Conceptualizing Social Mathematical Empowerment: What is the Curricular Connection?

Alicia C. Gonzales
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CONCEPTUALIZING SOCIAL MATHEMATICAL EMPOWERMENT: WHAT IS THE CURRICULAR CONNECTION

A Dissertation Presented

by

ALICIA C. GONZALES

Submitted to the Graduate School of the
The University of Massachusetts Amherst in partial fulfillment
of the requirements for the degree of

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Teacher Education and Curriculum Studies
CONCEPTUALIZING SOCIAL MATHEMATICAL EMPOWERMENT: WHAT IS THE CURRICULAR CONNECTION?

A Dissertation Presented

by

ALICIA C. GONZALES

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College of Education
DEDICATION

For my niblings, niblings-to-be, and my Celina.

Tristen, Xander, Augustine, Ian (Nuggy), Arya, Kaia/Micah, and Juniper. Remember that this life we live and society as we know it is just a giant Rube Goldberg machine that we use to meet our basic needs (food, shelter, and love). Some people benefit from this machine more than others, but the device is just a big game. And working as a collective, you have the power to change it to be more humanizing and respectful of our neighbor beings on this floating rock we call home (to push back). It will not happen overnight, and maybe not even in your lifetimes. But, doing this dissertation during the height of the COVID-19 pandemic made me realize that if the country we live in can implement policy change based on people's interests within two weeks, significant change is possible. I love you deeply. This work is for you.

In Lak'ech

Tú eres mi otro yo. You are my other me.
Si te hago daño a ti, If I do harm to you,
Me hago daño a mi mismo. I do harm to myself.
Si te amo y respeto, If I love and respect you,
Me amo y respeto yo. I love and respect myself.

-Luis Valdez
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“My life amounts to no more than one drop in a limitless ocean. Yet what is any ocean, but a multitude of drops?”

-- David Mitchell, Cloud Atlas

To Mom and Dad. You pushed my education since the time I was a tiny kiddo. Somehow you instilled in me a love of learning, probably because I didn't have to do housework if I had a book in my hand. It took me forever to finish this thing, but you always asked the right questions. Thank you for supporting Russell and me when we needed feeding, housing, loving, beans, chile, and money, and for asking any other questions. Mom, a special thank you, you gave me the hands that wrote this.

How to talk to a grad student over the holidays

Instead of asking...  Ask...

When are you going to get a real job?  What shows have you been binge watching recently?
What are your plans for after?  I found some money. Would you like it?
How is your thesis going?  Would you like this free food?
When will you be done?  <just stand there silently>

-- Original source unknown

Celina, Danica, and Krista. You are my best friends and my seeesters. Thank you for making me laugh until I pee, giving me my nephews, and dealing with my “Sassiness”. I love you.

Tristen, Xander, Auggie, Nugget, Arya, Kaia/Micah, and Juniper. Thank you for being you. You inspire me; you motivate me; you give me purpose. I want to be you all when I grow up. I have known you since you were babies, and it has been remarkable to watch you grow into strongly opinionated older versions of yourselves. I believe in the power of change because of you. Auntie Sassy loves you more than life itself.

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to, the happy place, dancing, games, and everything else I can't remember. You made it bearable.

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To my love affair. My clay. To every artist I've come in contact with the past eight years in my journey with mud, thanks for helping me develop my relationship with dirt. It taught me so much about life.

To Halsey, your music and lyrics were the perfect mix of cadence, depression, resentment, change and hope. Maybe I can see you in concert someday. Your work was the soundtrack that kept me company during the night's long hours.

To my addiction. Coffee, thanks for being my something to look forward to in the morning.

And last but not least, to my Big Bear. My love, my partner, and my forever friend. Thank you for constantly pushing me to be the best version of myself. For thinking I could do this even before I did. For moving with me across the country, back, and back again. For cooking and unloading the dishwasher. Life is better when you're in it. Thank you for reading my shitty drafts, arguing about framing and language, and letting me be me. I love you more than you will ever know. We did it. Alicia C. Gonzales de Santistevan, Ph.D. and Russell R. Reeves, M.D. Now, let's plan a trip to Iceland.
ABSTRACT
CONCEPTUALIZING SOCIAL MATHEMATICAL EMPOWERMENT: WHAT IS THE CURRICULAR CONNECTION?

FEBRUARY 2023

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Directed by: Professor Martina Nieswandt.

As a result of everchanging goals and visions for mathematics education in the United States, this study builds on an initial theoretical framework of empowerment and further conceptualizes one domain, namely Social Mathematical Empowerment [SME]. I used concept analysis to define processes and outcomes more precisely and identified SME as a multi-layered, dialectic construct. This approach results in two components of action and awareness with various instructional and theoretical approaches. I then applied that conceptualization and created an analytical tool to examine mathematics teaching materials to engage learners in SME. This study adds to the existing knowledge base by providing a clearer understanding of the antecedents, effects, and approaches that promote the dialectical components of the eight dimensions of SME and provides evidence for how teaching materials can promote SME. It also points to further research regarding new conceptualizations and implications for teaching and learning.
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CHAPTER 1

INTRODUCTION

My Perspective

One might ask why the fuss about identity and power if those terms are already starting to be used in mainstream mathematics education? Don’t all teachers want to empower their students, and doesn’t that require at some point attending to their identities? The answer lies partly in who defines these terms, how these terms are being used, and for what purposes. That is, these terms mean different things to different people, partly because people have different reasons for including them in their research and everyday work. (Gutiérrez, 2013, p. 44)

I begin this dissertation with my historical and personal perspective on power through mathematics education, and how I came to study social mathematical empowerment in my dissertation. In elementary school, I did well in school and mathematics, and using the framework I will further explore in this dissertation was what I could call “mathematically empowered.” I performed well on standardized tests, achieved As in mathematics courses and was aware that my ability to navigate mathematics as a youngster would serve me well later in life and open job opportunities (socially empowered through mathematics). In college, my mathematical empowerment tanked, and I no longer had a firm grasp of mathematics in my current school trajectory. However, I still understood the importance of mathematics in school to serve me in the future. I could be considered to have one of the dimensions of social mathematical empowerment (further more referred to in this dissertation as SME). Namely the dimension of taking action that studying mathematics or science would better prepare me
to enter into the workforce and that it would be more likely that I could find employment if I studied mathematics or science. I knew that the demand for teachers of liberal arts were declining. As a student in high school or as a student in my undergraduate, I did not have an awareness of the other domains of SME, but I knew that mathematics was helpful in an economic sense. As I continued studying mathematics, I was able to traverse the paths of mathematical empowerment from within the walls of school, to the social; what happens beyond school walls and what I explore in this dissertation. Beyond this is understanding that I was able to be a part of the system of learning and understanding and was able to question the foundations of learning and knowledge itself (epistemological empowerment). The above-mentioned story is only my story of understanding power in mathematics.

Jumping to more recent years, and during my study of mathematics education in graduate school, the term empowerment surfaced more heavily in policy documents and the education literature as one of the intended outcomes of mathematics education. On the surface, empowerment as an outcome makes sense, just like specific mathematical standards outcomes, but this power framing has caused much consternation. I started to find myself questioning the intended outcomes of mathematics in schools. I wanted to more deeply understand what we thought students should know, be able to do, or not do due to mathematics as a school requirement. I felt that empowerment, a recently surfaced intended outcome of school mathematics within the social realm, hadn’t yet been interrogated fully. And if a field has theoretical outcomes, then they need to be fully explicated. My dissertation begins with an argument of empowerment as an essential outcome of mathematics education. It outlines the different ways that empowerment has
surfaced. Then I outline how my dissertation fills a gap in mathematical empowerment’s conceptualization. I argue that empowerment in mathematics is important and focus on why the development of SME is an important avenue of study in and of itself. I further conceptualize the concept of SME and then create an analysis framework to analyze teaching materials using the lens of SME. I end my dissertation with a discussion about what I conceptualized and found in teaching materials and outline future research and implications.

**The Importance of Social Mathematical Empowerment [SME]**

Research studies show that mathematics serves as a gatekeeper to advancement in career and life trajectories (Engineering & Council, 2014; Stinson, 2004). As students progress through school, they receive the message that mathematics is a discipline that depends on ability rather than effort and that this discipline is meant for some and not for all (Gutiérrez, 2013; Stinson, 2004). Such messages have especially an effect on females and minorities, as seen in their low representation in STEM disciplines and careers. This effect, called the leaky pipeline effect, is related to emphasized ways of knowing and doing mathematics (Boaler & Greeno, 2000). Opening the gate and changing the perception from “mathematics is not for me” and “I am not good at mathematics” to “I am able to learn and understand mathematics” and “mathematics is important for my career” demand a shift from focusing solely on access and achievement opportunities to stressing identity and power (Gutiérrez, 2009) as goals of mathematics education. More recently, Principles to Actions and Common Core State Standards (National Governors Association Center for Best Practices and Council of Chief State School Officers, 2010; The National Council of Teacher of Mathematics, 2014) argue that school mathematics
should rest on beliefs and practices that empower all students to participate in mathematics and to achieve in ways that could not be predicted or correlated with student characteristics such as gender, race, or socio-economic status. Empowerment as individual student ability has also been heavily cited as one of the goals of Teaching Mathematics for Social Justice and Critical Mathematics Education, two theoretical approaches aiming to develop students’ sociopolitical consciousness in addition to just teaching mathematics content (E. R. Gutstein, 2016; Kokka, 2015; Stinson et al., 1985).

Even more recently, math educators stressed that mathematics courses should go beyond content knowledge, orient students to resolving world problems, and prepare them to function in society (Bakker et al., 2021). Moving from a theory of empowerment to empowerment in practice still needs work. The concept of SME, one domain of mathematical empowerment which I will expand upon later (Ernest, 2002), seems like a viable approach to address the demands of opening the gate and changing students’ perception of who can do mathematics and why mathematics is important; that mathematics is embedded in our day-to-day life as much as part of the work life. Ernest (2002) defines SME as the “ability to use mathematics to better one's life chances in study and work and to participate more fully in society through critical mathematical citizenship” (p.1). It involves the “gaining of power over a broader social domain, including the worlds of work, life, and social affairs.” (Ernest, 2002, p. 1). Human interests and values determine what mathematics is pursued, how it is pursued, and who pursues it (Ernest, 2009). Thus, teaching and learning of mathematics require an understanding that mathematics is not neutral, that mathematics is more than just procedures, and that mathematics can and has been used for change in a broader political
sense (defense budgets, COVID-19 pandemic, and understanding of climate issues to name a few) (Ernest, 2002; Gutiérrez, 2002). The mathematics taught in schools, along with national frameworks, textbooks, and teaching materials, are guided by underlying goals and views of people, which can and should be subject to change or critical analysis. As stated earlier, in my dissertation, I am examining the concept of SME and how this concept is addressed in teaching materials. Paul Ernest is the philosopher of math education who began this work in empowerment in mathematics education explicitly using these terms. He theoretically explored what power through mathematics education can look like. My initial framework follows Ernest (2002). My definition of SME is the development of the bidirectional interactions between the utilitarian aspect of SME and the critical citizenship aspect of SME: it is the multiple ways students see and perceive the place of mathematics in the world and in their life and how students see their abilities to change their world using mathematics.

This dissertation study is slightly nontraditional in that it is primarily a conceptual study (Baldwin & Rose, 2009) documenting and presenting the progress that has been made toward understanding social mathematical empowerment. Through my use of concept analysis, I advance both the evolutionary and theoretical understanding of SME and thus create a stronger conceptual understanding for future research. I use this conceptual understanding to examine teaching materials that have the potential to address SME. I then explore the work of providing first directions for teachers on criteria for selecting teaching materials for social mathematical empowerment and, thus, make better

---

1 Ernest uses the term continuum, but continuum implies a continuous sequence in which the ends are not distinct from each other.
informed instructional decisions. A robust conceptual underpinning helps teachers and researchers consider the purposes of teaching and learning mathematics.

I use this argument as the basis for pursuing these two research questions:

**RQ1** How can SME be conceptualized?

**RQ2** How do mathematics teaching materials address/align with concepts of SME?
CHAPTER 2

FRAMING THE DISSERTATION STUDY

This chapter outlines the framing of the study. I begin by stating my underlying assumptions of learning, state definitions for the domains of empowerment, and then articulate the framework for why teaching materials support the development of SME and why we should study both SME and teaching materials. Finally, this chapter ends with a restatement of my rationale and research questions.

Sociocultural Understanding of Learning

A sociocultural theory of learning underlies this study and what Anna Sfard terms as participationist (Mahn & John-Steiner, 2013; Sfard, 2007; Vygostsky, 1978). This learning theory influences my understanding of empowerment and power as part of development and learning. “The developmental transformations are the results of two complementary processes, that of individualization of the collective and that of the collectivization of the individual” In other words, by using this underlying theory of learning I am saying that learning of mathematics and learning in general is by nature not just an individual activity” (Sfard, 2006, p. 571). In other words, the individual learner is connected to other learners relationally and to other systems. They cannot be disconnected. They can be conceptualized separately, but they are always connected and interrelated by the nature of our being social creatures. We are human creatures, embedded within historical and political structures, they influence us, and we in turn can influence those systems. These two processes (the individualization and the collectivization) are dialectically interrelated and as a consequence … are in a constant
flux, resulting from inevitable modifications that happen in these bi-directional transitions.” (Sfard, 2006, p. 7).

**Domains of Mathematical Empowerment and Definitions**

Ernest (2002) distinguishes three different domains of empowerment in mathematics and their uses. These three domains are mathematics, social, and epistemological. Though my dissertation focuses on SME, I briefly define all three domains of empowerment. To foreshadow the findings of my dissertation, this was my beginning definition of SME I use here but you will see the shifting definition and conceptualization emerge as a result of my findings.

**Mathematical Empowerment [ME]**

ME “[c]oncerns the ‘acquisition’ of the facts, skills, concepts and conceptual structures of mathematics, and the general strategies of problem solving” (Ernest, 2002, p.1), and using and applying these capabilities. Furthermore, mathematical empowerment encompasses meta-cognitive skills (e.g., planning, monitoring progress, making effort calculations; Ernest, 2002, p.1) as well as reading and making meaning of mathematics texts and symbols from a semiotic perspective (Ernest, 2002).

**Epistemological Empowerment [EE]**

EE “concerns the individual’s growth of confidence not only in using mathematics but also a personal sense of power over the creation and validation of knowledge” (Ernest, 2002, p.1). Learners might realize that they are a part of their own knowledge creation, of which mathematics is a part. In mathematics, it might look like a realization that students are actively understanding and building their knowledge and that there is not just a pre determined mathematics to be learned.
Social Mathematical Empowerment [SME]

I build on Ernest’s (2002) definition of SME but add my own understanding of SME as a dialectical relationship with multiple related concepts. First, it is the numerous ways students see mathematics's applicability, usefulness and criticality beyond school walls. Second, SME is both outcome and process. It “concerns [both] the ability to use mathematics to better one’s life chances in study and work on one end of the [relationship] and to participate more fully in society through critical mathematical citizenship” (Ernest, 2002, p.1) on the other end. Ernest argues that it could be helpful to understand empowerment in mathematics education within three domains. Although at the same time he argues that these are conceptually interrelated. So why do I pursue SME on its own? Below I outline my reasons for this.

Historically, much emphasis has been placed on the study of ME, focusing on doing well in “school mathematics.” The study of typical school mathematical learning and progressions (number sense to algebra to geometry to calculus) is well studied, although it still has room for more research regarding how students come to know and understand domains specific mathematics concepts (Cai, 2017; Grouws, 2006). High school mathematics achievement predicts college matriculation, graduation, and early career-earnings (National Research Council, 2013a). These outcomes are often correlated with performance on standardized tests (Dossey et al., 2016).

EE, as it is conceptualized here, is the idea of seeing oneself as a “doer and creator” of knowledge and is seen as related to identity. Studies have shown that students’ different experiences in school influence their achievement and identities as learners (Boaler, 2002; Muis, 2004; Nasir, 2002), and few studies have explored the concept of
EE and its development over time (Boaler & Selling, 2017a). Although an important overall part of mathematical empowerment, EE is not the focus of this study because I see the need for a long-term approach to understanding how it builds and develops (potential for future research). The question of which domain of empowerment should come first or is necessary before the others develop has also been posed by researchers (Madusise & Mwakapenda, 2014) and the wondering if EE encompasses both ME and SME.

For the reasons above, the study of SME as a distinct construct is what I pursue in this dissertation. SME, the idea that one takes mathematics and applies it to their life and then use it to critique the world around them, can potentially be fostered through teaching materials and approaches. SME is distinct from ME because its focus is not on mastering the signs and symbols of mathematics or problem-solving in an immediate “what happens in school.” The focus is not on procedural skills or methods for solving a problem such as: Can a student recognize that distance problems might be solved with the distance formula? Or can a student prove why a geometrical relationship holds true given certain axioms? Although incredibly important in and of itself, solely stressing ME is not enough; because many of the procedural skills we focus on in schools are absent of meaning for students. In contrast, what SME does is take into account the applicability of mathematics beyond the classroom walls through awareness of social issues, and applicability of mathematics to “usefulness, and critical citizenship” both personal and collectively. It also recognizes that mathematics is a human endeavor subject to critique.

Foreshadowing my dissertation findings, I find other related constructs useful for developing SME, both theoretical and instructional such as teaching mathematics for
social justice, critical mathematics education, or the role of talk within instructional approaches. I also pose teaching mathematics for social justice teaching materials as useful for the development of SME, but I do not equate those constructs to SME in this dissertation. My reasons for this include the following: first, if empowerment as a term and a construct is deemed important enough to surface in the work of policy and mathematics education discourse, then it deserves exploration as its own construct. That is not to say that many scholars don’t equate empowerment with other constructs, such as agency. My dissertation framing does not seek to equate related constructs but rather to explore how empowerment is equated with other concepts through the researchers’ lens. Secondly, methodologically speaking, empowerment is what I consider a non-mature construct (Morse et al., 1996) in the field of mathematics education, meaning that is is not distinctly defined with clear boundaries. As a result, I chose to narrow the focus on empowerment and limit snowballing in my literature search. For example, suppose I had included agency as a search term, a reader might ask, “what about choice? That also seems to be related to agency?” or “locus of control”. Though these constructs are important and worthy of study, including them in my dissertation research would have gone beyond the scope of a dissertation and my research question of how can SME be conceptualized? Thirdly, and related to why I did link teaching math for social justice and empowerment is that there are no “course in empowerment” materials in mathematics education as there is a “course in algebra” or a “course in geometry,” but there are materials that promote empowerment through teaching mathematics for social justice. For this reason, I chose social justice teaching materials as part of my data set because the authors linked those two constructs together.
The Role of Teaching Materials for Mathematics as a Social Activity

The idea of mathematics as a human endeavor and activity is traced to Freudenthal, who argued that mathematics should be taught “as mathematizing” (Gravemeijer & Terwel, 2010). In this view, the mathematics in curriculum materials should be present not as math to be learned but as mathematics as an activity, or in a word, materials should support the activity mathematizing. Mathematizing as a concept is hard to grasp, but for Freudenthal, mathematics is not a product to be learned, but the activity of doing mathematics. For curriculum development, this implies that “the instructional activities should capitalize on mathematizing as the main learning principle” (Gravemeijer, 1994, p. 446). What mathematizing is conceptually, is a bit hard to grasp. Mathematizing is that first and foremost, mathematics is a human activity.

For Freudenthal (1968), and compatible with my theoretical view, is to first ask the question of usefulness. Elaborating on this view of mathematization is the further classification of two types of mathematization; horizontal and vertical. In his view, the mathematics of schools should actually have some “use” value beyond the application of schooling and that the way that it should be taught is as process. He made this distinction.

Horizontal mathematization leads from the world of life to the world of symbol.

In the world of life, one lives, acts (and suffers); in [vertical mathematization] symbols are shaped, reshaped and manipulated, mechanically, comprehendingly, reflectively. The world of life is what is experienced as reality… These worlds can expand and shrink—also at one another’s expense (Freudenthal, 1991, pp. 41–42).
However, this horizontal and vertical view of mathematization has vaguely marked limits, as our realities have vaguely marked boundaries. Something could go beyond the world of life and another to the world of symbols. For example, natural numbers can be viewed within the world of life, but the abstract numbers are in the world of symbol (Freudenthal, 1991). The mathematics learned in schools is concerns itself with being familiar with the “tool of mathematics” and how to use it, but Freudenthal argues, but the mathematics learned in school doesn’t usually have a clear path on how to use it. For example: when was the last time you used a geometric proof in the way it is taught to do anything of use in your life?

From a sociocultural framing, we imagine that as mathematics becomes useful, and is mathematized by learners, so too do learners bring their own views on the mathematics that they need to solve unfamiliar problems. Mathematics now goes beyond just a tool; it is transformed into a way to view and shape the world by people and through people. This mathematics is understood as socio (related to society), human-constructed, and in need of humans to craft it as a discipline. Although becoming more popular and understood as important in recent years, these sociological and sociopolitical views are constantly in flux and under attack (Gutierrez, 2018) but supported broadly by the governing professional organizations of mathematics. For example, Gutierrez’s (2017) work was criticized when she argued that mathematics operates as whiteness because it centers European standards, perpetuates privilege and centers some voices and some knowledge but not others. She also claims that teachers need to understand its political nature. Another example was Rubel’s (2017) perspective, where she argues beyond a dominant perspective of mathematics to talk about equity-directed practices to
help teachers develop deep understanding of social realities experienced by learners and how mathematics uses the tools of whiteness to reproduce social inequities. For more examples see (No Title, n.d.). This view of mathematics, as rooted in culture, activity, and politics and how it perpetuates privilege, calls us to think more deeply about how we can promote mathematics as human activity rather than the counter-mathematics narrative continues of mathematics to be learned. The teacher then becomes not just one who teaches but who is also taught and develops through a dialogic and dialectical process and part of the more extensive mathematical system (Freire, 2018). They are also a learner. One way to think about this is through the materials where students learn what mathematics is and who can do it. It also calls us to think about the role of the teacher, students, and teaching materials. This calls for restructuring and examining teaching materials and the view of the teacher and student's role in a classroom, particularly for ways in which mathematics as activity and for empowerment.

**Prior Research on Curriculum Analysis**

The study of curriculum historically distinguished three types of curriculum to reflect the outcomes for teaching and learning: the intended curriculum, the implemented curriculum, and the achieved curriculum (Reys et al., 2010). Other scholars used four levels: the ideal, the written, the teacher intended, and the enacted curriculum (Remillard & Heck, 2014; Stein et al., 2007). The distinction between types of curriculum levels is what we want to happen in the classroom, the visions or ideals of what occurs (intended or ideal), what happens in the classroom (enacted and implemented) and what occurs as a result of what happens in the classroom and how did that impact learning or outcomes (enacted or achieved).
The aims and goals of curriculum, often presented as standards or policy documents (Reys et al., 2010), are translated into what I refer to in this dissertation as teaching materials, sometimes called curriculum materials or intended curriculum materials. Teaching materials are resources designed to support or supplement instruction and are part of the overall concept of curriculum and work to serve as the link of curricular goals to instructional plans. Valverde (2002) describes teaching materials (e.g., handouts, pictures, manipulatives) and textbooks as mediators between the intended and implemented curriculum and offer a lens that reveals particular visions and purposes of schooling (Valverde et al., 2002).

One current example of this political controversy within materials is the curriculum controversy in Florida, where mathematics textbooks are banned for including controversial social and political topics for fear of “indoctrinating students” (https://www.fldoe.org/newsroom/latest-news/florida-rejects-publishers-attempts-to-indoctrinate-students.stml). Another example is the Common Core State Standards. The aim was “a unified set of standards and goals” to prepare students for standardized tests, primarily due to the policy implications of No Child Left Behind (Geer, 2018). These aims and goals were then translated into teaching materials to meet the “standards”. In this way, teaching materials serve as a vehicle to move from intention to enactment to larger societal and political goals. The implementation of materials does not necessarily mean that the implementation of materials or standards will come to fruition but it dramatically increases the chance of encountering these learning opportunities (Remillard et al., 2014).
Although the enactment of materials, meaning what happens in the world of the learning environment is also critical, there is an argument that scholars should examine the relationships that textbook materials themselves encourage because curriculum materials are tools that can support the teacher’s goal of introducing students to practices and language of the mathematical community (Herbel-Eisenmann, 2007).

The teaching materials I analyze in this study are lessons intended for learners of secondary mathematics. These lessons come from two books. The first *High School Mathematics Lessons to Explore, Understand and Respond to Social Injustice* claims to align with mathematical standards and empowerment, while the second *Thinking Mathematically* is also aligned with mathematical standards it additionally states, to align with some of Florida’s banned instructional topics and includes what students should be learning (learning outcomes or objectives) as well as examines theoretical and pedagogical approaches that support those learning outcomes (Valverde et al., 2002).

There are many ways that teaching materials can be analyzed. In mathematics education there has been a focus on analyzing intended curricular materials for cognitive demand, culturally responsive teaching, teacher noticing of the potential of materials to create curricular spaces, etc. (Arbaugh & Brown, 2005; Drake et al., 2015; Land et al., 2019). My use of the term teaching materials in this work refers to written materials that contain mathematical content. Herbel Eisenmann’s work and her analysis of teaching materials provided guidance for how to imagine teaching materials as an “objectively given structure”, meaning that the structure and ideas of the written text, and not what happens when a person interacts with it, are the unit of analysis. I do not mean that teaching materials are objective in their presentation. Still, in this way I was able to look
for the potential of materials to support the SME conceptualization that I found for SME.
In addition, Land and colleagues (2018) helped me to frame what I mean by potential.
Where Herbel Eisenmann (2017) looked more at the sentence level of materials, while
Land and colleagues looked for open spaces in materials. But their code development
process helped me imagine what looking for SME in materials could be.

My study addressed the alignment/addressing of SME in teaching materials. I
focused on identifying different opportunities to take up dimensions of SME.
Unfortunately, I found no such curriculum analysis tool to support me in analyzing the
two textbooks I chose for this analysis, so I created one. I used the textual features
(standards, overview of the lesson, general structure of what students were going to be to
do) to look for potential ways that teaching materials can support the enactment of SME.
My tool contained two organizational features. Namely, what SME dimensions or
features did the materials align with (as an outcome and a process), and how did they
align (what strategies did the text use to support the outcome or process)? This content
analysis and curriculum analysis helped to answer my second research question: How do
mathematics teaching materials address/align with concepts of SME?

**My Conceptual Model for how Learning and Materials Support SME**

To meld these ideas together and create a visual, conceptual frame to situate my research
questions, I took inspiration from the *imagery (the picture)* of Broffenbner’s (1977)
ecological systems model. Although Broffenbner’s model shows development from the
outward inward, his visual model helped me create a visual model of my SME framing
and learning as the dialectical relationship of development and the learner's influence on
understanding of the development and learning of mathematics as a dialectical sociocultural and political process, I present a visual model of mathematical empowerment as a starting point for this dissertation study (see Figure 1). The model includes the individual learners' and collection of learners and the collection of individuals at the center. The inner ring of the model contains mathematical formal and informal schooling experiences. Social Mathematical Empowerment [SME] is depicted in the third and fourth rings from the center. The third ring from the model's center is mathematics's role in life and the broader social world. The fourth ring from the center includes the aims of teaching mathematics and the objectives of learning mathematics. Finally, the fifth ring consists of the creation and validation of knowledge building. The highlighted rings show my focus in my dissertation and the parts of the framing I explore.
Mathematical Empowerment (ME), Social Mathematical Empowerment (SME), and Epistemological Empowerment (EE)

These external systems, depicted by the bidirectional arrows in the figure, exert influence over the learner’s schooling and life experience. Empowerment acts the opposite, where the learner exerts influence over the systems. These systems, distinct in this model, are related by their influence over each other.

Mathematical empowerment (ME) is the development of gaining power over the mathematical domains indicated in Figure 1 as a bi-directional arrow between the ring of schooling and the collective. Social mathematical empowerment (SME) is indicated by a bi-directional arrow between the learner/s and the role of mathematics in life and the

Figure 1. Model of Empowerment through Mathematics, with SME highlighted. Mathematical Empowerment (ME), Social Mathematical Empowerment (SME), and Epistemological Empowerment (EE)
broader social world. A bi-directional arrow indicates epistemological empowerment (EE) through the outer rings of the model (the aims of teaching math (the utilitarian) and the objectives for learning math (the critical) and the creation and validation of knowledge).

The learner's opportunities to engage with SME is likely influenced by their experiences with mathematics in and out of school. Through schooling, we have the opportunity to grapple with our ideas of mathematics, what it is, who can do it, who can use it, how we can use it, and who can create it. Teaching materials have the potential to guide the learner to grapple with and embrace the ideas of mathematics. In this way, hopefully, learners can also bump up against and travel along the path of asking the same questions of knowledge and understanding in general.

To summarize, this framework for thinking of mathematical empowerment is built on existing research conceptualizing it (Ernest, 2002) and a broader understanding of learning and development and RQ2 lenses in Figure 1 depict where I focus in my dissertation. Below are my research questions. Figure 1 depicts my research questions as the two lenses.

**RQ1**  How can SME be conceptualized?

**RQ2**  How do mathematics teaching materials address/align with concepts of SME?

**Prior Research on SME**

Rubin and Babbie (2014) argue that clear conceptualization and operationalization should be formed to investigate certain outcomes or variables. Mathematics education research that focuses on empowerment in mathematics education as an outcome does not
providing a coherent conceptualization of empowerment. Instead, the literature shows a variety of descriptions and definitions and with varying details. Boaler (1997) definition of empowerment is an example of a quite narrow perspective. She views it as “the improved performance of girls scores to boys at levels of school mathematics as well as the amount of girls that take advanced levels of mathematics courses. (Boaler, 1997,, p. 327). In contrast, Hassi and Laursen (2015) outline in their study of undergraduate mathematics students a view of empowerment as personally relevant and consisting of three forms: self-empowerment, cognitive empowerment, and social empowerment. They emphasize personal empowerment, which includes students increased reflection, sense of self, agency, responsibility for their own and others’ learning, and their willingness and ability to contribute to in-class mathematics knowledge and problem solving (p. 318).

Gutierrez (2013) argues that empowerment is tied to students' identities and that citizen can "discern for themselves which kinds of questions can be answered using mathematics and which cannot.” Stinson (2004) argues that empowerment is related to consciousness (Freire, 1970), and he uses Ernest’s (2002) framing to discuss empowering mathematics. While the previous examples focus on the learner, Wright (2017a) argues that for mathematics learning to become genuinely empowering, teachers need to reflect on the relationships they build with students to help students develop and make sense of their learning situations.

The variety of definitions calls for a refined understanding of empowerment, which then have implications for instructional approaches achieving empowerment. This shared understanding and conceptualization can help instructors come to common
understandings when teaching for equity approaches, when thinking about power
dynamics.

Conventional approaches to teaching mathematics bring images of the teacher at
the front of the room doing most of the mathematical work, students responding with pre-
determined answers, and students working on practice problems devoid of context. This
“banking” education (attributed to Freire’s conception of empowerment) (Freire, 2018) of
teaching approaches reproduces social and classroom inequities taken for granted and
unquestioned. Applications of mathematics to systems like grocery spending or taxes
accept that those calculations or applications are themselves rooted in injustice. Questions
like “Should we have to pay for food, shelter, and healthcare?” aren’t undertaken. And
students are left to believe that a system in which performance on standardized tests,
quickness, or individualistic thinking is typical, and that if they are not successful in this
system, it is justified. Students further accept that only certain groups of students might
benefit (Frankenstein, 1983). The notion that if you work hard in school earns you the
ability to have your basic human needs met, rather than wondering if success in school
should not be tied to narrow performance on a narrow range of assessments. And the
pushing back on these external systems does not occur—a non-resistance sets in. Wright
(2017a) argues that schools are smaller microcosms of capitalist economies that need a
non-resistant workforce. To disrupt inequitable structures, teachers and students need to
be made aware of and be asked to reflect on their underlying knowledge assumptions
about their own views of mathematics, like believing mathematical ability is innate
(Wright, 2017).
Teaching with empowerment requires an awareness of the systems of oppression at work at one end of the dialectical relationship and with the goal of action to create social change at the other end. Following Ernest (2002), I view the learning outcomes of SME existing along a dialectical relationship consisting of awareness, utility, action, and critical mathematics citizenship. Utilitarianly, mathematical success gives students more opportunities in work and social affairs. The underpinnings of social and political decision making comprise critical mathematics education on the critical side of the dialectical relationship. “It means independent thinking” (Ernest, 2002, p. 4). The socially empowered learner will be able to understand and begin to question important questions about a broad range of social uses and abuses of mathematics.

This dissertation argues that critical instructional and theoretical approaches are tied to SME. These approaches draw attention to the culture of classrooms as activity oriented. These approaches focus on social, political, economic realities, and lived experiences. Within these instructional approaches, students are involved in both the capacity to be critical and mathematical (Tan et al., 2012). However, these approaches to teaching mathematics are not without their dilemmas. A study conducted by Brantlinger (2013a) found that teaching mathematics for social justice with high school students created resistance from students. Students believed it was not good preparation for the study of mathematics, did not view what they learned as mathematics, and thought the content would better fit within another discipline. Brantlinger (2013) wonders how the inclusion of critical approaches could better be implemented to be responsive to students' needs and lived experiences while better preparing teachers to enact those approaches.
To summarize: The aim of empowering learners mathematically, socially, and epistemologically stresses the potential of teaching and learning mathematics beyond working with fractions and solving equations to permit students to apply their mathematical knowledge to real-life situations and to use critical thinking and problem-solving abilities to identify and interpret claims made as truthful, false, or misleading. Taking a step further, SME has the potential to address inequity. SME helps students view the world with a critical mindset, imagine how the world might be a more socially just, equitable place, and lead to engagement in personal and social transformation (Tan et al., 2012).

Ernest (2002) provides a framework in which to begin with regarding SME and its components. These domains are listed below

- Better one’s chances in life or study
- Critically understand the use of mathematics in society: to identify, interpret, evaluate, and critique the mathematics embedded in social, commercial, and political systems.
- Be aware of how and the extent to which math permeates everyday life
- Have a sense of mathematics as a central element of art, culture, life, and an understanding of the historical and technological aspects of which mathematics permeates
- Be aware of the historical developments of mathematical symbols and abstractions, their theories, concepts, and problems
- Understand that there are multiple views of the nature of mathematics and controversy over the philosophical foundations of its’ knowledge.
Scholars who have studied social mathematical empowerment, both the utilitarian and critical components, have conceptualized it using constructs like utility value (Dobie, 2019), critical consciousness, or critical mathematics citizenship (Rubel et al., 2016).

In the critical vein, being SME can be seen as an increased ability to organize and synthesize mathematical features presented to us in the news or media, of which the recent pandemic is an example, and to ask critical questions such as: What do we mean when the media outlets tell us to “flatten the curve”? Under what conditions can our current healthcare system operate, when is it over capacity, and what does that mean? Does doing well in mathematics imbue me with some sort of higher power, a window into the heavens? What types of mathematics is valued, and is mine valued?

Outcomes of SME in the utilitarian vein include: increased job and career opportunities; increased ability to continue with mathematical coursework and to engage in the sciences, where mathematics is often used as a tool; or the ability to engage in everyday life skills (gardening, driving, taxes, cooking, consumerism, taxes). These everyday life skills can also be part of the critical vein and demonstrate SME’s dialectical relationship. For example, as students enter a store, they might ask questions like: Why does it cost less to buy one brand of gardening soil or a lower quantity of soil, even if the amounts don’t increase or decrease proportionally? What does it mean for 20% of my income to be taxed, where does that income go, and does it match my intended priorities as a citizen? Why have I been successful or not at mathematics so far, who has not been successful, and why? Is there a way that this can be studied mathematically? Are there other ways for me to show what I know mathematically, and why are these particular
types of knowing more valued and privileged than others, like the symbolic versus the visual?

Although the study of SME has begun, we still have more work to do, some of which this dissertation attends to.

**How this Study Fills a Gap and Restatement of the Research Questions**

For decades, mathematics educators promoted and argued for access to high-quality mathematics and the opportunities that mathematics affords. In 1989 The National Council of Teacher of Mathematics released their first standards document of specific national, professional standards for school curriculum in the discipline. This document set forth the goals of mathematics education as providing opportunities for all, preparing students for the workforce, success in school mathematics, and growing in confidence in mathematics (The National Council of Teacher of Mathematics, 1989).

This focus on access and achievement for students was a first step towards applying what students learned in their mathematics classes to aspects of their lives such as the humanities, the life sciences, the social sciences, while simultaneously opening the doors to taking more advanced mathematics courses. Building on these goals, the National Research Council (2013) stressed the need to prepare students to cope with the multiple demands of workplaces that require mathematical skills such as problem solving, reasoning, and defending solutions and conjectures (The National Council of Teacher of Mathematics, 2014). More recently, scholars pushed for not only access and achievement outcomes but also identity and power outcomes. The purpose of education should not be to integrate marginalized people into the existing society but rather to change society so that all are included. Thus education should help students analyze
oppression and critique inequities, highlighting how these issues connect to their lives and challenge those inequitable structures (Bartell, 2013, p. 131). Mathematics teaching materials have never been without controversy. The field of mathematics education is rife with terms like “new math,” “old math,” “Common Core math,” and “traditional.”

Around the 1950s, at least in the US, we saw the rise of what is typically thought of as the path of secondary mathematics (i.e., Algebra, Geometry, Trigonometry, Pre-Calculus) to prepare for College Calculus (Usiskin, 2010). I argue that we should question the mathematical understandings taught in schools, as any set of standards and curriculum should always be up for debate. However, removing materials because they engage students in critical thinking and engaging with their affective selves seems dangerous.

As I mentioned, this controversy over materials surfaced in the mainstream media with the case of Florida. In Florida, the Commissioner of Education rejected 54 textbooks to be approved for use in schools because they “indoctrinate” students through references to Critical Race Theory inclusions of the Common Core, and unsolicited Social Emotional Learning in Mathematics,

April 15, 2022

Today, Commissioner of Education Richard Corcoran approved Florida’s initial adoption list for mathematics instructional materials properly aligned to Florida’s Benchmarks for Excellent Student Thinking (B.E.S.T.) Standards. The approved list followed a thorough review of submissions at the Department, which found 41 percent of the submitted textbooks were impermissible with either Florida’s new standards or contained prohibited topics – the most in Florida’s history. Reasons for rejecting textbooks included references to Critical Race Theory (CRT), inclusions of Common Core, and the unsolicited addition of Social Emotional Learning (SEL) in mathematics. The highest number of books rejected were for grade levels K-5, where an alarming 71 percent were not appropriately aligned with Florida standards or included prohibited topics and unsolicited strategies. Despite rejecting 41 percent of materials submitted, every core mathematics course and grade is covered with at least one textbook. (https://www.fldoe.org/newsroom/latest-news/florida-rejects-publishers-attempts-to-indoctrinate-students.stml).
Briefly, critical race theory is a framework in education that argues that inequities continue to exist in U.S. education because we exist in a society based on racism (Ladson-Billings Tate, William F., 1995). Socio-emotional learning is the philosophy that promoting emotional and interpersonal skills in schools helps students integrate into school, work, and life (Yorke et al., 2021).

Why is this such a high-stake case that it made the news nationwide? This sentiment was expressed eloquently by Gutierrez, where she pushes us to think more deeply about equity and whether dominant and critical mathematics be coordinated to achieve equity goals. She argues,

The learning of dominant mathematics may serve as an entrance for students to critically analyze the world (using mathematics) and being able to critically analyze the world with mathematics may be an entrance for students to engage in dominant mathematics. It may be that as reform mathematics empowers marginalized students to freely reason about the world around them, it will come to be associated less with dominant mathematics and more with critical mathematics in its aims. In this sense, reform mathematics may serve as a necessary catalyst for the ability to address broader equity goals (2002, p. 152).

What is worrisome about the Florida case is that even dominant (mainstream) mathematics is too “radical” for students to grapple with. Implementing the vision of and call for mathematical empowerment demands a comprehensive “redo” of mathematics – an unpacking of mathematical empowerment into accessible national and state standards and frameworks. This unpacking requires specific expectations and learning goals across K-12; sample lesson plans, assessments, and concrete tasks; unique teaching methods; as
well as specific goals for teacher preparation programs and professional development courses. Focusing on SME my dissertation first conceptualizes SME, unpacking its specific constructs and creating a structure that is applied to review existing curricula on the presence or absence of SME. Thus, my dissertation answers the following two research questions (RQ):

RQ1  How can SME be conceptualized?

RQ2  How do mathematics teaching materials address/align with concepts of SME?
CHAPTER 3

METHODOLOGY

This chapter gives an overview of the research questions and methods used to answer the research questions. It ends with a summary of the methodology and a reminder of the research questions. A reminder, my research questions are RQ1) How can SME be conceptualized? And RQ2) How do mathematics teaching materials address/align with the concepts of SME?

To answer my research questions, I used an exploratory qualitative sequential approach (Morse, 2010). I used the qualitative findings from the first phase to help develop the analysis tool to conduct the second phase. I answered the first research question using conceptual analysis. This conceptual analysis informed the study of the data for the second research question. I answered my second research question using a content analysis.

Answering RQ 1

A Brief History of Concept Analysis as a Method

Concept analysis as a method has a long history with different underlying philosophies, traditions, and ontological assumptions and commitments. Mathematics education has typically employed conceptualization studies when related to learning mathematical concepts like in imagining typical mathematics learning trajectories for arithmetic or exponential growth (Lester & Steffe, 2013; Lobato et al., 2012; Thompson, 1994)) and for me and my framing, related to ME. Many of these analyses are rooted in the work of von Glaserfeld (1995) who argued that we must pay a great deal of attention to the mental operations of learners. However, other concept analysis methods better fit
this study's research questions. Unfortunately, those examples are not as present in the context of mathematics education, at least in the way that I needed them. As a result, I turned to the healthcare disciplines to find more explicit guidance and structure on how to examine the field for concept and conceptual evolution.

Within the healthcare field, although more numerous in their application of concept analysis, there is a range of methods informed by differing underlying philosophies. In this research, my methods aligned with evolutionary and reductionist approaches, which means I looked for claims and outcomes within context-specific cases. Those claims and consequences can be discussed and evaluated based on previously agreed-upon decisions. In this approach, we can see how changes in meaning occur over time or how other references, consequences, and related concepts are used. These are not without their strengths and weakness. The strength is that the methods of concept analysis that are evolutionary and reductionist in nature help to shed light on what is known and how the field knows it, but a weakness is I do not have a claim to understand SME as seen in the mind of students (Beckwith et al., 2008).

**The Importance of Conceptualization**

According to Baldwin and Rose (2009), there are several reasons why concept analysis is justifiable as a dissertation research method. Firstly, when a concept emerges, its meaning is often unclear, and concept analysis can help clarify concepts that emerge. Second, there should be a shared understanding of meaning throughout the profession to help develop models and analysis tools. This is further refined, and more insights emerge. Thirdly, concepts are the building blocks of theory. Clarifying concepts allows us to operationalize them further. Concepts represent the agreed meaning we assign to terms,
and sometimes only have meaning or are clear for a subgroup of people. Sometimes
concepts can be grouped into related dimensions (Rubin & Babbie, 2014). Although
Baldwin and Rose (2009) focus on concept analysis in nursing terminology, the same
applies to my research. I am conducting a concept analysis because the general concept
of empowerment in mathematics education has emerged as a focus, but there remains
some ambiguity surrounding the conceptual meaning. It is often unclear what
empowerment means in both a theoretical and practical sense. As stated previously, I
focused on one domain of mathematical empowerment, SME, so my concept analysis
uses this SME as the focus.

Adopting concept analysis as a dissertation method to answer the first research
question meant I had to decide how I would go about creating conceptual order. I
intended to illuminate SME and its dimensions so that SME could be steered forward
with more confidence and clarity (Young et al., 2020). I searched for common use and
meanings of the concept acknowledging it as context-bound. My view of concept
analysis is that this analysis is not an end point but is part of a continuing process of
development and progress. Concepts “provide the ability to categorize, organize, label,
discuss, and consequently to study the phenomenon of interest within the discipline”
(Rodgers et al., 2018a, p. 456). I will further elaborate in the following subsections.
According to Morse and colleagues (1996), an in-depth literature review is intrinsic to
concept analysis and can represent the entire approach to concept analysis.

Concept analysis is a method of inquiry resulting in clarification, identification,
and meaning of words. Several methods of concept analysis exist, but each depends on
the nature of the research question (Baldwin & Rose, 2009; Saldaña, n.d.)
Unfortunately, there is limited guidance for how to analyze concepts in mathematics education, thus, turning to healthcare literature, where concept analysis as a method is more clearly described and defined, was how I undertook this first research question.

**Trustworthiness**

Maintaining trustworthiness in qualitative research is an integral part of the process. One way I maintained trustworthiness was to be explicit in how I conducted my analysis. I am not suggesting that the methods I chose are the only ways to do these research questions, but I felt that these made the most sense given my research questions. As such, I maintain trustworthiness by offering transparency of my data analysis procedures (Saldaña, 2021). Morse and colleagues (1996) assisted me in thinking of how to maintain trustworthiness during concept analysis. When concepts are immature or little is known about the idea, the first step is developing a *skeletal framework*. For my concept analysis, I had some information about the essential characteristics or attributes of the concept. My skeletal framework drew heavily on Ernest (2002) and the original six dimensions for this research. My skeletal framework helped to focus my inquiry at the early stage, it provided internal structure to the study and allowed analysis to proceed as I created conceptual order.

**How I Created Conceptual Order**

The clarification of concepts is a continuous process. Therefore, concept clarification is an ongoing element in data collection. To answer RQ1 two sub questions guided my concept analysis through the in-depth literature view. (1) How have
mathematics education researchers framed the concept of SME, analyzed it, and reported on it? (2) How are the dimensions of SME conceptualized and reported?

**The Little Data Set of the Original Six Dimensions and Action/Awareness**

In qualitative research, the researcher should always be open to what might be present. Although answering my research questions meant I mainly relied on deductive approaches, I used an inductive approach to derive two themes of the original six dimensions put forth by Ernest (2002). The original set of dimensions (the skeletal framework, Morse et al., 1996) was the data set that allowed one layer of the findings to emerge. The sub-question that guided this inductive analysis of the little data set of the original dimensions was: Was there a way to think about empowerment differently than Ernest (2002) presented it? This sub-question led to part of the findings of SME, the action and awareness layer, which I present in Chapter 4.

**The Start Set and Establishing the Filtering Criteria**

I established the literature set through a variety of activities. My first search activity (stage 1) included using Google Scholar to find the relevant set of literature that cited Ernest (2002), which resulted in 163 hits. Additionally, I used search terms related to my conceptual framework (Mathematics OR Math) AND (Empower OR Empowerment) AND students AND (Students) in the ERIC and Psych Info databases. I included peer- and non-peer-reviewed articles to be inclusive of epistemologies and methods. From this search, 277 reports emerged in ERIC and 196 in Psych Info. I limited my search to research published after 1989, when NCTM Principles and Standards was released. This year saw the beginning of the “standards movement” in mathematics education to promote systemic improvement in mathematics education (National Council
of Teachers of Mathematics Reston, Va., 1989). The standards called attention to the potential of mathematical education to stop the stratification of learners in mathematics and create equity related to mathematical opportunity. Listed in the 1989 document are links to how this new vision or new goals would lead to student empowerment. Although Ernest (2002) was the first to explicitly formalize the concept of empowerment in math education, the 1989 standards document were likely the first to introduce these empowerment ideals and components of equity goals in math education.

Abstracts of a total of 636 articles were downloaded and entered into an excel spreadsheet. I read through each abstract and initially screened them using the criteria of whether they fit the six original dimensions or if they dealt with social empowerment as an outcome in some way. I used my initial conceptual framework of SME Ernest (2002) (bullets below): the development of the bidirectional interactions between the utilitarian aspects of SME and the critical citizenship aspect of SME: the multiple ways students see and perceive the place of mathematics in the world and in their life and how students see their ability to change their world using mathematics or if they could be included in other potential dimensions of empowerment (e.g., did they discuss empowerment as an outcome, but not align clearly with the dimensions above?). Notes and short rationales were entered into the excel document to build my first set of inclusions. (see my screening codebook in Appendix A.

- Better one’s chances in life or study
- Critically understand the use of mathematics in society: to identify, interpret, evaluate, and critique the mathematics embedded in social, commercial, and political systems.
- Be aware of how and the extent to which math permeates everyday life
- Have a sense of mathematics as a central element of art, culture, life, and an understanding of the historical and technological aspects of which mathematics permeates
• Be aware of the historical developments of mathematical symbols and abstractions, their theories, concepts, and problems
• Understand that there are multiple views of the nature of mathematics and controversy over the philosophical foundations of its knowledge.

Non-inclusion was related to did the article focus more on in school mathematical problem solving? For example, one article was excluded based on its emphasis on developing place value understandings with elementary students, so this article was removed as dealing more with mathematical empowerment than the social.

From the initial review and removing some abstracts due to being duplicates, I arrived at N=232 abstracts (Stage 2) for the next review stage (Stage 3). These were coded as yes or maybe inclusions. Analyzing 232 articles, books, chapters, or dissertations was beyond the scope of this study. As a result, I needed to narrow down the inclusions. I chose to look for specific language related to my framework to do this systematically. For this review (Stage 3), I focused on specific terminology each article used in the abstract (power and social) precisely because I was interested in how the authors were conceptualizing those terms—I wanted to understand how social, mathematics, and power were being held together conceptually. This would help me focus my understanding on how the authors conceptually tied together empowerment, mathematics, and social.

Stage 3 of review eliminated further articles, resulting in a set of N=75 pieces. From here, I quickly read through each piece and abstract to eliminate those that did not deal with the concept of empowerment significantly, eliminated duplicates, or disregarded those not focused on mathematics specifically. Some inclusions were also behind paywalls—but this only made up a small portion (just one article). This N=75
changed through this iterative process and I arrived at N=44 articles (Stage 4). After looking at the Stage 4 (see Appendix B) set I noticed that there were no papers that dealt specifically with the following two domains:

- Be aware of the historical developments of mathematical symbols and abstractions, their theories, concepts, and problems
- Understand that there are multiple views of the nature of mathematics and controversy over the philosophical foundations of its’ knowledge.

After reviewing the prior cuts, I found that there was one article that I had initially coded in these historical aspects, but it did not deal with the concept in a way that made sense to include in the review. Therefore, I also chose to review prior cuts (Stage 2, Stage 3, Stage 4) to include more of the “multiple views” and include or point to 9 additional articles or books (Stage 5), resulting in a total of 53 (Final Stage) articles to do the in-depth concept analysis. You can find the list of articles, dissertations, book chapters, or conference proceedings in Appendix C.

Sometimes I had to revisit the previous stages to find or validate my results during the conceptualization process and where I made additional methodological choices. As described above, two of the domains of SME were not present in the Stage 4 literature set so I went back to the original set and identified articles dealing with these domains. However, in the “being aware of historical developments,” there was a shortage of results. For this reason, I point to potential future research in this area (which I point to in my Chapter 6). To foreshadow this future research briefly The International Commission on Mathematics Instruction in 1998 created a study group to study the “role of history of mathematics in teaching and learning of mathematics,” which included discussion on
why, how and when to use history in mathematics teaching, strategies to be used for the
effective use of history at school, university and in teacher preparation and state of the art
(in research and in practice).

Table 1 summarizes the steps of this review process.

<table>
<thead>
<tr>
<th></th>
<th></th>
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</tr>
</thead>
<tbody>
<tr>
<td>Stage 2</td>
<td>Reading through each abstract and creating a filtering criterion for possible inclusion</td>
<td>N=232</td>
</tr>
<tr>
<td>Stage 3</td>
<td>Included abstracts that contained either power or social</td>
<td>N=75</td>
</tr>
<tr>
<td>Stage 4</td>
<td>Read pre-final set and then narrowed down based on coding criteria. Noticed that there were no articles in two dimensions, revisited originals set to verify this was the case.</td>
<td>N=44</td>
</tr>
<tr>
<td>Stage 5</td>
<td>I did a brief google search for the historical aspects—only to give the reader an understanding of what was out there regarding specific historical considerations.</td>
<td>N=44 + 9 from reviewing previous cuts= Final set= 53</td>
</tr>
</tbody>
</table>

**How I Conducted My Concept Analysis**

I coded and analyzed the final set of articles (Stage 5) using both an iterative and
sequential process. My method drew on different related approaches, but I as
predominantly guided by Baldwin and Rose (2009), Rodgers’ (2018a), and Morse’s (1996) description of concept analysis. I list my process sequentially below.

1) Select a concept. For this dissertation work, I identified the concept as Social Mathematical Empowerment, as I outlined in Chapter 2. To remind the reader, empowerment is still not well defined in mathematics education research. However, the concept of mathematical empowerment has been theorized by Ernest (2002), and other researchers have used his work. I looked for ways in which Ernest’s concepts or related constructs were taken up by authors.

2) Determine the aims or purpose of the analysis. My aims with this concept analysis were an increased understanding of SME as a process and an outcome. The purpose of this analysis was tied to an understanding of how researchers were understanding social empowerment in and through mathematics education. This is what I call SME. This finding of process and outcome as the main central core of SME subsumes all coding of dimensions, theoretical and instructional approaches, and of antecedents and effects.

3) Find uses of the concept. Some concept analyses use computer software to identify all possible uses of the concept. Since there are no simple guidelines for undertaking this type of analysis, I worked to identify data relevant to the dimensions of SME. This type of analysis yields qualitative data. I looked for clues within the literature that helped me to understand how the author defined SME and how they represented it, talked about it, or studied it. I did most of this work in NVIVO, used the Find command, looking for specific phrases that would provide clues (where were the authors mentioning empowerment, what did it look
like for them?). I also looked through the documents when this way of searching did not yield any clues.

4) During the analysis process I wrote analytic memos which helped to maintain trustworthiness. In these memos I answered the questions, how could I conceptualize SME more than Ernest (2002) by looking more deeply and attending to what is in front of me in the data. Throughout the process, I asked myself how can I conceptualize SME more than Ernest (2002) by looking more deeply and attending to what is in front of me. Many theories are provisional (Saldaña, 2021). For example, one reflective, analytic memo of economic framing was discussed more deeply in the discussion. These reflective, analytic memos helped to form the basis of the finding of SME as a multilayered construct, with process and outcomes, and, helped to form the two more prominent themes of action and awareness.

a. An example of RQ1 Analysis (Turner (2003)): First, I examined the abstract and made note that the goal of this study was to build “students capacity to (a) view the world with a critical mind set and imagine how the world might become a more socially just, equitable place, and (b) identify themselves as powerful mathematical thinkers who construct rigorous mathematical understandings, and who participate in mathematics in personally and socially meaningful ways” (p.iv). This aligned with the SME dimension of “Critically understand the use of mathematics in society: to identify, interpret, evaluate, and critique the mathematics embedded in social, commercial, and political systems. “She called this
dimension Critical Mathematical Agency (OUTCOME) and argued that it could be developed (PROCESS) through Teaching Mathematics for Social Justice (INSTRUCTIONAL APPROACH) (p.9-10).

5) Collect data relevant to identify the contextual basis of the dimension. In my case, I looked in the articles for the context of the dimensions reported. Were authors always tying a particular dimension to certain theoretical or instructional approaches? Did they have ways to talk about a dimension within context?

6) Because I see SME as related and part of a larger concept system, SME may have its location in other concept systems. Therefore, other types of concepts may also be involved in clarifying that particular concept. This is why the antecedents, and the effects of the concept were important to look at. In the case of Turner (2003), she understood critical mathematical agency as action-oriented, as students being able to act upon their world in some way, so this was coded as AC2, versus just AC1.

7) As findings were developed, the antecedents and effects that occurred before and those that happened as a result of were noted. These antecedents and effects are also described in my RQ1 findings (Ch 4).

8) I looked for implications, hypotheses and implications for further developments related to SME. One of those implications I have already listed above, is that historical developments and how they relate to SME need to be developed. Though articles were initially coded to align with certain dimensions based on the abstract, articles fit within multiple codes through the analysis process.
One of the implications of further development in SME leads to my next research question. I examined teaching materials to look for addressing/alignment of SME dimensions.

**Answering RQ 2**

**Teaching Materials Selected to Answer the Research Question**

A reminder to the reader that Research question 2 is: How do mathematics teaching materials address/align with the concepts of SME? Analyzing the breadths of teaching materials geared for secondary students would go beyond the scope of this study. I purposefully selected a textbook that claimed to have as its focus empowerment of students and another textbook from the list of books that the state of Florida blacklisted. The second book was blacklisted because of inclusion of topics such as: Critical Race Theory, inclusions of Common Core, and Social Emotional Learning in mathematics that the Florida department of education argued should not be taught in mathematics classrooms (see here for the news briefing [https://www.fldoe.org/newsroom/latest-news/florida-rejects-publishers-attempts-to-indoctrinate-students.stml](https://www.fldoe.org/newsroom/latest-news/florida-rejects-publishers-attempts-to-indoctrinate-students.stml)).

I will briefly define these blacklisted simple terms, Critical Race Theory is a framework that proposes that race continues to be a significant tool for creating inequity in the United States (Ladson-Billings Tate, William F., 1995). Social Emotional Learning focuses on the social and emotional development of social and emotional skills (Hoffman, 2009). The Common Core State Standards provide a grade level review of what students should know after each grade in the United States (National Governors Association Center for Best Practices and Council of Chief State School Officers, 2010).
Although they don’t provide the detailed reviews for each of their materials that are on the not recommended list, you can find examples of what they argue is inclusion on their website (https://www.fldoe.org/newsroom/latest-news/florida-rejects-publishers-attempts-to-indoctrinate-students.stml)

The first book included in my analysis, "High School Mathematics Lessons to Explore, Understand, and Respond to Social Justice" [HSML], is a collection of lessons geared for high school mathematics students to empower them to engage in mathematics. Berry, Conway, Lawler, Staley, and colleagues (2020) utilize a “strong background grounded in Standards-Based Mathematics Instruction, Complex Instruction, Culturally Relevant Pedagogy, and Critical Mathematics Education” (Berry III et al., 2020, p. 17).

In their introduction, the authors refer to Ernest’s (2010) concept of mathematical empowerment and specifically lay out their claims as to how these teaching materials support the empowerment of students in the social sense. They also provide their underlying philosophy of mathematics as a human and social activity. They see empowerment and social justice as intertwined concepts. For these reasons, I hypothesized that these teaching materials had the potential to align with the dimensions of SME. Below I quote from the introductory material to provide the reader with an understanding of why I chose these materials.

…Mathematics is a human activity that should cultivate in students a sense of wonder, beauty, and joy (NCTM, 2018). Furthermore, students’ mathematical activity empowers them to identify, interpret, evaluate, and critique the mathematics embedded in social, scientific, commercial, and political systems. Empowered students examine the claims made in the private and public sectors,
and the pronouncements of public interest (Ernest, 2010). Students can become empowered when they have access to deep, rigorous mathematics that offers opportunities to understand and use mathematics in their world. This is achieved when the mathematics they study in school integrates topics that help them investigate and understand social injustices and equity (Stinson & Wager, 2012).

For example, tasks that address income distributions, sustainability, mortality rates, taxing structures, or lending practices present opportunities for empowerment and social justice. (p.17)

The lessons in the book are built around three key goals addressing aspects of SME as I list below:

- Students will see how mathematics applies to their lives and be empowered to use it to change the issues that affect them most.
- Students will become more engaged in city, state, regional, and community grassroots to address social disparities.
- Students will have enhanced discourse around difficult topics. (Berry III et al., 2020, p. 1)

The second book I chose for inclusion is “Thinking Mathematically” [TM] authored by Ron Blitzer (Blitzer, 2019) from Florida’s list of not recommended textbooks. This book is an integrated mathematics 9-12 textbooks, and scored “not recommended” for both subject specific standards score and the inclusion of special topics (a reminder that the special topics that were banned were: inclusions to critical race theory, social emotional learning, and common core state standards). It is a more typical mathematics textbook (distributed by Pearson), and I believe it would serve as a good case comparison
of if and how materials were aligned to the dimensions of SME I found in the first research question. Another way to ask this is, does Thinking Mathematically actually align with some SME dimensions like Florida makes the case that it does?

Both books offer potential examples of opportunities to engage students in ways that will increase their SME. Before I present the findings, I hypothesized that the materials in HSML could support the development of SME. However, this was just conjecture based on a brief analysis of the introductory materials. For this reason, I needed more information using the findings for RQ1 to assist with the answering of RQ2. Below I describe how I conducted my analysis of the materials in these two mathematics textbooks.

**Using Content Analysis to Analyze Teaching Materials**

Content analysis is a research tool used to determine the presence of certain words, themes, or concepts within qualitative data. Because of this, I chose content analysis as a method to answer RQ2. Mayring (2014) offers specific guidance for qualitative content analysis methodology. There are different models for how to conduct content analysis, one stemming from the counting and sorting of text elements—informed by a positivist view of reality. This is not the method of content analysis I am using in this study. Although consideration for possibility of replication of analysis is important, I use content analysis with the understandings that text and meaning are created and constructed for particular contexts. I build upon Hardy, Harley, and Phillips’ (2004) and Mayring (2014) for a general framework of conducting a qualitative content analysis. These are the processes I followed.
1. Determine the material. The first step in analysis was to determine the material. As stated above I chose two collections of materials (two books) that were to be the materials analyzed. Both books were made up of collections of lessons, with introductory materials. I analyzed all student directed lessons in HSML, but I did not analyze all lessons in TM. For TM I chose the samples from the second book by their alignment of Essential Mathematics Concepts (National Council of Teachers of Mathematics, 2018) with HSML. These Essential Mathematics Concepts weren’t essential to the choice of the materials.

2. Deal with meaning and the circumstances of origin of the materials. In my case, these two textbooks were constructed in a particular context and were chosen for a particular reason which I named above. In the case of TM and HSML, the materials are intended for high school mathematics learners.

3. Deal with the unit of analysis. My unit of analysis was lessons. I looked at the lesson in each book, and the larger book as a whole to glean meaning. To find alignment to SME dimensions I looked at the lesson objectives. To find approaches, I looked within the text of the lesson. To find relations to other dimensions I looked at other text within the materials.

4. Deal with categories. In my case, the categories were pre-existing results from RQ1.
   a. RQ1 produced 8 dimensions, along with antecedents/outcomes and pedagogical strategies I used to help me distinguish the difference between dimensions. SME Dimensions. (AC1,
AC2, AW1, AW2, AW3, AW4, AW5, ADD8). What these dimensions are, and mean will become more apparent in the next Chapter as I present the findings from RQ1.

b. Antecedents/Effects

C. Theoretical and Instructional Approaches

5. Data analysis. These marking of passages in the teaching materials meant running through the materials in large chunks, and those cases were assigned to the specific categories with reasons for that coding. Table 2 lays out an example of how I did this with my codebook. I listed the source material, if that material addressed dimensions of SME, how I saw those dimensions addressed by specific passages in the text and the theoretical/instructional approaches that were used.

6. The final step was summarizing the themes and categories. For my purposes I provide descriptive results of those themes and categories and their alignment. I present the ways the textbooks took up alignment of SME.
Table 2. My Deductive Coding Scheme

<table>
<thead>
<tr>
<th>RQ2. How do secondary mathematics teaching materials align/address components of SME?</th>
<th>Does the teaching material align with SME? Does it address underlying concepts of SME?</th>
<th>Which dimensions of SME does the teaching material address?</th>
<th>How and why? Or how not and why not?</th>
</tr>
</thead>
<tbody>
<tr>
<td>Source</td>
<td>HSMLSJ Lesson 5.3</td>
<td>Yes</td>
<td>Understand that there are multiple views of the nature of mathematics and controversy over the philosophical foundations of its’ knowledge.</td>
</tr>
<tr>
<td>Evidence from</td>
<td>P.91</td>
<td>P.91</td>
<td></td>
</tr>
<tr>
<td>Mathematical Standards that it Addresses</td>
<td>Number</td>
<td></td>
<td></td>
</tr>
<tr>
<td>What pedagogical features does it contain?</td>
<td>LAUNCH EXPLORE SUMMARIZE</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Evidence from</td>
<td>p.91</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
In Summary

In summary, RQ 1) How can SME be conceptualized? was answered using a concept analysis. RQ2) How do teaching materials align/address SME concepts? was answered using qualitative content analysis. The next chapters provide findings for each of the research questions and discussions. If you are more interested in why I chose the materials I did, I have a completely different non-example alignment in Appendix D.
This chapter presents the findings for RQ1) How can SME be conceptualized? To answer this research question, I used an exploratory qualitative approach. In summary, the literature analysis revealed SME as a multilayered concept impacted by antecedents (conversation and the role of other people, training of teachers, and aims of teaching and learning mathematics). Across the layers, SME affects the individual learner and the collective of learners. The first layer stresses SME as both process and outcome, and process and outcome as dialectically related. The second layer points to action and awareness, to which eight dimensions are attributed (third layer), expanding Ernest’s (2002) six dimensions of SME by two more. Results revealed a fourth layer conceptualizing SME through theoretical and instructional approaches. In the following, I use examples from the set of literature to support my findings. At the end of this chapter, I discuss the findings and how those findings will be used to answer RQ2 in Chapter 5. My visual representation of this multilayered SME is depicted in Figure 2 which will be explained while describing the different layers of SME.
Layer 1: SME as Process and Outcome

Many conceptual analyses often start with a dictionary definition of each word of the concept to provide a mutual understanding. Many conceptual studies (Baldwin & Rose, 2009; Brush et al., 2011; Morse et al., 1996; Nuopponen, 2010; Yazdani & Shokooh, 2018) address how a concept is used and defined in other fields or locations because spoken and written concepts often influence the cultural understanding of that concept. For example, the word biscuit conjures a different meaning for people in the United States than for people in the United Kingdom. In the US, a biscuit is considered a bread type, slightly fluffy, soft, and savory; in the UK, a biscuit is viewed as a hard, dried, baked good, somewhat sweet. A dictionary definition helps to identify these different meanings of the same word depending on cultural contexts. Though my analysis
focused on how researchers in mathematics education defined, conceptualized, and used SME, a more general understanding of the three defining terms of SME – social, mathematics, and empowerment – through a dictionary definition revealed the first layer of SME, process, and outcomes, in all three concepts. Merriam Webster’s definition of social is “to or involving activities in which people spend time talking to each other or doing enjoyable things with each other.” Mathematics is defined as “the science of numbers and their operations, interrelations, combinations, generalizations or abstractions of space”; and empowerment as the “act or action of empowering someone or something: the granting of the power, right or authority to perform various acts or duties” (https://www.merriam-webster.com/dictionary/). The first defining word of SME, social as “relating to other people” is also a significant part of what distinguishes it from mathematical empowerment (achievement in school mathematics) and epistemological empowerment (relating to identity), the other two domains of Ernest’s (2002) model of empowerment through mathematics.

Researchers have used the concept of empowerment to explore relationships between individuals and their environments (Zimmerman, 1995). For example, in education, empowerment is often associated with the work of Freire (Freire, 2018). In his seminal text, Pedagogy of the Oppressed, Freire expresses the need to empower disenfranchised individuals by taking control over their learning and developing a deeper understanding of their position within a community through active participation and engagement. Although Freire’s work or other research in health sciences or psychology on empowerment (Freire, 2018; Perkins & Zimmerman, 1995; Rappaport, 1990) was not part of the data set for this study, their understanding of empowerment supported my
view of SME as both an outcome and a process - as the dialectical relationship between individuals, the collective, and their contexts.

SME is both outcome and process. Ernest (2002) stresses that SME “concerns [both] the ability to use mathematics to better one’s life chances in study and work on one end of the [relationship] and to participate more fully in society through critical mathematical citizenship” (p.1) on the other end. This dialectical relationship of process and outcome meant that a dictionary definition of SME would fall short and the need to interrogate further process and outcome and how they come to be. In this way, I have begun the work of translation from theory to practice within a particular developmental framework of learning. To help me show you why SME can be conceptualized in this way I lean on my learning and development framework. While constructing a general development trajectory, Vygotsky discussed two central elements: the dialectical waving of individual and social processes of learning and development. That human activity “takes place in a social and historical context and is shaped by, and helps shape, that context” (Mahn & John-steiner, 2013, p. 28) (Mahn & John-steiner, 2013, p. 28; Vygostsky, 1978). None of the articles that aligned with SME framed their alignment of SME as “complete.” Rather, the framings were primarily qualitative and supported the complexity of SME as multilayered. One of Ernest’s limitations in his work was his understanding that his frame of empowerment was primarily individualistic. Applying this developmental framework to Ernest’s, particularly with SME, allows us to understand better how SME is both process and outcome. This finding shows complexity and clarity when imagining what SME is and what it is not. Overall, the articles in the literature set point towards the process and outcome phases. No articles claim that this
SME process is done, or is finished, but present stories of complexity, of learners developing SME through action and awareness. The set of research inclusions is included in Appendix D.

**Layer 2: Action and Awareness of SME and Layer 3: Dimensions of SME**

Delving deeper into Ernest’s framework of SME (the little data set of the six dimensions described in Chapter 3), two different components emerge from his six dimensions: Action and Awareness. Action (AC) means achieving a goal; or doing something. Awareness (AW) means knowledge or perception of a situation. Using this approach to Ernest’s six dimensions resulted in five dimensions fall under awareness (AW): what is perceived, what is understood, and what a person’s sense is), and two falling under action (AC) (the act of doing something) (Table 3 below). An additional dimension (ADD) is present, which contains both awareness and action parts. This additional dimension emerged only from one article, so does not have as many connections to other codes as the others.

**Table 3. SME Awareness (AW) and Action (AC) Components and their Aligned Dimensions**

<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>SME Dimensions</th>
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<tbody>
<tr>
<td>AW1</td>
<td>Be aware of how and the extent to which math permeates everyday life.</td>
</tr>
<tr>
<td>AW2</td>
<td>Critically understand the use of mathematics in society</td>
</tr>
<tr>
<td>AW3</td>
<td>Have a sense of mathematics as a central element of art, culture, life, and an understanding of the historical and technological aspects of which mathematics permeates</td>
</tr>
<tr>
<td>AW4</td>
<td>Be aware of the historical developments of mathematical symbols and abstractions, their theories, concepts, and problems</td>
</tr>
<tr>
<td>AW5</td>
<td>Understand that there are multiple views of the nature of mathematics and controversy over the philosophical foundations of its’ knowledge.</td>
</tr>
<tr>
<td>AC1</td>
<td>Bettering one’s chances in life or study.</td>
</tr>
</tbody>
</table>
AW also implies thinking of a concept, whereas AC means an expression of a thought, goal, or intent. The words: sense, aware, and understanding are primarily individualistic. Applying this to Ernest’s six dimensions shows that four of them were developed within the individuals’ minds and are part of AW—shown in Table 3 as AW1 to AW5. The dimension AC1 included the word “better”. Bettering implies an action: to do something to improve (AC1). The sixth dimension exists in both of those components. Finally, the SME dimension AW2 implies recognition, “Critically understand the use of mathematics in society: to identify, interpret, evaluate, and critique the mathematics embedded in social, commercial, and political systems.” It means careful thinking. There is another part of this dimension, “the interpretation, the evaluation, and the critiquing of the embedded mathematics.” Interpretation, evaluation, and critiquing all serve to take knowledge and perform an action; thus it was aligned with the AC component (AC 2).

During my analysis of the literature set, I discovered one additional dimension aligned with SME that was not present in Ernest’s original framework. This dimension is what I call Experiencing beauty, joy, and pleasure through mathematics (ADD8). This finding emerged from an examination of an article entitled Recreational Mathematics—Only for Fun? Sumpter (2015), explicitly examines recreational mathematics from two perspectives. She first explores how the concept appears in educational policy documents from various countries (China, England, Finland, India, Japan, Singapore, and Sweden) and in comparison, to the USA by asking: “Can [recreational mathematics] be a tool for

<table>
<thead>
<tr>
<th>AC2</th>
<th>Critically understand the use of mathematics in society: to identify, interpret, evaluate, and critique the mathematics embedded in social, commercial, and political systems.</th>
</tr>
</thead>
<tbody>
<tr>
<td>ADD8</td>
<td>Experiencing beauty, pleasure, and joy through mathematics.</td>
</tr>
</tbody>
</table>


social empowerment?.” She defines recreational mathematics as doing mathematics for fun; as “an activity where the person wants to solve the problem based on positive motivation (most likely intrinsic); the activity is linked with positive emotions, and it has some educational dimensions, although topics may range over different mathematical areas” (Sumpter, 2015a, p. 123). She found that in contrast to the USA, these seven countries’ curriculum policy documents contained the intended aims of recreational mathematics. ² She argues that mathematics should be fun; it should educate all people on an individual and social level. She views this as linked to SME.

Layer 4: Theoretical and Instructional Approaches of SME Dimensions

Layer 4 emerged from the literature analysis and by asking: How are researchers further conceptualizing these eight dimensions, and what do they entail? These findings move beyond SME as process/outcome, action/awareness, and detail the eight dimensions using theoretical and instructional approaches. Studies addressed these eight dimensions as theoretical and/or instructional approaches. Theoretically, an SME dimension is conceptualized as a theoretical approach—in that it deals with the dimension in a potentially abstract way. For example, Scott-Parker and Baron-Nugent (2019) viewed AC1 (bettering one’s life or study) as economic empowerment and proposed teaching for social justice (theoretical approach) to achieve AC1. Instructional means that a SME dimension is described through practice. For example, Scott-Parker and Baron-Nugent (2019) also provide concrete examples of how to achieve AC1

² The United States does not have a national curriculum. The closest US national standards in mathematics are the Common Core State Standards in Mathematics. Those are not adopted by all states and are not adopted by the state of Florida although they have influenced multiple state frameworks, such as the state of Massachusetts.
(bettering one’s life or study) through role models that demonstrated a successful career in a STEM field (better one’s life). Figure 1 shows this fourth layer of SME near the dimension below the action and awareness component.

shows a summary of which study addressed which SME dimension. While some articles addressed only one of the dimensions, others addressed more than one and some pointed to the larger process outcome themes. In the following these findings are presented for each dimension.

<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Dimension</th>
<th>Literature Set</th>
</tr>
</thead>
<tbody>
<tr>
<td>AW1</td>
<td>Be aware of how and the extent to which math permeates everyday life.</td>
<td>(Foley, 2016; Johansen, 2006; Madusise &amp; Mwakapenda, 2014; Planas &amp; Civil, 2009)</td>
</tr>
<tr>
<td>AW2</td>
<td>Critically understand the use of mathematics in society</td>
<td>(Kokka, 2015; Turner, 2003; Tutak et al., 2011)</td>
</tr>
<tr>
<td>AW3</td>
<td>Be aware of the historical developments of mathematical symbols and abstractions, their theories, concepts, and problems</td>
<td>(Furinghetti, 2005; Johansen, 2006; Tzanakis et al., 2002)</td>
</tr>
<tr>
<td>AW4</td>
<td>Understand that there are multiple views of the nature of mathematics and controversy over the philosophical foundations of its’ knowledge.</td>
<td>(B. Lawler, 2012; Martin, 2009a; Stemhagen, 2007)</td>
</tr>
<tr>
<td>AW5</td>
<td>Have a sense of mathematics as a central element of art, culture, life, and an understanding of the historical and technological aspects of which mathematics permeates</td>
<td>(Unfried &amp; Canner, 2019)</td>
</tr>
</tbody>
</table>
| AC1          | Bettering one’s chances in life or study. | (Chehayl, 2008; Chen, 2010; de Oliveira Souza et al., 2020; Ernest, 2015; Lahana, 2016; Mujtaba & Reiss, 2015; Paudel, 2020; Planas & Civil, 2009; Scott-Parker &
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<tr>
<td><strong>AC2</strong></td>
<td>Critically understand the use of mathematics in society: to identify, interpret, evaluate, and critique the mathematics embedded in social, commercial, and political systems.</td>
<td>(Frankenstein, 1990; E. Gutstein, 2003; Hartman &amp; Kahn, 2017; Jett et al., 2015; Jong &amp; Jackson, 2016; Karaali &amp; Khadjavi, 2019; Killpack &amp; Melón, 2016; Kokka, 2015; Kurtuma &amp; Romano, 2015b; Leonard, 2010; Lucey &amp; Tanase, 2012; Madusise &amp; Mwakapenda, 2014; Mhlolo &amp; Schäfer, 2012; Tutak et al., 2011; Wright, 2017a)</td>
</tr>
<tr>
<td><strong>ADD8</strong></td>
<td>Experiencing beauty, pleasure, and joy through mathematics.</td>
<td>(Sumpter, 2015a)</td>
</tr>
</tbody>
</table>

**AW1: Be aware of how and to what extent mathematics permeates daily life**

This section describes how researchers addressed AW1 (Foley, 2016; Madusise & Mwakapenda, 2014; Sumpter, 2015a). Researchers stress that schools need to “legitimize the daily mathematics in students’ lives” (AW1) through the use of culturally relevant pedagogy theoretical approach and specific instructional approaches. This AW1 is also referred to as numeracy (Johansen, 2006) Teachers would need to implement the use of dialogue, and deepen their understanding of students’ daily lives, though not only just bringing in cultural artifacts, but through deepening their understanding of students cultural backgrounds. The following examples from the literature support these findings.

Madusise & Mwakapenda’s (2014) study asked, “To what extent can school mathematics be used to understand cultural activities of life? “ (Madusise & Mwakapenda, 2014, p. 9)in a South African village. They wanted to understand the impact of critical mathematics pedagogy, a theoretical approach, and utilized particular strategies embedded with that approach, and its impact on learners through cultural dance. Students’ knowledge of bounded sequences was used to understand the need for limiting the number of steps in each dance, and their knowledge of properties of shapes...
and different transformations was used to read and understand the decorations on Ndebele paintings and beadings on Venda traditional clothes, which led to deeper understanding of the paintings and beadings.” (Madusise & Mwakapenda, 2014, p. 152).

I pull a quote from one of their participants that shows how culturally based lessons aligned with SME dimension AW1. Participants in their study argued that they used to see cultural activities as just ending at the village, but they can now see how those activities can lead them to understand what they learn at school and how they can use mathematics to understand those activities (Madusise & Mwakapenda, 2014, p. 154).

Another quote from the article illustrates how the authors saw this AW1 development.

“In the past, although the teachers emphasized that they knew activities taking place at the cultural village, knowing them, and understanding them now appear to be two different things. The use of mathematical knowledge adds more appreciation of the cultural activities […] The deeper the mathematical knowledge one has, the deeper he or she will get to understand the cultural activities.

Another study showed how “talk” supported, or negatively supported AW1. Foley (2016), looked at younger girls’ mathematical activities and recognition of those activities through the use of scrapbooks and interviews. She used scrapbooks, an instructional strategy based on an instructional approach of problem solving, to help students develop AW1. Her results show how AW1 developed as she examined children’s scrapbooks and the links between mathematics in child’s home life and school life.
“There were many heartening examples of home-based mathematical activity being recognized as such, particularly amongst the games and hobbies of Jasmine, the many and varied family-based activities of Hetty and the recognition of mathematics as a problem-solving tool within the pictures and annotations included by Taylor…[but]…the data suggests that there is scope for more frequent and consistent links being made between the two predominant locations for learning in a child’s life, home and school.” (p.161)

Foley also uses other literature to provide specific instructional strategies to build AW1. “One possible bridge supported by the data is the use of cultural artefacts as recommended by Bonotto and Basso (2001); receipts, recipes or travel guides are examples of the kind of artefacts that might be used as a starting point for classroom-based mathematical enquiry, and the presence of these within the higher- but not the lower-attaining children’s scrapbooks suggest the potential benefits of making these links explicit for all children.” (p. 161)

**AW2: Critically understand the use of mathematics in society**

Researchers that addressed AW2 talked about AW2 as critical reflection and critical consciousness. They argue that instructional approaches like critical mathematics literacy, culturally relevant teaching, democratic mathematics, and ethnomathematics were tied to building AW1 (Hartman & Kahn, 2017; Jett et al., 2015; Jong & Jackson, 2016; Karaali & Khadjavi, 2019; Killpack & Melón, 2016; Kokka, 2015; Kurtuma & Romano, 2015a; Leonard, 2010; Lucey & Tanase, 2012; Madusise & Mwakapenda, 2014; Mhlolo & Schäfer, 2012; Turner, 2003; Tutak et al., 2011; Wright, 2017a). Tutak and colleagues (2011) point out that these approaches, although all related, have differing
underlying philosophies (2011). In the following paragraphs I define these and then provide specific literature examples that help to illustrate this finding.

Thinking of AW2 as critical consciousness, critical reflection (Kokka, 2015), and critical mathematics agency (Turner, 2003). As the authors describe them, these concepts are the mathematical understandings that enable people to live in new ways and consider new ways to change systems. In other words, these understandings means looking at the world around you and understanding your and others’ place in it; it is helping learners to believe that they are conscious actors in the world and society. These scholars draw on Gutstein’s work to help them illustrate AW2. Gutstein’s (2003) concept of reading the world using mathematics is “developing a sociopolitical consciousness of the conditions and contexts of one’s life but with the specific use of mathematics. In means using mathematics to understand inequities between different social groups and to understand explicit discrimination based on race, class, gender, language, and other differences. Further it means to dissect and deconstruct media and other forms of representation and to use mathematics to examine these various phenomena both in one’s immediate life and in the broader social world and to identify relationships and make connections between them” (E. Gutstein, 2003, p. 45) . AW2 emerged, and students asked, “What else have I been lied to about in my education?” (Gutstein, 2003, p.54).

I now point to the discussion of the different theoretical approaches brought to light by Tutak and colleagues (2011). Critical mathematics education, in theory helps learners question how things are done and how they could be done differently (developing AW2) (Tutak et al., 2011). Critical mathematics education stems from a Marxist framework (the dialectical framework that is based on the conflicts between the
powerful and subjugated) and examination of hegemony (the process in which leaders exert control). Teaching mathematics for social justice is an approach embedded within critical mathematics education. Gutstein (2003) used teaching mathematics for social justice to develop AW2 with secondary mathematics learners. Teaching mathematics for social justice develops this critical consciousness through engaging students in using mathematics to study and analyze social contexts. He used specific social contexts like race, gentrification, and wealth distribution to develop AW2 (map projection project). He argued AW2 developed overtime and argues

“Students overwhelmingly showed evidence of connecting mathematical analysis to deeper critiques of previous assumptions. On the map projection project, all but one student felt that something was amiss in their education after using mathematics to see that standard classroom maps misrepresented the true size of countries (Gutstein, 2001, as cited in Gutstein, 2003).

Critical mathematics education approaches may not always lead to AW2. A negative result of AW2 development is shown by Brantlinger (2013b) where AW2 did not emerge through instructional approaches embedded in similar theoretical approaches used by Gutstein (2003). He argued that “Several but not all, of the higher achieving mathematics students continued to resist [critical mathematics] projects from the final weeks because they did not see them as good preparation for college mathematics” (p. 1075).

A potential instructional avenue to remedy a negative result of AW2 is by building community with learners. Kokka (2015) noticed that one way to help learners to gain AW2, was through building community. Many times, she argues that to help
students build this consciousness, both teachers and students need to have a shared awareness of each other’s lived experiences. One way to build this shared awareness is through community introduction and community tours. “The community tour is intended for students to guide their teachers, many of whom do not live in and did not grow up in students’ communities (p.16).” In addition, she argues for specific instructional approaches to help develop. These include problem-based learning, complex instruction, and co constructed goals for learning.

Ethnomathematics, another theoretical approach to promote AW2 is to work “against Eurocentrism in the mathematics classroom and to include the voice of other cultures and the contributions of those cultures to mathematics” (Tutak et al., 2011, p. 70). This approach strives to reconstruct mathematics by using authentic mathematics from students’ lives and using history and culture to promote AW2 (Frankenstein, 1990). Instructional approaches where students reflect on ethnomathematics could be writing, class discussions or helping learners reconceptualize their mathematical knowledge.

Culturally Responsive Teaching is another theoretical approach that can promote AW2. Gay (as cited in Tutak, 2011) identified five elements of culturally responsive teaching as ‘developing a knowledge base about cultural diversity, including ethnic and cultural diversity content in the curriculum, demonstrating caring and building learning communities, communicating with ethnically diverse students, and responding to ethnic diversity in the delivery of instruction’ [35, p. 106].

Lucey and Tanase (2012) provide a joint instructional framework of these instructional and theoretical approaches (Lucey & Tanase, 2012). They argue theoretically for AW2 through Democratic Mathematics. Democratic Mathematics is the
development of socialization skills among students, so they are prepared to evaluate and respond to different social ideas using mathematics. They argue that a dependency on textbooks that promote problem mechanics inhibits creativity, but discussion and democratic processes help students understand mathematical connections within their lives. They also include a framework to incorporate instructional strategies to promote AW2 using cooperative learning. Although not intended to be a fixed guide, they do suggest steps to AW2. These steps below correspond to Gutstein (2003), NCTM (2009), and Borich (2011), as cited in Lucey and Tanse, 2012)

1. Step 1, Specifying the Goal, provides direction to the endeavor. The students in a classroom determine the topic and agree upon behavior standards and outcomes expectations.

2. Step 2, Structuring the Task, defines the cooperative process because it concerns the characteristics of the groups and members’ responsibilities. Decisions about group size, composition, responsibilities, incentives, and time spent on tasks may be done through teacher/student interfacing. As guided by the student experience with cooperative learning, teachers may present a recommended structure and invite discussion about its modification.

3. Step 3, Teaching and Evaluating the Cooperative Process, ensures that students realize that cooperative learning requires a different set of behavioral expectations than do traditional instructional methods. Communication about thoughts and emotions should occur in respectful manners that acknowledge all persons’ points of view.
4. Step 4, Monitoring the Group Performance, represents a difficult step for teachers because it requires a focus on the learning process. Teachers emphasize facilitation of student inquiry, guiding their efforts, but not providing solutions. Rather, the emphasis involves encouraging community and building the cooperativeness needed to achieve the common goal.

5. Step 5, Debriefing, provides an opportunity for students to discuss the experience in learning the content and processes. It may be the most crucial step in that provides students with an opportunity to discuss reasons for successes and ideas for revisions and next steps. It should be an opportunity for encouraging further exploration and reflection on previous efforts. (p.10)

In summary, instructional approaches that develop AW2 have no manuals, but a variety of theoretical approaches to look towards to develop AW2. Educators must go beyond just including contrived cultural examples to promote AW2 or begin with social justice issues that may not be relevant to their students live. In summary to develop AW2 also means being prepared for effects of AW2, such as tension and resistance

**AW3: Be aware of the historical developments of mathematical symbols and abstractions, their theories, concepts, and problems**

This section presents the lack of both theoretical and instructional approaches of AW3. I remind the reader that I had to figure out how to deal with this negative result methodologically. I decided to do a cursory search using Google Scholar to share findings regarding how this awareness of historical developments has been conceptualized with learners and the field. This was not specifically linked to SME by the authors, but I think it could be a starting point for future research.
I found an International Conference for Mathematics Education (ICME) study book named “Role of the History of Mathematics in the Teaching and Learning of Mathematics.” This book was the product of a conference specifically addressing the historical aspects of mathematics education. In my finding section, I focus on what was learned by reading Tzanakis and colleagues (Tzanakis et al., 2002). Theoretically, they argue that there are five main areas in how AW3 may be supported, enriched, and improved through integrating the history of math into the educational process (AW3) (p. 203). These are

1. The learning of mathematics
2. The development of views on the nature of math and math activity,
3. The background of teachers and their pedagogy
4. The affective dispositions towards math
5. And the appreciation of math as a cultural-human endeavor

They then go on to make explicit some of the reasons for integrating history into math education and further make explicit instructional approaches of AW3. First by learning history, learning mathematical topics, and developing a deeper awareness of both the math itself and the contexts in which that mathematics has been done. They provide a list of instructional approaches of ways to explore this in classroom teaching, including but not limited to:

- Historical snippets
- Research projects based on historical texts
- Primary sources
- Historical packages
- Errors, alternative conceptions, changes of perspective
- Historical problems
- Instruments
- Experiential activities
- Plays
- Films or other visual means
- Outdoors
- The internet

**AW4: Understand that there are multiple views of the nature of mathematics and controversy over the philosophical foundations of its’ knowledge.**

This section presents how researchers have addressed AW4. These examples show how theoretically and instructionally AW4 has been explored with learners. AW4 – is an interesting dimension of SME because it requires pushing past a view of mathematics that learners might hold or that we as learners might hold. It might even require a reimagining of the definition I began with in the start of these findings. What are some of the potential differing views and controversies present in relation to SME and what are some of those views of mathematics that learners could gain awareness of? Stemhagen (Stemhagen, 2004) addresses AW4 of mathematics as a series of evolving, constructed tools to solve genuine problems, and those tools are evaluated according to how well they help solve genuine problems. This development of AW4 begins with the instructional question of “why do we have to learn this?” Ethnomathematics, a theoretical approach that Lawler (2012) offers another view of mathematics as simulacra and uses his work with learners to support this theoretical position. Instructional approaches to build
differing views of nature of mathematics is not as present. Ethnomathematics also has the potential to develop AW4 through sharing and developing the contributions of students mathematics to the table (Kokka, 2015; Tutak et al., 2011)

For example, what does a different view of mathematics look like? Lawler (2012) conducted work on reconstituting mathematics as a simulacrum. He argues that this definition of mathematics is in our minds. He compares it to simulacra. Simulacra, loosely defined, is an imitation of a person or thing. Lawler proposes AW4 as “Mathematics as a fabrication and an untruth and that we forget that “there is not a mathematics to be learned” but that reality is unknowable and as such, any singular knowing represents a viability rather than a truth. Knowledge, [sic] one ‘s way of knowing, must fit reality, but does not represent reality.” He argues for a different type of thinking of mathematics as being “inside the heads of learners.”

Instructionally, he presents evidence to development for AW4. Lawler says, “While mathematics had an important social role, these subjects maintained a strong notion that it was for them to determine a truth to the knowing, particularly evident in Fisk. The disagreement of another served as a catalyst for further inquiry and drive to make meaning, rather than passive acceptance of proof from external authority. These students drew heavily on interaction with others to confirm their own knowing, to feel justified to speak of confirmed facts” (B. Lawler, 2012, p. 208). This model case show how it is possible that the nature of mathematical activity and knowing wasn’t locked in place and that it was up to the students to develop their own knowing and their own truths. This suggestion argues for “a mathematics” that could be learned through critical thinking. “That in and of itself would be a worthy mathematical goal.” Then achievement
would be redefined as being, thinking, and interacting which are constructs which are non-hierarchical.

**AW5: Have a sense of mathematics as a central element of art, culture, life, and an understanding of the historical and technological aspects of which mathematics permeates**

This section presents the findings for how researchers have addressed AW5. This expression of AW5 was addressed through pedagogical approaches like teaching mathematics for social justice, critical mathematics, and dialogue. The art component of AW5 is noticeably absent from the literature in relation to SME. Below I present a powerful example of what this AW5 can look like with students or learners, specifically related to the historical and technological aspects. Unfried and Canner (2019) present an example of how AW5 can look with their learners as they describe the course they teach through the lens of social justice (Unfried & Canner, 2019). They point to a specifical pedagogical instructional strategy they have used they find to be important in the development of AW5.

Perhaps the single most influential piece of the course that defines student perspectives on the relationship between mathematics and social justice, aside from the projects, is the inclusion of the book Weapons of Math Destruction: How Big Data Increases Inequality and Threatens Democracy, by Cathy O’Neil [14]. This book discusses how the use of mathematical models in various sectors of society has actually increased injustice, whether intentionally or not. For example, O’Neil discusses how algorithms have provoked injustices seen in the housing crisis, recidivism rates, for-profit colleges, and more. Students read one chapter a
week, respond to prompts given through our online learning management system, and discuss the chapters in class. The book chapter topics create the framework for the weekly class discussions on social justice issues. We see how mathematics directly influences society on a massive scale in many different fields. Unfortunately, this influence is not always positive, which is the focus of the book. We also read an article about using data for good, which highlights the work of former chief data scientist D.J. Patil [7]. Here, the readings remind students that the immense power of mathematics means that there is an immense opportunity to use math for good; that is, to better society. (p. 223-225)

**AC1: Better one’s chances in life or study.**

This section of findings presents overall theoretical view of AC1. Overall, the results in this section point to the focus on improving participation in the labor force and economic advantage through the lens of employability. Those approaches include functional numeracy (Ernest, 2015) and economic empowerment (Johnson, 2018) especially for females and minoritized learners to gain a potential for increase in social class. This is the theoretical approach of AC1. One of the ways in which this bettering of one’s chances in life or study is through functional numeracy (Ernest, 2015). This is defined as the ability to deploy mathematical and numeracy skills adequate for the general employment and functioning in society (Foley, 2016; Johnson, 2018).

This idea of AC1 as economically empowering through choice of college majors and careers, decreasing the gender gap in regards to the amount of women who pursue STEM majors and careers is promoted through approaches like including role models in curriculum, or by including relevant interesting topics into the curriculum for
underrepresented learners (Johansen, 2006; Mujtaba & Reiss, 2015; Murphy-Graham, 2007; Paudel, 2020; Planas & Civil, 2009; Scott-Parker & Barone-Nugent, 2019).

Increasing motivation and engagement through interest and inclusion of role models were seen as instructional approaches to promote AC1 through engagement by seeing potential career pathways or seeing how STEM skills could lead to real world impact.

A theoretical argument is often made for AC1 in that it will increase the chances for students as they move forward, that it minimize opportunity gaps. However, there are few studies that follow this trajectory over the long term and examine this approach over the long hall. For this reason, I chose to include one particular article that came to mind. This article was not in the final literature set of articles chosen to include in the results. But I include it because I know it as one of the few articles, specifically in mathematics education that looks at what happens to learners as a result of engagement in different mathematical experiences in high schools and the use of their “human agency” (Pickering, 1995), a related term that emerges when we think of empowerment. This study asked the question: Did the differing forms of identity and expertise that students had developed at school persist into their working lives and impact their use of mathematics in life? Their results showed that of the 63 participants that participated 65% of the Phoenix Park – the non-traditional mathematics school where students were able to enact their human agency—participants moved upward in their social class. Their interviews included questions about whether their school mathematics classes had been helpful in preparing them for their lives. The adults from Amber Hill (traditional, procedurally focused school) talked about mathematics through its separated content
areas. However, adults from Phoenix Park talked about the multidimensional mathematics that they had learned.

**AC2: Identify, interpret, evaluate, and critique the mathematics embedded in social, commercial, and political systems.**

This section of my findings describes how researchers have addressed AC2—the action component of the dimension we just explored. This expression of AC2 is addressed theoretically as “writing the world” (and not just understanding it), which means to be a part of remedying unjust situations, a way to transform their lives and the lives of others (de Oliveira Souza et al., 2020; E. Gutstein, 2003). Instructionally, AC2 is addressed through many of the same approaches as AW2, but critical literacy and critical mathematics agency specifically align with the action component. They are the ability to understand mathematics’ role in society and how to use it for AC2; it’s to use one’s capital to change the status quo (Lahana, 2016; Leonard, 2010). It also requires different research approaches (Martin, 2009) to understand AC2.

An example of AC2 is in Wright’s (2017b) article, where he describes the interplay between teachers as students to develop AC2. I present the case of Brian, one teacher learner who started to develop different perspectives, and how they used it in an action way by critiquing school mathematics and by taking action on how he used it.

“Brian developed an increasingly critical perspective on mathematics education, beginning to appreciate how schools perpetuate injustices. He began to advocate the importance of students developing critical understanding, for example, through appreciating how to think independently and develop their own arguments. He also argued that school mathematics had a vital role to play in
countering myths students were exposed to regularly in the media.” (Wright, 2017, p. 518)

Creative insubordination (de Oliveira Souza et al., 2020) as an instructional approach is also argued to promote this AC2 to promote positive action and promote change. This approach helps develop this AC2. These strategies include pressing for explanation, to counter with evidence, to share both practice and outcome of investigation, to position and justify actions within the policies that have the potential to restrict practice, to find allies, to turn issues into morals ones, and to fly under the radar (Gutierrez, 2016, as cited in de Oliviera Souza et a., 2020).

Culturally Sensitive Research Approaches, a way to re-examine research in mathematics education explored by Martin (2019). He critically analyzed research methods that have traditionally been used in mathematics education (i.e., constructivism, comparative studies, etc.). First, he argues that richer characterizations of African American experiences. These research methods include the use of qualitative methods where the researcher relies heavily on firsthand experiences by African Americans (p. 22) and efforts of methods to uncover power relations that minimize those experiences. Then follow culturally sensitive data approaches and interpretations. AC2. He pushes sociocultural theories of learning to account for race and racism in students experience instead of the way that race is usually used to disaggregate data and create a hierarchical explanation versus where we make comparisons rather that treating whiteness as a consideration in these student’s mathematical experiences and outcomes (Martin, 2009b). Finally, I present a quote from his work that applies more broadly to all students regarding the development of AC2.
My research has shown that to the degree that their African American status, identities, and prior knowledge are valued, devaluated, or challenged in schools and by teachers, African American students respond by engaging in complex meaning making practices and agency related behaviors in response to real and perceived racism and assaults on African American identity (p. 23).

In other words, the development of AC2, part of SME, and understanding it also rests on the field modifying its approaches to studying it.

**ADD8: Pleasure and Joy through Recreational mathematics.**

The last dimension of SME I present is pleasure and joy through recreational mathematics (ADD8). This result emerged as an additional dimension of SME primarily from a special case of a research article. In this article, Sumpter (2015) aligns social empowerment with recreational mathematics and see the parallel of doing mathematics for the sake of pleasure is one way to understand mathematics as a social endeavor, but also a way to equalize the playing field so to speak. This recreational mathematics becomes a tool to educate all people and not just some people because of the way in which it manifests itself through joy and play (Sumpter, 2015b). Interestingly, this ADD8 is present in other national curriculum documents but not present in the United States Common Core Frameworks. For example, in the Chinese curriculum there is particular attention paid to Affection and Attitudes, where standards suggest that teachers should pay attention to emotions encountered during mathematical learning. In Finland, there is attention to joy. In India there is a focus on enjoyment (Sumpter, 2015) This additional dimension falls under both the awareness and action components, it is being aware of
mathematics as evoking positive individual and group emotion, but also as a way to share mathematical knowledge (action) with others.

**Antecedents and Effects of SME**

Defining antecedents of a concept help to further illustrate a concept (Baldwin & Rose, 2009; Rodgers et al., 2018b). These antecedents as the events or attributes that arise prior to a concept’s occurrence. What are the antecedents of dimensions of SME? What would be important for the development of SME? What must happen before? Based on the review of the literature, I found that there were three general themes important for engagement in SME: Conversation and the Role of Other People, Training of Teachers, and an Examination of the Aims of Teaching and Learning Mathematics.

**Conversation and the Role of Other People**

The act of engaging all students in their learning process and in conversations is crucial to developing and fostering SME (Murphy, 2013). The role of other people, notably teachers, families, and friends have a significant influence on the development of empowerment as it relates to the dimensions outlined above (Foley, 2016; Lucey & Tanase, 2012; Murphy, 2013; Uribe-Flórez et al., 2014; Wilding-martin, 2011).

The role of communicating with others, including the learners in many of these studies was important in the development of dimensions of SME. Some examples from articles include “conversation plays an emancipatory role, fostering democratic citizenship by developing both mathematical and social empowerment in students” (Wilding-martin, 2011). “A key premise of this doctoral study is the realization that teaching, and learning relate to communication and mutual understanding. Hence the study recognizes the importance of the child in learning in the classroom and how
communication in a classroom context, including collaborative group work, involves reciprocity and empathy. As such, although the study focused on learning, agency resonates throughout in considering the child taking charge of their talk.” (Murphy, 2013, p. 289)

**Preparation of teacher learners to promote SME**

Another antecedent of SME includes the preparation of teachers. I do not mean that teachers are not learners themselves but in order for SME dimensions to be addressed, preparation of teachers must occur. How can dimensions of SME be developed without first describing it as a concept, and then trying to share that concept with those who work with learners in schools. Bringing the complexity of SME to light with those who work with learners helps us to develop the dimensions. Though Martin argues on teacher development specifically related to African American students, his argument can be broadened to include how we can think of preparation of teachers to develop SME. Is having mathematics content preparation enough if one of the goals of mathematics education is to develop SME? Teachers should be aware of all their students’ social realities, the challenges they face and benefits they experience depending on their race/ethnicity, socio-economic background, sex, sexual identity, religious beliefs and how these determine their students’ identity:

“in addition to having mastery of the mathematics they will teach, teachers should (a) develop a deep understanding of the social realities experienced by African American students, (b) take seriously one’s role in helping to shape the racial, academic, and mathematics identities of African American learners, (c) conceptualize mathematics not just as a school subject but as a means to empower
African American students to address their social realities and life conditions, and (d) become agents of change who challenge research and policy perspectives that construct African American children as less than ideal learners. (Martin, 2009a, p. 27)”

Teachers will also need knowledge of instructional approaches included here as important to developing the dimensions of SME. Kokka (2015) discusses some of these in relation to AC2. Teachers will need to be taught about problem solving approaches, complex instruction, ways to incorporate creative insubordination or teaching mathematics for social justice. They will need to think deeply about their ideas of mathematics and consider other perspectives like in the case of AW4.

Examination of the Aims of Learning Mathematics

What are and what should be the aims of teaching and learning mathematics? An examination of aims is an important antecedent in the development of SME, in that in order for SME to develop with learners, aims and goals of learning mathematics and SME need to be present in policy documents (Ernest, 2002)

Change of Attitudes

A potential effect of SME would be a change in school environments. They could become places that helped to enact action and awareness and we would see a change in beliefs around what mathematics is and who might be able to do it. Students would be able to assert their intentions, they would understand how mathematics supports a productive powerful elite and removes power from black and brown children (de Oliveira Souza et al., 2020; Jett et al., 2015; Karaali & Khadjavi, 2019; Martin, 2009b; Wilding-martin, 2011)
Tension and Resistance

An unintended effect of SME might be the resistance and tension. We see that resistance and tension have the potential to occur as a result of changes to curriculum or the ways students experience mathematics in schools (Brantlinger, 2013). Gutstein (2006) argues that the tension can be negotiated through conversation or discussion and that the benefits of pedagogical strategies such as RCM outweigh the tensions that occur.

Discussion and Implications for SME

In summary, my concept analysis of the set of literature provided a new conceptualization of SME as a multi-layered concept. I added both complexity to the concept of SME, but also clarity in how we can think about SME more clearly as it can be thought of as different components and different dimensions. In the following I discuss these findings and outline implications for addressing RQ2.

This dissertation study was framed with the introductory problem that there is a need for increased clarity when we talk about empowerment in mathematics education, and specifically the need for an understanding of social empowerment within mathematics education. As a result of increased attention to this underexplored outcome of mathematics education, I took up a conceptualization of SME. Embedded within my dissertation framing, I used Ernest’s (2002) original dimensions of SME to as the guiding analytical framework. My goal was to move from the abstract to the more concrete in relation to each of the eight dimensions that became part of my findings, to bring light to the antecedents and effects of the development of SME.

There is a metaphor that graduate students often hear as they prepare to conduct research. It’s the idea of inviting people to the dinner party and having a conversation
with the people at your dinner party to figure out how you can add and build on their work. There are certain conceptual and theoretical frames that we bring to the table to try to understand the phenomena under study. My approach to answering RQ1 was slightly different. My dinner party was set in a restaurant, and I visited tables briefly to better understand the multitude of conversations being had at the tables.

The purpose of this part of my dissertation study was to help the field get a sense of what was happening in the social realm of mathematics empowerment work. As Confrey (2017) writes, there are different reasons why one might engage in research in mathematics education—to inform, to deform, or to reform the field in some way. My study aimed to inform. Although it was not exhaustive, my findings helped to point to how those visions of empowerment were being taken up by researchers, studied, and where more work needs to be done.

To begin with, my results show that SME is multilayered concept with specific eight dimensions situated in its antecedents/consequences, process/outcome, and action/awareness components.

The analysis of the set of literature showed that AC2, AW3, AW4, AW5, and ADD8 are noticeably absent in research related to how scholars are conceptualizing them as related to SME dimensions. AC2 although theoretically studied, needs to be linked more deeply to how critical understanding can be seen and understood. This points to the need for further research in these areas.

In other words, if action is part of SME, how can we recognize and measure action when we see it? If action is part of empowerment, how do we develop it and include it in our policy documents? ADD8 the additional new dimension of SME is also
quite under-represented, e.g., only one study addresses it (Sumpter, 2015a) and CCSS does not recognize this SME dimension in its frameworks (a result from Chapter 5). Furthermore and as, I will show and discuss in Chapter 5, more work needs to be done to see effects of SME, particularly given the mismatch between the antecedents of policy and professional recommendations and theoretical and instructional approaches.

Even more questions emerge, if there are eight dimensions of SME, do they all have equal weight and importance and do some dimensions need to be undertaken before others in order to develop SME? It would make sense to me that awareness components would need to be developed before action, but that is a future research question.

**The case of AC1: Should we rethink it as a dimension of SME?**

My findings show that AC1 is link to ideas of economic empowerment, gender equality, and increased usefulness as related to participation in the current society (Foley, 2016; Lucey & Tanase, 2012; Murphy, 2013; Uribe-Flórez et al., 2014; Wilding-Martin, 2011), but after looking at this dimension in light of this new SME framing makes me wonder if AC1 should still be a dimension based on my theoretical commitments of SME as social, having to do with other people. If you recall, I began this dissertation with the framing that one of the reasons I pursued mathematics education was specifically to better my own chances for economic and career success. As I grappled with the findings of the other SME dimensions, I noticed that AC1 was unlike the others because it focused on the individual and not the collective. It feels to me that if AC1 is developed, then something breaks down in the social realm. The idea of “bettering one’s life chances in life and study” appears to be a worthy individual and necessary goal for how to empower students through mathematics; through awareness, perceived ability, and actual ability to
act on the world. But if we argue for mathematics as a way to give a better economic chance for students, implicit in the argument is that there are still some students who will not have the same chance. In other words, we are arguing that students should study mathematics because it will give them access to what is already a limited set of opportunities. However, during the writing of those findings I walked away with a sense that this rationale, one of economic success, should not be one of the goals of a mathematics education. Instead, drawing from work by David Labaree (2009) and Catherine Yeh (2018) I argue that bettering one’s chances in life and study is a goal that goes against the very nature of achieving SME. Mathematics can help some, but not all, and if it could help all then it wouldn’t be about chances.

Yeh argues,

Given its privileged position, mathematics education plays a key role in generating a discourse that considers students as human capital. Students are seen as future producers and consumers of the economy in which mathematics expertise is regarded as essential for economic growth and for the nation to compete effectively at a global level.

It is because

…education accomplishes what we want rather than what we say. We ask it to promote social equality, but we want it to do so in a way that does not threaten individual liberty or private interests. We ask it to promote individual opportunity, but we want it to do so in a way that does not threaten the integrity of the nation or the efficiency of the economy.
Scholars in science education have made similar arguments, more firmly rooted in discourse analysis and economic framing of … questioning the economized goals of science education (Bencze & Carter, 2011; Hoeg & Bencze, 2017). Applying this to mathematics education, I argue that we must move away from the goal that a socially empowering mathematics education can at the very least benefit the individual, that an individual student’s economic life and opportunity is a worthwhile goal in and of itself.

In Support of Affect, ADD8

Although ADD8, the affective dimension, emerged from only one research study, it stands to be included because of its prevalence across country standards and its recent emergence in newer documents (National Council of Teachers of Mathematics, 2018), which indicate the raising importance to paying attention to the affective facets of teaching and learning mathematics and thus, going beyond cognitive aspects and addressing the whole person. The initial code given to this article was “recreational mathematics” and was subsumed under the code alternative conceptions of SME. After revisiting the codes I realized that this code didn’t fit well. It addressed something new, affective aspects of learning and teaching that were not part of the other dimensions and therefore, it seemed important to stress this by creating its own dimensions. Social empowerment is not just a “cognitive” pathway; developing SME includes affect.

The legitimacy of this additional dimension of SME is further supported by research on goals of mathematics education, in which authors claim that affect is not just an individual phoenomenon but a social one (Bakker et al., 2021; Debellis & Goldin, 2006; Middleton et al., 2017). Further exploration is needed to address how to integrate
this dimension into teaching and learning of mathematics leading to SME (Bakker et al., 2021).

**In Summary**

As a researcher in mathematics education, I push the field to think deeply about these SME dimensions, and particularly question the goal of social mathematical empowerment as a gate-keeper to opportunity (Rubel, 2017; Stinson, 2004) and a means to stratify learners as human capital (Freire, 2018; Labaree, 2009) A more worthwhile AC1 dimension might be: Bettering quality of life for all. This pursuit of further research makes me excited, to explore potential new dimensions with learners in the classroom and in theory.

I have pointed to new directions in research related to the conceptualization of SME, theoretically and instructionally, and I use the next chapter to do some of this work through RQ2) How do teaching materials address/align with dimensions of SME?
CHAPTER 5

RQ 2 FINDINGS: TEACHING MATERIALS ALIGNMENT TO SME

This chapter presents the findings for RQ2) How do mathematics teaching materials address/align with SME concepts? This research question was answered using qualitative content analysis and applying the conceptualization of SME as developed through RQ1. This content analysis aimed to describe how teaching materials within a textbook titled “High School Mathematics Lessons to Explore, Understand, and Respond to Social Injustice” [HSML], and that had as one of their intended goals “empowerment of students”, align with the multiple layers of SME, in comparison to a more traditional mathematics textbook titled “Thinking Mathematically” [TM]. In each book I analyzed materials whose mathematical learning goals fall within the four essential concepts of number, algebra and functions, statistics and probability, and geometry and measurement (National Council of Teachers of Mathematics, 2018). In total I analyzed 52 lessons within those materials, along with supplementary and each books’ introductory chapter materials. I also looked for evidence of SME within the mathematics curriculum frameworks from Florida and Massachusetts, along with the Common Core State Standards and most recent Catalyzing Chance recommendations. I expanded the scope of analysis from just the lessons because my findings from RQ1 necessitated that I also look at the “antecedents” of SME, which included standards and frameworks.

This chapter is divided into three sections. The first section focuses on alignment to antecedents, and the second on alignment to dimensions of SME, and the third, looks at how those dimensions are developed through theoretical and instructional approaches of SME in the materials. At the end of the chapter, I discuss these findings. Overall, my
findings point to the process, components, and dimensions of SME in various ways throughout lessons and the accompanying supplementary materials. The sets of lessons from HSML and TM do not address all eight dimensions of SME. The aims and goals of each set of materials also lacked attention to SME, except for recent recommendations from NCTM. As a reminder, Table 5 below lists the eight dimensions and their alignment to the either awareness (AW) or action (AC).

**Table 5. SME Dimensions and Alignment to the Components of Awareness (AW) and Action (AC)**

<table>
<thead>
<tr>
<th>Components</th>
<th>SME Dimensions</th>
</tr>
</thead>
<tbody>
<tr>
<td>AW1</td>
<td>Be aware of how and the extent to which math permeates everyday life.</td>
</tr>
<tr>
<td>AW2</td>
<td>Critically understand the use of mathematics in society</td>
</tr>
<tr>
<td>AW3</td>
<td>Have a sense of mathematics as a central element of art, culture, life, and an understanding of the historical and technological aspects of which mathematics permeates</td>
</tr>
<tr>
<td>AW4</td>
<td>Be aware of the historical developments of mathematical symbols and abstractions, their theories, concepts, and problems</td>
</tr>
<tr>
<td>AW5</td>
<td>Understand that there are multiple views of the nature of mathematics and controversy over the philosophical foundations of its’ knowledge.</td>
</tr>
<tr>
<td>AC1</td>
<td>Bettering one’s chances in life or study.</td>
</tr>
<tr>
<td>AC2</td>
<td>Critically understand the use of mathematics in society: to identify, interpret, evaluate, and critique the mathematics embedded in social, commercial, and political systems.</td>
</tr>
<tr>
<td>ADD8</td>
<td>Experiencing beauty, pleasure, and joy through mathematics.</td>
</tr>
</tbody>
</table>

**SME Antecedents Present in Teaching Materials**

The antecedents I identified of SME from RQ1 included training of teachers, conversations and the role of other people, and examination of aims and goals of learning materials. I also looked for inclusion of the action/awareness layer of SME. To find evidence for aims and goals I looked briefly at curriculum frameworks from Florida, Massachusetts, Catalyzing Change recommendations and the CCSS related to mathematics aims and standards. (National Council Of Teachers Of Mathematics, 2000;
National Governors Association Center for Best Practices and Council of Chief State School Officers, 2010). In the text itself this mean I looked for paragraphs or chunks of paragraph language that aligned with a particular SME dimension or antecedent. I did not fully interrogate each of these documents standard by standard, but I did look for ways in which the standards support development of SME in the overall vision and philosophies of potential. How did the standards and frameworks show their commitment to promoting SME, not just mathematics concepts? Below I present a table with the overall themes found that shows which antecedent theme it addressed, the source of evidence, if it was addressed in any way, and how. For example, training of teachers is an important antecedent to the development of SME with learners, so how do the frameworks address this antecedent?

Overall the three broader themes presented in Table 6 for this section are that 1) standards and frameworks primarily focus on SME dimension AC1, college and career readiness, 2) there is limited support for how teachers should develop SME dimensions, although components of SME are present in the standards and goals of what students should learn, and 3) the importance of the role of conversation between teachers and students are present through the language within the documents.
### Table 6. Antecedents of SME present in materials

<table>
<thead>
<tr>
<th>Antecedents of SME, (Broader Theme from RQ1)</th>
<th>Source of evidence</th>
<th>Overall finding for RQ2</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>Examination of Aims and Goals</td>
<td>CCSS, FL frameworks, MA frameworks, Catalyzing Change Recommendations, TM and HSML introductory materials.</td>
<td>Broader theme is on development of AC1 in standards and frameworks, except for in Catalyzing Change materials and HSML introductory materials. ADD8 and AC1 are more present in these.</td>
<td><strong>Experience Wonder, Joy, and Beauty</strong>&lt;br&gt;High school mathematics can potentially cultivate in students a sense of wonder, beauty, and joy—and doing so is an important but often neglected purpose for teaching mathematics. The ability to reflect on a phenomenon and experience awe is a distinctly human activity, and seeing the world through a mathematical lens can help students experience wonder and beauty in the world in unexpected places. A student with a deep understanding of mathematics can connect counting strategies with Pascal’s triangle or appreciate the role of Fibonacci numbers in nature (Catalyzing change, ebook, no page number)</td>
</tr>
<tr>
<td>Training of Teachers</td>
<td></td>
<td>“The standards define what all students are expected to know and be able to do, not how teachers should teach. While the standards focus on what is most essential, they do not describe all that can or should be taught. A great deal is left to the discretion of teachers and curriculum developers.” (MA frameworks, p. 13)</td>
<td></td>
</tr>
<tr>
<td>The Role of Other People</td>
<td></td>
<td>“Students should be given opportunities to discuss math’s relevance to everyday life and their interests and potential careers with teachers, parents, business owners, and employees in a variety of fields such as computer science, architecture, construction, healthcare, engineering, retail sales, and education. From such discussions, students can learn that a computer animator uses linear algebra to determine how an object will be rotated, shifted, or altered in size. Policy analysts use statistics to monitor and predict state, national or international healthcare use, benefits, and costs…” (MA frameworks, p.9)</td>
<td></td>
</tr>
</tbody>
</table>
Briefly, the CCSS are a set of K-12 standards that lay out what “students should be able to do and understand in their study of mathematics” (CCSS, p. 3). Beyond including mathematics standards, the CCSS also lays out practice standards that are processes and proficiencies. The eight practice standards are 1) Make sense of problems and persevere in solving them 2) Reason abstractly and quantitatively 3) Construct viable arguments and critique the reasoning of others 4) Model with mathematics 5) Use appropriate tools strategically, 6) Attend to precision 7) Look and make use of structure and 8) Look for and express regularity in repeated reasoning. The mathematical practice standards embedded in CCSS are the closest tie to alignment of SME as the mathematical practice standards. But overall, the larger goal of SME is to develop AC1. The purpose of these standards is to develop college and career readiness, AC1. As stated in the introductory materials “No set of grade-specific standards can fully reflect the great variety in abilities, needs, learning rates, and achievement levels of students in any given classroom. However, the Standards do provide clear signposts along the way to the goal of college and career readiness for all students.” (p. 4) Although other dimensions of SME might also be viewed as important as developing AC2 through modeling, those dimensions are viewed as supportive of AC1.

Looking at state-specific standards showed a slightly different story, but alignment is still focused on AC1 of SME. MA standards describe a vision of college and career readiness. The standards provide at the minimum “knowledge, skills, and practices necessary to enter into and succeed in … college or career” (p. 9). Although MA frameworks discuss the frameworks are “critical to college, career, and civic readiness, they do not define the “whole” of readiness. Students require a wide-ranging, rigorous
academic preparation and attention to such matters as social, emotional, and physical
development and approaches to learning.” MA frameworks address alignment to
antecedents of SME (training of teachers, and conversation and the role of other people)
(p.14).

From such discussions, students can learn that a computer animator uses linear
algebra to determine how an object will be rotated, shifted, or altered in size.
They can investigate how public policy analysts use statistics to monitor and
predict state, national or international healthcare use, benefits, and costs (AC2).
Students who meet the standards develop persistence, conceptual understanding,
and procedural fluency; they develop the ability to reason, prove, justify, and
communicate. They build a strong foundation for applying these understandings
and skills to solve real world problems (AC2). These standards represent an
ambitious prekindergarten to grade 12 mathematics program that ensure that
students are prepared for college, careers, and civic life (AC1).

(https://www.doe.mass.edu/frameworks/math/2017-06.pdf, p. 9).

Similarly, but less broad in their focus on dimensions of SME are Florida’s
standards for mathematics. Florida standards stress their vision for development of AC1
as a mathematics education “supports student success in the workforce and prepares them
for the jobs of tomorrow” (Thinking et al., n.d., p. 118)

In contrast to the CCSS Standards for Mathematical Practice, the National
Council of Teachers of Mathematics in their more recent 2018 Key Recommendations
and Catalyzing Change Series for High School Mathematics (National Council of
Teachers of Mathematics, 2018) are more aligned with SME and include both action and
awareness components. For example, students should be able to critique the world using mathematics and challenge inequitable mathematics structures (AC2). They also attend to ADD8 of SME and recommend that students “should experience the wonder, joy, and beauty of mathematics.” See Figure 3 for the location of the codes within materials noted in green.

Figure 3. Catalyzing Change Recommendations and SME alignment
I also looked for antecedents of SME in HSLM and TM’s introductory materials. What are the aims of the materials and how do they align to SME and how do the authors talk about the teacher’s role (role of other people) to support SME? I found that both sets of materials address SME and the role of teachers in fundamentally different ways.

HSML’s stated purpose focuses on the action components (AC) of SME: “We intend for this book to support you and your students to move from questions like ‘When/where/how am I going to use this?’ to ‘What can we do about this?’” (p. 11). In contrast, TM’s purpose is primarily on the awareness aspects – to “show students how mathematics can be applied to their lives in interesting, enjoyable, and meaningful ways” (p. vii). The primary purpose of the text within the introductory materials for TM are around awareness (AW). In HSML, the teacher is framed as a co-learner, while in TM, the teacher is framed as someone who brings awareness to students.

I present some of the text from the teaching materials in Figure 4 to show as an example how the awareness versus action component.

HSML on page 12 also has awareness aims present within the text materials, but those aims are to promote action.
<table>
<thead>
<tr>
<th>SME Alignment</th>
<th>HSML</th>
<th>TM</th>
<th>SME Alignment</th>
</tr>
</thead>
<tbody>
<tr>
<td>AC</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>We intend for this book to support you and your students to move from questions like “When/How/Where am I ever going to use this?” to questions like “What can we do about this?” (Berry III et al., 2020, p. 11)</td>
<td>My primary purpose in writing the book was to show students how mathematics can be applied to their lives in interesting, enjoyable, and meaningful ways (Blitzer, 2019, p. vii)</td>
<td></td>
</tr>
<tr>
<td>AW/AC</td>
<td>When children learn that mathematics can be used as a tool to help them understand, explore, and investigate social situations, they are empowered to see themselves as active agents in a world of change. We hope that the lessons and the critical call for action contained in this book highlight how each and every student is capable of mathematical learning and can be empowered to use mathematics for change in their own and others’ lives. (p. 12)</td>
<td>1. To help students acquire knowledge of fundamental mathematics, 2. To show students how mathematics can solve authentic problems that apply to their lives, 3. To enable students to understand and reason with quantitative issues and mathematical ideas they are likely to encounter in college, career, and life, 4. To enable students to develop problem-solving skills, while fostering critical thinking, within an interesting setting. (p. vii)</td>
<td></td>
</tr>
</tbody>
</table>

Figure 4. Antecedents of SME present in Aims of Teaching Materials
SME Dimensions Present in Teaching Materials

This section presents whether and how the set of materials addressed the different dimensions of SME. The most common alignment in both set of materials was through the awareness and action dimensions (AW1 and AC1) of critically understanding the use of mathematics in society: to identify, interpret, evaluate, and critique the mathematics embedded in social, commercial, and political systems. The other dimensions were either only addressed in one set of the materials or not at all. Table 7. provides an overview of which lesson in HSML and TM address which dimension. Appendix E contains the RQ 2 codebook and more detailed examples of how these lessons were coded in regards to alignment.

Table 7. SME Dimensions and their Alignment in the Teaching Materials

<table>
<thead>
<tr>
<th>SME Dimension Addressed</th>
<th>HSML Lessons</th>
<th>TM Lessons</th>
</tr>
</thead>
<tbody>
<tr>
<td>AW1</td>
<td>Lessons 6.2, 7.1, 7.2, 7.3, 7.4, 7.5, 7.6, 7.7</td>
<td>Lessons 10.1, 10.2, 10.3</td>
</tr>
<tr>
<td>AW3</td>
<td>Lessons 7.3, 10.2, 10.3</td>
<td></td>
</tr>
<tr>
<td>AW4</td>
<td>Lessons 7.2</td>
<td></td>
</tr>
<tr>
<td>AW5</td>
<td>Lessons 5.3</td>
<td></td>
</tr>
<tr>
<td>AC1</td>
<td>Lessons 5.1, 5.4, 6.6.</td>
<td>11.1, 11.2, 11.3, 11.4, 11.7, 11.8</td>
</tr>
<tr>
<td>AC2</td>
<td>All lessons present in HSML. Each lesson ends with a section called “Taking Action”</td>
<td></td>
</tr>
<tr>
<td>ADD8</td>
<td>Not addressed in</td>
<td>10.4, 10.5</td>
</tr>
</tbody>
</table>
To identify the aim of a lesson and to align it with one of the eight dimensions of SME, I looked primarily in the lesson introduction and in the objectives section. The following two examples, the first from TM and the second from HSML, illustrate this process.

Lesson 7.3: Systems of Linear Equations in Two Variables, is part of the TM teaching materials. This lesson aligned with the standard “Algebra and Functions Essential Concepts.” Student objectives were listed as 1) determine whether ordered pairs were solutions to systems, solving linear equations by substitution, graphing, addition, and identifying systems that do not or have one solution. The introduction to the lesson uses an example that may resonate with the students – the relationship between college students identified as procrastinators and non-procrastinators and their physical illness experienced throughout the semester. Using a graphic representation of this problem the lesson continued not with an investigation into this problem but with a symbolic and procedural focused introduction to solving systems of linear equations (how do I work with the symbolic representations of the problem?). The lesson started with examples and step by step instructions of how to solve linear equations and with worked out solutions. After a few example solutions, students were then asked to practice - do one themselves. An application of linear solutions followed examples with no context. Students were shown how to find break-even points with a book-led or potentially teacher-led example of break even and profit points (AW1). See Appendix F for the full lesson.
The introductory material in Figure 6 of lesson “5.4: Estimated Wealth Distribution in the United States and in the World” from HSML. Stated as an overall objective for students to explore and define the meaning of wealth distribution by an examination of how wealth is divided in the US (AW2). More specific objectives related to understanding of power structures and general inequities that align with AW2, AC1 and AC2. During the lesson students look at graphs, distribution models, and research data on their own related to the distribution of wealth (AC2). At the end of the lesson, they are asked to engage in discussion around the use of mathematics in their life and in the world (AC2), and to identify, perceive, and discuss wealth inequality in their
everyday lives and experiences (AC1). See Figure 6 for introductory materials and Appendix G for the full lesson.

![Figure 6. Snippet from Lesson 5.4 from HSML](image)

**How SME was addressed**

In terms of how SME was addressed, teacher directed and text directed approaches were aligned with awareness and student focused approaches were aligned with action dimensions.

All the lessons in the book TM followed a similar structure with a teacher directed instructional approach: a hook to engage students with the materials such as the application of mathematics to either an everyday life experiences historical scenario, or a
potential critical application. The TM lesson’s instructional approach can be described through three phrases: example, repeat, and explore. In education, this pattern is called Initiation Respond Evaluate. The teachers show the students some content or material, the students then respond, then the teacher evaluates and students’ practice. The first exercises present in the materials were concept checks, then practice problem solving, followed by critical thinking exercises, and sometimes finished with two to three group problem solving at the end of a chapter.

The materials in HSML had instructional strategies that followed a student directed and typical launch explore summarize pattern. Although there were lessons where the three-act task model was used, this structure is similar to the launch explore summarize model. They both serve to engage students through problem solving, dialogue and discussion, and reflection and action. In this type of instructional approach, the launch helps to encourage students to delve into a task. After the launch, it is the students role to engage in the problem solving. The summarize uses conversation to promote, reflect, and plan for action related to what the learners were exploring.

HSML also provided resources for teachers to help them integrate this instructional approach into their classrooms. These approaches also aligned with the development of SME as we saw in the findings for RQ1. Although I hadn’t fully access to the annotated instructor’s edition of TM for my analysis, from what I was able to glean, the instructor’s material did not present substantially different material than what was in the student textbook. The teacher materials included lecture slides, example problems, sets of practice problems, and answers to the exercises (see page x).
In sum, while the HSML materials aligned with the antecedents and effects of SME, the TM materials did not align with these. Teachers following the TM materials as presented directed the instruction, initiated and led the conversation, and neither teachers nor students were prepared to deal with tension and resistance as an effect of potential controversial or emotional topics (e.g. college students identified as procrastinators and non-procrastinators and their physical illness experienced throughout the semester) (TM p. xi). In contrast, HSML (p. 54) provided strategies for managing discourse (e.g., four corners) and potential tension in the classroom (e.g., town hall circle, journaling), in particular for helping students as they work through potentially contentious or emotional topics as seen in Figure 7.
Answering RQ2 provided insight into how SME antecedents and effects, the components of awareness and action, and its eight dimensions were present in traditional and non-traditional teaching materials. In terms of the aims and goals in materials related to SME, it appeared that most of the materials had alignment to the dimensions. Most of the materials aligned with some dimensions of SME, but materials in TM were missing the addressment of components of conversation and the role of other people. Where was the collaboration?

Figure 7. Strategies for Managing Effects of SME

**Discussion and Implications for SME Teaching Materials**

Answering RQ2 provided insight into how SME antecedents and effects, the components of awareness and action, and its eight dimensions were present in traditional and non-traditional teaching materials. In terms of the aims and goals in materials related to SME, it appeared that most of the materials had alignment to the dimensions. Most of the materials aligned with some dimensions of SME, but materials in TM were missing the addressment of components of conversation and the role of other people. Where was the collaboration?
Although we cannot study the effects directly, the materials in HSML provided resources for the teachers to address the effects of SME (tension and resistance and change of attitudes through discussion and particular instructional strategies in classrooms).

I was mostly surprised that four dimensions of SME were also minimally addressed or missing in both of the books (AW3, AW4, AW5, and ADD8). Some studies have shown that specific instructional approaches increase the critical understanding of the use of mathematics in society or how to create actional change (Frankenstein, 1990; Jett et al., 2015; Karaali & Khadjavi, 2019; Leonard, 2010; Tutak et al., 2011; Wright, 2017a). Some of these theoretical approaches include TMSJ, CM, RME, and Ethnomathematics, focusing on seeing the world through a critical eye. Given that HSML focuses on Teaching Mathematics for Social Justice, enriching its materials with other theoretical approaches might potentially address the missing dimensions of SME such as AW3, AW4, AW5 and ADD8. In other words, TMSJ is tied to AC2, approaches need to be developed to build links between those and AW3, AW4, AW5, and ADD8.

Take ADD8 as an example, an addition present in Catalyzing Change Recommendations-part of the intended standards—how does that experiencing of joy and pleasure get translated into teaching materials and then studied as enacted. What does that look like? The work of Amy Noelle Parks might be a good starting point—and she argued that even researching the work of joy needs more work—(Parks, 2020). None of the lessons I looked at examine mathematics through any of the recreational modes offered by Sumpter (2015b) at least in the way that I saw it. Do materials need to incorporate more room for games or recreational mathematics as tied to helping develop
not only ADD8 but also AW5 the understanding that there are different ways to view mathematics?

What does this mean for SME antecedents and effects? The antecedents of SME are training of teachers, dialogue, and examination of the aims. For empowerment to continue as a stated goal of mathematics education, national and state standards, and policy documents put forth by NCTM, AMTE, TODOS, etc. will need to be examined to identify which SME dimensions are included and which are excluded. Is there for example, an emphasis on awareness but without any examples or opportunities for action, do instructional approaches provide teachers with support for challenging discussions or is there a misalignment between the call for addressing certain dimensions but instructional strategies focus on traditional teacher-directed approaches that are less likely to achieve empowerment? Is there sufficient systematic research exploring the effects of certain instructional approaches to achieve SME in its multilayered structure?

Not to be taken for granted is the antecedent of conversation and the role of other people. Having and opening up dialogue with teachers and the broader community (parents, teachers, neighbors, really anyone!) is difficult but necessary. Many people still hold the belief that mathematics has nothing to do with the social world. But it does, it has everything to do with the social world. It is by its very nature social, it was “created” as a cultural enterprise. There are no “right answers”! We see this taking place in Florida currently, though this is not an isolated case. Florida’s department of education, through the words of their governor, has taken the stance that mathematical content is factual, there is nothing to debate about: “Two plus two equals four,” DeSantis said at an event in Ocala. “It’s not two plus two and let’s have a struggle session over that.”
Many people, regardless of their education level, might agree with DeSantis view. But there is another argument to be made. Mathematics is a human endeavor and is imbued with social realities - an argument that I have quite often with my family. Push DeSantis a little further and we can find that in fact, in modular arithmetic, 2+2 does not always have to equal 4. In a set that contains 0,1,2, we find that 2+2=1. It just so happens that we teach our students in one of many number systems. Usually, students are only limited to the positive real numbers up until later elementary schools (National Governors Association Center for Best Practices and Council of Chief State School Officers, 2010). There are also other counter examples that use temperature, velocity, and time and where addition follows different rules. There are no right answers.
I began my dissertation with applying a personal perspective of mathematics as useful for economic earning and job career potential as one of the reasons to study mathematics and viewed SME as an approach of empowering learners, of opening the gate to economic and higher education opportunities (Boaler & Greeno, 2000) for particularly underrepresented minorities. Since such a goal was not explicitly articulated in policies and curricula, a more explicit conceptualization of SME was necessary to achieve empowerment. My RQ1 addresses this. Through the systematic analysis of literature, I walk away with a different framing of SME and its connection to teaching and learning mathematics outcomes, and which gives the field of mathematics education a better understanding of empowerment in mathematics related to the social domain. SME is as multilayered concept containing different components (awareness and activity) that address different dimensions and which can be conveyed through theoretical and practical approaches. Furthermore, various antecedents are central in developing SME such as teacher training, goals of materials, or policy aims. There are also potential effects that are likely to be the fall out of action and awareness of learners as they engage in the work of SME.

Developing SME through teaching has potential with the teaching materials that explicitly take up and address the different domains of SME. My RQ2 looked at teaching materials claiming to work towards empowerment and an example of a common mathematics high school textbook. I found that both teaching materials as well as policy aims (antecedents) did align with some domains of SME, but some were missing such as
joy or understanding of multiple perspectives of SME. I also found that the development of SME is likely to produce some tension (effects) for learners and teachers. How to navigate that tension is an area of future research.

As a result of my research and in contrast to my earlier argumentation, I argue that framing the importance of empowerment through an economic lens is counter productive if mathematics educators strive towards humanizing mathematics and social mathematics (Gutiérrez, 2017; National Council of Teachers of Mathematics, 2018). Policy documents more closely resemble a commitment to the SME domains (arguments are being made that we should attend to more than access and opportunity) (Gutiérrez, 2002), but studies in mathematics education still use research approaches that do not tie to the intended social outcomes of mathematics education (Martin, 2009b).

Based on my findings for RQ1 I argue that social empowerment still needs further conceptualization in some domains: historical awareness and the understanding that there are multiple views and controversy over the philosophical foundations of knowledge. Based on my findings for RQ2 I wonder how we can include missing dimensions in teaching materials? I push the field of mathematics education to consider moving beyond mathematics for bettering one’s chances in life or study through job opportunities, as currently emphasized in Florida and instead broaden the goals for the study of mathematics to include all eight dimensions. This means working through differing perspectives and beliefs that people hold related to what mathematics is and how mathematics is tied to power. Teachers are also learners, embedded within the larger conceptual system I explored much earlier in this dissertation. Teachers are people too and have beliefs that may align with DeSantis view. Future studies should investigate
how teachers have come to understand the power of mathematics through the social---
which aspects they have not considered? How was this awareness built? What theoretical
approaches and instructional approaches do we still have left to consider regarding SME?
We also need to have more studies to better understand what happens after students
experience SME aligned materials in schools, and how SME is developed. One of the few
eamples is Boaler and Selling’s (2017b) (cite) longitudinal examination of what
happens when students experience mathematics in fundamentally different ways in
schools. They found that adults who had experienced mathematics through problem
solving had higher mathematics achievement and increased positive math identity. Those
were not the focus of this study, although they are related to mathematical empowerment
and epistemological empowerment.

Common threads resonate throughout the conceptualization of SME: dialogue is
critical, training for teachers is essential, and examination of the goals of instruction is
necessary. Engaging teachers and the larger community in discussion around their ideas
of the power of mathematics is a way to start the conversations of how to address SME
in learning goals, in the development of teaching materials as well as teacher training.
The materials I analyzed in this study are a good exemplary teaching materials for
teachers interested in socially empowering their students. Critical, social justice, and
ethnomathematics approaches (Kokka, 2015; Tutak et al., 2011) promote some of the
development of SME, namely the critical understanding, but lack the connection between
the AW3 domain of joy and beauty in doing mathematics as an activity.

There were limitations to this study. One is that I had a particular framing to
answer my research questions. I chose one set of specific materials that had potential to
engage students in SME, along with another set of materials in a high school mathematics textbook that was banned based on its inclusion of social and critical issues. Thus, the results of the analysis of the two sets of materials are not generalizable to other books or frameworks. However, they provide an analytical template for how to continue this work.

One can argue that the set of literature selected for answering my RQ1 was limited as a result of the chosen keywords for the literature search. However, these were determined by my framework. With a change in emphasis, other keywords such as mathematical agency, critical consciousness, joy, recreation might be fruitful in making more links. These concepts emerged as I wondered how SME was being talked about at the dinner tables.

Future studies could conceptualize potentially related concepts to SME such as “critical reflection” and “critical consciousness” (Gutstein, 2003) and views of mathematics (Lawler, 2012). This result of SME can be explored further with teachers and learners as well as operationalized and “measured.” (E. Gutstein, 2003; B. Lawler, 2012). More research could also examine the trajectory of SME domains and if there are orders in how they develop. My study shows that there are so many different ways researchers are talking about SME but we need to be more explicit, what constructs are tying it to, how are we providing evidence for it, how are we thinking of developing it? Future research can also investigate how SME is linked to the other domains of mathematical power. The results of this study call to researchers to be explicit when talking about power; how do they conceptualize it, and which constructs do they tie it to?

From a practical perspective, future research could assess how teachers may use the analytical tool I used to examine lessons for SME domains. Questions to asks could
be: Does it help them to evaluate whether the lessons actually address SME domains, are they some missing, are the instructional strategies appropriate for their students, etc.?

Finally, future research should shift to the learner. Do they develop the goals of the different dimensions of SME? How do they articulate awareness of SME? Do their perceptions of SME align with the goals of the teaching materials? And from a developmental perspective it would be interesting to ask whether awareness is needed before action can begin, and what happens later in life beyond “mathematical achievement” or job success, or how will learners push back against the systems? (Boaler & Selling, 2017a).

Overall, it may be unlikely that these eight domains, through awareness and action, will come to fruition during my lifetime. Partly because SME is both process and outcome, the work is never done. But, as we see in this current political, pandemic, and overall global climate, people can make a significant change in very little time—so I hold out hope!
## APPENDIX A

### STAGE 1 SCREENING BOOK FOR RQ1

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Research on perceptions of school environment:</strong> A recent study reveals that teacher effectiveness is closely related to the effectiveness of school integration and overall student satisfaction. This study first reports the validation of both the Principal’s Instruction Questionnaire (PIQ) and the Computer Attitude Scale (CAS) questionnaires in schools. The second objective was to investigate associations between teachers’ perceptions of the principal’s instructional behavior and their attitudes towards the use of information and communication technology in teaching environment. The third objective was to investigate associations between their role in the principal’s instructional behavior, their attitudes towards the use of information and communication technology in teaching environment. The study involved 46 teachers in seven government and government aided schools in Singapore. The PIQ and CAS were also shown to be reliable and valid. Qualitative and quantitative research methods were used to present study.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Climate change is one of the most urgent human concerns. Mathematics, in various forms of complexity, is used to communicate climate change to scientists, students, and the public. This study examines the role of climate change in mathematics education. We interviewed 20 mathematics teachers who were involved in teaching mathematics in Singapore. The teachers were asked to describe their understanding of climate change and its impact on their teaching. They were also asked about their strategies for addressing climate change in their classrooms. The study revealed that although the concept of climate change in mathematics was introduced as an environmental issue, the teachers varied in their understanding of the concept.</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Focused on teachers**

**Res`**

**N**

**Res**

**N**

**Res**
## APPENDIX B

### STAGE 4 SET SCREENING BOOK FOR RQ1

<table>
<thead>
<tr>
<th></th>
<th>Alternative Constructions of Social Mathematical Empowerment</th>
<th>(Martin, 2009b; Murphy, 2013; Wilding-Martin, 2011; Wilding-Martin, 2009)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>Be aware of how and to what extent mathematics permeates every day life</td>
<td>(Sumpter, 2015b)</td>
</tr>
<tr>
<td>0</td>
<td>Be aware of the historical developments of mathematical symbols and abstractions, their theories, concepts and problems</td>
<td>None</td>
</tr>
<tr>
<td>7</td>
<td>Better one’s chances in life or study</td>
<td>(de Oliveira Souza et al., 2020; Foley, 2016; Johnson, 2018; Lahana, 2016; Mijtapa &amp; Reiss, 2015; Paudel, 2020; Scott-Parker &amp; Barone-Nugent, 2019)</td>
</tr>
<tr>
<td>18</td>
<td>Critically understand the use of mathematics in society: to identify, interpret, evaluate and critique the mathematics embedded in social, commercial, and political systems.</td>
<td>(Hartman &amp; Kahn, 2017; Jett et al., 2015; Jong &amp; Jackson, 2016; Karaali &amp; Khadjavi, 2019; Killpack &amp; Melón, 2016; Kurtuma &amp; Romano, 2015a; B. R. Lawler et al., 2017; Leonard, 2010; Lucey &amp; Tanase, 2012; Madusise &amp; Mwakapenda, 2014; Mendez et al., 2020; Mhlobo &amp; Schäfer, 2012; Russell, 2013; Shocker et al., 2016; Turner, 2003; Tutak et al., 2011; Wright, 2017a)</td>
</tr>
<tr>
<td>1</td>
<td>Have a sense of mathematics as a central element of art, culture, life, and an understanding of the historical and technological aspects of which mathematics permeates</td>
<td>(Unfried &amp; Canner, 2019)</td>
</tr>
<tr>
<td>0</td>
<td>Understand that there are multiple views of the nature of mathematics and controversy over the philosophical foundations of its’ knowledge.</td>
<td>None</td>
</tr>
<tr>
<td>14</td>
<td>Uses empowerment as an outcome</td>
<td>(Brantlinger, 2013b; Chehayl, 2008; Chen, 2010; Ernest, 2015; Frankenstein, 1990; Han &amp; Leonard, 2017; M. L. Hassi &amp; Laursen, 2015; Hlalele &amp; Tsotetsi, 2016; Maoto, 2011; Murphy-Graham, 2007; Planas &amp; Civil, 2009; Rosales,</td>
</tr>
<tr>
<td></td>
<td>2010; Sutherland &amp; Ridgway, 2017; Uribe-Flórez et al., 2014)</td>
<td></td>
</tr>
<tr>
<td>---------------------------</td>
<td>--------------------------------------------------------------</td>
<td></td>
</tr>
<tr>
<td>Total=</td>
<td>44</td>
<td></td>
</tr>
</tbody>
</table>
## APPENDIX C

### STAGE 5 RQ1

<table>
<thead>
<tr>
<th>Understand that there are multiple views of the nature of mathematics and controversy over the philosophical foundations of its’ knowledge.</th>
<th>(B. R. Lawler, 2011; Martin &amp; Gholson, 2012; Stemhagen, 2004, 2007)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Critically understand the use of mathematics in society: to identify, interpret, evaluate and critique the mathematics embedded in social, commercial, and political systems.</td>
<td>(Kokka, 2015)</td>
</tr>
<tr>
<td>Be aware of the historical developments of mathematical symbols and abstractions, their theories, concepts and problems</td>
<td>(Furinghetti, 2005; Johansen, 2006; Nolan &amp; Weston, n.d.; Tzanakis et al., 2002)</td>
</tr>
<tr>
<td>Total= 9</td>
<td></td>
</tr>
</tbody>
</table>
APPENDIX D

A NON EXAMPLE

Using a non-example for the argument of why the materials I’ve chosen seems promising. My purpose was not to examine materials that ask students to satisfy the same expectations where the context is slightly more interesting, while the general goal is still procedural fluency. My purpose was to examine materials that have the potential to empower students beyond just procedural fluency and mathematical skills.

My non-example teaching material was taken from the 8th module Connecting Algebra and Geometry Lesson 8.4 from the Mathematics Vision Project, and which has been reviewed by the Massachusetts Department of Elementary and Secondary Education as meeting expectations through standards alignment and classroom application (see https://www.doe.mass.edu/instruction/curate/faq.html for more information). The mathematics standards’ focus is on linear functions and their relationships. The context of the specific lesson I reviewed is a story of Fernando and Mariah training for six weeks to run in a marathon through running laps around the track. Students have to extract information from the tabular representations and then translate those into algebraic and mathematical understandings. Using evidence from both the teacher notes and the student activities I determined that this material, although structured to engage students in mathematical practices and age-appropriate mathematical standards, does not go beyond procedural fluency and mathematical skills, which addresses components of Mathematical Empowerment but not SME. The goals for the activity do not go beyond the immediate problem, in this case, transforming an original function to a transformed function using algebraic notation. Although the lesson uses reform-based pedagogical
approaches (Launch, Explore, Summarize) it does not appear to align with (conceptualization in process) constructs of SME.

<table>
<thead>
<tr>
<th>RQ2: How do mathematics teaching materials address/align with the concepts of SME?</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Source</strong></td>
<td><a href="https://www.mathematicsvisionproject.org/uploads/1/1/6/3/11636986/m1_mod8_teh_52016f.pdf">https://www.mathematicsvisionproject.org/uploads/1/1/6/3/11636986/m1_mod8_teh_52016f.pdf</a></td>
</tr>
<tr>
<td><strong>Does the teaching material align with SME?</strong></td>
<td>No</td>
</tr>
<tr>
<td><strong>Which constructs of SME does the teaching material address?</strong></td>
<td></td>
</tr>
<tr>
<td><strong>How and why? Or how not and why not?</strong></td>
<td>This material, although structured to engage students in mathematical practices and age appropriate mathematical standards, does not go beyond procedural fluency and mathematical skills (conceptualized in this paper as ME). The reasons for this is because it does not go beyond the immediate problem, in this case, transforming an original function to a transformed function using algebraic notation.</td>
</tr>
</tbody>
</table>
8.4 Training Day

A Develop Understanding

Task

Fernando and Mariah are training for six weeks to run in a marathon. To train, they run laps around the track at Eastland High School. Since their schedules do not allow them to run together during the week, they each keep a record of the total number of laps they run throughout the week and then always train together on Saturday morning. The following are representations of how each person kept track of the total number of laps that they ran throughout the week plus the number of laps they ran on Saturday.

<table>
<thead>
<tr>
<th>Time (in minutes on Saturday)</th>
<th>0</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>Distance (in laps)</td>
<td>60</td>
<td>66</td>
<td>72</td>
<td>78</td>
<td>84</td>
<td>90</td>
</tr>
</tbody>
</table>

Mathematics Vision Project
mathematicsvisionproject.org
1. What observations can be made about the similarities and differences between the two trainers?

2. Write the equation, \( m(t) \), that models Mariah’s distance.

3. Fernando and Mariah both said they ran the same rate during the week when they were training separately. Explain in words how Fernando’s equation is similar to Mariah’s. Use the sentence frame:
   
   The rate of both runners is the same throughout the week; however,
   Fernando ________

4. In mathematics, sometimes one function can be used to build another. Write Mariah’s equation, \( m(t) \), by starting with Fernando’s equation, \( f(t) \).
   
   \[ f(t) = \]

5. Use the mathematical representations given in this task (table and graph) to model the equation you wrote for number 4. Write in words how you would explain this new function to your class.
8.4 Training Day – Teacher Notes

A Develop Understanding Task

**Purpose:** Students have had a lot of experience with linear functions and their relationships. They have also become more comfortable with function notation and features of functions. In this task, students first make observations about the rate of change and the distance traveled by the two runners. Using their background knowledge of linear functions, students start to surface ideas about vertical translations of functions and how to build one function from another.

**Core Standards Focus:**

- **F.BF.3** Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k f(x)$, $f(kx)$, and $f(x + k)$ for specific values of $k$ (both positive and negative); find the value of $k$ given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. (Note: Focus on vertical translations of graphs of linear and exponential functions.)

- **F.IF.9** Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.

- **F.BF.1** Write a function that describes a relationship between two quantities.

**Related Standards:** F.BF.1b, F.IF.1, F.IF.2, A.CED.3

**Standards for Mathematical Practice of Focus in the Task:**

- **SMP 1** – Make sense of problems and persevere in solving them

- **SMP 2** – Reason abstractly and quantitatively
SMP 4 – Model with mathematics

SMP 7 – Look and make use of structure

The Teaching Cycle:

Launch (Whole Class):
To begin this task, read the scenario as a whole group, then ask students to write their answer to the question: "What observations can be made about the similarities and differences between the two trainers?" After a couple of minutes, have students share their observations with a partner. Listen for students to discuss the meaning of the slope and the y-intercept in both situations. If needed, ask the group questions to clarify that the y-intercept is the number of laps each person runs during the week before they meet on Saturday morning and that the slope is the same in both situations. Since the purpose of the lesson is to see how one function can be built from another similar function, these are the two most important ideas to come out of the launch conversation.

Explore (Small Group):
As you monitor, listen for student reasoning about the relationship between the amount of laps run by Mariah and Fernando. Encourage students to explain their reasoning to each other using prior academic vocabulary while working through solutions to problems. If students are incorrect in their thinking, redirect their thinking by asking them to explain how their function relates to the situation.

Discuss (Whole Class):
During the monitoring phase, select students to share their results to strengthen the whole group understanding of the relationship between the 'original function' \((m(t))\) and the 'transformed function' \((f(t))\). You may wish to start the whole group discussion by choosing someone who has graphically shown Fernando and Mariah's graph on the same axes. Have this student share the relationship between the two graphs and press to bring out that at any given time, \(f(t)\) is always '20
laps more than \( m(t) \). Likewise, have a student share who can explain the relationship using a table.

After both representations (table and graph) are shown, ask the whole group to see what connections they can make between the equation, table, and graph.

**Aligned Ready, Set, Go: Connecting 8.4**
**APPENDIX E**

**RQ2 CODEBOOK**

<table>
<thead>
<tr>
<th>Broader Theme</th>
<th>Code</th>
<th>Definition</th>
<th>Example from Data</th>
<th>Explanation of Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>AW</td>
<td>AW1</td>
<td>using mathematics to understand inequities between different social groups and to understand explicit discrimination based on race, class, gender, language, and other differences: answers the question: what decisions does mathematics conceal, how can it help me evaluate mathematical claims?</td>
<td>Lesson 5.4 from HSML</td>
<td>using mathematics to understand inequities between different social groups and to understand explicit inequities based on race, class, gender, language, and other differences. Can also be seen in particular health contexts.</td>
</tr>
</tbody>
</table>

Though worldly assets can never be shared equally among all people, the drastic disparity in who holds a nation’s wealth can often be cause for concern. This multiday lesson helps students explore the estimated wealth distribution, first in the United States and then in other nations, ultimately comparing the United States and other countries. They first explore and define the meaning of wealth distribution by dividing the population of the United States into five equal portions and examining visually how wealth is divided. They then examine the wealth distribution of ten countries, including the United States and Sweden, and compare them in terms of equity and fairness. The activities provide a framework to analyze social injustice regarding how wealth distribution impacts society.
<table>
<thead>
<tr>
<th>AW</th>
<th>AW2</th>
<th>Lesson 7.3 from TM</th>
</tr>
</thead>
</table>
|    |     | Technology is now promising to bring light, fast, and beautiful wheelchairs to millions of disabled people. A company is planning to manufacture these radically different wheelchairs. Fixed cost will be $500,000 and it will cost $400 to produce each wheelchair. Each wheelchair will be sold for $600.  
| a. | Write the cost function, C, of producing x wheelchairs.  
| b. | Write the revenue function, R, from the sale of x wheelchairs.  
| c. | Determine the break-even point. Describe what this means. |

<table>
<thead>
<tr>
<th>AW</th>
<th>AW4</th>
<th>Lesson 10.2 from Thinking Mathematically</th>
</tr>
</thead>
</table>
|    |     | IN CHAPTER 1, WE DEFINED deductive reasoning as the process of proving a specific conclusion from one or more general statements. A conclusion that is proved to be true through deductive reasoning is called a theorem. The Greek mathematician Euclid, who lived more than 2000 years ago, used deductive reasoning. In his 13-volume book, Elements, Euclid proved over 465 theorems about geometric figures. Euclid’s work established deductive reasoning as a fundamental tool of mathematics. Here’s looking at Euclid! A triangle is a geometric figure that has three sides, all of which lie on a flat surface or plane. If you start at any point along the triangle and trace along the entire figure exactly once, you will end at the same point at which you started. Because the beginning point and ending point are the same, the triangle is called a closed geometric figure. Euclid used parallel lines to prove one of the most important properties of triangles: The sum of the measures of the three angles of any triangle is 180°. Here is how he did it. He began with the following general statement:  
<p>|    |     | This lesson draws awareness to historical aspects of mathematics by discussing how Euclid’s approach was integral in the geometry students typically learn in high school math. |</p>
<table>
<thead>
<tr>
<th>AW</th>
<th>AW5</th>
<th>What has it been useful for?</th>
<th>Lesson 5.3 from HSML</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Understanding that there are multiple views of the nature of mathematics and controversy over the philosophical foundations of its’ knowledge. It requires pushing past a view of mathematics that learners might hold or that we as learners might hold. Answers the question: are there other ways that we can view these representations of mathematics?</td>
<td>The lesson is a launch–explore–summarize instructional model and is intended to take approximately 180 minutes to complete across three class periods. Lesson 1: Students engage in sifting through lots of data in order to construct a matrix. They think about which data points suit themselves to a matrix organization and which data points do not. They also consider whether there are multiple ways to construct a matrix from a given data set.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>This piece of data points to pushing past the ideas the underlying assumptions we have regarding certain mathematical representations? This example asks is this the only way we can do this? What are other ways we can do this?</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>AW</th>
<th>AW3</th>
<th>What has it been useful for?</th>
<th>Lesson 10.3 from TM</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Have a sense of mathematics as a central element of art, culture, life, and an understanding of Bees use honeycombs to store honey and house larvae. They construct honey storage cells from wax. Each cell has the shape of a regular hexagon.</td>
<td>This lesson draws attention to the particular artistic patterns that are present in the world.</td>
</tr>
</tbody>
</table>

121
<table>
<thead>
<tr>
<th>AC</th>
<th>AC1</th>
</tr>
</thead>
<tbody>
<tr>
<td>the historical and technological aspects of which mathematics permeates. Answers the question: How can I see mathematics around me?</td>
<td>The cells fit together perfectly, preventing dirt or predators from entering. Squares or equilateral triangles would fit equally well, but regular hexagons provide the largest storage space for the amount of wax used.</td>
</tr>
</tbody>
</table>
| The focus on economical empowering through choice of college majors and careers, decreasing the gender gap in regards to the amount of women who pursue STEM majors and careers is promoted through approaches like including role models in curriculum, or by including relevant interesting topics into the curriculum for Lesson 5.1 in HSML | This lesson is intended to initiate and/or transform a class to begin including the exploration of social injustice and action toward social justice as a centerpiece of the curriculum. It is meant to help students mathematize the notion of social justice and see mathematics as a tool for change. Therefore, the action steps are meant to carry this mindset into the rest of the year’s learning. To set up this environment in the course after having gone through this lesson, use one or more of these action steps:  
• Have students create rules for discourse and understanding others’ perspectives. Use this exercise as an opportunity to have students come up with rules that will provide a safe space for sharing ideas and their own stories. Refer back to Chapter 3 for deeper insight.  
• Have students create objectives that align with social justice standards from This lesson aligns with better chances because it allows times for students to list and describe injustices in and around their experiences, it exposes students to the idea of mathematics as a tool for change. |
<table>
<thead>
<tr>
<th>AC</th>
<th>AC2</th>
<th>ADD8 (Joy)</th>
<th>J</th>
</tr>
</thead>
<tbody>
<tr>
<td>underrepresented learners: Answers the question: how will mathematics help me now or in the future?</td>
<td>Teaching Tolerance (2016). <strong>Begin mission statements for the class that seek to empower students to be agents of change and articulate additional goals for their mathematical classroom.</strong>  • Begin addressing social injustices that are occurring in their immediate environments. Allow time for students to list and describe injustices around them. Use Chapters 9 and 10 to help you include some of these injustices in your future student explorations.</td>
<td>Lesson 5.2 in HSML  Black and Latinx students are less likely to have access to course pathways leading to Advanced Placement mathematics courses. In this lesson, students use schools’ local course enrollment data to explore the racial distribution in course pathways that lead to Advanced Placement Calculus or Statistics to determine if any disparities exist in their school, district, or state.</td>
<td>Racial disparities are one way to understand justice disparities in society and to critique the world around us  I would expect to see something related to engaging students in mathematical games,</td>
</tr>
<tr>
<td>to be a part of remediing unjust situations, a way to transform their lives and the lives of others and the ability to understand mathematics’ role in society and how to use it for action: answers the question: how can I use mathematics to evaluate or critique the world around me?</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Not found in dataset</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
see the parallel of doing mathematics for the sake of pleasure is one way to understand mathematics as a social endeavor, but also a way to equalize the playing field so to speak. Answers the question: how is mathematics enjoyable, recreational or fun? or talking about how mathematics can be used for fun or enjoyment. No examples I could see were present in the data. I could anticipate examples such as using video games to explore algebraic functions, or discussing probabilities through playing chess.

<table>
<thead>
<tr>
<th>Instructional strategies</th>
<th>IS</th>
<th>Answers the question: what are students and teachers doing?</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Potential strategies:</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Initiate Respond Evaluate Launch Explore Summarize Three act task model Collaborative work Students work in groups</td>
</tr>
</tbody>
</table>

These examples show the strategies present in materials. The first is a collection of group exercises that comes at the end of the lesson. The second, present the lesson as a launch explore summarize lesson. The third presents an instructional strategy of presenting materials without

**Group Exercises**

50. The method currently used to apportion the U.S. House of Representatives is known as the Huntington-Hill method, and more commonly as the method of equal proportions. Research and present a group report on this method. Include the history of how the method came into use and describe how the method works.

51. Research and present a group report on a brief history of apportionment in the United States.
<table>
<thead>
<tr>
<th>Student discuss Presenting material but no student engagement</th>
<th>ABOUT THE LESSON</th>
</tr>
</thead>
<tbody>
<tr>
<td>“Bees use honeycombs to store honey and house larvae. They construct honey storage cells from wax. Each cell has the shape of a regular hexagon. The cells fit together perfectly, preventing dirt or predators from entering. Squares or equilateral triangles would fit equally well, but regular hexagons provide the largest storage space for the amount of wax used.“</td>
<td>The lesson is a launch-explore-summarize instructional model and is intended to take approximately 240 minutes to complete across four class periods. Lesson 1: Students are introduced to issues at the US border. Lessons 2 and 3: Students work in groups to research their topic and create a presentation.</td>
</tr>
<tr>
<td>(Students do not explore this as part of a problem within the teaching materials)</td>
<td>student engagement (building awareness).</td>
</tr>
</tbody>
</table>
APPENDIX F

LESSON 7.3 FROM TM

82. The relationship between Celsius temperature, $C$, and Fahrenheit temperature, $F$, can be described by a linear equation in the form $F = mC + b$. The graph of this equation contains the point $(0, 32)$: Water freezes at $0^\circ$C or at $32^\circ$F. The line also contains the point $(90, 212)$: Water boils at $100^\circ$C or at $212^\circ$F. Write the linear equation expressing Fahrenheit temperature in terms of Celsius temperature.

83. Use a graphing utility to verify any three of your hand-drawn graphs in Exercises 21–32.
84. Use a graphing utility to verify any three of your hand-drawn graphs in Exercises 33–40. Solve the equation for $y$ before entering it.

Systems of Linear Equations in Two Variables

Mathematical goals

7.3

WHAT AM I SUPPOSED TO LEARN?
After studying this section, you should be able to:
1. Determine whether an ordered pair is a solution of a linear system.
2. Solve linear systems by graphing.
3. Solve linear systems by substitution.
4. Solve linear systems by addition.
5. Identify systems that do not have exactly one ordered-pair solution.

Symptoms of Physical Illness

Researchers identified college students who generally were procrastinators or nonprocrastinators. The students were asked to report throughout the semester how many symptoms of physical illness they had experienced. Figure 7.27 shows that by late in the semester, all students experienced increases in symptoms. Early in the semester, procrastinators reported fewer symptoms, but late in the semester, as work came due, they reported more symptoms than their nonprocrastinating peers.

The data in Figure 7.27 can be analyzed using a pair of linear models in two variables. The figure shows that by week 6, both groups reported the same number of symptoms of illness, on average of approximately 3.5 symptoms per group. In this section, you will learn two algebraic methods, called substitution and addition, that will reinforce this graphic observation, verifying $(6,3.5)$ as the point of intersection.

Systems of Linear Equations and Their Solutions

We have seen that all equations in the form $Ax + By = C$, $A$ and $B$ not both zero, are straight lines when graphed. Two such equations are called a system of linear equations or a linear system. A solution to a system of linear equations in two variables is an ordered pair that satisfies both equations in the system. For example, $(3, 4)$ satisfies the system

$$
\begin{align*}
2x + y &= 7 \\
x - y &= -1
\end{align*}
$$

Thus, $(3, 4)$ satisfies both equations and is a solution of the system. The solution can be described by saying that $x = 3$ and $y = 4$. The solution can also be described using set notation. The solution set of the system is \{(3,4)\}—that is, the set consisting of the ordered pair $(3,4)$.

A system of linear equations can have exactly one solution, no solution, or infinitely many solutions. We begin with systems having exactly one solution.
SOLVING LINEAR SYSTEMS BY SUBSTITUTION

1. Solve either of the equations for one variable in terms of the other. (If one of the equations is already in this form, you can skip this step.)
2. Substitute the expression found in step 1 into the other equation. This will result in an equation in one variable.
3. Solve the equation containing one variable.
4. Back-substitute the value found in step 3 into the equation from step 1.
5. Check the proposed solution in both of the system’s given equations.

EXAMPLE 3
Solving a System by Substitution

Solve by the substitution method:

\[
\begin{align*}
\begin{cases}
y = -x - 1 \\
4x - 3y = 24.
\end{cases}
\end{align*}
\]

SOLUTION

Step 1 Solve either of the equations for one variable in terms of the other. This step has already been done for us. The first equation, \(y = -x - 1\), is solved for \(y\) in terms of \(x\).

Step 2 Substitute the expression from step 1 into the other equation. We substitute the expression \(-x - 1\) for \(y\) into the other equation:

\[
\begin{align*}
y = -x - 1 & \quad \Rightarrow \quad 4(-x - 1) = 24 \quad \text{Substitute } -x - 1 \text{ for } y. \\
4x - 3(-x - 1) & = 24.
\end{align*}
\]

This gives us an equation in one variable, namely

\[4x - 3(-x - 1) = 24.\]

The variable \(y\) has been eliminated.
SECTION 7.3  Systems of Linear Equations in Two Variables  441

Step 3  Solve the resulting equation containing one variable.

\[ 4x - 3(-x - 1) = 24 \]  This is the equation containing one variable.
\[ 4x + 3x + 3 = 24 \]  Apply the distributive property.
\[ 7x + 3 = 24 \]  Combine like terms.
\[ 7x = 21 \]  Subtract 3 from both sides.
\[ x = 3 \]  Divide both sides by 7.

Step 4  Back-substitute the obtained value into the equation from step 1.
We now know that the x-coordinate of the solution is 3. To find the y-coordinate, we back-substitute the x-value into the equation from step 1.

\[ y = -x - 1 \]  This is the equation from step 1.
\[ \text{Substitute 3 for x.} \]
\[ y = -3 - 1 \]
\[ y = -4 \]  Simplify.

With \( x = 3 \) and \( y = -4 \), the proposed solution is \( (3, -4) \).

Step 5  Check.  Check the proposed solution, \( (3, -4) \), in both of the system’s given equations. Replace \( x \) with 3 and \( y \) with -4.

\[
\begin{align*}
  y &= -x - 1 \\
  -4 &= -3 - 1 \\
  -4 &= -4
\end{align*}
\]
\[
\begin{align*}
  4x - 3y &= 24 \\
  4(3) - 3(-4) &= 24 \\
  24 &= 24
\end{align*}
\]

The pair \( (3, -4) \) satisfies both equations. The system’s solution set is \{ \( (3, -4) \) \}.

\[ \checkmark \] **CHECK POINT 3**  Solve by the substitution method:
\[
\begin{align*}
  y &= 3x - 7 \\
  5x - 2y &= 8
\end{align*}
\]

**EXAMPLE 4**  Solving a System by Substitution

Solve by the substitution method:
\[
\begin{align*}
  5x - 4y &= 9 \\
  x - 2y &= -3
\end{align*}
\]

**SOLUTION**

Step 1  Solve either of the equations for one variable in terms of the other.
We begin by isolating one of the variables in either of the equations. By solving for \( x \) in the second equation, which has a coefficient of 1, we can avoid fractions.

\[ x - 2y = -3 \]  This is the second equation in the given system.
\[ x = 2y - 3 \]  Solve for \( x \) by adding 2y to both sides.

Step 2  Substitute the expression from step 1 into the other equation. We substitute \( 2y - 3 \) for \( x \) in the first equation.

\[ x - 2y - 3 \]
\[ 5\left\{x \right\} - 4y = 9 \]
When we use the addition method, we want to obtain two equations whose sum is an equation containing only one variable. The key step is to obtain, for one of the variables, coefficients that differ only in sign. To do this, we may need to multiply one or both equations by some nonzero number so that the coefficients of one of the variables, \( x \) or \( y \), become opposites. Then when the two equations are added, this variable is eliminated.

**Solving Linear Systems by Addition**

1. If necessary, rewrite both equations in the form \( Ax + By = C \).
2. If necessary, multiply either equation or both equations by appropriate nonzero numbers so that the sum of the \( x \)-coefficients or the sum of the \( y \)-coefficients is 0.
3. Add the equations in step 2. The sum is an equation in one variable.
4. Solve the equation in one variable.
5. Back-substitute the value obtained in step 4 into either of the given equations and solve for the other variable.
6. Check the solution in both of the original equations.

---

**EXAMPLE 5**  
Solving a System by the Addition Method

Solve by the addition method:

\[
\begin{align*}
3x + 2y &= 48 \\
9x - 8y &= -24
\end{align*}
\]

**SOLUTION**

**Step 1** Rewrite both equations in the form \( Ax + By = C \). Both equations are already in this form. Variable terms appear on the left and constants appear on the right.

**Step 2** If necessary, multiply either equation or both equations by appropriate numbers so that the sum of the \( x \)-coefficients or the sum of the \( y \)-coefficients is 0. We can eliminate \( x \) or \( y \). Let’s eliminate \( x \). Consider the terms in \( x \) in each equation, that is, \( 3x \) and \( 9x \). To eliminate \( x \), we can multiply each term of the first equation by \( -3 \) and then add the equations.

\[
\begin{align*}
3x + 2y &= 48 & \text{Multiply by } -3 \\
9x - 8y &= -24 & \text{No change}
\end{align*}
\]

Add the equations. \(-14y = -168\)

**Step 3** Add the equations. \(-14y = -168\)

**Step 4** Solve the equation in one variable. We solve \(-14y = -168\) by dividing both sides by \(-14\).

\[
\begin{align*}
-14y &= -168 \\
\frac{-14y}{-14} &= \frac{-168}{-14}
\end{align*}
\]

\( y = 12 \) Simplify.

**Step 5** Back-substitute and find the value for the other variable. We can back-substitute 12 for \( y \) into either one of the given equations. We’ll use the first one.

\[
\begin{align*}
3x + 2y &= 48 & \text{This is the first equation in the given system,} \\
3x + 2(12) &= 48 & \text{Substitute } 12 \text{ for } y \\
3x + 24 &= 48 & \text{Multiply,} \\
3x &= 24 & \text{Subtract 24 from both sides,} \\
x &= 8 & \text{Divide both sides by } 3.
\end{align*}
\]

We found that \( y = 12 \) and \( x = 8 \). The proposed solution is \( (8, 12) \).
Step 6 Check. Take a few minutes to show that \((8, 12)\) satisfies both of the original equations in the system. The solution set is \(\{(8, 12)\}\).

**CHECK POINT 5** Solve by the addition method:

\[
\begin{align*}
4x + 5y &= 3 \\
2x - 3y &= 7,
\end{align*}
\]

**EXAMPLE 6** Solving a System by the Addition Method

Solve by the addition method:

\[
\begin{align*}
7x &= 5 - 2y \\
3y &= 16 - 2x.
\end{align*}
\]

**SOLUTION**

Step 1 Rewrite both equations in the form \(Ax + By = C\). We first arrange the system so that variable terms appear on the left and constants appear on the right. We obtain

\[
\begin{align*}
7x + 2y &= 5 & \text{Add 2y to both sides of the first equation.} \\
2x + 3y &= 16 & \text{Add 2x to both sides of the second equation.}
\end{align*}
\]

Step 2 If necessary, multiply either equation or both equations by appropriate numbers so that the sum of the \(x\)-coefficients or the sum of the \(y\)-coefficients is 0. We can eliminate \(x\) or \(y\). Let’s eliminate \(y\) by multiplying the first equation by 3 and the second equation by \(-2\).

\[
\begin{align*}
21x + 6y &= 15 \\
-4x + 6y &= -32
\end{align*}
\]

Step 3 Add the equations.

\[
17x = -17
\]

Step 4 Solve the equation in one variable. We solve \(17x = -17\) by dividing both sides by 17.

\[
\begin{align*}
17x &= -17 \\
x &= -1
\end{align*}
\]

Step 5 Back-substitute and find the value for the other variable. We can back-substitute \(-1\) for \(x\) into either one of the given equations. We’ll use the second one.

\[
\begin{align*}
3y &= 16 - 2(-1) \\
3y &= 16 + 2 \\
3y &= 18 \\
y &= 6
\end{align*}
\]

With \(x = -1\) and \(y = 6\), the proposed solution is \((-1, 6)\).

Step 6 Check. Take a moment to show that \((-1, 6)\) satisfies both given equations. The solution is \((-1, 6)\) and the solution set is \(\{-1, 6\}\).

**CHECK POINT 6** Solve by the addition method:

\[
\begin{align*}
3x &= 2 - 4y \\
5y &= -1 - 2x.
\end{align*}
\]
5

Linear Systems Having No Solution or Infinitely Many Solutions

We have seen that a system of linear equations in two variables represents a pair of lines. The lines either intersect at one point, are parallel, or are identical. Thus, there are three possibilities for the number of solutions to a system of two linear equations.

**THE NUMBER OF SOLUTIONS TO A SYSTEM OF TWO LINEAR EQUATIONS**

The number of solutions to a system of two linear equations in two variables is given by one of the following. (See Figure 7.29.)

<table>
<thead>
<tr>
<th>Number of Solutions</th>
<th>What This Means Graphically</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exactly one ordered-pair solution</td>
<td>The two lines intersect at one point.</td>
</tr>
<tr>
<td>No solution</td>
<td>The two lines are parallel.</td>
</tr>
<tr>
<td>Infinitely many solutions</td>
<td>The two lines are identical.</td>
</tr>
</tbody>
</table>

**EXAMPLE 7** A System with No Solution

Solve the system:

\[
\begin{align*}
4x + 6y &= 12 \\
6x + 9y &= 12.
\end{align*}
\]

**SOLUTION**

Because no variable is isolated, we will use the addition method. To obtain coefficients of \(x\) that differ only in sign, we multiply the first equation by 3 and the second equation by \(-2\).

\[
\begin{align*}
4x + 6y &= 12 \quad \text{Multiply by } 3 \quad 12x + 18y = 36 \\
6x + 9y &= 12 \quad \text{Multiply by } -2 \quad -12x - 18y = -24 \quad \text{Add} \quad 0 = 12.
\end{align*}
\]

The false statement \(0 = 12\) indicates that the system has no solution. The solution set is the empty set, \(\emptyset\).

The lines corresponding to the two equations in Example 7 are shown in Figure 7.30. The lines are parallel and have no point of intersection.

**CHECK POINT 7** Solve the system:

\[
\begin{align*}
x + 2y &= 4 \\
3x + 6y &= 13.
\end{align*}
\]
EXAMPLE 8  A System with Infinitely Many Solutions

Solve the system:
\[
\begin{align*}
    y &= 3x - 2 \\
    15x - 5y &= 10.
\end{align*}
\]

SOLUTION

Because the variable \( y \) is isolated in \( y = 3x - 2 \), the first equation, we can use the substitution method. We substitute the expression for \( y \) into the second equation.

\[
\begin{align*}
    y &= 3x - 2 \\
    15x - 5(y) &= 10 \\
    15x - 5(3x - 2) &= 10 \\
    15x - 15x + 10 &= 10 \\
    10 &= 10.
\end{align*}
\]

This statement is true under all values of \( x \) and \( y \).

In our final step, both variables have been eliminated and the resulting statement, \( 10 = 10 \), is true. This true statement indicates that the system has infinitely many solutions. The solution set consists of all points \((x, y)\) lying on either of the coinciding lines, \( y = 3x - 2 \) or \( 15x - 5y = 10 \), as shown in Figure 7.31.

We express the solution set for the system in one of two equivalent ways:

\[
\begin{align*}
    \{(x, y) \mid y = 3x - 2\} \quad \text{or} \quad \{(x, y) \mid 15x - 5y = 10\}.
\end{align*}
\]

GREAT QUESTION!

The system in Example 8 has infinitely many solutions. Does that mean that any ordered pair of numbers is a solution?

No. Although the system in Example 8 has infinitely many solutions, this does not mean that any ordered pair of numbers you can form will be a solution. The ordered pair \((x, y)\) must satisfy one of the system’s equations, \( y = 3x - 2 \) or \( 15x - 5y = 10 \), and there are infinitely many such ordered pairs. Because the graphs are coinciding lines, the ordered pairs that are solutions of one of the equations are also solutions of the other equation.

CHECK POINT 8  Solve the system:
\[
\begin{align*}
    y &= 4x - 4 \\
    8x - 2y &= 8.
\end{align*}
\]

LINEAR SYSTEMS HAVING NO SOLUTION OR INFINITELY MANY SOLUTIONS

If both variables are eliminated when solving a system of linear equations by substitution or addition, one of the following applies:

1. There is no solution if the resulting statement is false.
2. There are infinitely many solutions if the resulting statement is true.
Modeling with Systems of Equations: Making Money (and Losing It)

What does every entrepreneur, from a kid selling lemonade to Mark Zuckerberg, want to do? Generate profit, of course. The profit made is the money taken in, or the revenue, minus the money spent, or the cost.

**REVENUE AND COST FUNCTIONS**
A company produces and sells \( x \) units of a product. Its **revenue** is the money generated by selling \( x \) units of the product. Its **cost** is the cost of producing \( x \) units of the product.

- **Revenue Function**
  \[ R(x) = (\text{price per unit sold})x \]

- **Cost Function**
  \[ C(x) = \text{fixed cost} + (\text{cost per unit produced})x \]

The point of intersection of the graphs of the revenue and cost functions is called the **break-even point**. The \( x \)-coordinate of the point reveals the number of units that a company must produce and sell so that money coming in, the revenue, is equal to money going out, the cost. The \( y \)-coordinate of the break-even point gives the amount of money coming in and going out. Example 9 illustrates the use of the substitution method in determining a company’s break-even point.

**EXAMPLE 9 Finding a Break-even Point**

Technology is now promising to bring light, fast, and beautiful wheelchairs to millions of disabled people. A company is planning to manufacture these radically different wheelchairs. Fixed cost will be \$500,000 and it will cost \$400 to produce each wheelchair. Each wheelchair will be sold for \$600.

a. Write the cost function, \( C \), of producing \( x \) wheelchairs.

b. Write the revenue function, \( R \), from the sale of \( x \) wheelchairs.

c. Determine the break-even point. Describe what this means.

**SOLUTION**

a. The cost function is the sum of the fixed cost and variable cost.

\[ C(x) = 500,000 + 400x \]

b. The revenue function is the money generated from the sale of \( x \) wheelchairs. We are given that each wheelchair will be sold for \$600.

\[ R(x) = 600x \]

c. The break-even point occurs where the graphs of \( C \) and \( R \) intersect. Thus, we find this point by solving the system:

\[
\begin{align*}
C(x) &= 500,000 + 400x \\
R(x) &= 600x 
\end{align*}
\]

or

\[
\begin{align*}
y &= 500,000 + 400x \\
y &= 600x
\end{align*}
\]

Using substitution, we can substitute \( 600x \) for \( y \) in the first equation:

\[
600x = 500,000 + 400x
\]

Subtract \( 400x \) from both sides:

\[
200x = 500,000
\]

Subtract \( 500,000 \) from both sides:

\[
x = 2500
\]

Divide both sides by \( 200 \).

133
Back-substituting 2500 for \( x \) in either of the system's equations (or functions), \( C(x) = 500,000 + 400x \) or \( R(x) = 600x \), we obtain
\[
R(2500) = 600(2500) = 1,500,000.
\]
We have \( R(x) = 600x \).

The break-even point is \((2500, 1,500,000)\). This means that the company will break even if it produces and sells 2500 wheelchairs. At this level, the money coming in is equal to the money going out: $1,500,000.

**Figure 7.32** shows the graphs of the revenue and cost functions for the wheelchair business. Similar graphs and models apply no matter how small or large a business venture may be.

The intersection point confirms that the company breaks even by producing and selling 2500 wheelchairs. Can you see what happens for \( x < 2500 \)? The red cost graph lies above the blue revenue graph. The cost is greater than the revenue, and the business is losing money. Thus, if they sell fewer than 2500 wheelchairs, the result is a loss. By contrast, look at what happens for \( x > 2500 \). The blue revenue graph lies above the red cost graph. The revenue is greater than the cost, and the business is making money. Thus, if they sell more than 2500 wheelchairs, the result is a gain.

**CHECK POINT 9** A company that manufactures running shoes has a fixed cost of $300,000. Additionally, it costs $30 to produce each pair of shoes. They are sold at $80 per pair.

\[ a. \] Write the cost function, \( C \), of producing \( x \) pairs of running shoes.

\[ b. \] Write the revenue function, \( R \), from the sale of \( x \) pairs of running shoes.

\[ c. \] Determine the break-even point. Describe what this means.

The profit generated by a business is the money taken in (its revenue) minus the money spent (its cost). Thus, once a business has modeled its revenue and cost with a system of equations, it can determine its profit function, \( P(x) \).

**THE PROFIT FUNCTION**

The profit, \( P(x) \), generated after producing and selling \( x \) units of a product is given by the profit function
\[
P(x) = R(x) - C(x),
\]
where \( R \) and \( C \) are the revenue and cost functions, respectively.

The profit function for the wheelchair business in Example 9 is
\[
P(x) = R(x) - C(x)
= 600x - (500,000 + 400x)
= 200x - 500,000.
\]

The graph of this profit function is shown in **Figure 7.33**. The red portion lies below the \( x \)-axis and shows a loss when fewer than 2500 wheelchairs are sold. The business is "in the red." The black portion lies above the \( x \)-axis and shows a gain when more than 2500 wheelchairs are sold. The wheelchair business is "in the black."
APPENDIX G

LESSON 5.4 FROM HSML

LESSON 5.4: ESTIMATED WEALTH DISTRIBUTION IN THE UNITED STATES AND THE WORLD

Enrique Ortiz

ECONOMIC INEQUALITY

Though worldly assets can never be shared equally among all people, the drastic disparity in who holds a nation’s wealth can often be cause for concern. This multiday lesson helps students explore the estimated wealth distribution, first in the United States and then in other nations, ultimately comparing the United States and other countries. They first explore and define the meaning of wealth distribution by dividing the population of the United States into five equal portions and examining visually how wealth is divided. They then examine the wealth distribution of ten countries, including the United States and Sweden, and compare them in terms of equity and fairness. The activities provide a framework to analyze social injustice regarding how wealth distribution impacts society.

DEEP AND RICH MATHEMATICS

This lesson extends students’ middle-school understanding of distributions using quantiles while comparing distributional shape. Students use graphs to provide models and illustrate of social injustice related to wealth distribution. They use modeling with mathematics and statistics to examine and understand empirical conditions and to improve their decision-making process.

ABOUT THE LESSON

The lesson is a launch-explore-summarize instructional model and is intended to take approximately 110 minutes to complete across two class periods.

Lesson 1: Students investigate wealth distributions represented by quantiles, then create an ideal wealth distribution and a self-perceived wealth distribution of the United States.

Lesson 2: Students use their understanding of wealth distributions to grapple with related graphics and discover the wealth distribution of the United States.

SOCIAL JUSTICE OUTCOMES

- I am aware of the advantages and disadvantages I have in society because of my membership in different identity groups, and I know how this has affected my life. (Justice 14)

- I understand that diversity includes the impact of unequal power relations on the development of group identities and cultures. (Diversity 10)

MATHEMATICS ESSENTIAL CONCEPTS

- Numbers—Quantitative reasoning includes, and mathematical modeling requires, attention to units of measurement. (N.2)

- Statistics and Probability—Distributions of quantitative data (continuous or discrete) in one variable should be described in the context of the data with respect to what is typical (the shape, with appropriate measures of center and variability, including standard deviation) and what is not (outliers), and these characteristics can be used to compare two or more subgroups with respect to a variable. (VSD.2)

MATHEMATICAL PRACTICES

- Reason abstractly and quantitatively.

- Model with mathematics.
Resources and Materials
- Student Resource 1, Wealth Distribution Grid (2 per group)
- Worksheet 1, Activity 1: Perceptions of Wealth Distribution in the United States of America (1 per student)
- Worksheet 2, Activity 2: Wealth Distribution Around the World (1 per student)
- Rulers
- 100 beans per student group
- Devices with internet access
- Video, “Wealth Inequality in America,” bit.ly/2IPjxl
- You may be interested in another SJML on wealth distribution by Michael Langewel (Gutstein & Peterson, 2015).

Lesson 1 Facilitation
Introduction to Wealth Distribution
During this lesson, students use prior knowledge of percent, percentiles, stacked bar graphs, and interpreting graphs as they deepen their understanding of distributions and connect this to wealth distribution in the United States.

Launch (10 minutes)
- Begin the lesson with a whole-group discussion of wealth. Ask, What is wealth, and how do we measure it?
  - Wealth is defined as a measure of the value of all the assets of worth owned by a person, community, company, or country.
- Tell students, Today we will use our skills with percent, creating graphs, and interpreting graphs to explore the distribution of wealth in the United States and, later, around the world.

Explore (35 minutes)
- Move students into groups of three or four and distribute a copy of the Wealth Distribution Grid found on the companion website along with two sets of 100 beans per group.
- Share that each shaded part on the worksheet represents 20% of the population and each square represents 1% of the population.
- Tell students to label the bottom shaded 20% “poorest” and the top 20% “wealthiest.”
• Have students take out a packet of beans and inform them that the beans represent all of the wealth in a particular country.

• Have students place 20 beans into each of the five quintiles to represent a completely equal distribution of wealth.

• Have students briefly discuss in their groups the following question: Do you think this is an accurate representation of the US population, where 20% of the people are in the wealthiest group and 20% in the poorest group? Explain.

• Optional: Present students with the following example of a possible wealth distribution and an inappropriate distribution. Have students determine which is mathematically possible and which is not. Have students model the distributions using the beans.

<table>
<thead>
<tr>
<th>Sample Scenario</th>
<th>Country A</th>
<th>Country B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Top 20% (Top one fifth, wealthiest):</td>
<td>30%</td>
<td>15%</td>
</tr>
<tr>
<td>Second 20%:</td>
<td>25%</td>
<td>20%</td>
</tr>
<tr>
<td>Third 20%:</td>
<td>20%</td>
<td>30%</td>
</tr>
<tr>
<td>Fourth 20%:</td>
<td>15%</td>
<td>20%</td>
</tr>
<tr>
<td>Bottom 20% (Bottom one fifth, poorest):</td>
<td>10%</td>
<td>15%</td>
</tr>
</tbody>
</table>

Post or project the sample scenario.
Be careful in word phrases "wealth distribution" and "distribution of wealth." The phrase "distribution of wealth" may cause improper inference or political associations to socialism by those involved.

- Ask, How should we place the beans on the Wealth Distribution Grid to model the United States? How should we place the beans to ensure an equitable and just wealth distribution?
- Look for students who may inappropriately assign more beans to lower quintiles. Providing an example with inappropriate assignments of beans may be useful to develop a deeper understanding of the context of the situation and mathematics as a whole class. Do this by putting fewer beans in a higher quintile than a lower quintile. This will naturally occur as students attempt to address an ideal and equitable distribution.
- Use the following questions to facilitate the learning during the launch:
  + Why are there different colored/shaded sections of the diagram? Why are they the same size?
  + If any given fifth of the graph has the same number of people, and the people in one fifth have a higher or lower relative wealth compared to those in other fifths, what must be true? Why? (A higher fifth must have more total beans than lower fifths because the number of people is equal and each has more wealth than the people in the lower fifths.)
  + Why do you believe having the beans allocated like this is more equitable?
  + What are the benefits and drawbacks of a more unequal wealth distribution?
  + What makes you believe that this represents the wealth distribution of the United States?
- Pass out Worksheet 1: Activity 1: Perceptions of Wealth Distribution in the United States of America
  + Students may want to use their diagrams to continue making sense of proper wealth distributions; however, students should convert their bean graphs to a stacked bar graph. Students will need to divide their rectangles up proportionally using a ruler.
- Monitor student work for misconceptions moving from the bean graph to the stacked bar graph representation. Often, students do not have percents that sum to 100 and don’t divide evenly when they begin.

**Summary (15 minutes)**

- Tell students that tomorrow they will explore the actual wealth distribution of the United States.
- Provide an opportunity at the end of this lesson for students to share with the whole class their predictions for the current wealth distribution of the United States and what they believe is the ideal wealth distribution.
• Use the following prompts to connect the mathematics and social injustices:
  + Why are there always more or fewer beans in each quintile?
  + How do we measure the wealth of these quintiles?
  + How do we determine what is fair?
  + What are the benefits and drawbacks of these distributions? (Highlight the most equal allocation of wealth and the most skewed distribution for comparison.)

**LESSON 2 FACILITATION**

Connecting Wealth Distribution to the United States

**Launch (5 minutes)**
• Begin class by having students connect to the prior day’s activity with an initial prompt:
  + What were the commonalities and differences between each of our beliefs on how the wealth is distributed in the United States? What were the important mathematical constructs of modeling this distribution?
• Use a think-pair-share format to highlight student beliefs and important mathematics.
• Say, *Today we are going to determine the actual wealth distribution of the United States.*

**Explore (30 minutes)**
• Distribute Worksheet 2, Activity 2: Wealth Distribution Around the World, which represents the wealth distribution in three different countries.
• Have students indicate which country they think is represented by each pie chart and why they would or would not like to live in each one.
• Allow approximately 15 minutes for students to come to a consensus in small groups.
• If time is available, allow students to justify their solutions by researching the internet. See http://www.infoplease.com/countries.html for information on countries of the world.
• Allow groups time to share which country they believe represented the United States and why. Select groups that may have differing opinions and have them justify their reasoning.
• Disclose the correct answers to question 2, but remind students that the emphasis is not on getting the right answer but justifying their solutions.
Pie chart for Country A represents “Freedom” (an equality utopia that does not exist).

Pie chart for Country B represents Sweden.

Pie chart for Country C represents the United States.

Have students find the percent of the students in class who indicated that they wanted to live in Countries A, B, and C (which happens to be the United States), and then ask them these questions:

- Why is this true?
- What are your general reactions to the findings in this activity?

Have students compare their original beliefs of the United States with their new information, and ask them these questions:

- How does the graph compare with the pie chart for the United States?
- Are you surprised by the differences, if any? Why?

Have students model using the 10 x 10 grid and 100 beans from the first lesson or a stacked bar graph. This will require breaking a bean into small pieces.

- What is income?
- What is the difference between equality and inequality?
- What is the difference between equity and equality?
Summary (15 minutes)

- Ask students the following questions:
  - How do you explain the actual wealth distribution in the United States?
  - What do you think about these results? What inference can you make about people’s perceptions of wealth distribution in the United States?

- Watch the YouTube video “Wealth Inequality in America” (bit.ly/2IXp4bL) about wealth distribution.

- Present the figure that summarizes what Gudrais’s (2011) and Norton and Ariely’s (2011) respondents said would be ideal (leftmost stacked bar graph), how they estimated wealth was currently distributed (middle stacked bar graph), and the actual distribution of wealth in the United States (far right stacked bar graph).
• SME components: critical thinking, critique, action points, discourse, and talk

• Ask students these questions:
  + Why do you think the general population’s understanding of wealth distribution differs so much from actual results?
  + Why do you think the ideal situation is hard to obtain?

TAKING ACTION

• Facilitate discussion about differences between the five subgroups. Use questions like these to help:
  + How would you define wealth?
  + How do you know if somebody is wealthy?
  + How do you know if somebody is not wealthy?
  + What things can you do if you are wealthy?
  + What things might you not be able to do if you are not wealthy?

• Highlight how the top one-fifth group had more wealth than all the other groups combined.

• Highlight what it looks like to be in the lower 20% of wealth holders in the United States.

• Emphasize the difficulty of sharing for the lower groups and why this may be the case.

• Discuss the importance of these different groups and their implications for society and wealth distribution in the United States. Use questions like these to help:
  + Does it make any difference if a lot of US wealth is concentrated in the hands of a few people?
  + Would you prefer to live in a country where wealth is distributed evenly?
  + Do you perceive wealth inequality in the United States? Explain.
  + Why might it be difficult to perceive wealth inequality in your everyday experiences?

• Have discourse around the difficulty of having a uniform distribution—equal wealth for all. Consider the following discussion questions during discourse:
  + Could an equal wealth distribution encourage people to contribute to the workforce and hold less desirable jobs? If so, how?
  + Does it make any difference if a lot of US wealth is concentrated in the hands of a few people?
  + Would you prefer to live in a country where wealth is distributed evenly?
• Connect students to the power of mathematics to understand their world. Use questions like these to help:
  + How can you use mathematics to explore social injustice or other real-life situations?
  + How can you use mathematics to illustrate how wealth distribution in the United States affects people's quality of life?
  + What type of graphs can you use to understand real-life issues?
  + How can you use graphs to understand real-life situations? What are some examples?

• Provide an opportunity for students to complete a short written piece with the following prompt: What is wealth distribution? How may this be or not be an important social issue?

• Assign students homework to discuss this mathematics lesson with an adult. Have students share their created graphics to help enlighten others regarding the disparity.
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