DEFORMATIONS OF GEOMETRICALLY FRUSTRATED ELASTIC SHEETS

A Dissertation Presented
by
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The wrinkling and buckling of thin solids are common phenomena in our daily life and can be observed in many situations, such as crumpled papers, stretched plastics, compressed metals, clothes on our bodies and even furrowed human skin. Understanding of these phenomena has therefore long drawn interest of scholars. In this thesis, we discuss two buckling problems numerically and analytically. First, we study the wrinkling mechanism of stretched sheets with clamped edges. A central puzzle underlying this canonical example of “tensional wrinkling” has been the origin of compressive stress, which eventually leads to buckling instability. We elucidate
the source of the compression as the relative extension of the clamped edge in comparison to the bulk of the sheet, and show how it gives rise to buckling instability. Distinguishing between a “near-threshold” parameter regime, in which the stress is well approximated by the planar, unwrinkled state, and a “far-from-threshold” regime, where wrinkles have a strong, non-perturbative effect, we address the transition of the morphology from a buckling-like to wrinkling-like pattern. Our work reveals that the stretch-induced wrinkling also arises from a common compressional confining geometry rather than from Poisson’s contraction as argued in previous research, thereby elucidating the conceptual similarity between this problem and other wrinkling problems. The second phenomenon we address is the emergence of defects and amplitude modulations in non-ideal patterns, where the locally-favorable wrinkle wavelength is inconsistent with the geometry imposed by confining forces. With such a problem, we seek to push the understanding of wrinkles from ideal cases to a case that is closer to natural situations. We propose a relatively simple theoretical model to study the wrinkling patterns in such a scenario. Finally, we include a technical chapter that elaborates on the numerous subtleties that must be taken into consideration by practitioners who seek to use the “Surface Evolver” algorithm for numerical simulations of elastic sheets, e.g. how to properly disable features designed for liquid surfaces, how to define the reference configuration and deal with refinements, and so on.
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CHAPTER 1
INTRODUCTION TO ELASTICITY OF THIN SOLIDS

1.1 3D Elasticity

1.1.1 Strain

The theory of elasticity deals with the deformations of solids in the presence of stress, namely when the body transforms from its reference configuration (initial configuration) to an actual configuration under forces. The strain is a characterization of deformation. Let us consider a material element initially located at \( r \) that is transformed to \( r'(r) \). Notice that \( r'(r) \) is actually a vector field. As we want to describe the deformations, we consider how a line element \( dr \) transforms.

\[
dr' = r'(r + dr) - r'(r) \\
= \frac{\partial r'}{\partial r} dr \\
= F dr
\]

(1.1)

where \( F = \frac{\partial r'}{\partial r} \) is called deformation gradient, or the transformation matrix between the reference configuration and actual configuration. Eq. 1.1 is the most fundamental way to represent the deformations. However, it contains not only information on the elastic deformation, but also rigid body translations and rotations, which are not the
focus of elasticity theory. What we actually care about is how the distance between material elements change, so here rises the definition of strain,

\[
dr' - dr^2 = dr F^T F dr - dr^2
\]

\[
= dr (F^T F - 1) dr
\]

\[
= dr 2 \varepsilon dr
\]

where

\[
\varepsilon = \frac{1}{2} (F^T F - 1)
\]

is the strain tensor, specifically the Green-Lagrange strain tensor. Notice that \( \varepsilon \) is symmetric. In actual calculations, one would find it easier to write the strain tensor in terms of displacement vector \( u \), which is defined as the difference between \( r'(r) \) and \( r \), \textit{i.e.}

\[
u = r'(r) - r
\]

We can easily derive

\[
\varepsilon = \frac{1}{2} \left( \frac{\partial u}{\partial r} + \left( \frac{\partial u}{\partial r} \right)^T + \left( \frac{\partial u}{\partial r} \right)^T \frac{\partial u}{\partial r} \right)
\]

In Cartesian coordinate systems, when \( r = (x, y, z) \) or \( (x_1, x_2, x_3) \), the components of the tensor \( \varepsilon \) can be written as

\[
\varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \frac{1}{2} \frac{\partial u_k}{\partial x_j} \frac{\partial u_k}{\partial x_j}
\]
where the Einstein summation is assumed. One thing worth mentioning here is that the second order term $\frac{\partial u_k}{\partial x_j} \frac{\partial u_k}{\partial x_j}$ is often neglected in small strain cases. i.e.

$$\varepsilon_{ij} \approx \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad \text{when } |\varepsilon_{ij}| \ll 1$$ (1.7)

To get a sense about the physical meaning of the strain tensor, consider a simple uniaxial stretching, e.g. $\mathbf{r} = (x, y, z)$ and $\mathbf{r}' = (\lambda x, y, z)$, which indicates that the material is uniformly stretched ($\lambda > 1$) or compressed ($\lambda < 1$) along the $x$ direction. In this case, all components of the strain tensor are 0 except $\varepsilon_{xx} = \frac{\lambda^2 - 1}{2}$. For small strains ($|\varepsilon_{ij}| \ll 1$), we can get

$$\lambda - 1 \approx \varepsilon_{xx}$$ (1.8)

which is the stretching or compression ratio of the material in $x$ direction.

1.1.2 Stress

For continuum mechanics, the material is not composed of discrete particles, so it is not reasonable to consider point forces. Instead, we should analyze the action of forces on small volume elements. There are two types of such forces, one is the “body” force, which directly acts on every part of the element, like gravity. The other one is interior force, e.g. two volume elements pushing each other though their boundary. Now let us focus on the interior force and consider a volume element at $\mathbf{r}$ with a boundary $S$, and suppose the force acting on its boundary element $dS$ is $f dS$. Then the total force acting on the volume element is

$$\mathcal{F} = \int_S f dS$$ (1.9)
Notice that $f$ does not only vary with position $r$, but also with normal direction $n$ of $dS$. To get the force density (called traction vector) on arbitrary surface element with normal direction $n$, we can decompose $f$ by the direction of $n$

$$f_i = \sigma_{ij} n_j \quad (1.10)$$

where $\sigma_{ij}$ is a tensor, which is called Cauchy stress tensor. By this definition, $\sigma_{ij}$ is the force density ($force/area$) directed to $i$-th direction acting on surface element with normal direction $j$. If we consider a small cube whose three adjacent edges are parallel to the coordinate axes (Fig. 1.1), then we can find that the diagonal terms of the stress tensor are normal to the surfaces of the cube, the non-diagonal terms are shearing the cube, hence they are called normal stress and shear stress respectively.

Another thing we could notice in Fig. 1.1 is that the shear stress generates torque. If the material is at equilibrium, the total toque should be zero. Take $z$-direction as
an example,
\[
\frac{dx}{2} \sigma_{xy} dy dz - \frac{dy}{2} \sigma_{yx} dx dz + \frac{dx}{2} \sigma_{xy} dy dz - \frac{dy}{2} \sigma_{yx} dx dz = 0 \quad (1.11)
\]
and consequently \( \sigma_{xy} = \sigma_{yx} \), we can therefore conclude that the stress tensor is symmetric,
\[
\sigma_{ij} = \sigma_{ji} \quad (1.12)
\]
Furthermore, at mechanical equilibrium the total force acting on the element should be zero too, \( i.e. \)
\[
\mathcal{F} = \int_S \sigma_{ij} n_j dS = \iiint \frac{\partial \sigma_{ij}}{\partial x_j} dV = 0 \quad (1.13)
\]
So we get another constraint about the stress tensor
\[
\frac{\partial \sigma_{ij}}{\partial x_j} = 0 \quad (1.14)
\]
### 1.1.3 Hookean Elasticity

Similarly to the force-displacement relation of springs, \( i.e. \) the Hooke’s law, \( \mathcal{F} = k \Delta x \), there is also such a relation between the stress and strain in continuum mechanics. Generally the stress is a function of strain,
\[
\sigma = g(\varepsilon) \quad (1.15)
\]
For small strain and isotropic materials, one can expand \( g(\varepsilon) \) at \( \varepsilon = 0 \) and keep the first order of the Taylor series and to get the linear elasticity or Hookean elasticity.
\[
\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2 \mu \varepsilon_{ij} \quad (1.16)
\]
where $\lambda$ and $\mu$ are constants of the material, called Lamé coefficients. However in most cases, it is more convenient to rewrite Eq. 1.16 as

$$\sigma_{ij} = \frac{E}{1 + \nu} \left( \varepsilon_{ij} + \frac{\nu}{1 - 2\nu} \varepsilon_{kk}\delta_{ij} \right)$$ (1.17)

where $E$ is called Young’s modulus and $\nu$ is called Poisson’s ratio, such that

$$\lambda = \frac{E\nu}{(1 + \nu)(1 - 2\nu)} \quad \text{and} \quad \mu = \frac{E}{2(1 + \nu)}$$ (1.18)

### 1.1.4 Nonlinear Elasticity

Nonlinear elasticity is a rather complex topic, but it will be of little relevance to this thesis, hence we will discuss it only very briefly here.

There are multiple ways to express strain, and the Green-Lagrangian strain (Eq. 1.6) we introduced above is usually used in linear elasticity. Here we introduce another way, using the three invariants of the left Cauchy–Green deformation tensor $\mathbf{B} = \mathbf{F}\mathbf{F}^T$,

$$I_1 = \text{Tr} \mathbf{B} = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$$ (1.19a)

$$I_2 = \frac{1}{2} \left( (\text{Tr} \mathbf{B})^2 - \text{Tr} (\mathbf{B}^2) \right) = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2$$ (1.19b)

$$I_3 = \det \mathbf{B} = \lambda_1^2\lambda_2^2\lambda_3^2 = J^2$$ (1.19c)

where $\lambda_i, \{i = 1, 2, 3\}$ are eigenvalues of the deformation gradient $\mathbf{F}$, and $J = \lambda_1\lambda_2\lambda_3$ is the determinant of $\mathbf{F}$. $J$ has an obvious physical meaning, being the volume change of the material. The volume is conserved when $J = 1$, corresponding to
incompressibility. In practice the three invariants $I_1$, $I_2$ are normalized by the volume change, and $I_3$ is replaced by $J$ i.e.

$$
\bar{I}_1 = J^{-2/3}I_1, \quad \bar{I}_2 = J^{-4/3}I_2
$$

(1.20)

The three invariants are now $\bar{I}_1$, $\bar{I}_2$, $J$.

Instead of starting from strain-stress relation, hyperelasticity starts from strain energy density. For isotropic material, the polynomial form of energy is (see Ref. [3] for details)

$$
e_{el} = \sum_{p+q=1}^{N} C_{pq}(\bar{I}_1 - 3)^p(\bar{I}_2 - 3)^q + \sum_{m=1}^{N} \frac{1}{D_m}(J - 1)^{2m}
$$

(1.21)

Where $C_{pq}$ and $D_m$ are material constants. Notice that when there is no deformation, $\lambda_i = 1$, so $\bar{I}_1 = 3$, $\bar{I}_2 = 3$, $J = 1$ means strain free.

There are several named models in this context, the simplest is *neo-Hookean* model, in which only $C_{10}$ and $D_1$ are kept, others are set to 0. Another more complex is *Mooney–Rivlin* model, when one more parameter $C_{01}$ is kept. The stress-strain response is usually complicated for these hyperelastic models, we will not discuss further in this thesis.

1.1.5 Elastic Energy

The elastic energy is given by the scalar product of the force and displacement. In continuum material, this implies that the energy density is given by the variational form

$$
\delta e_{el} = \sigma_{ij}\delta \varepsilon_{ij}
$$

(1.22)
For linear elasticity, there is the relation

$$\sigma_{ij} \delta \varepsilon_{ij} = \delta \sigma_{ij} \varepsilon_{ij}$$

(1.23)

So that the elastic energy density for Hookean material can be written as

$$e_{el} = \frac{1}{2} \sigma_{ij} \varepsilon_{ij}$$

(1.24)

1.2 Elasticity of Thin Plates

The focus of this thesis is thin plates, for problems in which the thickness of the plate is much smaller than its other two dimensions. In this chapter we will derive the basic concepts and equations of thin plates from 3D elasticity introduced in previous sections.

1.2.1 Bending and Stretching Energy

Consider the thin plate in Fig. 1.2 whose thickness is $h$, and it is originally flat. Assume the thickness dimension is $z$, the other two dimensions are $x$ and $y$. We still use $u_x$ and $u_y$ to denote the displacement in $x$ and $y$ directions (in-plane displacement), but, instead of $u_z$, let us denote the displacement in $z$ direction as $\zeta$ (out-of-plane displacement, also called deflection). To specialize for thin plates, we simplify the 3D elasticity under the assumption that the thickness is small, and furthermore, specialize for situations in which the upper and lower surfaces are free,

$$\sigma_{zx}(z = \text{surfaces}) = \sigma_{yz}(z = \text{surfaces}) = \sigma_{zz}(z = \text{surfaces}) = 0$$

(1.25)
Combining with the equilibrium condition Eq. 1.14, we notice that the stress derivations are all in same order, and since $h$ is much smaller than other scales, the $\sigma_{iz}$ deviate little from their respective surface values in Eq 1.25. So in nearly all cases we neglect $\sigma_{iz}, i = \{x, y, z\},$

\[
\sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0
\]  

(1.26)

Plugging it back Eq. 1.17, we can reduce $\varepsilon_{iz}, i = \{x, y, z\}$ and get the Hookean elasticity for thin plates,

\[
\sigma_{xx} = \frac{E}{1 - \nu^2} (\varepsilon_{xx} + \nu \varepsilon_{yy}), \quad \sigma_{yy} = \frac{E}{1 - \nu^2} (\varepsilon_{yy} + \nu \varepsilon_{xx}), \quad \sigma_{xy} = \frac{E}{1 + \nu} \varepsilon_{xy}
\]  

(1.27)

The strain expressions in Eq. 1.6 remain intact except that we don’t have $\varepsilon_{iz}$ any more, and $u_z$ is replaced by $\zeta$.

\[
\varepsilon_{\alpha\beta} = \frac{1}{2} \left( \frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha} \right) + \frac{1}{2} \left( \frac{\partial u_x}{\partial x_\alpha} \frac{\partial u_x}{\partial x_\beta} + \frac{\partial u_y}{\partial x_\alpha} \frac{\partial u_y}{\partial x_\beta} \right) + \frac{1}{2} \left( \frac{\partial \zeta}{\partial x_\alpha} \frac{\partial \zeta}{\partial x_\beta} \right)
\]  

(1.28)
Note that we use indices \( \alpha, \beta \) instead of \( i, j \) to indicate \( \{x, y\} \) rather than \( \{x, y, z\} \).

When strain is small, the second order term of in-plane displacement, \textit{i.e.} the second paranthetical term in the right side of Eq. 1.28 is often neglected, as the first order term will dominate.

Until now, we have not yet eliminated the \( z \) dependence in strain, stress and displacement, \textit{e.g.} in Eq. 1.28, the \( \varepsilon_{\alpha\beta} \) may depend on \( z \). However, since \( h \) is small with respect to lateral size, we can do Taylor expansion around the mid surface (\( z = 0 \) in reference configuration) of the plate and keep only the leading order. We start with the displacement \( u_\alpha \),

\[
u_\alpha(x, y, z) \approx u_\alpha(x, y, 0) + z \frac{\partial u_\alpha(x, y, 0)}{\partial z}
\]

(1.29)

With the previous assumptions \( \sigma_{iz} = 0 \), we have

\[
\varepsilon_{\alpha z} = \frac{1}{2} \left( \frac{\partial u_\alpha}{\partial z} + \frac{\partial u_z}{\partial x_\alpha} \right) + \frac{1}{2} \frac{\partial u_k}{\partial x_\alpha} \frac{\partial u_k}{\partial z} = 0
\]

(1.30)

In this thesis, it will be sufficient for us to focus on small strain and small slope approximation, \textit{i.e.}

\[
|\varepsilon_{\alpha\beta}| \ll 1 \quad \text{and} \quad \left| \frac{\partial \zeta}{\partial x_\alpha} \right| \ll 1
\]

(1.31)

Additionally, keeping the leading order in \( z \), the dependence on \( z \) of \( \zeta \) can be neglected too, namely, \( \zeta(x, y, z) = \zeta(x, y) \), then we can get

\[
u_\alpha(x, y, z) \approx u_\alpha(x, y) - z \frac{\partial \zeta(x, y)}{\partial x_\alpha}
\]

(1.32)
With this result, we can further get the expressions for strain,

\[ \varepsilon_{\alpha\beta}(x, y, z) = \varepsilon_{\alpha\beta}(x, y) - z \frac{\partial^2 \zeta(x, y)}{\partial x_\alpha \partial x_\beta} \]  

(1.33)

Now we actually have separated the in-plane contributions and out-of-plane contributions (from \( \zeta \)). Plugging Eq. 1.27 and Eq. 1.33 in Eq. 1.24 and integrating out \( z \) from \(-\frac{h}{2}\) to \( \frac{h}{2}\), we finally get the energy density for thin plates,

\[ e_{el}^{(2D)} = \frac{Eh}{2(1+\nu)} \left( \text{Tr} \left( \varepsilon^2 \right) + \frac{\nu}{1-\nu} (\text{Tr} \varepsilon)^2 \right) \]

\[ + \frac{Eh^3}{24(1-\nu^2)} \left( (\Delta \zeta)^2 - 2(1-\nu) \left( \frac{\partial^2 \zeta}{\partial x^2 \partial y^2} - \left( \frac{\partial^2 \zeta}{\partial x \partial y} \right)^2 \right) \right) \]  

(1.34)

Where \( \varepsilon \) is a two-dimensional tensor. One can notice that the energy is composed of in-plane strain and deflection, we call the in-plane part stretching energy or strain energy, and the deflection part bending energy. It is useful to define the stretching modulus and bending modulus of thin plates,

\[ Y = Eh \quad \text{and} \quad B = \frac{Yh^2}{12(1-\nu^2)} \]  

(1.35)

Moreover, we notice that \( \Delta \zeta / 2 \) and \( \frac{1}{2} \left( \frac{\partial^2 \zeta}{\partial x^2} \frac{\partial^2 \zeta}{\partial y^2} - \left( \frac{\partial^2 \zeta}{\partial x \partial y} \right)^2 \right) \) are mean curvature \( H \) and Gaussian curvature \( K \) respectively in small slope approximation. So we finally find

\[ e_s = \frac{Y}{2(1+\nu)} \left( \text{Tr} \left( \varepsilon^2 \right) + \frac{\nu}{1-\nu} (\text{Tr} \varepsilon)^2 \right) \]  

(1.36a)

\[ e_b = B \left( 2H^2 - (1-\nu)K \right) \]  

(1.36b)
The total stretching energy $U_s$ and bending energy $U_b$ are just the integral of $e_s$ or $e_b$ over the mid-plane surface of the plate. We thus reduced 3D problem to 2D, such that when we discuss strain, stress in thin plates problems, we need not consider the $z$ dependence.

**1.2.2 The Föppl–von Kármán Equations**

Similarly to the derivation of the elastic energy, we can derive also the force balance equation of a solid plate at mechanical equilibrium (e.g. through Euler-Lagrange analysis of the energy functional), and continue the process of approximation in $z$ direction. Alternatively, we can also get the expression for $\sigma_{\alpha\beta}(x, y, z)$, then with the equilibrium equation Eq. 1.14, and integrating out $z$, we can get the equilibrium equations for thin plates,

$$B\Delta^2 \zeta - h \left( \sigma_{xx} \frac{\partial^2 \zeta}{\partial x^2} + \sigma_{yy} \frac{\partial^2 \zeta}{\partial y^2} + 2\sigma_{xy} \frac{\partial^2 \zeta}{\partial x \partial y} \right) = 0 \quad (1.37a)$$

$$\frac{\partial \sigma_{\alpha\beta}}{\partial x_\beta} = 0 \quad (1.37b)$$

Eq. 1.37a and Eq. 1.37b are called first and second Föppl–von Kármán equation (FvK) respectively. Notice the appearance of $h$ in Eq. 1.37a, and remember that the $\sigma_{\alpha\beta}$ is still in dimension of force per area, which is not what we want for a 2D theory, so usually we redefine $\sigma_{\alpha\beta}$,

$$\sigma^{(2D)}_{\alpha\beta} = h \sigma^{(3D)}_{\alpha\beta} \quad (1.38)$$

From now on, we will mean $\sigma^{(2D)}_{\alpha\beta}$ when mentioning $\sigma_{\alpha\beta}$. The Hookean elasticity Eq. 1.27 needs to be modified too, $E$ should be replaced by $Y$. 

12
\[ \sigma_{xx} = \frac{Y}{1 - \nu^2} (\varepsilon_{xx} + \nu \varepsilon_{yy}), \quad \sigma_{yy} = \frac{Y}{1 - \nu^2} (\varepsilon_{yy} + \nu \varepsilon_{xx}), \quad \sigma_{xy} = \frac{Y}{1 + \nu} \varepsilon_{xy} \]  
\[ \text{(1.39)} \]

Another technique often used here is the *Airy potential* or *Airy stress function*, denoted as \( \Phi \), which can represent \( \sigma_{\alpha\beta} \) by,

\[ \sigma_{xx} = \frac{\partial^2 \Phi}{\partial y^2}, \quad \sigma_{yy} = \frac{\partial^2 \Phi}{\partial x^2}, \quad \sigma_{xy} = -\frac{\partial^2 \Phi}{\partial x \partial y} \]
\[ \text{(1.40)} \]

With Eq. 1.39, the in-plane equilibrium Eq. 1.37b is automatically satisfied. Eq. 1.37 can be written as (see Ref [1] for details),

\[ B \Delta^2 \zeta - \left( \frac{\partial^2 \Phi}{\partial y^2} \frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \Phi}{\partial x^2} \frac{\partial^2 \zeta}{\partial y^2} - 2 \frac{\partial^2 \Phi}{\partial x \partial y} \frac{\partial^2 \zeta}{\partial x \partial y} \right) = 0 \]
\[ \text{(1.41a)} \]

\[ \Delta^2 \Phi + Y \left( \frac{\partial^2 \zeta}{\partial x^2} \frac{\partial^2 \zeta}{\partial y^2} - \left( \frac{\partial \zeta}{\partial x \partial y} \right)^2 \right) = 0 \]
\[ \text{(1.41b)} \]

Eq. 1.41a and 1.41b respectively describe force balance in the normal (out-of-plane) and tangent (in-plane) directions on a unit area of the plate’s midplane.

### 1.3 Buckling and Wrinkling

After describing some fundamentals of elasticity, let us consider two examples which introduce two important concepts central to the topic of this thesis.

#### 1.3.1 Buckling

The first example is the most simple bending problem, called *Euler buckling*. We take a rectangular sheet with width \( W \) and confine it uniformly to a slightly shorter...
scale $W - \Delta$, here $\Delta \ll W$. Fig. 1.3 shows the side view, the length of the sheet is not important here.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig1.3}
\caption{The side view of a uniformly confined rectangular sheet, whose original width is $W$}
\end{figure}

From the second FvK equation Eq. 1.41a, we have

$$B \frac{\partial^4 \zeta}{\partial x^4} - \sigma_{xx} \frac{\partial^2 \zeta}{\partial x^2} = 0$$

(1.42)

With the boundary conditions

$$\zeta(x = 0) = \zeta(x = W) = 0$$

(1.43)

Notice that in Eq. 1.43 we assume $\zeta(x = W) = 0$, rather than $\zeta(x = W - \Delta) = 0$. For small confinement $\Delta \ll W$, these two configurations only amounts to corrections at high orders of $\Delta/W$. The solution to Eq. 1.42 is a sine wave,

$$\zeta(x, y) = A \sin \left( \frac{n\pi x}{W} \right)$$

(1.44)
Where \( n \in \mathbb{N} \). We can also get the compressive stress,

\[
\sigma_{xx} = -B\frac{n^2\pi^2}{W^2}
\]  

(1.45)

which is a constant. Now consider the strain

\[
\varepsilon_{xx} = \frac{\sigma_{xx}}{Y} = \frac{\partial u_{xx}}{\partial x} + \frac{1}{2}\left(\frac{\partial \zeta}{\partial x}\right)^2
\]  

(1.46)

Obviously, the term \( \left(\frac{\partial \zeta}{\partial x}\right)^2 \) is undulating and therefore, to make \( \sigma_{xx} \) a constant, \( \frac{\partial u_{xx}}{\partial x} \) must cancel the undulating part. Hence we get

\[
\varepsilon_{xx} = -\frac{n^2\pi^2A^2}{4W^2}
\]  

(1.47)

Using the definition of strain Eq. 1.2, we obtain

\[
W - \Delta = \int_0^W \sqrt{1 + 2\varepsilon_{xx}}dx \approx W - \frac{n^2\pi^2A^2}{4W}
\]  

(1.48)

Thus, we get

\[
\Delta = \frac{n^2\pi^2A^2}{4W}
\]  

(1.49)

Here an interesting observation pops out. If we calculate the width of the buckled sheet in actual configuration, which is taken as a starting point for energy minimiza-
tion, one would find Eq. 1.49 indicates the arclength is equal to original width of the sheet at leading order of $\Delta/W$, i.e.

$$\int_{0}^{W-\Delta} \sqrt{1 + \left(\frac{\partial \zeta}{\partial x}\right)^2} \, dx \approx W$$

which means the sheet is “inextensible”.

As for the mode number $n$, we consider the energy and find the lowest mode $n = 1$ minimizes the energy Eq. 1.36,

$$U = \int (e_s + e_b) \, dx = \frac{\pi^2 B \Delta n^2}{W^2}$$

finally we come to the solution,

$$\zeta(x, y) = \frac{2}{\pi} \sqrt{\Delta W} \sin \left(\frac{\pi}{W} x\right)$$

Notice that the wavelength is totally determined by the width,

$$\lambda = 2W$$

There should be a critical value or threshold $\Delta_c$, such that when $\Delta < \Delta_c$, the sheet remains flat. To get it, we can consider the compression stress $\sigma_{xx}$ in planar
state, when this is larger than the buckling stress in Eq. 1.42, the sheet will buckle, so

\[
\sigma_{xx}^{(\text{planar})} = -\frac{Y\Delta_c}{(1-\nu^2)W} = \sigma_{xx}^{(\text{buckle})} = -B\frac{\pi^2}{W^2}
\]

(1.54)

Usually we prefer dimensionless parameter, so we normalize \(\Delta\) by \(W\). From Eq. 1.54 we thus get the critical value \(\Delta_c\),

\[
\tilde{\Delta}_c = \frac{\Delta_c}{W} = \frac{\pi^2 \overline{h}^2}{12W^2} = \frac{\pi^2}{12} \overline{h}^2
\]

(1.55)

We note that the above analysis, which echoes the common wisdom in the physics literature on solid mechanics, address in fact a “far-from-threshold” regime, namely, \(\Delta \gg \Delta_c\). Here, Eq. 1.45 means that the stress \(\sigma_{xx}\) is determined as a Lagrange multiplier that enforces the inextensibility constraint. A more rigorous analysis of the “near-threshold” regime, i.e. \(\Delta \simeq \Delta_c\), involves a linear stability analysis of the planar state, such that \(\sigma_{xx}\) is taken as its \(\Delta\)-dependent value at the planar state, rather than being dependent on \(A\) and \(n\). The distinction between a “near-threshold” and “far-from-threshold” analysis will be a central part of the discussion in this thesis.

1.3.2 Wrinkling

Now let us add some complexity to the previous buckling example, and consider a sheet attached a \textit{substrate} with a stiffness \(K\). See Fig. 1.4, we still confine the sheet like the previous problem, the sheet also tries to buckle, but the difference here is that the substrate will “drag” or “push” the sheet so that it cannot buckle too much. One will see the wrinkling pattern shown in the figure.
Figure 1.4: The sheet floating on substrate with stiffness $K$. One can think of the sheet connected to many springs whose spring constant is $K$. A classic example of substrate in reality is a liquid subphase, in which case the stiffness comes from liquid pressure and $K = \text{density of liquid} \times \text{gravitational acceleration}$ [29]. When the sheet is compressed, it will wrinkle.

The substrate resistance adds another force to the first FvK equation $K\zeta$,

$$B \frac{\partial^4 \zeta}{\partial x^4} - \sigma_{xx} \frac{\partial^2 \zeta}{\partial x^2} + K\zeta = 0 \quad (1.56)$$

which may also be derived from energy variation. We still consider the boundary conditions Eq. 1.43, and note that the general solution Eq. 1.44 remains valid, but the energy becomes,

$$U = \Delta \left( \frac{\pi^2 n^2 B}{W^2} + \frac{KW^2}{\pi^2 n^2} \right) \quad (1.57)$$

The energy is minimized when

$$n = \left( \frac{K}{B} \right)^{1/4} \frac{W}{\pi} \quad (1.58)$$

Or in terms of wavelength $\lambda$

$$\lambda = \frac{2W}{n} = 2\pi \left( \frac{B}{K} \right)^{1/4} \quad (1.59)$$
Note that Eq. 1.59 reflects the local balance between the bending force and the resistance from substrate, thus it is also called *local lambda law*. The compression stress is

\[ \sigma_{xx} = -2\sqrt{BK} \]  

which is also the critical compression stress as we analyzed in the buckling problem, so the critical confinement is,

\[ \tilde{\Delta}_c = 2(1 - \nu^2)\sqrt{BK}\frac{1}{Y} \]  

1.3.3 Buckling v.s. Wrinkling

Comparing the buckling and wrinkling phenomena, we can notice two key physical differences. First, the wavelength of buckling is merely limited by the size of the sheet (*e.g.* the width), while in wrinkling, the wavelength is governed by the local lambda law and thus depends on the thickness (*i.e.* \( \lambda \sim (B/K)^{1/4} \sim h^\alpha \), \( \alpha > 0 \), where we use \( \alpha \) instead of a specific power because the stiffness \( K \) can also depend on \( h \) in more complicated problems.). Second, the compressive stress in wrinkling is usually much larger than in buckling due to the resistance of substrate, and consequently, the threshold confinement \( \Delta_c \) of wrinkling is much larger too. Table 1.1 is a summary of these differences.
<table>
<thead>
<tr>
<th>Buckling</th>
<th>Wrinkling</th>
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<tbody>
<tr>
<td><img src="image1" alt="Diagram of Buckling" /></td>
<td><img src="image2" alt="Diagram of Wrinkling" /></td>
</tr>
</tbody>
</table>

Table 1.1: Compare of buckling and wrinkling.

1.4 Outline of the dissertation

In chapter 2, we explain how wrinkling happens when a rectangular sheet with clamped transverse edges is subject to longitudinal tension. We point out that the origin of the transverse compression is relative edge extension and the near-threshold instability is buckling type. Next, we address the far-from-threshold regime in this system.

In chapter 3, we address different wrinkling patterns under the existence of spatially varying wavelength. We develop a simple model, in which a rectangular sheet is floating on a substrate with a stiffness gradient. In our simulations, we have observed aligned defects and amplitude modulations. Motivated by these observations we seek to develop an effective theory for “non-ideal” wrinkling patterns.

In chapter 4, we collect all technique topics of elastic simulations with *Surface Evolver*. As surface evolver is not designed for elastic surfaces, simulating elastic sheets needs some nontrivial workarounds. We introduce these techniques and give example implementations of simulations in chapters 2 – 3.
CHAPTER 2
STRECH-INDUCED WRINKLING

2.1 Introduction

In this chapter we will introduce a familiar but nontrivial wrinkling problem — the stretch induced wrinkling, which happens when a rectangular sheet is subjected to longitudinal tension with two transverse edges clamped (see Fig. 2.1a). The emergent pattern is an array of wrinkles parallel to the tension [12]. Recalling the buckling and wrinkling instabilities we introduced in Sect. 1.3, which appear in response to compression, one may readily conclude that the wrinkles parallel to the stretching axis \( \hat{x} \), appear due to a compressive component of the stress tensor, along the transverse axis, \( \hat{y} \). However, while a transverse contraction (i.e. negative strain, \( \varepsilon_{yy} < 0 \)) in response to longitudinal tension, \( \sigma_{xx} \approx T > 0 \), is the essence of the Poisson’s effect, the appearance of transverse compression (i.e. \( \sigma_{yy} < 0 \)) is far less obvious. Indeed, if the pulled edges were not clamped, the whole sheet would have contracted uniformly in the transverse direction, the stress would have been perfectly uniaxial and tensile everywhere (i.e. \( \varepsilon_{yy} \propto -\nu T \), where \( \nu \) is the Poisson ratio, and \( \sigma_{xx} = T \), \( \sigma_{yy} = \sigma_{xy} = 0 \)), and the planar, unwrinkled state, would have been stable. Hence, the emergence of transverse compression is necessarily a boundary effect, which may exist only near the clamped edges. While numerical and analytical
Figure 2.1: (a) Schematic drawing of “model A”: a rectangular sheet with width $W$ and length $L$, subjected to longitudinal tensile loads, $T$ (force/length) that pull on the two short edges, $x = \pm L/2$, while the long edges are free. The short edges are clamped, such that both normal (out-of-plane) displacement, $\zeta(x = \pm L/2, y)$, and transverse (in-plane) displacement, $u_y(x = \pm L/2, y)$, vanish, and their longitudinal displacements are $u_x(x = \pm L/2, y) \approx \pm \tilde{T} L/2$. (b) Schematic drawing of “model B”: a similar sheet is subjected to uniform stretching of its two short edges, such that $\partial_y u_y(x = \pm L/2, y) = \alpha$, while $\zeta(x = \pm L/2, y) = u_x(x = \pm L/2, y) = 0$.

studies of the planar (unwrinkled) state did reveal the presence of zones with small transverse compression close to the clamped edges [24, 38], the physical mechanism underlying this boundary effect remains elusive.

Even more puzzling than the mere existence of transversely compressed zones in the planar state is the ensuing elastic instability. Rather than forming a buckling pattern, characterized by a thickness-independent topography that relieves the transverse compression, the stretched sheet appears to develop a highly corrugated topography, whereby the characteristic wavelength $\lambda$ of transverse undulations has been reported to be proportional to the square root of the sheet’s thickness $h$ [14]. Such a thickness-dependent wavelength resembles the wrinkling problem we discussed in Sect. 1.3.2, suggesting the deformation is a wrinkling. However, there is no physical substrate attached to the sheet which is necessary in wrinkling.
Realizing that the observed wrinkle pattern in this system cannot be described by a standard “post-buckling” approach, in which the out-of-plane deflection is assumed a perturbation of the planar state [58], numerous researchers employed non-Hookean models, attempting to capture the elastic response of the stretched sheet at finite \((O(1))\) strain [39, 27, 34, 25, 52, 40, 66, 43, 32]. However, while certain aspects of this problem do indeed stem from non-Hookean response (most notably, the reentrance of a stable planar state when the exerted tensile strain exceeds a finite value, typically 0.3 – 0.4 [27, 34, 66, 52, 40], the transition from the near-threshold localized buckling shape at \(T \gtrsim T_c\) (NT) to a spatially extended wrinkle pattern at the far from threshold regime, \(T \gg T_c\) (FT), does not stem from non-Hookean response. Instead, the FT wrinkle pattern can be fully characterized by the framework of Hookean elasticity, in which the stress tensor (averaged throughout the thickness of the sheet) has linear dependence on the corresponding strain tensor, but the nonlinear effect of the out-of-plane deflection on the strain within the sheet is taken into consideration. In fact, this “mechanically linear” (i.e. Hookean stress-strain relationship), yet “geometrically-nonlinear” (i.e. rotationally-invariant displacement-strain relationship) approach to elasticity underlies the Föppl-von Kármán (FvK) equations that we introduced in the previous chapter, and was shown to describe quantitatively fully developed wrinkle patterns in a variety of examples [60, 47, 57, 56, 44, 17, 23, 7].

The essential reason that a Hookean, geometrically-nonlinear framework suffices to explain the fully developed wrinkle pattern was noted in a seminal 2003 paper of Cerda & Mahadevan (CM) [13]. Since for very thin sheets the threshold tensile load may be arbitrarily small – more precisely, \(T_c \sim Y \left(\frac{t}{W}\right)^2\) [67], where \(Y\) is the stretching
modulus – the FT regime $T \gg T_c$ is reached while the characteristic strain $(T/Y)$ remains very small, such that Hookean response is a valid approximation everywhere in the deformed sheet. Motivated by this observation, these authors introduced a model to describe the Hookean FT regime, $T_c \ll T \ll Y$, assuming that the formation of wrinkles affects a strong, non-perurbative deviation of the stress field from the planar stress. The mechanism invoked by the CM model is strictly distinct from standard “post-buckling” analysis, which assumes that the planar stress is only mildly perturbed (and therefore characterizes the buckled shape in the NT regime, $T \gtrsim T_c$). In the CM model wrinkles are assumed to expand through the whole length $L \gg W$ of the sheet, rather than being confined to the transversely-compressed zones of the planar state, and the wrinkle wavelength $\lambda$ and amplitude $A$ are determined by effective rules that interweave mechanics and geometry, yielding:

$$\lambda \sim (B/K_{\text{eff}})^{1/4} = \left(\frac{Y}{T}\right)^{1/4}L^{1/2}t^{1/2}$$  \hspace{1cm} (2.1a)$$

$$A \sim \sqrt{\nu \frac{T}{Y}} \lambda$$  \hspace{1cm} (2.1b)$$

where $K_{\text{eff}} = T/L^2$ is the stiffness of a tension-induced “effective substrate” that governs the resistance to out-of-plane deflection. Briefly, Eq. (2.1a) is essentially the local lambda law Eq. 1.59, but now the “substrate resistance” is given by the tension together with the amplitude suppression at two short edges. Equation (2.1b) follows from a second, “transverse inextensibility” assumption: “As the sheet wrinkles in the $y$ direction under the action of a small compressive stress, it satisfies the condition of
According to this assumption, wrinkles do not emerge to relax (transverse) compressive stress but rather to prevent transverse strain, \( \epsilon_{yy} \approx -\nu T/Y \) (which is the transverse contraction of the sheet in the planar, unwrinkled state of the stretched sheet).

The CM model [13] attracted a remarkable level of interest and provoked research activity that far exceeded its original realm of application. Specifically, the proposal that wrinkle patterns in thin solid bodies should be considered far-from-threshold phenomena and correspondingly be analyzed through a theoretical framework that is sharply distinct from traditional post-buckling methods inspired a multitude of experimental and theoretical studies in ultathin sheets subjected to confinement by capillary effects or other forces [64, 45, 4, 59, 63, 2, 30, 41, 16, 22, 62, 21, 61]. In particular, these studies provided strong support to the reasoning underlying CM principle (2.1a) that determines the wrinkle wavelength: a balance between the bending modulus and the stiffness of an effective substrate, which may be an actual foundation, or induced by a boundary load or curved topography that imply tension perpendicularly to the compressed axis [35, 44].

Nevertheless, the validity of the second CM principle (2.1b) has been challenged by observations that the wrinkle amplitude in experiments and simulations is substantially smaller than this prediction [39, 27, 34] (even at the Hookean regime, where the amplitude is observed to increase with applied tension [43]). Furthermore, the mere rationale of the transverse inextensibility assumption underlying Eq. (2.1b) is confounding. According to this assumption, wrinkles emerge to prevent the transverse contraction in the bulk of the stretched sheet (i.e. away from the clumped
edges) and one would thus expect to observe wrinkling even if the pulled edges were not clamped, in which case the whole sheet contracts transversely. Putting it in more formal terms, according to Hookean mechanics a vanishing transverse strain in a sheet under longitudinal tension \( \sigma_{xx} = T \) implies transverse tensile stress \( \varepsilon_{yy} = 0 \implies \sigma_{yy} = \nu T \), whereas a vanishing transverse compression implies a transverse contractive strain \( \sigma_{yy} = 0 \implies \varepsilon_{yy} = -\nu T / Y \). Hence, the CM assumption of vanishing transverse strain appears to be at odds with the Poisson effect, which posits that the minimization of elastic energy is attained by eliminating transverse stress. Thus, paradoxically, for a sheet under longitudinal tensile load \( T \gg T_c \), the CM Eq. (2.1b) implies that the elastic energy of a wrinkled state is larger than the corresponding energy of a planar state!

Let us then summarize the current state of this tension-induced wrinkling problem. There are still some basic puzzles concerning the instability of the planar state, and some misunderstanding that has been sowed by the CM model despite its success in describing some essential aspects of the far-from-threshold behavior. In this chapter, we revisit the problem, and address the planar state, its instability, and the transition from the NT regime, \( T \gtrsim T_c \), to the far from threshold regime. Following this part, we seek to clarify the obscure aspects of the CM model through a theoretical framework, known as “FT analysis” [20, 21], that has been applied successfully for studying various wrinkling problems – a systematic expansion of the FvK energy around the singular limit of a hypothetic, infinitely bendable sheet, which cannot accommodate any compressive stress, and its stress field is the subject of tension field theory (TFT) [65, 55, 36, 48, 54]. A central part of this approach is that the
transverse (compressive) stress, rather than the transverse strain, vanishes with the bendability of the sheet, yielding a “slaving condition” between the wrinkle amplitude and wavelength. A primary tool that we employ in our studies is numerical simulations with Surface Evolver (SE) [8], which we find to be an excellent method for finding the planar state as well as the energetic minimum of fully developed wrinkled states in the far-from-threshold regime of very thin sheets. Combining theoretical considerations and numerical simulations we offer a modified version of the CM model for the Hookean FT regime in this problem, which is compatible with the Poisson effect’s rationale, and revise accordingly its central prediction, Eq. (2.1a).

Starting in Sect. 2.2 with numerical and analytical study of the planar state, we show that the ultimate cause of transverse compression is the extension of the clamped edges relative to the transversely contracted bulk of the sheet. We elucidate this subtlety by analyzing a specific set-up (Fig. 2.1b) and demonstrate how transverse compression in a rectangular sheet with free long edges can occur even without exerting longitudinal tensile load. In Sect. 2.3 we address the instability of the planar stress and show that it is essentially an Euler buckling, whose spatial extent is restricted to the transversely-compressed zones of the sheet. A direct corollary of this observation is that the emergence of wrinkles in this set-up, whereby the wavelength vanishes with the thickness of the sheet, does not occur at the near-threshold regime \( T \gtrsim T_c \); instead, it may only be observed in the far-from-threshold regime, \( T \gg T_c \).

In Sect. 2.4 we provide evidence for the transition from near-threshold buckling to far-from-threshold wrinkling.
Then, starting from Sect. 2.5, we turn our focus to FT wrinkling. We describe the general principles of TFT and the corresponding FT analysis of the wrinkle pattern, and provide a revised version of the amplitude-wavelength ratio (Eq. 2.1a) in terms of a “confinement function”, $\Phi^{TFT^2}(x)$, that emanates from TFT and characterizes the fraction of transverse arclength that must be “wasted” by wrinkles in order to ensure an asymptotically compression-free stress field in the stretched sheet. In Sect. 2.6 we present results of our numerical simulations in the FT regime, showing that the emergence of wrinkles comes in tandem with an intricate collapse of the transverse compressive stress, whereby the compression level vanishes asymptotically (as $T/T_c \to \infty$ while $T \ll Y$) in comparison to the corresponding planar state, but the spatial extent of the transversely compressed zones is increased. These numerical results substantiate the rationale underlying the FT analysis and highlight similarities and differences with other tensional wrinkling phenomena. In Sect. 2.7 we describe numerical simulations of a hypothetic sheet with finite stretching modulus ($Y \gg T$) but no bending modulus ($B = 0$), which allows us to obtain numerically the tension field limit of a compression-free stress field. We extract from these simulations the confinement function $\Phi^{TFT^2}(x)$, and show how it encapsulates the intrinsic geometrical nonlinearity that stems from infinitesimal out-of-plane deflections on the in-plane transverse strain, even though the exerted longitudinal tensile strain $T/Y$ may be arbitrarily small (so that Hookean mechanics is valid). In Sect. 2.8 we turn to discuss the various aspects of the wrinkle pattern, specifically the wavelength $\lambda$, and the amplitude-wavelength ratio. We elucidate some subtlety in evaluating the dependence of the effective, tension-induced stiffness, on the width $W$ and length $L$.
of the sheet. While our numerical simulations strongly support the dependence of λ on the tensile load $T$ and bending modulus $B$ of the sheet, we argue that the length’s dependence predicted in the CM model, Eq. (2.1a), may not necessarily be valid for $L \gg W$. In Sect. 2.9 we conclude with a summary of results and a discussion of open questions.

2.2 The planar state

The observation that longitudinal tension does not induce transverse compression if the short edges are unclamped (or alternatively if $\nu \leq 0$) suggests that the primal cause for transverse compression is neither uniaxial tension nor positive Poisson ratio, but rather a relative extension of the short edges in comparison to the bulk of the sheet. In order to elucidate this geometrical-mechanical effect, we contrast in this section the planar state of our set-up, hence called “model A” (Fig. 2.1A), with the planar state of another system, called “model B” (Fig. 2.1B), in which a relative extension of the edge is imposed directly on a rectangular sheet, with arbitrary Poisson ratio and no longitudinal tension.

For the planar state, we can solve the equation for the Airy potential in Eq. 1.41b, while here the deflection $\zeta = 0$, while here the deflection $\zeta = 0$,

$$\Delta^2 \Phi = 0$$  \hspace{1cm} (2.2)

2.2.1 model A versus model B

The mathematical description of our original set-up (“model A”, Fig. 2.1A) consists of a non-homogeneous BC:
\[
\int_{-\frac{W}{2}}^{\frac{W}{2}} \sigma_{xx}(x,y)dy = TW, \quad (2.3a)
\]

expressing the fact that a force \(TW\) is pulling each of the short edges outward (and applies also for any \(-\frac{1}{2}L \leq x \leq \frac{1}{2}L\) by force balance consideration). Additionally, there are four homogeneous BCs:

1. at \(y = \pm \frac{1}{2}W\): \(\sigma_{yy} = \sigma_{xy} = 0\) \hfill (2.3b)
2. at \(x = \pm \frac{1}{2}L\): \(u_y = 0\); \(\frac{\partial u_x}{\partial y} = 0\). \hfill (2.3c)

The first two equations (2.3b) reflect the fact that the long edges are free, namely, \(\sigma_{ij}n_j = 0\), where \(\hat{n} = \pm \hat{y}\) is the outward normal to the long (undeformed) edges, respectively. The last two equations (2.3c) imply that the short edges are displaced as rigid, inextensible sticks, pulled apart along the \(\hat{x}\) axis, such that their displacement is given by \(u_x = \text{const}\) and \(u_y = 0\).

With the aid of Eq. 1.40 and the BCs 2.3a-c may be converted to a set of four BCs for the Airy potential:

1. at \(y = \pm \frac{1}{2}W\): \(\Phi = 0\) \hfill (2.3d)
2. and \(\frac{\partial \Phi}{\partial y} = \pm \frac{1}{2}TW\) \hfill (2.3e)
3. at \(x = \pm \frac{1}{2}L\): \(\frac{\partial^2 \Phi}{\partial x^2} - \nu \frac{\partial^2 \Phi}{\partial y^2} = 0\) \hfill (2.3f)
4. and \(\frac{\partial^3 \Phi}{\partial x^3} + (2+\nu) \frac{\partial^3 \Phi}{\partial x \partial y^2} = 0\). \hfill (2.3g)
Equation (2.3d) follows from the first part of (2.3b), which implies that \( \Phi(x, y = \pm \frac{1}{2}W) = C_0 + C_1x \), with arbitrary constants \( C_0, C_1 \), upon choosing the natural gauge: \( C_0 = C_1 = 0 \). Integrating \( \sigma_{xy}(x, y = \pm \frac{1}{2}W) \) over \( x \) one readily obtains from the second part of Eq. (2.3b) that \( \partial_y \Phi(x, \pm \frac{1}{2}W) \) is independent on \( x \), and Eq. (2.3e) is thus obtained directly from Eq. (2.3a), which is valid – as explained above – for any \(-\frac{1}{2}L \leq x \leq \frac{1}{2}L\). Using the strain-displacement relations (1.6), Eq. (2.3f) follows directly from the first part of Eq. (2.3c), whereas Eq. (2.3g) follows from the second part of (2.3c) after some tedious, but straightforward algebraic manipulations.

As for the BCs for “model B”, they are quite similar to the BCs of “model A”, with only two differences. First, the tension is now absent. Second, the transverse edges are extended, \( i.e. \) the edges have a positive strain \( \varepsilon_{yy}(x = \pm \frac{1}{2}L) = \alpha > 0 \), such that:

\[
\int_{-\frac{W}{2}}^{\frac{W}{2}} \sigma_{xx}(x, y) dy = 0 \quad (2.4a)
\]
\[
\frac{\partial u_y(x = \pm \frac{1}{2}L)}{\partial y} = \varepsilon_{yy}(x = \pm \frac{1}{2}L) = \alpha \quad (2.4b)
\]

Converting to Airy potential,

at \( y = \pm \frac{1}{2}W \) :
\( \Phi = 0 \) \quad (2.4c)
and \( \frac{\partial \Phi}{\partial y} = 0 \) \quad (2.4d)

at \( x = \pm \frac{1}{2}L \) :
\[
\frac{\partial^2 \Phi}{\partial x^2} - \nu \frac{\partial^2 \Phi}{\partial y^2} = \alpha Y \quad (2.4e)
\]
and \( \frac{\partial^3 \Phi}{\partial x^3} + (2 + \nu) \frac{\partial^3 \Phi}{\partial x \partial y^2} = 0 \). \quad (2.4f)
Note that the bi-harmonic equation 2.2 is linear, and the two models both have a single non-homogeneous BC (Eq. 2.3a for “model A”, Eq. 2.4b for “mode B”), therefore the stress field of each model is unique up to a scale factor. Namely, denoting the planar stress field for model A under a given exerted longitudinal tension \( T \) by \( \sigma_{ij}(x, y; T) \), and the planar stress field for model B under a given edge extension \( \alpha \) by \( \sigma_{ij}(x, y; \alpha) \), we have that:

\[
\text{model A : } \frac{\sigma_{ij}(x, y; T_2)}{T_2} = \frac{\sigma_{ij}(x, y; T_1)}{T_1} \\
\text{model B : } \frac{\sigma_{ij}(x, y; \alpha_2)}{\alpha_2} = \frac{\sigma_{ij}(x, y; \alpha_1)}{\alpha_1}
\]  

(2.5)

2.2.2 Numerical Simulations

We employ SE to study the planar state of the two models, implementing an equilateral-triangular mesh of density \( 6.95 \times 10^5 \) \((\text{total area/cell area})\). The SE built-in “linear_elastic” method is adapted for computing the strain energy, and “star_perp_sq_mean_curvature” and “star_gauss_curvature” to compute bending energy.

For model A, we consider a sheet with a relatively large length-to-width ratio, \( \frac{L}{W} = 8 \), a Poisson ratio \( \nu_A = 0.32 \), and some exerted longitudinal tension \( T \) whose actual numerical value is not important as long as it is sufficiently small for consistency with Hookean elasticity (see Eq. 2.5). For model B, we consider a sheet with the same length-to-width ratio, \( \frac{L}{W} = 8 \), and Poisson ratio \( \nu_B = \nu_A \) or \( \nu_B = 0 \). Since in model B \( u_{yy} = 0 \) in the bulk, we make the extension of the short edge relative to
the bulk identical to model A (where \( u_{yy} = 0 \) at the clamped edge and \(-\nu_A T / Y\) in the bulk), by choosing the edge extension parameter in model B to be \( \alpha = \nu_A T / Y \).

![Figure 2.2](image)

**Figure 2.2:** (a) The transverse component of the planar stress, evaluated from our SE simulations along the midline, \( \sigma_{yy}(x, y = 0) \), of the stretched sheet (model A, dashed-dotted red curve) and the corners-pulled sheet (model B, with \( \nu_B = 0 \), solid blue curve). Also plotted is the analytic solution (dashed green). (b) A plot analogous to (a) for the corners-pulled sheet (model B) but with Poisson ratio \( \nu_B = \nu_A \) (red) rather than \( \nu_B = 0 \) (dashed blue) exhibits an almost indistinguishable profile of the transverse stress. Note this is from analytical solution for semi-infinite sheet where the clamped edge is at \( x = 0 \).

While the longitudinal stress components, \( \sigma_{xx}(x, y) \), of the two models are obviously distinct, Figs. 2.2 shows that the transverse stress, \( \sigma_{yy}(x, y) \) in the two models is essentially identical. Furthermore, the direct effect of the Poisson ratio is negligible, as can be seen by comparing the transverse stress of model B with \( \nu_B = \nu_A \) and \( \nu_B = 0 \). The transverse stress is positive (tensile) in the vicinity of the short edges, becoming compressive at a distance \( \approx 1.5 \cdot W \) from each short edge, and remains compressive over a strip of length \( \sim W \), after which it vanishes exponentially.
Figure 2.3: The identical transverse stress profile in model A and model B indicates that the ultimate cause for a transversely compressed zone in the planar state is a relative extension of the short edge with respect to the bulk. This may be a direct outcome of pulling the short edge outward (model B) or the collective effect of imposing longitudinal tension, clamping the short edge, and positive Poisson ratio (model A).

Our numerical solution of the planar stress in the two models indicates that the essential cause of transverse compression in a rectangular sheet is the extension of the short edge relative to the bulk. As the schematic in Fig. 2.3 shows, this effect can be attained directly (as is the case in model B) even for a sheet with $\nu = 0$ with no longitudinal tensile load, or indirectly – as in our original model A – by applying longitudinal tension and clamping the short edges of a sheet with positive Poisson ratio.

2.2.3 Analytical solution

2.2.3.1 Equivalence of model A and model B

The identity of the transverse stress components in models A and B can be understood by decomposing the Airy potential of model A into “bulk” and “edge”
terms:

\[
\text{model A : } \Phi(x, y) = \Phi_b(x, y) + \Phi_e(x, y)
\]

where: \( \Phi_b(x, y) = \frac{1}{2} T (y^2 - 1) \). \hspace{1cm} (2.6a)

If the short edges were not clamped (i.e. if the first part of Eq. (2.3c) had been replaced by \( \partial_y^2 u_y = 0 \), such that \( u_{yy}(x = \pm \frac{1}{2}L, y) \) may be nonzero), then \( \Phi_e = 0 \), and the resulting stress, associated only with \( \Phi_b \), would have been constant: \( \sigma_{xx} = T, \sigma_{yy} = \sigma_{xy} = 0 \). However, clamping implies that \( \Phi_e(x, y) \neq 0 \), since it must satisfy the nonhomogenous set of BCs:

\[
\text{at } y = \pm \frac{1}{2}W : \quad \Phi_e = 0 \quad \hspace{1cm} (2.6b)
\]

\[
\text{and } \quad \frac{\partial \Phi_e}{\partial y} = 0 \quad \hspace{1cm} (2.6c)
\]

\[
\text{at } x = \pm \frac{1}{2}L : \quad \frac{\partial^2 \Phi_e}{\partial x^2} - \nu_A \frac{\partial^2 \Phi_e}{\partial y^2} = \nu_AT \quad \hspace{1cm} (2.6d)
\]

\[
\text{and } \quad \frac{\partial^3 \Phi_e}{\partial x^3} + (2 + \nu_A) \frac{\partial^3 \Phi_e}{\partial x \partial y^2} = 0. \quad \hspace{1cm} (2.6e)
\]

Remarkably, the BCs (2.6b-2.6e) are identical to the BCs satisfied by the Airy potential \( \Phi(x, y) \) of model B (2.3d-2.3g), with \( \nu_B = \nu_A \), and edge extension \( \alpha = \nu_AT/Y \)!

This observation immediately explains our numerical result: the planar stress field of the original problem (model A) is identical (up to a constant, purely uniaxial stress, \( \sigma_{xx} = T, \sigma_{yy} = \sigma_{xy} = 0 \)) to the stress field in a sheet whose short edges are pulled outward, and no longitudinal tension.
2.2.3.2 The origin of transverse compression

The above discussion reveals that the origin of transverse compression in a longitudinally-stretched sheet whose short edges are clamped is the “edge-induced” potential $\Phi_e(x, y)$ in the decomposition (2.6a), or equivalently the Airy potential of our model B. It is thus natural to seek a solution using a basis of eigenfunctions of the bi-harmonic equation:

$$
\Phi_{i,c}(x, y) = e^{-\frac{p_i x}{W}} \cos(p_i \frac{y}{W}),
$$
$$
\Phi_{i,s}(x, y) = e^{-\frac{p_i x}{W}} \frac{y}{W} \sin(p_i \frac{y}{W}),
$$
(2.7a)

where $\{p_i\}_{i=1}^\infty$ is a discrete set of (generally complex) eigenvalues, which must be determined through the BCs (2.4c-2.4f), and the symmetry: $\Phi(x, y) = \Phi(x, -y)$ has been exploited.

Assuming $\{\text{Re } p_i\} > 0$, the basis functions (2.7a) describe deformations that decay as $x/W \to \infty$, hence – supplemented by the analogous set of functions ($p_i \to -p_i$) that describes deformations that decay as $x/W \to -\infty$ – this basis is useful to describe deformations of a very elongated sheet, namely, $L/W \gg 1$. In the following calculation we consider this limit (which may be loosely called an “infinitely long” sheet). Our numerical simulations show that for $L/W$ larger than 3-4, the stress profile is nearly indistinguishable from the one obtained by a calculation based on the basis functions (2.7a) and the assumption $L/W \gg 1$. Let us consider then $-L/2 < x < 0$ and express the solution $\Phi_e(x, y)$ through the basis functions (2.7a):
\[
\Phi_e(x,y) = \text{Re}\left\{ \sum_i C_i \left( \Phi_i^{e,c}(x,y) + A_i \Phi_i^{e,s}(x,y) \right) \right\}, \quad (2.7b)
\]

In order to determine the eigenvalues \( \{p_i\} \) and the sequence of coefficients \( \{A_i\}, \{C_i\} \), we must apply the BCs (2.6b-2.6e). This is a rather tedious process, which requires an inverse Laplace transform [5], and we thus defer it to appendix A.1. Nevertheless, it is useful to note that the eigenvalues \( \{p_i\} \) are determined by the equation:

\[
p_i + \sin p_i = 0 , \quad (2.7c)
\]

and in turn determine the coefficients \( \{A_i\} \) through the simple relation:

\[
A_i = -2 \cot \frac{p_i}{2} , \quad (2.7d)
\]

while obtaining the sequence \( \{C_i\} \) requires an explicit evaluation of an inverse Laplace transform. The numerical values of \( \{|p_i|\} \) and \( \{|C_i|\} \) are plotted in Fig. 2.4 (for \( 1 \leq i \leq 20 \), where the order is determined by increasing \( |p_i|'s \)). Remarkably, Eqs. (2.7c, 2.7d) reveals two facts on whose importance we will elaborate below:

(a) All eigenvalues \( p_i \) are non-real numbers. (b) the eigenvalues \( p_i \) and the coefficients \( A_i \) do not depend on the Poisson ratio \( \nu_B \), and approach a well-defined limit values as the length-to-width ratio \( \frac{L}{W} \to \infty \).

In Table 1 we report the values of the three eigenvalues \( p_i \) with the smallest (positive) real parts, which govern the sum (2.7b), together with the corresponding values of \( A_i \) and \( C_i \). Note that \( C_i \) is proportional to \( \alpha \), so that upon normalizing by \( \alpha \) they depend only on the Poisson ratio.

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Table 2.1: The values of the three leading poles \( p_i \), and corresponding pairs of coefficients \( A_i, C_i \) in the expansion, Eq. (2.7b), for the Airy function. The values of the coefficients \( C_i \) are normalized by the edge extension \( \alpha \) (model B) or \( \nu_A T / Y \) (model A). Note that the pole \( p_i \) and coefficients \( A_i \) are independent on the Poisson ratio.

<table>
<thead>
<tr>
<th>( i )</th>
<th>( p_i )</th>
<th>( A_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( i = 1 )</td>
<td>4.212 + 2.251j</td>
<td>0.332 + 1.78j</td>
</tr>
<tr>
<td>( i = 2 )</td>
<td>10.713 + 3.103j</td>
<td>0.168 + 1.94j</td>
</tr>
<tr>
<td>( i = 3 )</td>
<td>17.073 + 3.551j</td>
<td>0.111 + 1.97j</td>
</tr>
</tbody>
</table>

\[ \nu_B = 0.32 \times 10^{-3} \quad \nu_B = 0 \times 10^{-3} \]

| \( C_1 \) | 109 – 213j | 131 – 229j |
| \( C_2 \) | –22.9 + 13.5j | –26.1 + 11.7j |
| \( C_3 \) | 8.68 – 2.84j | 9.49 – 1.52j |

Figures 2.2a shows that approximating the Airy potential through the first term in the sum (2.7b) (green dashed line), \( \text{i.e.} \)

\[
\Phi_e(x, y) \approx \text{Re}\{C_1 e^{-p_1 x W} \left[ \cos\left(\frac{p_1 y W}{2}\right) - 2 \cot\left(\frac{p_1}{2}\right) \frac{y}{W} \sin\left(\frac{p_1 y W}{2}\right) \right]\} \tag{2.7e}
\]

matches already very well the transverse stress obtained in the numerical solution at the vicinity of the short edges of a sheet with \( \frac{L}{W} = 8 \). Furthermore, the analytic solution of the edge-induced Airy function \( \Phi_e(x, y) \), given by Eq. (2.7b) with the numerical values of \( \{p_i, A_i, C_i\} \) in Table 2.1, provides some valuable insights into the mechanism by which transverse compression develops in an elongated sheet.
Figure 2.4: The magnitudes of the first 20 eigenvalues $|p_i|$ and coefficients $|C_i|$, show that $|p_i|$ increases in constant increments while $|C_i|$ is decreasing quickly, and thus only the leading term contributes significantly to the solution.

- First, Table 1 indicates that the dependence on Poisson’s ratio, which stems only from the sequence $\{C_i\}$, is very weak. This observation, which has been noted already in our numerical analysis, substantiates the rationale illustrated in Fig. 2.3 – the primary cause of transverse compression is the extension of the short edges relative to the bulk, rather than the Poisson ratio of the sheet.

- Second, the unavoidable presence of transversely compressed zone in a sufficiently long sheet is a direct consequence of the fact that all eigenvalues $\{p_i\}$, namely, roots of Eq. (2.7c), are complex. The implication is revealed by evaluating $\sigma_{yy}$ from the approximated Airy potential (2.7e) along the centerline ($y = 0$) of a semi-infinite sheet:

$$\sigma_{yy}(x, y = 0) \propto e^{-p_1^{(r)} \frac{x}{W}} \cos[p_1^{(i)} \frac{x}{W} + g] ,$$

$$g = \tan^{-1} \left( \frac{C_1^{(i)} p_1^{(r)} + 2C_1^{(r)} p_1^{(i)} p_1^{(r)} - C_1^{(i)} p_1^{(i)} p_1^{(r)}}{C_1^{(r)} p_1^{(i)} + 2C_1^{(i)} p_1^{(i)} p_1^{(r)} - C_1^{(r)} p_1^{(i)} p_1^{(r)}} \right)$$

(2.7f)
where the superscripts \((i)\) and \((r)\) refer to the imaginary and real parts, respectively.

It is evident from the first line of Eq. (2.7f) that the imaginary component of the root, \(p_1^{(i)} \neq 0\), gives rise to negative (i.e. compressive) transverse stress at 
\[ -\frac{1}{2}L + d^*W < x < -\frac{1}{2}L + d^*_mW \quad \text{and} \quad \frac{1}{2}L - d^*_mW < x < \frac{1}{2}L - d^*W, \]
where:
\[
\begin{align*}
d^* &= (\pi/2 - g)/p_1^{(i)} \approx 0.646 \\
d^*_m &= d^* + \tan^{-1}(p_1^{(i)}/p_1^{(r)})/p_1^{(i)} \approx 0.864.
\end{align*}
\]

\[ (2.7g) \]

- Third, Eq. (2.7g), indicates that the response of a rectangular sheet whose short edges are extended relative to the bulk, can be classified into three types, depending solely on the aspect ratio, \(L/W\):

(I) For \(L/W < 2d^*\) there is no transverse compression. Here, the transverse stress, which is obviously tensile at the far edges \((x = \pm \frac{1}{2}L)\), does not have enough room to vary significantly, hence the whole sheet is under pure (biaxial) tension.

(II) For \(2d^* < L/W < 2d^*_m\) there is a single transversely-compressed zone located around the center of the sheet. Here, the sheet is sufficiently elongated such that the transverse stress has enough room to approach negative values away from the tensed edges, but not to overturn and decay to zero. Hence, the two compressive zones, generated by each of the tensed edges, are merged into a single one.
(II) If $\frac{L}{W} > 2d^*_m$ the sheet is long enough such that there are two transversely-compressed zones, each of them starts at a distance $d^*W$ from a tensed edge, and extends over a length $\propto W$. We note that since the eigenvalue $p_1$ is complex, the Airy potential $\Phi_e(x, y)$, Eq. (2.7e), gives rise to additional transversely-compressed zones, away from the clamped edges. However, the exponential decay of $\Phi_e(x, y)$ with the corresponding distance $(|x \pm L/2|)$ implies that the magnitude of compression in these zones is much smaller in comparison to the transverse compression in the first zone. Since the stability of the planar state to buckling instability is determined by the maximal compression, these additional compressive zones have thus little effect on the mechanics.

Fig. 2.5 shows the transverse stress profile, obtained from our simulations for several representative values of the aspect ratio $L/W$, supporting the above classification into three regimes. We note that the actual values of $d^*$ and $d^*_m$ obtained from our simulations are rather close, but not identical, to the theoretical prediction, Eq. (2.7g). An obvious reason for this discrepancy is that the values of $d^*, d^*_m$, reported in Eq. (2.7g), are obtained from an analytic solution of the transverse stress in a semi-infinite sheet (i.e. $\frac{L}{W} \to \infty$), and we may thus expect corrections of $O(\frac{W}{L})$ to this predictions. From this viewpoint, Fig. 3 indicates that those corrections to are in fact surprisingly small. Thus, while the above classification has been noted before by numerous workers (e.g. [39]), our analytic approach elucidates the origin of this classification through the complex values of the eigenvalues $\{p_i\}$ of the bi-harmonic...

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Figure 2.5: Our SE simulations of the planar state in Hookean sheets with various aspect ratios, $\tilde{L} = L/W$. In order to conveniently show the values of $d^*$ and $d^*_m$, we shift the $x$ axis, $x \rightarrow x + \frac{L}{2}$, such that the left clamped boundary (not shown) is at $x = 0$, and show only the left half of the sheet. The transverse stresses of different $L/W$ suggest the existence of a parameter regime (i) where the stress may be purely tensile ans the planar state is thus stable ($\frac{L}{W} < 2d^*$); (ii) where transverse compression exists around the middle of the sheet ($2d^* < \frac{L}{W} < 2d^*_m$); and (iii) where transverse compression exists in two zones ($\frac{L}{W} > 2d^*_m$). For $d^*$, the numerical values extracted from our SE simulations is $d^* \approx 0.625$, whereas for $d^*_m$ the extracted value is $d^*_m \approx 0.8 – 1.0$ (the uncertainty is a consequence of a very shallow local maximum of $\sigma_{yy}(x, 0)$.) x
equation under the BCs (2.4c,2.4d).

- Finally, we note that employing the basis functions (2.7a) yields a rapidly converging sequence and thereby the compact expression (2.7e) that describes quantitatively the stress field throughout the whole sheet. This global approach, which has been employed broadly for solving the bi-harmonic equation in viscous fluid mechanics and linear elasticity problems [5, 53] is thus advantageous to an approximation using “corner functions” [17] that does not explain the emergence of transverse compression.

2.3 Buckling Instability

In section 1.3, we introduced the basic natures of buckling and wrinkling instabilities, and listed the differences between them in Table 1.1. In this section, we will invoke these concepts to elucidate the instability in this problem. Before that, let us briefly review several other related cases.

2.3.1 Instability under uniform compression

Except the buckling and wrinkling we mentioned in Sect. 1.3 (Fig 2.6a-b), there is a variant of instability under uniform, uniaxial compressive load \( \sigma_{yy} = -\sigma_0 \), which is depicted in Figs. 2.6c-d. Here, the amplitude is suppressed at the edges \( x = \pm \frac{1}{2} \ell \), such that a shape \( A \cdot \zeta_\lambda(x, y) \) that undulates over a characteristic scale \( \lambda \) along the compressive axis (\( \hat{y} \)), must vary also along the \( \hat{x} \) axis, thereby being penalized also by the bending cost of the corresponding curvature, \( \propto A/\ell^2 \). For a given \( \lambda \), a planar
Figure 2.6: A schematic of a rectangular sheet under uniaxial compressive load, $\sigma_{yy} = -\sigma_0$. (a) A classic version of the Euler buckling instability, where the edges, $x = \pm \ell/2$, are free, and the unstable mode consists of a single undulation ($\lambda \sim W$), regardless of the length $\ell$. (b) Attachment to an elastic substrate ("Winkler foundation") of stiffness $K_{\text{sub}}$ enhances resistance to undulations and thereby the threshold $\sigma_c$, thereby reducing the wavelength. (c-d) If the amplitude is suppressed at the edges $x = \pm \ell/2$, the instability mode is affected by the length $\ell$, such that the threshold $\sigma_c$ is increased in comparison to (a) and the near-threshold undulation wavelength $\lambda_c$ is decreased. For $\ell \ll W$ the near-threshold pattern consists of periodic undulations of wavelength $\lambda_c \sim \ell$. (e) The presence of longitudinal tension, $\sigma_{xx} = T$, acts as an "effective substrate" of stiffness $K \sim T/\ell^2$ [12], affecting further reduction of the wavelength and correspondingly enhancement of the threshold value $\sigma_c$. 
state becomes unstable only if the compressive load exceeds, \( \sigma^*_0(\lambda) \sim B(1/\lambda^2 + \lambda^2/\ell^4) \), and if \( \ell \ll W \) we find that the planar state first becomes unstable to undulations of wavelength, \( \lambda_c \sim \ell \), at a threshold, \( \sigma_c \sim B/\ell^2 \sim Y(h/\ell)^2 \). Furthermore, if the sheet is subjected also to a tensile load \( \sigma_{xx} = T \) along the longitudinal (\( \hat{x} \)) axis (Fig. 2.6e), there will be yet another energetic penalty for undulations, \( \sim T(\frac{4}{\ell})^2 \), which is analogous to the energy implied by an actual (Winkler) substrate, \( K_{\text{sub}} \propto T/\ell^2 \).

Putting together the effects of a real substrate, amplitude-suppressing boundaries, and longitudinal tension, we find that, for a given wavelength \( \lambda \), the planar state of a rectangular sheet under uniform compression in the transverse (\( \hat{y} \)) axis becomes unstable if the compressive load \( \sigma_0 \) exceeds:

\[
\sigma^*_0(\lambda) \sim B \frac{1}{\lambda^2} + \left( K_{\text{sub}} + T \frac{1}{\ell^2} + B \frac{1}{\ell^4} \right) \lambda^2 ,
\]

(2.8a)

and hence the instability is characterized by a wavelength

\[
\lambda_c \sim \min \left( W , \frac{B}{K_{\text{sub}} + T \ell^{-2} + B \ell^{-4}} \right)^{1/4} \]

(2.8b)

and occurs as the compressive load exceeds a threshold

\[
\sigma_c \sim \max \left( BW^{-2}, \sqrt{B(K_{\text{sub}} + T \ell^{-2} + B \ell^{-4})} \right).
\]

(2.8c)

(The basic examples discussed in section 1.3 of Euler buckling and wrinkling in substrate-supported sheet are described by Eqs. 2.8b, 2.8c with \( T = \ell = 0 \).)
2.3.2 Why is the instability buckling-like?

Let us turn back now to our problem – where the transverse stress induced by the relative edge in nonuniform (Figs. 2.2, 2.5), namely $\sigma_{yy}(x, y)$ varies along both $\hat{x}$ and $\hat{y}$ axes. One can still perform a linear stability analysis of the planar stress to infinitesimal deflections, $\zeta(x, y) = A \cdot g_\lambda(x, y)$, which undulate with a characteristic wavelength $\lambda$ along the $\hat{y}$ axis and an infinitesimal amplitude $A$. However, the lack of translation symmetry of the planar state implies that the eigenfunctions, $g_\lambda(x, y)$, of the corresponding (linearized) energy functional are not simply sinusoidal Fourier modes. Nevertheless, as we explain below the physical mechanisms that determine the critical wavelength $\lambda_c$ and the threshold $\sigma_c$ for the uniform compression problem, Eqs. (2.8a-2.8c), are analogous to those that govern the instability of the nonuniform planar stress in our problem, allowing us to gain valuable insights.

Let us consider first model B, where the magnitude $\sigma_0 \propto \alpha \cdot Y$ of the transverse compressive stress is induced directly by the edge extension parameter, $\alpha$, and the stretching modulus $Y$. Here, there is no longitudinal tension ($T = 0$), and – since our sheet is unsupported (i.e. $K_{sub} = 0$) – nor there is a real substrate effect. Since the transverse compression in the planar state is limited to a narrow zone in the sheet, the compressive stress $\sigma_0^\star(\lambda)$ above which a undulation of wavelength $\lambda$ becomes favorable is subjected to the effect of an amplitude-suppressing boundaries discussed above (last term in Eq. 2.8a). However, since the length of the compressive zone is proportional to the sheet’s width (i.e. $\ell \propto W$), the overall effect on the critical wavelength $\lambda_c$ and threshold value $\sigma_c^\star \propto \alpha_c Y$ is inconspicuous, and we find the scaling:
model B: \( \alpha_c \sim \frac{B}{YW^2} \sim \left( \frac{h}{W} \right)^2 \); \( \lambda_c \sim W \).

(2.9)

Turning now to our original problem (model A), where the magnitude of the transverse compressive stress \( \sigma_0 \propto T \), we recognize an additional contribution to \( \sigma_0^*(\lambda) \), Eq. (2.8a), due to the energetic cost for deflection over a length \( \ell \sim W \) along the tensile axis. However, since \( \sigma_0 \) is also proportional to the longitudinal tension \( T \), we find that the minimal value of \( T \) for which Eq. (2.8a) is satisfied is again realized when the wavelength \( \lambda \) is a finite, thickness-independent fraction of the sheet width, implying:

model A: \( T_c \sim \frac{B}{W^2} \sim Y \left( \frac{h}{W} \right)^2 \); \( \lambda_c \sim W \).

(2.10)

Thus, notwithstanding the narrowness of the compressive zone and the presence of longitudinal tension in it, inspection of Eq. (1.53, 1.54) and Eqs. (2.9, 2.10) reveals that the instability of the planar shape caused by relative edge extension exhibits the typical behavior of the classic Euler buckling instability, namely, a thickness-independent critical “wavelength” \( \lambda_c \) set by the sheet geometry, and a threshold load value that scales quadratically with the thickness-width ratio.

Figure 2.7 shows that the predicted buckling-like behavior, characterized by the scaling rules (2.9, 2.10), is confirmed by our simulations. In Fig. 2.7c, threshold values (\( T_c \) for model A and \( \alpha_c \) for model B) were obtained for a range of sheet thicknesses by carefully probing intervals of the control parameters (\( T \) and \( \alpha \), respectively), and then plotted vs. the aspect ratio \( \frac{h}{W} \), showing an excellent agreement with the predicted scaling behavior. Apart from their identical scaling behavior, the threshold value of the dimensionless control parameter \( T_c/Y \) in model A is larger than its counterpart \( \alpha_c \) in model B, in accord with the enhanced resistance of the former to buckling.
Figure 2.7: (a-b) The deformation \( \zeta(x, y) \) of the stretched sheet (model A, in Fig. 2.1 a) and the corners-pulled sheet (model B, in Fig. 2.1 b) in the near-threshold regime, \( \tilde{T} \approx 2\tilde{T}_c(\epsilon) \). For each model we show a topographic map and a corresponding transverse cross-section, \( \zeta(x = \frac{1}{2}L - x_{\text{max}}, y) \). (c) The instability threshold \( \tilde{T}_c \), of a long stretched rectangular sheet with clamped edges (model A with \( L = 8W \)), as obtained from our SE simulations, plotted vs. the thickness, \( \frac{t}{W} \). The threshold value is shown to be proportional to \( (\frac{t}{W})^2 \), with a proportionality constant that depends weakly on the aspect ratio \( \frac{W}{L} \) for \( L \gg W \). This result indicates that the instability is an Euler-like buckling, due to a compressed zone of width \( \sim W \), where the compression is, \( \sigma_{yy} \sim -T \) (see text). (d) The transverse profile of the shape (at \( x \approx x_{\text{max}} \), where the compression is maximal), plotted close to threshold, indicates that the critical wavelength, \( \lambda_c \), is a finite, thickness-independent fraction of the sheet width, in accord with Euler buckling instability. (e) As the exerted tension \( T \) is increased beyond \( T_c \), the energetically-favorable wavelength \( \lambda \) becomes smaller (in comparison to \( \lambda_c \)), and develop explicit dependence on the sheet thickness. This buckling-to-wrinkling trend is consistent with a transition from near-threshold to far-from-threshold behavior envisioned in [12]. From top to bottom (looking at center \( y/W = 0 \)), \( T/T_c \approx 66, \ 118, \ 266, \ 1074, \ 4324 \).
due to the effect of longitudinal tension in the transversely compressed zone. The enhanced resistance to undulations is reflected also in the near-threshold pattern (Fig. 2.7a-b). While both models exhibit near threshold a buckling \( i.e \) thickness-independent undulation) pattern, such that the wavelength \( \lambda_c \) is a finite fraction of the width \( W \), this fraction is smaller in model A (by a factor of \( \approx \frac{1}{3} \)) in comparison to its counterpart in model B.

2.4 Far-from-threshold Wrinkling

Upon increasing the control parameter \( T \) in model A substantially above its threshold value (2.10), our simulations (Fig. 2.7e) show that the near-threshold buckling pattern undergoes two dramatic changes. First, undulations expand (along the \( \hat{x} \) axis) beyond the transversely-compressed zone of the planar state. Second, the characteristic wavelength \( \lambda \) becomes substantially smaller than its threshold value \( \lambda_c \sim W \). For model B, the analogous process of increasing \( \alpha \) beyond threshold (2.9) does lead to expansion of the deflected zone, but not to any significant reduction in the characteristic undulation wavelength.

Inspecting the considerations underlying the critical wavelength \( \lambda_c \) (2.8b), one may notice that the only way in which the planar state is explicitly affecting the wavelength is through the length \( \ell \sim W \) of the transversely-compressed zone. Assuming that even when the control parameter exceeds considerably the threshold value the wrinkle wavelength \( \lambda \) is affected by the stress distribution through the length \( \ell \) of the actual compressive zone, the rule (2.8b) can be generalized to:
\[ \lambda \sim \min \left( W, \left( \frac{B}{K_{\text{sub}} + T\ell_\ast^{-2} + B\ell_\ast^{-4}} \right)^{1/4} \right), \quad (2.11) \]

where \( \ell_\ast \) is the *actual* length of the compressive zone (at a given, post-threshold value of the control parameter) rather its length in the planar state.

With the generalized version (2.11) of the wavelength rule, one may immediately notice the difference between models A and B. In the former, the presence of longitudinal tension eventually dominates the wavelength, hence:

\[ \text{model A} \ (T \gg T_c) : \ \lambda \sim \left( \frac{B}{T\ell_\ast^{-2}} \right)^{1/4} \quad (2.12) \]

such that at a fixed value of \( T/Y \), the wavelength \( \lambda \) vanishes with the sheet thickness, signifying a transition from buckling \( (\lambda \sim W \text{ at } T \approx T_c) \) to wrinkling \( (\lambda \sim t^{1/2} \ll W \text{ for } T \gg T_c) \). The scaling rule of Cerda & Mahadevan Eq. 2.1a [12] is obtained by assuming that the transversely-compressed zone extends throughout the whole sheet, *i.e.* \( \ell_\ast \sim L \) in Eq. (2.12). In contrast, for model B, the absence of longitudinal tension implies that the pattern does not undergo a similar buckling-to-wrinkling transition as the control parameter \( \alpha \) exceeds the threshold value.

The above heuristic argument deserves a healthy dose of skepticism. Why does the transversely-compressed zone expand when the sheet is driven away from threshold? Why is it justified to approximate the energetic cost (per area) imposed on undulations by the longitudinal tension as \( T(A/\ell_\ast)^2 \)? In the following sections, we will address these questions as a part of our attempt to resolve the confusing aspects of the CM model.
2.5 The FT analysis

2.5.1 Overview

We denote $\tilde{()}$ a dimensionless version of a physical parameter or variable $(·)$, where stresses (integrated over the thickness $t$ of the sheet) are normalized by the stretching modulus $Y$, and lengths are normalized by the width $W$. The problem is to find the displacement field, $\mathbf{u} = (u_x, u_y, \zeta)$ that minimizes the enthalpy

$$U = E - \text{Work} \quad (2.13)$$

where the elastic energy $E$ and Work are given by

$$E = \frac{1}{2} \int dxdy \ B(\nabla^2 \zeta)^2 + \sigma_{ij} \varepsilon_{ij}$$

$$\text{Work} = 2 \cdot T \cdot W \cdot u_x(x = L/2, y). \quad (2.14)$$

Note that since we consider small-strain conditions ($\tilde{T} \ll 1$), we could simplify the above equations in two ways: first, mechanically – by assuming a Hookean stress-strain relation (Eq. 1.27), and second, geometrically – by assuming a small-slope deflection from the plane ($|\nabla \zeta| \ll 1$) and correspondingly using the strain-displacement relation (Eq. 1.6), and approximating the mean curvature by $\frac{1}{2} \nabla^2 \zeta$. We also took advantage of the symmetry $x \leftrightarrow -x$. In this FvK framework, the nonlinear response emanates solely from the geometrically-nonlinear coupling of out-of-plane displacement to the strain tensor in the sheet, of which the most important component for our problem is:

$$\varepsilon_{yy} = \partial_y u_y + \frac{1}{2}(\partial_y \zeta)^2. \quad (2.15)$$
This relation shows that even for large in-plane transverse displacement, it is possible for the corresponding strain to be arbitrarily small by tuning suitably the deflection from the plane, namely,

\[(\partial_y \zeta)^2 \approx -2\partial_y u_y \Rightarrow |\varepsilon_{yy}| \ll |\partial_y u_y|.

Using our normalization convention, one readily finds that the physics is governed by 3 dimensionless groups:

\[
\tilde{T} = \frac{T}{Y}; \quad \epsilon = \frac{B}{T W^2}; \quad \tilde{L} = \frac{L}{W}.
\] (2.16)

The parameter \(\tilde{T}\) is the characteristic tensile strain imposed on the sheet in the longitudinal axis \(\hat{x}\); the parameter \(\epsilon\) is recognized as the inverse of the “bendability”[20], and the parameter \(\tilde{L}\) is the aspect ratio. We focus on the “corner” in parameter space \((\tilde{T} \ll 1, \epsilon \ll 1, \tilde{L} \gg 1)\), namely – the Hookean, yet geometrically-nonlinear response of long, highly bendable ribbons.

In Sect. 2.3.2 we showed that the planar state (i.e. \(\zeta = 0\)) becomes unstable and develops a buckling pattern (with a wavelength \(\lambda_c \approx W/3\)) when the exerted tension exceeds a threshold value \(T_c \sim Y (h/W)^2\). Notably, when expressing the system through the dimensionless groups (2.16), the threshold occurs along a “vertical” line \((\epsilon_c, \tilde{T})\) in the parameter plane \((\epsilon, \tilde{T})\), where:

\[
\epsilon_c \approx 10^{-6}.
\] (2.17)

for any \(\tilde{L}\) larger than about 4. Hence, for the rest of this section, we will refer to the threshold through the value \(\epsilon_c\) of the dimensionless parameter \(\epsilon\). (A reader who
finds it more convenient to associate a threshold with the value of the tensile load, may readily convert: \( T_c \approx \epsilon_c \cdot B/W^2 \).

Underlying the NT analysis, which is valid for \( \epsilon \ll \epsilon_c \), there is an expansion:

\[
U(\tilde{T}, \epsilon) = U_{\text{plane}}(\tilde{T}) + \Delta U
\]  

(2.18)

where \( U_{\text{plane}}(\tilde{T}) \) is the enthalpy of the planar state, which does not depend on the bending modulus (hence is \( \epsilon \)-independent), and \( \Delta U \) is negative for \( \epsilon < \epsilon_c \) such that \( |\Delta U| \sim (\epsilon - \epsilon_c)^2 \ll U_{\text{plane}}(\tilde{T}) \) for \( \epsilon \ll \epsilon_c \). The buckling shape can be found by minimizing \( \Delta U \), assuming a perturbation with infinitesimal amplitude and negligible correction to the planar stress.

The basic premise of the FT framework is a description of the deformed sheet for a regime in the parameter space \((\epsilon^{-1}, \tilde{T})\) far beyond the threshold line, \( i.e. \epsilon \ll \epsilon_c \). This is done through an expansion of the elastic energy around the singular limit \( \epsilon \to 0 \) for a fixed geometry \((\tilde{L})\) and tensile load per thickness \((\tilde{T})\). For an experimenter whose set-up comprises a single sheet, \( i.e. \) fixed thickness and aspect ratio \( \tilde{L} \), on which the exerted tensile load is gradually raised or lowered, thereby changing smoothly both \( \tilde{T} \) and \( \epsilon \), such an approach may sound as an obscure mathematical trickery. Nevertheless, we shall show that this theoretical framework bears invaluable advantages for actual computations as well as for conceptual understanding.

Underlying the FT analysis (for a sheet with a given \( \tilde{L} \)) there is an expansion:

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\[ U(\tilde{T}, \epsilon) = U_{\text{TFT}}(\tilde{T}) + U_{\text{sub}}(\tilde{T}, \epsilon) \]

s.t. \( \frac{U_{\text{sub}}}{U_{\text{TFT}}} \to 0 \) as \( \epsilon \to 0 \), \quad (2.19)

where the “dominant” contribution \( U_{\text{TFT}}(\tilde{T}) \) is obtained by solving tension field theory for a hypothetical sheet with finite stretching modulus and zero bending modulus, and \( U_{\text{sub}}(\tilde{T}, \epsilon) > 0 \) is a subdominant contribution to the energy, associated with the direct energetic cost of wrinkling: bending the film and deforming the substrate. Crucially, \( U_{\text{TFT}}(\tilde{T}) < U_{\text{plane}}(\tilde{T}) \), hence – for any finite \( \tilde{T} \) and sufficiently small \( \epsilon \), it is the FT expansion (2.19), rather than its NT counterpart (2.18), which provides a reliable evaluation of the energy, and whose minimization should be used for characterizing the deformation.

The energetic hierarchy (2.19) entails three principles that comprise the FT expansion:

(a) an asymptotic, compression-free stress field;

(b) “slaving” the wrinkle amplitude to its wavelength;

(c) a “wavelength rule”.

In the rest of this subsection we explain these general principles, specializing to our model system. In Secs. 2.6-2.8 we demonstrate through numerical simulations how these elements govern the wrinkle pattern in our problem.

2.5.2 Asymptotic compression-free stress field

In the limit \( \epsilon \to 0 \), the stress tensor in the wrinkled sheet approaches a compression-free limit value. That is, for \( 0 < \epsilon \ll \epsilon_c \) the stress tensor can be approximated as:
\[
\sigma_{ij}(\mathbf{x}; \tilde{T}, \epsilon) \approx \sigma^{(\text{TFT})}_{ij}(\mathbf{x}; \tilde{T}),
\]

where the principal components of the tensor \(\sigma^{(\text{TFT})}_{ij}(\mathbf{x}; \tilde{T})\) are non-negative everywhere in the sheet. The approximation symbol indicates \(O(\epsilon^\beta)\) corrections, with \(\beta > 0\). A central premise of TFT is that the compression-free stress field \(\sigma^{(\text{TFT})}_{ij}(\mathbf{x}; \tilde{T})\) is a well-defined tensor, obtained directly through the energy minimization procedure underlying the dominant part \(U_{\text{TFT}}(\tilde{T})\) in Eq. (2.19) – allowing the deformation to have any (wrinkly, highly curved) out-of-plane component while ignoring its energetic cost. This amounts to solving the force balance equations for the stress tensor, subject to non-negativity of its principal components.

Being independent on the small parameter \(\epsilon\), the tensor \(\sigma^{(\text{TFT})}_{ij}(\mathbf{x}; \tilde{T})\) characterizes the smoothly-varying, gross features of the wrinkle pattern, the most basic of them is the extent of the wrinkled zone. The TFT solution marks two regions:

\[
\text{unwrinkled : } \frac{1}{2}L - x^* < |x| < \frac{1}{2}L \\
\text{wrinkled : } |x| < \frac{1}{2}L - x^*. \tag{2.21}
\]

where \(x^*\) denotes the extent of the transversely-tensed, unwrinkled zone near each clamped edge. In the unwrinkled zone, near the clamped edges, \(\sigma^{(\text{TFT})}_{ij}(\mathbf{x}; \tilde{T})\) is characterized by two positive (i.e. tensile) principal components; in the wrinkled, central region, only one principal component is positive and wrinkles undulate along the axis perpendicular to the corresponding principal direction. For set-ups characterized by some spatial (e.g. axial [20, 33, 63, 22] or translational [16]) symmetry, this direction is typically determined by the underlying symmetry, whereas in our problem the
clamping of the short edges breaks translational symmetry. Nevertheless, we expect
the deviation of the principal directions from \( \hat{x}, \hat{y} \), correspondingly, to be at most
\( O(\tilde{T}) \), and since we consider only \( \tilde{T} \ll 1 \), we ignore such deviations when analyzing
our numerical simulations.

Notably, the actual stress field \( \sigma_{ij}(x; \tilde{T}, \epsilon) \) is not compression-free, but rather
comprises a small residual compressive (i.e. negative) stress component in the per-
pendicular axis at the wrinkled zone (\( \hat{y} \)). Nevertheless, this residual stress compo-

dent vanishes as \( \epsilon \to 0 \). The absence of residual compression from the TFT stress
field, \( \sigma_{ij}^{(TFT)}(x; \tilde{T}) \), is intimately related to the fact that \( \sigma_{ij}^{(TFT)}(x; \tilde{T}) \) determines
only the gross features of the pattern but carries no information on the fine features,
specifically the wrinkle wavelength \( \lambda \). Finally, let us note that the extent of the
wrinkled zone, which is determined by \( x^* \) in Eq. (2.21), may depend on \( \tilde{T} \) and \( \tilde{L} \)
(even though the actual dependence turns out to be rather weak), but not on \( \epsilon \). This
independence on \( \epsilon \) of all TFT-derived expressions is crucial for understanding the
amplitude-wavelength “slaving” condition, which we discuss next.

### 2.5.3 Amplitude-wavelength slaving condition

Since TFT ignores the energetic cost associated with out-of-plane deflection \( \zeta(x) \),
any contraction of length is facilitated by “wasting” the excess length through some
\( \zeta(x) \). Specifically, this means that the transverse strain, \( \varepsilon_{yy}(x) \), Eq. (2.15), “de-
couples” from the corresponding derivative, \( \partial_y u_y(x) \), of the transverse displacement
(as long as the latter is contractive, i.e. negative). On the other hand, compat-
ibility of the TFT stress field (RHS of Eq. 2.20) with the limit value of the stress
in a Hookean sheet (LHS of Eq. 2.20) requires that \( 0 \approx \sigma_{yy} = Y(\varepsilon_{yy} + \nu\varepsilon_{xx}) \) and
\[ T \approx \sigma_{xx} = Y(\varepsilon_{xx} + \nu \varepsilon_{yy}). \] As a consequence, the TFT solution implies a “slaving” condition for all feasible out-of-plane deflections:

\[ \varepsilon_{yy} = \partial_y u_y + \frac{1}{2} (\partial_y \zeta)^2 \implies \frac{1}{2} (\partial_y \zeta)^2 = -\partial_y u_y - \nu \bar{T}. \quad (2.22) \]

Let us define:

\[ \Phi^2(x) \equiv \frac{1}{2W} \int_{-W/2}^{W/2} (\partial_y \zeta)^2 \, dy \quad (2.23) \]

\[ \tilde{\Delta}(x) \equiv \frac{1}{W} [u_y(x, -W/2) - u_y(x, W/2)] \quad (2.24) \]

where \( \tilde{\Delta}(x) \) is the contractional transverse displacement and \( \Phi^2(x) \) is the corresponding “confinement function”, namely, the excess length wasted by out-of-plane deflections (normalized by the width \( W \) of the undeformed sheet). From Eq. (2.22) we obtain that in the TFT solution, these are related through the relation:

\[ \Phi^{TFT^2}(x) = \tilde{\Delta}^{TFT}(x) - \nu \bar{T}, \quad (2.25) \]

Similarly to the convergence of the stress to the compression-free TFT value, Eq. (2.20), the functions \( \Phi(x) \) and \( \tilde{\Delta}(x) \) of Hookean, bendable sheets converge to their respective TFT values in the limit \( \epsilon \to 0 \), hence, for \( \epsilon \ll 1 \) we have that:

\[ \Phi(x; \bar{T}, \epsilon) \approx \Phi^{TFT}(x; \bar{T}) ; \tilde{\Delta}(x; \bar{T}, \epsilon) \approx \tilde{\Delta}^{TFT}(x; \bar{T}) \quad (2.26) \]
(Note that since $\tilde{\Delta}^{TFT}(x)$ and $\Phi^{TFT}(x)$ are determined by TFT, they vanish at $x \to x^*$, Eq. (2.21)). Using a common wrinkling ansatz for the out-of-plane displacement:

$$\zeta(x, y) \approx A(x) \cdot g\left(\frac{y}{W}\right) \cdot \cos\left(\frac{2\pi y}{\lambda(x)}\right)$$  \quad (2.27)

with: $g(0) = 1$ ,

where $A(x)$ and $\lambda(x)$ are, respectively, the wrinkle “amplitude” and “wavelength”, and $g(\xi)$ is a slowly-varying “envelope” (such that $(g'(\xi) \sim O(\epsilon^0))$, Eqs. (2.23,2.26) imply a “slaving” of the ratio between wrinkle amplitude and wavelength of actual sheets ($0 < \epsilon \ll \epsilon_c$) to the TFT value (of hypothetic sheets with $\epsilon = 0$):

$$\frac{A}{\lambda} \approx C \cdot \Phi^{TFT}(x)$$  \quad (2.28)

(where $C$ is some numerical constant, which does not depend on $\epsilon$ or $\tilde{T}$).

As long as Eq. (2.23) is satisfied, one may consider $\sigma_{ij}^{(TFT)}(x; \tilde{T})$ as the stress field in a hypothetic sheet characterized by finite stretching modulus and Hookean stress-strain relation but zero bending modulus. We note by passing that since we consider $\tilde{T} \ll 1$, the integrand in Eq. (2.23) is $\frac{1}{2}(\partial_y \zeta)^2 \approx [\sqrt{1 + (\partial_y \zeta)^2} - 1]$, that is the portion of the transverse arclength “wasted” by out-of-plane undulations.

Equations (2.23,2.25) highlight two intimately-related flaws in the original Cerda-Mahadevan model [12]:

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Firstly, Equation 2 of Ref. [12] invokes an equality of the excess length wasted by wrinkles and the transverse displacement of the free edges, namely:

\[
\text{CM assumption (I) : } \int_{\frac{W}{2}}^{\frac{W}{2}} dy \left[ \sqrt{1 + (\partial_y \zeta)^2} - 1 \right] \approx u_y(x, \frac{W}{2}) - u_y(x, -\frac{W}{2}) \\
\text{or : } \Phi^2(x) = \tilde{\Delta}(x) .
\]

Contrasting this with Eqs. (2.25,2.28), we see that CM assumption ignores the transverse strain, \(-\nu \tilde{T}\), which exists in fact also in the fully-developed, compression-free wrinkled state.

Secondly, Cerda-Mahadevan assumed that the transverse displacement of the free edges, \(\tilde{\Delta}(x) \cdot W\), is identical to its counterpart in the planar state, namely,

\[
\text{CM assumption (II) : } \tilde{\Delta}(x) \sim \nu \tilde{T} .
\]

However, Eq. (2.25) shows that in order for wrinkles to exist away from the clamped edges the in-plane transverse displacement of the free edges, \(\tilde{\Delta}(x) \cdot W\), must exceed the Poisson value, \(\nu \tilde{T} W\). This is crucial for understanding the very mechanism by which transverse compressive stress is relieved from the planar state: further transverse shrinking of the planar projection of the deformed sheet (in comparison to the
planar state) is necessary in order to “make room” for wrinkles.

### 2.5.4 Effective substrate and wrinkle wavelength

For given geometry and loading (i.e. given \( \bar{L}, \bar{T} \)), there are infinitely many functions \( \zeta(x, y) \) that are compatible with Eq. (2.23) and are therefore legitimate candidates to describe the wrinkle pattern. For a given \( 0 < \epsilon \ll \epsilon_c \), this degeneracy is lifted by minimizing the residual, sub-dominant contribution \( U_{\text{sub}} \) in Eq. (2.19), associated with the explicit energy cost of out-of-plane deformations in the functional (2.13), subject to the slaving constraint (2.28). One sub-dominant contribution is the bending energy, \( \propto B\kappa_{yy}^2 \), where \( \kappa_{yy} \approx \partial_y^2 \zeta \) is the curvature due to wrinkly undulations, whereas other contributions are often gathered into an “effective substrate” term, \( \propto K_{\text{eff}}\zeta^2 \) [12, 44], with an effective stiffness:

\[
K_{\text{eff}} = K_{\text{sub}} + K_{\text{curv}} + K_{\text{tens}}.
\] (2.31)

The various parts of \( K_{\text{eff}} \) correspond to a real substrate attached to the sheet (\( K_{\text{sub}} \), e.g. a heavy liquid bath [37, 49, 28, 47, 61] or a compliant solid [6], see Fig. 2.6b), a curvature imposed along the axis perpendicular to wrinkly undulations (\( K_{\text{curv}} \) [35, 44]), and also a tensile load exerted along that axis through the boundaries (\( K_{\text{tens}} \) [12], see Fig. 2.6e).

In order to elucidate the simultaneous effect of bending rigidity and effective substrate, it is useful to consider the ansatz (2.27) and amplitude-wavelength slaving (2.28). One may note that the bending energy becomes \( \sim B\Phi^{TFT^2}/\lambda^2 \), whereas the
effective substrate energy is $\sim K_{\text{eff}} \Phi_{TFT}^2 \lambda^2$, favoring, respectively, large and small wavelength. Such a constrained minimization of the sub-dominant energy yields the scaling relations:

$$\lambda \sim \left( \frac{B}{K_{\text{eff}}} \right)^{1/4} \text{ and } \sigma_{\text{res}} \sim -\frac{B}{\lambda^2}$$

(2.32)

where the residual compressive stress in the undulatory axis ($\sigma_{\text{res}} = \sigma_{yy}$ in our problem) is obtained by treating it as the Lagrange multiplier associated with the slaving constraint (2.28). As was pointed out by Cerda & Mahadevan [12], $K_{\text{tens}}$ is the only effective stiffness, among the three terms on the RHS of Eq. (2.31), which is operative in our system, making it a primary example – along with the axisymmetric Lamé set-up [26, 10, 18] – for “tensional wrinkling” phenomena. Nonetheless, a quantitative evaluation of $K_{\text{tens}}$ and $\lambda$ beyond the scaling level (2.1a,2.32) requires some subtle considerations, on which we elaborate below.

In the CM model, the wrinkle amplitude is assumed to vary smoothly between the two clamped edges (where $A = 0$), and the tension-induced stiffness is therefore estimated as $K_{\text{tens}} \sim T/L^2$, corresponding to the resistance of a stretched string to deflection (see Eq. (2.1a) and the subsequent paragraph). More recently [44] it was pointed out that a quantitative estimate of the tension-induced stiffness must take into consideration the actual gradient of the TFT confinement function $\Phi_{TFT}^2(x)$, Eq. (2.25), so that a more accurate expression for the tension-induced stiffness is:

$$K_{\text{tens}}(x) \approx \frac{\sigma^2_{TFT}(x)}{\ell^2_{\parallel}(x)} \; ; \; \ell_{\parallel}(x) \equiv \left| \frac{\Phi_{TFT}(x)}{\Phi'_{TFT}(x)} \right|$$

(2.33)
where $\sigma_{TFT}^T(x)$ is the tensile component of the TFT stress tensor ($\sigma_{TFT}^T \approx T$ in our problem), \textit{i.e.} \textit{along} the wrinkles. A spatially-varying stiffness may give rise to a spatially-varying wavelength $\lambda(x)$ [44], which requires the proliferation of localized defects, where wrinkles are “born” or “terminate”. However, if the necessary defects are too costly energetically [57, 31, 61], the pattern may consist of a spatially-uniform wavelength, determined through a \textit{global} balance of bending and effective substrate energies, whereby an effective stiffness $\bar{K}_{\text{tens}}$ is obtained by integrating over the excess energy associated with the variation of the wrinkle amplitude along the tension direction:

$$\lambda \approx C_1 \left( \frac{B}{\bar{K}_{\text{tens}}} \right)^{\frac{1}{4}}; \quad \bar{K}_{\text{tens}} = T \frac{\int_{\frac{L-x^*}{2}}^{\frac{L-x^*}{2}} dx \Phi_{TFT}'(x)^2}{\int_{\frac{L-x^*}{2}}^{\frac{L-x^*}{2}} dx \Phi_{TFT}(x)^2} \quad (2.34)$$

where $x^*$ marks the end of the wrinkled zone, Eq. (2.21), and the numerical prefactor $C_1$ is determined by the envelope function $g(y/W)$ of the wrinkle ansatz (2.27).

Even if the TFT confinement function $\Phi_{TFT}(x)$ is known analytically, evaluation of the integral for $\bar{K}_{\text{tens}}$ in (2.34) is hindered by a logarithmic divergence, since $\Phi_{TFT}(x) \sim \sqrt{|x - (\frac{L}{2} - x^*)|}$, near the end of the wrinkled zone [20, 4]. A similar difficulty is in axial geometries [19, 57], where it was found that regularization gives rise to a smooth wrinkle “foot” (\textit{i.e.} a “boundary layer” around $x = \pm (\frac{1}{2}L - x^*)$). In Sect. 2.8 we will discuss a similar effect found upon applying Eq. (2.34) to our problem.
2.6 The collapse of compression

We employ Surface Evolver for numerical simulations, focusing now on the FT regime, namely, sheets with finite, small bending rigidity, such that $0 < \epsilon \ll \epsilon_c$. A characteristic example of such a fully-developed wrinkle pattern is shown in Fig. 2.8. We continue to consider a sheet with a relatively large length-to-width ratio, $\tilde{L} = 8$, Poisson ratio $\nu = 0.4$, and thickness $t/W = [5 \times 10^{-6}, 4 \times 10^{-5}]$, and vary the exerted tensile load $\tilde{T}$.

Figure 2.9a shows the profile of the transverse stress at the midline $\sigma_{yy}(x, y = 0)$, for a sequence of values of $\epsilon \approx (9.3 \times 10^{-4}, 0.0037, 0.015) \cdot \epsilon_c$. The collapse of transverse compression upon decreasing $\epsilon$ is featured by three prominent motifs:

- First, the maximal level of transverse compression, $\max |\sigma_{yy}(x, y = 0)|$, realized at a distance $x_{\text{max}}$ from each of the clamped edges, decreases substantially from the planar value ($\approx 0.005 \cdot T$). Furthermore, it vanishes upon decreasing $\epsilon$, $\max |\sigma_{yy}| \sim \epsilon^{1/4}$ (Fig. 2.9b).

- Second, the longitudinal extent of the zone with significant transverse compression also decreases substantially in comparison to the planar state. One way to quantify this effect is by considering the distance $d = |x_{\text{max}} - x^*|$, between the points
Figure 2.9: (a) The transverse component of the stress tensor along the midline, \( \sigma_{yy}(x, y = 0) \) for \( \bar{T} = 0.01 \). Shown here are profiles of the planar stress (which is stable when \( \epsilon < \epsilon_c \)) and the stress in a wrinkled state for a few values of the bendability parameters \( \epsilon/\epsilon_c = 9.3 \times 10^{-4}, 0.0037, 0.015 \). As the bendability increases \( (\epsilon \to 0) \) we observe a spatially-nonuniform collapse of the compressive stress. (b) The dependence of the maximal compression \( \max_x \{ -\langle \sigma_{yy}(x, y) \rangle_y \} \) on \( \epsilon \) vanishes at a rate proportional to \( \epsilon^{1/4} \). (c) The extent of the compressive zone, \( d = |x_{max} - x^*| \), evaluated as the distance between the points of maximal transverse compression \( (x = \frac{1}{2}L - x_{max}) \) and zero transverse stress \( (x = \frac{1}{2}L - x^*) \). The value of \( d \) in the planar state is shown in the dashed horizontal line.
at which the transverse stress becomes negative and reaches its maximal negative value. Figure 2.9c shows that \( d \) too vanishes, albeit at a much slower rate than \( \max |\sigma_{yy}| \), namely: \( d \sim \epsilon^{1/9} \).

- Third, as is shown in Fig. 2.10, the longitudinal extent \( x^* \) of the transversely-tensile zones next to the clamped edges, is smaller than its counterpart in the planar state, approaching a finite value as \( \epsilon \to 0 \).

![Figure 2.10: While the compression level in the wrinkled state vanishes asymptotically as \( \epsilon \to 0 \), the transversely-compressed zones gets somewhat closer to the clamped edges. We plot here the extent, \( x^*(\epsilon) \), of the transversely-tensile zone next to each of the clamped edges (i.e. the distance from the clamped edge at which \( \sigma_{yy}(x, y = 0) \) changes sign for given \( \bar{T} = 0.01 \)). The red circle shows the asymptotic value, \( x^* \), extracted from the TFT simulations in Sect. 2.7 of a sheet with no bending resistance, and the dashed horizontal line indicates the corresponding length in the planar stress. Inset: \( x^*(\epsilon) \) converges to the TFT value \( x^* \) with a residual \( \sim \epsilon^{1/3} \).](image)

A central result of our SE simulations is presented in Fig. 2.11a, where we plot the energy \( U(\epsilon) \) (for \( \bar{T} = 0.01 \)) as a function of \( \epsilon \). In accord with the scenario described by Eq. (2.19), this plot shows that the energy is reduced from the value \( U_{\text{plane}} \) of
the planar state (dashed horizontal line), such that the energy gain, $U_{\text{plane}} - U(\epsilon)$, associated with the formation of a fully-developed wrinkle pattern, approaches a finite value as $\epsilon \to 0$, which we call $U_{\text{plane}} - U^{TFT}$, attributing it to the prevalence of tension field theory in the high bendability limit, $\epsilon \to 0$. Assuming that the sub-dominant energy, $U_{\text{sub}} = U(\epsilon) - U^{TFT}$, is determined by a work of a virtual compressive load, whose magnitude is equal to the residual compressive stress $\sim \max |\sigma_{yy}|$, that exists in a zone of length $d$, one may expect that $U_{\text{sub}} \sim d \cdot \max |\sigma_{yy}| \sim \epsilon^{13/36}$ (where we used the scaling relations extracted from out simulations in panels b and c of Fig. 2.9.). This is rather close to the scaling extracted from direct evaluation of the energy, $U_{\text{sub}} \sim \epsilon^{1/3}$ (Fig. 2.11b).

Taken together, these numerical observations reverberate the universal scenario outlined in Sect. 2.5.1 for NT-FT transition between the parameter regime, $\epsilon \lesssim \epsilon_c$, which is governed by the transversely-compressed planar stress, and the regime, $\epsilon \ll \epsilon_c$, where the fully-developed wrinkle pattern enables the stress field to approach a distinct, compression-free profile, thereby entailing a finite, $\epsilon$-independent energetic gain, $U_{\text{plane}} - U^{TFT}$.

2.7 The limit of compression-free stress field

The observations described in the preceding section provide strong evidence to the prevalence of an asymptotic, compression-free stress field, which underlies key features of the fully-developed wrinkle pattern. Nonetheless, such a stress field, which is the subject of TFT, can be realized only by a hypothetical sheet with no bending rigidity, and hence cannot be attained by simulating a physical Hookean sheet (i.e.
Figure 2.11: (a) The energy $U(\epsilon)$ of the wrinkled state for a given tensile load, $\tilde{T} = 0.01$, and several values of $\epsilon$. The red circle indicates the asymptotic energy $U^{TFT}$, extracted from the TFT simulations in Sect. 2.7, and the dashed horizontal line is the energy $U_{\text{plan}}$, of the corresponding planar stress. (b) the difference $U(\epsilon) - U^{TFT}$, plotted versus $\epsilon$, indicates that the subdominant energy $U_{\text{sub}} \sim \epsilon^{1/3}$.

$\epsilon > 0$), no matter how small $\epsilon$ is. In this section we seek to resolve this hurdle through SE simulations of precisely such a hypothetical sheet, free of bending rigidity, from which we extract directly the asymptotic stress field and the constraints imposed on the wrinkle pattern.

For a sheet with no bending rigidity, but finite stretching modulus $Y$, only tensile stress can be accommodated at mechanical equilibrium. Furthermore, since curvature comes at no energetic cost, even an infinitesimal amount of compression is fully relaxed by energy-free, out-of-plane undulations. The only (non-physical) mechanism limiting the scale of such undulations is the mesh size used in the simulation. Hence, as the mesh is made denser, the shape appears to be rougher. Nevertheless, we show in App. A.2 that the increasing corrugation does not affect the macro-scale features.
Figure 2.12: (a) The confinement function $\Phi^2(x)$ for $\tilde{T} = 0.01$, extracted by computing the excess length (RHS of Eq. (2.23) in the SE simulations. The red curve is extracted from the TFT solution (i.e. a sheet with no bending resistance), and the other curves are extracted from the simulations described in Sect. 2.6 for several values of $0 < \epsilon < \epsilon_c$. (b) The transverse contraction of the planar projection, $\tilde{\Delta}(x)$, extracted by evaluating the RHS of Eq. (2.25), from the same simulations as in panel A. We subtract $\nu \tilde{T}$ from the computed $\tilde{\Delta}(x)$, in order to allow easy comparison with the computed confinement function $\Phi^2(x)$ in A, and thereby examining the compression-free constraint, Eq. (2.25), in the TFT limit (red curves, $\epsilon = 0$), and how this condition is approached as $\epsilon \to 0$. (c) The computed value of the confinement function at the center of the sheet, $\Phi^2(x = 0)$, plotted versus $\epsilon$, exhibits convergence to the value $\Phi^{TFT^2}(x = 0)$ of TFT solution (red dot). Inset: the convergence to $\Phi^{TFT^2}(x = 0)$ is characterized by a residual, $\sim \epsilon^{1/2}$. (d) The transverse contraction of the planar projection at the center of the sheet in the TFT solution, $\Delta^{TFT}(x = 0)$, plotted versus $\tilde{T}$, exhibits a nonlinear dependence on $\tilde{T}$. For reference, we show also the analogous quantity extracted from the planar state, which (for sufficiently large $\tilde{T}$) is given by the Poisson value $\nu \tilde{T}$.
of the deformation, nor does it affect the stress components, all of which appear to converge to well-defined values, independent on the mesh density.

Our SE simulations of the TFT solution enable us to compute directly the dominant energy, $U^{TFT}$ in Eq. (2.19), denoted by red circle in Fig. 2.11, rather than by extrapolating the limit value, $U^{TFT} = \lim_{\epsilon \to 0} U(\epsilon \to 0)$ from results of SE simulation at finite values of $\epsilon$. This is crucial for our ability to compute the scaling, $U(\epsilon) - U^{TFT} \sim \epsilon^{1/3}$ of the subdominant energy (inset of Fig. 2.11), which we discussed above, as well as the asymptotic extent of the wrinkled zone in the sheet (red circle in Fig. 2.10) and how it is approached as $\epsilon \to 0$ (inset of Fig. 2.10).

The most valuable reward for solving numerically the compression-free stress is a direct computation of the conjugated excess length, namely, the confinement function, $\Phi^{TFT^2}(x)$, as well as the transverse contraction of the planar projection, $\tilde{\Delta}^{TFT}(x)$, from Eqs. (2.23) and (2.24), respectively. Let us elaborate on several important insights that are revealed in Fig. 2.12.

- The mere existence of a well-defined confinement function $\Phi^{TFT^2}(x)(\tilde{T}, \tilde{L})$ (red curve in Fig. 2.12a) proves the basic premise of the FT framework underlying the CM model (Eq. 2.1b). Namely, for a given geometry (i.e. $\tilde{L}$), the exerted load $\tilde{T}$ determines the transverse arclength wasted by out-of-plane deflections, thereby enforcing a finite, $\epsilon$-independent ratio between the asymptotically-vanishing wavelength and amplitude of wrinkles. The numerically-evaluated confinement function in Fig. 2.12 is analogous to similar constructs in analytically-tractable models [20, 33, 62, 16].
Figure 2.13: (a) Transverse profile of the wrinkles, measured at the center of the sheet, $\zeta(x = 0, y)$ for $\tilde{T} = 0.01$ and several values of $\epsilon$. The profiles are made discernible by shifting them vertically, and multiplying their amplitude by an arbitrary factor. (b) Normalizing each profile by the maximal amplitude, $\zeta(0, 0)$, the wrinkly profiles appear to be confined to the central half of the sheet width, and be enveloped by a slowly-varying function $g(y/W)$ that becomes constant at $|y| < \frac{1}{2}W$ as $\epsilon \to 0$. (c) Plotting the amplitude-wavelength ratio (extracted from their respective values at the center) versus $\epsilon$, for a given value of $\tilde{T} = 0.01$, we find that the ratio is not affected by $\epsilon$, in accordance with the FT framework. (d) Dividing the amplitude-wavelength ratio by the TFT confinement function, $\Phi_{\text{TFT}}^2(x = 0)$ (Fig. 2.12), we find a weak dependence on $\tilde{T}$. Here the amplitude $A = \zeta(0, 0)$ and corresponding wavelength $\lambda$ are determined at the center of the sheet. This discrepancy with the FT framework may be attributed to the modulations of the amplitude.
• As we argued in Sect. 2.5.3, the collapse of transverse (compressive) stress does not require an equality of $\Phi^2(x)$ (2.23) and $\Delta(x)$ (2.24), which was postulated in the original CM model [12], and would have implied a vanishing transverse strain. Instead, collapse of transverse compression requires $\Phi^2(x) - \Delta(x) \approx \nu\tilde{T}$, Eqs. (2.25, 2.26). The apparent equality of the red curves in Fig. 2.12a and Fig. 2.12b supports Eq. (2.25), thereby proving that underlying the fully-developed wrinkle pattern there is a collapse of transverse compression (stress) rather than vanishing transverse strain.

• An important property of TFT, revealed by Figs. 2.12b and 2.12c, is that $\Delta^{TFT}(x) \sim \tilde{T}^{4/5} > \nu\tilde{T}$. Namely, the planar projection of the deformed sheet is narrower than its counterpart in the planar state (which is in turn directly determined by the Poisson effect). Furthermore, the observation that the transverse contraction $\Delta^{TFT}(x)$, as well as the confinement function $\Phi^{TFT}(x)$, are not proportional to $\tilde{T}$, indicates that TFT is a nonlinear theory of the in-plane strains. This is notable, since TFT has a similar formal structure to the planar state solution, which is obviously linear in $\tilde{T}$. Namely, both theories amount to minimizing an energy functional, expressed solely through a quadratic (Hookean) form of in-plane displacement field $(u_x, u_y)$, where TFT is supplemented by the compression-free constraint (Sect. 2.5.2). The observation that $\Delta^{TFT}(x)$ is not proportional to $\tilde{T}$ points to the obscure way by which the compression-free constraint on the TFT stress field (2.20) embodies the geometrical nonlinearity (2.15), even though the actual out-of-plane displacement...
\( \zeta(x, y) \) is absent from the TFT calculation.

- The rest of the curves in Figs. 2.12a,2.12b (i.e. other than the red solid) show the analogous quantities, extracted from the finite-\( \epsilon \) SE simulations that were described in the preceding section. In accord with Figs. 2.9,2.10,2.11, which indicated convergence to the TFT limit as \( \epsilon \to 0 \), we observe that the constraints imposed by TFT on the transverse contraction and wasted arclength are reached asymptotically by physical sheets upon increasing their bendability, thereby proving Eq. (2.26). Figure 2.12d suggests that the convergence of these features to the TFT limit values is \( \propto \epsilon^{1/2} \), somewhat more rapidly than the convergence of the stress, energy, and the longitudinal extent of the wrinkled zone to their respective TFT values.

### 2.8 The wrinkle pattern

In Sect. 2.4 we described in qualitative terms the transition from buckling in the NT regime to wrinkling in the FT regime. The numerical analysis of the TFT solution described in Secs. 2.6-2.7 enable us to address the wrinkling pattern quantitatively, following the prescription laid out in Sect. 2.5.4. Aiming to examine the validity of the CM scaling law, Eq. (2.1), we start by comparing our numerical observation with the ansatz (2.27), and then proceed to address the wavelength \( \lambda \). As we will argue below, our SE simulations enable us to analyze how \( \lambda \) varies with bendability and tensile load (i.e. \( \epsilon^{-1} \) and \( \tilde{T} \)), but not the manner in which the wrinkle pattern varies with \( \tilde{L} \), a task that requires substantial computational power that is beyond the scope of this thesis. A consequence of this shortcoming is that we cannot address
directly the scaling relation $\lambda \sim L^{1/2}$, predicted in the CM model (2.1a). We explain the rationale of this prediction from the perspective of the FT framework, and discuss how future simulations of sheets with $\tilde{L} \gg 1$ may support or revoke this predicted scaling.

### 2.8.1 Wrinkling ansatz and the amplitude-wavelength slaving condition

Figure 2.13a shows the transverse profile of the deformed sheet at the center, $\zeta(x = 0, y)$, for a given value of $\tilde{T} = 0.01$ and several values of $\epsilon < \epsilon_c$. (Note that amplitudes are not up-to-scale, in order to make the profiles fit into a single figure). Two noteworthy features are: *(i)* the characteristic wavelength increases with $\epsilon$; *(ii)* the wrinkle amplitude is modulated across the width of the sheet, reaching a maximal value at the centerline ($y = 0$). The transverse modulation of the amplitude is further highlighted in Fig. 2.13b, where we re-plot the wrinkle profiles, normalizing each of them by its maximal amplitude, $\zeta(x = 0, y = 0)$.

The numerical finding shown in Fig. 2.13b supports the wrinkle ansatz (2.27), suggesting that: *(i)* the transverse-confined zone in the sheet does not extend throughout the whole width, but is instead limited to the central half of the width. *(ii)* the transverse undulations of the wrinkle amplitude reflect a slow convergence to an $\epsilon$-independent envelope. More precisely:

$$\zeta(x, y) \approx \Phi^{FT}(x) \cdot g(y) \cos\left(\frac{2\pi y}{\lambda}\right) \cdot (1 + O(\epsilon^3))$$

where:

$$g(y) \approx \Theta(|y| - \frac{W}{2}).$$

(2.35)
Here, $\Theta(y)$ is the Heavyside function, and $\beta$ is some positive constant, whose actual value is beyond the scope of this thesis. Notwithstanding its asymptotic convergence to a Heaviside function, the spatial variation of the envelope function $g(y)$ at any finite $\epsilon$ is smooth in comparison to superimposed wrinkles. For infinitesimal $\epsilon$, the envelope Equation (2.35) describes an amplitude that does not vary along the transverse direction, and its discontinuity at $y = \pm W/2$ suggests that only the central portion of the sheet, $|y| < W/2$, is subject to residual compression.

In Fig. 2.13c we plot the amplitude-wavelength ratio (which we determine for each profile through the largest amplitude, $\zeta(0,0)$), for the various profiles in Fig. 2.13a). In accordance with the basic paradigm of the FT framework (Sect. 2.5.3), we find that this ratio is essentially independent on the bending modulus of the sheet (i.e. $\epsilon$). Furthermore, dividing the amplitude-wavelength ratio at a given $\tilde{T}$ by $\Phi^{TFT^2}(x = 0, \tilde{T})$, and plotting the result versus $\tilde{T}$ (Fig. 2.13d), we find a good agreement with the amplitude-wavelength slaving condition (2.28) that we obtained in Sect. 2.5.3. (The slight deviation from constancy, $\sim \tilde{T}^{0.06}$, may be attributed to the modulation of the amplitude across the width, and to the fact that we determine the amplitude-ratio only through the central wrinkle). Notably, the nonlinear dependence of $(A/\lambda)^2$ on $\tilde{T}$ even though $\tilde{T} \ll 1$ and the simulated sheets are Hookean is in clear contradiction to Eq. (2.1b) of the CM model. Thus, Fig. 2.13d highlights the two intimately-related drawbacks in Eq. (2.1b), which we mentioned already in our discussion in Sect. 2.5.3:

(i) The amplitude-wavelength ratio is determined by the collapse of transverse compressive stress, hence by the TFT solution, and not by a vanishing transverse strain.
Figure 2.14: (a) The wavelength $\lambda$ (measured at the center of the sheet) for a fixed value of $\tilde{T} = 0.01$ and a few values of $\epsilon$ (log-log plot) exhibits the scaling $\lambda \sim \epsilon^{1/4}$ predicted by the CM model (Eq. 2.36). (b) Plotting $\epsilon^{-1/4} \lambda$ versus $\tilde{T}$, we find a nearly constant value, indicating that the exerted tensile strain $\tilde{T}$ does not affect the wavelength.

(ii) The amplitude-wavelength ratio is nonlinear function of the exerted strain $\tilde{T}$ even for $\tilde{T} \ll 1$, thereby reflecting the geometrically nonlinear nature of TFT.

### 2.8.2 How do bending rigidity and tension affect the wavelength?

In order to analyze the wrinkle wavelength $\lambda$, it is useful to express the prediction (2.1a) of the CM model using the three dimensionless groups, $\epsilon, \tilde{T},$ and $\tilde{L}$:

$$\text{CM prediction (dimensionless)} : \frac{\lambda}{W} \sim \epsilon^{1/4} \cdot \tilde{L}^{1/2}. \quad (2.36)$$

Notably, the CM model predicts that $\lambda$ depends on the ratio between the bending modulus and exerted tensile load (through $\epsilon$), as well as the rectangular shape (through $\tilde{L}$), but is indifferent to the exerted tensile strain $\tilde{T}$. Recalling our discussion in Sect. 2.5.4, we note that the dependence of $\lambda$ on $\epsilon$ follows directly from
Eq. (2.34), since the integral expression for $K_{\text{tens}}$ is fully determined by the TFT confinement function, $\Phi^{\text{TFT}2}(x)$, which – being a product of TFT – can depend only on $\tilde{T}$ and $\tilde{L}$. However, the dependence of $\lambda$ on $\tilde{T}$ and $\tilde{L}$ may be more complicated, since – as we have seen already in analyzing the amplitude-wavelength ratio – the geometrical nonlinearity underlying TFT may impart a nonlinear dependence of $\Phi^{\text{TFT}}(x)$ on these parameters. Being limited to a single value of $\tilde{L} = 8$, our SE simulations enable us to address the dependencies of $\lambda$ on $\epsilon$ and $\tilde{T}$, but not on $\tilde{L}$ (on which we will comment in the following subsection).

Figure 2.14 shows the wavelength $\lambda$, extracted from our SE simulations. For consistency, we determine $\lambda$ in each wrinkled sheet as $|y^+ - y^-|$, where $y^+$, $y^-$ are the closest points to the center at which the deflection vanishes, i.e. $\zeta(x = 0, y^\pm) = 0$. In Fig. 2.14a we focus on a single value of $\tilde{T} = 0.01$ and plot $\lambda$ versus $\epsilon$, finding an excellent agreement with the CM prediction (2.36). In Fig 2.14b we plot $\lambda \cdot \epsilon^{-1/4}$ versus $\tilde{T}$, and find no apparent dependence on $\tilde{T}$, again in excellent agreement with the prediction of the CM model. This finding indicates that although the magnitude of the TFT confinement function $\Phi^{\text{TFT}2}(x)$ is a nonlinear function of the exerted tensile strain, $\tilde{T}$, its spatial variation along the sheet is barely affected by $\tilde{T}$.

Attempting to obtain a quantitative test for the prediction (2.34), one may naturally seek to employ the confinement function $\Phi^{\text{TFT}2}(x)$ found in our numerical solution of the TFT in Sect. 2.7 (Fig. 2.12d,d) for several values of $\tilde{T}$, and evaluate the corresponding integrals that define $K_{\text{tens}}$. However, as we indicated in Sect. 2.5.4 this scheme is readily stymied due to the logarithmic divergence of the integral in the numerator of $K_{\text{tens}}$. (Note that, as is evident in Fig. 2.12a,
\( \Phi^{TF} T^2(x) \propto x - (\pm \frac{1}{2} L - x^*) \), at the vicinity of the boundary of the transversely-confined zone, yielding \( \Phi^{TF} T'(x)^2 \propto [x - (\pm \frac{1}{2} L - x^*)]^{-1} \). This divergence indicates that another physical effect, which is not accounted for in the balance of bending and stretching energies underlying Eq. (2.34), becomes significant at \( |x| \lesssim \frac{1}{2} L - x^* \). A similar phenomenon has been found in tensional wrinkling of an annular sheet (the Lamé problem) [20], where numerical simulations showed that divergence is inhibited through the formation of a partially-compressed boundary layer (whose width decreases slowly with \( \epsilon \)) [57], although another regularization mechanism that involves wrinkle cascades has also been proposed [4]. While a suitably regularized calculation of the integral in Eq. (2.34) is beyond the scope of this thesis, we note that the solution of the Lamé set-up suggests that sufficiently far from threshold the wavelength retains the scaling \( \lambda \sim \epsilon^{1/4} \) as if the integral in Eq. (2.34)) was convergent (albeit with a numerical prefactor whose evaluation requires regularization). Consequently, since our numerical results support the scaling \( \lambda \sim \epsilon^{1/4} \) (Fig. 2.14a), we conclude that both integrals in Eq. (2.34) are dominated by the bulk of the wrinkled region rather than by the vicinity of its boundaries.

2.8.3 How does the sheet’s length affect the wavelength?

We have seen above that the dependence of the wrinkle wavelength on the elastic moduli \( (B \text{ and } Y) \) and the exerted tensile load \( (T) \), expressed through the dimensionless parameters \( \epsilon \) and \( \tilde{T} \), agrees very well with the prediction of the CM model (Eqs. 2.1a, 2.36). While our simulations do not allow us to test directly the dependence of \( \lambda \) on \( \tilde{L} \), we elaborate here on the rationale of the CM prediction \( \lambda \sim \tilde{L}^{1/2} \).
from the perspective of the FT analysis, and discuss the asymptotic limit $\tilde{L} \to \infty$ in the Hookean FT regime (assuming fixed values of $\tilde{T} \ll 1$ and $\epsilon \ll \epsilon_c$).

Recalling that the tensional stiffness $K_{\text{tens}}$ in Eq. (2.34) is a product of TFT and thus independent on $\epsilon$, and assuming that both integrals in the denominator and the numerator are dominated by the bulk of the transversely-confined zone, we consider the “asymptotically long” limit, $\tilde{L} \gg 1$. We may envision (at least) two different scenarios for the outcome of TFT in this limit:

- A spatially-uniform confinement:

  \begin{align*}
  \text{Scenario A :} \\
  \Phi_{\text{TFT}}(x)^2 &\sim \tilde{L}^\alpha \\
  \Phi_{\text{TFT}}'(x)^2 &\sim \tilde{L}^{\alpha-2}
  \end{align*}

  with an exponent $\alpha > 0$.

- A spatially-nonuniform confinement:

  \begin{align*}
  \Phi_{\text{TFT}}(x)^2 &\sim \begin{cases} 
  C_0(\tilde{T}) \cdot f(\tilde{x}/W) & \tilde{x} < C \cdot W \\
  C_1(\tilde{T}) & \tilde{x} > C \cdot W
  \end{cases} \\
  \Phi_{\text{TFT}}'(x)^2 &\sim \begin{cases} 
  \frac{1}{W^2} C_0(\tilde{T}) f'(\tilde{x}/W)^2 & \tilde{x} < C \cdot W \\
  0 & \tilde{x} > C \cdot W
  \end{cases}
  \end{align*}

  (2.38)
where \( \tilde{x} = |x \pm (\frac{1}{2}L - x^*)| \) is the distance from the end of the transversely-confined zone, \( C \) is some constant, \( C_0(\tilde{T}) \) and \( C_1(\tilde{T}) \) vanish as \( \tilde{T} \to 0 \), and \( f(\xi) \) is some function such that \( f'(\xi)^2 \) is integrable as \( \xi \to 0 \).

The rationale underlying scenario A, which echos an assumption made in the CM model, is that the stretched sheet “feels” the clamping at the short edges everywhere in the wrinkled zone, even though the sheet is arbitrarily long. The rationale underlying scenario B is that for \( \tilde{L} \gg 1 \), the confinement varies spatially only in a region close to the clamped edges, whose extent is indifferent to the length of the sheet (and hence must scale with the width \( W \)). While inspection of our numerical TFT solution (Fig. 2.12a) seems to support scenario A, we emphasize that we cannot rule out scenario B, or even more complicated scenarios, since our simulations do not explore sufficiently broad interval of values of \( \tilde{L} \).

Assuming scenario A, one readily note that Eq. (2.34) yields \( K_{\text{tens}} \sim \frac{T}{L^2} \), whereas model B yields \( K_{\text{tens}} \sim \frac{T}{LW} \). Consequently, we find that:

\[
\begin{align*}
\text{Scenario A : } & \lambda/W \sim \tilde{L}^{1/2} \\
\text{Scenario B : } & \lambda/W \sim \tilde{L}^{1/4}
\end{align*}
\]

Once again we find that the nature of the confinement function \( \Phi^{TFT^2}(x) \), which is derived from the geometrically-nonlinear TFT and is strictly distinct from the planar state, may affect a noticeable departure from the prediction of the CM model. Numerical simulations of sufficiently long sheets will help to elucidate the length dependence of the wrinkle wavelength.
While the confinement function is derived from TFT, which totally ignores the bending rigidity of the sheet, the wrinkling of physical, highly bendable sheet (i.e. $0 < \epsilon \ll \epsilon_c$), cannot be described by any of the scenarios in Eq. (2.39) for arbitrarily long sheets. To see this, note that $\lambda$ is trivially bounded by the sheet width $W$. Thus, for any $\epsilon > 0$ there exists a maximal length:

- Scenario A: $\bar{L}_{\text{max}}(\epsilon) \sim \epsilon^{-1/2}$
- Scenario B: $\bar{L}_{\text{max}}(\epsilon) \sim \epsilon^{-1}$

(2.40)

such that for a sheet longer than $\bar{L}_{\text{max}}(\epsilon) \cdot W$, the energetically-favorable deformation is no longer a parallel array of wrinkles that occupy most of the sheet. The nature of the deformation in such highly bendable but “superlong” sheets ($\epsilon \ll \epsilon_c$, $\bar{L} > \bar{L}(\epsilon)$) is an interesting question for future studies, even though it may not be easily accessible for experiments.

2.9 Discussion

2.9.1 Phase diagram

Figure 2.15 delineates a schematic “phase diagram” of the stretched Hookean sheet. Considering a given, large value of $\bar{L}$, the diagram we plot in Fig. 2.15a is spanned by the two dimensionless parameters, $\epsilon^{-1}$ and $\bar{T}$, Eq. (2.16). Below the vertical threshold line, $\epsilon > \epsilon_c$, Eq. (2.17), the planar state is stable. (In Fig. 2.15b we re-plot the same diagram using as two independent parameters $\bar{T}$ and $W/t$, in which the threshold is a curve $\bar{T}_c \sim (t/W)^2$). For $\epsilon \lesssim \epsilon_c$, our analysis in Sect. 2.3 revealed that the deformation is characterized by a buckling mode, whose wavelength
Figure 2.15: (a) A “phase diagram” (for a fixed, sufficiently large $\tilde{L}$), spanned by the bendability $\epsilon^{-1}$ and exerted tensile strain $\tilde{T}$. The threshold occurs at a vertical line, $\epsilon = \epsilon_c$ (Eq. 2.17). Close to the threshold line, $\epsilon \lesssim \epsilon_c$ (NT regime), the sheet exhibits a buckling pattern with a wavelength $\lambda \approx W/3$, localized in the compressed zones of the planar stress (see Fig. 2.7d). For $\epsilon \ll \epsilon_c$ (FT regime), the pattern consists of wrinkles (wavelength $\lambda \sim W\epsilon^{1/2}$), which expands throughout the whole sheet. (b) Re-plotting the above phase diagram where the axes are now the normalized thickness $W/t$ and $\tilde{T}$, the threshold occurs at a curve $\tilde{T}_c \sim (t/W)^2$.

is independent on $\epsilon$ ($\lambda \approx W/3$), and whose spatial extent is limited to the transversely compressed zone of the planar state. Such a deformation is properly described by standard NT approach – linear stability analysis and post-buckling methods. When $\epsilon \ll \epsilon_c$, the deformation becomes a wrinkle pattern which expands throughout most of the sheet, with a wavelength that vanishes as $\lambda \sim \epsilon^{1/4}$. Although we try to keep the phase diagram, Fig. 2.15, simple, we note that our discussion in Sec. 2.8.3 suggests that yet another curve should be added at $\epsilon^{-1} \sim \tilde{L}$ or $\epsilon^{-1} \sim \sqrt{\tilde{L}}$ (see Eq. 2.40). In Fig. 2.15a such a curve is a vertical line, to the right of which the deformation pattern is no longer described as parallel wrinkle that extend uniformly between the clamped edges of the sheet.
While studies of other model systems revealed a pronounced variation of the deformation between distinct wrinkle patterns in the respective NT and FT regimes, the stretched rectangular sheet is exceptional, exhibiting a transition from a regular buckling mode to fully-developed wrinkle pattern. Notably, this dramatic morphological transition is driven by a minute energetic gain. This has been hinted already in Sect. 2.2.2, where we showed that the maximal transverse compression in the planar state is barely a half percentile of the exerted longitudinal tensile load. Figure 2.11 shows that the energetic gain of the TFT limit (which provides a lower bound for the energy of the fully wrinkled state) may be a tiny fraction of the elastic energy of the corresponding (unstable) planar state.

2.9.2 Summary and open questions

The main accomplishment reported in the section part of this chapter (Secs. 2.6-2.8) is a numerical demonstration that the fully-developed wrinkle pattern observed upon stretching a thin rectangular sheet is described by the FT framework (Sect. 2.5) – a singular expansion of the Hookean elastic energy around the TFT solution of where the small parameter is the inverse bendability \( \epsilon \) – similarly to other problems in which such a description is amenable for analytic calculations. In addition to elucidating that the formation of wrinkles is governed by the collapse of transverse compressive stress, rather than transverse strain, our SE simulations further elucidate the geometrically-nonlinear nature of the fully-developed wrinkle pattern. In Eq. (2.1b) of the CM model, the geometrical nonlinearity has been incorporated by invoking that the amplitude-wavelength ratio is independent on the bending mod-
ulus (i.e. the dimensionless parameter $\epsilon$). However, our analysis shows that this $\epsilon$-independent ratio, given by the confinement function derived from the compression-free TFT solution, is itself a nonlinear function of the exerted tensile strain $\tilde{T}$ even for arbitrarily small $\tilde{T}$. This observation illuminates yet another subtle manifestation of the geometrical nonlinearity (Eq. 2.15) underlying FT analysis.

The analysis we presented here is based on numerical simulations, yielding numerous observations on the TFT solution ($\epsilon = 0$) and on the wrinkle pattern ($0 < \epsilon \ll \epsilon_c$). One may wish to explain these observations by developing and analyzing a simplified, analytically-tractable model. For the benefit of a motivated reader, we close this chapter by highlighting some of these unexplained observations.

2.9.2.1 TFT solution

• We found that TFT yields a nonlinear dependence of macroscale features on the exerted tensile strain, most notably the transverse contraction of the planar projection, $\tilde{\Delta}^{TFT} \sim \tilde{T}^{4/5}$. We interpreted this finding as a signature of the geometrically nonlinear nature of TFT, even though – similarly to the planar state (which predicts linear dependence on $\tilde{T}$) – it depends explicitly only on the in-plane displacement field.

Is it possible to obtain the exponent $\frac{4}{5}$ analytically?
Our numerical solution of TFT is limited to a single length ($\tilde{L} = 8$), hence hampering our ability to make predictions for $\tilde{L} \gg 1$ even at a qualitative level (e.g. discerning between scenarios A and B in SubSect. 2.8.3).

*It is possible to predict the qualitative nature of the TFT solution for $\tilde{L} \gg 1$ without simulating long sheets?*

In our analysis we employed a semi-one-dimensional (1D) approach, whereby we extracted from simulations central features, such as the confinement function $\Phi^2(x)$ and the planar transverse contraction $\tilde{\Delta}(x)$, by integrating over the width of the sheet. However, the observed wrinkle patterns (Fig. 2.13) hint at nontrivial spatial structure of the TFT solution, whereby transverse confinement is restricted to the central half of the sheet.

*What gives rise to an apparent “half-width” rule?*

### 2.9.2.2 Wrinkle pattern

In our SE simulations we found various power laws that characterize the convergence of the residual (transverse compressive) stress, as well as various macroscale features of the wrinkle pattern in physical, highly bendable sheets ($0 < \epsilon \ll \epsilon_c$), to the respective TFT values.

*Is it possible to obtain analytic expressions for the exponents in the power laws in Figs. 2.9b-c, 2.10, 2.11, 2.12c?*

Our semi-1D analysis falls short of accounting for the slowly-varying envelope that modulates the wrinkle amplitude along the transverse axis (Fig. 2.13). Although
amplitude modulations induced by geometrical frustration have been observed in more symmetric set-ups [61], our problem appears to be different, since clamping the pulled edges violates transnational symmetry and may thus the cause of a non-periodic pattern.

Is it possible to predict how non-symmetric boundary conditions affect non-periodicity of wrinkle patterns?
CHAPTER 3
MECHANISMS FOR SPATIALLY-VARYING WAVELENGTH

3.1 Introduction

In the previous chapter, we discussed a problem in which the wrinkle wavelength and direction are nearly uniform. However in many cases, the wrinkling patterns are more complicated due to confinement topography or substrate, e.g. the wrinkling in a shriveled apple (Fig. 3.1a) and the wrinkling of a spherical shell floating on liquid (Fig. 3.1b). In these two examples, the wrinkling patterns exhibit nontrivial distributions of localized defects (circled parts in Fig. 3.1), revealing two main sources of the complexity. The former one is the existence of multiple wavelengths $\lambda$ and the latter one is a spatially varying director $\hat{n}$, which denotes the axis along which the wrinkles undulates.

It is a challenge to understand the emergence and distribution of defects. Previous research has been focused on most simple cases. In Ref [61], Tovkach et.al. presented a model for the case of spatially varying director by considering a thin sheet attached to a spherical substrate. When the stiffness of the substrate is large enough, the emergent wrinkles form domains whose directors approximately follow the azimuthal direction $\hat{\theta}$. As two domains are meeting each other, aligned defects
Figure 3.1: The examples of wrinkling with varying wavelength or direction. (a) A shriveled apple, the wrinkling wavelength changes due to nonuniform substrate [12]. (b) A thin spherical shell floating on a liquid, the complicated geometrical incompatibility induces varying wrinkling directions.[61]

appear in between (see Fig. 3.2a). In addition, these authors observed modulations of wrinkle amplitude along $\hat{\theta}$ (Fig. 3.2b). In order to explain these phenomena, they presented a model of wrinkles based on the multiple-scale perturbation theory[61].

For the other source of defects, namely, the existence of multiple wavelengths, J. Schleifer et.al tried to modify the uniform wrinkling of a rectangular sheet as showed in section 1.3.2. Instead of a uniform sheet, they used a surface with gradually changing thickness along the longitudinal direction, in which case, the wavelength is expected to change with the thickness according to the local lambda law Eq. 1.59. With this set-up, they observed a complex distribution of defects (see Fig. 3.3) [51].

Motivated by J. Schleifer et.al’s experiment, we propose a theoretical model to describe this phenomenon and seek to characterise the types of wrinkling patterns. We expect the emergent pattern to depend on factors such as the steepness of the
gradient of the average wavelength, and whether the wrinkles experience a tension along their length. Our model is depicted Fig. 3.4, a rectangular sheet under uniaxial confinement in transverse direction $\hat{y}$ which is similar to the problem in section 1.3.2, but instead of a uniform substrate, now the substrate’s stiffness is varying over the longitudinal direction $\hat{x}$, i.e. the stiffness $K$ is a function of $x$. A tension $T$ may be exerted along $\hat{x}$ axis.

In order to find the emergent patterns of this model, we performed numerical simulations with Surface Evolver to find the energetically-favorable deformation of the surface, subject to periodic boundary condition in the $\hat{y}$ direction. In Sect. 3.2, we introduce our set-up and discuss a wrinkle-flod transition, which we seek to avoid in the current study. In Sect. 3.3, we present our simulations of two limit cases of a
Figure 3.3: (a) The experiment in [51], the surface is uniformly confined in horizontal direction, the thickness of the surface varies vertically. This causes the wavelength changes over vertical direction, and defects emerge. (b) The distribution of the defects.

“jumping” (sharply varying) and a moderately-varying stiffness with $x$, respectively, suggesting that the jumping case is akin to the clamped boundary effect, and the moderately-varying case has well-structured distributions of defects, then we focus on narrow ribbons under small tensions.

3.2 Elements of Numerical Simulations

3.2.1 Overview

We commence with some general considerations. We expect the fundamental rule, local lambda law Eq. 1.59 tp remain valid at some average sense, since it reflects the local balance between bending and substrate resistance, even though it is not possible
Figure 3.4: A rectangular sheet under confinement in transverse direction is floating on a substrate with varying stiffness. The original length is $L$, width is $W$. Upper: The top view of the sheet, the width is confined to $W - \Delta$, we can also exert a tension $T$ in longitudinal direction. Lower: The side view of the sheet, the stiffness is varying.

to be satisfied everywhere. Hence, we define the preferred wavelength $\bar{\lambda}$ by the local lambda law Eq. 1.59,

$$\bar{\lambda}(x) \equiv 2 \pi \left( \frac{B}{K(x)} \right)^{1/4}$$

(3.1)

We also define the corresponding, energetically-preferable wavenumber $\bar{n}(x) = W/\bar{\lambda}(x)$.

In our simulations we choose $K(x)$ such that

$$\bar{n}(x = 0) = \bar{n}_0 \quad \text{and} \quad \bar{n}(x = L) = \bar{n}_L > \bar{n}_0$$

(3.2)

where we manually choose a increasing function of $\bar{n}(x)$. Under this constraint, $\bar{\lambda}(x)$ can be set to different functions, e.g. $\bar{\lambda}(x) \sim x$, $\bar{\lambda}(x) \sim \frac{1}{x}$ and so on.
We also find it helpful to imply a periodic boundary condition (BC) in the transverse direction, \textit{i.e.} the deflection, \( \zeta \), satisfies

\[
\zeta(x, y) = \zeta(x, y + W)
\]  

(3.3)

Periodic BC are useful for two reasons. First, other BCs like fixed or free BC may lead to unnecessary difficulties for analytic calculations. Second, we can observe well-behaved patterns even in narrow ribbons, since boundary layers are eliminated.

### 3.2.2 Wrinkle-fold

From the analysis in section 1.3.2, we know that wrinkling happens when \( \tilde{\Delta} > \tilde{\Delta}_c \). However, there is also an upper bound for \( \tilde{\Delta} \) beyond which wrinkles transition to folds (See Fig. 3.5), we denote this limit as \( \tilde{\Delta}_f \), which is given in [42],

\[
\tilde{\Delta}_f = \left( \frac{\bar{\lambda}}{W} \right)^2 = \frac{1}{\bar{n}^2}
\]  

(3.4)

Since our focus here is on defects in wrinkle patterns, we must keep \( \tilde{\Delta} \) smaller than the minimum of \( \tilde{\Delta}_f \). Consequently, \( \tilde{\Delta} \) should satisfy

\[
\tilde{\Delta}_c < \tilde{\Delta} < \min_x \tilde{\Delta}_f(x)
\]  

(3.5)

In our case, \( \min_x \tilde{\Delta}_f(x) \) is determined at the right end of the sheet,

\[
\min_x \tilde{\Delta}_f(x) = \frac{1}{\bar{n}_L^2}
\]  

(3.6)
Figure 3.5: The transitions of different regimes when $\Delta$ changes. In this chapter, we want to focus on wrinkling regime $\Delta_c < \Delta < \Delta_f$. When $\Delta > \Delta_f$, the surface starts to fold.

Notice that, To avoid folding, Eq. 3.5 is a natural choice for numerical study. But for experiments, it is not necessary to intentionally enforce such a condition, since the soft substrate has nonlinear response under large deflections, which can also prevent folding. In our simulations, we employ also other ways that may be useful to avoid folding when $\tilde{\Delta} > \tilde{\Delta}_f$ to enable compare with experiments.

### 3.2.3 Coarse-grained analysis

In the case $d\tilde{\lambda}/dx \ll 1$, we can conduct theoretical analysis using coarse graining. Following the results in section 1.3.2, we make the following assumptions,

$$
\zeta(x, y) = A(x) \cos \left( \frac{2\pi}{\tilde{\lambda}(x)} y \right), \quad A(x) = \frac{\sqrt{\Delta}}{\tilde{\lambda}(x)} = \frac{\sqrt{\Delta}}{\pi}
$$

(3.7)
which is not valid everywhere, but we expect it to be satisfied in a coarse-grained manner if tension is sufficiently small. Note that \( d\bar{\lambda}/dx \ll 1 \) implies \( \partial \bar{\kappa}/\partial x \sim A'(x) \sim \bar{\lambda}'(x) \ll 1 \), thanks to which, we can neglect the bending in \( x \) direction. The undulation of the amplitude \( A(x) \) also causes an excess stress \( \sigma_{xx}(x) \sim Y(\partial \bar{\kappa}/\partial x)^2 \), which introduced a nonlinear term that is hard to deal with. So we want the exerted tensile load \( T \) to dominate this excess contribution to \( \sigma_{xx} \), but we do not want it stretch out the local defects, whereby, the bending energy should dominate strain energy caused by tension. In order to set proper parameters under these assumptions, we first write the three energies: bending and substrate deformation, strain due to exerted tensile load, and strain due to deflection, respectively,

\[
e_b = \frac{B}{2} \left( \frac{\partial^2 \zeta}{\partial y^2} \right)^2 + \frac{K(x)}{2} \zeta^2 \tag{3.8a}
\]

\[
e_s = \frac{T}{2} \left( \frac{\partial \zeta}{\partial x} \right)^2 \tag{3.8b}
\]

\[
e_d = \frac{Y}{2} \left( \frac{1}{2} \left( \frac{\partial \zeta}{\partial x} \right)^2 \right)^2 \tag{3.8c}
\]

Then we perform coarse graining, i.e. integrate the energy over a period and set,

\[
\int_0^{\bar{\lambda}} e_b dy \gg \int_0^{\bar{\lambda}} e_s dy \gg \int_0^{\bar{\lambda}} e_d dy \tag{3.9}
\]

Plugging the assumption Eq. 3.7, we get the following result

\[
\frac{T}{\sqrt{BK}} \lambda'(x)^2 \ll 1 \tag{3.10a}
\]

\[
\frac{T}{Y \Delta \lambda'(x)^2} \gg 1 \tag{3.10b}
\]
The parameter regime (3.10) is our main target in this problem, and will be addressed in the following sections. Note that, if we have preset \( B, K(x), \tilde{\Delta} \), Eq. 3.10a requires \( T \) to be small, while Eq. 3.10b requires \( T \) to be large, so the best choice \( T \) is the “middle” value, which we denote as \( T_0 \).

### 3.3 Simulations

We employ Surface Evolver (SE) for numerical simulations with similar methods as in the previous chapter. As for the substrate and periodic BC, we defer the details to the last chapter introducing SE techniques.

#### 3.3.1 Cascades under jumping stiffness

For \( \frac{d\lambda}{dx} \gg 1 \), we consider an extreme case, a jump of stiffness \( K \) in the substrate, which is corresponding to a jump of preferred wavenumber \( \tilde{n} \) at some \( x_0 \), i.e.

\[
K(x < x_0) = K_1, \quad K(x > x_0) = K_2 \quad (3.11a)
\]

\[
\tilde{n}(x < x_0) = \tilde{n}_1, \quad \tilde{n}(x > x_0) = \tilde{n}_2 \quad (3.11b)
\]

The pattern we observe in this case is similar to “wrinkling cascades” that are induced by amplitude-suppression edges. Fig. 3.6a is an experiment from [29], wrinkling cascades appear near a constrained edge of a sheet that is floating on liquid. One may view such cascades, whereby the wrinkle wavelength changes very rapidly,
Figure 3.6: Wrinkling with a substrate of jumping stiffness. (a) Wrinkling cascades near the constrained edge of a polystyrene sheet floating on water [29]. (b-c) Simulations of jumping wavelength when $\bar{n}_0 = 3$, $\bar{n}_1 = 24$. (b) is with small tension $T \sim 4\sqrt{BK_2}$. (c) is with large tension $T \sim 400\sqrt{BK_2}$. 
as an accumulation or “condensation” of many defects. Fig. 3.6(b-c) show our simulations of jumping stiffness, where $n_1 = 3$, $n_2 = 24$. The surface in (b) is weakly stretched with a tension $T \sim 4\sqrt{BK}$, while in (c) the tension is 100 times larger. One can notice that cascades in (c) is similar to the boundary effects in (a). The underlying reason is that, the constraint implied by clamping a boundary is equivalent to attaching the wrinkled portion of the sheet to an infinitely stiff substrate, so it also leads to a jump of stiffness between the constrained and free portions. Moreover, the surface tension of the liquid bath is stretching the sheet, playing the role of $T$ in our simulation.

### 3.3.2 Defects under gradual changes of stiffness

We are more interested in case of the gradually changing stiffness and perform simulations with a relatively wide sheet of aspect ratio $L/W = 0.3$. We start with a linear $\bar{\lambda}(x)$ and set $\bar{n}_0 = 24$, $\bar{n}_L = 48$, in which case $\bar{\lambda}'(x) = 0.0695$. Figure 3.7a shows the wrinkling pattern of such a configuration within wrinkling regime $\Delta = 0.9/\bar{n}_L^2$. There emerge aligned defects which are similar to the case of spatially-varying $\hat{n}$ in Fig. 3.2. Through the cross section at $x = center$ in Fig. 3.7c, we can also see amplitude modulations along $\hat{y}$ axis, in accord with the results of [61].

The defects appear to be highly ordered which is quite different from the experimental observations in Fig. 3.3, Motivated by this difference, we tried different configurations and did not find a disordered distribution of defects inside the wrinkling regime Eq. 3.5. Suspecting that such a disorder may appear for sufficiently large amplitude, we turned to study the regime $\hat{\Delta} > \hat{\Delta}_f$. In order to avoid folding,
Figure 3.7: The wrinkling patterns for gradually varying stiffness, when $\bar{\lambda}(x) \sim \sqrt{x}$, $\bar{n}_0 = 24$, $\bar{n}_L = 48$, $T = 0$. (a) The natural wrinkling regime when $\Delta_c < \bar{\Delta} < \min_x \Delta_f(x)$, there appear aligned defects. (b) $\bar{\Delta} > \min_x \Delta_f(x)$, but we prevent the folding artificially by constraining the wave slopes at two ends. (c-d) The cross sections of (a) and (b) at $x = \text{center}$.

we artificially constrain the slope of wrinkling wave at two ends $x = \{0, L\}$ not exceeding the slope of a sinusoidal wave, i.e.

$$\left| \frac{\partial \zeta(x = 0 \text{ or } L, y)}{\partial y} \right| < \frac{2\pi A(x = 0 \text{ or } L)}{\bar{\lambda}(x = 0 \text{ or } L)} = 2\sqrt{\bar{\Delta}}$$

(3.12)

In practice, this constraint is not a hard limit in simulations, instead, we apply an energy penalty when it is violated. In Surface Evolver, this can be implemented with the method “edge\_scalar\_integral”. The result is in Fig. 3.7b, which shows a disordered distribution of defects, and there is no obvious amplitude modulation as Fig. 3.7d shows. Nevertheless, we must point out that this can be a local stable state or an artificial result.
Let us turn back to our focus, the result within wrinkling regime in Fig. 3.7a. The next thing we want to confirm is how $\lambda(x)$ affects the wrinkling patterns. For this purpose, we tried two other different $\lambda(x)$ — $\lambda(x) \sim \frac{1}{x}$ and $\lambda(x) \sim \sqrt{x}$, while keeping other parameters unchanged. Fig. 3.8 shows the comparison of the three, revealing that $\lambda(x)$ does not affect the fundamental features of the wrinkling patterns, e.g. the number of defects and defects lines, or the width of the defects lines $W$. What is affected is the distance between adjacent defects in one line. Thanks to this feature, we can choose to focus on the linear $\lambda(x)$.

![Figure 3.8: The wrinkling patterns for different $\lambda(x)$ while other parameters are same, $L/W \approx 0.3$, $n_0 = 24$, $n_L = 48$, $T = 0$. The patterns are very similar, except that the distances between adjacent defects are different. This implies the $\lambda(x)$ does not affects the fundamental features. (a) $\lambda(x) \sim x$ (b) $\lambda(x) \sim \frac{1}{x}$ (c) $\lambda(x) \sim \sqrt{x}$](image)

### 3.3.3 Distance between defects

If we zoom into one defects line in Fig. 3.7a (see Fig. 3.9), we naturally face two other questions. First, what is the distance between the adjacent defects $d$. Second, does there exist a preferred width $W$ for the defects domain, such that if
the sheet is one $\tilde{W}$ wider, there will be another defects line. The second question is highly non-trivial, it is intimately related to the modulations of amplitude, we do not offer a credible solution, but will discuss it in the discussion and give an asymptotic approach.

![Figure 3.9: Zooming into one defects line whose width we note as $\tilde{W}$, the distance between the adjacent defects is $d$.](image)

To solve the first question, we start with a simple case — a narrow ribbon exhibiting a single line of defects. We choose aspect ratio $L/W = 15.4$, $L = 80$, and in order to remain in the regime we proposed in Eq. 3.10, we set $\bar{n}_0 = 3$, $\bar{n}_L = 24$, and $T = T_0 \approx 4\sqrt{K_{LB}}$. Fig. 3.10 shows the simulation with such a configuration, it indeed has a single line of defects.

In order to describe the wrinkling pattern quantitatively, we employ Fourier analysis. In this approach, the deflection can be written as a summation of Fourier modes

$$\zeta(x, y) = \sum_{n=-\infty}^{\infty} f_n(x)e^{i\phi_n y}$$  \hspace{1cm} (3.13)
Figure 3.10: The simulation of a sheet with $L/W = 15.4$, $\bar{n}_0 = 3$, $\bar{n}_L = 24$, $\bar{\lambda}(x) \sim x$ and $T = T_0$.

Where $q_n = \frac{2\pi n}{W}$. In principle, $n \in (-\infty, \infty)$, but inspection of the Fourier spectrum suggests that significant modes are in the range $[n_0, n_L]$. Figure 3.11a shows all $f_n(x)$ together with the shape of the surface, revealing two remarkable features. One is that every mode $f_n$ dominates an interval in between two defects, where $\bar{n}(x) \approx n$ on average, which follows the local lambda law. The other is that every cross point of two adjacent modes is in the vicinity of some defect, which offers an effective way to determine the defect positions, namely, we can define the position of a defect to be the cross point of its adjacent modes. Then the distance between two adjacent defects $d$ can be easily extracted.

Expecting $d$ to be a function of $\bar{\lambda}$ and tension, we performed four simulations with different tensions, and compared the results in Figure 3.11b, which unravels:

$$d \sim \frac{\bar{\lambda}^2}{W}$$  \hspace{1cm} (3.14)

Surprisingly, $d$ is independent of $T$, indicating that $d$ is purely determined by the geometric structure. This result can be explained by considering the wavenumbers of adjacent modes. Since the wavenumber changes by one when crossing a defect,
Figure 3.11: The Fourier modes of \( f_n(x) \). From left to right (see peaks), \( n \) changes from 3 to 24. The cross points of adjacent modes are the positions of defects.

\[
\begin{align*}
0 & \leq x < W \\
\lambda(x) & = W
\end{align*}
\]

Figure 3.12: The schematic plot shows two adjacent modes. We assume that, wavenumbers is \( n \) at \( x \), and wavenumber is \( n+1 \) at \( x+d \).

It is reasonable to assume that, if at some position \( x \) the wavenumber is \( n \), then at \( x+d \) the wavenumber should be \( n+1 \) (see Fig. 3.12), \( i.e. \)

\[
\begin{align*}
(3.15a) & \quad n\bar{\lambda}(x) = W \\
(3.15b) & \quad (n+1)\bar{\lambda}(x+d) = W
\end{align*}
\]

Subtracting Eq. 3.15b from Eq. 3.15a and expanding \( \bar{\lambda}(x+d) \) at \( x \), we can easily get

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\[ d \approx \frac{\bar{\lambda}^2}{W\bar{\lambda}'(x)} \]  

(3.16)

Since this analysis does not depend on a specific \( \bar{\lambda}(x) \), it should be generally available for different \( \bar{\lambda}(x) \). In our case, \( \bar{\lambda}(x) \sim x \) so that \( \bar{\lambda}'(x) = const \), which immediately leads to Eq. 3.14. For the case of multiple lines of defects, we do similar analysis, but now replacing the \( n + 1 \) in Eq. 3.15b with \( n + m \), where \( m \) is the number of defects lines, \( i.e. m\bar{W} = W \). If \( \bar{W} \gg \lambda \), we easily get

\[ d \approx \frac{\bar{\lambda}^2}{W\bar{\lambda}'(x)} \]  

(3.17)

### 3.4 Discussion

As we hinted earlier, the mechanism that determines \( \bar{W} \) is a non-trivial question. Here we give an asymptotic result based on simulations. First, we notice that the defects line will cause the tilting of wrinkles, \( i.e. \) the wrinkling direction is not exactly \( \hat{y} \), but some direction \( \hat{n} = a\hat{x} + b\hat{y}, (a, b \neq 0) \), which means the wrinkles undulates not only along \( \hat{y} \), but also along \( \hat{x} \). The undulation period in \( \hat{x} \) is \( \lambda \), and the period in \( \hat{y} \) is 2\( d \), which is determined by distance between defects (See Fig. 3.13). Considering the tilting is symmetric with respect of defects lines, we can write the tilted undulations of one side. We assume \( \lambda, \bar{W} \) are approximately constants within two adjacent defects so that the deflection can be written of

\[ \zeta(x, y) \sim A \sin \frac{2\pi}{\lambda_m} y \sin \left( \frac{2\pi}{\lambda} y + \frac{\pi}{d} x + \delta \right) . \]  

(3.18)
where $\lambda_m = 2\tilde{W}$ is the modulation period, and $\delta$ is a constant phase. We furthermore assume that the sheet is incompressible in the $\hat{y}$ direction,

$$\int \left( \frac{\partial \zeta}{\partial y} \right)^2 dy \approx \Delta, \quad (3.19)$$

which gives

$$A \approx \frac{1}{\pi} \sqrt{\frac{2\Delta}{W} \frac{\lambda^2 \lambda_m^2}{\lambda^2 + \lambda_m^2}} \quad (3.20)$$

Plugging Eq. 3.18 and Eq. 3.20 to Eq. 3.8a-b and integrating $y$ out, we can get the bending and tension energies

$$e_b = \frac{4B\pi^2 \Delta}{\lambda^2} + \frac{K\Delta \lambda^2}{4\pi^2} + \frac{20B\pi^2 \Delta}{\lambda_m^2} + O\left( \frac{\lambda^4}{\lambda_m^4} \right) \quad (3.21a)$$

$$e_s = \frac{T\Delta \lambda^2}{4d^2} + O\left( \frac{\lambda^2}{\lambda_m^2} \right) = \frac{T\Delta \lambda^2 \lambda_m^2 \bar{\lambda}^2}{16\lambda^4} + O\left( \frac{\lambda^2}{\lambda_m^2} \right) \quad (3.21b)$$

Notice where we assume $\lambda \ll \lambda_m$. Eq. 3.21 is an interesting result. The bending energy $e_b$ is composed of two parts, the first two terms are essentially identical.
Figure 3.14: Here shows three deflection patterns with same configuration except tension, (a) $T = \frac{T_0}{2}$ (b) $T = T_0$ (c) $T = 2T_0$. We can notice that there are more defects lines when tension is larger. Though the defects lines are not well structured, we can still count the defects line and claim that $\bar{W} \sim T^{-\alpha}$, where $\alpha$ is small. If we naively count the numbers of defects lines $n_m$ and assume $\bar{W} = W/n_m$, we can get $\alpha \approx 0.29$.

to the energy of a uniform wrinkling with wavelength $\lambda$, while the third term is from the modulation of amplitude. Recalling that we assume here the parameter regime Eq. 3.9, the dominating energy is $e_b$, so when minimizing the energy, the wavelength is still determined by balancing wrinkling and substrate energy, which gives $\lambda \approx 2\pi \left( \frac{B}{K} \right)^{1/4} = \bar{\lambda}$. As for $\lambda_m$, we balance sub-dominating energies $e_s$ and the modulation energy,

$$\bar{W} = \frac{\lambda_m}{2} \approx \left( \frac{5B}{T} \right)^{\frac{1}{2}} \left( \frac{2\pi \bar{\lambda}}{\lambda'} \right)^{\frac{1}{2}} \quad (3.22)$$
Eq. 3.22 looks like another “local lambda law” that governs the modulation width. It is hard to test it numerically, because our working regime Eq. 3.9 is quite narrow. For this purpose, one needs a very large sheet to run multiple simulations. However, we can still provide a partial conformation for Eq. 3.22 by examining the scaling behavior, $\tilde{W} \sim T^{-\alpha}$, where $\alpha$ is a small positive power. In Fig. 3.14, we show results of three simulations of a relatively wide sheet with different tensions $\{T_0, T_0, 2T_0\}$ and keep other parameters identical. It is obvious that not every simulation has highly ordered defects lines, i.e. the defects line are not identical to each other and not evenly spaced. In which case, the modulation width is hard to define, but we can keep our logic simple and naively count the numbers of defects lines which are $n_m \approx \{6, 7, 9\}$ respectively. Then we use $\tilde{W} = \frac{W}{n_m}$ to make a regression and obtain $\alpha \approx 0.29$. Although the evidence provided by this analysis is admittedly weak, the key point is that $\alpha$ is small and positive, which reflects that our result Eq. 3.22 is meaningful.

3.5 Summary

In this chapter, we presented a simple model that can generate wrinkles with spatially-varying wavelength, which gives rise to distributions of “defects”. Such a model offers a relatively simple way to study the relationship between defects distribution and wavelength variation. With our simulations, we showed that a jump of stiffness of a substrate can generate wrinkling cascades, similarly to a boundary effect. We showed that defects are well aligned in a natural wrinkling regime, and this behavior is not affected by how the stiffness varies so long as $\bar{\lambda} \ll 1$. However,
the disordered distribution seen in the experiment is not observed in simulations and it possibly appears under conditions with larger compression beyond the wrinkling regime. We examined the distance between defects, and showed that it is primarily determined by the wavelength and gradient of substrate stiffness. Finally, we discussed the distance between defects lines and gave a reasonable asymptotic result and verified $\bar{W} \sim T^{-\alpha}$ where $\alpha$ is a small and positive power.
CHAPTER 4
SIMULATION OF ELASTIC SHEETS WITH SURFACE EVOLVER

Surface Evolver (SE) is a program originally designed for modeling liquid surfaces. It was later extended to support elastic surface modeling as well [8, 9]. However, it is not that straightforward and requires some effort.

In comparison to another widely used software in elastic surface simulation, Abaqus, SE is free and open source, and is based on energy or Lagrangian minimization. Such a feature makes it easier to be improved or extended. Users can define their own energies in input files or directly in the source code. In one of the examples in this chapter you will see how easy it is to implement a substrate with varying stiffness.

We start with the basics of SE in Sect. 4.1, and then illustrate in the following sections how to implement the simulations that appeared in the previous chapters (Chap. 2-3).

4.1 Basics of Surface Evolver

In this section, we will introduce a few fundamentals of SE so that readers can get a quick idea to understand the later sections of this chapter, but keep in mind
that this section is not intended to be a complete tutorial of SE. Readers should refer to the SE manual for more details [9].

4.1.1 Idea

Surface Evolver searches the equilibrium of a surface by minimizing the energy, e.g. the surface tension, strain energy or bending energy. To model the surface, it employs the idea of the finite element method and subdivides the surface into small triangles [8]. As the sizes of the triangles are small enough compared to the emergent structures (e.g. the radius of curvature), the surface can be well approached.

4.1.2 Input files

SE is a command-line based program, there is no graphical user interface. The users has to define the surface structures, energies etc. in a text file, which typically consists of the following three parts,

1. The first part defines various parameters, attributes and quantities. It may also include declarations of procedures and functions.

2. The second part describes the geometry of the surface, including lists of vertices, edges and facets.

3. The third part contains the commands that are read after the first two parts. This part is not necessary, you can always input the needed information interactively in the command prompt. Usually, it is used to set initial conditions and define reusable procedures and functions.
4.1.3 Mesh

A mesh is a discretization of a surface. The mesh structure is defined by a list of geometric elements, including vertices, edges and facets. All energies are carried by these elements, e.g. the surface tension is bound to each facet and is given by the facet area. Every element has an integer index and can be referred by its index.

- **Vertices**: A vertex is a point defined by a coordinate \((x, y, z)\).

- **Edges**: An edge is a directed line segment connecting two vertices.

- **Facets**: A facet is a triangle composed of three edges, it also has a direction which is specified by the direction of circled edges according to the right-hand rule.

Note that a body enclosed by facets can be defined, but this is beyond the scope of this thesis.

4.1.4 Refinement

Normally it is hard to write out a dense mesh with many facets. SE solves this problem with refinement. The user can initially give a sparse mesh that is sufficient to describe the geometry of the surface, then the mesh can be refined by adding new vertices at the midpoints of all edges, and each facet is subdivided into four new facets by connecting new vertices (see Fig. 4.1). We can always get a dense mesh after a few refinements.
4.1.5 Parameters

A parameter is a global variable that can be defined in the first part of the input file with the keyword \texttt{PARAMETER},

\begin{verbatim}
PARAMETER <name> = <value>
\end{verbatim}

or it can be defined implicitly by assignment operation in the command prompt, e.g. \texttt{len := 8; wid := 4}, a defined parameter can also be modified in this way.

A parameter can be defined as a degree of freedom in energy minimization by the keyword \texttt{OPTIMIZING\_PARAMETER}. A parameter and an optimizing parameter can be toggled with the commands \texttt{unfix} and \texttt{fix}.

4.1.6 Attributes

Every geometric element has its own attributes, some of which are defined internally by Surface Evolver, for example, every element has an attribute \texttt{id} which is its
integer index, every vertex has an attribute `edges` which is a list of edges starting or ending with it.

Users can also define additional attributes for elements in the following way,

```
define <element_name> attribute <attribute_name> <type[n]>
```

where `<element_name>` can be `vertex`, `edge` or `facet`, `<attribute_name>` is the name given by users, `<type[n]>` can be `integer` or `real` and the optional integer `[n]` indicates the attribute is a length-`n` array.

### 4.1.7 Methods and quantities

A named `method` is a way to calculate a physical value in a geometric element, such as the area, curvature, strain and so on. The sum of such a physical value of all elements is called a `quantity`. So the definition of a quantity must be associated with a method, a typical way is

```
quantity <quantity_name> <quantity_type> [modulus <modulus_value>]
method <method> [global] [parameters]
```

where `<quantity_name>` is a name specified by the user. `<quantity_type>` can be `energy`, `conserved` and other two other rarely used types, `energy` is the target to be minimized, `conserved` is a conserved quantity such as the center of mass in the absence of external force. `[modulus <modulus_value>]` is the optional modulus, which is 1 by default if omitted. `<method>` is the method to calculate this quantity, a full list of predefined methods can be found in the manual. `[global]` is also optional, it indicates that all corresponding elements are counted for this quantity, otherwise users must specify which elements are counted.
4.1.8 Boundaries

A *boundary* is a parametric border, vertices can move along this boundary, *e.g.* user can define a straight line, specified vertices will stay on this line and can slide along it.

\[
\text{boundary } <\text{boundary\_id}> \text{ parameters } <\text{number\_of\_parameters}>
\]

\[
x1: f1(p1, p2)
\]

\[
x2: f2(p1, p2)
\]

\[
x3: f3(p1, p2)
\]

where *boundary\_id* is an integer identifier, *number\_of\_parameters* is the number of parameters used to define the boundary, *p1, p2* are the parameters, and *f1, f2, f3* are user-defined functions. Both vertices and edges can lie in a boundary. For edges on boundaries, the new vertices from refinements will automatically lie in the boundary.

4.1.9 Constraints

Users can constrain the motion of vertices to follow an equality or inequality by defining a *constraint*,

\[
\text{constraint } <\text{constraint\_id}> [\text{nonpositive}|\text{nonnegative}]
\]

\[
\text{formula: } f(x,y,z) = 0
\]

where *constraint\_id* is an integer identifier, *f* is a custom function of *x, y, z*, [nonpositive|nonnegative] is for defining inequalities, nonpositive means \( f(x,y,z) \leq 0 \), while nonnegative means \( f(x,y,z) \geq 0 \), the constraint is an equality if not specified.
4.1.10 Minimization of energy

Surface Evolver minimizes the energy by gradient descent or Hessian iteration. To get the ideas, let us consider an energy $U(x)$, where $x$ is a vector whose dimension is the degree of freedom, e.g. it can include positions of all vertices. The idea of gradient descent is to move the $x$ along the opposite direction of the gradient, i.e. $x \rightarrow x - \gamma \nabla U(x)$, where $\gamma$ is the step size and is usually chosen empirically. Hessian iteration is to solve the equation $\nabla U(x) = 0$ with Newton’s method, $x \rightarrow x - H^{-1} \nabla U(x)$, where $H$ is the Hessian matrix. Compared to gradient descent, Hessian iteration does not require empirical hyper-parameters and is usually takes less time to approach the minimum, but it requires much more computer memory because it computes the full 2D Hessian matrix, which is much larger than the 1D gradient vector. In addition, Hessian iteration is more likely to stuck at local minima, so users had better use it after many gradient descent steps. In Surface Evolver, the command g is for gradient descent and the command hessian_seek is for Hessian iteration. Hessian iteration usually takes less time to approach the minimum compared to gradient descent [9].

4.2 Energies used in this thesis

This thesis focuses on elastic sheets, so in this section, we will introduce the implementation of related energies, including strain, bending and substrate energies.

Before starting the details, we should emphasize that SE is designed for liquid surfaces, there are two related features we do not want. First, the surface has a surface tension by default, which does not exist in elastic sheets. Second, vertices are constrained to move only in the direction perpendicular to the surface, because
the parallel movement will not change the surface tension. But in elastic sheets, parallel movement is important as it generates strain. Therefore, to simulate elastic sheets, it is crucial to turn off these two behaviors by adding the following code in the third part of the input file,

```plaintext
set facet tension 0 // disable surface tension
hessian_normal off // allow parallel movement
```

4.2.1 Elastic energy

Surface Evolver has implemented a method `linear_elastic` to calculate Hookean elastic energy, but users must take care of the reference configuration by themselves. For triangular meshes, the deformation gradient can be derived from the Gram matrix of edge vectors. Considering two edges starting from a vertex in a triangular facet, denote them as \( \mathbf{e}_1, \mathbf{e}_2 \) before strain and \( \mathbf{e}_1', \mathbf{e}_2' \) after strain (see Fig. 4.2). Define matrices \( \mathbf{S} = (\mathbf{e}_1 \ \mathbf{e}_2) \), \( \mathbf{S}' = (\mathbf{e}_1' \ \mathbf{e}_2') \), the deformation matrix satisfies

\[
\mathbf{S}' = \mathbf{F} \mathbf{S}
\]  

(4.1)
Eq. 4.1 is actually a finite version of Eq. 1.1, which means that $F$ transforms unstrained line elements into strained line elements. So $F = S'S^{-1}$, then we can get the strain tensor from Eq. 1.3, $\varepsilon = \frac{1}{2}(S^{-T}S'\varepsilon S^{-1} - 1)$. However, for computational efficiency, Surface Evolver uses an alternative definition of the strain tensor

$$C = S^T\varepsilon S^{-T} = \frac{1}{2}(S'^TSS'^{-T} - 1)$$

$$= \frac{1}{2}(G'G^{-1} - 1)$$

(4.2)

where $G = S'^T$ is the Gram matrix of edges. Since $\text{Tr} C = \text{Tr} \varepsilon$ and $\text{Tr} C^2 = \text{Tr} \varepsilon^2$, $\varepsilon$ can be replaced by $C$ in the strain energy Eq. 1.36a. The advantage of Eq. 4.2 is that we only need to provide $G$ in the reference configuration which is symmetric. Compared to $S$, there are only three independent entries which are fed to Surface Evolver as an array named $\text{form}_\text{factors} = \{e_1 \cdot e_1, e_1 \cdot e_2, e_2 \cdot e_2\}$.

$G$ will be automatically created by Surface Evolver from the initial surface geometry if the user does not define $\text{form}_\text{factors}$ by hand. Unfortunately, however, in most cases the mesh needs to be refined, and SE does not handle the $\text{form}_\text{factors}$ of new vertices from refinements, so users must take care of the $\text{form}_\text{factors}$ by themselves. Here we demonstrate how this can be handled. First, we need to enable elastic parameters by defining two attributes and a quantity of facets,

```
define facet attribute poisson_ratio real
define facet attribute form_factors real[3]
quantity stretch energy method linear_elastic global
```

where the Possion’s ratio can be set in the third part of the input file by,
The key is how we can set `form_factors` and deal with refinements. Remember that the refinement will create new vertices at the midpoints of edges, so the reference coordinates of a new vertex are just the average of the two endpoints of the old (before bisecting) edge. So, first, we need to know all the reference coordinates of vertices, this can be done by defining an attribute of length-three array. Second, after each refinement, we need to distinguish new vertices and edges from old ones. A feature of SE is helpful here: all user-defined attributes are automatically set to 0 if not specified. So we can define an integer attribute of vertex and edge and set it to be non-zero before refinement, it will be zero for new vertices and edges from refinements as they have not been set by the user. The code is

```plaintext
define vertex attribute ref_coord real[3]
define vertex attribute old_vid integer
define edge attribute old_eid integer
```

where `ref_coord` are the reference coordinates and `old_vid`, `old_eid` indicate whether a vertex or edge is old (exists before a refinement).

From the definition of `form_factors`, we can write a `procedure` to set them for all facets,

```plaintext
set_form_factors := {
    foreach facet ff do {
        set ff.form_factors[1]
    }
}
```

set ff.form_factors[2]

set ff.form_factors[3]
}
}

As for the refinement, we can write another procedure to deal with the reference coordinates and form_factors of new elements,

refine_basic := {

set vertex old_vid id; // can set to any non-zero integer
set edge old_eid id;
refine edges;
// set ref coordinates for all new vertices
foreach vertex vv where old_vid == 0 do {

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4.2.2 Bending energy

The bending energy Eq. 1.36b is composed by two parts: squared mean curvature and Gaussian curvature. They are implemented as methods `star_perp_sq_mean_curvature` and `star_gauss_curvature`, it is relatively easy to turn on them, first we need define two energies,

\[
\text{quantity bend energy method star_perp_sq_mean_curvature global}
\]

\[
\text{quantity gbend energy method star_gauss_curvature global}
\]

then the user must set the moduli correctly. From Eq. 1.36b, the modulus of squared mean curvature is \(2B\), and the modulus of Gaussian curvature is \(-B(1-\nu)\). Surface Evolver has no concept of physical thickness, the thickness only plays an implicit in
these two moduli implicitly. It is a good idea to define a global parameter to record the thickness,

\[
\text{PARAMETER thicknezz = <value> // do not use thickness}
\]

We use `thicknezz` instead of `thickness` because `thickness` is an internal keyword of Surface Evolver, which is used for graphical display purpose, but you can also choose any other reasonable name. Actually, if you are not sure whether a name is an internal keyword, e.g. “length”, “width”, it is a good practice to use other alternative names. Finally, we define a procedure with a parameter to set the thickness, which modifies the bending modulus,

```
procedure set_thickness(real th) {
    local pr;
    thicknezz := th; // modify the record
    pr := facets[1].poisson_ratio;
    bend.modulus := thicknezz^2 / 6 / (1 - pr^2);
    gbend.modulus := -thicknezz^2 /12/(1 + pr);
    recalc; // do not forget to recalc if something has changed
}
```

Defining the bending energy as global will cause a problem: Surface Evolver cannot properly handle the curvature at boundaries properly. For mean curvature, it assumes that all vertices are fully closed by edges, i.e. the vertex is surrounded inside by edges. If a vertex is on a boundary, its boundary edges are treated as if they were in mirror symmetry planes, which is often not true. For Gaussian curvature, Surface Evolver calculates it by \(2\pi - \text{sum of six angles around the vertex}\), which is
true for inner vertices, but introduces a fake constant into the energy for boundary vertices.

The best way to deal with the boundary problem is to neglect the boundary values, i.e. define the two energies as non-global, and count only the vertices inside. As long as the mesh is dense enough, the error is on the order of edge length/size of surface, which is negligible. To implement this, we need to distinguish the boundary edges, so we define an attribute `border` for edges,

```plaintext
define edge attribute border integer
```

We set all boundary edges with `border = 1` when creating the mesh. After refinements, the new boundary edges will inherit this value, so there is no extra work for refinements. However, we still need to count the new vertices from refinements for bending energy, this can be done by adding the following code to our refinement procedure,

```plaintext
refine_basic := {
  ... // code for ref coordinates
  foreach vertex vv do {
    // if edges of vertex are not border, add to bending calc
    if max(vv.edge, border) == 0 then {
      set vv bend;
      set vv gbend;
    }
  }
  ...
  // other code
}
```

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For Gaussian curvature, there can be other ways to deal with. First, the Gaussian curvature plays little role in many wrinkling problems, since the integral of the Gaussian curvature is actually a boundary term from the Gauss-Bonnet theorem, so users can choose to ignore the Gaussian curvature. Second, if the Gaussian curvature plays an important role in your problem, you can also try the `gauss_curvature_integral` method, which calculates the Gaussian curvature integral with the Gauss-Bonnet theorem, but we will not discuss it in this thesis.

4.2.3 Substrate energy

The substrate energy is just an integral of a resistance energy over the surface, it can be easily implemented with the method `facet_scalar_integral`. For a uniform substrate, we can do

```plaintext
quantity subenergy energy method facet_scalar_integral global
scalar_integrand: z^2
```

The modulus should be set to $K/2$. A non-uniform substrate is similar, e.g. the substrate we used in chapter 3, where $\bar{\lambda}(x) = l_s + l_f x$, when $K(x) = \left(\frac{2\pi}{\bar{\lambda}(x)}\right)^4 B$,

```plaintext
quantity subenergy energy method facet_scalar_integral global
scalar_integrand: 1 / (ls + lf*x)^4 * z^2
```

Since the wavenumber has a clearer physical meaning, we control the preferred wavenumber at two ends of the surface, a procedure like the following can handle it,

```plaintext
// set preferred wavenumber at left and right ends
```
procedure set_ksub(real lnum0, real lnum1) {
    num0 := lnum0;
    num1 := lnum1;
    w10 := ww / lnum0; // ww is the constrained width
    w11 := ww / lnum1;
    ls := w10;
    lf := (w11 - w10) / lngth; // lngth is the length
    subenergy.modulus := bend.modulus * 4 * pi^4; // B=bend.modulus/2
    recalc;
}

We can call this procedure like set_ksub(3, 24), which sets preferred the wavenumbers at two ends to be 3 and 24 respectively.

4.2.4 Tension

Surface Evolver can only handle energies, a tension exerted on a boundary must be transformed into an energy whose gradient is the tension. Here we introduce a way to apply a force to the boundary of a rectangular sheet, and the tension can be freely distributed, e.g. in chapter 2, there is a tension $T$ which gives $\int \sigma_{xx} dy = TW$ rather than $\sigma_{xx}(x = \text{ends}) = T$, where $\sigma_{xx}(x = \text{ends}, y)$ is an unknown function of $y$.

We fix the left boundary of the rectangular sheet and apply a force on the right boundary, which will stretch the sheet and move the right boundary to the right. First we need to define a boundary so that the boundary can move as a rigid line,
boundary 1 parameters 1
x1: rightEndX
x2: p1
x3: 0

where the boundary is \( x = \text{rightEndX} \). Note that the vertices can still slide on this boundary. Then we add an energy as negative work,

\[
\text{quantity righttension energy method vertex_scalar_integral}
\]

\[
\text{scalar_integrand: } -(x - f\text{Len}) * wd \quad /\!/ \text{wd is the original width}
\]

where \( f\text{Len} \) is a global parameter that should be the estimated “final length” of the sheet, \( i.e. \) the length after stretching. Though, theoretically, the choice of \( f\text{Len} \) has no effect on the physics since it only changes the choice of the zero point, which only introduces a constant to the energy. Numerically, we do not want a large constant to cover the main energy, so it is best to set \( f\text{Len} \) as the final stretched length. This energy can be added to any vertex in the boundary, its modulus should be tension \( T \). At last, \( \text{rightEndX} \) should be optimizable, so we declare it as an \text{OPTIMIZING_PARAMETER},

\[
\text{OPTIMIZING_PARAMETER rightEndX = lgth} \quad /\!/ \text{initially set to the length}
\]

4.3 Typical Meshes

We will introduce two typical mesh geometries used in this thesis — rectangular and annular meshes, which are the most commonly used geometries when studying 2D elasticity.
4.3.1 Rectangular mesh

As we have mentioned, Surface Evolver works best with equilateral triangles. Fortunately, a rectangular mesh can be filled with mostly equilateral triangles. See Fig. 4.3a for example, we can align the equilateral triangles with the length of the rectangle. In this way, the bulk can be filled with equilateral triangles, only a small region near two ends is filled with right triangles.

For this mesh structure, we want a refinement to produce a mesh like Fig. 4.3b, which preserves a similar structure. However, noticing that the two short base edges (thick black) of the right triangles are not divided, we cannot directly use the default refinement method, since it divides all edges. Thanks to the fact that the base edges are the shortest, an easy way to solve this problem is to select dividing edges by length. The threshold length should be longer than the base edges and shorter than the heights, which suggests a safe choice: the middle value of these two lengths. To do this, we first define a variable `gridsize` to keep track of the height of the triangles,

\[
\text{PARAMETER gridsize} = \text{<initial_height>}
\]

Then the length of the base edges would be \( \frac{1}{\sqrt{3}} \times \text{gridsize} \). We can do selected refinement by,

\[
\text{refine edges where length} > (1 + 1/\sqrt{3})/ 2 \times \text{gridsize};
\]

After this, you may find that the resulting mesh is quite different from Fig. 4.3b, but is similar to Fig. 4.3c which looks very messy. The reason is that SE connects new vertices randomly, so the generated mesh is probably not the best. Fortunately, it offers a procedure called `equiangulate` to solve this problem, which tries to adjust
Figure 4.3: An example of a rectangular mesh and its refinement. (a) A simple rectangular mesh composed of mostly equilateral triangles and some right triangles near two ends. (b) The refined mesh of (a). To preserve the well-structured arrangement of the equilateral triangles, the base edges (thick black edges) of the right triangles should not be divided. (c) SE randomly connects new vertices from refinements, the mesh can be disordered as shown here. To generate a mesh like (b), a procedure called *equiangulation* is required.
edges to make triangles as close as possible to equilateral triangles. So, it is always preferable to do such equiangularization after refinements. With all the above information, we can add the following code to the `refine_basic` procedure we previously mentioned to conveniently handle refinements of a rectangular mesh.

```plaintext
refine_basic := {
    ... // pre code
    refine edges where length > (1 + 1/sqrt(3)) * gridsize;
    gridsize /= 2;
    ... // other code
    u; // equiangulate all edges
}
```

### 4.3.2 Annular mesh

Compared to rectangular meshes, making an annular mesh of high quality is much harder, it can be an independent research project. Fortunately, there is a great algorithm “DistMesh” developed by Persson and Strang that can generate high-quality triangular meshes [46]. The basic idea is to treat the mesh edges as springs with a preferred length $l_0$. If the length of an edge is not $l_0$, there would be an energy penalty, so the lowest energy state would prefer a uniform mesh with all edges of the same length, which constructs equilateral triangles. “DistMesh” was
originally implemented with Matlab\textsuperscript{1}, then translated to Python by Nico Schlömer\textsuperscript{2}.

\begin{figure}[h]
\centering
\subfigure[]
\includegraphics[width=0.45\textwidth]{uniform_mesh.png}
\subfigure[]
\includegraphics[width=0.45\textwidth]{non_uniform_mesh.png}
\caption{Mesh structures generated by “DistMesh”. (a) A uniform mesh. (b) A non-uniform mesh with linearly increasing edge length along the radius.}
\end{figure}

Figure 4.4a shows an example of an annular mesh generated by “DistMesh”, it is quite well-shaped, denser mesh can perform even better. In some problems of annular or circular geometries, the curvature and strain are usually concentrated near the inner boundary or center, \textit{e.g.} the developable cones \cite{11, 15}. Such problems require a very dense mesh to properly simulate the strain / curvature focus near the center. However, the curvature and strain near the outer boundary are small, it is not necessary to maintain such a dense mesh over the whole surface. If we use a uniform mesh, then the mesh density should be adapted to the needs of the inner

\textsuperscript{1}http://persson.berkeley.edu/distmesh/
\textsuperscript{2}https://github.com/nschloe/dmsh

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boundary, so there will be tons of elements and simulations will slow down severely. One way to solve this problem is to adopt a non-uniform mesh, we can generate a mesh whose density decreases with the radius, which is supported by “DistMesh”. Figure 4.4b shows an example of linearly increasing edge lengths with radius. Our simulations with this kind of mesh structure show highly coincident results with our Matlab numerical results. These two different simulation mechanisms prove for each other.

An annular or circular mesh also requires special care for refinements. Since the new vertices are at the middle of old edges, the refinements cannot make the inner or outer boundary closer to a circle, so we must adjust their reference coordinates to make the new vertices lie exactly on the circular boundaries after refinements (see Fig. 4.5). The first step is to distinguish the vertices on the boundaries, we can define two integer edge attributes inborder and outborder and set them to 1 for inner and outer boundary edges respectively. Then we can add the following code to the refinement procedure,

refine_basic := {
    ...
    local temp_r;
    foreach vertex vv where old_vid == 0 do {
        ...
        if max(vv.edge, inborder) > 0 then {
            temp_r := in_r / sqrt(vv.ref_coord[1]^2 + vv.ref_coord[2]^2);
            vv.ref_coord[1] := temp_r * vv.ref_coord[1];
        }
    }
}
Figure 4.5: Here shows vertices and edges (blue) that lie on a circular boundary. The new vertices (red empty circles) from the refinement needs to be projected onto the boundary (red dots).

\begin{verbatim}
vv.ref_coord[2] := temp_r * vv.ref_coord[2];
// can set inclusion here
} else if max(vv.edge, outborder) > 0 then {
    temp_r := out_r / sqrt(vv.ref_coord[1]^2 + vv.ref_coord[2]^2);
    vv.ref_coord[1] := temp_r * vv.ref_coord[1];
    vv.ref_coord[2] := temp_r * vv.ref_coord[2];
};
};
... // other code
\end{verbatim}
where $\text{in}_r$ and $\text{out}_r$ are radius of inner and outer boundaries respectively.

Constrained by the language, the Matlab and Python implementations of “DistMesh” are slow, it takes quite a long time to create a large surface. One improvement people can do is to implement this algorithm with Surface Evolver, as it also supports the string model. This idea can be a future work.

4.4 Boundary conditions

In this section, we introduce two commonly used boundary conditions: fixed boundary and periodic boundary. Note that boundaries are free by default, so we do not need to do any extra work for free boundaries.

4.4.1 Fixed boundary

Surface Evolver supports to set fixed vertices and edges. A fixed vertex will not move during energy minimization, a new vertex generated by refinements will be fixed if it is on a fixed edge. To set a fixed boundary, the easiest way is to fix all vertices and all edges on that boundary when creating the mesh, and we do not need to worry about refinements later.

If we know the boundary satisfies a condition, we can also fix it by commands, *e.g.* fix the boundary at $\text{ref}\_coords[0] = 0$,

```
fix vertex where ref\_coords[0] == 0
```

but we need do this after each refinement.
4.4.2 Periodic boundary conditions

The periodic boundary condition is relatively complicated for elastic sheets, especially when refinements are needed. We will show how to set a periodic boundary condition in the defects problem of chapter 3.

For convenience, let us agree on several notations: \( v_i \) represents a vertex with \( \text{id} = i \); \( e_{i,j} \) represents an edge starting with \( v_i \) and ending with \( v_j \); \( f_{i,j,k} \) represents a facet enclosed by \( e_{i,j}, e_{j,k}, e_{k,i} \). Consider a rectangular mesh in Fig. 4.6a, whose length (horizontal dimension) is \( L \) and width (vertical dimension) is \( W \). We define the horizontal axis as \( \hat{x} \) and the vertical axis as \( \hat{y} \), with the origin is at the bottom left corner. If we want a periodic boundary condition in the \( \hat{y} \) direction, then the upper and lower vertices and edges should be “sticked” together. In Surface Evolver, this is implemented by reusing the vertices and edges of one boundary. See Fig. 4.6b, we define \( v_1, v_2, v_3 \) at the lower boundary (solid red), the upper boundary (dashed red) reuses these vertices. Readers can image that the surface is mapped onto a cylinder, and the edges \( e_{11,1}, e_{12,1} \) etc. are wrapping around the cylinder in the other direction, this kind of edges are so called wrapping edges.

Surface Evolver can do periodic boundary conditions in two basis directions for surfaces \(^3\), which is equivalent to mapping the surface to a torus. And consequently, this mechanism is called torus modeling. To enable it, we must first define the basis vectors in the first part of the input file,

\[ \text{torus} \]

\(^3\)Surface Evolver can do three-direction periodic BC for bodies, but we focus on surfaces.
Figure 4.6: (a) A typical rectangular mesh composed of mostly equilateral triangles and some right triangles at two boundaries. (b) The upper and lower boundaries (red) are “sticked” as one boundary. (c) An example of a wrapping edge $e_{12,2}$ that wraps in the direction of the basis vector.

```
periods // 3 basis vectors in 3D
1 0 0
0 upEndY 0 // length of the vector is the period
0 0 1
```

where we defined three basis vectors $b_1 = (1, 0, 0), b_2 = (0, upEndY, 0), b_3 = (0, 0, 1)$, and upEndY is the “upper end” of the sheet, which is usually the width if there is no compression. A basis vector $b_i$ gives the direction and length of a period, i.e. for any field $\varphi(x)$ in the surface satisfies $\varphi(x) = \varphi(x + b_i)$. In our example, we only want a period in the $\hat{y}$ direction, so $b_1, b_3$ are useless here, but they are required by the
syntax. Next, we need to indicate how the edges are wrapping with three candidate symbols *, +, -, whose meanings are respectively,

- **not wrapping**
- + wrapping in the direction of the basis vector, so that the basis vector is added to the edge vector.
- - wrapping in the opposite direction of the basis vector, so that the basis vector is subtracted from the edge vector.

See the example in Fig. 4.6c, considering the edge \((v_{12}, v_2)\), the original edge not wrapping is \(e'_{12,2}\), and the edge \(e_{12,2} = e'_{12,2} + b_2\) is wrapping in the direction of \(b_2\).

The corresponding code is,

```plaintext
... // first part and vertex list
edges
... // other edges
<edge_id> 12 1 * + *
... // following code
```

where ** + * indicates that \(e_{12,1}\) is wrapping in the direction of \(b_2\) and is not wrapping in directions of \(b_1, b_3\). Other wrapping edges can be defined similarly.

At last, as a result of wrapping, our previous algorithm for calculating **form_factors** fails at the wrapping facets, *i.e.* those have one or more wrapping edges like \(f_{12,1}\), \(f_{12,13,2}\). For these facets, the reference coordinates of the boundary vertices \(v_1, v_2, v_3\) should be corrected by adding the basis vector in the reference configuration. To do this, we need to distinguish the wrapping edges and the boundary vertices. Edges
have an internal attribute named \texttt{wrap} that indicates whether an edge is wrapping. But we must find our own way to distinguish boundary vertices, which can be easily done by defining an extra attribute for boundary edges. Nevertheless, for our example, we know the boundary vertices are at \( y = 0 \), so we can deal with \texttt{form_factors} in following way,

\[
\text{set\_form\_factors\_periodic} := \{
\begin{align*}
\text{local } & \text{vp1;} \\
\text{local } & \text{vp2;} \\
\text{local } & \text{vp3;} \\
\end{align*}
\begin{align*}
\text{foreach } & \text{facet } ff \text{ do } \{ \\
\text{ // not wrapping facet} & \\
\text{if } & \text{max(ff.edge, wrap)} == 0 \text{ then } \{ \\
\text{ ... } & \text{// code of previous non-periodic version} \\
\} & \text{ else } \{ \text{ // wrapping facet} \\
\text{ // wd is original width} & \\
\text{vp1} & := (ff.vertex[1].ref\_coord[2] == 0) * \text{wd}; \\
\text{vp2} & := (ff.vertex[2].ref\_coord[2] == 0) * \text{wd}; \\
\text{vp3} & := (ff.vertex[3].ref\_coord[2] == 0) * \text{wd}; \\
& \text{set } ff.\text{form\_factors}[1] \\
& \text{set } ff.\text{form\_factors}[2]
\end{align*}
\]
If we do not need refinements, then it is completed up to here. But if we do, we must correct the reference coordinates of the new vertices generated by refinements at the wrapping edges. Again, we add the basis vector to the boundary vertices,

refine_basic_periodic := {
  ...
  foreach vertex vv where old_vid == 0 do {
    ...
    if max(vv.edge where old_eid != 0, wrap) > 0 then {
    }
  }
};

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where we add half of the width instead of the width to the \texttt{ref_coord[2]} if the vertex is on a wrapping edge, because the coordinates are averaged from two ends of the original edge, e.g. in Fig. 4.6b, a refined vertex at the middle of $e_{12,2}$ has coordinates averaged from $v_{12}$, $v_{2}$, \textit{i.e.} $(v_{12} + v_{2})/2$, while only the coordinates of $v_{2}$ need to be corrected by adding the basis vector, which is the width in our case, so it needs to be divided by 2.

The implementation of periodic condition of other mesh structures can be more complicated, but the idea is similar. It is important to verify the implementation by checking the strain energy of the wrapping facets, because it is easy to make mistakes in such a complex procedure.

### 4.5 Summary

In this chapter, we introduced essential details for the simulations of elastic sheets with SE. We started with the basics of SE and gave the implementations of elastic energies, including strain, bending, and substrate energies. Then we illustrated the method of constructing high-quality meshes for rectangular and circular sheets and explained how to properly handle corresponding refinements. At last, we illuminated two commonly used boundary conditions, fixed and periodic for rectangular sheets. With these information, we hope SE can be more accessible for elastic simulations.
In the stretch-induced wrinkling problem (Chap. 2), we explained the origin of the transverse compression, which actually comes from the relative edge extension. The stretching, clamping and Poisson’s contraction together generates a relative edge extension, which acts like a normal compressional confining geometry, and the buckling or wrinkling occurs to release this compression, similarly to the mechanism of the Euler buckling or uniform wrinkling we discussed in Chap. 1. Thus, the stretch-induced wrinkling is not solely a tensional effect and does not necessarily rely on Poisson’s effect; An identical compressive stress profile can also be generated by replacing the tensile load on the edge with a manually applied edge extension even when the Poisson’s ratio of the sheet is zero. A potential study of interest is to apply this setup, in which, the compression is decoupled from the tension (does not depend on the tension), and the tension only plays a role of effective substrate. Consequently, this problem would have two independent control parameters – confinement and effective substrate stiffness, which make it a typical wrinkling problems like the uniform wrinkling. I would expect the instability type to become a wrinkling instability due to this decoupling. Another interesting study is to further understand how an edge extension generates a compression. My intuition is that the compression comes from
Figure 5.1: A schematic of the “scissoring effect”. (a) If the upper and lower halves are independent, they can act like a part of scissors. (b) Imaging that we cut a part of the sheet along the middle line, then extend the uncut edge, if we can see the two halves scissor, we can confirm that there is indeed “scissoring effect” in edge extensions.
a “scissoring effect”. See Fig. 5.1(a), whereby the upper and lower halves of the sheet can act like a pair of scissors, as if they were independent parts, but since they are in one sheet, the “scissoring effect” is more subtle. A simple way to verify this idea is to cut a part of the sheet into two halves along the longitudinal direction and then extend the uncut transverse edge to check whether the two halves “intersect” (see Fig. 5.1b). This intuition can also explain how an edge contraction gives rise to an extensional zone. Finally, one may wish to further explore the wrinkling extent we have mentioned in the discussion section of Chap. 2. Since the transverse compression zone is limited to a small region near the clamped edges, the wrinkling does not necessarily extend to the whole sheet so long as the bending modulus is finite. A theoretical prediction of the extent can give insights into how wrinkles evolve out of the confinement zone.

In Chap. 3, we proposed a theoretical model to study the wrinkling patterns with spatially-varying wavelength. It is a small step towards understanding more complicated real-world wrinkling phenomena. A crucial aspect of this problem is to figure out the distribution of the emergent defects, and we find that they tend to align with a line density that is determined by the preferred wavelength and its gradient (Eq. 3.17). As for the distance between the defects lines, we only gave an asymptotic result whose validity requires a more elaborate numerical study. A further study of this distance should be conducted, especially for cases without tension, since the tension in our solution is a “trick” to simplify the analysis, but the mere existence of defects does not depend on the tension. Additionally, future work should concentrate on modified models with more complicated substrates, e.g. eliminating the
wrinkle-to-fold transition by adding a nonlinear term to the substrate stiffness, which would allow for the study of wrinkling patterns beyond the parameter regime limited between wrinkling and folding thresholds. A potential finding could be the unstructured (or unaligned) distribution of defects that were observed in the experiments. Furthermore, one can make the substrate stiffness vary along both longitudinal and transverse directions. In this case, according to Eq. 3.16, the number of defects in every defects line cannot be nearly identical as shown in our simulations (Fig. 3.8), which breaks the symmetries of defects lines and thus the defects distribution may be more interesting.
APPENDIX

APPENDIX OF CHAPTER 2

A.1 Solution of planar stress

In this section, we describe the detailed solution of model B with BCs (2.4c-2.4f). In our analysis we follow closely the calculation of J.P. Benthem[5], who employed Laplace transformation to compute the stress at the clamped edge. Here we go beyond Benthem solution by performing an inverse Laplace transformation, which is required to evaluate the stress in the whole sheet.

A.1.1 The singularities in the corners

Let us consider first the vicinity of one corner of the sheet in Fig. ?? A, where one edge is clamped and the other one is free, and solve Eq. 2.2 in polar coordinates \((r, \theta)\), see Fig. A.1, which depicts the left bottom corner.

Figure A.1: A corner of a rectangular sheet. We set the corner as the origin of polar coordinates.
\[
\left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right)^2 \Phi = 0 \quad (A.1)
\]

The BCs Eq. (2.3b, 2.3c) now become

- free boundary (\( \theta = 0 \)) : \( \sigma_{r\theta} = \sigma_{\theta\theta} = 0 \), \hspace{1cm} (A.2a)
- clamped boundary (\( \theta = \frac{\pi}{2} \)) : \( u_r = u_\theta = 0 \). \hspace{1cm} (A.2b)

The general solution of Eq. A.1 is

\[
\Phi(r, \theta) = r^{\gamma+1} (D_1 \sin(\gamma + 1)\theta + D_2 \cos(\gamma + 1)\theta)
+ D_3 \sin(\gamma - 1)\theta + D_4 \cos(\gamma - 1)\theta) \quad (A.3)
\]

Notice that Eq. (A.3) has 4 unknowns \( D_i \), such that the four BCs (A.2) give rise to a nonzero solution only if the determinant of the corresponding 4x4 matrix vanishes, yielding:

\[
\sin^2 \left( \frac{\gamma \pi}{2} \right) = \frac{4 - \gamma^2(1 + \nu)^2}{(3 - \nu)(1 + \nu)} . \quad (A.4)
\]

For a sheet with positive Poisson’s ratio, \( \nu > 0 \), the solution of Eq. (A.4) is \( \gamma < 1 \), such that:

\[
\sigma \sim \frac{\partial \Phi}{\partial r^2} \sim r^{\gamma - 1} , \quad (A.5)
\]

and hence the components of stress tensor diverge at the corner (i.e. \( r \to 0 \)). For convenience, we choose to focus on a specific value, \( \gamma = 3/4 \), which corresponds to
Figure A.2: The configuration of a semi-infinite sheet \((L \to \infty)\). We set the width \(W = 2\) for convenience, the left boundary \(x = 0\) is clamped and the two boundaries \(y = \pm 1\) are free.

\[ \nu = \frac{17 + 4\sqrt{2} - 8\sqrt{7}}{4\sqrt{2} - 1} \approx 0.32. \] For this value, the the boundary values of \(\sigma_{\theta\theta}\) and \(\sigma_{r\theta}\) are given by [5]:

\[ \sigma_{\theta\theta}(\theta = \frac{\pi}{2}) = c_1 r^{-\frac{1}{4}}, \quad \sigma_{r\theta}(\theta = \frac{\pi}{2}) = -c_2 r^{-\frac{1}{4}} \]

\[ c_1 = \sqrt{8 + 3\sqrt{7}}c_2 \]  

(A.6)

**A.1.2 Laplace transformation**

Let us address now the Airy potential in a semi-infinite sheet (Fig. A.2). For convenience, we set the width \(W = 2\) \((i.e. y \to \frac{y}{2W}, x \to \frac{x}{2W})\), and transform the origin of the coordinate system to the middle of left edge, such that the left clamped boundary is \(x = 0\) and the two free boundaries are \(y = \pm 1\). To solve Eq. (2.2), we employ Laplace transformation,

\[ f(p, y) = \int_0^\infty \Phi(x, y)e^{-sx}dx, \]  

(A.7)

such that the transformed Eq. (2.2) becomes
\[ s^4 f + 2s^2 \frac{\partial^2 f}{\partial y^2} + \frac{\partial^4 f}{\partial y^4} = s^3 \Phi(0, y) + s^2 \frac{\partial \Phi}{\partial x}(0, y) \]
\[ + s \left( \frac{\partial^2 \Phi}{\partial x^2}(0, y) + 2 \frac{\partial^2 \Phi}{\partial y^2}(0, y) \right) \]
\[ + \frac{\partial^2 \Phi}{\partial x^3}(0, y) + 2 \frac{\partial^3 \Phi}{\partial x \partial y^2}(0, y) \]

(A.8)

With the up-down symmetry \( \Phi(x, y) = \Phi(x, -y) \), the singularities at corners, Eq. (A.6), and BC (2.4c), we can assume \cite{5}

\[ \frac{\partial^2 \Phi}{\partial y^2} = \sigma_{xx}(0, y) = c_1 \{(1 - y)^{-\frac{1}{4}} + (1 + y)^{-\frac{1}{4}} - 2^{-1/4}\} \]
\[ + \sum_{n=1} a_n \cos(q_n y) \]
\[ \frac{\partial^2 \Phi}{\partial x \partial y} = -\sigma_{xy}(0, y) = c_2 \{(1 - y)^{-\frac{1}{4}} - (1 + y)^{-\frac{1}{4}} + y 2^{-1/4}\} \]
\[ + \sum_{n=1} b_n \sin(k_n y) \]

(A.9)

where \( q_n = (2n - 1)\frac{\pi}{2}, k_n = n\pi \) and \( a_n, b_n \) are constants that must be computed. Equation (A.9) and the BCs (2.4e,2.4f) enable us to express all boundary terms in the right side of Eq. (A.8) in terms of the unknowns \( c_1, c_2 \) and \( a_n, b_n \), whereas the BCs (2.4c,2.4d) become:

\[ f(s, y = \pm 1) = 0 \] \hfill (A.10a)
\[ \frac{\partial f}{\partial y}(s, y = \pm 1) = 0 \] \hfill (A.10b)

The solution \( f(s, y) \) to Eq. (A.8) can now be expressed in the following form \cite{5}:

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\[
f(s, y) = g(s, y) - 2g(s, 1) \frac{(s \cos s + \sin s) \cos sy + s(\sin s) y \sin sy}{\sin 2s + 2s} + 2 \frac{\partial g}{\partial y}(s, 1) \frac{(\sin s) \cos sy - (\cos s) y \sin sy}{\sin 2s + 2s}
\]  
(A.11)

where \( g(s, y) \) is the particular solution of the non-homogeneous ODE (A.8), while the left two terms correspond to solutions of the homogeneous equation (i.e. replacing the right side with 0), which enable satisfying the BCs (A.10). The function \( g(s, y) \) is relatively complicated, and in order to express it compactly we define an “auxiliary” function:

\[
R(q; s, y) = \frac{1}{4s^{4+2q}} \left( e^{isy}(-is)^q ((-1 + is) \Gamma(1 + q; is) + (1 - isy) \Gamma(1 + q; isy) - \Gamma(2 + q; isy) + \Gamma(2 + q; -isy))
\right)
\]  
(A.12)

where \( \Gamma(x; y) \) is the incomplete gamma function. Then \( g(s, y) \) can be written as
\[ g(s, y) = \frac{s^3 c_1}{1.3125} \left( R \left( \frac{7}{4}; s, 1 - y \right) + R \left( \frac{7}{4}; s, 1 + y \right) \right) \]
\[ - \frac{4}{3} s^2 c_2 \left( R \left( \frac{3}{4}; s, 1 - y \right) + R \left( \frac{3}{4}; s, 1 + y \right) \right) \]
\[ + s(2 + \nu)c_1 \left( R \left( -\frac{1}{4}; s, 1 - y \right) + R \left( -\frac{1}{4}; s, 1 + y \right) \right) \]
\[ - \frac{1}{4} \nu c_2 \left( R \left( -\frac{5}{4}; s, 1 - y \right) + R \left( -\frac{5}{4}; s, 1 + y \right) \right) \]
\[ + \left( -s^3 c_1 2^{-\frac{5}{4}} + s^2 c_2 2^{-\frac{5}{4}} \right) \frac{s^2 y^2 - 4}{s^6} \]
\[ + \frac{c_1}{s} \left( 2^{\frac{5}{4}} - \frac{2^2}{1.3125} \right) - \frac{2 + \nu}{s^3} c_1 2^{-\frac{1}{4}} \]
\[ - \frac{\nu}{s^4} c_2 2^{-\frac{1}{4}} + \frac{\alpha}{s^3} + \frac{b_0}{s^2} \]
\[ + \sum_{n=1} \frac{-s^3 q_n^2 + s(2 + \nu)}{(s^2 - q_n^2)^2} a_n \cos q_n y - \frac{s^2 k_n + \nu k_n}{(s^2 - k_n^2)^2} b_n \cos k_n y \]  
(A.13)

where \( b_0 \) is an integration constant:

\[ \frac{\partial \Phi}{\partial x}(0, y) = \int \frac{\partial^2 \Phi}{\partial x \partial y}(0, y) dy + b_0. \]  
(A.14)

### A.1.3 Solution of Laplace transform and its inversion

Next we must solve all unknowns \( a_n, b_n, c_1, b_0 \) that define the function \( g(s, y) \), Eq. (A.13), substitute it in Eq. (A.11) to obtain the Laplace transform \( f(s, y) \), and perform the inverse transform

\[ \Phi(x, y) = \frac{1}{2i} \int_{c-i\infty}^{c+i\infty} f(s, y) e^{sx} ds \quad (c > 0) \]  
(A.15)

to obtain the Airy potential \( \Phi(x, y) \).
Recalling that our analysis here assumes a semi-infinite sheet, for which we expect the stress potential \( \Phi(x, y) \) vanishes exponentially away from the clamped edge at \( x = 0 \), we note that the Laplace transform \( f(s, y) \) must not have any poles at the right half of complex plane \( \text{Re}(s) > 0 \). Inspection of Eqs. (A.11, A.13) reveals that the only poles originate from the contribution of the homogeneous solutions of Eq. (A.8), namely, complex numbers \( s \) which solve the nonlinear equation:

\[
\sin 2s + 2s = 0. \tag{A.16}
\]

This equation has an infinite sequence of solutions in the right half plane, \( \{s_i \text{ and } \bar{s}_i; i \in \mathbb{N}, \text{Re}(s_i) > 0, \text{Im}(s_i) > 0 \} \), which can be arranged in increasing order, \( |s_{i+1}| > |s_i| \). Requiring the residues of \( f(s, y) \) at \( s_i \) and \( \bar{s}_i \) to vanish, yields an infinite sequences of algebraic equations

\[
\text{Res} f(s_i, y) = 2 \left( \frac{\partial g}{\partial y}(s_i, 1) - g(s_i, 1) \sin^2 s_i \right) G(s_i, y) = 0, \tag{A.17}
\]

where we used Eqs. (A.11, A.13) and defined

\[
G(s_i, y) = \frac{(\sin s_i) \cos s_i y - (\cos s_i) y \sin s_i y}{4 \cos^2 s_i}. \tag{A.18}
\]

Equations (A.17, A.13), together with the complementary set of conjugate equations for the residues \( \text{Res} f(\bar{s}_i, y) = \overline{\text{Res} f(s_i, y)} \), determine the unknowns \( a_n, b_n, c_1, b_0 \) in Eq. (A.13), and thereby the Laplace transform \( f(s, y) \) through Eq. (A.11). In order to obtain a numerical solution of this system of equations, we truncate the sequence.
of unknowns, and keep only the 1000 leading unknowns (ordered by increasing magnitude of the corresponding poles $|s_i|$).

Furthermore, in order to carry out the integration in Eq. (A.15), we can close the contour in the left half complex plane $\text{Re}(s) < 0$ and compute it through the sum of residues of $f(s, y)$. Noticing that the poles of $f(s, y)$ in the complex $s$ plane are solutions of Eq. (A.16), one readily realizes that the poles in the left half of the complex plane are $\{-s_i, -\bar{s}_i\}$, and consequently:

$$
\Phi(x, y) = \sum_{i=1}^{\infty} \text{Res} \left( (f(s, y)e^{sx}; -s_i) + \text{Res} (f(s, y)e^{sx}; -\bar{s}_i) \right)
= 4\text{Re} \sum_{i=1}^{\infty} \left( g(-s_i, 1) \sin^2 s_i - \frac{\partial g}{\partial y}(-s_i, 1) \right) G(s_i, y)e^{-s_ix}
$$

(A.19)

Finally, recovering our transformation $y \rightarrow \frac{2y}{W}, \ x \rightarrow \frac{2x}{W}$ and define $p_i = 2s_i$, we can get Eq. (2.7b)

$$
\Phi(x, y) = \text{Re} \left\{ \sum_i C_i \cdot \left( \Phi^{i,c}(x, y) + A_i \Phi^{i,s}(x, y) \right) \right\},
C_i = \frac{\sin \frac{p_i}{2}}{\cos^2 \frac{p_i}{2}} \left( g\left(-\frac{p_i}{2}, 1\right) \sin^2 p_i - \frac{\partial g}{\partial y}\left(-\frac{p_i}{2}, 1\right) \right)
$$

(A.20)

A.2 TFT simulations

In simulating a sheet with no bending rigidity, any compression gives rise to an infinitely corrugated shape, limited only by the mesh size. In order to check that these simulations provide the TFT solution reliably, we performed simulations with a sequence of mesh densities, starting with the “base” density $\rho_n^{(0)} = 6.95 \times 10^5$, used in most of our simulations, then increasing the density to $4\rho_n^{(0)}$ and to $16\rho_n^{(0)}$. Figure A.3
Figure A.3: Various macroscale properties indicate convergence of our TFT simulations (a Hookean sheet with no bending rigidity) upon increasing the mesh density $\rho_n$ beyond the standard value $\rho_n^{(0)} = 6.95 \times 10^5$ used in most of the simulation: (a) the transverse contraction $\tilde{\Delta}_{TFT}(x)$, (b) the energy $U_{TFT}$, (c) the distance $x^*$ of the the transversely-contracted zone from the clamped edges. In both B and C, the differences ($|x^*(\rho_n) - x^*(\rho_n^{(0)})|$ and $|U_{TFT}(\rho_n) - U_{TFT}(\rho_n^{(0)})|$), respectively) are much smaller than the corresponding differences from the values of the respective observables for sheets with finite bending rigidity ($0 < \epsilon \ll 1$).
shows the numerical values of several macroscale features, which are predictable by TFT, for these mesh densities values. The variation among these different meshes is a tiny fraction ($\lesssim 10^{-3}$) of the characteristic differences between the TFT value and the finite-$\epsilon$ simulations, from which we extract the scaling laws in Figs. 2.10,2.11).
BIBLIOGRAPHY


