagger + \text{c.c.}$, also known as the two-mode squeezing drive [60], has long been a workhorse in quantum optics, stabilization of relatively large pair coherent states requires either a strong (non-unitary) pair-photon loss mechanism or a strong (unitary) cross-Kerr interaction (relative to the single-photon decay rates). These two possible strategies are analogous to the stabilized “dissipative cat” [57, 77] and “Kerr cat” [78, 59], respectively, in the single-mode two-photon processes. In fact, the unitary and dissipative effects play the roles of real and imaginary components, respectively, of the restoring force and are mutually compatible (see Table 2.1). Inspired by both types of cat-state stabilization, our technique combines both effects while generalizing the coherent state stabilization to the two-mode scenario. The 3D cQED architecture is ideal for realizing this driven dissipation process due to the availability of strong coupling between modes, the four-wave-mixing capability of the Josephson junction, and the wide range of mode lifetimes achievable in the same system.

As shown in Fig. 3.1(a), our system contains two cylindrical post cavity modes a and b (with single photon loss rates $\kappa_a/2\pi = 0.30$ kHz and $\kappa_b/2\pi = 0.74$ kHz), a stripline resonator r (with decay rate $\kappa_r/2\pi = 1.0$ MHz) used for readout and as a Markovian reservoir, and an ancilla transmon. The device architecture is similar to that in Wang et. al. [51] except that the two cavity posts share the same elliptical cavity body to allow strong transmon-cavity couplings with a relatively small transmon antenna. The cavity/resonator modes have annihilation operators $\hat{a}, \hat{b}, \hat{r}$ and the ancilla, treated as a two level system, is represented by the Pauli operators. The leading order terms of the system Hamiltonian in the rotating frame are:

$$\frac{\hat{H}_0}{\hbar} = - \sum_{\hat{\phi} = \hat{a}, \hat{b}, \hat{r}} \frac{\chi_{q\phi}}{2} \hat{\phi}^\dagger \hat{\phi} \hat{\sigma}_z - \sum_{\hat{\phi} = \hat{a}, \hat{b}} \frac{K_\phi}{2} \hat{\phi}^\dagger \hat{\phi} \hat{\sigma}_z - K_{ab} \hat{a}^\dagger \hat{a} \hat{b}^\dagger \hat{b}$$

where $\chi_{q\phi}$ are dispersive couplings between the ancilla and the other modes, $K_{a,b}$ describes the self-Kerr of the storage cavities, and $K_{ab}$ is the cross-Kerr between the cavities. Considering the limit where the reservoir has a small occupation $\bar{n}_r \ll 1$, 

34
at all times other than readout, we ignore $4^{th}$ order terms involving $\hat{r}$. The device is measured in a dilution refrigerator at a nominal base temperature of 20 mK.

To implement the pair photon drive and dissipation, we drive our system with two tones: a strong off-resonance pump at the exact frequency to coherently convert reservoir photons with pairs of photons in a and b, and a weak drive on resonance with the reservoir (Fig. 3.1b,c). Under these drives, the Hamiltonian gains an interaction term under the rotating wave approximation:

$$\hat{H}_{\text{int}}/\hbar = g_{ab} \hat{a}^{\dagger} \hat{b}^{\dagger} \hat{r} + \epsilon_d \hat{r}^{\dagger} + \text{h.c.},$$

where $\epsilon_d$ is the amplitude of the reservoir drive, and $g_{ab}$ is the four-wave mixing rate activated by the off-resonance pump. We can adiabatically eliminate the dynamics of the short-lived reservoir mode [57], as demonstrated in Chapter 2, to obtain our desired form of the Lindblad master equation that stabilizes the manifold of pair coherent states as described in Table 2.1:

$$\partial_t \rho = -\frac{i}{\hbar} \left[ (\epsilon_{ab} \hat{a}^{\dagger} \hat{b}^{\dagger} - K_{\text{eff}} \hat{a}^{\dagger} \hat{a} \hat{b} + \text{h.c.}), \rho \right] + \kappa_{ab} D[\hat{a} \hat{b}] (\rho),$$

where we have approximated all the Kerr terms as one $K_{\text{eff}}$ term (valid in the limit $\gamma \gg \delta$), and assumed that the transmon stays in its ground state. Our pair photon drive $\epsilon_{ab} = -2i g_{ab} \epsilon_d / \kappa_r$ and pair photon dissipation $\kappa_{ab} = 4|g_{ab}|^2 / \kappa_r$ both scale with the four-wave mixing rate while only the pair photon drive scales with the reservoir drive.

### 3.2 Device Architecture and Transmon Fabrication

The system consists of two post cavities dispersively coupled to a fixed-frequency transmon ancilla that is again dispersively coupled to a stripline $\lambda/2$ resonator used for readout. The posts and the sapphire chip with the transmon and stripline resonator are placed inside a high-quality aluminum body. This high-quality superconducting body shields the interior from magnetic field lines that can degrade the coherence of
Figure 3.2. Pair coherent state characterization: population and time dynamics. (a) Populations of entangled cavity states $|0,0\rangle$ through $|4,4\rangle$ after 15 $\mu$s of pumping sweeping the frequency of the cavity drive. We vary the difference between the mixing and reservoir drives while holding the sum constant to ensure that excess reservoir photons do not spoil the pumping condition. Cavity populations are measured by a number splitting [3], selective, ancilla transmon spectroscopy measurement after playing cw mixing and drive tones. (b) Transmon spectroscopy after 15 $\mu$s of pumping at $\omega_d = 0$. Dark vertical bars correspond to $\delta = 0$ states while light bars correspond to error states caused by single photon loss. (c) Population measurements performed starting in vacuum and pumping with both cw tones for a variable time. 1QuTiP fits of this data are used to extract the pair photon pumping rate $\epsilon_{ab} = 88 kHz$. (d) Population measurements performed starting with the $t = 15 \mu$s pair coherent state and playing only the mixing tone for variable time. This measurement demonstrates the pair dissipations ability to remove pairs of photons significantly faster then the single photon loss rate. QuTiP fits of this data are used to extract the pair photon dissipation rate $\kappa_{ab} = (11.85 \mu s)^{-1}$ which is significantly faster than either cavities single-photon loss rate.
the qubit. The ancilla and the cavity modes each have their own drive ports. The readout signal is collected from a port coupled strongly with the stripline resonator.

The transmon ancilla and stripline resonator are constructed by depositing thin film aluminum on the sapphire wafer. The transmon has a single Al-AlO$_x$-Al Josephson junction and is fabricated by the Dolan bridge method. The fabrication takes place in a clean room facility with electron-beam lithography carried out by 30 keV JEOL JSM-7001F SEM, and evaporation/oxidation is carried out with a Plassys MEB550S evaporator.

### 3.3 Measurement Setup

The microwave setup consists of separate LO drive and IQ mixer combos to produce the FWM tone, and drive the ancilla and readout modes. These three tones are all sent to the input port for ancilla mode (a port physically close to the transmon with a strong coupling). The Alice and Bob storage modes each have an input port. The setup is standard, with signal generators being local oscillators for the IQ mixer and AWG, creating I and Q signals for sidebands, and the RF output then enters the fridge. The Bob drive is generated with sideband modulation of signal generated for reservoir, FWM pump ($\omega_r - \omega_a - \omega_b$) and Alice modes in order to achieve the same drive phase while simultaneously displacing both the cavities.

### 3.4 Measurements of Pair Photon Population Dynamics

To characterize the pair photon driven dissipative dynamics, we first measure the two-cavity photon number distributions. To do this we perform spectroscopy of the ancilla transmon, whose frequency is shifted by $-\chi_{a,b}$ for every photon in cavity a, b. A frequency-selective rotation of the ancilla at a detuning $\Delta \omega_q = -n_a \chi_a - n_b \chi_b$ maps the probability of being in Fock state $|n_a, n_b \rangle$ to the ancilla excitation, which can be read out. To find an optimal condition to create a pair-coherent state in the presence of the ac-Stark shift, we sweep the frequency of the four-wave mixing pump while measuring populations of various $\delta = 0$ states (Fig. 3.2a). As shown in Fig. 3.2b, after 15 $\mu$s of pumping with drive rates of $g_{ab} = 57.9$ kHz and $\epsilon_d = 760$ kHz, we
observe the full spectroscopy of the ancilla (Fig. 3.2b), which illustrates the Poisson-like distribution of a pair coherent state. We can also track these photon populations over time to understand how the system converges to a quasi-steady state with a pair photon drive and pair photon dissipation (Fig. 3.2c) or how it decays under the pair photon dissipation (Fig. 3.2d).

Fitting these time-domain data of pair photon population dynamics to numerical simulations of EQ 3.5, we extract pair photon dissipation rate $\kappa_{ab}/2\pi = 13.4$ kHz (from Fig. 3.2d) and subsequently the pair photon drive rate $\epsilon_{ab}/2\pi = 97$ kHz (from Fig. 3.2c). Together with the effective cross-Kerr of the system ($K_{\text{eff}}/2\pi = 85$ kHz), these rates are significantly faster than the undesirable single photon loss rates $\kappa_a$ and $\kappa_b$. Therefore, we experimentally create a quasi-steady pair coherent state $|\gamma = 1.55, \delta = 0\rangle$ under these driven dissipation conditions before single photon loss eventually decoheres the state by altering $\delta$. Quantum coherence of this quasi-steady state will be analyzed by joint Wigner tomography. Discrepancies between its population distributions and an ideal pair coherent state are due to small drive detunings and the 6th order Hamiltonian terms.

We note that different combinations of $n_a$ and $n_b$ can in principle result in similar dispersive shift to the ancilla, causing ambiguity in our measurement of cavity photon population. However, for our system with $3\chi_{qa} \approx \chi_{qb}$, this ambiguity arises only when the underlying Fock state $|n_a, n_b\rangle$ deviates from its expected $\delta$ by at least 4, which is highly unlikely to occur over time scales shorter than the single photon losses.

A prominent feature of the population dynamics in Fig. 3.2c is the oscillations between states that are then damped to a steady state, which can be viewed as the ground state of the driven dissipative system (whose effective Hamiltonian includes the drive, the Kerr, and the non-Hermitian dissipative term). Starting in a vacuum state and turning on drive tones, we non-adiabatically project the system into a superposition of the eigenstates of the driven Hamiltonian (all with $\delta = 0$), which oscillate due to their different eigen-energies. These oscillations are dictated by the Kerr (and 6th order number-conserving) terms, and indicate that the system dynamics are in the under-damped regime ($\kappa_{ab} < K_{\text{eff}}$). Nevertheless, the pair
Figure 3.3. Pair coherent state with $\Delta \neq 0$. (a) Level diagram of states with $\delta = -1, 0, 1$ connected by single photon loss operators. Each branch experiences the pair photon drive and dissipation stabilizing pair coherent states of equal complex amplitude $\gamma$. (b),(c) Number splitting experiment of the $\delta = 1$ ($\delta = -1$) pair coherent state. This measurement is performed by preparing a $|10\rangle$ ($|01\rangle$) Fock state with SNAP gates [4] and then performing the exact same procedure as shown in Fig. 3.2b creating the pair coherent distribution of $|\gamma, \delta = 1\rangle$ ($|\gamma, \delta = -1\rangle$). For each plot, dark vertical lines indicate the desired states while light vertical lines mark error states due to single photon loss or imperfect initial state prep.

Photon dissipation plays a crucial role of relaxing the system from higher energy states over a time scale of $1/\kappa_{ab}$ to the ground state – a pair coherent state. Although a dissipationless system (i.e. with pair-photon drive and cross-Kerr only) would also support pair coherent states as its driven ground state, application of the pair-photon loss bypasses the need for adiabatic ramping of the pump tones (as in the Kerr-cat experiment [59]) which would have led to a slower process than demonstrated here for our device parameters. This hybrid implementation of dissipation and Hamiltonian stabilization is analogous to recent proposals of combining Kerr cat qubits with engineered dissipation to further improve robustness against unwanted excitations [1, 79, 80].

3.5 Stabilized Manifold with Free $\delta$

An important property of the pair photon driven dissipation is that it conserves the photon number difference $\delta$. To be clear, we do not mean that the process
stabilizes the quantum state to a particular value of $\delta$. Quite the contrary: the process stabilizes pair coherent states of any $\delta$, allowing $\delta$ to be the only degree of freedom that is inherited from arbitrary initial values and may allow quantum operations in the presence of stabilizing pumps. For example, the $|\gamma = 1.55, \delta = 0\rangle$ state demonstrated in Fig. 3.2 is the unique steady state of the two-cavity system within the $\delta = 0$ subspace, and its $\delta$ is inherited from the initial vacuum state before the pair-photon pumping is applied. To create $\delta = 1$ or $-1$ pair coherent states, we can use a SNAP gate [4] to prepare a $|10\rangle$ or $|01\rangle$ initial state before applying the same pumping conditions.

More generally, the pair photon dynamics of the form of EQ 3.5 confine the two-cavity state to a quantum manifold (Hilbert subspace) spanned by a series of pair coherent states with a fixed $\gamma = \frac{e_{ab}}{\kappa_{ab}^{1/2} - K_{\text{eff}}} \ Amplified$ and an arbitrary integer $\delta$. Any superpositions (or mixtures) of these pair coherent states are steady states supported by the driven dissipation and stabilized against dephasing or (non-jump) energy decay in either cavities. For example, a $(|\gamma, 1\rangle + |\gamma, -1\rangle)/\sqrt{2}$ state can be created from an initial state of $(|10\rangle + |01\rangle)/\sqrt{2}$ and is stable under the pair-photon driven dissipation. This manifold confinement can be compared to the stabilization of a cat state manifold using single-mode two-photon processes (see Table 2.1). Instead of a 2d manifold spanned by cat states with photon number parity $\Pi = \pm 1$, here the manifold, in principle, has infinite dimensions with $\delta \in \mathbb{Z}$. Instead of parity flips between even and odd cat states, single photon loss shifts $\delta$ of pair coherent states between neighboring integers. At infinite pumping time, due to single photon losses, we expect that the state becomes a mixed state of different values of $\delta$, but that the state projected to individual $\delta$ remains a pure pair coherent state.

3.6 Measurement of Photon Number Difference

Since single photon loss is a fundamental decoherence channel in superconducting cavities, quantum non-demolition (QND) measurement of photon number difference (PND) followed by autonomous or digital feedback is a crucial component of the pair-cat QEC code [69]. Existing proposal of PND measurement
requires an exact negative $\chi$-matching condition, $\chi_{qa} = -\chi_{qb} = \chi$. For a transmon ancilla, this negative $\chi$ matching condition does not occur naturally but can be achieved with additional strong off-resonant pumps [81]. It should be noted that managing multiple strong off-resonant pumps in cQED devices without incurring instabilities remains a challenge [82].

Here we introduce an alternative method of PND measurement without $\chi$ matching: We apply a comb of number-selective $\pi$-pulses to excite the ancilla at frequencies corresponding to the dispersive shifts for all cavity states (below a reasonable truncation) with a targeted $\delta$. Subsequent readout of the ancilla informs whether the PND of the two-cavity state is equal to $\delta$ or not. For instance, we could superimpose pulses at $\omega_q - \chi_{n,n}$, where $0 \leq n \leq 5$, to inquire whether the system is in $\delta = 0$. In Fig 3.2b,c we use this measurement to probe the populations in the $\delta = \pm 1$ states, which shows the effect of single-photon loss. This technique of PND measurement is in principle QND, as long as the phases of the $\pi$-pulses are tuned to be effectively equal after compensating for any ac Stark shift and Kerr effects. It can then be used to repetitively monitor single photon loss from a state with known $\delta$ so long as $|00\rangle$, $|10\rangle$, and $|01\rangle$ are number resolved and a recovery operation can be applied before the next photon loss occurs. Therefore, it provides a practical avenue for QEC of single photon loss in the pair cat code, relaxing a demanding requirement for implementation. Our experimental setup does not have a quantum-limited parametric amplifier to carry out high-fidelity single-shot readout necessary to demonstrate the QNDness of this PND measurement. However, we find an important use of the PND measurement for performing tomography of our two cavity system.

### 3.7 Using PND for State Tomography

To demonstrate non-classicality and quantify quantum coherence of the pair coherent state in our experiment, we measure the joint Wigner function [8]

$$W_J(\alpha, \beta) = \frac{4}{\pi^2} Tr[H_a(\alpha) H_b(\beta) P_J H_b(\beta)^\dagger H_a(\alpha)^\dagger],$$

(3.6)
Figure 3.4. **Pair coherent state tomography.** (a) Joint Wigner tomography of the $\delta = 1, 0, -1$ states after 15 $\mu$s of pumping. Below are corresponding theoretical plots for ideal pair coherent states with $\delta = 1, 0, -1$ after 1 $\mu$s of Kerr evolution. (b) Tomography of the $\delta = 0$ state with $|\alpha| = |\beta| = .3$ sweeping over the angle of both displacements. Position of the peaks and valleys along the diagonal cut gives the phase of $\gamma$. (c) Joint Wigner tomography measurement protocol. After preparing a pair coherent state with two tone pumping, we displace the cavities and then simultaneously and selectively rotate either all even PND states or all odd PND states. (d) Phase and amplitude of gamma over time as the state stabilizes. Phase of $\gamma$ is inferred from cuts of (b) at different times. $|\gamma|$ is calculated from the value that best fits the population data in Fig. 3.2c.
where $\hat{P}_T = e^{2\pi i (\hat{n}_a + \hat{n}_b)}$. In previous publications this involves a complex procedure [51], which is made difficult due to unequal spacing of dispersive shifts that we suspect is caused by level crossings. Instead we use the PND measurement as a novel method of measuring joint parity.

Due to the previously described feature of our system that $3\chi_a \approx \chi_b$, measurements of $\delta = 0$ probe not only all states where $\delta = 0$ but also all states for which $\delta \equiv 0 \pmod{4}$. This mod4 behavior is due to different cavity states $|n_a, n_b\rangle$ inducing the same dispersive shift to the ancilla, $\chi(n_a, n_b) \approx \chi(n_a - 3, n_b + 1) \approx \chi(n_a + 3, n_b - 1)$. Instead of mapping one $\delta$ population to the ancilla, we can use a comb containing multiple $\delta$ value detunings to map multiple $\delta$ populations to the ancilla. With the mod4 behavior, a measurement of $\delta = 0, 2$ probes the population of every even PND state while a measurement of $\delta = 1, -1$ probes the population of every odd PND state (see Fig. 3.4b). By taking the the difference between these experiments we measure the PND parity of our state. Note that while the term photon number difference indicates how the measurement is performed, PND parity is exactly identical to joint photon number parity. The primary limitations of this measurement are the photon number truncation and the breakdown of $3\chi_a \approx \chi_b$ for large values of $\delta$. To minimize these two imperfections, we limit the absolute value of our cavity displacements to ensure that the maximal photon number and the maximal PND is sufficiently small after cavity displacement.

Measuring PND over the 4d phase space of both cavities, complex displacements $\alpha$ and $\beta$, gives us a sufficient data set to perform density matrix reconstruction. We are currently performing and processing this experiment to extract the fidelity of our pair coherent states. Sampling this 4d space is experimentally intensive, so to confirm the stability of the phase over time we measure just a 1d cut, sweeping the combined cavity phase with fixed differential cavity phase as in Fig. 3.4c. Inferring the amplitude of $\gamma$ from our population measurements and the phase from the 1d cut, we can track the value of $\gamma$ over time, demonstrating state stabilization.
3.8 Pump Tuneup Procedure

To generate and stabilize pair coherent states we need to first find the correct pumping condition. This involves setting both the frequency and power of the reservoir and mixing drives as well as the two photon drive phase. By measuring $\omega_a$, $\omega_b$ and $\omega_r$ through system calibration we have a good initial guess as to the frequency of both drives. We want to set the mixing drive power as high as possible without causing significant unwanted consequences to the system, such as ancilla heating, as we want to maximize $\kappa_{ab}/\kappa_{a,b}$. The tuneup procedure involves an iterative guess-and-check for the reservoir and mixing power.

Our first step of the check procedure is to find the correct mixing frequency to enable the pair photon dissipation $\kappa_{ab}$. To do this we first need a pre-sequence that prepares some state with photon pairs. The exact state is not critical so long as the state is largely $\delta = 0$ and does not have a large $|00\rangle$ population. To generate this state we set the reservoir drive on resonance and sweep the mixing drive frequency. Once we have this pre-sequence we use it for a spectroscopy measurement, sweeping the mixing drive frequency and measuring the $|00\rangle$ population at fixed time. Pumping with just the mixing drive (not reservoir) enables the pair photon dissipation that we want to maximize. The mixing frequency that maximizes the $|00\rangle$ population also maximizes the pair photon dissipation. Fortunately, the reservoir drive causes minimal Stark shifts to the system so the ideal mixing frequency is the same with or without the reservoir drive on. Having found the ideal mixing frequency, we sweep the reservoir drive frequency, looking for a state that most closely matches a pair coherent state in Fock state populations. This procedure gives us both frequencies for any choice of mixing and reservoir power. We then proceed to increase the reservoir power with this procedure until we achieve the largest manageable pair coherent state possible with our system and mixing power constraint.
Table 3.1. System parameters.
†The transmon $T_2^*$ reflects any $1/e$ decay time of Ramsey oscillations.
‡ During pumping the equilibrium excited state population rises to $\sim 7\%$ largely due to the strength of the mixing tone.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Transmon frequency</td>
<td>$\omega_q/2\pi$ 5378 MHz</td>
</tr>
<tr>
<td>Transmon anharmonicity</td>
<td>$\alpha_q/2\pi$ 204 MHz</td>
</tr>
<tr>
<td>Transmon $T_1$</td>
<td>$T_{1q}$ 40 $\mu$s</td>
</tr>
<tr>
<td>Transmon $T_2^*$ Ramsey</td>
<td>$T_{2q}^*$ 18 $\mu$s †</td>
</tr>
<tr>
<td>Transmon $T_2$ Echo</td>
<td>60 $\mu$s</td>
</tr>
<tr>
<td>Transmon $</td>
<td>e\rangle_q$ population</td>
</tr>
<tr>
<td>Reservoir frequency</td>
<td>$\omega_r/2\pi$ 7409 MHz</td>
</tr>
<tr>
<td>Reservoir-transmon coupling</td>
<td>$\chi_{r/2\pi}$ 3.19 MHz</td>
</tr>
<tr>
<td>Reservoir $T_1$</td>
<td>$1/\kappa$ 160 ns</td>
</tr>
<tr>
<td>Cavity A frequency</td>
<td>$\omega_A/2\pi$ 4072 MHz</td>
</tr>
<tr>
<td>Cavity A-transmon coupling</td>
<td>$\chi_q/2\pi$ 1.89 MHz</td>
</tr>
<tr>
<td>Cavity A anharmonicity</td>
<td>$K/2\pi$ 5 kHz</td>
</tr>
<tr>
<td>Cavity A $T_1$</td>
<td>$T_{1A}$ 530 $\mu$s</td>
</tr>
<tr>
<td>Cavity A $T_2$</td>
<td>$T_{2A}$ 400 $\mu$s</td>
</tr>
<tr>
<td>Cavity A $</td>
<td>1\rangle$ population</td>
</tr>
<tr>
<td>Cavity A-B coupling</td>
<td>$\chi_{AB}/2\pi$ 48 kHz</td>
</tr>
<tr>
<td>Cavity B frequency</td>
<td>$\omega_B/2\pi$ 6094 MHz</td>
</tr>
<tr>
<td>Cavity B-transmon coupling</td>
<td>$\chi_q/2\pi$ 6.26 MHz</td>
</tr>
<tr>
<td>Cavity B anharmonicity</td>
<td>$K/2\pi$ 71 kHz</td>
</tr>
<tr>
<td>Cavity B $T_1$</td>
<td>$T_{1B}$ 216 $\mu$s</td>
</tr>
<tr>
<td>Cavity B $T_2$</td>
<td>$T_{2B}$ 200 $\mu$s</td>
</tr>
<tr>
<td>Cavity B $</td>
<td>1\rangle$ population</td>
</tr>
</tbody>
</table>
3.9 Simulation for Fitting Rates

To find the rates of two-photon pumping $\epsilon_{ab}$ and two-photon dissipation $\kappa_{ab}$ from the time domain data of pumping the pair-coherent state from vacuum state as in Fig.3.2c and its decay after 20$\mu$s of pumping as in Fig.3.2d, respectively, we run a QuTiP simulation to solve the following equation,

$$\partial_t \hat{\rho} = -i[\hat{H}_{Kerr} + (\epsilon_{ab} \hat{a} \hat{b} + h.c.), \hat{\rho}] + \kappa_{ab} D[\hat{a} \hat{b}] \hat{\rho}$$

$$+ \kappa_a D[\hat{a}] \hat{\rho} + \kappa_b D[\hat{b}] \hat{\rho}$$

This master equation represents a system after adiabatically eliminating the reservoir and working in the rotating frame of the cavities. Initially, we run a spectroscopy experiment to find the population of different Fock states after 20$\mu$s of pumping with the drives and use that as an initial state to simulate the evolution of pair coherent states under the two-photon dissipation without the drive term in the Hamiltonian, to fit to the time domain data of Fig.3.2d and find the $\kappa_{ab}$ rate. We then use $\kappa_{ab}$ as a fixed parameter and introduce the two-photon pumping drive rate in the Hamiltonian to fit the time domain data of Fig.3.2c.

3.10 Effective $\gamma$ and Effective Kerr

To determine the amplitude ($\gamma$) of the pair coherent state generated by the system, we must look at the Hamiltonian of the system and find the steady state solution of the time evolution. We can start with our original Hamiltonian with the $\hat{a} \hat{b}$ dissipator as written in EQ 2.12.

We can make a few simplifications, by defining the relative difference between $\hat{a}^\dagger \hat{b}^\dagger \hat{a} \hat{b}$ and $\hat{a}^\dagger \hat{a} \hat{a} \hat{a}$ as $\frac{\left|\hat{a}^\dagger \hat{a} \hat{a} \hat{a} - \hat{a}^\dagger \hat{b} \hat{a} \hat{b}\right|}{\hat{a}^\dagger \hat{a}^\dagger \hat{a} \hat{a}} |\gamma, \delta\rangle$ which scales as $\frac{|\delta - 1|}{n_a - 1}$ (with $\frac{|n_b - n_b - \delta - \delta^2|}{n_b - n_b - 2n_b - \delta - \delta^2}$ $\approx \frac{|\delta - 1|}{n_b - 1}$ when $n_b \gg \delta$ for the $\hat{b}$ analog), where $n_a$ ($n_b$) is the expected photon number in cavity a (b), which is very small if we work in the limit $n_a, n_b \gg \delta$. Working in this limit allows us to safely make the approximation of grouping all the Kerr terms into one term, $K_{eff} = K_{ab} + K_{aa}^2 + K_{bb}^2$, multiplied by the $\hat{a}^\dagger \hat{b}^\dagger \hat{a} \hat{b}$ operator. We can further simplify the time evolution of the Hamiltonian by grouping some of the terms.
in the Lindblad dissipator into the Hamiltonian itself, allowing us to write a simplified equation for the density matrix ($\rho$) time evolution:

$$\dot{\rho} = -i(\hat{H}_{eff}\rho - \rho\hat{H}_{eff}^\dagger) + \kappa_{ab}\hat{a}\hat{b}\rho\hat{a}^\dagger\hat{b}^\dagger$$

(3.8)

$$\hat{H}_{eff} = \epsilon_{ab}\hat{a}\hat{b} + \epsilon_{ab}^*\hat{a}^\dagger\hat{b}^\dagger + K_{eff}\hat{a}^\dagger\hat{b}^\dagger\hat{a}\hat{b} - i\kappa_{ab}\hat{a}^\dagger\hat{b}^\dagger\hat{a}\hat{b}$$

(3.9)

Assuming we have a pair coherent state, we can write the density matrix time evolution as:

$$\dot{\rho} = (\epsilon_{ab}\gamma - \epsilon_{ab}^*\gamma^* + i\kappa_{ab}|\gamma|^2)\rho$$

$$+ (\epsilon^* + K_{eff}\gamma - i\frac{\kappa_{ab}}{2}\gamma)\hat{a}^\dagger\hat{b}^\dagger\rho$$

$$- (\epsilon + K_{eff}\gamma^* + i\frac{\kappa_{ab}}{2}\gamma^*)\rho\hat{a}\hat{b}$$

(3.10)

Setting the terms that contain $\hat{a}^\dagger\hat{b}^\dagger\rho$ or $\rho\hat{a}\hat{b}$ to 0 gives us the following expression for the expected value of $\gamma$:

$$\gamma = \frac{\epsilon^*}{\frac{i\kappa_{ab}}{2} - K_{eff}}$$

(3.11)
Figure 3.6. Density matrix reconstruction (a)-(f) Comparison of the 2-D Wigner cuts from experiment (left) against the reconstructed density matrix (right) for a) Re(α) vs. Re(β), b) Re(α) vs. Im(α), c) Re(α) vs. Im(β), d) Im(α) vs. Im(β), e) Im(α) vs. Re(β), and f) Re(β) vs. Im(β) respectively. (g) Color plot of the reconstructed density matrix with only the elements corresponding to the δ = −1, 0, 1 states.
3.11 Prediction of Distribution of $\delta$ States from Loss

Given a pair coherent state that is allowed to freely evolve with single photon loss from each cavity while being stabilized by two photon loss, we would like to be able to predict the distribution of the populations of each $\delta$ state by finding the $C_{\delta}$ coefficients in the following equation:

$$\rho(t = \infty) = \sum_{\delta = -\infty}^{\infty} C_{\delta} |\gamma, \delta\rangle \langle \gamma, \delta|$$  

(3.12)

We are working in the regime where the two photon loss stabilization is much stronger than the single photon loss ($\kappa_{ab} >> \kappa_a, \kappa_b$), so we can assume that after a single photon loss event happens, creating a change in $\delta$ by 1, the state can be treated as though it is effectively instantaneously stabilized to a pair coherent state of the new $\delta$ while maintaining the state amplitude $\gamma$. With this assumption in place, we can treat the problem as a simple equilibrium solution to the different delta states and their respective probabilities to jump to neighboring states based on the photon populations in each cavity, as illustrated in Fig. 3.5. We can represent the photon populations at a given $\gamma$ and $\delta$ ($\bar{n}_a(\gamma, \delta)$) in cavity $a$ with the following expression [65]:

$$\bar{n}_a(\gamma, \delta) = \frac{N_{\gamma,\delta}^2 |\gamma|^2}{N_{\gamma,\delta+1}^2} + \delta,$$

$$N_{\gamma,\delta} = ((i|\gamma|)^{-\delta} J_\delta(2i|\gamma|))^{-1/2}$$  

(3.13)

where $J_\delta$ is the Bessel function of the first kind of order $\delta$. We can now construct the rates of photon loss $\kappa_i \bar{n}_{a/b}(\gamma, \delta)$ from cavity $a, b$. Treating each state with an outgoing rate due to photon loss and an incoming rate from photon loss of neighboring states allows us to create the following equilibrium equation for each $\delta$ value:

$$C_{\delta}(\kappa_a \bar{n}_a(\gamma, \delta) + \kappa_b \bar{n}_b(\gamma, \delta)) - C_{\delta+1}(\kappa_a \bar{n}_a(\gamma, \delta)) - C_{\delta-1}(\kappa_b \bar{n}_b(\gamma, \delta)) = 0$$  

(3.14)
Numerically solving this set of coupled equations for all the desired $\delta$ states yields a solution for the distribution of $C_\delta$ values.

### 3.12 Density Matrix Reconstruction

To reconstruct the density matrix from a set of Wigner tomography slices with varying displacements on both cavities, we implemented a general N-mode numerical density matrix reconstruction method outlined in Reinhold [83]. We can treat the Wigner tomography data as general measurements $M$, where we can express the measurement outcomes $x$ in terms of the density matrix $\rho$ as:

$$x = Tr(M\rho) = \sum_{ij} M_{ij}\rho_{ji} \equiv \langle\langle M|\rho\rangle\rangle$$  \hfill (3.15)

Taking $N_{exp}$ measurements gives us the following relation:

$$\vec{x} = (\sum_{i}^{{N_{exp}}} x_i \langle\langle M_i|\rho\rangle\rangle \equiv \mathcal{M}|\rho\rangle) \hfill (3.16)$$

Since $\mathcal{M}$ will not generally be a square matrix, we can take a pseudo inverse to get the closest approximate solution using the Moore-Penrose pseudo-inverse:

$$\text{minimize } |\mathcal{M}|\rho\rangle - \vec{x} | \rightarrow |\rho_{\text{least-sq}}\rangle = (\mathcal{M}^T\mathcal{M})^{-1}\mathcal{M}^T\vec{x} \hfill (3.17)$$

To constrain the physicality of the reconstruction, we added a Lagrange multiplier constraint to the reconstruction that fixes the density matrix on diagonal population values to what was observed in the transmon spectroscopy data plotted in Fig. 3.2b:

$$\begin{pmatrix} \mathcal{M}^T \mathcal{M} & C^T \\ C & 0 \end{pmatrix} \begin{pmatrix} \vec{\rho} \\ \vec{\lambda} \end{pmatrix} = \begin{pmatrix} \mathcal{M}^T \vec{x} \\ d \end{pmatrix} \hfill (3.18)$$
where $C\tilde{\rho} = \tilde{d}$ represents the constraints. Since the input diagonals are all real and with the trace equaling one, further physicality constraints on the matrix, such as positive definite eigenvalues and $Tr(\rho) = 1$, are unnecessary.

### 3.13 Summary and Outlook

There is a growing number of cQED experiments motivated by the fields of optical and atomic systems being performed. Implementing interesting bosonic states with the convenience of microwave systems and the flexibility of superconducting circuitry has brought new life to interesting states such as the pair coherent state. These cross-field inspirations have pushed research towards bosonic QEC and have brought a change of pace in a field dominated by the race to a useful surface code.

The pair cat code, a promising new addition to the zoo of bosonic error correcting codes, offers a fault tolerant solution for many of the issues faced by single mode codes. Our experimental realization of a pair coherent state, demonstration of the first cross cavity dissipator, and introduction of a novel PND measurement are all important steps towards a pair cat code. Further experiments with single-shot readout and FPGA capabilities are required to demonstrate that our PND measurement is indeed QND. To suppress propagation of ancilla errors, this measurement will also likely need $\chi$ matching between the two storage modes [84]. Fully implementing the code will require upgrading the pair photon dissipation to a 4-photon cross cavity dissipator, a step involving a leap in experimental complexity. More work is also needed to implement strong off-resonant drives in the system without causing heating, either through the use of new ancillas or better filtering.
CHAPTER 4
AUTONOMOUS QUANTUM STATE TRANSFER

4.1 Introduction

The goal of Quantum State Transfer (QST) is analogous to its classical counterpart: take stored information and physically move it. For classical information we have many technological options ranging from simple wires inside a computer, to coaxial cables connecting internet nodes, to free propagating light waves that carry wifi, to loading it on a flash drive and flying it across the globe. The fundamentals of the problem are the same irrespective of the technology and while each method has its advantages and disadvantages, it is clear that the number of options has profound value for solving the range of modern communication challenges. All of these methods also have a common concern: the need for information transfer to be lossless, or at least, be lossless over the course of multiple attempts.

For quantum systems the problem of state transfer has the same need for lossless communication, but this is complicated by the delicate nature of quantum states and the difficulty of redundantly encoding information. The loss here is largely due to dissipative processes where the information decays due to errors such as the accidental absorption of an excitation or the collapse of a superposition resulting from an environmental measurement. With the growing understanding that dissipation can be engineered for interesting processes [74] that go beyond what is possible in unitary quantum dynamics [85] it’s interesting to consider how it can be useful for QST.
4.2 Reciprocal? Hermitian?

When we consider ideal quantum systems, we often think of a system entirely free of any dissipation. For this system, the swapping of information would rely on operations that exchange excitations, and therefore information, between two modes. These Hermitian processes are by definition reciprocal; reversing the operation will reverse the state transfer and restore the initial state. The precision of the external control decides the fidelity of the state transfer, as slightly too much or too little swapping will leave information behind. This undesired reciprocality can be avoided through measurement and its back-action on quantum states, but this only limits the transfer fidelity with the measurement fidelity. New operations are necessary to solve these problems.

Instead of viewing dissipation as an antagonist to QST, we consider its advantages. Dissipative systems allow for non-reciprocality, as was discussed in Chapter 3, where a number of initial conditions all converged to the same manifold of final states. Here we move beyond the idea of stabilizing a manifold of states and instead use dissipation as a map connecting meaningful initial states to useful final states.

4.3 Existing Directional Channels

Dissipation itself is not necessary for directionality. Work has been done on minimizing back action in quantum systems [86, 87, 88] to enable directionality after measurement. The idea of a directional channel [89] or cascaded system [90] is hardly new. Experimental results have demonstrated QST can be lossless [91], easily generalized to a wide range of applications [89, 92], and highly valuable for novel quantum simulations [93, 94] but these results suffer from errors incurred on the final storage location due to their direct coupling to some communication channel [90]. The common solution, as mentioned before, is to use a time-dependent control to couple the final storage to a directional channel for a small period of time [95, 96, 97]. With this in mind it is clear that an autonomous implementation of QST is a logical next step.
4.4 Directional Pumping Channel

A common theme of this work is the need to autonomously and directionally transfer from a set of initial states to a set of final states. For this we use directional pumping channels (DPC). A DPC is a coherent process that transforms an initial state into a target final state modified by the addition of one excitation to a Markovian reservoir $|\psi_i\rangle |g\rangle_{res} \rightarrow |\psi_f\rangle |e\rangle_{res}$. The decay of this reservoir excitation is non-reciprocal, leaving the system in the target final state $|\psi_f\rangle |g\rangle_{res}$. If this reservoir decay happens faster than the other time scales of the system, we can adiabatically eliminate the dynamics of the reservoir, realizing a jump operator of the form $\hat{J} = |\psi_f\rangle \langle \psi_i|$. In cQED an excellent tool for accomplishing this is junction-mediated four wave mixing (FWM), which allows us to effectively engineer arbitrary Hamiltonian terms of up to third order in mode creation and annihilation operators.

By combining multiple DPCs that use the same reservoir we can create a dissipative map that connects any number of initial and final states, written as

$$\hat{J} = \sum_k \sqrt{\Gamma_k} |\psi^k_f\rangle \langle \psi^k_i|$$

for rates $\Gamma_k$ connecting initial states $|\psi^k_i\rangle$ to final states $|\psi^k_f\rangle$. There is no requirement that initial and final states are unique, and this map need not be one-to-one and onto, but in this thesis all examples will share that feature.

It may not be immediately obvious why using a DPC can accomplish beyond the promises of Optimal Control Theory (OCT) pulses [5]. A DPC has two desirable characteristics: it is non-reciprocal and autonomous. The first is especially convenient for quantum state transfer, as normal unitary transformations run the risk of converting final state population to initial state with any timing errors. In addition a DPC can be run time-independently and without any active feedback measurements. When we use the same reservoir mode for multiple DPCs and match the transition rates, we can construct a directional map that coherently transfers multiple states simultaneously. Because we can excite the same reservoir for
Figure 4.1. A schematic diagram of AQST. Encoded quantum state $|\psi\rangle$ is spontaneously emitted from a subsystem $A$ and fully absorbed by another subsystem $B$. This is realized via a directional coupling channel which is blind to $|\psi\rangle$.

multiple DPCs, the reservoir photon emission carries no information about which DPC path was taken, preventing collapse of a quantum state.

4.5 Autonomous Quantum State Transfer: Objective

We have an arbitrary single qubit state $|\psi\rangle$ that resides in mode Alice $|\psi\rangle_A$. We want to transfer this state to mode Bob $|\psi\rangle_B$ with no change to the state. While not strictly necessary, Bob will start in vacuum $|\text{vac}\rangle_B$, which could be any state. We likewise will need to leave Alice in some final state $|\text{vac}\rangle_A$. To be autonomous this transfer must not disturb an already transferred state. We can write these requirements as

\[
|\psi\rangle_A |\text{vac}\rangle_B \rightarrow |\text{vac}\rangle_A |\psi\rangle_B \\
|\text{vac}\rangle_A |\psi\rangle_B \rightarrow |\text{vac}\rangle_A |\psi\rangle_B
\] (4.2)

4.6 Minimum System Size, 2x2 or 3x2?

Looking at EQ 4.2 it is not immediately obvious how big our Hilbert space needs to be for this operation. Its obvious that $|\psi\rangle$ requires two states to hold a qubit of information so both Alice and Bob must have at least two states each. Writing out this equation again as a DPC with our initial state stored in $\alpha |0\rangle_A + \beta |1\rangle_A$. 

55
\begin{equation}
J_{QST} = \sqrt{\Gamma} \left[ |\text{vac}\rangle_A |0\rangle_B \langle 0|_B + |\text{vac}\rangle_A |1\rangle_B \langle 0|_B \langle 1|_A \right],
\end{equation}

where $|\text{vac}\rangle_A$ is yet undefined. For $|\text{vac}\rangle_A = |0\rangle_A$ or $|1\rangle_A$ one of our final states will still be affected by this DPC breaking the autonomous requirement. To solve this issue we introduce a third level to Alice, which we use as our vacuum state $|\text{vac}\rangle_A = |2\rangle_A$ making our Hilbert space 3x2. We can now rewrite EQ 4.3 as

\begin{equation}
J_{QST} = \sqrt{\Gamma} \left[ |2\rangle_A |0\rangle_B \langle 0|_B + |2\rangle_A |1\rangle_B \langle 0|_B \langle 1|_A \right]
\end{equation}

### 4.7 Quantum State Transfer: cQED Scheme

To transfer one quantum bit of information we require two DPCs that both excite the same reservoir. To achieve this we transfer information from a three-level qutrit (mode A) to a two-level qubit (mode B) (Fig. 4.2). For our scheme we use a transmon as the qutrit and the lowest two Fock states of a harmonic oscillator as the qubit (no higher Fock states are excited in the scheme). We also employ a two level reservoir (mode R) constructed from a transmon-like structure that is strongly coupled to a transmission line. The circuit diagram is shown in Fig. 4.2a, which includes the necessary ancilla qubit and readout resonators to characterize the system, prepare initial states, and readout the final state.

The initial logical qubit is encoded in the $|e\rangle$ and $|f\rangle$ states of qutrit A while the final logical qubit is the $|g\rangle$ and $|e\rangle$ states of cavity B. Fig 4.2b details the two DPCs. Due to the nature of the cQED Hamiltonian the photons emitted from the two paths have slightly different frequencies that result from the dispersive coupling between reservoir R and cavity B $\chi_B$. To combat this we drive the FWM transitions to slightly detuned virtual states (shown by the dotted lines). This does incur a phase error although it scales as $(\kappa/\chi_b)^{-2}$, which is favorable for our parameters.

#### 4.7.1 Driven Josephson Circuit Hamiltonian

To understand our system we perform the same derivation as in Chapter 2 but with two main differences. There are two drive tones instead of one and there are two
Figure 4.2. Implementation of AQST in circuit QED. (a) Effective circuit diagram including a transmon qutrit $A$, a storage cavity $B$, a reservoir transmon $R$, and auxiliary elements for state preparation and readout. (b) Energy-level diagram that shows the state transfer paths. The quantum state initially encoded in $A$ is driven to a pair of virtual states by slightly-detuned Rabi drives with equal rate $\Omega$ (straight arrows), and subsequently decay to the final states in $B$ by reservoir dissipation (twisted arrows). (c) Numerical results of transferring an equator state, including decoherence, showing fidelity of instantaneous state $\rho$ against target state $\sigma$, $\mathcal{F} = [\text{Tr}(\sqrt{\rho} \sqrt{\sigma})]^{1/2}$ as a function of time during the transfer. Simulation parameters correspond to $A$ and $R$ having frequencies of 5.6 GHz and 8 GHz, anharmonicities of 78 MHz and 210 MHz, and a dispersive ($ZZ$) coupling of $\chi_a = 4.2$ MHz between them. Different color curves are simulated for different $\chi_b$ and their corresponding optimal $\kappa$. Inset shows the ideal-case infidelity due to rotation of the virtual states in the drive frame, which scales as $(\kappa/\chi_b)^{-2}$. 

(a) Effective circuit diagram including a transmon qutrit $A$, a storage cavity $B$, a reservoir transmon $R$, and auxiliary elements for state preparation and readout. (b) Energy-level diagram that shows the state transfer paths. The quantum state initially encoded in $A$ is driven to a pair of virtual states by slightly-detuned Rabi drives with equal rate $\Omega$ (straight arrows), and subsequently decay to the final states in $B$ by reservoir dissipation (twisted arrows). (c) Numerical results of transferring an equator state, including decoherence, showing fidelity of instantaneous state $\rho$ against target state $\sigma$, $\mathcal{F} = [\text{Tr}(\sqrt{\rho} \sqrt{\sigma})]^{1/2}$ as a function of time during the transfer. Simulation parameters correspond to $A$ and $R$ having frequencies of 5.6 GHz and 8 GHz, anharmonicities of 78 MHz and 210 MHz, and a dispersive ($ZZ$) coupling of $\chi_a = 4.2$ MHz between them. Different color curves are simulated for different $\chi_b$ and their corresponding optimal $\kappa$. Inset shows the ideal-case infidelity due to rotation of the virtual states in the drive frame, which scales as $(\kappa/\chi_b)^{-2}$. 

57
junctions instead of one. Both of these changes are relatively easy to accommodate as the effect of the different tones, and different junctions, can be considered separately and then recombined later.

The system Hamiltonian incorporating two microwave drives with angular frequencies $\omega_1, \omega_2$ and amplitudes $\epsilon_1, \epsilon_2$ can be written as [57]

$$\hat{H} = \hbar\omega_A \hat{a}^\dagger \hat{a} + \hbar\omega_B \hat{b}^\dagger \hat{b} + \hbar\omega_R \hat{r}^\dagger \hat{r} - \sum_{i=I, II} E_{ji} \left[ \cos \left( \hat{\varphi}_i \right) + \frac{\varphi_i^2}{2} \right]$$

$$+ \sum_{k=1,2} \hbar\epsilon_k \cos(2\omega_k t)(\hat{r} + \hat{r}^\dagger)$$

where $\hat{a}^\dagger, \hat{b}^\dagger$ and $\hat{r}^\dagger$ are creation operators of LC oscillator modes that are closely associated with $A, B$ and $R$. $E_{ji}$ is the Josephson inductance of junction $i$ (= I or II). The phases across the junctions I and II are given by

$$\hat{\varphi}_i = \phi_{Ai}(\hat{a}^\dagger + \hat{a}) + \phi_{Bi}(\hat{b}^\dagger + \hat{b}) + \phi_{Ri}(\hat{r}^\dagger + \hat{r}),$$

where $\phi_{Xi}$ is the zero-point flux fluctuation of mode $X$ (=A, B, or R) across junction $i$. We use junction I as a resource of three-body nonlinear coupling with relatively large $\phi_{Ai}\phi_{Bi}\phi_{Ri}$ product. junction II provides anharmonicity to make $A$ a usable qutrit, whose coupling to $B$ is negligible (i.e. $\phi_{Bi} \approx 0$) and contributes relatively little to the state transfer process. Because the junctions I and II are located within transmons $R$ and $A$ respectively, $\phi_{Ri}, \phi_{AII} \gg$ all other $\phi_{Xi}$’s.

We perform two unitary transformations to the system, instead of just one,

$$\hat{U}_{1,2} = \hat{D}(-\xi_{1,2} e^{-i\omega_{1,2} t}),$$

where we can define the amplitudes of these displacements.

$$\xi_{1,2} \approx i\epsilon_{1,2}/(\kappa_A + i(\omega_A - \omega_1,2)),$$
These values are dependent on the detunings between the reservoir mode and the drive tones, the reservoir linewidth, and the strength of the drive tones. Keeping terms up to 4th order in the expansion of the junction energies, our Hamiltonian now appears as:

\[
\hat{H} = \hbar \omega_A \hat{a} \hat{a} + \hbar \omega_B \hat{b} \hat{b} + \hbar \omega_R \hat{r} \hat{r} - \sum_{i=I,II} \frac{E_{ji}}{24} [\phi_{Ai} \hat{a} + \phi_{Bi} \hat{b} + \phi_{Ri} (\hat{r} + \tilde{\xi}_1 + \tilde{\xi}_2) + h.c.]^4. \tag{4.9}
\]

For simplicity of the derivation, we assume that the drive is equally effective in coupling to Junctions I and II. This is not essential for the experiment as Junction II contributes very little to the conversion Hamiltonian in the state transfer.

The 4th order expansion in EQ (4.9) contains a large number of terms, but if we apply rotating-wave approximations (RWA), the only stationary terms are diagonal terms (which preserve excitation numbers in all modes) and off-diagonal terms that convert excitations between specific modes if certain frequency-matching conditions are satisfied. In this case, we choose drive frequencies \( \omega_1 \) and \( \omega_2 \) close to the following frequencies

\[
\omega_1 \approx \omega_B + \omega_R - \omega_A \tag{4.10}
\]
\[
\omega_2 \approx 2 \omega_A - \omega_R \tag{4.11}
\]

Under RWA, we have:

\[
\hat{H} = \hat{H}_0 + \hat{H}_{SS} + \hat{H}_{Kerr} + \hat{H}_{conv} \tag{4.12}
\]

\[
\frac{\hat{H}_0 + \hat{H}_{SS}}{\hbar} = (\omega_A - \delta \omega_A) \hat{a} \hat{a} + (\omega_B - \delta \omega_B) \hat{b} \hat{b} + (\omega_R - \delta \omega_R) \hat{r} \hat{r} \tag{4.13}
\]

\[
\frac{\hat{H}_{Kerr}}{\hbar} = -\alpha_A \hat{a} \hat{a}^2 - \alpha_B \hat{b} \hat{b}^2 - \alpha_R \hat{r} \hat{r}^2 - \chi_{AB} \hat{a} \hat{b} \hat{b} \hat{r} - \chi_{AR} \hat{a} \hat{r}^2 \hat{r} - \chi_{BR} \hat{b} \hat{r} \hat{r} \tag{4.14}
\]

\[
\frac{\hat{H}_{conv}}{\hbar} = (\Omega_1 e^{-i \omega_1 t} \hat{a} \hat{b} \hat{r} + \Omega_2 e^{-i \omega_2 t} \hat{a} \hat{a}^2 \hat{r} + h.c.) \tag{4.15}
\]
Here $\hat{H}_0$ is the original linear contributions of the modes, $\hat{H}_{SS}$ contains the Stark shifts caused by both drives and the Lamb shift caused by junction anharmonicity:

$$\delta\omega_X = \sum_{i=I,II} \frac{E_{ji}}{\hbar} \left( \phi_{Xi}^2 \phi_{Ri}^2 |\bar{\xi}_i|^2 + \phi_{Xi}^2 \phi_{Ri}^2 |\bar{\xi}_i|^2 + \frac{1}{2} \phi_{Xi}^4 \right)$$  \hspace{1cm} (4.16)

$\hat{H}_{Kerr}$ contains the anharmonicities (self-Kerr, $\alpha_X$) of the modes and the dispersive shifts (cross-Kerr, $\chi_{XY}$) between the modes:

$$\alpha_X = \sum_{i=I,II} \frac{E_{ji}}{2\hbar} \phi_{Xi}^4$$  \hspace{1cm} (4.17)

$$\chi_{XY} = \sum_{i=I,II} \frac{E_{ji}}{\hbar} \phi_{Xi}^2 \phi_{Yi}^2$$  \hspace{1cm} (4.18)

$\hat{H}_{conv}$ describes the targeted four wave mixing terms with the Rabi drive rates

$$\Omega_1 = \sum_i \frac{E_{ji}}{2\hbar} \xi_1 \phi_{Ai} \phi_{Bi} \phi_{Ri}^2$$  \hspace{1cm} (4.19)

$$\Omega_2 = \sum_i \frac{E_{ji}}{2\hbar} \xi_2 \phi_{Ai}^2 \phi_{Ri}^2$$  \hspace{1cm} (4.20)

To implement the protocol, $\xi_1$ and $\xi_2$ are chosen to satisfy $\Omega_1 = \Omega_2 \equiv \Omega$, and $\omega_1$, $\omega_2$ are chosen specifically as:

$$\omega_1 = \left[ (\omega_B - \delta\omega_B) + (\omega_R - \delta\omega_R) - \chi_{BR} \right] - \left[ \omega_A - \delta\omega_A \right] + \delta_1$$  \hspace{1cm} (4.21)

$$\omega_2 = \left[ 2(\omega_A - \delta\omega_A) - \alpha_A \right] - \left[ \omega_R - \delta\omega_R \right] - \delta_2$$  \hspace{1cm} (4.22)

$\delta_1$ ($\delta_2$) represents a small detuning of the drive tone 1 (2) from the transitions $|eg,g\rangle \rightarrow |ge,e\rangle$ ($|fg,g\rangle \rightarrow |gg,e\rangle$) after accounting for Stark shifts.

Now we apply the following transformation to go into a rotating frame:
\[ |0\rangle_B \equiv |gg, g\rangle = |gg, g\rangle \] (4.23)

\[ |1\rangle_B \equiv |ge, g\rangle = e^{-i\omega_B} |ge, g\rangle \] (4.24)

\[ |0\rangle_A \equiv |fg, g\rangle = e^{-i(2\omega_A - \alpha_A)} |fg, g\rangle \] (4.25)

\[ |1\rangle_A \equiv |eg, g\rangle = e^{-i\omega_A} |eg, g\rangle \] (4.26)

\[ |0\rangle_e \equiv |gg, e\rangle = e^{-i(\omega_R - \alpha_e)} |gg, e\rangle \] (4.27)

\[ |1\rangle_e \equiv |ge, e\rangle = e^{-i(\omega_R + \omega_B - \chi_b - \delta_1)} |ge, e\rangle \] (4.28)

where \( \chi_b \equiv \chi_{BR} \) is the simplified notation for the dispersive shift between B and R and \( |gg, g\rangle \) is a reduced notation for \( |g\rangle_A |g\rangle_B |g\rangle_R \). The system Hamiltonian EQ (4.12), within the Hilbert space of the 6 relevant states, is transformed to:

\[
\hat{H} = (\chi_b + \delta_1) |ge, e\rangle \langle ge, e| + \delta_2 |gg, e\rangle \langle gg, e| + \Omega (|ge, e\rangle \langle eg, g| + |gg, e\rangle \langle fg, g| + h.c.)
\] (4.29)

The reservoir loss operator \( \hat{L} = \sqrt{\kappa R} \) is transformed to:

\[
\hat{L}_{rot} = \sqrt{\kappa R} (|gg, g\rangle \langle gg, e| + e^{i(\delta_2 - \delta_1 - \chi_b)t} |ge, e\rangle \langle ge, e|)
\] (4.30)

The-time dependent phase factor in \( \hat{L}_{rot} \) indicates a dephasing effect due to the energy difference of the reservoir emission for logical \( |0\rangle \) versus \( |1\rangle \). To eliminate this error, we may choose detunings \( \delta_1 = -\chi_b/2 \) and \( \delta_2 = \chi_b/2 \) to make \( \hat{L}_{rot} \) stationary:

\[
\hat{L}_{rot} = \sqrt{\kappa R} (|gg, g\rangle \langle gg, e| + |ge, e\rangle \langle ge, e|)
\] (4.31)

Effectively, we drive the two sets of transitions through nearby virtual states to compensate for the dispersive shift of the real states. These symmetrically chosen detunings also ensure equal rates \( (= \sqrt{\Omega^2 + (\chi_b/2)^2} \) for the two detuned Rabi drives. In the main text and in the discussions below, we refer to all states in the rotating frame directly omitting the use of the “tilde” signs.
4.8 Simulation Parameters

Simulations of the cQED implementation were performed with QuTiP, a python-based master equation solver. This allows us to simulate both Hamiltonian terms and any potential jump/loss operator. Our system is composed of a 3-level system (A), and two 2-level systems (B and R) making a 12-level Hilbert space. We simulate under the rotating-wave approximation in the rotating frame defined by EQ (4.23)-(4.28), canceling out all bare mode energies. Initial states are encoded in a pure superposition of \(|fg, g\rangle\) and \(|eg, g\rangle\). The initial states are coupled to the states \(|gg, e\rangle\) and \(|ge, e\rangle\) through the four wave mixing (FWM) process, simulated through off-diagonal Hamiltonian terms with amplitude \(\Omega\) (EQ 4.29). These two intermediary states are slightly detuned from the bare states by \(\pm \chi_b\), as reflected by appropriate diagonal elements for these two states (EQ (4.29)) and other, non-computational states. The loss acting upon the reservoir mode is then stationary with respect to the intermediate states and can be simulated with a time-independent operator \(\hat{L}\) as in EQ (4.31).

In order to faithfully assess the potential of this scheme in cQED, we must consider a number of errors that could spoil the process fidelity. The errors here are simulated with loss operators: 1) Both A and B have intrinsic loss as well as Purcell loss introduced by couplings to R. 2) Loss out of the second excited state in A is simulated with twice the rate as loss from the first excited state. 3) The reservoir can be excited by a hot environment due to non-ideal thermalization, which is simulated with a rate \(\Gamma_{\uparrow}\). This is expected to be by far the dominant dephasing mechanism in A and B.

Simulation parameters were selected to be realistic with transmons in a coaxial 3D cavity architecture [55]. For all simulations we start with the two junction energies \((E_{Ji})\) and the zero-point fluctuation (ZPFs) across each junction from each mode \((\phi_{Xi})\) as shown in Table 4.1. Junction energies and the ZPFs, except \(\phi_{BI}\), are kept constant throughout all simulations. These parameters uniquely define the mode frequencies, anharmonicities and dispersive couplings as shown in Table 4.2 (with the exception of \(\omega_B\)) assuming A and R are transmon-like modes with no additional linear inductance. One can reduce \(\omega_A\) and \(\omega_R\) without impacting any other Hamiltonian
terms by introducing a linear inductance. Frequencies of all objects were kept in the 4-8 GHz range, convenient for most experimental setups, but they play no explicit role in the simulation because of the rotating-wave approximation.

We sweep the dispersive coupling between $B$ and $R$, $\chi_b$, by changing $\phi_{BI}$. For each $\chi_b$, because the relaxation rate of the reservoir can be engineered at will, it is always beneficial to maximize $\xi_{1,2}$ and hence the FWM rate $\Omega$ to the extent possible. $\xi$ is limited in practice by heating effects due to higher-order Josephson non-linearity [98], and we consider a conservative upper bound of $\xi_{\text{max}} = 0.3$, compared with $\xi \approx 0.35$ in Gao et. al. [99] and $\xi \approx 0.5$ in Rosenblum et. al. [81]. Because the linear cavity $B$ has the relatively small junction ZPFs ($\phi_{BI}$ and $\phi_{BII}$), larger $\xi_1$ than $\xi_2$ is needed to achieve $\Omega_1 = \Omega_2$, as indicated by EQ (4.20). Therefore, we always maximize $\Omega_1$ by maximizing $\xi_1$ and then choose $\xi_2$ accordingly, i.e. $\xi_2 < \xi_1 = 0.3$. For each $\chi_b$, after maximizing $\Omega$, we then sweep over values of $\kappa$ to maximize the state transfer fidelity. Increasing $\chi_b$ increases the transfer speed but has drawbacks we need to consider. Larger $\chi_b$ causes an increase in Purcell loss through the reservoir, an increase in dephasing from thermal shot noise of the reservoir, and an increase in intrinsic infidelity. We report overall state transfer fidelity as an average of the fidelities of the six cardinal points on the logical bloch sphere (between $|0\rangle_L$ and $|1\rangle_L$).

The transfer scheme requires a $3 \times 2$ system in addition to the reservoir mode, but only half of these states are inside the logical space of the transfer. States outside the useful computational space include $|eg,e\rangle$, $|ee,g\rangle$, $|ee,e\rangle$, $|fg,e\rangle$, $|fe,g\rangle$, and $|fe,e\rangle$. For these states we only have diagonal Hamiltonian terms, no couplings to other modes. Notably, none of these states are the result of acting on any of the computational states with any single loss operator, so energy relaxation will not bring us outside of the logical space. Because of its short $T_1$ time, for typical non-ideal thermalization the reservoir will have a significant $\Gamma_\uparrow$ rate, which can bring us to the states $|fg,e\rangle$ and $|eg,e\rangle$ and quickly relax back. This is accounted for in the simulation by considering a rate $\Gamma_\uparrow$ to be $\kappa/100$, corresponding to a thermal population of 1% for the reservoir, comparable to dissipative modes in Chen et. al. or Touzard et. al. [51, 76]. This effectively captures the dominant
Table 4.1. Simulation parameters. Josephson energy of Junctions I and II, and the zero point fluctuations (ZPF) across each of them due to excitations in modes A, B, and R. ZPF of B mode across junction I is a variable parameter in the simulation.

<table>
<thead>
<tr>
<th></th>
<th>Junction I</th>
<th>Junction II</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_J/2\pi$</td>
<td>40 GHz</td>
<td>56 GHz</td>
</tr>
<tr>
<td>$\phi_A$</td>
<td>0.03</td>
<td>0.23</td>
</tr>
<tr>
<td>$\phi_B$</td>
<td>0.0025-0.0141</td>
<td>0.002</td>
</tr>
<tr>
<td>$\phi_R$</td>
<td>0.32</td>
<td>0.01</td>
</tr>
</tbody>
</table>

dephasing effects in both A and B, and we do not consider any additional dephasing that A and B may experience. This is because the internal dephasing rate for fixed-frequency transmons or linear cavities, if there is any, is much smaller than other error rates in our simulation, and dephasing due to other peripheral (i.e. readout) modes can also be minimized by choosing relatively slow rates for them.

We do not account for a $\Gamma_\uparrow$ in A or B as we expect them to be negligibly small excluding the possibility of accessing the states $|fe, g\rangle$, $|ee, g\rangle$, $|ee, e\rangle$ and $|fe, e\rangle$. In addition, we have neglected the leakage error out of the 12-dimension Hilbert space, but its leading contribution from spurious transition of R to its second excited state is estimated to be less than 0.2%.
**Table 4.2. Experimental proposal.** Frequencies, nonlinear couplings ($\chi$ matrix /2π), and relaxation times of modes A, B and R used in our simulation. For the $\chi$ matrix, diagonal terms are the mode anharmonicities and off-diagonal terms are dispersive frequency shifts. $\chi_{BR}$ and the reservoir lifetime (= 2π/κ) are variable parameters in the simulation. The loaded $T_1$ time of A and B (including the Purcell effect) are varied accordingly. The $T_1$ time of $|f\rangle_A$ is assumed to be half of that of $|e\rangle_A$ (which is listed here).

*The frequency of cavity B can be chosen arbitrarily with no effect to the simulation.

<table>
<thead>
<tr>
<th></th>
<th>Frequency  $\omega/2\pi$</th>
<th>Nonlinear coupling</th>
<th>Intrinsic $T_1$</th>
<th>Loaded $T_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>5.9 GHz</td>
<td>78 MHz</td>
<td>50 $\mu$s</td>
<td>14-42 $\mu$s</td>
</tr>
<tr>
<td>$B$</td>
<td>6.5 GHz*</td>
<td>0.01 MHz</td>
<td>800 $\mu$s</td>
<td>80-500 $\mu$s</td>
</tr>
<tr>
<td>$R$</td>
<td>8.0 GHz</td>
<td>4.0 MHz</td>
<td>210 MHz</td>
<td>0.05-0.7 $\mu$s</td>
</tr>
</tbody>
</table>
CHAPTER 5
AUTONOMOUS QUANTUM ERROR CORRECTION

Driven Dissipation Delivering Decisive Decrease in Decoherence

5.1 Introduction

Quantum error correction (QEC) is currently the most pressing challenge in the creation of a functional quantum computer. Although the lifetime of information stored in various physical qubits continues to improve [100], the ability to correct inevitable errors in information storage and computation is essential. To accomplish this a common strategy is to encode the logical information in multiple physical qubits. In this setup errors can be detected through some non-demolition measurement and active correction cycle [101, 102]. Existing experimental implementations of QEC are based on an active process of measurement and adaptive recovery operations [101, 102, 103, 25, 104, 105] that incur additional hardware overhead and propagating errors.

This strategy has been extended to bosonic systems where the logical qubit can be encoded in the Fock states of a quantum harmonic oscillator [6]. Bosonic qubits have recently shown promise for a similar form of QEC based on fast feedback and active error correction [25] but suffer from errors introduced by the QEC protocol as well as requiring complex hardware needed for active measurement. In this work we use a very similar encoding but propose a new continuous form of QEC. By using DPCs we can construct a directional jump operator that stabilizes the odd Fock state manifold of the oscillator, effectively protecting it against single photon loss.
5.2 Error Correction

There are a wide variety of possible quantum error correction protocols and codes but what is common to all is the Knill-Laflamme condition [106]. The statement of this condition is that for all possible errors if two different errors occur we must be able to distinguish which error occurred. Consider a logical state \( |\psi_L\rangle \), which is a superposition of code states \( |k_L\rangle \) with coefficients \( c_k \). We subject the system to a number of errors \( \{\hat{E}_a\} \) that transform the density matrix as follows

\[
\hat{\rho} = |\psi_L\rangle \langle \psi_L| \rightarrow \hat{\rho}' = \sum_a \hat{E}_a |\psi_L\rangle \langle \psi_L| \hat{E}_a^\dagger \tag{5.1}
\]

The Knill-Laflamme condition states that the encoded information can be recovered from \( \hat{\rho}' \) if and only if

\[
\langle j_L| \hat{E}_b^\dagger \hat{E}_a |i_L\rangle = C_{ab} \delta_{ij} \tag{5.2}
\]

for all errors \( \hat{E}_a \) and all codeword states \( |i_L\rangle, |j_L\rangle \). Here the \( \delta_{ij} \) enforces that the codewords cannot overlap, they must be orthogonal. The \( C_{ab} \) enforces that the error rates must be the same for every logical state as these coefficients only depend on the errors that occurred and not the states effected.

5.3 Code and PReSPA

For our physical system we use the same experimental package as for the pair coherent state project although we only use one of the two storage cavities. As shown in Fig. 5.1b our system consists of one transmon ancilla \( q \), a high Q storage cavity \( A \) and a readout/reservoir resonator \( R \). For our experiment we use the following Truncated 4-component Cat (T4C) code [107] written in the Fock states of cavity \( A \):

\[
|0_L\rangle = C_1 |1\rangle + C_5 |5\rangle, \quad |1_L\rangle = C_3 |3\rangle + C_7 |7\rangle \tag{5.3}
\]

where \( C_1, C_3, C_5, C_7 = \sqrt{0.35}, \sqrt{0.9}, \sqrt{0.65}, \sqrt{0.1} \) are chosen to balance the mean photon number in the logical states \( \langle 0_L| \hat{n} |0_L\rangle \sim \langle 1_L| \hat{n} |1_L\rangle \). Looking at some key...
features of this code we can understand its motivation. The balance of the average photon number in these states ensures that when a photon loss event happens (or does not happen) neither state is more likely given only that information. A single-photon loss event will map these states onto the even Fock states of the oscillator making up our error states.

\[ |0_E\rangle = C_0 |0\rangle + C_4 |4\rangle, \quad |1_E\rangle = C_2 |2\rangle + C_6 |6\rangle, \quad (5.4) \]

where our coefficients each pick up a bosonic factor of \( \sqrt{n} \) before being renomalized. It is important that there is no overlap between code and error states \( \langle x_C | x_E \rangle = 0 \) for \( x \in [0, 1] \). This ensures that we cannot mistake the error state for a code state itself. If a second photon loss event occurs then we do get overlap with our code states, destroying the quantum information. We will no longer be able to reliably distinguish between a code state with no errors and one that experienced two errors. This means that as soon as a photon is lost we must quickly correct the error before it can happen again and the rate of correction versus photon loss will likely be the deciding factor in the success of QEC.

To correct the loss of a photon we propose a simple procedure: add a photon. This idea, while simple, is difficult to perform without destroying the quantum information. The two steps required for photon addition (irrespective of method) are 1) the determination of whether a photon loss has occurred and 2) the photon addition. The existing method of performing this type of procedure [25] involves measuring the parity of the system to check whether the photon loss has occurred. This solves step 1 but requires high fidelity, single-shot readout, that doesn’t destroy the state (QND), and fast feedback hardware that allows real time reaction based on the information. While it is becoming more common, experiments that fulfill this entire list are rare and all these requirements are potential points of failure. After performing the measurement and determining that an error has (or has not) occurred, a calculated corrective pulse is applied to the system to
Figure 5.1. Autonomous quantum error correction: concept and protocol.

(a), A classical bit is stored in isolated points in configuration space, which are local energy minima stabilized by dissipation in all directions. In comparison, a logical qubit $|\psi\rangle$ is encoded in a continuous two-dimensional code space $C$ designed in a way that both natural errors (which map the states to the error space $E$) and engineered dissipation are only allowed perpendicular to $C$. (b), Schematic of the circuit QED device composed of storage cavity $A$, transmon ancilla $q$ and reservoir resonator $R$. (c), AQEC scheme against single-photon loss illustrated in a level diagram (not to scale). The level indices refer to $A$, $q$, $R$ sequentially. A continuous wave (cw) “transmon comb” is applied to resonantly excite the transmon ancilla with a Rabi rate $\lambda$ (magenta arrows), selectively targeting the four even-parity states (red levels) when $\chi_q \gg \lambda$. Similarly, a cw “mixing comb” targets the $|2n, e, 0\rangle \leftrightarrow |2n + 1, g, 1\rangle$ transitions with an equal Rabi rate $\Omega$ (black arrows). Both combs are composed of four tones equally spaced by $2\chi_q$ as indicated by the varying slopes of the magenta and black transitions. Spontaneous decay of the reservoir $R$ converts the quantum state back to the code space (blue levels) without leakage of which-path information; note the identical slopes of the light-green arrows. The numbered labels show the error and recovery sequence 1-4 for one of the four parallel paths (using the initial state $|1g0\rangle$ as an example).
effectively add a photon and potentially correct for other types of errors (Kerr, measurement backaction, etc.).

The fundamental achievement of this work is performing both steps without measurement, without active control, and without even gaining knowledge of how many times a photon was lost, if at all. To do this we require a directional jump operator from the even states back to the corresponding odd states, which we name Parity Recovery by Selective Photon Addition (PReSPA):

\[ \hat{\Pi}_{eo} = |1\rangle \langle 0| + |3\rangle \langle 2| + |5\rangle \langle 4| + |7\rangle \langle 6|. \]  (5.5)

We can construct this operator from four DPCs (See section 3.4) all with the same reservoir, which together implement this transition. Because we require this operator to be selective (we do not want to add photons to odd code states, only even error states) we need a means for increasing the selectivity of our DPCs. To accomplish this we add an intermediary state to the pumping channel as shown in Fig. 5.1c. A first tone excites the \( |e\rangle \langle g| \) transition of the transmon, which is selective on the cavity A photon number due to the large dispersive shift between transmon and cavity A. The second transition is a more standard FWM transition that converts that transmon excitation into the target final state accompanied by a reservoir excitation. Going back to our photon addition objective, the selective transmon drive maps the parity of the cavity onto the transmon, which holds the information required for step 1) the determination of whether a photon loss has occurred. Then the FWM drive accomplishes the role of a normal DPC by non-reciprocally adding a photon to the system, conditional on the transmon being excited.

### 5.4 PReSPA Spectroscopy and Rates

Implementing PReSPA requires initial parameter guesses supplemented with experimental corrections. We make the initial assumption that the comb frequency spacing \( \eta = 2\chi_q \) and that the \( |0e0\rangle \leftrightarrow |1g1\rangle \) (mixing) transition frequency is \( \omega_A + \omega_R - \omega_q \). Because our strong off-resonant drives cause a Stark shift to the transmon frequency, a detuning \( \Delta \) needs to be added to the mixing comb
Figure 5.2. Characterization of the PReSPA operator: photon population conversion. (a) Control pulse sequence of a transmon spectroscopy measurement to infer the cavity photon distribution after PReSPA: We initialize cavity $A$ to a specific initial state using an Optimal Control Theory (OCT) pulse [5], apply PReSPA for a variable time $t$, apply a spectrally-selective $\pi$-pulse to transmon $q$ at a variable detuning $\Delta \omega_q$, and measure the transmon excitation probability $P(\Delta \omega_q, t)$. (b) Transmon spectroscopy data $P(\Delta \omega_q, t)$ for cavity $A$ initialized in vacuum. The bright feature is shifted from $\Delta \omega_q = 0$ to $-\chi_q$ over time, showing the $|0\rangle_A \rightarrow |1\rangle_A$ conversion. (For $0 < t < 20 \mu s$ an additional delay time of $20 \mu s$ is inserted between PReSPA pumps and the transmon $\pi$-pulse to improve clarity of the spectroscopy data by allowing the partially-excited transmon to relax.) (c) Cuts of (b) at $t = 0$ and $25 \mu s$ (grey dashed line). (d), (e) $P(\Delta \omega_q)$ for $A$ initialized in an even-parity cat state at (d) $t = 0$ and (e) $t = 25 \mu s$. All four spectroscopy peaks corresponding to even photon numbers (red) are shifted by $-\chi_q$ after $t = 25 \mu s$ indicating odd photon numbers (blue). (f) Probability of achieving the target cavity state $|2n + 1\rangle_A$, as measured by $P(t)$ for fixed $\Delta \omega_q = -(2n + 1)\chi$, for cavity $A$ initialized in $|2n\rangle_A$. Error bars reflect standard error of the mean. These four time-domain curves are fitted using a numerical model of the cascaded pumping process, resulting in $\Omega = 92, 88, 87, 85$ kHz; and $\lambda = 28, 27, 27, 26$ kHz, respectively. The inset shows a block of the cavity process $\chi$ matrix for $25 \mu s$ of PReSPA. The matrix elements $\chi_{nn',n'n'}$ are calculated from transmon spectroscopy measurements from all pairs of initial Fock states $|n\rangle_A$ and final Fock states $|n'\rangle_A$.
evenly-spaced tones that can drive all transitions on resonance. To best match the experimentally measured transitions we calibrate $\eta$ and $\Delta$ through a set of spectroscopy measurements. Experimentally we choose $\eta = 2.679$ MHz and $\Delta \approx 2.9$ MHz (calibrated on a daily basis) to ensure no pair of tones are farther than 10 kHz ($\approx 2\chi_q'$) off-resonant to their corresponding transitions.

To match the dissipative processes across the four parallel paths in PReSPA, we measure the probability of photon addition, over time, to each of the four even states as described Fig. 5.2. By fitting the curves of photon population and transmon excitation probability we can extract $\Omega_n$ and $\lambda_n$ for each of the four conversion paths. The fit model considers the quantum dynamics across the four relevant levels ($|2n, g, 0\rangle$, $|2n, e, 0\rangle$, $|2n + 1, g, 1\rangle$ and $|2n + 1, g, 0\rangle$) in a two-stage pumping process. It is simulated with QuTiP and includes the small detuning of the drives due to $\chi_q'$, the decay of Cavity $A$ and Reservoir $R$, and the relaxation, dephasing, and spurious two-frequency switching behavior (see notes in Table 5.1) of the transmon.

From Eqs. (5.11, 5.12) we can also estimate $\Omega_n$ and $\lambda_n$ from the amplitudes and phases of the microwave tones. We find experimentally that achieving equal $\Omega_n$ and $\lambda_n$ for the four conversion paths requires significantly different microwave amplitudes within each comb, in quantitative agreement with theoretical predictions (table 5.2). There is a discrepancy in a global pre-factor in the magnitude of $\Omega_n$, possibly caused by the coarse estimates of zero-point fluctuation parameters ($\phi/s$) from measurable experimental parameters (i.e. frequencies and dispersive shifts).

### 5.5 Hamiltonian and Driven System

The Hamiltonian of the circuit QED system can be derived using the perturbation theory [52] that adds weak anharmonicity to three harmonic oscillator modes corresponding to the Cavity $A$, transmon ancilla $q$ and the stripline resonator $R$: 
Figure 5.3. PReSPA spectroscopy. (a), (b) Control pulse sequence for two-dimensional (2d) spectroscopy to find the resonance conditions for the PReSPA mixing comb and transmon comb. We prepare an even-parity Fock state ($|0\rangle$, $|2\rangle$, $|4\rangle$, or $|6\rangle$), apply PReSPA for a fixed time ($12 \mu s$) with varying detunings of the transmon comb ($\Delta_q$) and the mixing comb ($\Delta_m$) in an attempt to activate dissipative photon addition. After a $1 \mu s$ wait time for the reservoir to relax, we either (a) selectively $\pi$-pulse the transmon conditioned on Cavity $A$ being in the targeted final state ($|1\rangle$, $|3\rangle$, $|5\rangle$, or $|7\rangle$) or (b) skip this pulse (for a background measurement), and proceed to read out the transmon state. The difference between the two measurements informs the likelihood of successful photon addition. (c), 2d PReSPA spectroscopy data: likelihood of photon addition as a function of the comb detunings ($\Delta_q$ and $\Delta_m$) for the $|0\rangle$ to $|1\rangle$ transition. Note that the linewidth of the four-wave-mixing transition is an order of magnitude greater than that of the transmon excitation due to the short reservoir $T_{1r}$. We can repeat this procedure to find all four sets of transition frequencies. (d), Cartoon spectrum of PReSPA drive frequencies. Four transmon drives, left, and four mixing drives, right, compose PReSPA. The colored ticks indicate the actual transition frequencies while the vertical black bars show the microwave drive frequencies in PReSPA. The transmon drive for the $|0\rangle$ to $|1\rangle$ conversion process is approximately at the Stark-shifted transmon frequency, $\omega_q - \Delta_{\text{Stark}}$, and the $|0\rangle$ to $|1\rangle$ mixing drive is near $\omega_A + \omega_R - \omega_q + \Delta_{\text{Stark}}$. Because of the equal frequency spacing $\eta$ in each comb and the unequal frequency spacing between the transitions with different photon numbers (due to the $6^{th}$-order non-linearity, $\chi'_q$), not all drives can be placed exactly on resonance. Experimentally, we settle for $\eta$ slightly greater than $2\chi_q$, and $\Delta_q = \Delta_m$ slightly smaller than $\Delta_{\text{Stark}}$ to compensate for the effect of $\chi'_q$. 

73
\[
\frac{\hat{H}}{\hbar} = \tilde{\omega}_q \hat{q}^\dagger \hat{q} + \tilde{\omega}_A \hat{a}^\dagger \hat{a} + \tilde{\omega}_R \hat{r}^\dagger \hat{r} - \frac{E_J}{\hbar} \left( \cos \hat{\phi} + \frac{\hat{\phi}^2}{2} \right) \tag{5.6}
\]

where \(\tilde{\omega}_q, \tilde{\omega}_A, \tilde{\omega}_R\) are the frequencies of the eigenmodes of the linearized system, \(\hat{q}, \hat{a} \) and \(\hat{r}\) their lowering operators, \(E_J\), the Josephson energy of the junction, and \(\hat{\phi} = \phi_q \hat{q} + \phi_A \hat{a} + \phi_R \hat{r} + \text{h.c.}\) the phase operator of the Josephson junction. \(\phi_q, \phi_A, \phi_R\) are the mode zero-point phase fluctuations across the junction for modes \(q, A\) and \(R\), respectively.

Implementing PRcSPA requires two frequency combs: one comb exciting the transmon, and one mixing comb converting that transmon excitation into excitations in \(A\) and \(R\). This mixing comb activates a four-wave mixing process using the nonlinearity of the Josephson junction. To account for the four mixing tones with frequencies \(\omega_m\) in our Hamiltonian, we make a unitary transformation by displacing the qubit annihilation operator \(\hat{q} \to \hat{q} + \sum_m \xi_m e^{i \omega_m t}\), and the phase across the junction is (see e.g. Supplementary materials of Ref. [57]):

\[
\hat{\phi} = \phi_q \hat{q} + \phi_A \hat{a} + \phi_R \hat{r} + \sum_{m=0}^{3} \phi_q \xi_m e^{i \omega_m t} + \text{h.c.} \tag{5.7}
\]

Expanding the cosine, going to a rotating frame, and employing a dispersive transformation that cancels 2nd order terms, the relevant 4th and 6th order terms include non-driven terms (without \(\xi_m\)) and driven terms. The non-driven terms, generic to most circuit QED systems, are:

\[
\frac{\hat{H}_{\text{nd}}}{\hbar} = -\chi_q \hat{q}^\dagger \hat{q} \hat{a}^\dagger \hat{a} - -\chi_r \hat{q}^\dagger \hat{r}^\dagger \hat{r}^\dagger \hat{r} - \chi_{Ar} \hat{a}^\dagger \hat{a} \hat{r}^\dagger \hat{r} - \frac{K}{2} \hat{a}^\dagger \hat{a}^\dagger \hat{a}^\dagger \hat{a}^\dagger - \frac{\chi'_q}{2} \hat{q}^\dagger \hat{q} \hat{a}^\dagger \hat{a}^\dagger \hat{a}^\dagger \hat{a}^\dagger, \tag{5.8}
\]

where \(\chi_q, \chi_r, \chi_{Ar}\) are the dispersive couplings between the transmon and Cavity \(A\), between transmon and reservoir \(R\), and between \(A\) and \(R\), respectively. \(K\) is the self-Kerr of Cavity \(A\), and \(\chi'_q\) is the 6th order non-linearity. We treat the transmon as a two-level system since the higher excited states are not accessed in this experiment.
Likewise, for all operations except readout, $R$ is either in the ground or first excited state and we ignore its higher-order terms.

By setting $\omega_m = \omega_a + \omega_r - \omega_q + m\eta$ we get stationary, or slowly rotating, four-wave mixing terms:

$$
\frac{\hat{H}_{\text{mix}}}{\hbar} = -\frac{E_J}{\hbar} \phi_q^2 \phi_a \phi_r \sum_{m=0}^3 \xi_m e^{im\eta t} \hat{q} \hat{r}^\dagger + \text{h.c.,}
$$

(5.9)

where $\eta$ is the difference in frequency between each nearest pair of mixing tones. The four mixing tones will each individually Stark shift the transmon but we can simply absorb those into the transmon frequency for the rotating frame transformation mentioned above. There will also be slowly-rotating Stark shift terms that are a result of the cross terms of two different mixing tones:

$$
\frac{\hat{H}_{\text{Stark}}}{\hbar} = -\frac{E_J}{\hbar} \phi_q^4 \sum_{k=0}^{3} \sum_{l=0}^{3} \xi_k \xi_l e^{-i(l-k)\eta t} \hat{q}^\dagger \hat{q}.
$$

(5.10)

The four mixing tones acting together drive the four $|2n, e, 0\rangle \leftrightarrow |2n+1, g, 1\rangle$ ($n = 0, 1, 2, 3$) transitions (the black solid arrows in Fig. 1c). We can calculate the complex Rabi rate of these driven transitions under a weak-drive approximation of $|\xi_k|^2 \ll \hbar\eta/(E_J\phi_q^4)$:

$$
\Omega_n = -\frac{E_J}{\hbar} \sqrt{2n+1} \phi_q^2 \phi_a \phi_r \left[ \xi_n - \frac{E_J \phi_q^4}{\hbar \eta} \sum_{m;k:l\neq k} \frac{\xi_m}{(l-k)} \right] \times (\xi_k \xi_l^* \delta_{n-m,l-k} - \xi_k^* \xi_l \delta_{n,m,k-l}) + \mathcal{O}\left(\frac{E_J^2 \phi_q^4 \xi_n^4}{\hbar^2 \eta^2}\right).
$$

(5.11)

where $\delta$ is the Kronecker delta function. This rate is caused by two different mechanisms: a direct drive by one of the four tones that is on resonance, and a multi-tone parametric effect. This parametric effect arises from the time-modulation of transmon frequency by the Stark shift induced by pairs of mixing tones. When the detuning of one of the off-resonant mixing tones exactly matches the modulation
frequency, the transition can be parametrically driven by a combination of three tones. Depending on the relative phases of the three tones, these terms can contribute constructively or destructively to the transition rate. Due to the bosonic enhancement factor $\sqrt{2n + 1}$, higher photon number states typically require smaller $\xi_n$.

This calculation has been done with the assumption that the spacing between tones $\eta$ is constant but we could also consider non-even frequency spacing. In this case, terms that rotate very slowly with time would appear in the mixing rates, which would cause our PReSPA operator to undesirably change with time. We make the choice of keeping $\eta$ constant to avoid these complications.

A similar calculation can be done for the four selective transmon drives. To preserve path independence we use the same frequency spacing magnitude $|\eta|$ as the mixing drives but with the opposite sign. As the transmon drives are relatively weak, they do not cause a significant single-tone or multi-tone Stark shift but the frequency modulation caused by the mixing drives will allow for similar three-tone parametric mixing effects (two mixing tones and one transmon tone). We can write the rates for the four transmon transitions as:

$$\lambda_n = \Lambda_n - \frac{2E_J}{\hbar \eta} \sum_{m \neq n, k \neq l} \frac{\Lambda_m \xi_k \xi_l}{m - n} \delta_{n - m, k - l},$$

(5.12)

where $\Lambda_n$ are the bare transmon Rabi rates without the mixing drives. For either comb, the effect of this parametric mixing is a renormalization of the Rabi rates $\Omega_n$ and $\lambda_n$ from the Rabi rates by individual resonant tones (that are proportional to $\xi_n$ and $\Lambda_n$).

### 5.6 Characterization of PReSPA: State Populations

To experimentally evaluate PReSPA we first need to check if it delivers on its core promise: adding a photon. We can measure the cavity populations by performing a number-splitting experiment. Naturally the system starts in the ground state of the
Table 5.1. System parameters.

†The transmon $T^*_2$ reflects any $1/e$ decay time of Ramsey oscillations. The transmon displays a random switching behavior between two values of $\omega_q/2\pi$ 40 kHz apart, with a dwell-time split of approximately 85%:15%. The switching time scale is on the order of sub-seconds to seconds. All our experimental data reflects the averaged result from sampling the two ancilla frequencies.

‡ The cross-Kerr $\chi_{Ar}$ is derived from other measured parameters.

#Cavity A has a distinctive switching behavior between a regular state with stable $T_{2A}$ (380 ± 25 µs) and occasional “bad periods” lasting for 2-8 hours where $T_{2A}$ fluctuates wildly in the range of 200-340 µs. The reduced cavity coherence during these periods is not accompanied by any other changes of system parameters, and can be recovered by a Hahn echo pulse (with echo, cavity $T_{2A} \approx 390 \mu s$ at all times).

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Transmon frequency</td>
<td>$\omega_q/2\pi$</td>
</tr>
<tr>
<td>Transmon anharmonicity</td>
<td>$\alpha_q/2\pi$</td>
</tr>
<tr>
<td>Transmon $T_1$</td>
<td>$T_{1q}$</td>
</tr>
<tr>
<td>Transmon $T^*_2$ Ramsey</td>
<td>$T^*_2q$</td>
</tr>
<tr>
<td>Transmon $T_2$ Echo</td>
<td></td>
</tr>
<tr>
<td>Transmon $</td>
<td>e\rangle_q$ population</td>
</tr>
<tr>
<td>Reservoir frequency</td>
<td>$\omega_r/2\pi$</td>
</tr>
<tr>
<td>Reservoir-transmon coupling</td>
<td>$\chi_{r}/2\pi$</td>
</tr>
<tr>
<td>Reservoir $T_1$</td>
<td>$1/\kappa$</td>
</tr>
<tr>
<td>Cavity A frequency</td>
<td>$\omega_A/2\pi$</td>
</tr>
<tr>
<td>Cavity A-transmon coupling</td>
<td>$\chi_{q}/2\pi$</td>
</tr>
<tr>
<td>Cavity A-reservoir coupling</td>
<td>$\chi_{Ar}/2\pi$</td>
</tr>
<tr>
<td>Cavity A anharmonicity</td>
<td>$K/2\pi$</td>
</tr>
<tr>
<td>Cavity A 2nd order coupling</td>
<td>$\chi_{q}'/2\pi$</td>
</tr>
<tr>
<td>Cavity A $T_1$</td>
<td>$T_{1A}$</td>
</tr>
<tr>
<td>Cavity A $T_2$</td>
<td>$T_{2A}$</td>
</tr>
<tr>
<td>Cavity A $</td>
<td>1\rangle$ population</td>
</tr>
</tbody>
</table>
oscillator $|0\rangle$ so we can turn on PRESPA for variable amounts of time and confirm we end with a $|1\rangle$ state as in Fig. 5.2b,c. More generally we can prepare arbitrary initial states in our cavity through the use of OCT pulses [4] to see how any even state undergoes photon addition and arrives at the corresponding odd state. We can track this for superpositions (Fig. 5.2d,e) and for all the error states (Fig. 5.2f). While conversion efficiency is reduced for higher photon numbers (due to faster single-photon loss), it is important to match the shapes of these curves, or a PRESPA jump could incorrectly shift the code-state coefficients. When the photon is lost from the reservoir, all error states are projected instantaneously back into the code states. If the rate of transfer is different for the different branches then a reservoir loss will distort the coefficients.

Table 5.2. Comparison of calculated and measured PRESPA transition rates. Using calibrated amplitudes and phases of the transmon comb and mixing comb in the experiment, the complex-valued PRESPA transition rates ($\lambda_n$ and $\Omega_n$) can be calculated based on Eq (5.11, 5.12). Here the transmon comb amplitude $\Lambda_n$ is calibrated from ancilla Rabi oscillations, and the amplitudes of mixing tones are approximately converted to the dimensionless displacement parameter $\xi_n$ by measuring the Stark shift $\Delta_{\text{Stark}}$ induced by that single tone: $\xi \approx \sqrt{\Delta_{\text{Stark}}/2\alpha_q}$. The experimentally measured PRESPA transition rates result from fitting the time-domain dynamics of the photon-addition processes that do not contain phases. Note that $\Omega_0$ is intentionally set opposite to others in phase to suppress multi-tone mixing effects by destructive interference.

| index n | raw mixing amp (AU) | $\xi_n$ (approx) | $\Omega_n$ (kHz) (theory) | $|\Omega_n|$ (kHz) (fit) |
|---------|---------------------|-----------------|------------------------|---------------------|
| 0 ($|0\rangle$ to $|1\rangle$) | -1.22 | -0.058 | -125 | 92 |
| 1 ($|2\rangle$ to $|3\rangle$) | 1.00 | 0.048 | 127 | 88 |
| 2 ($|4\rangle$ to $|5\rangle$) | 0.64 | 0.030 | 127 | 87 |
| 3 ($|6\rangle$ to $|7\rangle$) | 0.49 | 0.023 | 124 | 85 |

| index n | raw transmon amp (AU) | $\Lambda_n$ (kHz) | $\lambda_n$ (kHz) (theory) | $|\lambda_n|$ (kHz) (fit) |
|---------|---------------------|------------------|------------------------|---------------------|
| 0 ($|0\rangle$ to $|1\rangle$) | -0.98 $e^{-0.43i}$ | -21.4 $e^{-0.43i}$ | -27 $e^{-0.37i}$ | 28 |
| 1 ($|2\rangle$ to $|3\rangle$) | 1.52 $e^{0.00i}$ | 33.1 $e^{0.00i}$ | 28 $e^{0.04i}$ | 27 |
| 2 ($|4\rangle$ to $|5\rangle$) | 1.27 $e^{0.02i}$ | 26.7 $e^{0.02i}$ | 28 $e^{0.02i}$ | 27 |
| 3 ($|6\rangle$ to $|7\rangle$) | 1.14 $e^{-0.35i}$ | 24.9 $e^{-0.35i}$ | 27 $e^{-0.34i}$ | 26 |
5.7 Characterization of PReSPA: Coherence

Characterizing a dissipative jump operator, especially a complex one, is an interesting problem without many experimental examples. Quoting the fidelity of a final state against the target state is simple but to characterize a jump operator we need to measure a process matrix. For our system, working with the full process matrix is prohibitively difficult so we focus on some key elements.

We observe the phase and coherence of even superposition states under PReSPA. In Fig. 5.4 we take equal population even-Fock superpositions and calculate the final state coherence and phase using Wigner density matrix reconstruction. By tuning the relative phase of our drives we can tune the phase of these final odd states. To maintain coherence we want the phase of a quantum trajectory that undergoes single photon loss and PReSPA recovery to match that of the trajectory with no single-photon loss.

5.7.1 Cavity and Transmon Ramsey under PReSPA

To characterize the phase evolution of various multiphoton superposition states under PReSPA, we perform a Ramsey-style experiment. This can be used both for measuring the frequency and coherence of odd-parity superposition states under PReSPA and for tuning up superposition phases in the PReSPA operator $\hat{\Pi}_{eo}$.

We measure the phase imparted in the photon addition process by starting with an equal superposition of two even states (e.g. $|0\rangle + |2\rangle$) and perform a PReSPA Ramsey at times greater than 25 $\mu$s, sufficient for nearly-complete even-to-odd conversion. By fitting the Ramsey oscillations, we can extract the phase of the PReSPA-converted state (e.g. $|1\rangle + |3\rangle$), which is calibrated against the phase extracted from another PReSPA Ramsey experiment on the corresponding odd-parity initial state (e.g. $|1\rangle + |3\rangle$). Phase and amplitude parameters of the microwave combs are adjusted to minimize the imparted phases of the photon-addition process.
Figure 5.4. Characterization of PReSPA operator: preservation of coherence. (a), Cavity Wigner tomography of six even-parity superposition states, $(|n\rangle_A + |m\rangle_A)/\sqrt{2}$, prepared by OCT pulses, as input states for PReSPA. (b), Wigner tomography of the six corresponding output states after 25 μs of PReSPA, which are converted approximately to odd-parity superpositions $(|n'\rangle_A + |m'\rangle_A)/\sqrt{2}$, with $n' = n + 1$, $m' = m + 1$. The Wigner function, a quasi-probability distribution in the oscillator phase space, is directly measured via photon-number-parity measurements after variable cavity displacements [6]. From each Wigner function we reconstruct the density matrix, and the most significant off-diagonal element is $\rho_{n,m}$ (or $\rho_{n',m'}$) that reflects the coherence between $|n\rangle$ and $|m\rangle$ (or $|n'\rangle$ and $|m'\rangle$). We also perform similar measurements with permutations of odd-parity superpositions as input states (not shown). The $\chi$ matrix block describing the coherence of the process can be computed by combining all the off-diagonal elements in these reconstructed density matrices. The result for the six key elements characterizing PReSPA coherence, $\chi_{nm,n'm'}$, are shown next to the vertical arrows and to a good approximation are equal to $\rho_{n'm'}/\rho_{n,m}$. The deviations of $\chi_{nm,n'm'}$ from unity reflects the infidelity of the PReSPA process.
PRReSPA Ramsey is further used to measure the phase accumulation rate of our logical code words. When measuring the process fidelity of error correction we work in a frame where $|0\rangle_L$ is stationary. By measuring the phase of $|1\rangle_L$ as a function of time we can calculate the correct decoding angle.

We also perform a regular qubit Ramsey experiment for the transmon in the presence of the mixing combs of PRReSPA (without exciting Cavity $A$). This allows measurement of the Stark shift of the transmon rapidly and accurately, because to a very good approximation, only the mixing comb contributes to the Stark shift of the transmon. This experiment further shows that the coherence time of the transmon is not affected by the strong off-resonant drives of PRReSPA (with $T_{2q}^* = 17 \mu s$).

5.8 Decoding and Cat Distortions

Now that we have demonstrated the stabilization of parity with PRReSPA, we consider how the state evolves under jump and no-jump evolution. No-jump is simpler, the Bayesian update biases towards the lower photon number states $|1\rangle$ and $|3\rangle$ and away from the higher states $|5\rangle$ and $|7\rangle$. The jump case has two components: the photon loss and the PRReSPA restoration. Because our system is a truncated cat-code, single-photon loss does not have exactly the same effect as for a non-truncated cat. We do not have an infinite number of higher Fock states being lowered towards our code states. The effect in our system is that the Bayesian update does bias towards the higher photon-number states. The PRReSPA correction, if done properly, gives no information about which state is more or less likely and does not change the state. We have two competing processes, the no-jump evolution that distorts the codewords towards $|1\rangle$ and $|3\rangle$ and the jump evolution that distorts the codewords towards $|5\rangle$ and $|7\rangle$. For any one trajectory the exact number of jumps taken over time $t$ will decide the codewords ratios. Our initial state $|\psi_0\rangle = x|0_L\rangle + y|1_L\rangle$ after $j$ jumps will be
Figure 5.5. Cavity Wigner and PReSPA Ramsey measurements. (a) Experimental Wigner function $W(\alpha)$ of $|0_L\rangle$, acquired by applying a cavity displacement operation $\hat{D}_\alpha = \exp(\alpha \hat{a}^\dagger - \alpha^* \hat{a})$ with variable complex amplitude $\alpha$ followed by an ancilla-assisted photon-number-parity measurement (which is composed of two $\pi/2$ pulses of the ancilla and a delay time of $\pi/\chi_q$ and an ancilla readout [6, 7]). The Wigner function rotates around the origin over time at a rate proportional to the frequency difference between $|1\rangle$ and $|5\rangle$ in the rotating frame of the experiment. (b), Measured Wigner function values at a fixed phase-space position (as indicated by the cross in (a), at $\alpha = 0.75$) as a function of time under PReSPA. Analogous to a qubit Ramsey measurement, this cavity PReSPA Ramsey experiment can be used to efficiently track the phase evolution of any two-component superposition states using the interference effect enabled by the coherent cavity displacement ($\hat{D}_\alpha$) before readout. The exponential envelope of the sinusoidal fit indicates the rate of decay for the coherence between $|1\rangle$ and $|5\rangle$ under the correction of PReSPA. Similar measurements are applied to various superposition states to provide direct calibration of the frequencies and phases of these states under PReSPA. PReSPA enhances the ability to use such Ramsey measurements at high photon numbers since it approximately preserves photon number distributions in the cavity.
Figure 5.6. Process $\chi$ matrix block for 25 $\mu$s of PReSPA. The matrix converts elements of input density matrices, top axis, to output density matrix elements, left axis, expressed in the Fock state basis. (a), Amplitude values of the $\chi$ matrix elements. The upper left block ($\chi_{nn,mm}$) describes the conversion of diagonal elements of the input and output density matrices, which is associated with transfer of photon occupation probabilities calculated from transmon spectroscopy experiments. The lower right block ($\chi_{nm,kl}$) describes the conversion of relevant off-diagonal elements of density matrices, which is calculated from Wigner tomography and density matrix reconstruction [8]. The greyed blocks are assumed to be zero due to the absence of interference between the four conversion paths in PReSPA. (b), Phase values of the $\chi$ matrix elements for the lower right block in (a). For best illustration of the PReSPA process, the phases are reported in the full rotating frame where all Fock states have zero energy. Values in grey are measured but not statistically significant as the corresponding amplitude value is not large enough. In this frame, as prescribed by EQ (5.5), PReSPA requires zero phase for the six $\chi_{nm,(n+1)(m+1)}$ elements representing the coherence of the even-to-odd conversion process, which is accomplished by our PReSPA calibration. The diagonal elements $\chi_{nm,nm}$ representing the preservation of odd-state superpositions should have zero relative phase by definition. Their systematic deviation from zero was caused by parameter drift in the experiment as that block of data was acquired at a later time than the earlier rotating-frame calibration.
\[ |\psi_j(t)\rangle = x n_{15}^j(t) \left( \cos \theta_j |u_0\rangle + \sin \theta_j |u_1\rangle \right) \]
\[ + y n_{37}^j(t) \left( \cos \varphi_j |v_0\rangle + \sin \varphi_j |v_1\rangle \right) \]  

(5.13)

where \( \theta_j \) and \( \varphi_j \) describe the angles of the states in the 1, 5 and 3, 7 planes. The states \( \{|u_0\rangle, |u_1\rangle\} \) and \( \{|v_0\rangle, |v_1\rangle\} \) denote orthonormal bases of the \( \mathcal{H}_{15} \) and \( \mathcal{H}_{37} \) subspaces, respectively. Amplitudes \( n_{15}^j(t) \) and \( n_{37}^j(t) \) represent how the probability of having 1,5 photons (or 3,7 photons) changes over time \( t \) given a number of jumps \( j \).

\[ |u_0\rangle = |0\rangle_L \]
\[ |u_1\rangle = |u_0\rangle_\perp \]
\[ |v_0\rangle = |1\rangle_L \]
\[ |v_1\rangle = |v_0\rangle_\perp \]  

(5.14)

This is simply a trajectory picture, the density matrix will be a mixed state containing all possible number of jumps.

\[ \rho(t) = \sum_{j=0}^\infty c_j(t) |\psi_j(t)\rangle \langle \psi_j(t)| \]

(5.15)

But all these states will have the same type of decomposition into \( |u_0\rangle, |u_1\rangle, |v_0\rangle, \) and \( |v_1\rangle \).

It may appear that information is lost by the mixing of states with different numbers of jumps but the phase and amplitude information of \( x \) and \( y \) remain in these different sized cats. To extract it we need decoding, a unitary transformation performed by an OCT pulse.

\[ |g\rangle \otimes |u_0\rangle \leftrightarrow |g0\rangle, \quad |g\rangle \otimes |u_1\rangle \leftrightarrow |g1\rangle, \]
\[ |g\rangle \otimes |v_0\rangle \leftrightarrow |e0\rangle, \quad |g\rangle \otimes |v_1\rangle \leftrightarrow |e1\rangle \]  

(5.16)
Using this decoding to map Eq. 5.15 onto the transmon we get a transmon density matrix of
\[
\hat{\rho}_q = \sum_j p_j \begin{pmatrix}
|x|^2 (n_j^{15})^2 & x^* y n_j^{15} n_j^{37} \cos \zeta_j \\
x y^* n_j^{15} n_j^{37} \cos \zeta_j & |y|^2 (n_j^{37})^2
\end{pmatrix},
\]
where \(\zeta_j = \theta_j - \varphi_j\).

The truncated cat states evolve with a deterministic collapse-revival cycle due to the self-Kerr interaction introduced by the Josephson junction \[108\]. This effect does not directly cause logical qubit decay; however, it imposes an inconvenience in that for an arbitrary time \(t\) (modulo \(\pi/K\) or 288 \(\mu s\)), a dedicated OCT decoding pulse is needed to account for the phase accumulation due to the Kerr effect. In our experiment, by accounting for an effective Z-rotation of the logical qubit after decoding, we can recycle the same decoding pulse every \(\pi/4K\) or 72 \(\mu s\).

To visualize this, in the rotating frame chosen to make \(|0_L\rangle\) stationary, the Kerr evolution of a T4C state (without decoherence) is:
\[
|\psi\rangle = x (C_1 |1\rangle + C_5 |5\rangle) + ye^{-2iKt} (C_3 |3\rangle + C_7 e^{8iKt} |7\rangle)
\]
(5.18)

In the experiment, we computed two different decoding pulses. The first decoding pulse, used for \(t = 0\) (mod 72 \(\mu s\)), implements the original transformation prescribed by Eq. (5.16):
\[
|g\rangle \otimes (C_1 |1\rangle + C_5 |5\rangle) \mapsto |g0\rangle,
|g\rangle \otimes (C_1 |5\rangle - C_5 |1\rangle) \mapsto |g1\rangle,
|g\rangle \otimes (C_3 |3\rangle + C_7 |7\rangle) \mapsto |e0\rangle,
|g\rangle \otimes (C_3 |7\rangle - C_7 |3\rangle) \mapsto |e1\rangle.
\]
(5.19)

The second decoding pulse, used for \(t = 36 \mu s\) (mod 72 \(\mu s\)), implements a modified transformation where the coefficient of \(|7\rangle\) acquires a extra minus sign in Eq. (5.19).
These two decoding pulses differ slightly in their performance. This is responsible for the alternating pattern in the state and the process fidelity of T4C code (as a function of time) in Fig 5.7b.

5.9 Experimental Setup

This experiment uses largely the same setup as the the pair coherent state experiment discussed in Chapter 3. The same cavity and sapphire chip are used but with a different amount of junction aging, leading to a different transmon frequency and different dispersive couplings. Likewise, in every fridge cool-down we find different coherence numbers due to a variety of factors that are poorly understood. The specific wiring setup is shown in Fig. 1.6. For this experiment we need only one of the two bosonic storage modes. We chose to use mode $a$ due to its more manageable dispersive coupling, lower anharmonicity and better coherence time.

5.10 Autonomous Error Correction

A tuned up PReSPA can be used as a tool for error correction. Using an Optimal Control Theory (OCT) pulse [5] we initialize cavity A into the six cardinal points of our logical bloch sphere (Fig. 5.7a). We can watch these states evolve with and without PReSPA and see that the features of the Wigner tomography, most notably its negative values that are a clear indication of a non-quantum state, are better preserved with PReSPA. To qualitatively assess our QEC we use another OCT pulse to decode the cavity state to the transmon for complete tomography. This decoding maps both the logical code word, $|0_L\rangle$ and its orthogonal vector, $C_5 |1\rangle - C_1 |5\rangle$, to the same transmon state $|g\rangle_q$ (likewise for $|1_L\rangle$ and $|e\rangle_q$).

We find that PReSPA boosts the process fidelity lifetime by more than a factor of 2 compared with the free evolution of the codewords. Unfortunately this is still not enough to pass the lifetime of the longest lived physical component encoding, the $|0\rangle$ and $|1\rangle$ states of cavity A. While this result does not achieve break even, it confidently demonstrates AQEC and shows that it can be achieved without the experimental complexity of other methods.
Figure 5.7. AQEC performance. (a)-(c), Cavity Wigner tomography for the six cardinal-point states of our Truncated 4-component Cat (T4C) code. We measure the states at (a) $t = 0$, (b) after $143 \, \mu s$ of free evolution, and (c) after $143 \, \mu s$ of AQEC using PReSPA. Center points of the Wigner function show a measurement of state parity, which is well-preserved under PReSPA. The Wigner functions have been manually rotated into the correct rotating frame for the ease of comparison. (d), Quantum process fidelity of information storage with the AQEC protocol (green) compared with other reference methods performed with the same physical system. We use each method to store quantum states corresponding to the six cardinal points of the Bloch sphere, wait for a variable time $t$ (or apply PReSPA for the AQEC curve), retrieve the state via the transmon using a decoding unitary, and perform quantum state tomography of the transmon. Process fidelity is calculated from the quantum state fidelity for the corresponding six measurements. All curves are fitted to the model $F(t) = .25 + Ae^{-t/\tau}$ to extract characteristic times and error bars are one standard error.
CHAPTER 6
CONCLUSION

6.1 Summary

Unitary dynamics of small quantum systems are finally transitioning from the realm of science to engineering. With the advent of optimal control theory [109, 4, 5] we can write a unitary operator and implement it with incredible accuracy. Measurement feedback is even being used to improve these unitary operators for increasingly complex systems [110, 111, 112, 113, 114]. While the most logical steps forward involve pushing for slightly better gate fidelities, slightly longer coherence times, and slightly larger quantum volumes, there is a clear need for new technologies and perspectives to achieve useful quantum computation. A major shift in approach is reconsidering dissipation from the source of problems to a resource that offers solutions. This shift is counter-intuitive and requires experimental demonstrations before gaining more interest from industry.

In this work I introduced three applications of reservoir engineering: manifold stabilization, state transfer, and error correction. Through my experimental work I demonstrate the feasibility of these applications within cQED systems. Specifically, multi-photon states are especially well suited for these applications. Engineering complex jump operators while using eignestates of those operators is a powerful combination. I also demonstrate the strength of autonomous methods in a field increasingly interested in measurement-based protocols. As the size of quantum computers increases, methods that reduce the complexity and overhead of measurement protocols will only become more valuable.
6.2 Prospective Projects and Future Directions

6.2.1 Dissipative OCT

In this work we use OCT pulses to implement unitary transformations. These pulses are now commercially available and will see more use in the future. That said, these pulses rely on unitary dynamics, which limits the range of operations available. In this work we carefully design schemes to perform specific non-unitary operations, but conceivably these operations could be designed with an optimization algorithm. As the field becomes more confident about the effect of strong drives on the system and as system modeling improves, there is an opportunity to dramatically increase the power of OCT pulses by explicitly including non-unitary dynamics.

6.2.2 Entangled Cat State Pumping and Stabilization

While separate stabilization of two cats in independent cavities is a simple extension of previous ideas, a two-cavity entangled cat can be dissipatively pumped and stabilized. The two-cavity two-component cat between cavities Alice (A) and Bob (B) has the following basis states:

\[
|0\rangle_L = |\alpha\rangle_A |\alpha\rangle_B + |\alpha\rangle_A |\alpha\rangle_B
\]

\[
|1\rangle_L = |\alpha\rangle_A |\alpha\rangle_B - |\alpha\rangle_A |\alpha\rangle_B
\]

(6.1)

The generation of these states has been realized in cQED [51] but not with an autonomous driven dissipative method. The operator that produces a dark state manifold spanned by the states above is \( \hat{O} = \hat{a}^2 + 2\hat{a}\hat{b} + \hat{b}^2 \). This will require 3 separate tones applied to the system with correct relative phase and amplitude. Fortunately each of these three operators (\( \hat{O} = \hat{a}^2 \), \( 2\hat{a}\hat{b} \), and \( \hat{b}^2 \)) can be tested individually before being combined.

6.2.3 AQEC V2

A key benchmark for any error correction procedure is having the procedure reach and exceed the break-even point. Break-even is the point where the lifetime of the
logical qubit exceeds the lifetimes of any of the physical qubits used to construct the error-correcting system. For our system we quantify the logical lifetime by measuring the process fidelity of the system. The longest-lived physical qubit for our system is Cavity $A$, the storage cavity. While this system isn’t a qubit (it is only slightly anharmonic), we can use the transmon to encode and decode a logical qubit in the two lowest Fock states of the oscillator $|0\rangle_L = |0\rangle_A$ and $|1\rangle_L = |1\rangle_A$. This encoding cannot be error corrected but has a long logical lifetime due to its low mean photon number $\bar{n} = \frac{1}{2}$ and relative simplicity.

The experiment documented [10] does not achieve break-even, making it a proof of concept rather than a practical encoding. This is the primary goal of a second generation AQEC experiment. We have an increased understanding of the parameters required to achieve this benchmark. From our results we understand that the physical error with the largest impact on the logical lifetime of our encoding is spontaneous transmon excitation. Some of this error is due to the 8 CW tones used to enable PReSPA and is intrinsic to the AQEC scheme, but we believe a large amount is due to the experimental setup. We believe better cryogenic thermalization will be a key factor along with more carefully filtered microwave pulse generation.

6.2.4 4th Order Dissipator

One common feature of many bosonic QEC schemes is a need for a 4th order dissipator. Both the mod4 cat code and the pair-cat code require these complex dissipators that have yet to be experimentally realized. Higher-order Hamiltonian terms have been realized [115] and this same process can likely be used to make the necessary dissipators.

6.2.5 AQST Experiment

The biggest challenge for this experiment is circuit fabrication, which includes three Josephson junctions, two of which need to have precise dimensions and are on the same sapphire chip. Another major challenge is that the coupling pins must be carefully cut to achieve the right reservoir dissipation rate. Tables 4.1 and 4.2
show the desired system parameters for ideal experimental conditions as simulated in QuTiP [116].

6.2.6 Cosmic Rays

Growing evidence points to cosmic ray bursts as a significant error source for cQED experiments. Beautiful experiments have documented this phenomenon [117], where the high-energy impact creates phonons in the substrate, which break Cooper pairs into quasiparticles. Quasiparticles cause energy relaxation in transmons, dramatically affecting coherence. Even worse, these events cause correlated errors in multiple qubits, making error correction extremely difficult. There are proposals for shielding [118, 119], quasiparticle trapping [120, 121, 122], and even phonon down-conversion [123] but fundamentally new solutions will likely be necessary.
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