RESOURCE ALLOCATION IN SUBSIDY WELFARE PROGRAMS: MANAGERIAL INSIGHTS FOR NONPROFITS, GOVERNMENTS, AND SERVICE PROVIDERS

RESOURCE ALLOCATION IN SUBSIDY WELFARE PROGRAMS: MANAGERIAL INSIGHTS FOR NONPROFITS, GOVERNMENTS, AND SERVICE PROVIDERS

A Dissertation Presented

by

WEI WEI

Submitted to the Graduate School of the
University of Massachusetts Amherst in partial fulfillment
of the requirements for the degree of

DOCTOR OF PHILOSOPHY

September 2023

Isenberg School of Management

RESOURCE ALLOCATION IN SUBSIDY WELFARE PROGRAMS: MANAGERIAL INSIGHTS FOR NONPROFITS, GOVERNMENTS, AND SERVICE PROVIDERS

A Dissertation Presented

by

WEI WEI

Approved as to style and content by:

Senay Solak, Co-chair

Priyank Arora, Co-chair

Anna Nagurney, Member

Hari Balasubramanian, Member

George R. Milne, Ph.D. Program Director
Isenberg School of Management

DEDICATION

To my parents.

ACKNOWLEDGMENTS

First and foremost, I would like to express my deepest gratitude to my co-advisors, Professors Senay Solak and Priyank Arora, for their invaluable guidance, continuous support, encouragement, and patience during my Ph.D. studies. I am fortunate to work with them and cannot appreciate them enough as they help me to learn and grow. I always feel lucky to have two advisors standing at different stages of their careers, as they provide help and set great examples of how to do good research, be a dedicated researcher, and improve interpersonal skills from different perspectives.

I am also grateful to my committee members, Professors Anna Nagurney and Hari Balasubramanian, for their time, suggestions, and valuable comments and inputs on my dissertation. I would also like to thank all the faculties, staff, and friends in the Isenberg School of Management for their help. Especially, I want to extend my gratitude to Zhangchen Hu for his generosity in sharing experiences in academia and for providing invaluable support as a great friend. In addition, I want to thank Professor Sarah Jinhui Wu. Without taking her operations management class and her support, I would never have boldly decided to switch to and explore this area and even pursue a Ph.D. degree.

I would like to thank my parents for providing me with the best they could, trusting me and giving me the freedom to explore, and always respecting my decisions. Especially, I am thankful to my mom for her unconditional love, support, and understanding, even though we sometimes have disagreements (like all moms and daughters). I appreciate my dad for being hard-working over the years so I could grow up comfortably, setting a role model as a devoted researcher, and keeping reminding me to be a hard worker.

Last, there are many other good people who are standing by my side and even changed the trajectory of my life, leading me to who I am and where I am now. I am indebted to them for their kindness and support.

ABSTRACT

RESOURCE ALLOCATION IN SUBSIDY WELFARE PROGRAMS: MANAGERIAL INSIGHTS FOR NONPROFITS, GOVERNMENTS, AND SERVICE PROVIDERS

SEPTEMBER 2023

WEI WEI

B.Sc., HEFEI UNIVERSITY OF TECHNOLOGY

M.Sc., FORDHAM UNIVERSITY

Ph.D., UNIVERSITY OF MASSACHUSETTS AMHERST

Directed by: Professor Senay Solak and Professor Priyank Arora

Subsidy welfare programs provide financial assistance to economically disadvantaged individuals and families to access essential and life-altering services (e.g., education, child care, and housing) that they might not otherwise have access to. Access to these services is considered critical to achieving a better and more sustainable future for all. As such, these high-quality services are directly related to several United Nations Sustainable Development Goals, which were adopted as "*a universal call to action to end poverty, save the planet and improve the lives and prospects of everyone, everywhere.*" In particular, the need for these affordable and high-quality services has been underscored during the COVID-19 pandemic to facilitate a safe and robust reopening of the economy. Inspired by this, in this dissertation, we construct,

analyze, and analytically solve a spectrum of resource allocation problems involving different participants within subsidy welfare programs, including local nonprofit organizations, government agencies, private service providers, and individuals and families. The models we construct aim to help different participants make better operational decisions that enable the generation of the most effective and/or equitable social outcomes under these programs. The dissertation consists of three studies addressing these decisions.

In the first study, we consider operational challenges faced by a local nonprofit organization that administers and manages the operations of a subsidy voucher program within its service area. Specifically, motivated by a child care subsidy voucher program, we develop an analytical model that incorporates details of the subsidy voucher offer process and that captures the challenges faced by a Child Care Resource and Referral Agency (CCR&R, a local nonprofit organization) when allocating funds for its outreach and provider services activities. We analyze how a CCR&R should allocate its limited funds between these two types of activities to ensure equitable access to child care across the different regions of its service area. We show that it might be optimal for the CCR&R to invest more funds in outreach in the region with a lower proportion of income-eligible families. This is especially true when: the external considerations (e.g., public transportation and infrastructure) in that region have a greater impact on a family's acceptance propensity; the marginal return of investment in outreach in that region is higher and abundant funds are available; the socioeconomic distress experienced by families in that region is significantly higher; or a large amount of funds is earmarked for outreach in that region. We contextualize our study for a CCR&R in Massachusetts and conclude that the proposed investment decisions can improve equity outcomes by 7.0%.

In the second study, we examine operational challenges faced by a government agency in a subsidy voucher program. Specifically, we delve deeper into another

important complexity within the subsidy voucher programs by studying how a government agency should allocate funds among several local nonprofit organizations. In a typical subsidy voucher program, a government agency (say, the *funding* agency) provides funds to multiple local nonprofit organizations (say, the *service* agencies) in order to enhance the accessibility and quality of subsidized services for beneficiaries residing in their local service areas. These service agencies invest in activities within their areas to generate social impact for beneficiaries by enhancing the quantity and quality of services at local providers. The funds allocation decisions in such a program are complicated by consideration of equity in social impact generated across different areas, intricate relationships among contextual factors in social impact generation, and information asymmetry between different entities. Considering that additional funds may become available for only one area, we develop a model to analyze how the funding agency's funds allocation decisions lead to the most overall social impact in an equitable manner. Our analysis shows how the funding agency should incorporate the within-area factors in addition to the between-area factors in its optimal allocation decisions. For instance, the funding agency should allocate more funds toward an area when it has a relatively balanced mix of subsidy-accepting and non-accepting providers, or outreach activity is more likely to yield a higher investment return and it has fewer non-accepting providers. Also, comparing the resulting outcomes under the equity-ensuring method with those under different funding methods, we find that: While an efficiency-focused method leads to a higher total social impact, it could lead to significantly high levels of inequity across the areas. Further, although a simple formula-based method could achieve greater total social impact while not severely sacrificing equity in certain situations, the equity-ensuring method always eliminates inequity while not severely sacrificing the total social impact under a wide range of values of contextual factors. Finally, using a case study based on Massachusetts' child care subsidy program, we illustrate that the proposed optimal decisions achieve equity

while enhancing overall social impact by approximately 3% versus current allocation decisions.

In the third study, we investigate operational challenges faced by a government agency in designing subsidy welfare programs and program participation decisions faced by service providers. Specifically, we help a government agency make a selection decision between two types of subsidy welfare programs–the *subsidy voucher* programs and the *contracted slot* programs. Although both programs are service-based and rely on the involvement of service providers for service delivery to the beneficiaries, they create social impact through different mechanisms. Under the subsidy voucher programs, beneficiaries have access to services from a large number of service providers (including a mix of high- and low-quality providers). Whereas, under the contracted slot programs, beneficiaries have access to services only from high-quality providers (even if at a fewer number of service providers). Since the government's goal is to deliver high levels of quantity and quality of services to the beneficiaries, which are both influenced by the service providers' participation decisions (based on their payoff-driven objectives), the government's mechanism selection decision becomes non-trivial. We develop a game-theoretical model setup to analyze how contextual factors impact the service providers' participation decisions in these two programs. Considering the interrelationship between service providers' decisions and contextual factors under each type of subsidy program, our analysis shows that providers are more willing to participate in the program when: the reimbursement rate is relatively high and the cost of managing the program is relatively low. Further, we compare the two programs in terms of the societal outcomes generated for the beneficiaries of the programs. To do so, we conduct numerical analysis using the child care context in Massachusetts and identify conditions under which the level of societal outcomes under a contracted slot program outperforms a subsidy voucher program or vice versa. For example, a contracted slot program generates higher societal outcomes than a

subsidy voucher program when: there are relatively more high-quality providers, and they have a relatively high capacity; the reimbursement rate in the contracted slot program is relatively high, and the demand for the high-quality providers' services in the private market is relatively low; or low-quality providers' capacity is relatively high, and the demand for the high-quality providers' services in the private market is relatively low. However, we find that the subsidy voucher program generally outperforms the contracted slot program when we evaluate the two programs based on the societal outcomes per total reimbursement expenditure by the government.

As one of the first few research studies to consider resource allocation in subsidy welfare programs, we help nonprofit organizations, government agencies, and for-profit service providers improve their operational decisions in order to benefit the beneficiaries of the programs. The results of this dissertation are expected to offer managerial insights to the participants within the subsidy welfare programs, increase equity, efficiency, and sustainability of the programs, and benefit society at large.

TABLE OF CONTENTS

LIST OF TABLES

Table **Page**

LIST OF FIGURES

Figure **Page**

CHAPTER 1

INTRODUCTION

1.1 Background on Subsidy Welfare Programs

Subsidy welfare programs play a critical role in fostering sustainable societal growth. Under these programs, economically disadvantaged individuals and families receive subsidies (i.e., monetary support that subsidizes fees at partnering service providers) that enable them to afford and access life-altering services, such as education, health care, child care, and housing. Through increasing stability, self-sufficiency, and skill levels of beneficiaries, these programs ensure sustainable growth for individuals and society. The U.S. federal and state governments spend a total of approximately $670 billion annually to fund various subsidy welfare programs (Urban Institute 2020, Department of Housing and Urban Development 2021). According to the latest census data, over 20 percent of the U.S. population is estimated to have participated in at least one subsidy welfare program (U.S. Census Bureau 2015), with reports indicating that these programs have halved the U.S. poverty rate in the past century (Trisi and Saenz 2019). Further, access to high-quality services is a key enabler for several United Nations Sustainable Development Goals (UN SDGs), which were adopted as *"a universal call to action to end poverty, save the planet and improve the lives and prospects of everyone, everywhere."* These goals include no poverty (SDG #1), quality education (SDG #4), gender equality (SDG #5), and reduced inequalities (SDG #10), among others.

Types of Subsidy Welfare Programs: In practice, subsidy welfare programs can be classified into two different types: (i) *subsidy voucher* programs, under which

beneficiaries who are offered a subsidy voucher can use it to cover a fixed amount of service fees at any voucher-accepting service providers that best suit their needs, and (ii) *contracted slot* programs, under which governments contract with service providers who guarantee a certain number of slots for eligible beneficiaries, and beneficiaries who are offered a contracted slot must use it to access services at a particular contracted slot provider (Matthews and Schumacher 2008, Adams et al. 2021, Bipartisan Policy Center 2021).

Among all subsidy welfare programs, subsidy voucher programs constitute a substantial portion of the overall portfolio (Schumacher et al. 2003). For example, let us consider child care subsidy welfare programs. Overall, the federal and state governments spend more than $20 billion to fund child care subsidy welfare programs annually. Approximately 90% of these funds are used to provide assistance to income-eligible (IE) families through subsidy voucher programs, enabling monthly care for over 1.4 million children (Rachidi 2017, Government Accountability Office 2019a).

Operations under Subsidy Voucher Programs: For reasons involving the convenience of managing operations at a local level and delegating day-to-day operations of these programs to agencies with local knowledge, subsidy voucher programs are organized in a bilevel structure (Isaacs et al. 2015, Government Accountability Office 2019b). In a typical subsidy voucher program, a government agency (say, the *funding agency*) provides the financial resources to multiple local nonprofit organizations[1] (say, the *service agencies*) in order to enhance the accessibility and quality of subsidized services for beneficiaries residing in their local areas. For example, in a typical child care subsidy voucher program, the state Department of Early Education and Care (i.e., the funding agency) provides funds to local Child Care Resource

[1]Throughout the dissertation, we use the term *nonprofits* to refer to these organizations. In the literature, the term *NPOs* is also commonly used for nonprofit organizations. Further, we use the terms *government* and *government agency* interchangeably.

and Referral Agencies (i.e., the service agencies) to help beneficiaries in their respective service areas. Likewise, in a typical rental subsidy voucher program, the state Department of Housing allocates funds to multiple local Public Housing Agencies. The service agencies utilize the allocated funds to invest in a range of activities to encourage and support local service providers (such as child care providers and landlords) in delivering high-quality life-altering services to the beneficiaries of the subsidy voucher programs. Examples of investments made by the service agencies to support providers include hiring staff to provide assistance with accreditation and billing-related paperwork, arranging training sessions for business and professional development, and conducting community outreach activities.

Operations under Contracted Slot Programs: Under the contracted slot programs, the government agency contracts directly with the service providers. Generally, service providers must meet certain requirements (e.g., quality standards) in order to secure a contract. Further, service providers commit to allocating specific numbers of slots from their capacity to serve beneficiaries in the programs. Due to this commitment, under many contracted slot programs, participating service providers receive a higher reimbursement rate than voucher-accepting providers. For instance, contracted slot providers in Georgia receive a reimbursement rate that is 50% higher than that of voucher-accepting providers (Morrisey and Workman 2020). Consequently, contracted slot providers are able to recruit and retain qualified staff in order to improve or maintain their quality levels (Dorn 2020). Regarding subsidy recipients, unlike those in subsidy voucher programs who have the freedom to choose any provider that accepts vouchers, recipients of contracted slots are obligated to utilize the slot at the designated provider when it becomes available.

In this dissertation, we study three novel operational problems related to subsidy welfare programs. The first two studies are based on the subsidy voucher programs, whereas the third study is based on subsidy voucher programs and contracted slot

programs. In the remainder of this chapter, we introduce the background, motivation, and research questions for each problem.

1.2 Resource Allocation by Service Agencies in Subsidy Voucher Programs

This section provides an overview of the background, motivation, and research questions for the first study, which is conducted in Chapter 3. In this study, we examine the resource allocation problems faced by a service agency in a subsidy voucher program. As mentioned previously, subsidy voucher programs are implemented in various domains, such as housing, child care, and education. For ease of exploration, we describe the resource allocation problem using the child care domain. Subsequently, in Section 6.1.2, we illustrate how this resource allocation problem can be generalized to other domains.

Child care is an important intervention for early childhood development and provides valuable support to families with young children (Richter et al. 2017). Past research has shown that providing access to high-quality child care can help improve the long-term outcomes (e.g., educational attainment, earning capacity, and reduced anxiety and depression) of children from economically disadvantaged backgrounds (Barnett and Masse 2007, Herbst 2017). Increasing the accessibility of child care helps parents in low-income families maintain employment or attend school, which in turn helps them break the cycle of poverty and contribute to the economy (Hartmann et al. 2003). While access to child care has these far-reaching positive effects on children, families, and society, it is prohibitively expensive throughout the United States (U.S.). On average, families in the U.S. spend $10,000 per year on child care, which forms as high as 60% of their annual income (CCAoA 2018). Unable to access affordable child care, approximately two million parents in the U.S. are forced to make career sacrifices every year, leading the U.S. economy to lose $28.9 billion

in wages annually (Schochet and Malik 2017, CCAoA 2018). To help people balance the demands of work, school, and parenting, the federal and state governments in the U.S. provide support to IE families through child care subsidy voucher programs.

For the administration and management of day-to-day operations of child care subsidy voucher programs, almost all the state governments in the U.S. (specifically, 47 states and the District of Columbia) partner with local service agencies. These service agencies are typically referred to as Child Care Resource and Referral Agencies (CCR&Rs). CCR&Rs are state-funded nonprofits tasked with providing services to IE families and child care providers in their designated areas. A typical CCR&R provides the following services: adding families that are in need of subsidized child care to a centralized waitlist; completing paperwork for families that are issued subsidy vouchers; increasing the supply of child care providers; supporting providers to ensure high-quality care; and collecting, analyzing and disseminating child care data, among others. A CCR&R offers an available voucher to IE families on the waitlist on a first-come, first-served (FCFS) basis.

1.2.1 Operational Challenges Faced by Service Agency

Several practitioner reports emphasize the need to address the low quantity and low quality of voucher-accepting providers in order to ensure that eligible families are able to access subsidized child care (Pilarz et al. 2016, Ullrich et al. 2019). Within the context of these supply-related concerns, the fact that IE families are often unable to accept offered vouchers is noteworthy (CCAoA 2018). Two principal factors cause this inability to accept vouchers. First, a low number of voucher-accepting providers that best suit IE families' practical needs leads to a low propensity of IE families accepting the offered vouchers. For instance, some providers may not operate during nontraditional hours, despite the high level of need for child care during such hours among IE families (Anderson et al. 2003). Also, immigrant IE families without cars

are less likely to enroll at providers outside their neighborhoods (Greenberg et al. 2016). Second, research has found that, since continuity and stability of child care are important factors for IE families, they often decline subsidy vouchers due to potential administrative hassles or errors (Speirs et al. 2015, Isaacs et al. 2016). For instance, Isaacs et al. (2016) state that a reduction of hassles faced by IE families at child care providers can significantly increase the chances that eligible families are able to accept offered vouchers.

Unable to accept the offered subsidy vouchers, such IE families continue to bear an avoidable financial burden. This burden could take the form of having to give up a job or drop out of school, using savings or retirement funds, or borrowing money at high-interest rates to cover child care (Marshall et al. 2013, Banerjee et al. 2017). Further, Malik et al. (2018) report a 3% lower maternal labor force participation in areas that have an insufficient number of voucher-accepting providers because mothers who would otherwise be able to accept and use vouchers cannot re-enter the workforce. These studies underscore how IE families' inability to accept available vouchers adversely impacts society by exacerbating economic and gender inequalities.

To reduce these negative impacts on IE families and society, CCR&Rs undertake two types of supply-enhancing activities: outreach and provider services. As part of *outreach*, CCR&Rs invest funds in various activities to increase the number of voucher-accepting providers across different regions within their service areas. The rationale of this approach is that the greater the number of voucher-accepting child care providers in a specific region, the more likely an IE family in that region will have access to a provider that suits their needs. The outreach activities can include identifying and collaborating with local stakeholders, creating public-private partnerships that stimulate local provider participation in subsidy voucher programs, and organizing events to inspire corporate voices to speak up for promoting voucher acceptance among local providers.

As part of *provider services*, CCR&Rs dedicate funds to reduce several potential administrative and ancillary challenges that may prohibit parents from using a subsidy voucher at voucher-accepting providers. CCR&Rs can signal and ensure better delivery of child care at voucher-accepting providers in their areas by offering several provider services (Pilarz 2018). These services can include the following: assisting providers with accreditation and quality improvement programs; conducting centralized training sessions for providers on health and safety, child development, and effective business practices; creating professional development opportunities for providers' workforce; and purchasing technological or software tools to streamline administrative processes such as reimbursements. Although these provider services require a significant upfront investment, they potentially benefit all the affiliated providers at a negligible additional investment. For instance, developing an online information repository, organizing webinars by industry experts on quality improvements, or purchasing a software package to automate exchange of information with providers will require a significant amount of investment by a CCR&R; however, this support will significantly improve service delivery at all providers in its area (e.g., see Illinois Department of Human Services 2017). So, while the investment in provider services is typically made at a service area level (unlike the regional outreach investment), its positive benefits accrue across all regions in the area.

However, as with most nonprofits, CCR&Rs have a limited amount of funds to allocate to these two types of activities. The levels of investments dedicated to regional outreach and provider services activities influence the supply (in terms of quantity and quality) of voucher-accepting providers, which leads to disparities in access to subsidized child care for the IE families residing in different regions of the service areas (Chaudry et al. 2011, Pilarz et al. 2016). For example, within the Western Massachusetts service area, the ratio of the number of voucher-accepting child care providers to the number of children on the waitlist differs greatly across different re-

gions, with Hampshire County having the highest ratio (0.28) and Western Worcester having the lowest ratio (0.08). Further, research finds that residents and businesses in certain localities (e.g., ones with higher educational levels, stronger shared identities, or close-knit communities) are more pro-social and, thus, are more likely to help in community projects (Putnam 2001, Bierhoff 2005). As a result, the marginal benefit of funds invested in outreach in those localities may be higher than in other localities. Moreover, there are region-based differences in the socioeconomic burden of distress faced by IE families when they are unable to accept offered subsidy vouchers. For example, Smith and Adams (2013) find that families in rural regions face more challenges and pay more when making child care arrangements than their urban counterparts.

1.2.2 Motivation & Research Questions

Equity or fairness is a measure that reflects the notion that "rewards, punishments, and resources should be distributed according to a combination of different criteria: merit, need, equality, and procedural" (Leventhal 1980a). Ideally, a CCR&R will split its budget to offer provider services in its service area and to conduct outreach in the regions within the service area, such that an IE family in any region can accept the offered voucher to access subsidized child care (see, for example, the CDBG Act of 2014 and the report by National League of Cities 2017). However, as noted above, families are often unable to accept offered vouchers, and the probability of acceptance may differ from one region to the other. This leads to disparities in the distress experienced by IE families across different regions when they are unable to accept offered subsidy vouchers. Thus, from an equity perspective, the allocation of the CCR&R's funds for provider services and outreach should minimize this inequity in distress experienced by IE families. The presence of various regional asymmetries–including transportation infrastructure and regulatory environment, marginal return

of outreach investment, and distress experienced by families–complicates the allocation decisions, and it becomes important to answer the following research questions: (i) *What are the optimal levels of investment by a CCR&R toward outreach and provider services activities that minimize the inequitable societal burden across different regions of its service area?*; and (ii) *How is this optimal allocation affected by activity- and region-specific characteristics?*

1.3 Resource Allocation by Funding Agencies in Subsidy Voucher Programs

This section provides an overview of the background, motivation, and research questions for the second study, which is presented in Chapter 4. While the first study is motivated by and developed within the domain of child care, the management of subsidy voucher programs is similar across various domains (e.g., housing, child care, and education) from the perspective of the funding agencies in different domains. As mentioned earlier, the funding agencies allocate funds to their local service agencies who make investment decisions in activities within their service areas to assist local service providers and beneficiaries (e.g., CCR&Rs in the child care subsidy voucher programs, which are studied in Chapter 3). In light of this, building upon the first study, the second study in this dissertation examines the resource allocation problem at a higher level from the perspective of a funding agency and investigates its funds allocation decisions among multiple service agencies.

1.3.1 Operational Challenges Faced by Funding Agency

The funds allocation decisions by the funding agency and the investment decisions by the service agencies generate a positive impact on the beneficiaries and society by making essential services at local providers accessible and affordable to the eligible beneficiaries. Naturally, a higher budget would allow a service agency to generate

a greater social impact in its coverage area by enhancing the quantity and quality of providers who deliver services to beneficiaries. However, due to the limited funds available to the funding agency (Hamm 2014, Rice 2016), allocating a higher budget to a particular service agency would limit the availability of funds for all other service agencies. In addition to the resource constraints, based on our conversations with practitioners and our reviews of the relevant reports, we learned that several other complexities make it challenging for the funding agency to decide how to allocate its scarce funds among the multiple service agencies. We explain the key complexities faced by the funding agency below.

The first complexity stems from the growing attention within these programs on ensuring equity in the social impact created across different service areas. Typically, given the one-to-many structure of the funds allocation problem under study (i.e., one funding agency working with many service agencies), the funding agency (which is at the upper level) aims to ensure that the most social impact is generated across all areas, whereas the service agencies (which are the lower level) use the allocated funds to maximize the social impact generated in their respective service areas. In recent years, funding agencies are increasingly aiming to ensure equity in the outcomes generated across different areas (e.g., Metropolitan Area Planning Council 2017, Hardy et al. 2018, Banghart et al. 2021). During our conversations with the Massachusetts Department of Early Education of Care, we learned that this consideration of "geographic equity" is an important concern for the funding agencies when making operational decisions, such as funds allocation decisions. Further, in its recommendation to the funding agencies on utilizing scarce resources toward welfare programs, the United Nations states that "policies that allow for the equitable tar-

geting of resources...are needed" (UNICEF 2015).[2] This growing attention on equity consideration in operational decision-making within these programs necessitates that the funds allocation problem must take into account the distinct goals targeted by the different entities–that is, impact-maximization by the funding agency *while caring about equity* across all service areas, versus impact-maximization by the service agencies in their respective areas.

The second complexity stems from the service focus of these programs. In contrast to directly distributing food or medical supplies to beneficiaries, the service agencies generate social impact for the beneficiaries through the provision of essential services (in exchange for subsidies) at the partnering local service providers. In that regard, the service agencies have the opportunity to influence the level of social impact generated in their areas by enhancing the number of service providers who are willing to accept beneficiaries enrolled in the subsidy programs, as well as the quality of service provided by the subsidy-accepting providers to the beneficiaries. As a result, it becomes critical for the funding agency to take into account the differences in the social impact based on the types of activities undertaken by the service agency (e.g., see different outcomes of quality-adjusted life-years under HIV treatment and prevention activities presented in Walensky et al. 2007), and based on different targets of each supply-enhancing activity (e.g., see differences in the number of providers accepting versus not accepting subsidies in Garboden, Rosen, Greif, DeLuca and Edin 2018, Department of Health and Human Services 2019). In other words, the service focus necessitates the funding agency to explicitly include relationships between various contextual factors in its social impact generation function.

[2]Equity is a measure that reflects the notion that "rewards, punishments, and resources should be distributed according to a combination of different criteria: merit, need, equality, and procedural" (Leventhal 1980b).

The third complexity arises from an information asymmetry between the entities at different hierarchical levels. The information asymmetry we focus on relates to the availability of financial funds, which is the central focus of the resource allocation problem under study. The funding agency provides budgets to the service agencies from funds made available by the legislative body for the planned horizon of the subsidy welfare program. However, in practice, there is a chance that additional funds could be approved by the related legislative body at some instance in the future during the planned horizon (Cameron et al. 2005, Lynch 2020, Department of Health and Human Services 2021, Maine Department of Health and Human Services 2023). Further, we learned from practitioners that the funding agency, which is at the upper level, is typically closer to the legislative body; hence, it possesses more information on the availability of these additional funds. Whereas the service agencies, which are at the lower level, possess limited to no such information. As a result, it is important to incorporate this asymmetry in information about the availability of such additional funds when the funding agency distributes its initial funds, and when the service agencies make their initial investment decisions. Moreover, these additional funds generally come with stipulations on their use, such as restricted for use in a particular service area only (e.g., Lynch 2020) or activity only (e.g., Maine Department of Health and Human Services 2023), which adds further complications to the funding agency's funds allocation problem.

1.3.2 Motivation & Research Questions

Despite the aforementioned complexities, many funding agencies allocate funds to service agencies using relatively simple formulas which combine service area related information. Specifically, under these formula-based methods, the amounts of funds made available to different service agencies are calculated based on a simple or weighted sum of proportions of multiple between-area factors (National Research

Council 2001, Dilger and Boyd 2014). These factors include the number of eligible beneficiaries, the number of local service providers, and the implementation cost of service activities, among others. Formula-based methods are popular mainly for their simplicity and ease of use. However, several researchers point out that these methods do not fully capture the distinct objectives of agencies at multiple levels and cannot completely reflect the intricate relationship between different contextual factors that impact the generation of social impact (Sargrad et al. 2020). Instead, these scholars encourage the funding agency to utilize scientific means of fund allocation by employing optimization tools (Epstein et al. 2005). Even practitioners and government executives underscore the need to utilize mathematical models for the distribution of intergovernmental funds. For instance, the U.S. Department of Housing and Urban Development recognizes "optimizing service delivery and decision-making" as one of its strategic goals (Department of Housing and Urban Development 2019).

Inspired by the call for developing scientific tools for the allocation of funds in subsidy welfare programs and based on learnings from our close interactions with practitioners (e.g., the Massachusetts Department of Early Education and Care), we study the funds allocation problem by incorporating all of the complexities discussed above. In particular, we take the perspective of a funding agency to answer the following two research questions: (i) *In a subsidy welfare program, how should the funding agency distribute its limited funds among various service agencies in order to maximize overall social impact while ensuring equitable outcomes across all areas?* and; (ii) *How do the societal outcomes under our optimization-based equity-ensuring method compare with those under other funding methods (specifically, one that focuses only on efficiency and not on equity and another that is based on a proportional formula)?*

1.4 Mechanism Selection for Subsidy Welfare Programs: Contracted Slots vs. Vouchers

This section provides an overview of the background, motivation, and research questions for the third study, which is presented in Chapter 5. The first two studies of this dissertation are based on a widely-popular and widely-used subsidy welfare program, namely, subsidy voucher programs. As mentioned previously, subsidy welfare programs are also offered as contracted slot programs. In contrast to subsidy voucher programs that offer beneficiaries access to a large number of subsidized slots at a wide spectrum of service providers (in terms of quality of service offered), contracted slot programs ensure beneficiaries access to only high-quality subsidized service through government contracting with service providers that offer high quality of service and reserve a fixed number of subsidized slots.

The practical evidence shows that the selection between the subsidy voucher programs and contracted slot programs varies across different state governments (Rachidi 2017). For example, let us consider child care subsidy programs. While most state governments in the U.S. (specifically, 31 states) serve all their beneficiaries through subsidy voucher programs, California serves almost 40% of its beneficiaries through the contracted slot programs (Department of Health & Human Services 2019). However, there is little work that can help a government scientifically select between the two programs. Therefore, in the third study, we compare a subsidy voucher program with a contracted slot program, examine service providers' participation decisions in each program, and study a government's selection decision between the two programs.

1.4.1 Operational Complexities in Designing Subsidy Welfare Programs

While both subsidy voucher programs and contracted slot programs contribute to positive social impact by providing economically disadvantaged individuals and families with affordable access to essential services (e.g., education, housing, and child

care), the government faces a complex decision when selecting which type of the programs to implement and how to implement (Weber and Grobe 2014, Bipartisan Policy Center 2021). This complexity arises from the fact that the underlying mechanisms through which the government can create the social impact for beneficiaries are quite distinct across the two programs.

The subsidy voucher programs create social impact by mainly offering a large number of service providers to the beneficiaries. This is because many service providers are willing to participate in the subsidy voucher programs, which is due to two main reasons. First, in the subsidy voucher programs, service providers serve beneficiaries only when they have a vacancy. Therefore, the providers have the assurance that their underutilized capacity (i.e., vacancies) could be filled by the beneficiaries using the voucher. Second, voucher-accepting providers receive support from governments to manage day-to-day coordination and engagement activities in the programs, including help with the reimbursement procedures, training sessions, and technical assistance, which reduces their administrative burden and costs of working with voucher-using individuals and families. Although subsidy voucher programs can lead to a high quantity of services (by virtue of no restriction on what type of service provider can participate in the programs) to the beneficiaries, the delivered services in these programs can range from high to low quality (by virtue of the participation of both high- and low-quality providers).

The contracted slot programs are an alternative to the subsidy voucher programs wherein the government could ensure a high quality of service delivery to beneficiaries, albeit at fewer service providers. This is achieved by requiring service providers to meet high-quality standards in order to participate in the contracted slot programs. By implementing this requirement, low-quality service providers are excluded from the program, thus ensuring that beneficiaries receive services from high-quality service providers. While contracted slot programs ensure that a high quality of ser-

vices is offered to the beneficiaries, these programs might attract fewer participating service providers, which can be attributed to the following two main reasons. First, contracted slot programs impose additional requirements on providers that could be cost-prohibitive for some providers to meet (Schneider et al. 2017). For instance, providers in these programs must allocate staff time and resources to apply for and manage contracts. Additionally, providers awarded contracts must use the state's centralized subsidy waitlist to identify eligible families for their contracted slots. These requirements lead to enhanced costs for service providers, which are in addition to their typical operational expenses. Such cost burdens could discourage service providers from participating, especially when the private market is very lucrative for these providers. Second, providers in contracted slot programs commit to allocating specific numbers of slots from their capacity that will be reserved for serving beneficiaries in the programs. This requirement of reserving a portion of their capacity introduces inflexibility to their operations, which could impact their ability to serve private market demand. Consequently, providers may be hesitant to allocate many slots to these programs.

Irrespective of the type of subsidy welfare program, the government's primary goal is to ensure a high level of both quantity and quality of service in the subsidy welfare programs (e.g., CCDF Act 45 CFR Part 98). Given that both types of programs–subsidy voucher programs and contracted slot programs–lead to different levels of quantity and quality of subsidized assistance to beneficiaries, which in turn implies different levels of generated social impact, it becomes critical for the government to select the type of subsidy program wisely. Moreover, since service delivery to the beneficiaries relies on the involvement of service providers in subsidy welfare programs, the government's selection decision is further complicated by service providers' (privately-optimal) participation decisions. In essence, the number of participating service providers and the quality level of these participating service providers directly

impact the quantity and quality levels of services received by beneficiaries. This implies that several key program-related and non-program-related factors interrelate with the service providers' participation decisions and the government's mechanism selection decision. Take reimbursement rate, for instance. At a higher reimbursement rate, more service providers are willing to participate in both subsidy welfare programs (Schneider et al. 2017, Morrisey and Workman 2020); however, the degree of improvement in social impact may be quite distinct in these two programs because of the differences in underlying mechanisms through which the reimbursement rate impacts the providers' payoffs and participation decisions. Additionally, since these service providers primarily serve private market demand, the private market characteristics may also influence service providers' participation decisions. For example, market-related factors, including market prices, demand for service providers' services in the private market, and competition in the market, could play a key role in providers' decisions to participate in the programs (Turner 2003, Garboden, Rosen, DeLuca and Edin 2018), and these factors may affect participation decisions differently across the two program types.

1.4.2 Motivation & Research Questions

As discussed above, a mechanism selection problem in this context should take into account service providers' participation decisions, which determine the level of quantity and quality of service delivered to the beneficiaries in the subsidy welfare programs. Our reviews of the related literature and conversations with government experts reveal that most state governments, in practice, select subsidy welfare programs without scientifically comparing the resulting outcomes of these two programs by incorporating service providers' participation decisions and examining how program- and non-program-related factors impact providers' participation decisions (Katz and Adams 2015, Rohacek and Adams 2017). Consequently, the lack of scientific com-

parison and the lack of consideration of participation decisions and related factors could significantly impact the overall societal outcomes (which is impacted by both quantity and quality of service provided) generated for the beneficiaries through the subsidy welfare programs.

Inspired by the need to incorporate and analyze service providers' participation decisions in the subsidy voucher programs and contracted slot programs, as well as to compare the resulting outcomes of these programs, we develop a game-theoretical model setup to incorporate the features of the programs discussed above and investigate service providers' participation decisions in these programs. In particular, we aim to answer the following research question: (i) *How do subsidy program parameters impact service providers' decisions to participate in each type of subsidy welfare program?* (ii) *Under what conditions do the societal outcomes under the contracted slot programs outperform the voucher programs, or vice versa?*

CHAPTER 2

LITERATURE REVIEW

In this chapter, we introduce the relevant literature on two main topics: (i) the role of government interventions in subsidy welfare programs; and (ii) nonprofits and socially responsible operations. Specifically, in the literature on subsidy welfare programs, we discuss relevant research from two streams: one focusing on government subsidies in various contexts and the other examining the operational challenges related to resource allocation decisions within these programs. Regarding the literature on nonprofits and socially responsible operations, we also discuss two streams: one involving resource allocation by nonprofits and the other examining supply-side decisions in socially responsible operations. After reviewing the literature on each stream, we demonstrate how this dissertation can contribute to the existing studies.

2.1 Literature on Subsidy Welfare Programs

In this section, we review relevant literature on subsidy welfare programs involving two major streams of research. First, we discuss current studies on subsidy welfare programs conducted in various contexts, as well as the distinctive contributions of this dissertation. Next, we provide a summary of the operational challenges associated with resource allocation decisions in subsidy welfare programs and explain how our study complements the existing literature.

2.1.1 Subsidy Welfare Programs in Various Contexts

Taylor and Xiao (2014), Levi et al. (2017), Yu et al. (2018), Alizamir et al. (2019), Martin et al. (2020), Nagurney et al. (2021), and Olsder et al. (2022) have studied

government support mechanisms in various settings, including health care, agriculture, humanitarian, technology, etc. Many focus on the health care setting. For example, Taylor and Xiao (2014) study whether a donor should subsidize malaria drugs through the purchases and/or the sales of the private-sector distribution channel. They formulate a model in which the donor wants to maximize average sales to customers subject to a budget constraint and determine the optimal size and type of subsidies dependent on the perishability of the product. They find that the donor should only subsidize purchases instead of sales when the products have a long shelf life. Levi et al. (2017) complement Taylor and Xiao (2014) by studying the setting of a central planner who aims to increase market consumption. The authors examine the effectiveness of uniform co-payments and derive the optimality condition for the uniform subsidies offered to malaria drug firms. Some scholars focus on the agriculture sector. Take Alizamir et al. (2019) for instance. They examine two subsidy welfare programs the U.S. government offers to farmers in order to protect and raise their income, including the Price Loss Coverage (PLC) program and the Agriculture Risk Coverage (ARC) program. They find that ARC may have unintended outcomes under which farmers may plant fewer acres, which leads to a lower crop supply. Also, even though one might expect ARC generally dominates PLC, they find that both farmers and consumers may be better off under PLC. Several studies evaluate subsidy programs that aim to alleviate poverty in emerging markets. For example, Yu et al. (2018) examine the government subsidy welfare programs for home appliances. Similarly, Besley and Kanbur (1988) study the food subsidies in the context of poverty alleviation. Several other papers investigate the utilization of subsidies in the humanitarian domain. For example, Nagurney et al. (2021) demonstrate how governments can employ subsidies as policy interventions to moderate the flow of human migrations and enhance societal welfare. They outline a procedure that assists governments in determining the subsidies for the different migrant classes and locations to achieve

the system-optimizing population distribution pattern across multiple locations and user-optimizing.

Several economists and sociologists have addressed the importance and examined factors that can improve short-term outcomes (e.g., performance on tests) and long-term outcomes (e.g., educational attainment and earning capacity) in education and child care subsidy welfare programs (Barnett and Masse 2007, Herbst 2017). However, only a few studies have considered operational challenges faced by organizations under these programs (Kretschmer et al. 2014, Slaugh et al. 2016). In particular, Kretschmer et al. (2014) study food distribution across different regions through school feeding programs. Using a theoretical framework, they state that incorporating equity considerations is necessary for strategic resource planning because remote regions are often the most impoverished and are logistically difficult to access. Also, a few recent papers analyzing the operational decisions of education systems have focused on issues of inequitable access (Bertsimas et al. 2019, Nguyen and Vohra 2019). Meanwhile, several reports by U.S. government agencies and media organizations call for studying education and child care programs. In particular, they emphasize the importance of increasing the number of voucher-accepting providers, better allocating limited funds for more outreach across different regions, and enhancing the participation of low-income families, all to reduce the societal burden resulting from inequitable access to subsidized child care (Department of Health and Human Services 2016, Keith 2019, Williams 2020). In their review of the gap between practitioner needs and academic publications, Besiou and Van Wassenhove (2020) find that operations management researchers have not paid much attention to the operational aspects of voucher programs.

Motivated by this, Chapters 3 and 4 of this dissertation analyze enhancing the positive impact of the subsidy voucher programs, especially those focusing on providing essential services. Unlike the studies mentioned earlier that concentrate on

product-based subsidy welfare programs such as vaccines, food, and green technology, Chapters 3 and 4 of this dissertation specifically examine service-based social subsidy welfare programs. Through these service-based subsidy welfare programs, the governments make essential services (e.g., housing and child care) affordable and accessible to economically disadvantaged people. However, the service nature of these programs makes allocation decisions of the social planner and participation decisions of beneficiaries and service providers more complicated as compared with those product-based programs. We will expand on this in the following section.

Specifically, in Chapter 3, we study how to split the limited financial resources between different activities to reduce discrepancies in the distress experienced by low-income families residing in different regions under a particular subsidy voucher program. To the best of our knowledge, Chapter 3 is the first study to analyze the impact of operational decisions on equity outcomes within subsidy voucher programs. Given resource constraints and region-specific differences in contextual parameters, such operational decisions constitute a resource allocation problem for the local agencies that administrate these programs. Building upon Chapter 3, Chapter 4 adds to existing studies by taking the perspective of a government agency that has the task of funding a subsidy welfare program across multiple service areas. This perspective allows us to focus on the following key challenge: any feasible distribution of financial resources (by the upper-level agency) is socially beneficial but leads to a different level of inequity in the societal outcomes generated (by the lower-level agencies) across different areas.

Although still focusing on the service-based subsidy welfare programs, Chapter 5 differs from Chapters 3 and 4. Instead of solely focusing on a specific type of service-based program (i.e., subsidy voucher programs), Chapter 5 stands at a different perspective by comparing two types of subsidy welfare programs, including subsidy voucher programs and contracted slot programs. The two programs create

social impact through different mechanisms–the subsidy voucher programs mainly offer a large number of service providers (including a mix of high- and low-quality) to the beneficiaries, while the contracted slot programs ensure beneficiaries receive only high-quality service (even if at a limited number of service providers). Furthermore, Chapter 5 investigates service providers' decisions regarding participation in these two programs and the government's selection decision between the two programs (to ensure a high quantity and quality of service delivered to the beneficiaries). As explained in Section 1.4, service providers are key enablers for achieving a high quantity and quality level of service delivered to the beneficiaries in the subsidy welfare programs. To our knowledge, Chapter 5 is the first to study and compare service providers' participation decisions in the two types of subsidy welfare programs.

2.1.2 Operational Challenges under Subsidy Welfare Programs

Besiou and Van Wassenhove (2020) underscore the need to study operational aspects of cash and voucher programs that are aimed at alleviating poverty and enhancing the lives of under-served beneficiaries. A series of papers have considered operational challenges within social subsidy welfare programs. For example, Cohen et al. (2016) examine the impact of consumer subsidy on the manufacturer's response by incorporating demand uncertainty in a newsvendor setting. They show that the government can miss the desired adoption target level if it ignores demand uncertainty when designing consumer subsidies. Taylor and Xiao (2014) consider one manufacturer selling malaria drugs to multiple heterogeneous retailers facing stochastic demand. Their analysis focuses on the placement of the subsidy by the central planner in the supply chain, comparing the possibility of subsidizing either sales or purchases (from the retailer's point of view). Whereas Levi et al. (2017) consider multiple heterogeneous manufacturers and focus on the effectiveness of uniform subsidies. While several of these papers study design of optimal subsidies offered to

manufacturers and/or consumers to increase access to lifesaving products, there are a few recent papers that analyze additional ways (other than the choice of subsidy levels) to enhance the positive impact of the subsidy welfare programs, especially those focusing on providing essential services. For instance, Arnosti and Shi (2020) study allocation of housing vouchers to low-income families to maximize social welfare.

Chapter 3 complements this latter set of studies by studying how to split the limited financial resources between different activities to reduce discrepancies in the distress experienced by low-income families residing in different regions. This is motivated by the unique challenges local agencies face when managing subsidy welfare programs. That is, even if being offered a voucher, IE families might not be able to accept the offered subsidy vouchers due to the low quantity and low quality of voucher-accepting providers (Pilarz et al. 2016, Ullrich et al. 2019). Unable to accept the offered subsidy vouchers, such IE families continue to bear an avoidable financial burden. This burden could take the form of having to give up a job or drop out of school, using savings or retirement funds, or borrowing money at high-interest rates to cover child care, which adversely impacts society by exacerbating economic and gender inequalities (Marshall et al. 2013, Banerjee et al. 2017).

A few case- and survey-based studies in the domain of public policy have highlighted ways to improve decision-making tools for intergovernmental transfers of funds in social welfare programs (McKeown 1996, Guthrie 2006, Banful 2011). These studies underscore the need to: (i) combine operational as well as geographical and demographic factors in mathematical programming applications; (ii) recognize the different sets of goals, constraints, and information pertaining to entities at different levels of governmental programs; and (iii) address inequity issues related to societal outcomes across different geographical areas. Chapter 4 adds to this set of studies by taking the perspective of a state agency that has the task of funding a subsidy welfare program across multiple service areas. This perspective allows us to focus on the following key

challenge: any feasible distribution of financial resources (by the upper-level agency) is socially beneficial but leads to a different level of inequity in the societal outcomes generated (by the lower-level agencies) across different areas.

Chapter 5 complements Chapter 3 and 4 by studying the operational decisions of both nonprofit entities (i.e., government agencies) and for-profit entities (i.e., service providers) within subsidy welfare programs. That is, government agencies select between subsidy voucher programs and contracted slot programs. Service providers decide whether they should participate in each type of subsidy welfare program or not. Further, Chapter 5 also captures the interconnectedness of the operational decisions of these two different entities. Government agencies rely on service providers to participate in the subsidy welfare program to generate positive social impact by offering beneficiaries affordable essential services. Service providers' participation decisions are impacted by program-related factors (e.g., the reimbursement rate for providers in the programs and service providers' cost of participating in the programs) that are decided by government agencies.

Given the societal objectives of the considered entities, this dissertation also relates to the literature stream that studies resource allocation in nonprofits. In the following section, we review this literature in detail.

2.2 Literature on Nonprofits and Socially Responsible Operations

There is a growing interest within the field of Operations Management (OM) to study various aspects of nonprofits and socially responsible operations (Besiou and Van Wassenhove 2015, Lee and Tang 2017, Berenguer and Shen 2020). In particular, an emerging body of work within OM sheds light on challenges faced by nonprofits that help alleviate economic burdens on society by providing critical products or services to under-served populations (Feng and Shanthikumar 2016). The review of

relevant literature involves two major streams of research. First, we describe current studies on the complexity of resource allocation by nonprofits and discuss how our study complements existing ones. Then, we summarize supply-side decisions in socially responsible operations and the uniqueness of this dissertation.

2.2.1 Resource Allocation by Nonprofits

There is an emerging stream of literature within OM that analyzes the distinct operational challenges faced by nonprofits in their quest to create the most social impact (Berenguer and Shen 2020). The studies in this stream have offered solutions related to various operational aspects, such as fundraising (Burkart et al. 2016), volunteer management (Urrea et al. 2019), project management (Devalkar et al. 2017), inventory rationing (Natarajan and Swaminathan 2017), and resource allocation (Kotsi et al. 2020). Considering a macro-level view, several studies (e.g., Arya and Mittendorf 2016, Kotsi et al. 2020) have analyzed how nonprofits should allocate their resources between fundraising, administration, and programs. While such studies offer interesting insights into when to prioritize program activities over fundraising and administrative tasks, they omit contextual details of how program activities create societal value. Considering the salient trade-offs nonprofits are forced to make, a few studies offer insights into nonprofits that face the challenge of allocating limited resources across various program activities (de Véricourt and Lobo 2009, McCoy and Lee 2014, Arora et al. 2022). For instance, McCoy and Lee (2014) study the equitable allocation of limited resources for conducting outreach across different sites within the service area. Further, Arora et al. (2022) consider the trade-off between investing funds in providing advisory support to help beneficiaries identify best-suited services and the delivery of each of those services. A few other papers capture competition among nonprofits for financial funds from donors into nonprofits' operational decisions. For example, Nagurney et al. (2018) build upon the work of Nagurney

et al. (2016) by developing a Generalized Nash Equilibrium network model. This model captures the competition among nonprofit organizations for financial funds and the delivery of relief items. Each nonprofit aims to maximize its utility, which depends on its financial gain from donations and the weighted benefit resulting from doing good through delivering relief items minus the related delivery costs. Chapter 3 complements these studies by examining how a nonprofit organization should incorporate geographical differences and economic burdens on economically disadvantaged families and society into its resource allocation problems.

Chapter 4 complements the following two sets of papers within this literature stream: First, the set of papers that focus on pro-social settings with more than one entity and study the effect of the interplay between decisions of those entities on the societal outcomes (Kraft et al. 2013, Arora and Subramanian 2019). For instance, by considering the information asymmetry between the donor and the nonprofit, Privett and Erhun (2011) and Sharma et al. (2021) analyze ways to improve societal outcomes by promoting transparency between the two entities. Second, the set of papers that consider earmarking of funds and study the impact of restricting the use of funds (say, for investments in a specific zone or activity only) on the operations of nonprofits (Toyasaki and Wakolbinger 2014, Besiou et al. 2014, Pedraza-Martinez et al. 2020). For instance, Toyasaki and Wakolbinger (2014) compare equilibrium under two fundraising modes, including allowing for earmarked funds and not allowing for earmarked funds, to examine the conditions under which each mode is preferable for donors and nonprofits.

In light of the distinct features of the subsidy welfare programs under study, the resource allocation model setup in Chapter 4 incorporates a unique combination of complexities–specifically, the interconnectedness of decisions of the funding and multiple service agencies (one-to-many setup), information asymmetry between entities at the two different hierarchical levels, and use of funds allocation decisions as a lever

to achieve equity in the system outcomes. Concerning information asymmetry, the context under study differs from the context of donors and nonprofit agencies studied by several previous studies, which typically consider that the donors have limited knowledge about the efficiency (in terms of investment returns) of nonprofit agencies. In this context, while the funding agencies (which are at the upper level) and the service agencies (which are at the lower level) have the same information about the efficiency of service agencies (see the Housing Act 310 ILCS 105 of 2005, and the Child Care CCDF Ruling 45 CFR 98 of 2016), the entities at different hierarchical levels have asymmetry of information on the chance of approval of additional funds in the future for the welfare program (McCarty 2014, Lynch 2020).

Further, by focusing on decisions that influence the supply-side of subsidy welfare programs, our dissertation adds to the nascent literature on the supply-side decisions in socially responsible operations, which are reviewed next.

2.2.2 Supply-Side Decisions in Socially Responsible Operations

Organizations that engage in socially responsible operations face unique challenges, owing to their distinct objectives and the significant gap between demand and supply of lifesaving products or services they offer (see Lien et al. 2014, Atasu et al. 2017, Martin et al. 2020). When the demand for these products or services is uncertain, a series of papers, including Urrea et al. (2019), Pedraza-Martinez et al. (2020), and Zhang et al. (2020), have examined how mechanisms can be designed to ensure the right amount or the right type of products provided to beneficiaries. When demand is known, several studies have analyzed allocation policies that maximize social or individual welfare, with supply assumed to be exogenous. For instance, in the context of offering an available organ to patients on a waitlist, Ata et al. (2016) consider geographic disparities, Bertsimas et al. (2013) consider fairness and flexibility, and Su and Zenios (2006) consider efficiency-equity trade-off to analyze health outcomes

resulting from different policies for organ allocation. Under the subsidy voucher programs studied in Chapter 3, while the demand for vouchers is known in the form of data collected by the government agencies, an IE family's acceptance of the available voucher can be influenced by investments of resources in supply-side activities. Thus, in Chapter 3, we focus on the following distinguishing elements of subsidy voucher programs. Since some low-income families cannot accept the offered vouchers (due to inadequate quantity and quality of the supply base), local nonprofit agencies can significantly reduce socioeconomic burdens through investments in provider services and outreach activities. Similarly, in Chapter 4, we focus on studying how government agencies can reduce inequitable societal outcomes generated across different geographic service areas through supply-side decisions, including funds allocation decisions made by government agencies and investment decisions made by nonprofits.

Most nonprofits depend on altruistic donations for their products or services,[1] which leads to several limitations and challenges for nonprofits when attempting to enhance their levels of supply (e.g., as studied in Solak et al. 2018, Arora and Subramanian 2019, and Mehrotra and Natarajan 2020). Except for a study by Mehrotra and Natarajan (2020), most related studies in this stream do not consider other socially beneficial activities undertaken by nonprofits, such as community outreach and partner development, to enlarge their supply of critical products or services. Mehrotra and Natarajan (2020) study the monetary incentive design problem of a healthcare provider in a developing country that aims to improve its delivery of services and encourage more patients to seek care. Whereas, in the supply-side decision problem considered in Chapters 3 and 4 of this dissertation, decision-makers (e.g., nonprofits or governments) need to allocate their limited financial funds between different types

[1]In addition to products or services, cash is an important supply item for nonprofits. We refer readers to Burkart et al. (2016) for a detailed review of fundraising and cash donations in humanitarian settings.

of supply-enhancing activities (instead of offering monetary incentives) to increase the quantity and quality of service delivered to IE families in their service areas.

In many cases, social planners rely on for-profit service providers to deliver products or services to the beneficiaries. Thus, the level of service or products received by the beneficiaries under such situations is directly influenced by service providers' participation decisions. In that regard, several papers examine the participation decisions of these for-profit entities. For example, Kotsi et al. (2022) study whether humanitarian organizations should provide refugee in-kind donations (e.g., food) or cash assistance that allows refugees to spend at local retail stores. They incorporate retailer's pricing decision and investigate the impact of retailer's market power on the welfare of refugees and local residents. Korpeoglu et al. (2023) study the market entry of a food cooperative, which is a nonprofit and aims to improve food supply in its local community. Specifically, they incorporate strategic interactions between this food cooperative and a for-profit retailer and analyze participation decisions of this food cooperative (e.g., pricing decisions and entry conditions). Chapter 5 contributes to existing literature and complements Chapter 3 and 4 by incorporating participation decisions by for-profit entities and examining the impact of contextual complexities on their decisions, which influence the level of quantity and quality of services to the beneficiaries in the programs.

CHAPTER 3

IMPROVING OUTCOMES IN CHILD CARE SUBSIDY VOUCHER PROGRAMS UNDER REGIONAL ASYMMETRIES

In this chapter, we develop an analytical model that incorporates details of the subsidy voucher offer process and captures the challenges faced by a CCR&R when allocating funds for its outreach and provider services activities in the child care subsidy voucher program. This chapter analyzes how a CCR&R should allocate its limited funds between these two types of activities to ensure equitable access to child care across the different regions of its service area. Analytical results are presented along with a case study which is based on a CCR&R in Massachusetts.

The remainder of this chapter is organized as follows. Section 3.1 describes the model of a CCR&R's resource allocation problem. The results are discussed in Section 3.2. Section 3.3 presents a case study based on a real-world example of the CCR&R that operates in Western Massachusetts. Table A.1 in Appendix A.1 summarizes the key notation of this chapter.

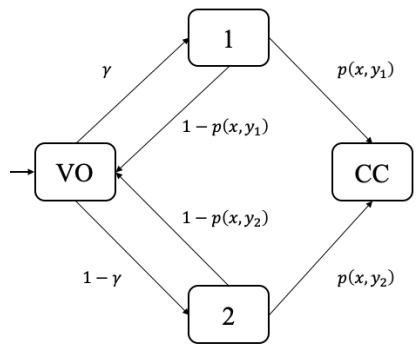
3.1 Model

In this section, we present our model of a CCR&R's allocation of funds between provider services and outreach activities in its designated service area. For these supply-enhancing activities, the CCR&R has limited funds, denoted by $F > 0$. Next, we describe the key elements of the model in detail.

3.1.1 Region- and Activity-Specific Elements of Model

Types of Regions in a Service Area: Differences in geography, income levels, and racial/ethnic compositions lead to differences in demand for vouchers across the various regions within a CCR&R's service area (Swenson 2008). Further, since the outreach activity is conducted at a local level, the extent of its impact is dependent on these region-specific differences. In order to capture the heterogeneity of regions within the CCR&R's service area, we consider that the service area comprises two types of regions, namely, region 1 and region 2. Let $\gamma \in (0, 1)$ denote the proportion of IE families in region 1 and $1 - \gamma$ denote the proportion of IE families in region 2 (for simplicity, we assume one child per eligible family). In Section 3.3, we present a case study to generalize our model and results for a service area with more than two types of regions.

Types of Supply-Side Activities: We consider that the CCR&R invests x amount of funds in provider services-related activities in its service area. As noted in Chapter 1.2, these activities are undertaken at the service area level (for all voucher-accepting providers in the service area) because of the large amount of funds required to, say, hire quality management consultants, and purchase attendance and workflow management software tools (Isaacs et al. 2016). Also, given that the investments in provider services are aimed at improving administrative and business practices at voucher-accepting providers, they are made at an area level to ensure a standardized quality of child care. However, since investments in outreach-related activities are targeted at promoting voucher-acceptance among local providers at a regional level, we consider that the CCR&R invests y_1 amount of funds for outreach in region 1 and y_2 in region 2.



Note: The node VO denotes the voucher offer to an IE family in the service area. The voucher is offered to a family in region 1 (2), denoted by node 1 (2), with probability γ $(1 - \gamma)$. The node CC denotes acceptance of the voucher with probability $p(x, y_i)$ by the family in region $i \in \{1, 2\}$.

Figure 3.1. Progression of a Voucher Offered by the CCR&R to IE Families in Different Regions

Voucher Offer to IE Families: When the state agency makes a voucher available for an IE family in the CCR&R's service area,[1] the CCR&R goes down the waitlist of the IE families until a family accepts the voucher and enrolls for child care at the voucher-accepting provider that best suits family's needs. Although the demand for vouchers in the service area is known in the form of a waitlist, the waitlist data is typically inaccurate, imperfect, and not up-to-date. Isaacs et al. (2015) emphasize that the poor quality of waitlist data on IE families makes it difficult to estimate the caseload of CCR&Rs (as the basis for making operational decisions) and suggest using the U.S. Census Bureau data instead. Accordingly, in our model, the proportions of IE families in regions 1 and 2 (γ and $1-\gamma$, respectively) also represent the probabilities that IE families that are offered the voucher reside in one region or the other. The probability that an IE family residing in region i accepts the offered voucher is denoted

[1]Due to budgetary constraints, state agencies typically fund a fixed number of vouchers. A voucher is made available for offering to an IE family only when an existing voucher-utilizing family's placement is terminated for failure to meet any eligible requirements (Pilarz et al. 2016, Massachusetts Department of Early Education and Care 2019). From the CCR&R's perspective, availability of a voucher is unknown, whereas the number of vouchers being utilized in its area remains at a fixed level.

by $p(x, y_i)$, where $i \in \{1, 2\}$. Accordingly, $1 - p(x, y_i)$ denotes the probability that the family is unable to accept the voucher, under which the CCR&R offers the voucher to the next eligible family in its service area. Figure 3.1 illustrates the progression of a voucher offered by the CCR&R to the IE families.

Acceptance Probability: Consistent with the discussion above, we consider that the probability $p(x, y_i)$ increases in the levels of investment made by the CCR&R in provider services as well as outreach activities, i.e., $\frac{\partial p(x,y_i)}{\partial x} > 0$ and $\frac{\partial p(x,y_i)}{\partial y_i} > 0$ for $i \in \{1, 2\}$. Further, investing more in provider services yields a large benefit when the investment in outreach is high, and vice versa (CCAoA 2018, p. 21–23). Based on our conversations with officials at different CCR&Rs, it is observed in practice that any increase in the likelihood of IE family's voucher acceptance that results from an increased provider services investment is greatly enhanced by an increased outreach investment. This is because improvements in administrative and business practices at voucher-accepting providers will be beneficial for the IE families (in the form of ease of doing business) when they are more likely to find the local voucher-accepting provider that best suits their needs. Mathematically, this implies that the probability $p(x, y_i)$ exhibits the supermodularity property: $\frac{\partial^2 p(x,y_i)}{\partial x \partial y_i} > 0$. Finally, it is reasonable to assume that the probability exhibits diminishing returns with respect to the CCR&R's investments in each of the two types of activities, i.e., $\frac{\partial^2 p(x,y_i)}{\partial x^2} < 0$ and $\frac{\partial^2 p(x,y_i)}{\partial y_i^2} < 0$.

A function of the form of the Cobb-Douglas function, $x^{\alpha} y_i^{\beta_i}$ with $0 < \alpha < 1$ and $0 < \beta_i < 1$, satisfies these mathematical properties. Within OM, this specific functional form is commonly used in situations in which the output is co-produced by two inputs (e.g., Roels et al. 2010, Andritsos and Tang 2018), which is true for this

context.[2] Also, it allows for the analytical characterization of the effect of different marginal returns of investments from different activities, which is an important consideration for nonprofits. Specifically, we consider $p(x, y_i) \doteq \min\{\underline{p}_i + p_{oi} x^{\alpha} y_i^{\beta_i}, 1\}$, in which $0 < \underline{p}_i < 1$ and $0 < p_{oi} < 1$ for $i \in \{1, 2\}$. The parameter $\underline{p}_i$ denotes the baseline acceptance probability in region i, i.e., the probability that an IE family accepts the offered voucher when the CCR&R invests no amount of funds toward either provider services or outreach in that region. In Section 3.2.3, we explore how the magnitudes of $\underline{p}_1$ and $\underline{p}_2$ affect the optimal investments.

External Environmental Factors: The parameters p_{o1} and p_{o2} capture the impact of external environmental factors on the probability that an offered voucher will be accepted by an IE family in regions 1 and 2, respectively. The term "external environmental factors" is commonly used in marketing parlance to refer to external legal, regulatory, and technical considerations that have an impact on the entity's decisions (Kotler and Armstrong 2017). In this context, in addition to the supply-enhancing activities of the CCR&R, several reports underscore the role of external environmental factors, such as transportation infrastructure and regulatory environment, in determining an IE family's ability to accept an available subsidy voucher. For instance, Chaudry et al. (2011) explain that a wider coverage of the local public transportation network, higher service frequency of buses, and greater lobbying efforts by social activists to reduce paperwork requirements of immigrant IE families can increase the IE family's propensity to accept the offered voucher. In essence, the parameter p_{oi} captures those considerations that are external to the CCR&R's two activities, yet they have an influence on the probability of an IE family's acceptance of an offered voucher in region i. Further, to analyze the effect of region-based asym-

[2] Our numerical studies show that insights obtained by using other functional forms that capture diminishing returns (e.g., exponential functions) are qualitatively similar to the ones obtained by using the Cobb-Douglas function.

metry in these external environmental factors on the optimal investments, we allow for p_{o1} to be different from p_{o2}. We consider $p_o \doteq \frac{p_{o2}}{p_{o1}}$ and refer to p_o as the *relative external environmental factor*.

Provider Services and Outreach Elasticities: The parameters α and β_i capture the elasticity of an IE family's probability of accepting a voucher with respect to levels of investment in provider services and regional outreach activities, respectively. To relate with practice the relative magnitudes of α and β_i, consider a service area in which parents in most IE families work on a daily-wage basis (e.g., a metro service area). In such an area, ease of doing business with child care providers (say, in terms of billing processes and limited unplanned shut-down of centers) is a more prominent factor when parents decide whether to accept a voucher for child care (i.e., $\alpha > \beta_i$) (Adams et al. 2003). In contrast, consider a service area with a significant lack of infrastructure. In such an area, the inconvenience of lengthy commute times plays a significantly greater role in determining a family's decision to accept or decline a voucher, as compared to the factors related to the ease of doing business with a child care provider (i.e., $\beta_i > \alpha$) (Walker and Reschke 2004).

Socioeconomic Burden of Distress: In practice, a family may decline an offered voucher due to an inadequate supply of voucher-accepting providers that best suit the family's child care needs. In such a scenario, the family faces a financial burden because a family member may have to give up a job, drop out of school, or use savings and retirement funds in order to take care of the child (Marshall et al. 2013, Banerjee et al. 2017). Meanwhile, society suffers from loss in productivity (e.g., absenteeism from work due to unavailability of reliable child care) and increased unemployment rates among parents in IE families (CCAoA 2018). Let $\xi_i > 0$ denote the socioeconomic burden of distress borne by an IE family in region i that is unable to accept an offered voucher. Further, to capture region-based differences in distress, we allow ξ_1 and ξ_2 to be different.

3.1.2 The CCR&R's Resource Allocation Problem

The CCR&R operates under the state mandate that allows it to go down the waitlist on a FCFS basis until an IE family accepts an available subsidy voucher (Massachusetts Department of Early Education and Care 2019). This implies that one of the IE families on the list is able to access child care using the available voucher. In that regard, any feasible distribution of funds among its supply-enhancing activities will generate a positive societal outcome (since the acceptance probability increases in x and y_i). With no constraint on its resources, the CCR&R should be able to build the supply-base to the extent that each IE family on the top of the waitlist can accept the subsidy voucher when it becomes available (i.e., $p(x, y_i) = 1$). However, due to the scarcity of resources and various regional asymmetries, the CCR&R's investment decisions lead to regional differences in the probability of accepting the offered voucher. Thus, in terms of equity, each feasible distribution of funds leads to varying levels of disparities in the distress experienced by IE families in different regions of the service area.

According to the World Health Organization, the term inequity refers to "avoidable or remediable differences among groups of people, whether those groups are defined socially, economically, demographically, or geographically" (WHO 2023). In our context, these differences arise when, because of CCR&R's levels of investments in each activity, an IE family in a given region is unable to accept an offered subsidy voucher. As per government regulations (the CDBG Act of 2014 and the CCDF Ruling 45 CFR 98.46(b) on September 30, 2016) and based on our interviews with experts in the field of child care subsidy voucher programs, the CCR&R must make operational decisions that ensure equitable access to affordable child care in different regions of its service areas. Moreover, several reports have underscored the need to address inequitable access to affordable child care across different parts of the nation (e.g., Malik et al. 2018).

Thus, the CCR&R should allocate its limited funds between x, y_1, and y_2 such that the resulting inequity in distress experienced by IE families (that are unable to accept the offered voucher) across different regions is minimized. We formulate an expectation based measure of inequity in distress to ensure that the allocation of funds is based on an assessment of disparities across IE families in different regions of its service area. Next, we describe the formulation of this measure, which is in line with the *relative equity* principle that views the FCFS policy as the most equitable policy.

Objective Function: We consider that the CCR&R decides on investment levels x, y_1, and y_2 such that when a voucher becomes available within the planning horizon, the expected measure of inequity in distress, denoted by $MI\,(x, y_1, y_2)$, is minimized. Recall that with probability γ, the CCR&R offers the available voucher to an IE family residing in region 1. With probability $1 - p\,(x, y_1)$, this family declines the voucher. In this scenario, since an IE family is unable to accept the subsidy voucher, the CCR&R increments the inequity measure by the magnitude of distress ξ_1. Next, the voucher is offered to an IE family in either region 1 (with probability γ) or region 2 (with probability $1-\gamma$). This voucher offer process continues till an IE family in the service area accepts the available voucher (see Figure 3.1). Using the probabilities γ and $p\,(x, y_i)$, and the magnitude of distress ξ_i, the expected value of the measure of inequity in distress–as a function of the CCR&R's investment levels–can be expressed as follows:

$$
\begin{aligned}
&MI\,(x, y_1, y_2) \doteq \\
&\gamma(1 - p\,(x, y_1))[\xi_1 + \underbrace{\gamma(1 - p\,(x, y_1))(\xi_1 + ...) + (1 - \gamma)(1 - p\,(x, y_2))(\xi_2 + ...)}_{\text{Per progression of voucher offer after the first rejection in region 1}}] + \\
&(1 - \gamma)(1 - p\,(x, y_2))[\xi_2 + \underbrace{\gamma(1 - p\,(x, y_1))(\xi_1 + ...) + (1 - \gamma)(1 - p\,(x, y_2))(\xi_2 + ...)}_{\text{Per progression of voucher offer after the first rejection in region 2}}],
\end{aligned}
$$

which can be re-written in a recursive form as

$$MI\,(x, y_1, y_2) =$$

$$\gamma(1 - p\,(x, y_1))[\xi_1 + MI\,(x, y_1, y_2)] + (1 - \gamma)(1 - p\,(x, y_2))[\xi_2 + MI\,(x, y_1, y_2)].$$

And, on simplification, we obtain the following:

$$MI\,(x, y_1, y_2) =$$

$$\underbrace{\left(\frac{\gamma\,(1 - p\,(x, y_1))}{\gamma p\,(x, y_1) + (1 - \gamma)p\,(x, y_2)}\right)}_{\text{Conditional rejection probability in region 1}} \xi_1 + \underbrace{\left(\frac{(1 - \gamma)\,(1 - p\,(x, y_2))}{\gamma p\,(x, y_1) + (1 - \gamma)p\,(x, y_2)}\right)}_{\text{Conditional rejection probability in region 2}} \xi_2. \tag{3.1}$$

The terms within parentheses in equation (3.1) capture the probability that an IE family in region i is unable to accept the offered voucher conditioned on the voucher being eventually accepted by an IE family in one of the regions. This implies that the expression for $MI\,(x, y_1, y_2)$ yields the expected distress value in the CCR&R's service area when the IE families in its different regions are unable to accept the offered voucher. In essence, the objective function ensures that the limited funds for the regional outreach activities are allocated proportionally between the two regions, wherein these proportions are governed by the contextual parameters. Furthermore, it turns out that the formulated measure $MI\,(x, y_1, y_2)$ captures the notion of proportional fairness. One of the first definitions of a proportional fairness objective was presented by Mo and Walrand (2000) in their model of equitable congestion controls. Specifically, they establish that a proportionally fair vector maximizes the (weighted) sum of logarithmic utility functions. In Lemma A.4 in Appendix A.5, we illustrate the equivalence of the vector of optimal levels of investments resulting from our objective function with their definition.[3]

[3]This equivalence also implies that the vector of optimal investments satisfies the four axioms pertaining to proportional fairness–namely, *Pareto Optimality, Symmetry, Affine Invariance,* and *Independence of Irrelevant Alternatives* (Bertsimas et al. 2011).

Optimization Problem: The measure $MI\left(x, y_1, y_2\right)$ obtains the highest value of $\frac{\gamma\left(1-\underline{p}_1\right)\xi_1+(1-\gamma)\left(1-\underline{p}_2\right)\xi_2}{\gamma\left(\underline{p}_1\right)+(1-\gamma)\left(\underline{p}_2\right)} > 0$ when $x = 0$ or $y_1 = y_2 = 0$. For a given amount of funds F, the CCR&R can minimize $MI\left(x, y_1, y_2\right)$ by solving the following optimization problem:

$$\min_{\{x,y_1,y_2\}} \quad MI\left(x, y_1, y_2\right) = \frac{\gamma\xi_1 + (1-\gamma)\xi_2 - \gamma\xi_1 p\left(x, y_1\right) - (1-\gamma)\xi_2 p\left(x, y_2\right)}{\gamma p\left(x, y_1\right) + (1-\gamma) p\left(x, y_2\right)} \tag{3.2}$$

$$\text{s.t.,} \quad x + y_1 + y_2 \leq F, \tag{3.3}$$

$$x \geq 0, y_1 \geq 0, y_2 \geq 0, \tag{3.4}$$

where the expression for the objective function is obtained by re-arranging terms in equation (3.1). The optimal investments in provider services and outreach activities are denoted by (x^*, y_1^*, y_2^*).

In the next section, we analyze the CCR&R's optimal levels of investments in each activity, and investigate the effect of contextual parameters on these optimal decisions. Proofs of the main analytical results are presented in Appendix A.4. Also, additional technical details are available in Appendix A.4. Throughout the rest of this chapter, the analysis is restricted to those combinations of parameters that ensure the probability $p\left(x, y_i\right)$ is less than unity, for $i \in \{1, 2\}$. This allows for a focus on non-trivial optimal decisions and on scenarios that are interesting as well as practically relevant. For instance, proof of Proposition 3.4 shows that $p\left(x^*, y_i^*\right) < 1$ when the amount of funds available to the CCR&R for supply-enhancing activities is low, which is true for most, if not all, CCR&Rs (CCAoA 2018).

3.2 Results

Proposition 3.1 shows how activity-specific characteristics impact CCR&R's optimal levels of investment in each activity. To highlight the relative role of provider

services and outreach elasticities in the allocation decision, we initially consider situations with no regional differences (specifically, $p_o = 1$, $\xi_1 = \xi_2$, and $\beta_1 = \beta_2 \doteq \beta$).

Proposition 3.1 *The CCR&R's optimal levels of investment for* $0 < \alpha < 1$ *and* $0 < \beta < 1$ *are:* $x^* = \frac{F\alpha}{\alpha+\beta} > 0$, $y_1^* = \frac{F\beta\gamma^{\frac{1}{1-\beta}}}{(\alpha+\beta)\left(\gamma^{\frac{1}{1-\beta}}+(1-\gamma)^{\frac{1}{1-\beta}}\right)} > 0$, *and* $y_2^* = \frac{F\beta(1-\gamma)^{\frac{1}{1-\beta}}}{(\alpha+\beta)\left(\gamma^{\frac{1}{1-\beta}}+(1-\gamma)^{\frac{1}{1-\beta}}\right)} > 0$. *In addition,*
(i) $x^* > y_1^* + y_2^*$ *if* $\alpha > \beta$*; and* $x^* \leq y_1^* + y_2^*$ *otherwise;*
(ii) There exists a threshold $\hat{\alpha} \doteq \beta\left(\gamma^{\frac{1}{1-\beta}} + (1-\gamma)^{\frac{1}{1-\beta}}\right)^{-1} \min\left\{\gamma^{\frac{1}{1-\beta}}, (1-\gamma)^{\frac{1}{1-\beta}}\right\}$ *such that,* $x^* < \min\{y_1^*, y_2^*\}$ *if and only if* $\alpha < \hat{\alpha}$.

The analytical finding in Proposition 3.1 that the inequity-minimizing split of funds depends on the relative magnitude of marginal returns is in line with the general view of the extant OM literature. As such, in practical terms, this finding is useful for nonprofits that aim toward generating societal impact in resource-constrained environments, a conclusion also noted by McCoy and Lee (2014) and Kotsi et al. (2020) in their baseline analyses of optimal investment decisions by nonprofits. In our context, it is particularly important because most CCR&Rs neglect outreach activities in the presence of scarcity of resources (CCAoA 2018, p. 22). Proposition 3.1 shows that when $\alpha < \beta$, it is optimal for the CCR&R to invest less in provider services than the total investment in outreach in regions 1 and 2. Moreover, when the provider services elasticity is below a threshold ($\hat{\alpha}$), it is optimal for the CCR&R to spend more on outreach in each region. This is because of the relatively higher impact of outreach on a family's propensity to accept an offered voucher.[4]

[4]It is possible that the CCR&R may possess knowledge only on the probability distributions of the elasticities α and β_i. In such a scenario, our extensive numerical studies show that: (i) the first-order insights (based on point estimates) generated in this chapter remain unchanged, and (ii) the CCR&R should hedge against a higher uncertainty in the estimate of elasticity of its investment in an activity by increasing its investment in that activity.

We next analyze the effects of various region-based asymmetries on the CCR&R's optimal levels of investments–specifically, we consider region-based asymmetries in: external environmental factors (Section 3.2.1); outreach elasticity (Section 3.2.2); IE family's distress (Section 3.2.3); and earmarking of funds for outreach in a region (Section 3.2.4). To simplify exposition and understand the pure effect of each asymmetry, we present our analytical findings by introducing one asymmetry in each subsection (with other factors considered to be symmetric). We later show in Section 3.3 that similar insights hold for a general setting.

3.2.1 Effect of Region-Based Asymmetry in External Environmental Factors

In this subsection, we study how differences in the external environmental factors in regions 1 and 2 (captured by the relative external environmental factor $p_o = {}^{p_{o2}}/_{p_{o1}}$) affect the optimal levels of investments in outreach activities in each region. Our analysis shows that the effect of p_o on the optimal outreach investments (y_1^* and y_2^*) depends on the mix of families in each region and outreach elasticity, as presented in the following proposition.

Proposition 3.2 *Define $\hat{p}_o \doteq \frac{\gamma}{1-\gamma}$. Then, (i) y_1^* is decreasing in p_o, and y_2^* is increasing in p_o; (ii) $y_1^* < y_2^*$ if and only if $p_o > \hat{p}_o$; (iii) y_1^* is unimodal with respect to β, and y_2^* increases in β when $p_o > \hat{p}_o$, and y_1^* increases in β, and y_2^* is unimodal with respect to β when $p_o \leq \hat{p}_o$.*

Proposition 3.2(i) shows that as p_o gets larger, i.e., when the external environmental factors in region 2 play an increasingly positive role as compared with those in region 1, the CCR&R should increase its outreach investment in region 2, at the expense of outreach investment in region 1. Next, when the effect of the external environmental factors in region 2 is significantly higher than in region 1 (i.e., when $p_o > \hat{p}_o$), Proposition 3.2(ii) shows that the CCR&R should invest more funds in

conducting outreach in region 2 than that in region 1. By doing so, it can further reduce the chances that the IE families in region 2 will reject offered vouchers, and as a result, improve the overall societal outcomes in the service area. This analytical finding implies that the optimal outreach investment in region 2 (y_2^*) can be greater than the optimal outreach investment in region 1 (y_1^*), even when the majority of IE families reside in region 1.

Also, in situations when $p_o > \hat{p}_o$, Proposition 3.2(iii) shows that the CCR&R should respond to an increase in the outreach elasticity β by increasing y_2^*. Whereas, an increase in β has a more nuanced effect on y_1^*. When β increases, but remains below a threshold, the optimal investment y_1^* increases. However, when β increases beyond a threshold, y_1^* undergoes a decrease. This happens because, for a significantly high marginal benefit of outreach investment, the CCR&R should optimally channel its funds toward outreach in region 2 instead of region 1 (since $p_o > \hat{p}_o$).

Further, based on expressions for optimal outreach investments, we obtain that the optimal balancing ratio of the CCR&R's investments in outreach activities, i.e., y_1^*/y_2^*, is equal to $\left(\frac{\gamma}{p_o(1-\gamma)}\right)^{\frac{1}{1-\beta}}$. This expression shows that the optimal outreach investments in both region are more balanced when the outreach elasticity is low. Whereas, when the outreach elasticity is high (β is large), the ratio of optimal investments becomes more skewed. Eventually, when $\beta \to 1$, the CCR&R should invest in outreach activity in only one region. The linear returns to scale reduce the proportional fairness objective to a utilitarian objective, and it becomes optimal for the CCR&R to conduct outreach in only the more "efficient" region (based on $\gamma \lesseqgtr p_o(1-\gamma)$).[5]

[5]In Appendix A.2, we analyze the CCR&R's funds allocation problem using a max-min fairness objective. Based on the preliminary analysis, we find that effects of the region-specific characteristics on the optimal investments are qualitatively similar under both proportional and max-min notions of fairness.

3.2.2 Effect of Region-Based Asymmetry in Outreach Elasticity

In this subsection, we analyze the effect of asymmetry in outreach elasticities on the CCR&R's optimal investments. Our analysis reveals that, when $\beta_1 \neq \beta_2$, the CCR&R should determine the optimal balancing ratio of investments in outreach activities in its two regions, not only by considering the proportion of IE families and the outreach elasticity in each region, but also by taking into account the available funds and provider services elasticity. This finding is characterized in the following proposition.

Proposition 3.3 *When the outreach elasticity is different for different regions, there exists a unique allocation strategy* $(x^*, y_1^*, y_2^*) > 0$*, such that* $x^* = F - y_1^* - y_2^*$*,* $y_2^* = \left(\frac{(1-\gamma)\beta_2}{\gamma\beta_1}\right)^{\frac{1}{1-\beta_2}} (y_1^*)^{\frac{1-\beta_1}{1-\beta_2}}$*, and* y_1^* *is the unique solution to the following equation:* $F - \left(1 + \frac{\alpha}{\beta_1}\right) y_1 - \left(1 + \frac{\alpha}{\beta_2}\right) \left(\frac{(1-\gamma)\beta_2}{\gamma\beta_1}\right)^{\frac{1}{1-\beta_2}} (y_1)^{\frac{1-\beta_1}{1-\beta_2}} = 0$*. Further, (i)* $\frac{\partial}{\partial F}\left(\frac{y_1^*}{y_2^*}\right) > 0$ *if and only if* $\beta_1 > \beta_2$*, and (ii)* $\frac{\partial}{\partial \alpha}\left(\frac{y_1^*}{y_2^*}\right) < 0$ *if and only if* $\beta_1 > \beta_2$*.*

Proposition 3.3 shows that, when the total amount of available funds increases (i.e., F increases), the CCR&R should increase its investment in outreach in the region with the higher outreach elasticity relative to the outreach investment in the other region. In contrast, when the provider services elasticity is high (i.e., α is large), it becomes optimal for the CCR&R to raise its investment in outreach in the region with the lower outreach elasticity relative to the region with the higher outreach elasticity. This is because, for a large α, the joint impact of investments in provider services and outreach activities on a family's acceptance probability would be higher when the levels of each investment are higher. This implies that the CCR&R would be able to overcome the adverse impact of smaller outreach elasticity in the region by investing more funds in outreach (in relative terms) in that region, when its provider services elasticity is high.

Another implication of asymmetry in outreach elasticities is that, under certain conditions, it may be optimal for the CCR&R to invest more funds in outreach activity

in the region with a lower proportion of IE families. Specifically, when the CCR&R has a sufficiently high amount of funds (F), it should invest more funds in outreach in the region with a higher outreach elasticity, even when a higher proportion of IE families reside in the other region.[6] This finding can be explained as follows: When the CCR&R has an abundance of resources, the effect of higher marginal returns of investment in outreach is more dominant in reducing the overall inequity as compared to the effect of having a higher proportion of IE families (since F is large). As a result, in such scenarios, it is beneficial to invest more funds toward outreach in a region with a higher marginal return than in a region with a higher proportion of IE families.

3.2.3 Effect of Region-Based Asymmetry in IE Family's Distress

In this section, we study the impact of asymmetry in socioeconomic burden of the distress experienced by IE families in regions 1 and 2. Without loss of generality, we consider the situations in which $\xi_2 > \xi_1$. To parse out the pure effect of asymmetry in distress, we conduct analysis by considering the absence of other activity- and region-specific differences. We later show in Section 3.3 that the generated insights are robust to situations in which other parameters are asymmetric.

Proposition 3.4 *Consider* $\alpha = \beta = 1/2$, $\gamma = 1/2$, $p_{o1} = p_{o2} \doteq p^o$, *and* $\underline{p}_1 = \underline{p}_2 \doteq \underline{p}$. *Then,* $(x^*, y_1^*, y_2^*) = \left(\frac{F}{2}, \frac{F}{2}\left(\frac{k_1^2}{k_1^2+k_2^2}\right) - \sqrt{\frac{F}{2}\left(\frac{k_1^2}{k_1^2+k_2^2}\right) - \frac{(p^o)^2F^3(1-\xi)^2}{2\left(k_1^2+k_2^2\right)}}, \frac{F}{2}\left(\frac{k_1^2}{k_1^2+k_2^2}\right) + \sqrt{\frac{F}{2}\left(\frac{k_1^2}{k_1^2+k_2^2}\right) - \frac{(p^o)^2F^3(1-\xi)^2}{2\left(k_1^2+k_2^2\right)}}\right)$, *where* $k_1 \doteq 2\left((\underline{p}+1)\xi - \underline{p} + 1\right) > 0$, $k_2 \doteq 2\big((1-\underline{p})\xi + 1 + \underline{p}\big) > 0$, *and* $\xi \doteq \frac{\xi_2}{\xi_1}$. *Also,* y_1^* *is decreasing in* ξ, *and* y_2^* *is increasing in* ξ.

Proposition 3.4 shows that a relative increase in the distress experienced by families in a particular region makes it optimal for the CCR&R to increase its investment

[6] We characterize these conditions for a special case when either β_1 or β_2 is equal to ½ (See details in Lemma A.9 in Appendix A.5). However, our numerical studies show that similar insights hold for any general case where $\beta_1 \neq \beta_2$.

in outreach in that region. This allows the CCR&R to increase the propensity of an IE family in such a region to accept an offered voucher, and, as a result, to minimize the adverse impact on the families in that region and on society. This finding also implies that, when the regional disparity in distress is large, it may be optimal for the CCR&R to allocate more funds to outreach in the region with the relatively higher socioeconomic burden of distress, even when the proportion of IE families in that region is lower. Consider that the majority of IE families reside in region 1. Then, per Proposition 3.4, it may be optimal for the CCR&R to invest a relatively higher amount of funds in outreach in region 2 as compared to region 1 when ξ_2 is substantially larger than ξ_1.

Further, we find that such a scenario (i.e., $y_1^* < y_2^*$, even when $\gamma > 1/2$) is more likely to arise in the following two situations. First, when the provider services elasticity α increases: This is because, when α is high, it is optimal for the CCR&R to increase investment in provider services by reducing investment in outreach in region 1 relatively more than the outreach investment in region 2 (since $\xi_2 > \xi_1$). Second, when the baseline acceptance probabilities increase: Consider that the baseline acceptance probability in region 1 $\left(\underline{p}_1\right)$ increases, i.e., the chance that an IE family in region 1 will reject an offered voucher reduces. As a result, the CCR&R should decrease outreach investment in region 1, and utilize those funds to increase its outreach investment in region 2 (since $\xi_2 > \xi_1$). The explanation for the effect of the baseline probability $\underline{p}_2$ is analogous.

3.2.4 Effect of Region-Based Earmarking of Funds for Outreach Activity

In most humanitarian and nonprofit settings, additional funds that are made available by a donor agency may be earmarked for investment in a specific activity only (Toyasaki and Wakolbinger 2014, Pedraza-Martinez et al. 2020). In our context, the CCR&R may receive funds that can only be invested in outreach in a certain

region (Keith 2019). Without loss of generality, we consider the case in which the CCR&R receives $\delta > 0$ additional funds to invest exclusively in outreach in region 1. Accordingly, the CCR&R's resource allocation problem described in Section 3.1 can be modified by replacing the previous resource constraint (in equation (3.3)) with the following two constraints: $x + y_1 + y_2 \leq F + \delta$ and $y_1 \geq \delta$. The latter constraint implies that the funds earmarked for outreach in region 1 cannot be invested in provider services or outreach in region 2. The analysis of this modified optimization problem shows that the earmarked funds have a non-trivial impact on the CCR&R's optimal decisions (as stated in the proposition below).

Proposition 3.5 *Suppose $\delta > 0$ units of additional funds are earmarked for outreach in region 1. Then, there exists a unique threshold $\bar{\delta} > 0$, such that: (i) When $\delta \leq \bar{\delta}$: $(x^*, y_1^*, y_2^*) = \left(\frac{(F+\delta)\alpha}{\alpha+\beta}, \frac{(F+\delta)\beta\gamma^{\frac{1}{1-\beta}}}{(\alpha+\beta)\left(\gamma^{\frac{1}{1-\beta}}+(1-\gamma)^{\frac{1}{1-\beta}}\right)}, \frac{(F+\delta)\beta(1-\gamma)^{\frac{1}{1-\beta}}}{(\alpha+\beta)\left(\gamma^{\frac{1}{1-\beta}}+(1-\gamma)^{\frac{1}{1-\beta}}\right)} \right)$ and; (ii) When $\delta > \bar{\delta}$: $y_1^* = \delta$, $x^* = F - y_2^*$, and y_2^* is the unique solution to the equation $(1-\gamma)(\alpha+\beta)y_2 + \gamma\alpha\delta^\beta y_2^{1-\beta} - (1-\gamma)\beta F = 0$.*

Further, y_2^* is increasing in δ if $\delta \leq \bar{\delta}$, and decreasing otherwise.

When the amount of earmarked funds is below the threshold $\bar{\delta}$, it remains optimal for the CCR&R to allocate the total funds $(F + \delta)$ between provider services and outreach activities in the same fashion as shown in Proposition 3.1. In contrast, when the amount of earmarked funds is greater than $\bar{\delta}$, it is optimal for the CCR&R not to invest anything more than the amount earmarked for investing in outreach in region 1. This is because, when the amount of earmarked funds for outreach in region 1 is sufficiently high, the CCR&R should utilize the remaining funds (F) to reduce the overall inequity outcomes through outreach in region 2 and improvement of service experiences delivered by the providers in its service area.

Another implication of the availability of the earmarked funds is that, when the amount of earmarked funds is substantially large, the CCR&R's optimal investment in

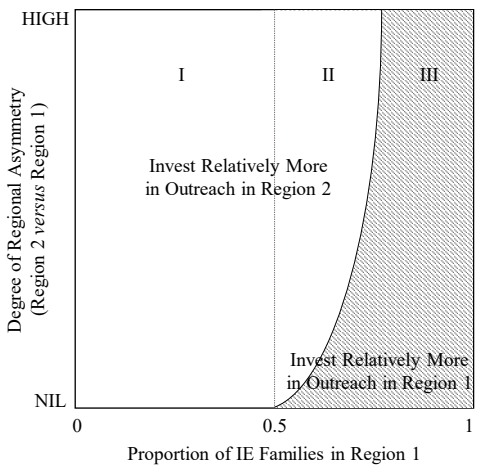
outreach in the non-earmarked region reduces. The intuition behind this implication is as follows: Recall that large amounts of investment in provider services and outreach in a region will lead to a significant improvement in the propensity of IE families to accept offered vouchers. As a result, with large amounts of earmarked funds for one region, the CCR&R can reduce the inequity outcomes in its area by increasing investment in provider services, even at the expense of investment in outreach in the other region. Thus, while additional funds always help reduce inequity outcomes, the funds earmarked for outreach in one region may crowd out the investment in outreach in the non-earmarked region. This finding on the unintended consequences of earmarked funds is consistent with findings in the existing literature on counter-productive effects of earmarked funds (Besiou et al. 2014, Arora et al. 2022). For instance, Besiou et al. (2014) show that earmarked funds for use in a specific zone can decrease the fleet size available for disaster response in the other zones.

Key Takeaways for CCR&Rs: Figure 3.2 summarizes the results discussed in Sections 3.2.1–3.2.4. In general, investing more funds in outreach in a region with more IE families is optimal (as depicted by portions I and III in Figure 3.2(a)). However, as depicted by portion II, it might be optimal for the CCR&R to invest relatively more funds in outreach in, say, region 2, even when the majority of IE families reside in region 1. This is especially true in the following situations, which are also presented in Figure 3.2(b):

(i) The external considerations (e.g., public transportation and infrastructure) in region 2 have a greater impact on a family's acceptance propensity;

(ii) The rate of return of investment in outreach in region 2 is higher and funds are in abundance;

(iii) The socioeconomic distress experienced by families in region 2 is significantly higher; or



(a) Optimal Outreach Investments

Under the following regional asymmetries, it is optimal to invest relatively *more* in outreach in Region 2, *irrespective* of the mix (γ) of IE families.

When Region 2 has *higher*:

(i) External considerations ($p_{o2} > p_{o1}$)

(ii) Marginal return of outreach investment ($\beta_2 > \beta_1$) and funds are in abundance (a large F)

(iii) Socioeconomic burden of distress ($\xi_2 > \xi_1$)

(iv) Earmarked funds for outreach ($y_2 > \delta > \bar{\delta}$)

(b) Impact of Regional Asymmetries

Figure 3.2. Key Takeaways for CCR&Rs

(iv) A large amount of funds is earmarked for outreach in region 2.

Additionally, portion II expands (and portion III shrinks) when the marginal return of investment in provider services is high, or when the baseline probability that an IE family accepts the offered voucher is high.

3.3 Practical Illustration Using a Case Study

In this section, we illustrate how the results discussed in Section 3.2 can apply in practice. We present a case study based on the real-world example of the CCR&R operating in the Western Massachusetts service area. We use data collected from the U.S. Census Bureau, the child care literature, and our interviews with multiple officials (director and front-line staff) at the CCR&R under study, to estimate values of the model parameters. Over a period of two years, we conducted 10 semi-structured interviews with multiple field experts to elicit the CCR&R-specific parameters.

Based on the U.S. Census Bureau's categorization of the U.S. landscape, we categorize the CCR&R's service area into three types of regions: urbanized (denoted

by 1), urban clusters (denoted by 2), and rural (denoted by 3). Table 3.1 lists the considered values of parameters and the sources used to estimate them. Appendix A.3 presents a generalized optimization model (for $n \geq 2$ different types of regions) and describes the steps taken to estimate the model parameters.

Table 3.1. Case Study: Estimated Values of Parameters

Parameter Value	**Sources/Methods (see details in Appendix A.3)**
$\gamma_1 = 0.27$, $\gamma_2 = 0.05$, $\gamma_3 = 0.68$	The U.S. Census Bureau data on number of households and household income, and the state's income eligibility criteria
$\alpha = 0.55$	Expert elicitation approach
$\beta_1 = 0.26$, $\beta_2 = 0.22$, $\beta_3 = 0.23$	Expert elicitation approach, child care literature, the local wage rate, and the local general election data on voting
$\underline{p}_1 = 0.3$, $\underline{p}_2 = 0.3$, $\underline{p}_3 = 0.2$	Field interviews
$p_{o1} = p_{o2} = 6 \text{ x } 10^{-7}$, $p_{o3} = 4 \text{ x } 10^{-7}$	Field interviews, and child care literature
$\xi_1 = 1$, $\xi_2 = 1.41$, $\xi_3 = 1.87$	The U.S. Census Bureau data on household income
$F = \$260,000$	The Massachusetts Department of Early Education and Care's data on the CCR&R's budget

Note: We also conduct sensitivity analysis by varying values of α from 0.25 to 0.85, β_3 from 0.20 to 0.30, ξ_2 from 1.41 to 7.05, ξ_3 from 1.87 to 9.35, and F from $\$260,000$ to $\$780,000$. See the summary description of model parameters in Table A.1 in Appendix A.1.

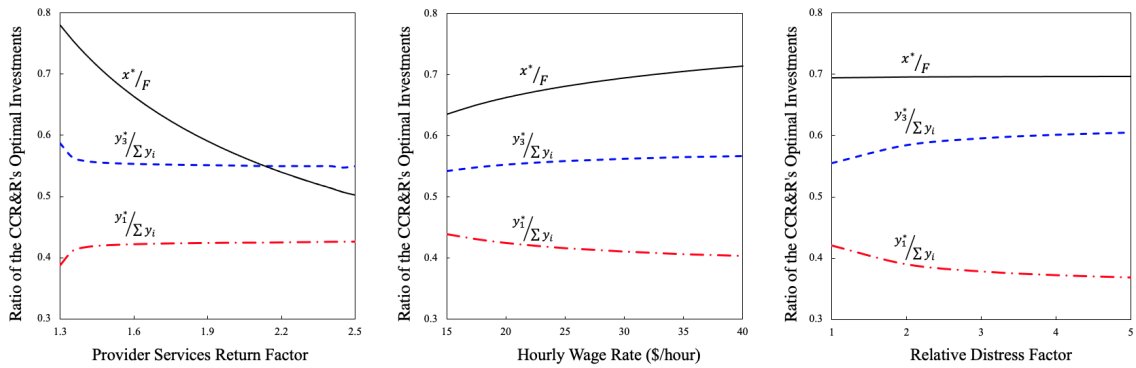
3.3.1 Optimal Investment Decisions for the CCR&R in Western Massachusetts

We solve the optimization problem by using the estimated parameters to compute the optimal levels of investments in provider services and outreach activities. We find that the optimal amounts of investment are as follows: $\$180,681$ for provider services (x^*), $\$33,362$ for outreach in region 1 (y_1^*), $\$1,972$ for outreach in region 2 (y_2^*), and $\$43,985$ for outreach in region 3 (y_3^*). For different values of the available funds F, we find that the optimal amounts of investments in provider services and outreach activities as a percentage of F remain almost identical: approximately, $x^*/F = 70\%$ and $(y_1^* + y_2^* + y_3^*)/F = 30\%$.

Notably, while the ratio of the distress faced by IE families scaled by the proportion of IE families and the external environmental factors in regions 3 and 1 $\left(\frac{\gamma_3 p_{o3} \xi_3}{\gamma_1 p_{o1} \xi_1}\right)$ is equal to 3.14, the ratio of optimal investments in outreach activities in regions 3 and 1 $\left(\frac{y_3^*}{y_1^*}\right)$ to minimize overall inequity outcomes is around 1.32. The difference in this magnitude arises because the outreach elasticity in region 3 is lower than the outreach elasticity in region 1. Further, even though the proportion of IE families residing in region 2 is quite low and the outreach elasticity in that region is the lowest, the optimal outreach investment in region 2 is a non-trivial percentage value of the CCR&R's total investment in all outreach activities in its service area (this is due to the fact that $\xi_2 = 1.41\xi_1$). This optimal investment y_2^*, although relatively smaller in magnitude, in practice can cover the expenses of a few visits to the region to identify and reach out to influential voices within the local community. The next subsection discusses how the optimal investments change as estimates of the model parameters vary.

We perform sensitivity analysis by varying estimates of provider services elasticity, outreach elasticities, and socioeconomic burdens of distress. As a summary measure of the CCR&R's allocation of funds, we consider the ratio of optimal investment in provider services to available funds, as well as the ratio of optimal investment in outreach in region i to total investment in outreach activities in all three regions, i.e., x^*/F and $y_i^*/\left(y_1^* + y_2^* + y_3^*\right)$, respectively. Figure 3.3 illustrates the findings, which are discussed below.

Estimation of provider services elasticity: The estimate of the provider services elasticity α is based on the relationship between increments in funding provided to the team focusing on provider services and the increase in the chance of an IE family accepting an offered voucher. The plots in Figure 3.3(a) depict changes in the CCR&R's optimal investments with respect to the required increment in investment of funds toward the service team to improve the acceptance chances from $1/5$ to $1/4$,



(a) Related to Estimation of α (b) Related to Estimation of β_3 (c) Related to Estimation of ξ_2, ξ_3

Note: Other parameters are same as listed in Table 3.1. Also, since the ratio $y_2^*/(y_1^* + y_2^* + y_3^*)$ is relatively smaller in magnitude and follows same behavior as the ratio $y_3^*/(y_1^* + y_2^* + y_3^*)$, we omit its plot for clarity of exposition.

Figure 3.3. Sensitivity Analysis of the CCR&R's Optimal Investments with Respect to Model Parameters

which we refer to as the provider services return factor. Naturally, the optimal investment in provider services is reduced when the provider services return factor increases (because α decreases as a function of this factor). Interestingly, we find that a higher portion of freed up resources (due to reduction in x^*) should be directed to outreach in the urbanized region rather than in the urban clusters and the rural regions (because β_1 is greater than both β_2 and β_3).

Estimation of outreach elasticity: The estimate of outreach elasticity β_3 is based on the information that the hourly wage rate of the personnel specializing in outreach-related events is \$30.2 per hour. The plots in Figure 3.3(b) depict changes in the CCR&R's optimal investments as the hourly wage rate changes. As expected, the optimal investment in provider services increases when the hourly wage rate increases (because the outreach elasticities decrease with this hourly wage rate). However, the decrease in the total optimal outreach investments (to free up resources to increases x^*) is such that the CCR&R takes away funds from its outreach investments in the urbanized and rural regions *only*. This can be explained as follows: Since the optimal amount of outreach investment in the urbanized clusters is already significantly

smaller (due to low γ_2), the effect of diminishing returns of the outreach investment is not as pronounced in the urbanized clusters as it is in the urbanized and rural regions. As a result, the CCR&R will find it optimal to use a portion of freed-up resources from y_1^* and y_3^* toward outreach investment in urbanized cluster y_2^*.

Estimation of distress: The estimate of distress ξ_i is based on the U.S. Census Bureau data on household income in region i. Additional reasons (other than the cost of living) can amplify these quantified values of an IE family's distress. The plots in Figure 3.3(c) depict changes in the CCR&R's optimal investments for larger values of the relative distress factors in the urban clusters and the rural region (specifically, when both ξ_2 and ξ_3 are amplified by a factor of 2, 3, and so on). We find that it becomes optimal for the CCR&R to increase its investment in outreach in the rural region, even at the expense of reducing its investment in outreach in the urbanized region despite the latter having both a larger outreach elasticity ($\beta_1 > \beta_3$) and a greater impact of external environmental factors ($p_{o1} > p_{o3}$).

We also perform sensitivity analysis by varying the amount of additional funds (δ) that are earmarked for conducting outreach in the rural region. We find that, when δ is greater than $\$52,000$, the optimal outreach investment in the rural region (y_3^*) is equal to the amount of the earmarked funds. Moreover, at such large values of δ, it becomes optimal to reduce the outreach investments in both the urbanized region and the urban clusters in order to invest more funds in the provider services. This specific finding with regard to the reduction in both y_1^* and y_2^* stands in contrast to the finding discussed above, wherein it is optimal for the CCR&R to use a portion of the freed-up resources (from the reduction in y_1^*) to increase y_2^*. This result is due to the complementary nature of the provider services and outreach activities. These findings can help donors make informed decisions on donating earmarked funds when earmarking might be unavoidable due to regulatory, accounting, or monitoring considerations of the donor entity.

3.3.2 Value of Optimal Allocation of Funds

To quantify the value of the proposed optimal allocation of funds, we calculate the percent value reduction in inequity outcomes by comparing values of the inequity measure under both the optimal and the current fund allocation strategies. The CCR&R currently allocates only 10% of the total working hours of an existing employee (working at a wage rate of \$15.44 per hour). Considering 2080 working hours in a year, the CCR&R's investment in outreach in its rural region (denoted by, say, $\hat{y}_3$) is \$3,211.52. In the current scenario, the interviews with the CCR&R officials indicate that the outreach investments in different regions are only proportional to the number of IE families residing in each region. Thus, the CCR&R's current investments in outreach in the urbanized ($\hat{y}_1$) and urban cluster ($\hat{y}_2$) regions are \$1,275.16 and \$236.14, respectively. Next, to calculate the improvement in $MI\left(x^*, y_1^*, y_2^*, y_3^*\right)$ relative to $MI\left(F - \hat{y}_1 - \hat{y}_2 - \hat{y}_3, \hat{y}_1, \hat{y}_2, \hat{y}_3\right)$, we run a simulation experiment using ranges of practical values for the activity- and region-specific parameters (formulated around the specific values estimated above). We find the CCR&R can achieve an average improvement of 7.0% in the equity outcomes, with a 95% confidence interval of [6.0%, 8.0%], by using the optimal allocation strategy instead of the current allocation strategy.[7]

As emphasized in SDGs, such a reduction in inequity in access to child care, especially for children in low-income families, plays a pivotal role in determining the future performance of the children and the lowering of the socioeconomic burden (Education Above All 2016). In particular, SDG #4 emphasizes the need to address

[7]We obtain these values of the percentage improvement in inequity outcomes by using uniform distributions for the model parameters. As a robustness check, we instead use censored normal distributions for these model parameters, and find that the CCR&R can improve the inequity outcomes, on an average, by 5.6%, with a 95% confidence interval of [4.7%, 6.5%]. Further, the magnitude of improvement (7.0% or 5.6%) is consistent with the improvement estimate of yearly societal benefits (in the range of 6.0-10.0%) in the study by Heckman (2012), which focuses on the effectiveness of child development programs, such as birth-to-five and preschool programs.

equity issues related to opportunities provided to rural children. In that regard, the estimated value of the improvement in equity outcomes in the CCR&R's service area is related to the SDG Indicator 4.2, which includes reducing inequity in "access to quality early childhood development, care, and pre-primary education" for rural versus urban recipients (SDG 2017). Furthermore, the magnitude of benefit of the proposed optimal allocation of funds should motivate all CCR&Rs (nearly 700 of them in the U.S.) to utilize this chapter to estimate the model parameters and calculate their optimal levels of investment in provider services and outreach in their service areas.

CHAPTER 4

ALLOCATION OF FUNDS IN BILEVEL SUBSIDY WELFARE PROGRAMS

In this chapter, we develop a bilevel, one-to-many, and forward-looking optimization model to analyze a funding agency's optimal funds allocation decisions in a subsidy voucher program. Our model includes the following three key features of such a program: (i) the funding agency's equity consideration, such that while all the involved entities are motivated to enhance the social impact generated by the programs, the funding agency's objective is to maximize the total expected social impact under an equity-based constraint, while the service agencies aim to maximize the social impact in their respective areas; (ii) the intricate relationship between the contextual factors (such as rates of investment return of activities and mix of different types of service providers) and the service agencies' investments that generate social impact; and (iii) the asymmetry of information, wherein the funding agency is better informed on the possibility of additional funds, if any, becoming available in the future. In our main analysis, we solve this model by considering a practical scenario in which additional funds may be approved by the funding source in the future for a certain service area only.

The remainder of this chapter is organized as follows. Section 4.1 outlines the model setup. Section 4.2 presents our main results on the optimal allocation decisions and the optimal outcomes. An illustrative case study is presented in Section 4.3. A summary of key notation in our study is provided in Table B.1 in Appendix B.1.

4.1 Model

In this section, we present our model for the allocation of funds in a subsidy voucher program by incorporating the bilevel hierarchical structure, wherein the funding agency distributes its available funds among all of its service agencies. We consider a stylized setup in which the funding agency works with two service agencies, with each service agency conducting provider-facing activities in its designated service area i, where $i \in \{1, 2\}$. We consider that the funding agency has a limited amount of financial resources, say $F > 0$, committed by the legislative body for investments in various activities during the planned horizon (e.g., a fiscal year) of the subsidy voucher program (Congressional Research Service 2015). Later in Section 4.3, we use a case study to show that insights based on our analytical results remain qualitatively similar for a general setup with more than two service areas.

In practice, additional funds are often expected to be sanctioned for a particular area at some future instances during the planned horizon. For example, given the exogenous activism and pressure to uplift certain minority communities, the legislative bodies could sanction additional funds to improve operations of the subsidy voucher program in a particular area (say, the one where a relatively larger proportion of beneficiaries reside) (Department of Health and Human Services 2021, Lynch 2020). To match with evidence and observations from practice, we consider not only a probabilistic availability of such additional funds, but also an information asymmetry between the funding agency and the service agencies about the availability of such funds. This is mainly because, due to the proximity to the funding source, the funding agency generally has more information on the likelihood and amount of additional funds that may become available. Accordingly, in our model, the funding agency considers there is a probability $p \geq 0$ that $f > 0$ amount of additional funds will be approved by the legislative body for investments in area 2 at a future

instance (therefore, no additional funds will be available with a probability $1 - p$).[1] However, we consider that the service agencies make their investment decisions without incorporating any such likelihood of additional funds becoming available.[2] While our main model and analysis consider that additional funds are for investments in a particular area only, our analysis shows that our main insights remain qualitatively similar under a setting where the additional funds are for investment in both areas for a particular activity only.

Next, we explain the mathematical formulations of the service agencies' social impact generation functions (in Section 4.1.1), and the funding agency's objective function and the equity consideration (in Section 4.1.2). Thereafter, we outline the optimization setup that captures the funds allocation problem in the bilevel subsidy voucher program (in Section 4.1.3).

4.1.1 Service Agencies' Model Elements

The provision of funds is a form of contract between the funding agency and the service agencies in terms of exchanging financial resources for positive social impact (Brest and Harvey 2008). A key feature of subsidy voucher programs under study is that service agencies rely on local service providers to improve the lives of beneficiaries in their respective areas. In particular, a service agency uses the funds received from the funding agency to invest in provider-facing activities – namely, quality improvement and outreach – with the goal of maximizing the total social

[1]The modeling choice of additional funds becoming available for area 2 is without the loss of generality. Later, in Section 4.2.2, we relate our findings on the optimal outcomes under different funding methods to the relative number of beneficiaries residing in area 2, which is often the motivation behind sanctioning of additional funds for an area.

[2]The modeling choice that the service agencies consider zero probability of additional funds becoming available is for analytical tractability only. All the analytical insights are qualitatively similar as long as there is information asymmetry between the entities at two different levels such that, the funding agency considers a probability p and the service agencies consider a probability $p' \leq p$ that f amount of additional funds will become available in the future.

impact generated in its area through those investments. Although the interventions under each of these two activities benefit the beneficiaries and society at large, they do so through distinct mechanisms. Specifically, investments in the quality improvement lead to a greater social impact by enhancing the quality of the essential services delivered to beneficiaries at subsidy-accepting providers; whereas, investments in the outreach lead to a greater social impact by increasing the ease of accessibility of the program for beneficiaries by enrolling additional providers to accept subsidies in that service area.[3]

Let B_i denote the budget provided by the funding agency from its available (initial) funds F to service agency $i \in \{1, 2\}$. Using the provided budget, we consider that service agency i invests X_i and Ψ_i amount of funds in quality improvement and outreach, respectively. Consistent with the view of the extant literature on return on investment in nonprofit settings (Privett and Erhun 2011, Sharma et al. 2021) and service settings (Gandjour 2010, Karaer et al. 2017), we consider that the return on investment toward each activity exhibits a diminishing rate of return. Further, we consider that the investment in different types of activities leads to different social returns. That is, quality improvement and outreach investments have different efficiencies in terms of how well they enhance the generated social impact. Accordingly, we model the social impact from investments in quality improvement and outreach activities as $\sqrt{\alpha X_i}$ and $\sqrt{\beta \Psi_i}$, where α and β denote how efficiently investments in quality improvement and outreach activities, respectively, lead to generation of social impact. In essence, in our model, α and β capture the rate of return of investment

[3]Our model can be generalized to more than two types of activities (wherein, the social impact generation function will have an additional term corresponding to each additional activity type). We focus on the aforementioned two types of provider-facing activities because several subsidy voucher programs often categorize providers into subsidy-accepting versus non-accepting when planning their operational activities (Garboden, Rosen, Greif, DeLuca and Edin 2018, Department of Health and Human Services 2019).

in quality improvement and outreach activities, respectively.[4] The use of square root functional form to capture diminishing returns in the generation of social impact is in line with the formulation of social impact generation functions in the related studies in the nonprofit settings. Note that the overall insights from our analysis remain qualitatively similar when using other functional forms that capture diminishing returns (e.g., exponential or logarithmic form).

Further, we allow for the return rate of investment in outreach activity to be uncertain. This is because, in practice, service providers' willingness to participate in subsidy voucher programs is impacted by multiple external factors that are outside the service agencies' purview (e.g., prosocial environment, providers' closeness to the community, or private child care market) and thus, it is typically difficult to assess investment returns of outreach activity (Putnam 2001, Bierhoff 2005). Accordingly, we consider that outreach investment return rate can be either β_H with a probability of ρ or β_L (where $\beta_L \leq \beta_H$) with a probability of $1 - \rho$.

As discussed earlier, we consider there is a likelihood that the legislative body could sanction additional funds f for investments in area 2 only. We next outline the social impact generated by the service agencies 1 and 2 if no additional funds become available in the future. Then we outline the social impact generated in area 2 if additional funds become available in the future.

Social Impact Generated If No Additional Funds Become Available: Let $V_i(\cdot)$ denote the social impact generated by service agency i through investments toward service providers operating in its designated area. $V_i(\cdot)$ can be expressed as a combination of expected social impact from investments in the two activities:

[4]Since all service agencies follow the government's mandate to deliver standardized subsidy assistance, we reasonably assume the same efficiency of investment in quality improvement and outreach activities in all areas, i.e., $\alpha_i = \alpha$ and $\beta_i = \beta$, $\forall i \in \{1, 2\}$. Later, we allow for asymmetry in the efficiency of investment in the activities across the two service areas, and find that all main insights remain qualitatively similar (see Appendix B.4.6).

$V_i\left(X_i, \Psi_i\right) \doteq q_i\sqrt{\alpha X_i} + (1-q_i)\left(\rho\sqrt{\beta_H \Psi_i} + (1-\rho)\sqrt{\beta_L \Psi_i}\right)$, where $0 < q_i < 1$. The parameter q_i appropriately weighs the contributions from investments in quality improvement and outreach activities in social impact generation. For example, contribution of investment in quality improvement would be higher than that in outreach (i.e., $q_i > 1/2$), when the number of service providers that accept subsidies is greater than the number of providers that do not yet accept subsidies (Department of Health and Human Services 2019).

Social Impact Generated If Additional Funds Become Available: Since the additional funds are restricted for investments in area 2 only, the social impact generated by service agency 1 is as given by the function $V_1\left(X_1, \Psi_1\right)$ outlined above. However, using the additional funds f, the service agency 2 makes additional investments in each of the two activities on top of the previously made investments. Accordingly, we consider that the service agency 2 invests χ_2 and ψ_2 in quality improvement and outreach, respectively, on top of the previous respective investments X_2 and Ψ_2. Let $v_2(\cdot)$ denote the overall social impact generated by service agency 2 using initial and additional budgets. Therefore, $v_2(\cdot)$ can be expressed as: $v_2\left(X_2, \Psi_2, \chi_2, \psi_2\right) \doteq q_2\sqrt{\alpha\left(X_2+\chi_2\right)} + (1-q_2)\left(\rho\sqrt{\beta_H\left(\Psi_2+\psi_2\right)} + (1-\rho)\sqrt{\beta_L\left(\Psi_2+\psi_2\right)}\right)$.

Before we proceed with describing the funding agency's model elements, two remarks are in order. First, the function $v_2(\cdot)$ assumes the previously-made investments (X_2 and Ψ_2) are already committed and cannot be altered by the service agency when the additional funds are made available. This is a reasonable assumption considering that the service agencies generally sign contracts and leases (and pay upfront) for the duration of the planned horizon (New England Farm Workers Council 2017). Second, the function $v_2(\cdot)$ also assumes that the outreach investment return rate remains to be uncertain when the additional funds become available. While this is a reasonable assumption (due to the features of outreach activity, as explained above), it is possible that the funding and service agencies will obtain a better understanding of the

outreach investment return when the additional funds become available (due to the temporal gap between the availability of F and f). To this end, we also examine the scenario under which this uncertainty gets resolved when future additional funds arrive (see Section B.4.4 in Appendix B.4).[5]

4.1.2 Funding Agency's Model Elements

The funding agency distributes its initial funds F between service agencies 1 and 2 with the goal of maximizing the overall expected social impact, while also ensuring the equitable social impact generation across service areas 1 and 2. We capture this goal of the funding agency by formulating its objective function and equity consideration as follows.

Total Expected Social Impact: Recall that the funding agency possesses information on the likelihood of additional funds f becoming available to area 2 in the future. Accordingly, the funding agency incorporates this information in the allocation of its initial funds F to maximize the total expected social impact across both service areas, which we denote by TSI. Mathematically, we have: $TSI \doteq V_1(\cdot) + (1-p)\,V_2(\cdot) + pv_2(\cdot)$, where $V_1(\cdot)$, $V_2(\cdot)$, and $v_2(\cdot)$ are as outlined in Section 4.1.1.

The expression of TSI assumes there is no time value associated with the generated social impact, i.e., there is no difference in delivering benefits to the beneficiaries sooner (using initial funds) than later (using additional funds). This is a reasonable assumption because, the temporal gap between allocation of initial funds and additional funds if available is quite small as compared to the lifelong nature of the benefits

[5]Although the social impact generation function captures key activity- and area-specific differences (e.g., as captured by α, $\beta_{\{H,L\}}$, ρ, θ_i, and q_i), a few practitioner reports highlight disparities in the number of service providers between different regions, say predominantly rural and predominantly urban regions, within a service area (Department of Health and Human Services 2023). Accordingly, we extend our model and analysis to also consider such regional differences between the areas (see Appendix B.4.5).

delivered under the considered programs (see Barnett and Masse 2007, Herbst 2017). Nevertheless, for completeness, we present an extension in Appendix B.4 (in Section B.4.2), where we use a discount factor to capture the time value of the generated social impact.

Expected Inequity: As discussed in Section 1.3, in addition to achieving a greater overall social impact, there is a growing attention by the funding agencies on ensuring equity in the outcomes generated across different geographic areas (here, service areas). To capture the funding agency's equity consideration, we first define a measure of inequity as 'the sum of the absolute deviations of the expected social impact generated between all pairs of service agencies based on initial and future additional funds allocation'. This inequity measure has been previously used by operations management scholars to capture the inequity in outcomes across different geographic areas in a variety of resource allocation problems, including organ allocation (Zenios et al. 2000), emergency medical services (Zhu et al. 2019), and facility location (Ohsawa et al. 2008). We denote the inequity in the expected social impact generated across areas 1 and 2 by I, which can be written as: $I \doteq \left| \frac{V_1(\cdot)}{\theta_1} - \frac{(1-p)V_2(\cdot)+pv_2(\cdot)}{\theta_2} \right|$. The parameter θ_i allows for a meaningful comparison between the social impact generated across different areas by normalizing the generated social impact in each area by the size of its pool of targeted beneficiaries. This is because, due to differences in geographic, demographic, and cost of living related factors, the mix of beneficiaries varies across different areas (Massachusetts Department of Early Education and Care 2017), and the funding agency needs to appropriately scale the generated social impact in different areas before comparing them. We refer to θ_i as the *volume adjustment factor* of service area i, such that $\theta_i > \theta_j$ implies that the size of the pool of beneficiaries in service area i is larger as compared to that of service area j.

Finally, we consider that the funding agency utilizes a constraint, given by $I = 0$, to ensure that social impact is generated in an equitable manner across the two service

areas. This constraint ensures perfect equity, which is consistent with the equity policy used in similar settings that focus on distribution of critical products and services (e.g., Orgut et al. 2018), and it is widely accepted that "the most equitable policy is one that completely eliminates outcome discrepancies across the various ... groups" (Zenios et al. 2000). However, later in Appendix B.4.3, we relax this constraint and modify it as $I \leq K$ to examine how the level of the maximum allowed inequity deviation $K \geq 0$ impacts the funding agency's optimal funds allocation.

4.1.3 Bilevel Funds Allocation Problem

Combining all of the aforementioned details, the funding agency's funds allocation problem can be expressed as the following sequential optimization model. For the ease of understanding, we describe these stages in the reverse order (since they are solved using the backward induction approach). If and when the additional funds f become available for the area 2, the service agency 2 makes two activity-specific investment decisions χ_2 and ψ_2 to maximize the overall social impact generated based on initial and additional funds. The optimal investment levels χ_2^* and ψ_2^*, for a given level of investments X_2 and Ψ_2 made using the initial funds provided by the funding agency, can be obtained by solving the following optimization problem:

$$\chi_2^*, \psi_2^* \in \underset{\{\chi_2^l, \psi_2^l\}}{\arg\max} \left\{ v_2\left(X_2, \Psi_2, \chi_2^l, \psi_2^l\right) : \chi_2^l + \psi_2^l \leq f, \quad \chi_2^l, \psi_2^l \geq 0 \right\}. \tag{4.1}$$

The funding agency's initial funds allocation problem–the distribution of F to provide budgets B_1 and B_2 to the service agencies 1 and 2, respectively–must incorporate the service agency 2's optimal investment decision using the additional funds f (if any). As given by the solution to equation (4.1), the optimal (additional) investment levels χ_2^* and ψ_2^* are functions of X_2 and Ψ_2 and hence, functions of B_1 and

B_2. Thus, the optimal budget levels B_1^* and B_2^* to service agencies 1 and 2 can be obtained by solving the optimization problem shown below:

$$\max_{\{B_1,B_2\}} TSI = V_1\left(X_1,\Psi_1\right) + \left(1-p\right)V_2\left(X_2,\Psi_2\right) + pv_2\left(X_2,\Psi_2,\chi_2^*,\psi_2^*\right), \tag{4.2}$$

$$\text{s.t.} \quad \left|\frac{V_1\left(X_1,\Psi_1\right)}{\theta_1} - \frac{\left(1-p\right)V_2\left(X_2,\Psi_2\right) + pv_2\left(X_2,\Psi_2,\chi_2^*,\psi_2^*\right)}{\theta_2}\right| = 0, \tag{4.3}$$

$$X_i,\Psi_i \in \underset{\left\{X_i^l,\Psi_i^l\right\}}{\arg\max}\left\{V_i(X_i^l,\Psi_i^l) : X_i^l + \Psi_i^l \leq B_i, \quad X_i^l,\Psi_i^l \geq 0\right\}, \quad \forall i \in \{1,2\}, \tag{4.4}$$

$$B_1 + B_2 \leq F, \tag{4.5}$$

$$B_i \geq 0, \quad \forall i \in \{1,2\}. \tag{4.6}$$

4.2 Results

We first present and discuss how the funding agency should take into account area- and activity-specific differences in its equity-ensuring funds allocation strategy (in Section 4.2.1). Thereafter, in Section 4.2.2, we examine how "valuable" our proposed equity-ensuring funds allocation strategy is. We do this by comparing and contrasting the resulting levels of equity and total social impact under that strategy with those under different funds allocation methods (specifically, the one that has no equity consideration, and the one that is based on a simple formula).

4.2.1 Optimal Budget Allocation Decisions

We solve the optimization problem outlined in Section 4.1–wherein the funding agency has equity consideration–to obtain the optimal budget allocated to each service agency, i.e., B_1^* and B_2^* as characterized in Lemma 4.1. Note that our analytical results are derived by setting $p = 1/2$, which is the most variable case in terms of the chances

of additional funds becoming available or not.[6] (Hereafter, we use the terms "higher," "lower," "increasing", and "decreasing" in the weak sense.)

Lemma 4.1 *Consider the optimization problem outlined in equations (4.2)-(4.6) (i.e., with the equity consideration). Suppose* $p = \frac{1}{2}$ *and* $f < \dot{f}$. *Then, the funding agency's optimal levels of initial funds to service agencies 1 and 2* (B_1^*, B_2^*) *are:* $B_1^* = \frac{\sqrt{N_1\left(4FN_1(F+f)\theta_2^2 - f^2\right)} + (2F+f)\theta_2 N_1}{4\theta_2 N_1\left(\theta_2^2 N_1 + 1\right)}$ *and* $B_2^* = F - B_1^*$, *where* $\dot{f} \doteq 4\theta_2^2 N_1 F > 0$ *and* $N_1 \doteq \frac{\alpha q_1^2 + \left(\rho\sqrt{\beta_H} + (1-\rho)\sqrt{\beta_L}\right)^2 (1-q_1)^2}{\alpha q_2^2 + \left(\rho\sqrt{\beta_H} + (1-\rho)\sqrt{\beta_L}\right)^2 (1-q_2)^2} > 0$. *Further,* B_1^* *increases in* f.

Lemma 4.1 formalizes the intuition that when the level of additional funds (f) that may become available for area 2 increases, the funding agency should increase the amount of budget allocated to area 1 (B_1^*) from its initial funds F in order to ensure equity in the social impact generated across both service areas. Increase in B_1^* (at the expense of B_2^*) with an increase in f keeps the expected inequity between the two areas to the minimum by taking into account the possibility of more funds being sanctioned during the planning horizon for area 2 only.[7] Using the optimal budget levels shown in Lemma 4.1, we next shine light on the effects of area- and activity-specific factors on the funding agency's optimal funds allocation strategy.

The proposition below characterizes the effect of the mix of subsidy-accepting versus non-accepting service providers in an area (captured by q_i), uncertainty in rate of return in outreach investment (captured by ρ), and rate of investment return

[6]This is required only for the analytical tractability of our sensitivity analyses of the optimal budget levels with respect to the contextual factors. In our case study, we numerically show that all insights discussed in Section 4.2.1 continue to hold for any value of p (e.g., see Figure 4.3).

[7]To examine the role played by information asymmetry in the funding agency's optimal funds allocation strategy, we also obtain the optimal budget decisions under the scenario when there is no information asymmetry between the funding and service agencies on the likelihood of additional funds becoming available. Comparing these with the optimal budget levels in Lemma 4.1, our analysis reveals that, under the information asymmetry scenario versus no information asymmetry scenario, the increase in B_1^* as f increases is more pronounced. This is because, when the service agencies have no or limited information on the likelihood of additional funds becoming available, the initial funds allocation is the lever at the funding agency's disposal to ensure equity while achieving the highest total expected social impact (additional details are presented in Appendix B.4.4).

in quality improvement activity (captured by α) on the optimal budget allocated from initial funds F to the service agency $i \in \{1, 2\}$.

Proposition 4.1 *Consider the optimal budget levels B_i^* and B_{3-i}^* $i \in \{1, 2\}$ in Lemma 4.1.*

(i) With Respect To Area-specific factors. *Denote a threshold $\hat{q}$. Then, B_i^* increases in q_i and B_{3-i}^* decreases in q_i if $q_i < \hat{q}$, and B_i^* decreases in q_i and B_{3-i}^* increases in q_i otherwise, where $\hat{q} \doteq \frac{\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2}{\alpha+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2}$.*

(ii) With Respect To Activity-specific factors. *B_i^* decreases in ρ and B_{3-i}^* increases in ρ if $q_i < q_{3-i}$, and B_i^* increases in ρ and B_{3-i}^* decreases in ρ otherwise. Further, B_i^* increases in α and B_{3-i}^* decreases in α if $q_i < q_{3-i}$, and B_i^* decreases α and B_{3-i}^* increases α otherwise.*

Proposition 4.1(i) states that to ensure equity in the social impact generated across both service areas, the funding agency should increase the funds allocated toward the area in which q_i is not quite low or high. Recall that q_i captures the relative weight of investments in quality improvement versus outreach in the social impact generation function. Thus, based on the practical interpretation of q_i, these results imply that as the mix of subsidy-accepting versus non-accepting service providers within an area becomes less skewed (say, q_i is closer to the threshold $\hat{q}$ rather than to either 0 or 1), the funding agency should increase the optimal budget allocated to the service agency serving that area. On the other hand, when the mix of providers becomes skewed (say, q_i is closer to either 0 or 1), the funding agency should decrease the budget provided to the service agency i and provide these funds to the other service agency instead. We explain the intuition behind this finding next.

Consider that the ratio of subsidy-accepting providers to non-accepting ones within a particular area i is high (i.e., $q_i \geq \hat{q}$). Given that the service agency i aims to maximize the social impact in its area, an increase in q_i implies that investing in quality improvement (instead of outreach) would generate more social impact within

its area. As a result, this service agency finds it optimal to take away funds from outreach to invest toward quality improvement. However, doing so would lead to a greater disparity between the social impact generated in these two areas, ceteris paribus. Consequently, the funding agency responds by reducing the budget allocated to this service agency. By reducing B_i^* in this situation, the funding agency is able to nudge service agency i toward achieving a more balanced split of its budget between the two types of activities within its area. This balanced split at the service agency level is assured because of the presence of diminishing returns on investments in provider-facing activities (note that diminishing returns have a salient effect when the amount of budget is smaller).

We also investigate how this non-monotonic effect of q_i on the optimal decisions interacts with the effect of the level of additional funds f expected to be available for area 2. Consider the optimal budget B_1^* allocated to area 1 under the equity consideration setting. We find that, when f increases, the largest increment in the optimal budget B_1^* happens when there is a less skewed mix of subsidy-accepting versus non-accepting service providers in area 1. This finding suggests that the level of additional funds and the balanced mix of service providers have a complementary effect on the initial budget allocated to the area that will not get additional funds, and this effect is primarily driven by the equity consideration of the funding agency.

Next, Proposition 4.1(ii) states that to ensure equity in the social impact generated across the two service areas, the funding agency should decrease the funds allocated toward the area that has a relatively lower (higher) proportion of subsidy-accepting (non-accepting) providers when the chance that outreach investment yields high investment return (ρ) increases. The intuition is as follows: When ρ becomes high, both service agencies find it optimal to increase investment in outreach. However, this increase in outreach investment is more beneficial for the service agency that serves the area with a relatively higher proportion of non-accepting providers

(who benefit from outreach investment), which can potentially lead to a wider gap in the social impact generated across the two areas. Therefore, by strategically reducing the optimal budget to that service agency, the funding agency is able to ensure a more balanced investment between the two types of activities, and hence, reduce the equity gap.

Finally, the results on the rate of investment return of the quality improvement activity (α) in Proposition 4.1(ii) follow from the fact that the interaction between the rate of investment return in quality improvement and the relative weight q_i is mathematically opposite of the interaction between the rate of investment return in outreach activity and the relative weight q_i in the social impact generation functions (see the expressions of V_1, V_2, and v_2 in Section 4.1.1). Therefore, we omit the related discussion for brevity.

In sum, these results show how the funding agency should take into account the differences between service areas (e.g., additional funds expected for an area only) and within service areas (e.g., mix of providers accepting subsidies versus those not accepting) in its funds allocation decisions. Also, they help understand how the optimal funds allocation decisions by the funding agency can serve as an important lever to reduce inequity in the social impact generated by different service agencies in their respective areas.

Note that all the aforementioned insights are drawn from our analysis of the optimization-based funds allocation method in which the funding agency aims to ensure equity in the social impact generated across both service areas. We conduct additional analysis to compare and contrast the results for an equity-ensuring funding agency in Proposition 4.1 with the corresponding results for an efficiency-focused funding agency–i.e., an agency that focuses only on maximizing the total expected social impact across both areas with no equity consideration. (See Lemma B.2 in Appendix B.2 for formal characterization.) This exercise reveals that the mix of

providers, the uncertainty in outreach investment return, and investment return rate of quality improvement have a completely opposite impact on the optimal budget levels when the funding agency considers versus does not consider equity within its funds allocation strategy. Overall, these comparative results signify that the funding agencies that plan to move away from their focus on efficiency only (by considering equity) should carefully adjust their funds allocation strategies to ensure maximum impact is generated in an equitable manner.

4.2.2 Comparisons of the Outcomes under Different Funds Allocation Methods

While the results above provide implications for the funding agency's optimal budgetary decisions, it is instructive to analyze how the societal outcomes–i.e., the levels of inequity and total social impact–resulting from equity-ensuring method compare against those resulting from efficiency-focused and formula-based methods. We discuss such comparative results below.

4.2.2.1 Equity-Ensuring Versus Efficiency-Focused Methods.

We first present our results on how the key parameters affect the gap between the resulting inequity outcomes and the difference between the resulting total social impact under the equity-ensuring (denoted by the use of Eq superscript) and efficiency-focused (denoted by the use of Ef superscript) methods of funds allocation. Note that both these methods are optimization-based, with the difference being that the former considers equity and the latter does not consider equity in the social impact generated in the two service areas.

All analytical results in this section are derived by setting $f = 0$. This simplification allows analytical tractability, as an analytical comparison of the optimal levels of equity and total social impact in our multi-stage optimization model becomes intractable when two additional activity-specific variables are introduced due to f.

However, to ensure the robustness of insights discussed in this section, we undertake extensive numerical analysis to establish that all the discussed insights continue to hold for $f > 0$ (e.g., see Figure 4.3).

Proposition 4.2 *There exists a unique threshold $\hat{\theta}_2 > 0$ such that, when $\theta_2 = \hat{\theta}_2$, the resulting inequity outcomes and the total social impact are equal under the equity-ensuring and efficiency-focused methods, i.e., $I^{Ef} = I^{Eq}$ and $TSI^{Ef} = TSI^{Eq}$, where $\hat{\theta}_2 \doteq \frac{1}{N_1}$, and N_1 is characterized in the statement of Lemma 4.1. Otherwise, the differences $I^{Ef} - I^{Eq} > 0$ and $TSI^{Ef} - TSI^{Eq} > 0$ decrease in θ_2 if $\theta_2 < \hat{\theta}_2$ and increase in θ_2 otherwise. Further, $\hat{\theta}_2$ increases in q_1 if $q_1 < \hat{q}$ and decreases in q_1 otherwise; $\hat{\theta}_2$ decreases in q_2 if $q_2 < \hat{q}$ and increases in q_2 otherwise, where $\hat{q}$ is defined in Proposition 4.1.*

The results in Proposition 4.2 indicate that the efficiency-focused funding agency enhances the total social impact at the expense of inequity outcomes. Specifically, we show that the optimal funds allocation strategy that focuses only on maximizing efficiency can lead to high levels of inequity in the social impact generated in the two services areas when the disparity in the size of the pool of beneficiaries across different areas is high. This is because, the efficiency-focused funding agency takes into account only the efficiency of the use of its funds, which is determined by the relationships between the investment return rate of different activities and the mix of service providers. Whereas, the equity-ensuring funding agency carefully adjusts its optimal funds allocation strategy to ensure that the optimal social impact generated across different areas, normalized by the size of their respective pool of targeted beneficiaries, are not significantly different from each other. Given this, in situations when θ_2 is sufficiently high or low compared to the threshold $\hat{\theta}_2$, the efficiency-focused funding method results in greater inequity.

Further, the results in Proposition 4.2 state how the threshold $\hat{\theta}_2$ changes with respect to the area-specific factor q_i. However, to fully understand the impact of

q_i on the comparisons of inequity outcomes and total social impact under the two optimization-based methods, we present the following lemma on how the differences $I^{Ef} - I^{Eq}$ and $TSI^{Ef} - TSI^{Eq}$ change with respect to q_i when there is a large disparity in the size of the pool of beneficiaries across different areas.

Lemma 4.2 *When θ_i is sufficiently small (large) compared with θ_{3-i}, the differences $I^{Ef} - I^{Eq} > 0$ and $TSI^{Ef} - TSI^{Eq} > 0$ first decrease (increase) and then increase (decrease) in q_i.*

Consider that the service area 2 has a relatively large number of beneficiaries (i.e., θ_2/θ_1 is sufficiently large) and has a mix of service providers that is quite skewed toward one type of service providers (say, q_2 is closer to zero or one). Then, when the mix of service providers becomes more balanced (i.e., when q_2 moves away from zero and one), the efficiency-focused funding agency should optimally take away funds from the budget allocated to area 2, which otherwise would get distributed more evenly between the two types of activities within the area 2 due to a balanced mix of service providers in this area. This will ensure that the most total social impact is generated, but at the expense of higher inequity outcomes. In contrast, the equity-ensuring funding agency should respond to the mix of service providers becoming more balanced by channeling more funds toward area 2 to maintain a balance in the social impact generated across the two areas (this boost is needed because area 2 has a large number of beneficiaries).

In sum, we show that our optimization-based model (as outlined in Section 4.1) helps the equity-ensuring funding agency generate the most overall social impact in an equitable manner. Without such equity consideration, the optimal funds allocation strategy results in a greater inequity especially when there is a large disparity in the size of the pool of beneficiaries across different areas. Moreover, in such situations, the level of inequity enhances further when the area with relatively more (fewer)

beneficiaries has a quite balanced (skewed) mix of service providers. Also, the above-discussed comparative results, based on the two optimization setups, indicate that the goals of efficiency and equity are in contrast with each other. In practice, several funding agencies use relative simple formulas to allocate funds among service areas, which could possibly achieve better efficiency while not sacrificing equity (to some extent). We investigate this next.

4.2.2.2 Equity-Ensuring Versus Formula-Based Methods.

As discussed in Section 1.3, for simplicity and ease of use, the funding agencies under many subsidy voucher programs rely on a simple or weighted sum of proportions of multiple between-area factors to allocate funds among service agencies serving those areas (National Research Council 2001, Dilger and Boyd 2014). Consistent with this observation, we define a benchmark formula-based method within our resource allocation problem setup to investigate whether and when the optimization-based equity-ensuring method leads to improved inequity outcomes and total social impact. Specifically, we consider that the funding agency obtains the proportion of its budget that should be made available to the service agency by using the relative size of the pool of beneficiaries across different areas. After receiving this budget, under the formula-based method, we consider that each service agency distributes it between different activities within its service area based on the proportion of subsidy-accepting and non-accepting service providers. Additional technical details of the resulting allocation decisions and outcomes are provided in Section B.2.2 in Appendix B.2.

Given that ensuring equity in the social impact generated is the central focus of our proposed optimization-based funds allocation strategy, we first analyze how the inequity outcomes compare between this equity-ensuring method (denoted by the use of Eq superscript) and the formula-based method (denoted by the use of Fo superscript). The proposition below presents how $I^{Fo} - I^{Eq}$ changes with respect

to the size of the pool of beneficiaries (θ_i) and the mix of subsidy-accepting and non-accepting service providers in service area $i \in \{1, 2\}$ (q_i).

Proposition 4.3 *Denote thresholds $\underline{\theta}$ and $\bar{\theta}$ as defined in the proof, where $0 < \underline{\theta} < \bar{\theta}$. Then,*

(i) When $\theta_i/\theta_{3-i} \leq \underline{\theta}$, the difference $I^{Fo} - I^{Eq} \geq 0$ first decreases and then increases in q_i;

(ii) When $\underline{\theta} < \theta_i/\theta_{3-i} < \bar{\theta}$, the difference $I^{Fo} - I^{Eq} \geq 0$ increases in q_i if $q_i \in \left(\underline{q}_i, \bar{q}_i\right) \cup (\dot{q}_i, 1)$ and decreases in q_i otherwise, where $\bar{q}_i = \frac{\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2}{\alpha+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2}$, and the existence of $\underline{q}_i$ and $\dot{q}_i$ are shown in the proof;

(iii) When $\theta_i/\theta_{3-i} \geq \bar{\theta}$, the difference $I^{Fo} - I^{Eq} \geq 0$ first increases and then decreases in q_i.

Proposition 4.3 states that, depending on the the relative size of the pool of beneficiaries in a service area, the mix of subsidy-accepting and non-accepting service providers in the service areas has a nuanced and non-monotonic effect on the gap between inequity outcomes under the formula-based method and the optimization-based equity-ensuring method. In particular, per Proposition 4.3(i) and (iii), when a service area has a relatively small (large) pool of beneficiaries, then the formula-based method versus the optimization-based method leads to a greater level of inequity when this area has a skewed (balanced) mix of service providers. The explanation for this implication stems from the following: To begin with, the role of the size of the pool of beneficiaries in determining the funds allocation strategy is aligned under both methods–specifically, the funding agency should allocate more funds toward the service area with a larger pool of beneficiaries. However, the optimization-based method ensures that the funding agency increases the budget allocated to the service area with the balanced mix of service providers to ensure the social impact is generated across the two in a balanced manner (as shown in Proposition 4.1). The formula-based method is unable to account for such optimal balancing of the budget levels to

curtail the gap in the social impact generated across both service areas (because it relies on a simple weight-driven distribution of funds that does not explicitly consider equity).

The results in part (ii) of Proposition 4.3 state that, when there is a moderate disparity in the pool of beneficiaries in the two service areas, the formula-based method leads to a greater deviation from equitable outcomes (achieved under the optimization-based method) when the service areas have either a very balanced or a very skewed mix of service providers. This is primarily driven by the key role played by the area-specific factor q_i in carefully adjusting the optimal funds allocation strategy (by increasing or decreasing the budget, respectively) to ensure that the most social impact is created in an equitable manner (as shown in Proposition 4.1). Our analysis reveals that this deviation of equitable outcomes under the formula-based method versus the optimization-based method is further pronounced when the rate of return of investing in quality improvement or outreach is high. In sum, all of these results underscore the value of using our proposed optimization-based method toward ensuring equitable allocation of funds among service agencies by the funding agency.

While this is an important finding, it still remains to be investigated how the formula-based method compares with the optimization-based method in terms of the total social impact generated across both areas. We answer the following specific question in our Proposition 4.4 below: While the difference of inequity outcomes is high when the service area 2 has a large pool of beneficiaries and the other service area has a skewed mix of service providers (see Proposition 4.3), does the formula-based method achieves higher total social impact but not significantly sacrifice equity under these situations? These specific considerations in Proposition 4.4–that is, $\theta_2 \geq \check{\theta}_2$ and $q_1 \leq q_2 = \frac{1}{2}$–are only for the ease of explanation of insights.

Proposition 4.4 *Consider* $\theta_2/\theta_1 \geq \check{\theta}$, $q_1 \leq q_2 = \frac{1}{2}$, *and* $\sqrt{\alpha} = \rho\sqrt{\beta_H} + (1-\rho)\sqrt{\beta_L}$, *where the existence of* $\check{\theta}$ *is shown in the proof.*[8]

(i) The difference $TSI^{Fo} - TSI^{Eq}$ *first increases and then decreases in* θ_2.

(ii) Denote a threshold $\dot{\theta}_2$ *as defined in the proof, then: (ii-a) When* $\theta_2/\theta_1 \leq \dot{\theta}$, $TSI^{Fo} \leq TSI^{Eq}$ *if and only if* $\tilde{q}_1 < q_1 < \tilde{\tilde{q}}_1$, *where* $0 < \tilde{q}_1 \leq \tilde{\tilde{q}}_1 \leq \frac{1}{2}$ *and; (ii-b) When* $\theta_2/\theta_1 > \dot{\theta}$, $TSI^{Fo} \geq TSI^{Eq}$ *for any* q_1, *where the existence of* $\tilde{q}_1$ *and* $\tilde{\tilde{q}}_1$ *are shown in the proof.*

Proposition 4.4 states the conditions under which the optimization-based method, as compared to the formula-based method, generates a higher total social impact across both service areas; however, we restrict the following discussion on highlighting those situations when the total social impact is higher under the formula-based method (despite it using a simple weight-driven method). One such situation arises when there is a significantly large disparity in the size of the pool of beneficiaries across service areas; see Proposition 4.4(ii-b). This is driven by the fact that, under a quite large disparity in the pool of beneficiaries, the optimization-based method focuses mainly on balancing equity in the social impact generated across the two areas. To ensure equity, the funding agency allocates a large amount of budget to the service area with a large pool of beneficiaries, but this does not translate to a higher total social impact because of the salient impact of diminishing returns of investment at such higher levels of budget. Another such situation (i.e., $TSI^{Fo} \geq TSI^{Eq}$) arises when the disparity in the size of the pool of beneficiaries between two areas is not too large (i.e., θ_2 has a moderate value) and the mix of service providers in service area

[8]The assumption $\sqrt{\alpha} = \rho\sqrt{\beta_H} + (1-\rho)\sqrt{\beta_L}$ allows to analytically parse out the *pure* effects of θ_i and q_i on the difference in the levels of total expected social impact under the two methods, without taking into account any activity-related differences. This is a useful scenario for the comparative analysis, because allocation decisions under the formula-based method are made based on area-related differences, as captured by the size of the pool of beneficiaries (θ_2) and mix of service providers (q_i) in our model. Later, our numerical analysis in the case study (in Section 4.3) shows that all main insights continue to hold when this assumption is relaxed.

1 is either very balanced or very skewed (i.e., q_1 is closer to $1/2$ or 0); see Proposition 4.4(ii-a). It follows a similar explanation as the situation mentioned above.

Next, combining the findings in Propositions 4.3 and 4.4, it can be concluded that, although the formula-based allocation method generates non-negligible inequity outcomes, it may enable the funding agency to generate greater total social impact (with only a moderate level of inequity outcomes) than the proposed equity-ensuring method. This is particularly true when the difference in the size of beneficiaries between areas is sufficiently large, and the area with a relatively large size of beneficiaries has a highly skewed distribution of service providers; or when both areas have a large proportion of one particular type of service providers, and the investment return rate of activity targeting that type of service providers is not too high. Overall, we have the following two practical takeaways for the funding agency: It is possible based on the contextual factors that an allocation of funds using a simple formula could achieve better efficiency while not severely sacrificing equity. On the flip side, our proposed optimization-based method ensures equity while not severely sacrificing efficiency under a wide range of values of contextual factors.

4.3 Practical Illustration Using a Case Study

In this section, we present a calibrated numerical study using data from the child care subsidy voucher program in Massachusetts to provide an illustration of how the insights discussed above can apply to practice. Under this program, the funding agency–Massachusetts Department of Early Education and Care (MEEC)–partners with different service agencies–CCR&Rs (i.e., Child Care Resource and Referral Agencies)–that serve IE families and local child care providers within their designated areas. These $n = 7$ service areas correspond to different geographic areas within Massachusetts, including Western, Central, Northeast, Southeast, Cape, Metro, and Boston, and we denote them by 1 to 7, respectively. MEEC has state-

issued annual funds (F) of approximately \$6.31 million, which it allocates among the seven areas using a formula described below.

Current Allocation Method: MEEC uses a formula-based method that combines several area-specific factors to distribute its funds among the seven service areas. These factors include number of IE families, number of child care providers that accept vouchers, and total number of child care providers. For a given area, MEEC obtains a percentage of its budget that should be made available to the CCR&R in that area as follows: It calculates the relative proportions pertaining to each of the above-mentioned factors in that area, and then computes the simple average (equal-weighted and linear sum) of these proportions to arrive at the requisite percentage value.[9] As can be noted, the proportion-based funding formula currently used by MEEC does not explicitly capture equity considerations by the funding agency and the interrelationships between contextual factors in the generation of social impact.

Numerical Illustration Setup: We illustrate how the MEEC should distribute its pool of funds among the seven service agencies using our optimization model setup that incorporates all of these above-mentioned complexities. We also compare and contrast the inequity outcomes and the total social impact under three distinct funding methods: current allocation (i.e., formula-based), optimization-based allocation with equity consideration (i.e., equity-ensuring), and optimization-based allocation without equity consideration (i.e., efficiency-focused). Further, we shed light on how the MEEC's funds allocation decisions and the two outcomes of interest are impacted by the different levels of the likelihood and amount of any future additional funds.

To estimate the values of the model parameters for our numerical illustrations, we use data provided by the MEEC and data collected from the U.S. Census Bureau, the

[9]In our stylized benchmark formula-based method, these proportions (here, percentages of the MEEC's budget that should be made available to CCR&Rs) are captured by the volume adjustment factor θ_i.

Table 4.1. Case Study: Estimated Values of Area-Related Parameters

Area-Related Parameters							
	Service Area (i)						
	Western (1)	Central (2)	Northeast (3)	Southeast (4)	Cape (5)	Metro (6)	Boston (7)
θ_i	1.192	1.125	1.215	1.146	1.000	1.022	1.187
q_i	0.593	0.253	0.316	0.412	0.464	0.342	0.416

Table 4.2. Case Study: Estimated Values of Other Parameters

Other Parameters					
$\alpha = 1.176\text{x}10^{-6}$	$\beta_H = 1.818\text{x}10^{-6}$	$\beta_L = 0.556\text{x}10^{-6}$	$\rho = 0.5$	$p = 0.5$	$f = \$600K$

Note: See the summary description of model parameters in Table *Note:* We also conduct sensitivity analysis by varying values of the parameters ρ, f, and p; see Figure 4.3.

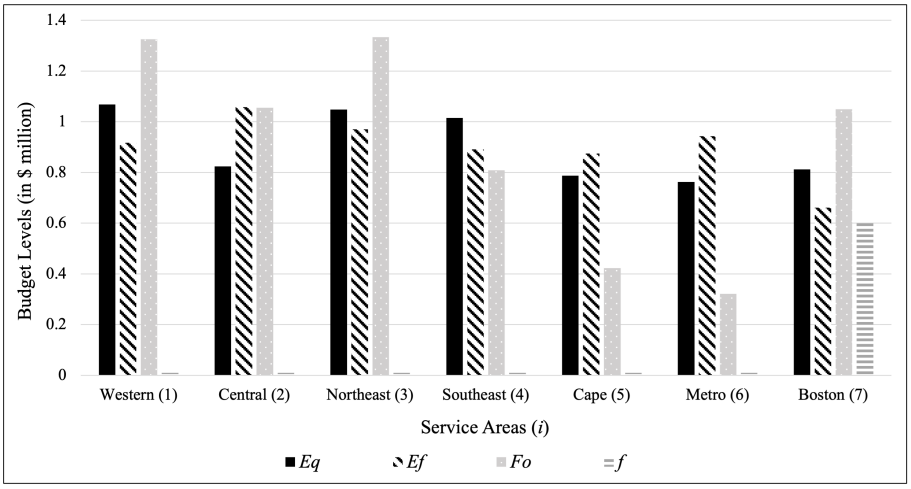
child care literature, and our interviews with multiple officials at the various CCR&Rs under study. Over a period of four years, we conducted about 14 semi-structured interviews with multiple field experts to elicit the CCR&R-specific parameters. For brevity, the detailed descriptions of the estimation of parameters are relegated to Appendix B.3. Tables 4.1 and 4.2 summarize the estimated values for the model parameters. Next, we discuss a few selected findings from our numerical analysis.

Findings Related to Budgets for Service Agencies: We solve the optimization problems outlined in Section 4.1 by using the estimated parameters, and obtain the optimal levels of funds toward each service area when: (i) no additional funds are expected to become available (i.e., $f = 0$); and (ii) $\$600,000$ additional funds are expected to become available for an area only (i.e., $f = 600,000$ for $i = 7$). Figure 4.1 depicts the budget levels provided by MEEC to the seven CCR&Rs under the three different funding methods. We first restrict our attention to the funding levels depicted by the first bar (i.e., corresponding to Eq) for each service area. We find that, to en-

sure equitable social impact among all seven service areas, MEEC should allocate a relatively higher level of funds from its pool of funds toward Western and Northeast areas, and a relatively lower level of funds toward Cape and Metro areas. This finding is driven in part by the fact that MEEC can reduce inequity across all service areas by allocating more funds to CCR&Rs serving a larger pool of beneficiaries (since θ_1 and θ_3 are relatively high). Also, this is driven by the insight presented in Section 4.2 that MEEC should allocate more funds toward areas that have a relatively balanced mix of subsidy voucher-accepting and non-accepting service providers (see the estimated values of q_i).

We next compare the budget levels depicted in the first and second bars for each service area in Figure 4.1 (i.e., Ef versus Eq). In contrast to the findings above, in the absence of equity consideration (i.e., when focus is on efficiency only), MEEC should allocate more funds to areas that have a skewed distribution of voucher-accepting and non-accepting providers (e.g., Central and Metro areas). In line with the insights presented in Section 4.2.1, these comparative findings highlight that MEEC should carefully adjust the budgets provided to CCR&Rs in the presence of equity considerations.

We finally compare the budget levels depicted in the first and third bars for each service area in Figure 4.1 (i.e., Eq versus Fo). We find that, while both Western and Northeast areas receive relatively higher levels of funds (from the initial pool of funds) under the equity-ensuring and current formula-based methods, MEEC should deviate from its current allocation of a large amount of funds to Northeast area, and instead provide more funds to Western area under the equity-ensuring method. The explanation for this difference is as follows: Since investment in outreach yields a higher expected return rate than quality improvement and Northeast area has relatively less subsidy-accepting providers, MEEC should allocate more initial funds



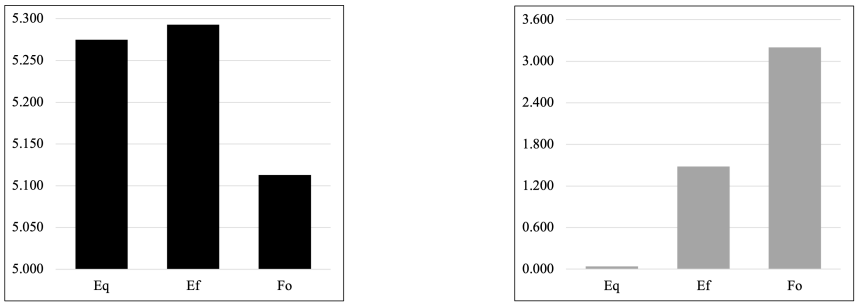
Note: Eq, Ef, and Fo refer to equity-ensuring, efficiency-focused, and formula-based funding methods, respectively. These budget levels are also summarized in Table B.2 in Appendix B.3.

Figure 4.1. Budgets for Service Areas under Different Funding Methods

to Western area in order to balance the social impact generated in these two areas (as shown in Proposition 4.1).

Findings Related to Inequity Outcomes and Total Expected Social Impact: Using the estimated parameters above, we also compare values of the inequity outcomes (I) and the total expected social impact (TSI) generated across all service areas under different funding methods. See Figure 4.2. In particular, we find that $I = 0$ and $TSI = 5.275$ under the equity-ensuring method as compared with $I = 3.199$ and $TSI = 5.113$ under the current method. Thus, the proposed optimal decisions help achieve equitable outcomes while enhancing the overall social impact by approximately 3% versus current allocation decisions.[10]

[10]Contrasting the societal outcomes under the equity-ensuring method with the corresponding outcomes under the efficiency-focused method (in Figure 4.2), we find that, when the MEEC considers equity, the decrease in the total social impact is quite small (5.275 versus 5.293) but the reduction in the inequity outcomes is sizeable (0 versus 1.481). This finding implies that, for the child care subsidy program under study, the proposed equity-ensuring method for funds allocation achieves equitable outcomes with only a relatively small loss in the overall social impact.



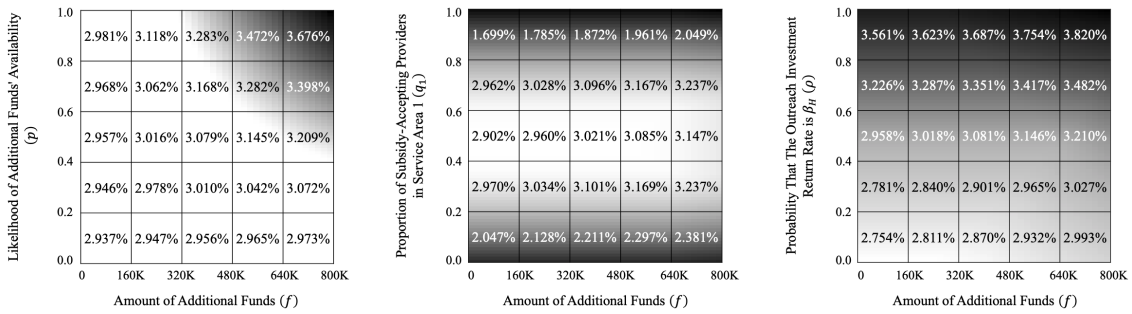
(a) Total Social Impact (TSI) (b) Inequity Outcomes (I)

Note: Eq, Ef, and Fo refer to equity-ensuring, efficiency-focused, and formula-based funding methods, respectively. Additional funds (f) and other model parameters are as estimated in Tables 4.1 and 4.2.

Figure 4.2. Comparing Outcomes under Different Funding Methods

Next, we perform sensitivity analysis with respect to a few key contextual parameters to remark on how the value (in terms of TSI and I) of using our proposed equity-ensuring method over the current formula-based method for allocation of funds changes. This analysis also helps generalize insights based on the formal results presented in Section 4.2.2.2, which are derived by considering simplifications for analytical tractability. For ease of interpretation, we use a combination of tabular and heat map formats to concurrently depict magnitudes of both TSI and I for different parameter values. In Figure 4.3, the numerical values present the percentage change in the magnitude of TSI if the funding agency uses the equity-ensuring method instead of the formula-based method for funds allocation. Whereas, the heat map presents changes in the magnitude of I such that a lighter (darker) hue captures a lower (higher) gap in the inequity outcomes under the formula-based method versus the equity-ensuring method (note that $I^{Fo} - I^{Eq} \geq 0$; see Proposition 4.3).

Consider Figure 4.3(a). We find that a higher total expected social impact is generated when the funding agency uses our forward-looking optimization setup that incorporates the asymmetry of information between MEEC and CCR&Rs on likelihood (p) and amount of additional funds (f) becoming available for Boston instead of



(a) With Respect to f and p (b) With Respect to f and q_1 (c) With Respect to f and ρ

Note: The percentage values are equal to $\frac{TSI^{Eq}-TSI^{Fo}}{TSI^{Fo}} \times 100\%$, where Eq and Fo refer to equity-ensuring and formula-based funding methods, respectively. The heat map corresponds to variations in the magnitude of I, such that a lighter (darker) hue captures a lower (higher) value of $I^{Fo} - I^{Eq}$.

Figure 4.3. Sensitivity Analysis of Outcomes under Equity-Ensuring and Formula-Based Funding Methods

the current formula-based funding method. Further, it can be seen that an increase in p or f exacerbates the inequity outcomes under the formula-based method (versus the equity-ensuring method). This is because, unlike the formula-based method, the equity-ensuring method helps generate the most social impact in an equitable manner by allocating less initial funds to Boston area, and instead increasing the level of initial funds allocated to all other areas when p or f increases.

Further, consider Figure 4.3(b). Given the enhanced attention of communities to make child care affordable and accessible in the far-flung rural areas, there is a practical possibility of an increase in the proportion of voucher-accepting service providers in the rural service area (say, Western). We find that as the proportion of voucher-accepting providers increases in the Western service area, the formula-based method (as compared with the equity-ensuring method) significantly hurts the inequity outcomes even though the gap between the overall social impact under both methods reduces. Similarly, we can see in Figure 4.3(c) that, as the likelihood of outreach investment return being high increases (say, due to enhanced awareness of how access to affordable child care is essential for sustainability for all), the equity-ensuring method performs better in terms of both I and TSI as compared with the formula-based

method. This, in particular, indicates the significance of incorporating differences in investment return rates and the diminishing nature of return of investment in the social impact generation functions (as also highlighted in Section 4.2).

CHAPTER 5

SUBSIDIZING SOCIAL WELFARE PROGRAMS: CONTRACTED SLOTS OR VOUCHERS?

In this chapter, we develop a game-theoretical model setup to study the mechanism selection for two types of subsidy welfare programs faced by a government agency, including the subsidy voucher programs and the contracted slot programs. Our model includes three key considerations in this selection: (i) the government agency's objective of providing both high-quantity and high-quality services to beneficiaries in subsidy welfare programs; (ii) the distinct mechanisms of creating social impact for beneficiaries of the two programs, with subsidy voucher programs primarily offering a large number of service providers (including a mix of high- and low-quality) to the beneficiaries and contracted slot programs ensuring beneficiaries receive only high-quality service (even if at a fewer of service providers); and (iii) the impact of program-related factors (e.g., cost of managing the programs) and non-program-related factors (e.g., demand for service providers in the private market) on service providers' participation decisions. This chapter analyzes whether and how service providers participate in these two programs and how the government selects the subsidy welfare program based on the resulting societal outcomes of the programs.

The remainder of this chapter is organized as follows. Section 5.1 outlines the model setup and presents our main results in a subsidy voucher program. Section 5.2 outlines the model setup and presents our main results in a contracted slot program. Later, in Section 5.3, we present a numerical illustration of comparisons of the societal outcomes under these two programs. A summary of key notation in this chapter is provided in Table C.1 in Appendix C.1.

5.1 Analysis of Subsidy Voucher Programs

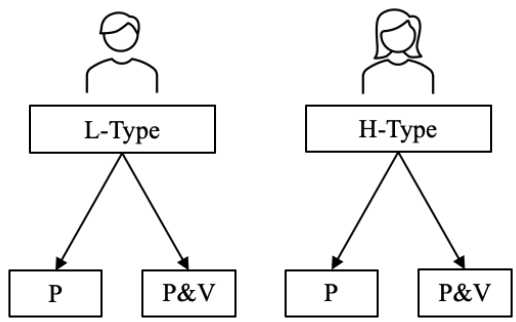
In this section, we model low- and high-quality service providers' decisions on participating in a subsidy voucher program in Section 5.1.1 and present related results in Section 5.1.2.

5.1.1 Modeling of Service Providers' Participation Decisions

Service providers vary in terms of the quality level and availability of the services they provide to their clients (Workman and Ullrich 2017, Schneider and Gibbs 2022, Howden-Chapman et al. 2023). Without loss of generality, we consider two types of service providers in the market: low-quality providers and high-quality providers. Further, we consider each low-quality provider to have a capacity of K_L and charge a market price of p_L per private-pay client. Each high-quality provider has a capacity of K_H and market price per private-pay client p_H.

Consider that the government offers a subsidy voucher program. Suppose a service provider chooses not to participate in this voucher program. In that case, it only serves clients from the private market, i.e., those who do not participate in government subsidy welfare programs. On the other hand, if a service provider opts to participate in this voucher program, it provides services to clients from both the private market and the voucher program. Hence, to decide whether it should participate in the subsidy voucher program, the service provider compares its expected payoff if it does not participate with if it participates in the voucher program. Figure 5.1 illustrates low- and high-quality service providers' participation decisions in a subsidy voucher program. Next, we describe the key elements of the model in detail.

Private Market Demand Distribution for Service Providers: We consider that the demand for service providers in the private market follows a uniform distribution with probability density function $g^{v}_{i\in\{L,H\}}(\cdot)$. Hereinafter, we refer to low-quality service providers by the use of L subscript and high-quality service providers by the use of H



Note: The nodes L-Type and H-Type denote low- and high-quality service providers, respectively. The node P denotes a service provider not participating in the subsidy voucher program, thus, only serving clients from the private market. The node P&V denotes a service provider participating in the subsidy voucher program, thus, serving clients from both the private market and the subsidy voucher program.

Figure 5.1. Service Providers' Participation Decisions in a Subsidy Voucher Program

subscript in the notations. Next, we describe the respective demand distribution for low- and high-quality service providers. Specifically, we denote d_L as the demand for each low-quality provider in the private market, which is uniformly distributed within the interval $\left[\underline{D}_L, \bar{D}_L\right]$. Denote d_H by demand for each high-quality provider in the private market, which is uniformly distributed within $\left[\delta\underline{D}_L, \delta\bar{D}_L\right]$. The parameter $\delta > 0$ represents the relative demand for high-quality providers compared to low-quality providers, such that $\delta > 1$ implies that the demand for each high-quality provider is higher than that for each low-quality provider.

Service Providers' Expected Payoff under No Participation: By combining the expected payoff of a low-quality provider when its demand in the private market is no more than its capacity ($d_L \leq K_L$) and when its demand exceeds its capacity ($d_L > K_L$), we obtain the expected payoff of each low-quality provider if it does not participate in the voucher program, denoted as $\Pi_L^{v0}(\cdot)$, as follows:

$$\Pi_L^{v0}(\cdot) = \int_{\underline{D}_L}^{K_L} p_L d_L g(d_L)\, dd_L + \int_{K_L}^{\bar{D}_L} p_L K_L g(d_L)\, dd_L.$$

To focus on the impact of program-related factors, we normalize service providers' cost of managing each client in the private market to zero. Our analysis shows that all the analytical insights remain qualitatively similar in the presence of such a cost.

Similar to the low-quality provider, we denote the expected payoff of a high-quality provider if it does not participate in the voucher program as $\Pi_H^{v0}(\cdot)$. It is given by:

$$\Pi_H^{v0}(\cdot) = \int_{\delta \underline{D}_L}^{K_H} p_H d_H g(d_H)\, dd_H + \int_{K_H}^{\delta \bar{D}_L} p_H K_H g(d_H)\, dd_H.$$

Service Providers' Expected Payoff under Participation: Suppose a low-quality provider participates in the voucher program. In that case, it will accept a voucher only if it still has available slots after serving clients in the private market, which occurs when $d_L < K_L$. Further, it receives reimbursement from the government for each beneficiary it serves in this program, which we denote by s_L^v. It is important to note that beneficiaries in the voucher program can choose any voucher-accepting provider. Therefore, even if a voucher-accepting provider has an available slot, it may not all be filled by beneficiaries. To account for this, we introduce the fill-out rate ρ_L, which represents the chance that a slot at a low-quality provider can be filled by beneficiaries. Consequently, the low-quality provider's expected reimbursement is calculated as the expected number of slots filled by beneficiaries multiplied by the reimbursement rate. Furthermore, in line with practical considerations, each voucher-accepting provider incurs a fixed cost associated with managing the voucher program. This cost encompasses expenses related to accreditation, paperwork management, or hiring additional employees responsible for reimbursement-related administrative tasks (Schneider et al. 2017). Accordingly, we denoted the cost of managing the offered subsidy voucher program by c_v. Taking all these factors into account, the expected payoff of a low-quality provider if it joins the voucher program, denoted as $\Pi_L^{v1}(\cdot)$, can be expressed as follows:

$$\Pi_L^{v1}(\cdot) = \Pi_L^{v0}(\cdot) + \int_{\underline{D}_L}^{K_L} s_L^v \left(K_L - d_L\right) \rho_L g(d_L)\, dd_L - c_v.$$

Similarly, we normalize the cost of managing each client, including those in the private market and voucher program, to zero.

Denote the expected payoff of the high-quality provider if it participates in the voucher program as $\Pi_H^{v1}(\cdot)$. Following similar steps, we have:

$$\Pi_H^{v1}(\cdot) = \Pi_H^{v0}(\cdot) + \int_{\delta\underline{D}_L}^{K_H} s_H^v \left(K_H - d_H\right) \rho_H g(d_H)\, dd_H - c_v.$$

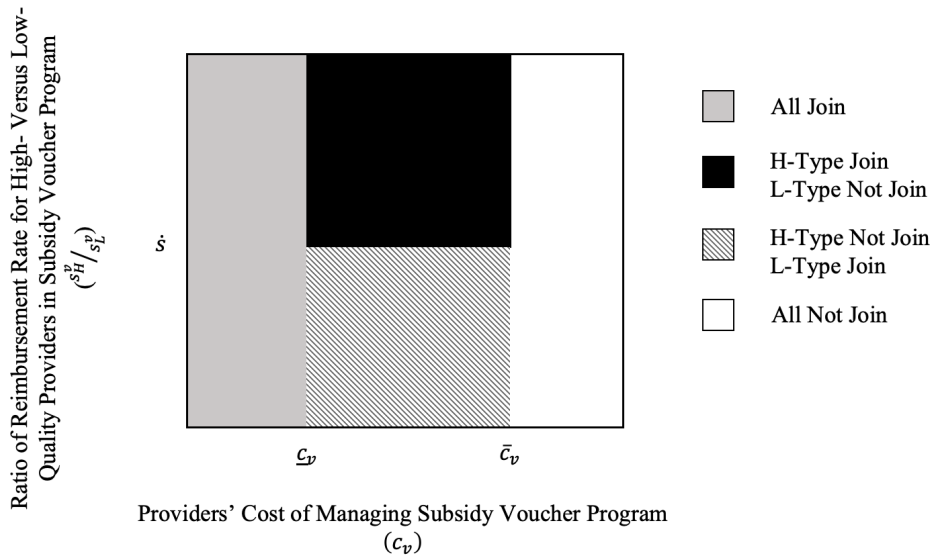
In the equation above, ρ_H represents the fill-out rate at a high-quality provider. In practice, the reimbursement rate for high-quality providers is higher than that for low-quality providers. Accordingly, we denote the reimbursement rate for a high-quality provider for each beneficiary it serves by s_H^v. Further, it is worth noting that since service providers adhere to governments' standardized guidelines when managing subsidy voucher programs, we reasonably assume that both low-quality and high-quality providers have the same cost of managing the voucher program.

Service Providers' Participation Decisions: Each low-quality provider aims to maximize its expected payoff. By comparing its expected payoff if it does not participate versus if it participates in the voucher program, i.e., $\Pi_L^{v0}(\cdot)$ versus $\Pi_L^{v1}(\cdot)$, a low-quality provider decides whether it should participate in the offered voucher program or not.

Similarly, by comparing $\Pi_H^{v0}(\cdot)$ versus $\Pi_H^{v1}(\cdot)$, a high-quality provider makes its participation decision.

5.1.2 Results

We solve the game theoretical model outlined in Section 5.1.1 to obtain low- and high-quality service providers' optimal participation decisions. We present our



Note: L-Type and H-Type represent low-quality and high-quality providers, respectively.

Figure 5.2. Impact of c_v and $s^v_{\{H,L\}}$ on Service Providers' Participation Decisions in Subsidy Voucher Program

results on the impact of the cost of managing the subsidy voucher program and reimbursement rate in the voucher program in Proposition 5.1 below. Service providers' participation decisions are also depicted in Figure 5.2.

Proposition 5.1 *Denote thresholds* $\underline{c}_v \doteq \min\left\{\frac{s_H^v \rho_H (K_H - \delta \underline{D}_L)^2}{2\delta\left(\bar{D}_L - \underline{D}_L\right)}, \frac{s_L^v \rho_L (K_L - \underline{D}_L)^2}{2\delta\left(\bar{D}_L - \underline{D}_L\right)}\right\}$, $\bar{c}_v \doteq \max\left\{\frac{s_H^v \rho_H (K_H - \delta \underline{D}_L)^2}{2\delta\left(\bar{D}_L - \underline{D}_L\right)}, \frac{s_L^v \rho_L (K_L - \underline{D}_L)^2}{2\delta\left(\bar{D}_L - \underline{D}_L\right)}\right\}$, *and* $\dot{s} \doteq \delta\left(\frac{\rho_L}{\rho_H}\right)\left(\frac{K_L - \underline{D}_L}{K_H - \delta \underline{D}_L}\right)^2$. *Then,*

(i) When $c_v \leq \underline{c}_v$*: All of the low- and high-quality service providers participate in the subsidy voucher program;*

(ii) When $\underline{c}_v < c_v < \bar{c}_v$*: All of the low-quality providers and none of the high-quality providers participate in the subsidy voucher program if* $s_H^v/s_L^v < \dot{s}$*, and none of the low-quality providers and all of the high-quality providers participate in the subsidy voucher program otherwise;*

(iii) When $c_v \geq \bar{c}_v$*: None of the low- and high-quality service providers participate in the subsidy voucher program.*

Proposition 5.1 states how the cost of managing the voucher program and the reimbursement rate for voucher-accepting providers can impact service providers' participation decisions. Specifically, if the cost of managing the voucher program is sufficiently low (high), both high-quality and low-quality providers decide to participate (not participate) in the subsidy voucher program. This can be explained as follows. In the subsidy voucher program, providers prioritize the private market and accept a voucher only when they have available capacity, which implies that service providers may be able to mitigate the risk of unmet demand by participating in the voucher program. Suppose the cost of managing the voucher program is low. In that case, service providers may choose to participate, even if the reimbursement rate is low, as they can still earn a relatively low expected payoff compared to earning nothing at all (if they choose not to participate). Suppose the cost of managing the voucher program is moderate (i.e., $\underline{c}_v < c_v < \bar{c}_v$), service providers' participation decisions become more nuanced, which also depends on the reimbursement rate for the service providers. Specifically, if the reimbursement rate for high-quality (low-quality) providers is relatively high, then high-quality (low-quality) choose to participate in the voucher program. Otherwise, they opt not to participate. This decision-making process reflects service providers' consideration of balancing the cost and benefit of participating in the program.

To examine how non-program-related factors impact service providers' participation decisions, we focus on provider-related factors, K_H and K_L. We find that, as the capacity of high- and low-quality providers increase (i.e., K_H and K_L), $\underline{c}_v$ and $\bar{c}_v$ increase. This implies that service providers are more likely to choose participation. The rationale behind this relationship is that with a larger capacity, service providers' need to use participation in the voucher program to hedge against uncertainty in the private market increases. Consequently, service providers become more willing to participate in the voucher program.

Overall, the analysis that we conduct in this section sheds light on the impact of program-related factors (e.g., the reimbursement rate) and non-program-related factors (e.g., the capacity of service providers) on service providers' participation decisions in a subsidy voucher program. Our results suggest that government can use the cost of managing the subsidy voucher program and the reimbursement rate as joint levers to influence the participation of high- and low-quality providers. Moreover, the government should also take into account the impact of non-program-related factors on providers' participation decisions.

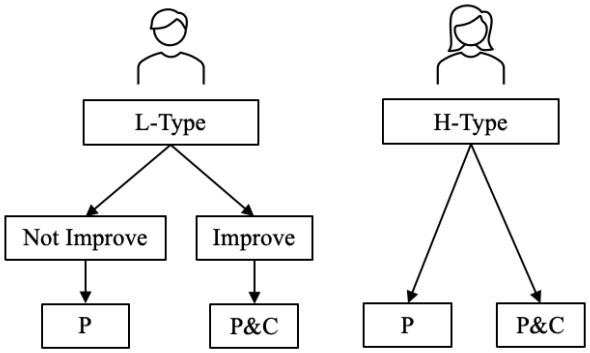
5.2 Analysis of Contracted Slot Programs

So far, our analysis has focused on subsidy voucher programs. Recall that governments also implement other subsidy welfare programs known as contracted slot programs. These programs are designed to ensure beneficiaries access to only high-quality service providers through government contracting with fewer service providers who reserve a fixed number of subsidized slots only for the beneficiaries in the contracted slot programs. Consider the same setting of low- and high-quality service providers outlined at the beginning of Section 5.1. In this section, we model low- and high-quality service providers' decisions on participating in a contracted slot program in Section 5.2.1 and present related results in Section 5.2.2.

5.2.1 Modeling of Service Providers' Participation Decisions

Consider that the government offers a contracted slot program. If a service provider opts not to participate in this contracted slot program, it only serves clients from the private market. Conversely, if a service provider chooses to participate in this contracted slot program, it serves clients from both the private market and the contracted slot program. Hence, to decide whether it should participate in a contracted slot program, the service provider compares its expected payoff if it does not

participate versus if it participates in the contracted slot program. Figure 5.3 illustrates low- and high-quality service providers' participation decisions in a contracted slot program. Next, we describe the key elements of the model in detail.



Note: The nodes L-Type and H-Type denote low- and high-quality service providers, respectively. The node P denotes a service provider not participating in the contracted slot program, thus, only serving clients from the private market. The node P&C denotes a service provider participating in the contracted slot program, thus, serving clients from both the private market and this contracted slot program.

Figure 5.3. Service Providers' Participation Decisions in a Contracted Slot Program

Private Market Demand Distribution for Service Providers: Recall that the government has quality requirements on service providers participating in the contracted slot programs. As a result, low-quality providers are not initially eligible to participate in the offered contracted slot program. In order to become eligible, they must enhance their quality levels to match those of high-quality providers. To align with observations from practice, we consider low-quality providers heterogeneous in terms of the costs of improving their quality levels. Such a variation arises from various external factors such as the availability of external support from umbrella agencies or the willingness of their employees to participate in quality improvement initiatives (Grunewald and Stepick 2022). We denote the cost of improving quality levels as $c_{improve}$ and assume it follows a uniform distribution, specifically $c_{improve} \sim U[0, C]$. Due to the difference in cost improvement, not all low-quality providers may be willing to participate in the contracted slot program. Let $f \in [0, 1]$ represent the proportion of low-quality providers who choose to improve their quality levels and

participate in the contracted slot program. We refer to these providers as low-to-high-quality providers. Consequently, the remaining $1 - f$ proportion of low-quality providers decide not to invest in improving their quality levels and continue only to serve private-pay clients.

After improving their quality, the low-to-high-quality providers in the contracted slot program receive the same level of demand as the high-quality providers. However, in contrast to the subsidy voucher program, wherein the mix of low- and high-quality providers in the market is unchanged, the presence of low-to-high-quality providers in the contracted slot program alters the mix of low- and high-quality providers. Specifically, with f proportion of low-quality providers improving their quality levels and participating in the contracted slot program, there are relatively fewer low-quality providers and more high-quality providers in the private market. As a result, the demand for each type of provider in the private market within the contracted slot program needs to be adjusted accordingly. We describe as follows.

For low-quality providers, with fewer low-quality providers in the market, each has less competition. Thus, the demand for each low-quality provider in the private market is higher in the contracted slot program than in the subsidy voucher program. Denote d_L^c by the demand for each low-quality provider in the private market in the contracted slot program. We have $d_L^c \sim U\left[\underline{D}_L + d_{lo}f, \bar{D}_L + d_{lo}f\right]$. The parameter $d_{lo} > 0$ appropriately adjusts the increased demand for each low-quality provider, with f proportion of low-quality providers improving their quality levels. Whereas, for high-quality providers, with more high-quality providers in the market, demand for each high-quality provider is diluted. Denote d_H^c by the demand for each high-quality provider in the private market in the contracted slot program. Accordingly, we have $d_H^c \sim U\left[\delta\underline{D}_L - d_{ho}f, \delta\bar{D}_L - d_{ho}f\right]$. Similarly, we introduce parameter $d_{ho} > 0$ to appropriately adjust the decreased demand for each high-quality provider, with f proportion of low-quality providers improving their quality levels.

Service Providers' Expected Payoff under No Participation: Consider f proportion of low-quality providers to improve their quality levels. Using the revised demand distributions for each low-quality provider as described above and by combining its expected payoff when its demand in the private market is no more than its capacity with its expected payoff when its demand in the private market is more than its capacity, we have a low-quality provider's expected payoff if it does not participate in the contracted slot program, denoted by $\Pi_L^{c0}(\cdot)$, as follows:

$$\Pi_L^{c0}(\cdot) = \int_{\underline{D}_L + d_{lo}f}^{K_L} p_L d_L^c g(d_L^c)\, dd_L^c + \int_{K_L}^{\bar{D}_L + d_{lo}f} p_L K_L g(d_L^c)\, dd_L^c.$$

Similarly, denote a high-quality provider's expected payoff if it does not participate in the contracted slot program by $\Pi_H^{c0}(\cdot)$. It can be written as follows:

$$\Pi_H^{c0}(\cdot) = \int_{\delta\underline{D}_L - d_{ho}f}^{K_H} p_H d_H^c g(d_H^c)\, dd_H^c + \int_{K_H}^{\delta\bar{D}_L - d_{ho}f} p_H K_H g(d_H^c)\, dd_H^c.$$

Service Providers' Expected Payoff under Participation: After investing c_{improv} to improve its quality level, a low-quality provider becomes a high-quality provider. This low-to-high-quality provider can enter into a contract with the government and serve beneficiaries in the contracted slot program. As part of this arrangement, the provider must reserve a certain number of slots only for beneficiaries in the contracted slot program. We denote the number of slots reserved by the low-to-high-quality provider as x_L^c. Due to the commitment to the contracted slot program, the low-to-high-quality provider serves clients from the private market using its remaining capacity (i.e., when $K_L - x_L^c > 0$). The provider receives reimbursement denoted as s_H^c for each beneficiary it serves. In addition to serving beneficiaries in the contracted slot program, the low-to-high-quality provider also serves clients from the private market using its remaining capacity, which occurs when $K_L - x_L^c > 0$. Participating in the contracted slot program, service providers incur a fixed cost of managing the

program, denoted as c_c. It is important to note that the cost of managing a contracted slot program is typically higher compared to managing a voucher program (Schneider et al. 2021). This difference arises from the fact that under voucher programs, the government often contracts with nonprofits to assist local voucher-accepting providers in managing administrative tasks related to the vouchers, thus alleviating a significant amount of effort and cost for the voucher-accepting providers. In contrast, contracted slot providers manage the program themselves, leading to higher associated costs.

Combining all those details, a low-to-high-quality provider's expected payoff if it participates in the contracted slot program, denoted by $\Pi_L^{c1}(\cdot)$, can be written as:

$$\begin{aligned}\Pi_L^{c1}(\cdot) =& s_H^c x_L^c + \int_{\delta \underline{D}_L - d_{ho} f}^{K_L - x_L^c} p_H d_H^c g(d_H^c)\, dd_H^c + \int_{K_L - x_L^c}^{\delta \bar{D}_L - d_{ho} f} p_H \left(K_H - x_L^c\right) g(d_H^c)\, dd_H^c \\ & - c_c - c_{improve}.\end{aligned}$$

The optimal number of slots reserved by this low-to-high-quality provider, x_L^{c*}, can be obtained by solving the following optimization problem:

$$x_L^{c*} \in \underset{\left\{x_L^{cl}\right\}}{\arg\max} \left\{ \Pi_L^{c1}\left(x_L^{cl}, f\right) : x_L^{cl} \geq 0, x_L^{cl} \leq K_L \right\}.$$

Since the high-quality provider already meets quality requirements for participation in the contracted slot program, it does not need to invest in improving its quality level. Therefore, its cost of improving quality is zero, i.e., $c_{improv} = 0$. Denote x_H^c the number of slots the high-quality provider reserves for the contracted slot program. Similar to the low-to-high-quality provider, the expected payoff for a high-quality provider participating in the contracted slot program, denoted as $\Pi_H^{c1}(\cdot)$, can be expressed as follows:

$$\Pi_H^{c1}(\cdot) = s_H^c x_H^c + \int_{\delta \underline{D}_L - d_{ho} f}^{K_H - x_H^c} p_H d_H^c g(d_H^c)\, dd_H^c + \int_{K_H - x_H^c}^{\delta \bar{D}_L - d_{ho} f} p_H \left(K_H - x_H^c\right) g(d_H^c)\, dd_H^c - c_c.$$

The optimal number of slots reserved by this high-quality provider, x_H^{c*}, can be obtained by solving the following optimization problem:

$$x_H^{c*} \in \arg\max_{\left\{x_H^{cl}\right\}} \left\{ \Pi_H^{c1}(\cdot) : x_H^{cl} \geq 0, x_H^{cl} \leq K_H \right\}.$$

Service Providers' Participation Decisions: Each provider aims to maximize its expected payoff. For each low-quality provider, it uses $x_L^{c*}(f)$ to compare its expected payoff if it does not participate versus if it participates in the contracted slot program, i.e., $\Pi_L^{c0}(\cdot)$ versus $\Pi_L^{c1}(\cdot)$, to decide whether it should participate or not. Similarly, for each high-quality provider, it uses $x_H^{c*}(f)$ to compare $\Pi_H^{c0}(\cdot)$ with $\Pi_H^{c1}(\cdot)$ to make participation decision. We denote by f^* the proportion of low-quality providers that improve their quality levels in equilibrium, which is an outcome of the aforementioned simultaneous game (in terms of participation and capacity reservation decisions) among the service providers.

5.2.2 Results

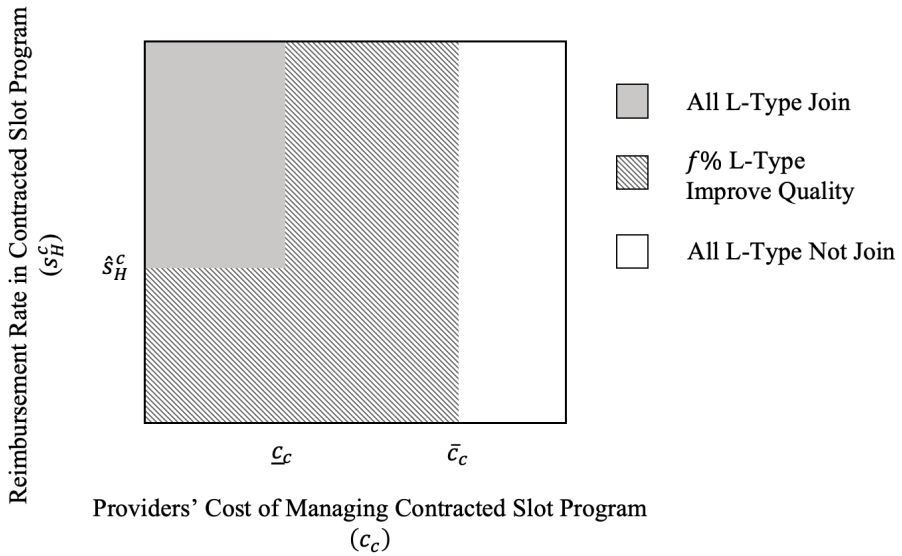
We solve the game theoretical problem outlined in Section 5.2.1 to obtain low- and high-quality service providers' optimal participation decisions and present in Lemma 5.1 and Proposition 5.2 below, respectively. To match with the observation from practice and for analytical tractability, we consider the reimbursement rate for providers in the contracted slot program relatively high (i.e., $s_H^c > \underline{s}_H^c$ as characterized in Lemma C.4.2 in Appendix C.4.). Our numerical analysis shows that the main insights still hold under a wide range of s_H^c.

Lemma 5.1 *Denote a threshold $\hat{c}_c$. High-quality service providers participate in the contracted slot program if and only if $c_c < \hat{c}_c$, where existence of $\hat{c}_c$ is defined in the proof.*

Lemma 5.1 above shows that the high-quality service providers' participation decisions in the contracted slot program depend on the cost of managing the contracted slot program–high-quality providers participate only when such a management cost is not too high. On the other hand, since the low-quality providers might invest in quality improvement to participate in the contracted slot program, their participation decisions are more nuanced. Proposition 5.2 and Figure 5.4 show how the cost of managing the contracted slot program and the reimbursement rate in the contracted slot program jointly impact service providers' participation decisions.

Proposition 5.2 *Denote thresholds $\underline{c}_c$, $\bar{c}_c$, and $\hat{s}_H^c$ as defined in the proof. Then,*
(i) When $c_c \leq \underline{c}_c$: All of the low-quality service providers participate in the contracted slot program if $s_H^c > \hat{s}_H^c$; $f \in (0,1)$ proportion of low-quality service providers participate in the contracted slot program otherwise.
(ii) When $\underline{c}_c < c_c < \bar{c}_c$: $f \in (0,1)$ proportion of low-quality service providers participate in the contracted slot program.
(iii) When $c_c \geq \bar{c}_c$: None of the low-quality service providers participate in the contracted slot program.

Proposition 5.2 shows that a certain proportion of low-quality providers are willing to enhance their quality levels and participate in the contracted slot program when the cost of managing the contracted slot program is not too high (i.e., $c_c < \bar{c}_c$). In the case where c_c is relatively low (i.e., $c_c < \underline{c}_c$), all low-quality providers would improve their quality levels and participate in the program if the reimbursement rate is sufficiently high ($s_H^c > \hat{s}_H^c$). However, as the cost of managing the program is sufficiently high (say, $c_c \geq \bar{c}_c$), none of the low-quality providers choose to participate, regardless of the reimbursement rate. We also examine how non-program-related factors impact low-quality service providers' participation decisions. To do so, we focus on market-related factors. Our analysis shows that when there is greater demand for high-quality providers in the private market (i.e., δ increases), $\underline{c}_c$ and $\bar{c}_c$ increase, which implies that

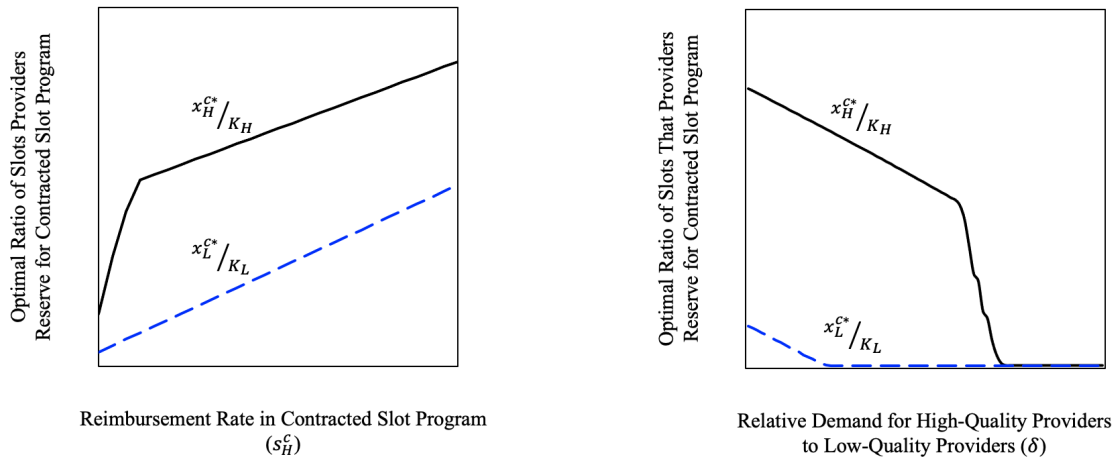


Note: L-Type and H-Type represent low-quality and high-quality providers, respectively.

Figure 5.4. Impact of c_c and s_H^c on Service Providers' Participation Decisions in Contracted Slot Program

low-quality providers are more inclined to improve their quality levels and participate in the contracted slot program. This is because, after improving their quality level, low-quality service providers can achieve a higher expected payoff from serving more clients in need of high-quality service as well as beneficiaries in the contracted slot program.

Combining Lemma 5.1 and Proposition 5.2, we conclude that the cost of managing the contracted slot program has a large impact on both high- and low-quality providers' participation decisions in the contracted slot program. In practice, one of the main challenges that providers face is that they have a much higher cost of managing a contracted slot program as compared with that of a subsidy voucher program. Providers participating in the contracted slot program are responsible for managing the waitlist, contacting eligible individuals and families to inquire about their interest in the available slot, and handling administrative tasks related to reimbursements, among other responsibilities (Schneider et al. 2021). Therefore, to encourage more providers to participate in the contracted slot program, aside from offering a rela-



(a) Related to Program-Related Factor s_H^c (b) Related to Market-Related Factor δ

Note: Here, we consider $K_H > K_L$. The scenario of $K_H \leq K_L$ follow similar pattern.

Figure 5.5. Sensitivity Analysis of the Optimal Ratio of Slots Service Providers Reserve for Contracted Slot Program with Respect to Program- and Market-Related Factors

tively high reimbursement rate, governments can also use the participation cost as an additional lever to influence providers' participation decisions. For example, governments can conduct training sessions to help providers streamline the administrative process, offer providers technical support to increase their efficiency, and even rely on the outside party to help providers alleviate the cost of managing the contracted slot program.

In the contracted slot program, aside from deciding whether it should participate in the program or not, each service provider also needs to decide how many slots it should reserve for the program if it participates. Therefore, we examine the impact of program-related factors and market-related factors on the optimal allocation of slots by high-quality providers and low-to-high-quality providers in the contracted slot program (i.e., x_H^{c*} and x_L^{c*}, respectively). As shown in Figure 5.5(a), as the reimbursement rate for the contracted slot program (s_H^c) increases, both x_H^{c*} and x_L^{c*} increase. This is because, unlike the uncertain demand in the private market, reimbursements that service providers receive from the government in the contracted

slot program are a relatively reliable and stable source of revenue. Therefore, when s_H^c is relatively high, the service providers are more willing to reserve more slots for the program to maximize their expected payoff. On the other side, despite the uncertainty of demand in the private market, Figure 5.5(b) reveals that when the demand for high-quality providers in the private market increases, both high-quality providers and low-to-high-quality providers reserve fewer slots from their capacities to the contracted slot program.

In summary, the results in this section suggest that the government can use program-related factors (e.g., reimbursement rate and cost of managing program) to encourage more high-quality providers to participate in the contracted slot programs as well as incentivize more low-quality providers to improve their quality levels and participate in the programs. Further, these program-related factors, along with non-program-related factors (e.g., demand for high-quality providers in the private market), can influence the degree of participation of service providers.

5.3 Practical Illustration Using a Case Study: Comparison of Subsidy Voucher Programs and Contracted Slot Programs

In the preceding two sections of this chapter, we examined service providers' participation decisions in a subsidy voucher program and a contracted slot program. To help the government agency evaluate these two types of subsidy welfare programs, we compare the resulting societal outcomes under these two programs using a case study in this section. First, we define the societal outcomes generated for the beneficiaries under each program.

Consider there are N_L number of low-quality providers and N_H number of high-quality providers in the private market. Each low-quality provider has a quality level of Q_L. Each high-quality provider has a quality level of Q_H. We evaluate each sub-

sidy welfare program based on the societal outcomes generated for the beneficiaries of the programs. For this, we define the resulting societal outcomes under a subsidy program by combining the levels of quantity and quality of services provided by subsidy-accepting providers to the beneficiaries in the program. We calculate the resulting societal outcomes in each type of subsidy welfare program as follows. For the subsidy voucher program, we multiply the number of each type of service provider (i.e., N_L or N_H), the quality level of each type of service provider (i.e., Q_L or Q_H), and the expected number of beneficiaries served by each type of service provider (i.e., $\int_{\underline{D}_L}^{K_L} (K_L - d_L)\, \rho_L g(d_L)\, dd_L$ or $\int_{\delta\underline{D}_L}^{K_H} (K_H - d_H)\, \rho_H g(d_H)\, dd_H$). For the contracted slot program, we multiply the number of low-to-high-quality providers or the number of high-quality providers (i.e., $f^* \times N_L$ or N_H), the quality level of high-quality providers (i.e., Q_H), and the number of slots that each low-to-high-quality or high-quality provider reserves for the program (i.e., x_L^{c*} or x_H^{c*})[1]. Denote societal outcomes generated under the subsidy voucher program and contracted slot program by SO_v and SO_c, respectively. Accordingly, we have:

$$SO_v = N_L Q_L \int_{\underline{D}_L}^{K_L} (K_L - d_L)\, \rho_L g(d_L)\, dd_L + N_H Q_H \int_{\delta\underline{D}_L}^{K_H} (K_H - d_H)\, \rho_H g(d_H)\, dd_H,$$

$$SO_c = f^* N_L Q_H x_L^{c*} + N_H Q_H x_H^{c*}.$$

5.3.1 Case Study Setup

We present a calibrated numerical study using child care context in Boston, Massachusetts, to compare societal outcomes in the subsidy voucher programs versus the contracted slot programs. In Boston, the Massachusetts Department of Early Education and Care (MEEC) offers IE individuals and families access to affordable child care through a subsidy voucher program and a contracted slot program. Based on

[1]Since our model considers that all low-to-high-quality providers are identical after they have improved their service quality, thus, we have that each of them reserves the same number of slots for the contracted slot program.

reports by child care literature, we categorize service providers in the Boston area into two types of service providers: low-quality providers and high-quality providers. Further, we use data collected from MEEC, the child care literature, and our interviews with managers in several daycare providers in Boston to estimate the values of the model parameters. The values of the parameters considered and the sources used for estimation are listed in Table 5.1. Appendix C.2 describes details of the steps to estimate the model parameters.

Table 5.1. Case Study: Estimated Values of Parameters

Parameter Value	**Sources/Methods (see details in Appendix C.2)**
$N_L = 725$, $N_H = 97$	MEEC, child care literature, interviews
$Q_L = 1$, $Q_H = 1.5$	MEEC, child care literature, interviews
$K_L = 60$, $K_H = 100$	MEEC, child care literature, interviews
$p_L = \$22,880$, $p_H = \$31,200$	Child care literature
$s_L^v = \$22,880$, $s_H^v = \$24,440$	MEEC
$s_H^c = \$29,380$	MEEC
$\rho_L = 0.35$, $\rho_H = 0.55$	Child care literature
$c_v = \$50,000$	Child care literature, child care accreditation associations
$\underline{D}_L = 40$, $\bar{D}_L = 70$	Child care literature, expert elicitation approach
$\delta = 1.6$	Child care literature, expert elicitation approach
$d_{lo} = 15.2$, $d_{ho} = 15.2$	Child care literature, expert elicitation approach

Note: We allow values of $c_{improve}$ to be uniformly distributed between $\$800,000$ to $\$1,000,000$, and c_c to be uniformly distributed $\$50,000$ to $\$200,000$.

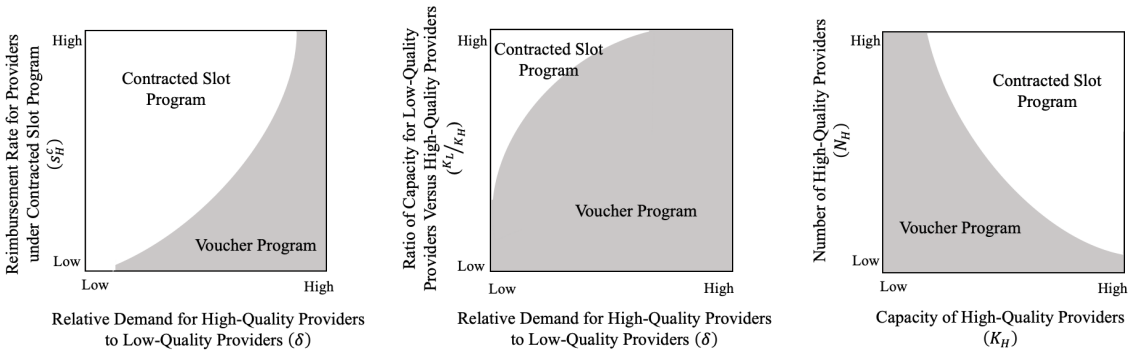
5.3.2 Numerical Findings

We solve the game-theoretical model using the estimated parameters to compute societal outcomes under the subsidy voucher and contracted slot programs–i.e., SO_v and SO_c as described above. Note that in our case study, we consider $c_{improve}$ and c_c to be uniformly distributed. For robustness, we run simulations with 300 iterations, where values of $c_{improve}$ and c_c parameters are drawn from the considered uniform distributions. Then, we compute the values of SO_v and SO_c by taking the average value

of the resulting societal outcomes under the subsidy voucher program and contracted slot program, respectively, under these 300 iterations.

We find that the societal outcomes under subsidy voucher and contracted slot programs are $SO_v = 2772.00$ and $SO_c = 5818.09$, respectively. These values suggest that the contracted slot program can generate more societal outcomes than the subsidy voucher program in the scenario considered in the case study. This can be explained as follows. On one side, the reimbursement rate for providers in the contracted slot program is relatively high compared to the subsidy voucher program. Hence, for high-quality providers, they are willing to participate and reserve as high as around 40% of their capacity (i.e., $x_H^{c*}/K_H \approx 40\%$) for the contracted slot program; For low-quality provider, similarly, almost 12% of them (i.e., $f^* \approx 12\%$) improve their quality and participate in the contracted slot program. On the other side, although both low- and high-quality providers participate in the subsidy voucher program, the voucher-accepting providers serve beneficiaries only when they have a vacancy after serving clients from the private market. Since the expected demand for child care for both types of providers in the private market is relatively high in Boston (e.g., $\underline{D}_L = 40$ and $\bar{D}_L = 70$ as compared with $K_L = 60$), voucher-accepting providers can only serve a limited number of beneficiaries in the subsidy voucher program, which limits the level of societal outcomes generated under the subsidy voucher program.

From a practical perspective, it is instructive to understand how the conclusion above (i.e., the contracted slot program is a more desirable outcome in our case study setup) changes with respect to a few key contextual factors. For this, we conduct sensitivity analysis by varying the values of a few program-, market-, and service-provider-related factors. These factors include the reimbursement rate for the contracted slot program, relative demand for high-quality providers to low-quality providers, capacity of providers, and number of providers in the market. Figure 5.6 illustrates the findings, which are discussed below. These numerical analyses capture scenarios un-



(a) Related to s_H^c and δ (b) Related to K_L/K_H and δ (c) Related to N_H and K_H

Figure 5.6. Comparison of Subsidy Voucher and Contracted Slot Programs Based on Societal Outcomes

der which the contracted slot program can generate more societal outcomes than the subsidy voucher program and vice versa. However, given a growing interest within the operations of subsidy welfare programs to implement the contracted slot programs, we restrict the following discussion to highlighting those situations when the level of societal outcomes is higher under the contracted slot program.

Impact of program- and market-related factors: We examine how program-related and market-related factors jointly impact the comparison of societal outcomes under the two programs. In particular, we focus on the reimbursement rate for the contracted slot program (s_H^c) and the relative demand for high-quality providers to low-quality providers (δ). As shown in Figure 5.6(a), the contracted slot program generates a relatively higher level of societal outcomes as compared with the subsidy voucher program when s_H^c is relatively high and δ is relatively low. This is because, with a relatively more stable and higher payoff for the reserved slots (as compared with the private market), the contracted slot program can attract more high-quality service providers to participate as well as incentivize more low-quality service providers to improve their quality levels and participate. Further, as explained in Section 5.2.2, when the demand for high-quality providers in the private market is relatively low, both high- and low-to-high-quality service providers reserve more slots for the con-

tracted slot program in order to maximize their expected payoff. As a result, with a greater improvement in the quantity side of service delivered in the contracted slot program, the contracted slot program can generate much higher societal outcomes than the subsidy voucher program.

Impact of market- and provider-related factors: We also investigate the joint impact of market-related and provider-related factors on the comparison of societal outcomes under the two programs. Specifically, we focus on the ratio of capacity for low-quality providers versus high-quality providers (K_L/K_H) and the relative demand for high-quality providers to low-quality providers (δ). Figure 5.6(b) shows that the contracted slot program can generate relatively more societal outcomes than the subsidy voucher program when K_L/K_H is relatively high, and δ is relatively low. We explain it as follows. When low-quality service providers have a relatively large capacity, there is an increased likelihood that they cannot full fill their capacity using clients from the private market. Therefore, to maximize their expected payoff, low-quality providers participate in the subsidy voucher program (as explained in Section 5.1.2) as well as improve quality levels (i.e., become low-to-high-quality providers) and participate in the contracted slot program. Although participating in both programs, low-to-high-quality service providers can further maximize their payoff by reserving more slots for the program. On the other hand, as explained above, high-quality service providers are willing to reserve more slots for the contracted slot program when δ is relatively low. As a result, the extent of improvement in the societal outcomes in the contracted slot program (due to improvement in the quantity side of services delivered to the beneficiaries) exceeds that of the subsidy voucher program.

Impact of provider-related factors: Finally, we examine how the characteristics of service providers impact societal outcomes under the two programs. To do so, we study the joint impact of the capacity of high-quality providers (K_H) and the number of high-quality providers in the market (N_H). The plot in Figure 5.6(c) suggests that

the contracted slot program can generate higher societal outcomes when K_H and N_H are relatively high. It can be explained as follows. As explained above, providers are more willing to participate in both subsidy welfare programs when they have a larger capacity, which allows them to mitigate the risk of the private market not fully filling their capacity. However, high-quality providers can reserve more slots under the contracted slot program as their capacity increases because the reserved slots bring guaranteed reimbursement from the government. Hence, similar to the explanations above, the contracted slot program can generate higher societal outcomes by offering the beneficiaries a higher level of quantity (through more participation) and quality (through quality requirements on the participating providers) of services. Moreover, the improvement of societal outcomes under the contracted slot program is further enhanced in the presence of a larger number of high-quality providers in the private market.

The aforementioned discussion is based on the societal outcomes delivered to the beneficiaries, who are the economically-disadvantaged and vulnerable entities in this context. In what follows, we remark on the relative benefit of the two types of subsidy welfare programs by considering the efficiency of the government's expenditure in the program. To do so, we compare two programs based on societal outcomes per total reimbursement expenditure. We calculate it by dividing societal outcomes in each type of program (as described above) by the total expected amount of reimbursement that government pays to participating service providers in each type of program. Denote societal outcomes per total reimbursement expenditure by SOE_v and SOE_c, respectively. Therefore, we have:

$$SOE_v = \frac{SO_v}{s_L^v N_L \int_{\underline{D}_L}^{K_L} (K_L - d_L)\, \rho_L g(d_L)\, dd_L + s_H^v N_H \int_{\delta \underline{D}_L}^{K_H} (K_H - d_H)\, \rho_H g(d_H)\, dd_H},$$

$$SOE_c = \frac{SO_c}{s_H^c \left(f^* N_L x_L^{c*} + N_H x_H^{c*}\right)}.$$

Using the estimated parameters, we compute societal outcomes per total reimbursement expenditure under the subsidy voucher program and contacted slot program in Boston, Massachusetts, and have $SOE_v = 0.0118$ and $SOE_c = 0.0050$. Therefore, we conclude that implementing a subsidy voucher program is more cost-efficient than a contracted slot program in the considered setup. To check the robustness of this finding, we also perform sensitivity analysis with respect to the same sets of factors as above to compare the societal outcomes per government total reimbursement expenditure under the subsidy voucher program and contracted slot program. Our analysis shows that the subsidy voucher program is more likely to have higher societal outcomes per expenditure than the contracted slot program under a large range of values of related factors. We find that this is more likely to happen when the difference in the quality level between low and high-quality providers is not too large (as shown in Table 5.1, $Q_L = 1$ and $Q_H = 1.5$). Conversely, our analysis shows that in a region with a large disparity in the quality level between high- and low-quality providers, the contracted slot program can generate higher societal outcomes per total reimbursement expenditure than the subsidy voucher program.

CHAPTER 6

CONCLUSIONS AND FUTURE RESEARCH

In this dissertation, we study a series of resource allocation problems that nonprofits, government agencies, and service providers face in subsidy welfare programs. Specifically, we conduct three studies in order to help these participants make better operational decisions that enable the generation of the most effective and/or equitable social outcomes under these programs. In this chapter, we describe conclusions and generalization of our studies in Section 6.1. In Section 6.2, we conclude with directions for future research.

6.1 Conclusions and Generalization of Results

In this section, we summarize three studies in this dissertation (conducted in Chapters 3, 4, and 5) and their managerial insights in Section 6.1.1. We discuss how these studies can be generalized to a broader context in Section 6.1.2.

6.1.1 Conclusions

In Chapter 3, we study optimal funds allocation decisions faced by a nonprofit that is tasked with providing services to its local beneficiaries and service providers in a subsidy voucher program. Using a child care subsidy voucher program as a motivated example, we help a CCR&R (a local nonprofit) decide how to allocate its limited funds between outreach and provider services activities so that beneficiaries in different regions within its service area can have equitable access to child care. We develop an analytical model which includes details of the subsidy voucher offer process and

captures the challenges faced by the CCR&R when allocating funds for these two types of activities. We show that when the marginal return of provider services investment is higher (lower) than the marginal return of outreach investment, the CCR&R should invest more (less) in provider services than in outreach. While, in general, investing more funds in outreach in regions with more IE families is optimal, this may not be the case when regional asymmetries are incorporated into the funds allocation problem. Specifically, a CCR&R should invest more funds in outreach in a region with a relatively lower proportion of IE families in the following situations: (i) when the external considerations (those exogenous to the CCR&R) in that region have a greater impact on the IE family's propensity of acceptance; (ii) when the marginal return of outreach investment in that region is higher, and abundant funds are available; or (iii) when the distress faced by IE families in that region is significantly higher.

We also extend our model and analysis to understand the impact of the infusion of additional funds earmarked for outreach in one region. The related analysis shows that a substantially large amount of such earmarked funds may crowd out a CCR&R's investment in outreach in the non-earmarked region, irrespective of the relative proportion of the IE families in each region. Further, we contextualize these insights within a practical setting using a case study of a service area in Massachusetts. Through this real-world application of our model, we conclude that the proposed investment decisions can improve equity outcomes by approximately 7.0%. Notably, this estimated value of the improvement in equity outcomes is directly related to the SDG Indicator 4.2, which includes reducing inequity in "access to quality early childhood development, care, and pre-primary education" for rural versus urban recipients (SDG 2017).

Most recently, the COVID-19 pandemic has underscored the need to ensure equitable access to affordable and high-quality child care in order to facilitate a safe and robust reopening of the economy. As a result, a CCR&R like this should appropri-

ately allocate its funds to offering provider services, such as cleaning and sanitation and other safety-related programs, and to conduct outreach, especially in rural regions (Johnson-Staub 2020, Williams 2020). We view this chapter as the first step in exploring the operational complexities on the supply-side of child care subsidy voucher programs. At a broader level, this chapter can also help any organization administering and operating subsidy voucher programs in other pro-social settings to allocate its limited funds across different supply-enhancing activities optimally. We provide a detailed discussion in Section 6.1.2.

In Chapter 4, we stand at a higher level by studying optimal funds allocation decisions faced by a government agency (whom we refer to as the funding agency) among several local nonprofits (whom we refer to as the service agencies) in a subsidy voucher program. The funding agency's funds allocation decisions in such a program are complicated by: (i) distinct goals considered by the funding and service agencies–i.e., impact-maximization while caring about equity across all service areas versus impact-maximization in their respective areas; (ii) the intricate relationship between contextual factors due to service focus of assistance–since the service agencies make investments to improve the quantity and quality of services received by beneficiaries at service providers, differences in the investment returns significantly affect social impact generation; and (iii) asymmetry of information–given its closeness to the funding source (such as legislative bodies), the funding agency is better informed on the possibility of future additional funds, if any, becoming available.

We introduce an optimization-based method that incorporates the above complexities and analyzes how the funding agency should allocate funds equitably among its partnering service agencies. We develop a bilevel, one-to-many, and forward-looking optimization model to analyze a practical scenario in which additional funds may be sanctioned by the funding source in the future (during the planned horizon) for a particular service area only. Our analysis reveals how the funding agency can use

funds allocation as a strategic lever to ensure equitable social impact across different areas. In particular, we recommend that the funding agency should allocate more funds toward a service area when: the level of additional funds expected to become available for the other area is higher, the mix of service providers (that are targeted by different activities) in this area is more balanced, or investment toward an activity by the service agency is more likely to yield a better rate of return and this area has fewer providers targeted by such an activity.

We also compare the societal outcomes–i.e., the levels of inequity and total social impact–under the optimization-based equity-ensuring method with those under the optimization-based efficiency-focused method (that has no equity consideration) and a formula-based method (that is based on proportions of contextual factors). Our comparative analysis reveals the following insights. First, the goals of efficiency and equity are in contrast with each other, such that the optimal funds allocation strategy based on the efficiency-focused method (versus the equity-ensuring method) results in a greater inequity, especially in the presence of a large disparity in the size of the pool of beneficiaries across different areas. Second, although the formula-based method could achieve greater total social impact than the equity-ensuring method while not severely sacrificing equity, our proposed equity-ensuring method help reduce inequity while not severely sacrificing the overall expected social impact under a wide range of contextual factors.

Further, we contextualize these insights in a practical setting using a real-world case study, which is based on the child care subsidy voucher program in Massachusetts. Our numerical studies reveal that, compared with the current formula-based method, the proposed optimization-based method can help the funding agency not only eliminate the inequity outcomes but also increase the overall social impact across all service areas by approximately 3%. Finally, we extend our model and analysis in several di-

rections (e.g., when considering regional asymmetries and maximum allowed inequity deviation) to show that our insights remain robust under alternative model settings.

In Chapter 5, we study the government agency's selection decision for subsidy welfare programs between the subsidy voucher programs and the contracted slot programs as well as service providers' participation decisions in each program type. Such a selection decision is complicated by: (i) each type of program creating social impact through different mechanisms–beneficiaries have access to services from a large number of service providers (including a mix of high- and low-quality providers) under the subsidy voucher programs while beneficiaries have access to services from only high-quality providers (even if at a fewer number of service providers) under the contracted slot programs; (ii) the government agency's goal of ensuring both high-quantity and high-quality services in the subsidy welfare program, which are impacted by the service providers' participation decisions (in order to maximize their expected payoff); and (iii) the intricate impact of contextual factors (e.g., reimbursement rate of the programs and demand for service providers' services in the private market) on service providers' participation decisions. We develop a game-theoretical model setup to analyze service providers' participation decisions in each type of subsidy welfare program and how contextual factors impact their decisions. Our analysis suggests that the government agency can use program-related factors as levers to influence service providers' participation decisions. In particular, service providers are more willing to participate in the program when the reimbursement rate is relatively high and the cost of managing the program is relatively low.

As the first study to incorporate service providers' participation decisions into the comparison of two different types of subsidy welfare programs, we compare the resulting societal outcomes for the beneficiaries of these programs through a case study based on child care subsidy programs in Massachusetts. In this case study, we examine how program-, market-, and service-provider-related factors impact the

comparison of societal outcomes under the two programs and identify conditions under which the level of societal outcomes under a contracted slot program outperforms a subsidy voucher program or vice versa. We find that a contracted slot program can generate higher societal outcomes under certain situations. For example, it happens when: (i) there are relatively more high-quality providers in the market, and high-quality providers have a relatively high capacity; (ii) the reimbursement rate for providers in the contracted slot programs is relatively high, and the demand for high-quality providers' services in the private market is relatively low; or (iii) low-quality providers' capacity is relatively high, and the demand for high-quality providers' services in the private market is relatively low. On the other side, our comparison of the two programs based on the societal outcomes per expenditure shows that when there is a relatively small disparity in the quality level between low- and high-quality providers, a subsidy voucher program is more likely to outperform a contracted slot program.

6.1.2 Generalization of Results

As summarized above, in this dissertation, we study three operational problems within subsidy welfare programs that are motivated by our close interactions with experts in the context of child care in the U.S. However, the models we formulate are general, and the generated insights can also be applied to subsidy welfare programs in other contexts. Next, we discuss the generalization of these studies.

In the first study (Chapter 3) of this dissertation, we look at a specific operational problem faced by nonprofits managing child care subsidy voucher programs. However, similar problem contexts exist in other subsidy welfare programs, such as those related to housing, maternity care, and education. Therefore, we outline the similarities between the problem context addressed in this study and other subsidy welfare programs and discuss how the insights from our study can be applied to

these programs. As addressed earlier, there is a supply-side concern in a child care subsidy voucher program–beneficiaries are often unable to accept offered vouchers due to the inadequate number of service providers in different regions and the low quality of service delivery by the voucher-accepting providers. Low acceptance of offered vouchers is also observed in several other subsidy voucher programs. For instance, Wykstra (2014) underscore the low utilization of housing vouchers offered to low-income families in the U.S. Also, Hatt et al. (2010) and Ir et al. (2010) provide empirical evidence on families' inability to accept maternity care vouchers offered to low-income households in Bangladesh and Cambodia, respectively. Also, Jones et al. (2006) discuss barriers faced by recipients in Tanzania to accept the available subsidy vouchers to purchase insecticide-treated bed nets to prevent malaria. Low acceptance in these contexts is due to supply-side concerns similar to child care context. In fact, Ir et al. (2010) recommend investing funds toward conducting outreach ("[to] allow the voucher recipients to select a provider convenient to them") and partner development ("to improve the quality of their services") to increase chances that beneficiaries accept the available vouchers for maternity care, and hence, mitigate the adverse impact on beneficiaries and the society.

Due to a similar problem context, our first study can help nonprofits operating in the aforementioned contexts make informed decisions on how to allocate their limited financial funds between different types of supply-side activities to increase the quantity and quality of service delivered to beneficiaries in their service areas. Note that our findings in the first study show that, when balancing investments in outreach investment across different regions within its area, the local nonprofit managing the child care voucher program under study should take into account regional asymmetries (e.g., differences in transportation infrastructure and regulatory environment, marginal return of outreach investment, and distress experienced by families). Given that region-based socioeconomic and demographic differences play a similar role in

contexts including housing, health care, and maternity care, the insights generated in the first study on the effects of regional asymmetries can also be applied to balancing investments in outreach activities across different regions in these contexts.

In the second (Chapter 4) and third (Chapter 5) studies of this dissertation, although their case studies are based on child care subsidy programs, problem contexts addressed in these two studies share similarities with those in several other subsidy welfare programs. First, consider the second study, wherein we analyze the funds allocation problem in a bilevel hierarchical subsidy voucher program. Similar funding structures can be observed in subsidy voucher programs in other domains. One such example is the rental subsidy voucher programs implemented by state governments in the U.S. to enable beneficiaries to afford apartments from private landlords. The government agencies spend more than $30 billion annually on these rental voucher programs, which assist 2 million households (Center on Budget and Policy Priorities 2009, Department of Housing and Urban Development 2023d). In a typical rental subsidy voucher program, the government (i.e., the funding agency) provides funds to several local Public Housing Agencies (PHAs) (i.e., the service agencies) to serve beneficiaries in their respective service areas (e.g., see Connecticut Department of Housing 2019). Similarly, these PHAs also undertake two types of provider-facing activities: quality improvement and outreach activities (Illinois Housing Development Authority 2017). Quality improvement activities conducted by these PHAs involve investing funds to provide ongoing assistance to enhance program quality (e.g., monitoring housing standards and providing landlords access to education and training sessions) (Illinois Housing Development Authority 2017). On the other hand, outreach activities by these PHAs include investing funds toward community outreach to attract more private landlords to participate in the rental subsidy voucher programs. Furthermore, in these rental programs, governments often use a formula-based allocation method to distribute their limited funds among different PHAs (e.g., see details

in Department of Housing and Urban Development 2023a), and additional funds may also be approved by the legislative body at some instance during the planned horizon (Lynch, 2020; HHS, 2021b). Hence, given these similarities between child care voucher programs and rental voucher programs (both of which offer critical essential services), the findings discussed in our second study can also help government agencies in rental voucher programs to allocate their limited budget among PHAs such that the most social impact can be generated across different service areas equitably.

Next, consider the third study, wherein we consider operational challenges faced by the government agency in selecting between two types of subsidy welfare programs. Similar program selection decisions also arise in several other subsidy welfare programs, for example, rental assistance programs. There are two different rental assistance programs: tenant-based rental assistance (TBRA) programs and project-based rental assistance (PBRA) programs. TBRA programs are similar to subsidy voucher programs under study in this dissertation, as eligible tenants participating in a TBRA program can choose to use offered vouchers at any participating property owners (Department of Housing and Urban Development 2023b). PBRA programs are similar to the contracted slot programs under study in this dissertation, as the governments contract directly with private property owners who guarantee to reserve a certain number of units for beneficiaries in the programs (National Housing Law Project 2023). Similarly, the two programs create social impact through different mechanisms. TBRA programs mainly offer a large amount of housing (including a mix of high- and low-quality housing) to the beneficiaries (Chinchilla et al. 2019). In contrast, PBRA programs ensure beneficiaries have access to only high-quality housing (even if at a limited amount of housing) as they usually locate in higher-quality neighborhoods and provide health service and support networks (Department of Housing and Urban Development 2017, Chinchilla et al. 2019). Similarly, governments also face complex challenges in selecting rental assistance programs in order

"to offer more housing options... [and] quality affordable homes" for their beneficiaries (Department of Housing and Urban Development 2023c). Therefore, the findings in our third study can be generalized to help government agencies understand how contextual factors can impact property owners' decisions on joining TBRA programs and PBRA programs and when TBRA programs can generate more societal outcomes than PBRA programs or vice versa.

6.2 Future Research

In this dissertation, while the operational problems in the subsidy welfare programs we study can be applied to a general setting (as discussed in Section 6.1), there are still several possible extensions for future research. Next, we discuss these extensions for each stakeholder, namely nonprofits, government agencies, and service providers.

First, we outline a few future research directions for the local *nonprofits* managing the subsidy voucher programs as studied in Chapter 3. In this chapter, we consider a local nonprofit has an equity-based objective such that beneficiaries can have equitable access to service across the different regions within the service area. While efficiency considerations are implicitly captured through marginal returns, future research can consider a multi-objective structure that explicitly combines equity and efficiency. For example, a multi-objective problem setup that considers the social benefits and operational costs of various supply-enhancing interventions can help understand the effect of other efficiency-related factors on investment decisions. Also, we consider the mandated FCFS allocation policy. Accordingly, our equity-based measure captures the resulting disparities between the families in different regions in terms of the likelihood of acceptance of an offered voucher. Future research can consider alternative allocation policies, wherein it might also be meaningful to capture disparities within the families in each region by using, say, a combination of mean

waiting time until acceptance and the likelihood of acceptance. Finally, we believe capturing the interplay between demand- and supply-side factors will be valuable to further improving societal outcomes.

Next, we discuss some future research directions for the *government agencies* managing the subsidy voucher programs as studied in Chapter 4. In this chapter, we study the funds allocation decisions of a government agency (whom we refer to as the funding agency) to several local nonprofits (whom we refer to as the service agencies). Consistent with studies that analyze the equitable allocation of resources within socially responsible operations (e.g., Zenios et al. 2000, Orgut et al. 2018), we consider that the funding agency aims to maximize overall social impact while ensuring equitable outcomes across all areas. However, in certain governmental programs (such as Property Assessed Clean Energy programs), the funding agency may partner with service agencies that aim to maximize their profits, which could introduce additional challenges stemming from a combination of revenue- and mission-oriented objectives. Therefore, future research can explore the funding agency's funds allocation decisions under such a multi-objective problem setup. Further, in emergency situations (such as the COVID-19 pandemic), our model can be used to equitably allocate relief funds among different service agencies so that they can assist their local service providers in providing healthy, safe, and uninterrupted services to beneficiaries. However, the setup may need to be modified as the social impact generation function in such emergency situations may exhibit an increased rate of investment return or include beneficiary-specific differences, which can be another interesting direction for future work. Finally, future researchers can also extend our optimization model to capture operational complexities due to unique aspects of such welfare programs in emerging economies (e.g., bureaucratic complexities and corruption).

Last, we provide several future research directions for the *government agencies* that make program selection decisions as well as the *service providers* that make

program participation decisions as studied in Chapter 5. In this chapter, we study how a government agency should select between two types of subsidy welfare programs– the subsidy voucher programs and the contracted slot programs and how service providers should make program participation decisions in each program type. We analyze service providers' participation decisions based on a given set of contextual factors without taking into account decisions made by other participants, for example, nonprofits. However, as studied in Chapters 3 and 4, nonprofits participate in and make allocation decisions in the subsidy voucher programs, which could add another layer of complexities to service providers' participation decisions. Therefore, future research can extend our work and explore the impact of the interplay between the decisions made by nonprofits, service providers, and government agencies. Further, in the subsidy voucher programs, the beneficiaries can use their vouchers at any service provider that suits their preferences. Such beneficiaries' choices are not explicitly captured in this chapter. (Instead, we capture them through parameters, the fill-out rates.) Hence, it would be interesting to incorporate such a choice by beneficiaries and examine its impact on service providers' participation decisions and the government agency's selection decision. Finally, while we identify conditions under which the level of societal outcomes under a contracted slot program outperforms a subsidy voucher program or vice versa, it would be interesting if future research could study how the government agency should design the optimal portfolio of subsidy welfare programs to maximize societal outcomes.

APPENDIX A

APPENDIX FOR CHAPTER 3: IMPROVING OUTCOMES IN CHILD CARE SUBSIDY VOUCHER PROGRAMS UNDER REGIONAL ASYMMETRIES

A.1 Notation

Table A.1. Description of Model Notation

Notation	
F	Funds available with the CCR&R
γ	Proportion of IE families that reside in region 1
α	Elasticity of family's probability of acceptance with respect to investment in provider services (referred to as provider services elasticity)
β_i	Elasticity of family's probability of acceptance with respect to investment in outreach in region i (referred to as outreach elasticity)
$\underline{p}_i$	Baseline acceptance probability in region i
p_{oi}	Impact magnitude of exogenous factors on acceptance probability in region i
ξ_i	Socioeconomic burden of distress faced by IE family residing in region i that is unable to accept the offered voucher
Decision Variables and Functions	
x	Level of investment in provider services
y_i	Level of investment in outreach in region i
$p(x, y_i)$	Probability of acceptance of voucher by an IE family in region i
$MI(x, y_1, y_2)$	Total expected value of the inequity measure

A.2 Optimal Investment Decisions and Outcomes under Max-Min Fairness Objective

Among the various notions of fairness, a proportional fairness objective has a higher promise of acceptability (Mo and Walrand 2000) and it is a relatively *cheaper*

notion, in terms of deviation from the resulting inequity under a fully efficient allocation scheme (Bertsimas et al. 2011). In this section, we analyze how the CCR&R should allocate its funds among various activities under a max-min fairness objective, which is another well-accepted notion of fairness in the literature.

Given that the CCR&R's aim is to minimize the inequity outcomes, the notion of max-min fairness–to maximize the minimum utility among all entities–translates into a *min-max* optimization problem in our context. Specifically, we consider that the CCR&R aims to minimize the highest socioeconomic burden of distress among its regions when the IE families are unable to accept the offered voucher. The CCR&R can obtain its optimal levels of investments in provider services and outreach activities, denoted by $\left(x^M, y_1^M, y_2^M\right)$, by solving the following optimization problem:

$$\min_{\{x,y_1,y_2\}} \quad \max \ \{MI_1\left(x, y_1, y_2\right), MI_2\left(x, y_1, y_2\right)\} \tag{A.1}$$

$$\text{s.t.,} \quad x + y_1 + y_2 \leq F, \tag{A.2}$$

$$x \geq 0, y_1 \geq 0, y_2 \geq 0, \tag{A.3}$$

where $MI_i\left(x, y_1, y_2\right)$ denotes the resulting distress in region i, for $i \in \{1, 2\}$, when the IE families in that region are unable to accept the offered voucher, conditional on the voucher being accepted by an IE family in either region 1 or 2. By following similar steps as described before equation (3.1) in Section 3.1, we have $MI_1 \doteq \left(\frac{\gamma(1-p(x,y_1))}{\gamma p(x,y_1)+(1-\gamma)p(x,y_2)}\right)\xi_1$, and $MI_2 \doteq \left(\frac{(1-\gamma)(1-p(x,y_2))}{\gamma p(x,y_1)+(1-\gamma)p(x,y_2)}\right)\xi_2$.

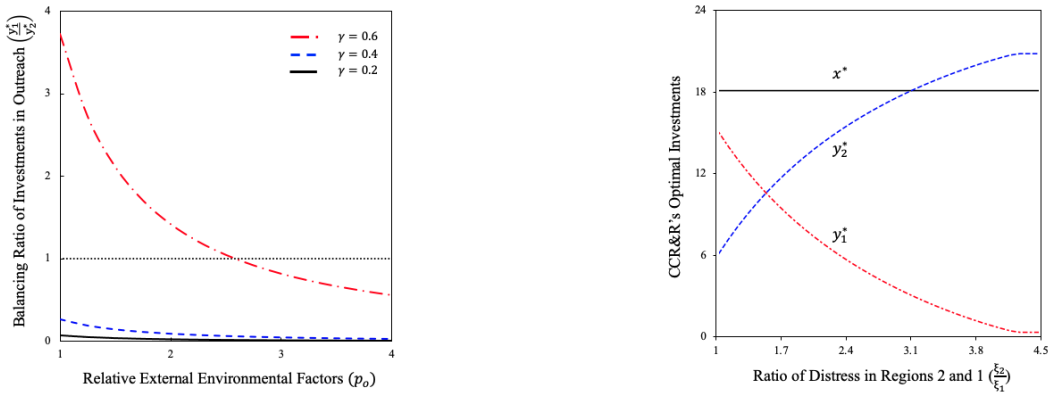
Lemma A.1 *For the optimization problem outlined in equations (A.1)–(A.3), there exists $\bar{\bar{F}} > 0$ such that, $MI_1\left(x^M, y_1^M, y_2^M\right) \neq MI_2\left(x^M, y_1^M, y_2^M\right)$ if $F < \bar{\bar{F}}$, and $MI_1\left(x^M, y_1^M, y_2^M\right) = MI_2\left(x^M, y_1^M, y_2^M\right)$ otherwise.*

Lemma A.1 shows that when the CCR&R has a sufficiently low amount of available resources (F is below a threshold), it is optimal to allocate funds in such a manner that

the resulting distress in one region is reduced to a larger extent than the resulting distress in the other region. However, when the CCR&R is not severely resource constrained, the optimal allocation of funds is such that it results in same levels of distress faced by IE families in both regions. This is because, when F is small, the max-min fairness objective forces the CCR&R to use these limited resource to prioritize reducing the distress in the region that has a higher proportion of IE families or higher socioeconomic burden of distress when IE families are unable to accept the voucher. In contrast, when F is high, the effect of diminishing returns of outreach investments becomes dominant and as a result, the CCR&R cannot further reduce distress in one region without a significant adverse impact on the other region.

In addition, the findings illustrated in Figures A.1(a) and A.1(b) are consistent with the insights generated through results in Propositions 3.1 and 3.4 in Section 3.2, respectively. Based on this preliminary analysis, we find that effects of the region-specific characteristics on the optimal levels of investment are qualitatively similar under both notions of fairness. Next, we compare and contrast the total distress faced by the IE families in the service area when funds are allocated based on proportional and max-min fairness objectives.

We find that, in the presence of regional asymmetries, the measure of inequity (i.e., the total expected distress experienced by the IE families that are unable to accept the voucher) has a lower optimal value under the proportional fairness objective as opposed to the max-min fair objective. This is because, under proportional fairness, the CCR&R allocated funds proportionally between the two regions by appropriately balancing the impact of regional asymmetries. In contrast, under max-min fairness, the CCR&R can prioritize outreach investment in one region, say, with higher proportion of IE families or higher socioeconomic burden of distress, after ensuring that the resulting distress in the other region is only at a modicum level.



(a) $\alpha = 0.2$ and $\beta = 0.3$ (b) $\alpha = 0.6$ and $\beta = 0.7$

Note: Other parameters: $F = 40$ and $\underline{p}_1 = \underline{p}_2 = 0.25$, with $\xi = 1$ in (a), and $\gamma = 0.6$ and $p_o = 1$ in (b).

Figure A.1. The CCR&R's Optimal Investments under Max-Min Fairness Objective

A.3 Additional Details of the Case Study

A.3.1 Generalized Optimization Problem

Consider that the CCR&R's service area comprises of $n \geq 2$ different types of regions. For a given region i, where $i \in \{1, 2, ..., n\}$, we denote the proportion of IE families by γ_i, the outreach elasticity by β_i, the amount of funds invested in outreach activity by y_i, the probability of an IE family accepting the offered voucher by $p(x, y_i)$, and the socioeconomic burden of distress faced by an IE family that is unable to accept the offered voucher by ξ_i, such that $p(x, y_i) = \underline{p}_i + p_{oi} x^{\alpha} y_i^{\beta_i}$ and $\sum_{i=1}^{n} \gamma_i = 1$. Thus, the CCR&R's resource allocation problem can be expressed as follows:

$$\min_{\{x, y_1, y_2, ..., y_n\}} \quad MI(x, y_1, y_2, ..., y_n) = \sum_{i=1}^{n} \left(\left(\frac{\gamma_i (1 - p(x, y_i))}{\sum_{j=1}^{n} \gamma_j p(x, y_j)} \right) \xi_i \right)$$

$$\text{s.t.,} \quad x + \sum_{i=1}^{n} y_i \leq F,$$

$$x \geq 0, y_i \geq 0, \text{for } i \in \{1, 2, ..., n\}.$$

We denote by $(x^*, y_1^*, y_2^*, ..., y_n^*)$ the optimal levels of investments in provider services and outreach activities in each region within the service area.

A.3.2 Estimation of Parameters

We use data from the U.S. Census Bureau, the child care literature, and our interviews with multiple CCR&R officials to estimate values of the model parameters needed to solve the above-mentioned optimization problem for the CCR&R that operates in the service area under study.

Types of Regions and Proportion of IE Families in Each Region. Based on various criteria, such as population density, land use, and distance between settlements, the U.S. Census Bureau categorizes the U.S. landscape into three types, namely urbanized, urban clusters and rural (Ratcliffe et al. 2016). Many federal assistance programs use this information on region type to determine eligibility for participation as well as funding levels for different regions across the U.S. (Hotchkiss and Phelan 2017). In the context of child care subsidy welfare programs, since the CCR&Rs operate under the mandate of the federal and state governments, we utilize the aforementioned information from the bureau to categorize the CCR&R's service area into $n = 3$ types of regions–urbanized (denoted by 1), urban clusters (denoted by 2), and rural (denoted by 3).

Next, the proportion of IE families residing in each of these three regions can be estimated by considering the number of households that satisfy the income criteria required for a family to be eligible for child care subsidy welfare programs in that state. For the Western Massachusetts service area, we combine information on the (i) number of households and median household income in each region (from the U.S. Census Bureau), and (ii) threshold household income for the family to be eligible for child care subsidy welfare programs in Massachusetts (Massachusetts Department of Early Education and Care 2019). Based on these information, we estimate the proportion of IE families residing in each region as: $\gamma_1 = 0.27$, $\gamma_2 = 0.05$, and $\gamma_3 = 0.68$.

Provider Services Elasticity. The parameter α in our model captures the marginal improvement in the propensity of an IE family accepting the offered subsidy voucher due to funds invested by the CCR&R in provider services. Hence, the estimation of the parameter is based on determination of this marginal improvement level through an expert elicitation approach. Based on our extensive interviews with the CCR&R official (at the director-level) for the Western Massachusetts service area, we estimate the provider services elasticity as follows: with a 50% increase in investment of funds toward the service team (that focuses on assisting providers to improve their quality of service delivery), the chance of an IE family accepting the offered voucher will increase from 1 out of 5 to 1 out of 4. Using this information in the ratio of probabilities of acceptance of the offered voucher when the team is made available 1.5 times the current level of funds, we calculate the provider services elasticity as $\alpha = \frac{\ln(5/4)}{\ln(1.5)} = 0.55$.

Outreach Elasticities. The parameter β_i in our model captures the marginal improvement in the propensity of an IE family residing in region i accepting the offered subsidy voucher due to funds invested by the CCR&R in outreach in that region. Thus, the procedure to estimate the values of these parameters is similar to the expert elicitation approach described above for the provider services elasticity.

Considering the rural region in the Western Massachusetts area, the CCR&R official states that by hiring an additional personnel who specializes in organizing and managing outreach related events (as opposed to the current allocation of requiring only 10% of the total working hours of an existing employee), the chance of an IE family accepting the offered voucher will be doubled. Using this information and the hourly wage rate of the requisite personnel ($30.2 per hour) and the existing personnel ($15.44 per hour), we calculate the outreach elasticity in rural region (i.e., region 3) for the Western Massachusetts service area as $\beta_3 \approx \frac{\ln(2)}{\ln(20)} = 0.23$. We next present a procedure for estimating the value of outreach elasticity in each region based

on a benchmark level determined for any one of the regions (in our case study, we determine it for the rural region).

Recall that the CCR&R's outreach activity in a region is aimed at increasing the number of voucher-accepting providers in that region. As part of the related literature that identifies factors that can help explain differences in marginal rates of return of investments in outreach activities in different regions, Bartels (2018) empirically finds that democratic voters are more favorable to the use of government's funds to provide child care subsidies to low-income families. Specifically, on a scale from zero (extremely unfavorable) to 100 (extremely favorable), on average, the democratic voters rate the government's child care subsidy program at 76.5 as opposed to the 39.5 rating by the republican voters. Combining this empirical evidence with the distribution of votes in the 2016 U.S. general election, which we obtain using data from the MIT Election Lab (`https://electionlab.mit.edu/`), we calculate the composite rating in favor of the government's child care subsidy program in each region as follows: (% Democratic votes in the region)$*$(76.5)+(% Republican votes in the region)$*$(39.5). For the Western Massachusetts area, the composite ratings are: 65.44 for the urbanized region, 56.97 for the urban clusters region, and 58.37 for the rural region. We consider that the composite ratings capture the relative willingness of local communities (and child care providers) to make available subsidized child care to the low-income families. Using these rating, we calculate the outreach elasticities in the urbanized region and the urban clusters as follows: $\beta_1 = \left(\frac{65.44}{58.37}\right) * \beta_3 = 0.26$ and $\beta_2 = \left(\frac{56.97}{58.37}\right) * \beta_3 = 0.22$.

External Environmental Factors. To obtain estimates of external environmental factors, we rely on inputs from experts who possess knowledge on these external environmental considerations that influence the IE family's propensity to accept the offered voucher. Research finds that improvements in public transportation in a region can enhance chances that an IE family may accept the offered subsidy voucher

(Chaudry et al. 2011). Therefore, investing funds toward increasing the number of buses and lobbying the government officials to expand the coverage area of public transportation in region i will lead to an increase in probability $p(x, y_i)$ through a larger p_{oi}, though such investments are exogenous to the CCR&R's investment decisions. For fixed amounts of investments in provider services and outreach activities, our interviews with experts and survey of the related reports (e.g, Chaudry et al. 2011 and Isaacs et al. 2015) reveal that an investment of \$2.5 million toward improvements in external environmental factors in the rural region will increase family's probability of acceptance by one point, which implies that $p_{o3} = \frac{1}{2.5\text{x}10^6} = 4 \text{ x } 10^{-7}$. We reasonably assume that the corresponding degree of improvement in acceptance probability will be higher in the two non-rural regions (because of presence of external network of charities that can potentially lower barriers for IE families to accept vouchers). Accordingly, we use $p_{o1} = p_{o2} = 6 \text{ x } 10^{-7}$.

Socioeconomic Burden of Distress. When the IE family remains in distress due to their inability to accept the offered subsidy voucher, costs are incurred not only by the individual family, but also by the society. This could be in the form of having to give up a job or drop out of school, using savings or retirement funds, borrowing money at high-interest rates in order to take care of the child, loss of productivity for the U.S. economy, among others. While it is difficult to completely and accurately quantify the overall distress faced by IE families, child care experts consider this socioeconomic burden to be correlated with the cost of living in the region where the IE family resides (Austermuhle 2015, CCAoA 2018). Therefore, we use the U.S. Census Bureau's data on household income in each region in the Western Massachusetts service area as a proxy for the socioeconomic burden of distress faced by IE families in that region. Dividing each of the three household income values by the household income value in the urbanized region (i.e., normalizing distress of IE families in the urbanized region to 1), we estimate $\xi_1 = 1$, $\xi_2 = 1.41$, and $\xi_3 = 1.87$. As shown in Proposition 3.4 in

Section 3.2, the ratios of distress in regions are sufficient to determine the CCR&R's optimal investments.

Other Parameters. The total budget of the CCR&R under study is \$1.75 million. Recall that the parameter F captures the amount of financial resources available for investments in the two supply-enhancing activities, i.e., provider services and outreach. Upon consultation with experts in the context of child care subsidy welfare programs, we use 15% of this total budget to estimate value of F as \$260,000. Further, based on our interviews with the front-line staff and the director at the CCR&R, we have that the IE family's probability of acceptance when there is no investment of funds in either provider services or outreach is about 20% in the rural region (i.e., $\underline{p}_3 = 0.2$). Based on the existing network of child care providers, these experts posit that the baseline acceptance probability is higher for IE families residing in the non-rural regions (urbanized and urban clusters). Thus, we reasonably assume $\underline{p}_1 = \underline{p}_2 = 0.3$.

A.4 Proofs of Analytical Results

Additional technical lemmas, which help us outline proofs below, are available in Appendix A.5.

Proof of Proposition 3.1. This proof uses Lemmas A.3 and A.5 (presented in Appendix A.5). Using Lemma A.3(i), we have $x = F - y_1 - y_2$. Substituting this in equation (3.2), we obtain $MI(y_1, y_2) \doteq \left(\gamma \underline{p}_1 + (1-\gamma)\underline{p}_2 + \Psi(y_1, y_2)\right)^{-1} - 1$, where $\Psi(y_1, y_2) \doteq \gamma\left(p_{o1}(F - y_1 - y_2)^{\alpha} y_1^{\beta}\right) + (1-\gamma)\left(p_{o2}(F - y_1 - y_2)^{\alpha} y_2^{\beta}\right)$. Further, using Lemma A.3(ii), we have that $x^* > 0$, which implies that $y_1 + y_2 < F$. For $y_1 + y_2 < F$ and $p_o = 1$, by Lemma A.5 it follows that there exist unique $y_1^* > 0$ and $y_2^* > 0$ that minimize $(\Psi(y_1, y_2))^{-1} - 1$. Next, since the constant term $\gamma \underline{p}_1 + (1-\gamma)\underline{p}_2$ is greater than 0 (because $0 < \gamma, \underline{p}_1, \underline{p}_2 < 1$) and $\Psi(y_1, y_2) \geq 0$, we can conclude that the aforementioned y_1^* and y_2^* minimize $MI(y_1, y_2)$. By applying the first-order

conditions on $MI\left(y_1, y_2\right)$ with respect to y_1 and y_2, we obtain expressions of y_1^* and y_2^* as presented in the statement of proposition. Also, $x^* = F - y_1^* - y_2^* = \frac{F\alpha}{\alpha+\beta}$.

Next, considering the optimal efforts derived above, we obtain $y_1^* + y_2^* = \frac{F\beta}{\alpha+\beta}$. Comparing this with x^*, we have $x^* > y_1^* + y_2^*$ if $\alpha > \beta$ and $x^* \leq y_1^* + y_2^*$ otherwise. Similarly, comparing the expression of $\min\left\{y_1^*, y_2^*\right\}$ with x^*, we have $x^* < \min\left\{y_1^*, y_2^*\right\}$ if and only if $\alpha < \hat{\alpha}$. $\square$

Proof of Proposition 3.2: This proof uses Lemmas A.6 and A.8 (presented in Appendix A.5). The optimal outreach investments y_1^* and y_2^* are characterized in Lemma A.6.

(i) Differentiating y_1^* and y_2^* with respect to p_o, respectively, we have $\frac{\partial}{\partial p_o} y_1^*\left(p_o\right) = -\frac{F\beta(p_o\gamma(1-\gamma))^{\frac{1}{1-\beta}}}{p_o(\alpha+\beta)(1-\beta)\left(\gamma^{\frac{1}{1-\beta}}+(p_o(1-\gamma))^{\frac{1}{1-\beta}}\right)^2} < 0$ and $\frac{\partial}{\partial p_o} y_2^*\left(p_o\right) = \frac{F\beta(p_o\gamma(1-\gamma))^{\frac{1}{1-\beta}}}{p_o(\alpha+\beta)(1-\beta)\left(\gamma^{\frac{1}{1-\beta}}+(p_o(1-\gamma))^{\frac{1}{1-\beta}}\right)^2} > 0$ (because $F > 0$, $0 < \alpha < 1$, $0 < \beta < 1$, $0 < \gamma < 1$, and $p_o > 0$).

(ii) Comparing the expressions of y_1^* with y_2^*, we have $y_1^* < y_2^*$ if and only if $p_o > \hat{p}_o$, where $\hat{p}_o \doteq \frac{\gamma}{1-\gamma}$.

(iii) Differentiating y_1^* with respect to β, we obtain $\frac{\partial}{\partial \beta} y_1^*\left(\beta\right) = \frac{FRq(\beta)}{(\alpha+\beta)^2(1-\beta)(1+R)^2}$, $\frac{\partial}{\partial \beta} y_2^*\left(\beta\right) = \frac{FR\underline{q}(\beta)}{(\alpha+\beta)^2(1-\beta)(1+R)^2}$, where $R \doteq \left(\frac{\gamma}{p_o(1-\gamma)}\right)^{\frac{1}{1-\beta}}$, $q(\beta) \doteq \alpha\left(1-\beta\right)\left(1+R\right) + \alpha\beta\ln\left(R\right) + \beta^2\ln\left(R\right)$, and $\underline{q}(\beta) \doteq \alpha\left(1-\beta\right)\left(1+\frac{1}{R}\right) + \alpha\beta\ln\left(\frac{1}{R}\right) + \beta^2\ln\left(\frac{1}{R}\right)$. Since $F > 0$, $0 < \alpha < 1$, $0 < \beta < 1$, $0 < \gamma < 1$, and $p_o > 0$, we have that signs of $\frac{\partial}{\partial \beta} y_1^*\left(\beta\right)$ and $\frac{\partial}{\partial \beta} y_2^*\left(\beta\right)$ depend on the sign of $q(\beta)$ and $\underline{q}(\beta)$, respectively. Next, we consider the following two cases based on the magnitude of p_o.

First, consider $p_o > \hat{p}_o$. We have $R < 1$, which implies that $\ln\left(R\right) < 0$. Using Lemma A.8(i), we have that there exists a unique $\hat{\beta} \in (0,1)$ such that $q(\beta) > 0$ if $\beta < \hat{\beta}$ and $q(\beta) \leq 0$ otherwise. Therefore, we can conclude that $\frac{\partial}{\partial \beta} y_1^*\left(\beta\right) > 0$ if $\beta < \hat{\beta}$ and $\frac{\partial}{\partial \beta} y_1^*\left(\beta\right) \leq 0$ otherwise. Next, using Lemma A.8(i), we have $\underline{q}(\beta) > 0$ for any β, which implies that $\frac{\partial}{\partial \beta} y_2^*\left(\beta\right) > 0$.

Second, consider $p_o \leq \hat{p}_o$. The proof for this case follows similar steps as for the case when $p_o > \hat{p}_o$. Therefore, we omit it for brevity. $\square$

Proof of Proposition 3.3. Using Lemma A.3(i) (presented in Appendix A.5), we have $x = F - y_1 - y_2$. Substituting $x = F - y_1 - y_2$, $p_o = 1$, and $\xi_1 = \xi_2$ in equation (3.2) in Section 3.1, and applying the first-order conditions, we obtain $y_2(y_1) = \left(\frac{(1-\gamma)\beta_2}{\gamma\beta_1}\right)^{\frac{1}{1-\beta_2}} y_1^{\frac{1-\beta_1}{1-\beta_2}}$. After replacing $y_2(y_1)$ in equation (3.2), and using similar steps as in the proof of Proposition 3.1, we have that the resultant objective function $MI(y_1)$ has a unique global minimum (y_1^*). Therefore, equating $\frac{\partial}{\partial y_1} MI(y_1)$ to zero, it follows that y_1^* is the unique solution to the equation stated within the proposition. Next, denote $\bar{R} \doteq \frac{y_1^*}{y_2^*} = \left(\frac{\gamma\beta_1}{(1-\gamma)\beta_2}\right)^{\frac{1}{1-\beta_1}} (y_2^*)^{\frac{\beta_1-\beta_2}{1-\beta_1}}$.

(i) Before analyzing $\frac{\partial}{\partial F}\bar{R}(F)$, we show that $\frac{\partial}{\partial F} y_1^*(F) > 0$ and $\frac{\partial}{\partial F} y_2^*(F) > 0$. Applying the Implicit Function Theorem to the equation that y_1^* solves, we obtain $\frac{\partial}{\partial F} y_1^*(F) = \left(\left(1+\frac{\alpha}{\beta_1}\right) + \left(1+\frac{\alpha}{\beta_2}\right)\left(\frac{(1-\gamma)\beta_2}{\gamma\beta_1}\right)^{\frac{1}{1-\beta_2}}\left(\frac{1-\beta_1}{1-\beta_2}\right) y_1^{\frac{\beta_2-\beta_1}{1-\beta_2}}\right)^{-1} > 0$, where the inequality holds because $0 < \alpha < 1$, $0 < \beta_1 < 1$, $0 < \beta_2 < 1$, and $0 < \gamma < 1$, and $y_1^* > 0$.

Similarly, differentiating y_2^* with respect to F, we obtain $\frac{\partial}{\partial F} y_2^*(F) = \left(\frac{(1-\gamma)\beta_2}{\gamma\beta_1}\right)^{\frac{1}{1-\beta_2}} \left(\frac{1-\beta_1}{1-\beta_2}\right)(y_1^*)^{\frac{\beta_2-\beta_1}{1-\beta_2}} \frac{\partial}{\partial F} y_1^*(F) > 0$, where the inequality holds because $0 < \alpha < 1$, $0 < \beta_1 < 1$, $0 < \beta_2 < 1$, and $0 < \gamma < 1$, $y_1^* > 0$, and $\frac{\partial}{\partial F} y_1^*(F) > 0$. Next, differentiating $\bar{R}$ with respect to F, we have $\frac{\partial}{\partial F}\bar{R}(F) = \left(\frac{\gamma\beta_1}{(1-\gamma)\beta_2}\right)^{\frac{1}{1-\beta_1}} \left(\frac{\beta_1-\beta_2}{1-\beta_1}\right)(y_2^*)^{\frac{\beta_1-\beta_2}{1-\beta_1}-1} \frac{\partial}{\partial F} y_2^*(F)$, from which we can conclude that the sign of $\frac{\partial}{\partial F}\bar{R}(F)$ depends on the sign of $(\beta_1 - \beta_2)$ (because $\frac{\partial}{\partial F} y_2^*(F) > 0$). Therefore, it follows that $\frac{\partial}{\partial F}\bar{R}(F) > 0$ if $\beta_1 > \beta_2$ and $\frac{\partial}{\partial F}\bar{R}(F) \leq 0$ otherwise.

(ii) Following similar steps as in (i) above, we can show that $\frac{\partial}{\partial \alpha} y_1^*(\alpha) < 0$, which helps show that $\frac{\partial}{\partial \alpha} y_2^*(\alpha) < 0$. Next, differentiating $\bar{R}$ with respect to α, we have $\frac{\partial}{\partial \alpha}\bar{R}(\alpha) = \left(\frac{\gamma\beta_1}{(1-\gamma)\beta_2}\right)^{\frac{1}{1-\beta_1}} \left(\frac{\beta_1-\beta_2}{1-\beta_1}\right)(y_2^*)^{\frac{\beta_1-\beta_2}{1-\beta_1}-1} \frac{\partial}{\partial \alpha} y_2^*(\alpha)$, from which we can conclude

that the sign of $\frac{\partial}{\partial\alpha}\bar{R}(\alpha)$ depends on the sign of $(\beta_2 - \beta_1)$ (because $\frac{\partial}{\partial\alpha}y_2^*(\alpha) < 0$). Therefore, it follows that $\frac{\partial}{\partial\alpha}\bar{R}(\alpha) < 0$ if $\beta_1 > \beta_2$ and $\frac{\partial}{\partial\alpha}\bar{R}(\alpha) \geq 0$ otherwise. □

Proof of Proposition 3.4. Denote $\xi \doteq \frac{\xi_2}{\xi_1}$ and $p^o \doteq p_{o1} = p_{o2}$. Before presenting the steps of the proof, we state the bound on F, which is needed to ensure $p\left(x, y_i\right) \leq 1$ for all the feasible values of x and y_i, when $\alpha = \beta = \frac{1}{2}$ and $\xi \neq 1$. We assume that $F \leq \frac{2\left(1-\underline{p}\right)}{p^o}$, because in the extreme cases (i.e., $\xi \to 0$ or $\xi \to \infty$), we have $p\left(x = \frac{F}{2}, y_i = \frac{F}{2}\right) = \underline{p} + \frac{p^o F}{2}$. The proof below uses Lemmas A.3 and A.10 (presented in Appendix A.5).

Using Lemma A.3(i), we have $x = F - y_1 - y_2$. Substituting $\alpha = \beta = \gamma = \frac{1}{2}$, $p^o = 1$, and $x = F - y_1 - y_2$ in equation (3.2) presented in Section 3.1, and applying the first-order conditions with respect to y_1 and y_2, we have $y_1 = \frac{F}{2} - y_2$, which implies that $x^* = \frac{F}{2}$. Using these relationships in equation (3.2), we obtain $MI\left(y_2\right) = \frac{4(1+\xi)(1-\underline{p}) - p^o\sqrt{2F}\left(2\xi\sqrt{y_2} + \sqrt{2F-4y_2}\right)}{8\underline{p} + \sqrt{2F}p^o\left(\sqrt{2F-4y_2} + 2\sqrt{y_2}\right)}$. Differentiating $MI\left(y_2\right)$ with respect to y_2, it can be shown that the extreme points of $MI\left(y_2\right)$, if any, must satisfy the equation $p^o F(1 - \xi)\sqrt{2F} - \left(k_1\sqrt{2F - 4y_2} - 2k_2\sqrt{y_2}\right) = 0$, such that $k_1 \doteq 2\left((\underline{p}+1)\xi - \underline{p} + 1\right) > 0$ and $k_2 \doteq 2\left((1-\underline{p})\xi + 1 + \underline{p}\right) > 0$ (inequalities hold because $0 < \underline{p} < 1$).

Using Lemma A.10, we have that only one of the two extreme points of the above-mentioned equation is a feasible solution. Thus, the extreme point, given by $\frac{F}{2}\left(\frac{k_1^2}{k_1^2+k_2^2}\right) + \sqrt{\frac{F}{2}\left(\frac{k_1^2}{k_1^2+k_2^2}\right) - \frac{(p^o)^2F^3(1-\xi)^2}{2\left(k_1^2+k_2^2\right)}}$, is the unique minimum y_2^* of $MI\left(y_2\right)$. Accordingly, y_1^* can be obtained using $\frac{F}{2} - y_2^*$. Next, differentiating y_1^* and y_2^* with respect to ξ, it follows that $\frac{\partial}{\partial\xi}y_1^*(\xi) < 0$ and $\frac{\partial}{\partial\xi}y_2^*(\xi) > 0$, in which both inequalities hold because $k_1 > k_2 > 0$, $0 < \underline{p} < 1$, $p^o > 0$, $F > 0$, and $\xi > 1$. □

Proof of Proposition 3.5. Let us denote, $\bar{\delta} \doteq \max\left\{\delta \middle| \frac{(F+\delta)\beta\gamma^{\frac{1}{1-\beta}}}{(\alpha+\beta)\left(\gamma^{\frac{1}{1-\beta}} + (1-\gamma)^{\frac{1}{1-\beta}}\right)} \geq \delta\right\}$ $= \frac{\beta\gamma^{\frac{1}{1-\beta}}F}{(\alpha+\beta)(1-\gamma)^{\frac{1}{1-\beta}} + \alpha\gamma^{\frac{1}{1-\beta}}}$. Consider the following two cases based on the magnitude of δ.

(i) Suppose $\delta \leq \bar{\delta}$. Following similar steps as in the proof of Proposition 3.1, we have that the solution of the first-order conditions for the optimization problem is as presented in the statement of proposition. Since $y_1^* \geq \delta$ (by definition of $\bar{\delta}$), this constitutes the optimal solution.

(ii) Suppose $\delta > \bar{\delta}$. In this scenario, the solution obtained in case (i) above no longer constitutes the optimal solution, because $y_1^* = \frac{(F+\delta)\beta\gamma^{\frac{1}{1-\beta}}}{(\alpha+\beta)\left(\gamma^{\frac{1}{1-\beta}}+(1-\gamma)^{\frac{1}{1-\beta}}\right)} < \delta$ (by definition of $\bar{\delta}$), which violates the constraint $y_1 \geq \delta$. Therefore, we re-solve the optimization problem after replacing $y_1 = \delta$. Substituting $y_1 = \delta$ and $x = F - y_2$ in equation (3.2) presented in Section 3.1, the objective function remains convex in y_2. Applying the first-order condition, we obtain that y_2^* is the unique solution to the equation stated in the statement of proposition.

Further, (a) When $\delta \leq \bar{\delta}$, differentiating y_2^*, as characterized in case (i) above, with respect to δ, we obtain $\frac{\partial y_2^*}{\partial \delta} = \frac{\beta(1-\gamma)^{\frac{1}{1-\beta}}}{(\alpha+\beta)\left(\gamma^{\frac{1}{1-\beta}}+(1-\gamma)^{\frac{1}{1-\beta}}\right)} > 0$ and; (b) When $\delta > \bar{\delta}$, applying the Implicit Function Theorem on the equation that y_2^* solves, we obtain $\frac{\partial y_2^*}{\partial \delta} = \frac{-\alpha\gamma\beta\delta^{\beta-1}y_2^*(\delta)}{(1-\gamma)(\alpha+\beta)\left(y_2^*(\delta)\right)^{\beta}+\gamma\alpha(1-\beta)\delta^{\beta}} < 0.$ □

A.5 Proofs of Lemmas Referred in Appendices A.2 And A.4

Proof of Lemma A.1: Suppose $\Omega \geq 0$. Then, the min-max optimization problem outlined in equations (A.1)–(A.3) presented in Appendix A.2 can be re-written as the following minimization problem:

$$\min_{\{x,y_1,y_2,\Omega\}} \quad \Omega \tag{A.4}$$

$$\text{s.t.,} \quad \xi_1\left(\frac{\gamma\left(1-p\left(x,y_1\right)\right)}{\gamma p\left(x,y_1\right)+(1-\gamma)p\left(x,y_2\right)}\right) \leq \Omega, \tag{A.5}$$

$$\xi_2\left(\frac{(1-\gamma)\left(1-p\left(x,y_2\right)\right)}{\gamma p\left(x,y_1\right)+(1-\gamma)p\left(x,y_2\right)}\right) \leq \Omega, \tag{A.6}$$

$$x + y_1 + y_2 \leq F, \tag{A.7}$$

$$x \geq 0, y_1 \geq 0, y_2 \geq 0, \Omega \geq 0. \tag{A.8}$$

Since $MI_1(x, y_1, y_2)$ and $MI_2(x, y_1, y_2)$ are decreasing in each of their arguments, we have that constraint in equation (A.7) binds. Using this, for any $x > 0$, it can be shown that $MI_1(.)$ and $MI_2(.)$ are decreasing in F. Also, $MI_1(F = 0) > 0$ and $MI_2(F = 0) > 0$. Combining these properties, we can conclude that there exists a unique $\bar{\bar{F}} > 0$ such that one of the two constraints (A.5) and (A.6) binds when $F < \bar{\bar{F}}$. In that scenario, we have either $MI_1(x, y_1, y_2) = \Omega > MI_2(x, y_1, y_2)$ or $MI_2(x, y_1, y_2) = \Omega > MI_1(x, y_1, y_2)$. Otherwise, when $F \geq \bar{\bar{F}}$, $MI_1(x, y_1, y_2) = MI_2(x, y_1, y_2) = \Omega$. The result stated in lemma (for the vector of optimal investments) follows from the fact that the optimization problem outlined in equations (A.4)–(A.8) minimizes Ω. □

Lemma A.2 *Consider* $g(s) \doteq (\alpha+\beta)(1+\alpha+\beta)s^2 - 2F\beta(1+\alpha+\beta)s + F^2\beta(1+\beta)$, *where* $0 < \alpha < 1$, $0 < \beta < 1$, *and* $F > 0$. *Then,* $g(s) > 0\ \forall s$.

Proof of Lemma A.2: The discriminant of $g(s)$ is equal to $-4F^2\alpha\beta(1+\alpha+\beta) < 0$, where the inequality holds because $0 < \alpha < 1$, $0 < \beta < 1$, and $F > 0$. This implies that $g(S) = 0$ has no real roots. Since the coefficient of s^2 in $g(S) = (\alpha+\beta)(1+\alpha+\beta) > 0$, we can conclude that $g(s) > 0\ \forall s$. □

Lemma A.3 *For the optimization problem outlined in equations (3.2)-(3.4) in Section 3.1, (i) the constraint in equation (3.3) binds, and (ii)* $x^* > 0$, *and* y_1^* *and* y_2^* *are not zero simultaneously.*

Proof of Lemma A.3: (i) For given y_1 and y_2, the objective function in equation (3.2) strictly decreases in x, and the left-hand side of the constraint (3.3) increases in x. Hence, the constraint (3.3) binds.

(ii) Suppose $x = \epsilon > 0$. Using equation (3.2), for given y_1 and y_2, we have $MI(\epsilon, y_1, y_2) < MI(0, y_1, y_2)$, where $MI(\epsilon, y_1, y_2) = \frac{\xi_1\gamma\left(1-\underline{p}_1-p_{o1}\epsilon^{\alpha}y_1^{\beta_1}\right)+\xi_2(1-\gamma)\left(1-\underline{p}_2-p_{o2}\epsilon^{\alpha}y_2^{\beta_2}\right)}{\gamma\left(\underline{p}_1+p_{o1}\epsilon^{\alpha}y_1^{\beta_1}\right)+(1-\gamma)\left(\underline{p}_2+p_{o2}\epsilon^{\alpha}y_2^{\beta_2}\right)}$ and $MI(0, y_1, y_2) = \frac{\xi_1\gamma\left(1-\underline{p}_1\right)+\xi_2(1-\gamma)\left(1-\underline{p}_2\right)}{\gamma\left(\underline{p}_1\right)+(1-\gamma)\left(\underline{p}_2\right)}$.

Thus, by proof by contradiction, we can conclude that $x^* > 0$. Using similar steps, we can show that y_1^* and y_2^* are not zero simultaneously. □

Lemma A.4 *Consider $\xi = 1$ and $\beta_1 = \beta_2 \doteq \beta$, i.e., the CCR&R's optimization problem in Section 3.1. For any given $x > 0$, the objective function in equation (3.2) in Section 3.1 is equivalent to the proportional fairness objective as defined in Mo and Walrand (2000).*

Proof of Lemma A.4: This proof uses Lemma A.3 above. Substituting $\xi = 1$, $\beta_1 = \beta_2 = \beta$, and $y_2 = F - x - y_1$ (by Lemma A.3(i)) in equation (3.2) in Section 3.1, the objective function, for any given $x > 0$, can be expressed as: $MI(y_1) = \left(\gamma\left(\underline{p}_1 + p_{o1}x^{\alpha}y_1^{\beta}\right) + (1-\gamma)\left(\underline{p}_2 + p_{o2}x^{\alpha}(F - x - y_1)^{\beta}\right)\right)^{-1} - 1$, which is convex in y_1. Next, applying first-order condition with respect to y_1 after substituting $p_o = \frac{p_{o1}}{p_{o2}}$, we have $\gamma p_{o1}{y_1^*}^{\beta-1} - (1-\gamma)\,p_{o2}\,(F - x - y_1^*)^{\beta-1} = 0$. Using $p_o = \frac{p_{o2}}{p_{o1}}$ and simplifying this equation, it follows that $y_1^* = \frac{w_1(F-x)}{w_1+w_2}$ and $y_2^* = \frac{w_2(F-x)}{w_1+w_2}$, where $w_1 \doteq (\gamma)^{\frac{1}{1-\beta}}$ and $w_2 \doteq (p_o(1-\gamma))^{\frac{1}{1-\beta}}$.

Next, consider $MI_o(y_1, y_2) \doteq w_1 \ln y_1 + w_2 \ln y_2$, where w_1 and w_2 are as defined above. Let us denote the vector that maximizes $MI_o(y_1, y_2)$ subject to $y_1 + y_2 \leq F - x$ by $\left(y_1^{PF}, y_2^{PF}\right)$. On solving this optimization problem, we have $y_1^{PF} \equiv y_1^*$ and $y_2^{PF} \equiv y_2^*$. Mo and Walrand (2000) generalize the concept of proportional fairness as follows: "a proportionally fair vector is one that maximizes the [weighted] sum of all the logarithmic utility functions." Since (y_1^*, y_2^*) maximize $MI_o(\cdot)$, using the aforementioned definition, we conclude that these optimal investments are proportionally fair.

Remark: The objective function in equation (3.2) in Section 3.1 ensures that the available funds for regional outreach activities are allocated proportionally between the two regions, wherein these proportions $\left(\frac{w_1}{w_1+w_2}\right.$ and $\left.\frac{w_2}{w_1+w_2}\right)$ are governed by contextual parameters, such as γ, p_{oi}, β_i, and ξ_i. Our analytical findings in Section 3.2 present and discuss the impact of these contextual parameters on optimal allocation decisions. $\square$

Lemma A.5 *Consider* $\Gamma(y_1, y_2) \doteq \frac{1}{\gamma p_o (F-y_1-y_2)^{\alpha} y_1^{\beta} + (1-\gamma) p_o (F-y_1-y_2)^{\alpha} y_2^{\beta}} - 1$, *where* $0 < \alpha < 1$, $0 < \beta < 1$, $0 < \gamma < 1$, *and* $0 < p_o < 1$. *For* $y_1 + y_2 < F$, $\Gamma(\cdot)$ *is jointly convex in* y_1 *and* y_2.

Proof of Lemma A.5: For a given y_2, twice differentiating $\Gamma(y_1, y_2)$ with respect to y_1, we have $\frac{\partial^2}{\partial y_1^2}\Gamma(y_1, y_2) \doteq \frac{p_o^2(\Gamma_1(y_1,y_2)+\Gamma_2(y_1,y_2)+\Gamma_3(y_1,y_2))}{\Gamma_4(y_1,y_2)} > 0$, where,

$$
\begin{aligned}
\Gamma_1(y_1, y_2) &\doteq \alpha y_2^{\beta}(1-\alpha)(1-\gamma)\left(\gamma y_1^{\beta} + (1-\gamma) y_2^{\beta}\right) > 0, \\
\Gamma_2(y_1, y_2) &\doteq 2\left(\alpha\left(\gamma y_1^{\beta} + (1-\gamma) y_2^{\beta}\right) - \beta\gamma(F-y_1-y_2) y_1^{\beta-1}\right)^2 > 0, \\
\Gamma_3(y_1, y_2) &\doteq \gamma y_1^{\beta-2}\left(\gamma y_1^{\beta} + (1-\gamma) y_2^{\beta}\right)\Big(\alpha(1-\alpha) y_1^2 \\
&\qquad + \beta(F-y_1-y_2)\left((1-\beta)(F-y_1-y_2) + 2\alpha y_1\right)\Big) > 0, \\
\Gamma_4(y_1, y_2) &\doteq (F-y_1-y_2)^{2-2\alpha}\left(p_o(F-y_1-y_2)^{\alpha}(\gamma y_1^{\beta} + (1-\gamma) y_2^{\beta})\right)^3 > 0,
\end{aligned}
$$

in which inequalities hold because $0 < \alpha < 1$, $0 < \beta < 1$, $0 < \gamma < 1$, $0 < p_o < 1$, and $y_1 + y_2 < F$. Since $\Gamma_1(y_1, y_2) > 0$, $\Gamma_2(y_1, y_2) > 0$, $\Gamma_3(y_1, y_2) > 0$, and $\Gamma_4(y_1, y_2) > 0$, we can conclude that $\frac{\partial^2}{\partial y_1^2}\Gamma(y_1, y_2) > 0$. Next, we show that the determinant of the Hessian of $\Gamma(y_1, y_2)$, denoted by $H(y_1, y_2)$, is greater than 0. Using expressions of $\frac{\partial^2}{\partial y_1^2}\Gamma(y_1, y_2)$, $\frac{\partial^2}{\partial y_2^2}\Gamma(y_1, y_2)$, and $\frac{\partial^2}{\partial y_1 \partial y_2}\Gamma(y_1, y_2)$, we obtain $H(y_1, y_2) = \frac{\alpha(1+\alpha+\beta)\left(\gamma y_1^{\beta} y_2 - (1-\gamma) y_1 y_2^{\beta}\right)^2 + \gamma(1-\gamma)(1-\beta) y_1^{\beta} y_2^{\beta} h(y_1,y_2)}{p_o^2 y_1^2 y_2^2 (F-y_1-y_2)^{2\alpha+2}\left(\gamma y_1^{\beta} + (1-\gamma) y_2^{\beta}\right)^4}$, where $h(y_1, y_2) = (\alpha+\beta)(1+\alpha+\beta)(y_1+y_2)^2 - 2F\beta(1+\alpha+\beta)(y_1+y_2) + F^2\beta(1+\beta)$. Denoting

$s \doteq (y_1 + y_2)$, by Lemma A.2 it follows that $h(y_1, y_2) > 0$. Using $h(y_1, y_2) > 0$, $0 < \alpha < 1$, $0 < \beta < 1$, $0 < \gamma < 1$, $p_o > 0$, and $y_1 + y_2 < F$, we have that $H(y_1, y_2) > 0$. Finally, combining $\frac{\partial^2}{\partial y_1^2}\Gamma(y_1, y_2) > 0$ and $H(y_1, y_2) > 0$, we can conclude that $\Gamma(y_1, y_2)$ is jointly convex in y_1 and y_2. □

Lemma A.6 *Suppose $\xi = 1$ and $\beta_1 = \beta_2 = \beta$. Then, the CCR&R's optimal levels of investment* $(x^*, y_1^*, y_2^*) = \left(\frac{F\alpha}{\alpha+\beta}, \frac{F\beta\gamma^{\frac{1}{1-\beta}}}{(\alpha+\beta)\left(\gamma^{\frac{1}{1-\beta}} + (p_o(1-\gamma))^{\frac{1}{1-\beta}}\right)}, \frac{F\beta(p_o(1-\gamma))^{\frac{1}{1-\beta}}}{(\alpha+\beta)\left(\gamma^{\frac{1}{1-\beta}} + (p_o(1-\gamma))^{\frac{1}{1-\beta}}\right)}\right) > 0$.

Proof of Lemma A.6. The proof follows similar steps as in the proof of Proposition 3.1 in Section 3.2. We therefore omit it for brevity. □

Lemma A.7 *Consider the optimal efforts (x^*, y_1^*, y_2^*) in Propositions 3.1, 3.3, and 3.5 in Section 3.2. For each set of the optimal efforts, there exists a unique $\bar{F} > 0$, such that both $\underline{p}_1 + p_{o1}(x^*)^\alpha (y_1^*)^{\beta_1}$ and $\underline{p}_2 + p_{o2}(x^*)^\alpha (y_2^*)^{\beta_2}$ are less than unity if and only if $F < \bar{F}$.*

Proof of Lemma A.7: We outline steps of proof below by using one set of optimal efforts. The proof for all other sets of optimal efforts follows similar steps, and we omit it for brevity. Consider the optimal efforts in Proposition 3.1 in Section 3.2. First, we show that $\underline{p}_i + p_{oi}(x^*)^\alpha (y_i^*)^{\beta_i}$ is increasing in F, for $i \in \{1, 2\}$. Since $\frac{\partial}{\partial F}x^*(F) > 0$ and $\frac{\partial}{\partial F}y_i^*(F) > 0$ (because $p_{oi} > 0$, $0 < \alpha < 1$, $0 < \beta_i < 1$, and $F > 0$), we have $\frac{\partial}{\partial F}p_{oi}(x^*(F))^\alpha (y_i^*(F))^{\beta_i} > 0$. Next, we show existence and uniqueness of $\bar{F}$. Since $p_{oi}(x^*(F = 0))^\alpha (y_i^*(F = 0))^{\beta_i} = \underline{p}_i < 1$ and $p_{oi}(x^*(F))^\alpha (y_i^*(F))^{\beta_i}$ increases in F, it follows $\max\left\{p_{o1}\left(x^*(F = \bar{F})\right)^\alpha \left(y_1^*(F = \bar{F})\right)^{\beta_1}, p_{o2}\left(x^*(F = \bar{F})\right)^\alpha \left(y_2^*(F = \bar{F})\right)^{\beta_2}\right\} = 1$, where $\bar{F} \doteq \left(\frac{1}{p_{o1}}\left(\frac{\alpha+\beta}{\alpha}\right)^\alpha \left(\frac{(\alpha+\beta)}{\beta}\left(\gamma^{\frac{1}{1-\beta}} + (1-\gamma)^{\frac{1}{1-\beta}}\right)\right)^\beta \zeta_1\right)^{\frac{1}{\alpha+\beta}} > 0$ and $\zeta_1 \doteq \min\left\{(1 - \underline{p}_1)\left(\frac{1}{\gamma}\right)^{\frac{\beta}{1-\beta}}, (1 - \underline{p}_2)\left(\frac{1}{1-\gamma}\right)^{\frac{\beta}{1-\beta}}\right\} > 0$.

This implies that $\underline{p}_1 + p_{o1}\left(x^*\right)^{\alpha}\left(y_1^*\right)^{\beta_1} < 1$ and $\underline{p}_2 + p_{o2}\left(x^*\right)^{\alpha}\left(y_2^*\right)^{\beta_2} < 1$ if $F < \bar{F}$, and $\max\{\underline{p}_1 + p_{o1}\left(x^*\right)^{\alpha}\left(y_1^*\right)^{\beta_1}, \underline{p}_2 + p_{o2}\left(x^*\right)^{\alpha}\left(y_2^*\right)^{\beta_2}\} = 1$ otherwise.

Similarly, using the optimal efforts in Lemma A.6, we have $\bar{F} \doteq \left(\frac{1}{p_{o1}}\left(\frac{\alpha+\beta}{\alpha}\right)^{\alpha}\left(\frac{(\alpha+\beta)}{\beta}\left(\gamma^{\frac{1}{1-\beta}} + \left(p_o(1-\gamma)\right)^{\frac{1}{1-\beta}}\right)\right)^{\beta}\zeta_1\right)^{\frac{1}{\alpha+\beta}}$ in that scenario. Next, using the optimal efforts in Propositions 3.3 and 3.5 in Section 3.2, the proof of existence and uniqueness of $\bar{F}$ follows similarly, however, $\bar{F}$ can only be implicitly defined in both these scenarios (because we do not have closed-form expressions of the optimal efforts). □

Lemma A.8 *Consider* $q(\beta) \doteq \alpha\left(1-\beta\right)\left(1+R\right) + \alpha\beta\ln\left(R\right) + \beta^2\ln\left(R\right)$ *and* $\underline{q}(\beta) \doteq \alpha\left(1-\beta\right)\left(1+\frac{1}{R}\right) + \alpha\beta\ln\left(\frac{1}{R}\right) + \beta^2\ln\left(\frac{1}{R}\right)$, *where* $0 < \alpha < 1$, $0 < \beta < 1$, $0 < \gamma < 1$, $p_o > 0$, *and* $R = \left(\frac{\gamma}{p_o(1-\gamma)}\right)^{\frac{1}{1-\beta}}$. *Then, (i) For* $p_o > \frac{\gamma}{1-\gamma}$, *there exists a unique threshold* $\hat{\beta} > 0$ *such that* $q(\beta) > 0$ *if and only if* $\beta < \hat{\beta}$ *and* $q(\beta) \leq 0$ *otherwise;* $\underline{q}(\beta) > 0$ *for any* β*; and (ii) For* $p_o \leq \frac{\gamma}{1-\gamma}$, $q(\beta) > 0$ *for any* β*; there exists a unique threshold* $\tilde{\beta} > 0$ *such that* $\underline{q}(\beta) > 0$ *if and only if* $\beta < \hat{\beta}$.

Proof of Lemma A.8: (i) First, we show that $q(\beta)$ is decreasing in β. Differentiating $q(\beta)$ with respect to β, we have $\frac{\partial}{\partial\beta}q(\beta) = \alpha\left(\ln\left(R\right)-1\right)\left(1+R\right) + \frac{\alpha\beta\ln\left(R\right)}{1-\beta} + \frac{\beta(2-\beta)\ln\left(R\right)}{1-\beta}$. When $p_o > \frac{\gamma}{1-\gamma}$, we have $R < 1$, which implies that $\ln\left(R\right) < 0$. Using $\ln\left(R\right) < 0$ and $0 < \beta < 1$, we have $\frac{\partial}{\partial\beta}q(\beta) < 0$. Next, we show existence and uniqueness of $\hat{\beta}$. We have that $\lim_{\beta\to 0} q(\beta) = \frac{\alpha}{1-\gamma} > 0$ and $\lim_{\beta\to 1} q(\beta) = -\infty$. Based on these values, and the above result that $q(\beta)$ is decreasing in β, we can conclude that there exists a unique $\hat{\beta} \in (0,1)$ such that $q(\beta) > 0$ if $\beta < \hat{\beta}$ and $q(\beta) \leq 0$ otherwise. Next, using $0 < \alpha, \beta < 1$, and $\ln\left(\frac{1}{R}\right) > 0$, we can conclude that $\underline{q}(\beta) > 0$. (ii) The proof for the case when $p_o \leq \frac{\gamma}{1-\gamma}$ follows similar steps as above. For brevity, we omit it. □

Lemma A.9 *Suppose* $\beta_2 = \frac{1}{2}$. *When* $\alpha < \beta_1$, *there exists* $\hat{F} \doteq \left(\frac{\left(\alpha+\frac{1}{2}\right)(1-\gamma)^2}{4^{1-\beta_1}\beta_1(\beta_1-\alpha)\gamma_2}\right)^{\frac{1}{\beta_1}} > 0$ *such that,* $y_2^* < \frac{F}{2} < y_1^*$ *if and only if* $F > \hat{F}$ *for all* $\gamma \in (0,1)$ *and* $\beta_1 \in (0,1)$.

Proof of Lemma A.9. Using $\beta_2 = \frac{1}{2}$, and following similar steps as in the proof of Proposition 3.3 in Section 3.2, we have y_1^* is the unique solution to the following equation (which crosses zero from above): $F - \left(1 + \frac{\alpha}{\beta_1}\right) y_1 - \left(\frac{(1+2\alpha)(1-\gamma)^2}{4{\beta_1}^2()^2}\right) {y_1}^{2(1-\beta_1)} = 0$. Considering $\alpha < \beta_1$, and substituting $y_1 = \frac{F}{2}$ in the equation above, we have that $y_1^* > \frac{F}{2}$ if and only if $F > \hat{F}$, where $\hat{F}$ is as defined in the statement of lemma. Further, since $x^* = F - y_1^* - y_2^* > 0$ (by Lemma A.3), we can conclude that $y_2^* < \frac{F}{2}$. □

Lemma A.10 *Consider $\xi > 1$, $k_1 > 0$, $k_2 > 0$, and $p^o F \leq 2\left(1 - \underline{p}\right)$. Then, there exists a unique $y_2 > \frac{F}{2}\left(\frac{k_1^2}{k_1^2+k_2^2}\right)$ that solves the equation $p^o F(1-\xi)\sqrt{2F} - \left(k_1\sqrt{2F - 4y_2} - 2k_2\sqrt{y_2}\right) = 0$.*

Proof of Lemma A.10: Since $\xi > 1$, we have $p^o F(1-\xi)\sqrt{2F} < 0$. Further, we can see that $\left(k_1\sqrt{2F - 4y_2} - 2k_2\sqrt{y_2}\right) > 0$ is decreasing in y_2. Using these two results in the equation $p^o F(1-\xi)\sqrt{2F} - \left(k_1\sqrt{2F - 4y_2} - 2k_2\sqrt{y_2}\right) = 0$, we have that at least one extreme point will exist if only if $p^o(1-\xi)F\sqrt{2F} \geq \min_{y_2}\left\{k_1\sqrt{2F - 4y_2} - 2k_2\sqrt{y_2}\right\} = -2k_2\sqrt{F/2}$. On simplification, we have that this always holds because $p^o F \leq 2\left(1 - \underline{p}\right)$. Next, re-arranging the terms in the equation, we have $p^o F(1-\xi)\sqrt{2F} = k_1\sqrt{2F - 4y_2} - 2k_2\sqrt{y_2}$. Since $p^o F(1-\xi)\sqrt{2F} < 0$ (because $\xi > 1$), we have that the extreme points must satisfy $k_1\sqrt{2F - 4y_2} < 2k_2\sqrt{y_2}$ or $y_2 > \frac{F}{2}\left(\frac{k_1^2}{k_1^2+k_2^2}\right)$.

Re-arranging terms in the equation, and taking square on both sides, we obtain a quadratic equation with two roots: $\frac{F}{2}\left(\frac{k_1^2}{k_1^2+k_2^2}\right) + \sqrt{\frac{F}{2}\left(\frac{k_1^2}{k_1^2+k_2^2}\right) - \frac{(p^o)^2F^3(1-\xi)^2}{2\left(k_1^2+k_2^2\right)}}$ and $\frac{F}{2}\left(\frac{k_1^2}{k_1^2+k_2^2}\right) - \sqrt{\frac{F}{2}\left(\frac{k_1^2}{k_1^2+k_2^2}\right) - \frac{(p^o)^2F^3(1-\xi)^2}{2\left(k_1^2+k_2^2\right)}}$. It can be seen that only the former root is greater than $\frac{F}{2}\left(\frac{k_1^2}{k_1^2+k_2^2}\right)$. □

APPENDIX B

APPENDIX FOR CHAPTER 4: ALLOCATION OF FUNDS IN BILEVEL SUBSIDY WELFARE PROGRAMS

B.1 Table

Table B.1. Description of Model Notation

Notation	
F	Initial funds available to the funding agency
f	Additional funds that may become available in the future
p	Probability that the additional funds become available in the future
α	Rate of return on investment in quality improvement activities
$\beta_{j=\{H,L\}}$	High (Low) rate of return on investment in outreach activities
ρ	Probability that the return rate of outreach investment is β_H
q_i	Activity-specific factor in service area i
θ_i	Volume adjustment factor in service area i
Decision Variables and Functions	
X_i	Service agency i's investment in quality improvement activities using initial funds
Ψ_i	Service agency i's investment in outreach activities using initial funds
χ_2	Service agency i's investment in quality improvement activities using additional funds
ψ_2	Service agency i's investment in outreach activities using additional funds
B_i	Service agency i's budget under initial budget allocation
$V_i(\cdot)$	Social impact generated through investments in service agency i's designated area using initial budget
$v_2(\cdot)$	Social impact generated through investments in service agency 2's designated area using initial and additional budget
I	Inequity in the expected social impact generated across service areas 1 and 2
TSI	Total expected social impact across service areas 1 and 2

B.2 Technical Details of Efficiency-Focused and Formula-Based Funding Methods

B.2.1 Efficiency-Focused Funding Method: Optimal Decisions and Outcomes

We present the budget allocation decisions under the efficiency-focused funding method and compare its resulting levels of inequity and total social impact with those under the equity-ensuring allocation method. The optimal levels of budget provided by the efficiency-focused funding agency from its initial budget, denoted by B_1^{Ef} and B_2^{Ef}, can be solved by the optimization problem outlined in equations (4.2)-(4.6) in Section 4.1 by removing the constraint in equation (4.3). They are characterized in Lemma B.1 below.

Lemma B.1 *Consider the optimization problem outlined in equations (4.2)-(4.6) without the constraint in equation (4.3) (i.e., without the equity consideration) presented in Section 4.1. Suppose $p = \frac{1}{2}$. Then, the funding agency's optimal levels of initial funds to service agency 1 and 2 $\left(B_1^{Ef}, B_2^{Ef}\right)$ are: B_1^{Ef} is the unique solution to the equation $B_1\left(\frac{1}{\sqrt{F-B_1}} + \frac{1}{\sqrt{F+f-B_1}}\right)^2 = N_1$ and $B_2^{Ef} = F - B_1^{Ef}$, where N_1 is as characterized in Lemma 4.1 presented in Section 4.2.*

Next, Lemma B.2 presents effects of several contextual factors, including q_i, ρ, and α, on the optimal level of initial funds allocated to service agency $i \in \{1, 2\}$ under the efficiency-focused funding method.

Lemma B.2 *Consider the funding agency has no equity consideration, then:*
With Respect To Area-specific factors. *(i) B_i^{Ef} decreases in q_i and B_{3-i}^{Ef} increases in q_i if $q_i < \hat{q}$, and B_i^{Ef} increases in q_i and B_{3-i}^{Ef} decreases in q_i otherwise, where $\hat{q}$ is defined in Proposition 4.2 in Section 4.2.*
With Respect To Activity-specific factors. *(ii) B_i^{Ef} increases in ρ and B_{3-i}^{Ef} decreases in ρ if $q_i < q_{3-i}$, and B_i^{Ef} decreases in ρ and B_{3-i}^{Ef} increases in ρ otherwise.*

Further, B_i^{Ef} decreases in α and B_{3-i}^{Ef} increases in α if $q_i < q_{3-i}$, and B_i^{Ef} increases α and B_{3-i}^{Ef} decreases α otherwise.

Finally, Lemma B.3 below presents differences in inequity outcomes and total expected social impact between the efficiency-focused (denoted by the use of Ef superscript) and equity-ensuring (denoted by the use of Eq superscript) funding methods.

Lemma B.3 *Consider $f = 0$. (i) The difference in the inequity outcomes, $I^{Ef} - I^{Eq}$, is $\left|\left(N_1 - \frac{1}{\theta_2}\right)\sqrt{\frac{F}{N+1}}\sqrt{\alpha q_2^2 + \left(\rho\sqrt{\beta_H} + (1-\rho)\sqrt{\beta_L}\right)^2 (1-q_2)^2}\right|$ and; (ii) The difference in the total expected social impact, $TSI^{Ef} - TSI^{Eq}$, is $F\left(\sqrt{(N_1+1)(N_1\theta_2^2+1)} - (\theta_2+1)\sqrt{N_1}\right)\sqrt{\frac{\alpha q_2^2 + \left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2(1-q_2)^2}{N_1\theta_2^2+1}} > 0$, where N_1 is as characterized in the statement of Lemma 4.1 presented in Section 4.2.*

B.2.2 Formula-Based Funding Method: Allocation Decisions and Outcomes

Based on the description of the formula-based funding method considered in Section 4.2.2.2, we have that the funding agency allocates initial funds (from F) to service agency i, denoted by B_i^{Fo}, based on the relative size of the pool of the families who need subsidy assistance in service area i. Therefore, we have $B_i^{Fo} = \left(\frac{\theta_i}{\theta_1+\theta_2}\right) F$. After receiving initial funds, each service agency distributes those funds between quality improvement and outreach activities based on the proportion of subsidy-accepting and non-accepting service providers in its area, denoted by X_i^{Fo} and Ψ_i^{Fo}, respectively. Therefore, we have $X_i^{Fo} = q_i \left(\frac{\theta_i}{\theta_1+\theta_2}\right) F$ and $\Psi_i^{Fo} = (1-q_i)\left(\frac{\theta_i}{\theta_1+\theta_2}\right) F$. Along the same lines, when additional funds f become available to service agency 2 in the future, service agency 2 distributes f between two provider-facing activities within its area proportionally, denoted by χ_2^{Fo} and ψ_2^{Fo}, such that $\psi_2^{Fo} = q_2 f$ and $\chi_2^{Fo} = (1-q_2) f$.

Finally, Lemma B.4 below presents differences in inequity outcomes and total expected social impact between the formula-based (denoted by the use of Fo superscript) and equity-ensuring (denoted by the use of Eq superscript) funding methods.

Lemma B.4 *(i) The difference in the inequity outcomes,* $I^{Fo} - I^{Eq}$*, is* $\left| \frac{1}{2\theta_2} \sqrt{\frac{1}{1+\theta_2}} \Big(2\theta_2 n_1 \sqrt{F} - n_2 \left(\sqrt{\theta_2 F} + \sqrt{\theta_2 F + (1+\theta_2) f} \right) \Big) \right|$ *and; (ii) The difference in the total social impact,* $TSI^{Fo} - TSI^{Eq}$*, is* $n_5 - n_4 \Big(2\sqrt{N_1 \left(n_3 + \sqrt{N_1} \theta_2 (2F + f) \right)} + \sqrt{n_6 - n_3} + \sqrt{n_7 - n_3} \Big)$*, where* N_1 *is as characterized in Lemma 4.1 presented in Section 4.2, and* $n_{i \in \{1..7\}}$ *is as characterized in the proof of this lemma.*

B.3 Additional Details of the Case Study

In this section, we describe the steps taken to estimate the model parameters that are needed to solve the optimization problems in the case study. We rely on the following three sources for inputs in our estimation steps: (i) Detailed data made available by the Massachusetts Department of Early Education and Care (MEEC); (ii) The child care literature; and (iii) Our interviews with managers at multiple Child Care Resource and Referral Agencies (CCR&Rs) in Massachusetts.

Volume Adjustment Factor (θ_i). The parameter θ_i captures the size of the pool of the families who need subsidy assistance in service area i relative to the area with the smallest size of pool of families who need subsidy. As per the MEEC's data on the number of families who need child care subsidy assistance in each area (tabulated based on the U.S. Census Bureau), service area 5 (i.e., Cape area) has the lowest number of such families. Therefore, we estimate $\theta_5 = 1.000$. Next, normalizing the number of families who need subsidy assistance in area $i = \{1, ..., 7\} \backslash \{5\}$ by the number of families who need subsidy assistance in area 5, followed by taking a natural logarithm of the resulting fraction (to account for high dispersion in the numbers of families across all areas), we have: $\theta_1 = 1.192$, $\theta_2 = 1.125$, $\theta_3 = 1.215$, $\theta_4 = 1.146$, $\theta_6 = 1.022$, and $\theta_7 = 1.187$.

Area-specific Factor (q_i). The parameter q_i captures the distribution of subsidy-accepting and non-accepting providers operating in service area i. Using the MEEC's

data on the total numbers of subsidy voucher-accepting and non-accepting child care providers in each service area, we estimate q_i as the percentage of subsidy voucher-accepting providers among all providers in area i. Accordingly, we have $q_1 = 0.593$, $q_2 = 0.253$, $q_3 = 0.316$, $q_4 = 0.412$, $q_5 = 0.464$, $q_6 = 0.342$, and $q_7 = 0.416$.

Rate of Return of Investment in Quality Improvement (α). The parameter α captures the efficiency of a service agency's investment in the quality improvement activity (i.e., it is a measure of how well the quality improvement investment enhances the resulting social impact). Based on our reviews of practitioner and governmental reports and using inputs provided by managers at different CCR&Rs in Massachusetts, we estimate that it costs a CCR&R approximately $\$850,000$ to assist providers with accreditation related paperwork, organize training sessions and workshops, and purchase software tools to help all providers in its area to achieve the highest child care quality rating on the state-approved quality rating system (Massachusetts Department of Early Education and Care 2017, Department of Health and Human Services 2021). Without loss of generality, we normalize the highest social impact by 1 (since it will end up as a scaling factor in expressions of $V_i(\cdot)$ and $v_i(\cdot)$, and hence, will not affect our analysis). Therefore, by using the relationship $\sqrt{\alpha \times 850,000} \approx 1$, we estimate the value of α as 1.176×10^{-6}.

Rate of Return of Investment in Outreach ($\beta_{\{L,H\}}$). Estimation of $\beta_{\{L,H\}}$ follows the similar steps as those of α above. Combining cost analysis reports and inputs from experts in the domain of child care subsidies, we estimate that it costs a CCR&R $\$550,000$ to $\$1,800,000$ toward outreach to advertise in local media sources, organize community development fairs, hire additional staff or specialists to travel to local communities, and communicate with local stakeholders routinely in order to propel all child care providers in its area to accept subsidies (Massachusetts Department of Early Education and Care 2005). Therefore, by using $\sqrt{\beta_L \times 1,800,000} \approx 1$ and $\sqrt{\beta_H \times 550,000} \approx 1$, we estimate $\beta_L = 0.556 \times 10^{-6}$ and $\beta_H = 1.181 \times 10^{-6}$. While

it is evidenced that outreach investment has uncertain rate of return (as discussed in Chapter 3), it is difficult to quantify the degree of uncertainty. For our numerical illustration in Chapter 3, we reasonable assume the maximum level of variance, which implies $\rho = 0.5$. Later, in the next section, we conduct sensitivity analysis by varying the estimate of ρ and a few other model parameters.

Amount and Probability of Future Additional Funds (f and p). Given the recent political spotlight on child care related issues (such as low availability and poor quality of care) and the prominent adverse effect of COVID-19 pandemic on child care in the U.S., there are several politician- and community-driven campaigns to ensure infusion of additional funds toward child care subsidy voucher programs in a particular area (that has been adversely impacted) or a particular type of activity (that can enhance quality of care); see Johnson-Staub (2020) and Lynch (2020). Our interviews with the MEEC executives revealed that they expect additional funds to become available during the current planned horizon of the program. However, they shared past instances of additional funds becoming available for the Boston service area (e.g., under housing rental program; Department of Health and Human Services 2021) or for quality improvement activity (e.g., under the child care subsidy welfare program in a neighboring state; Maine Department of Health and Human Services 2023), which motivated us to consider the following two practical scenarios in our numerical illustration: (i) area-only case, wherein the additional funds f are expected to become available for the Boston area only, and (ii) activity-only case, wherein the additional funds f are expected to become available for investment in quality improvement only across all seven areas. In practice, additional funds if available in the future are typically less than the funding agency's initial total budget. Here, based on our conversations with the MEEC executives, we assume the amount of future additional funds to be approximately 10% of F and estimate $f = \$600,000$. In line with our previous discussions, we consider the most variable case in terms of

chances of additional funds becoming available. That is, $p = 0.5$. For robustness and sensitivity analyses, Within our case study, we discuss sensitivity analysis based on varying the estimates of both f and p.

Summary of Optimal Allocation Decisions (B_i^*). Table B.2 presents the optimal levels of funds toward the seven service areas under different funding methods discussed in the study.

Table B.2. Case Study: Summary of Funding Levels

Panel A. No Additional Funds (in $ million)							
	Service Agency (i)						
	1	2	3	4	5	6	7
Optimal Funds (Eq)	1.023	0.789	1.003	0.972	0.754	0.731	1.045
Optimal Funds (Ef)	0.885	1.021	0.937	0.860	0.845	0.910	0.859
Funds under Current Method	1.326	1.055	1.334	0.809	0.423	0.321	1.049
Panel B. Area-only Additional Funds (in $ million)							
	Service Agency (i)						
	1	2	3	4	5	6	7
Optimal Initial Funds (Eq)	1.068	0.824	1.048	1.015	0.787	0.763	0.812
Optimal Initial Funds (Ef)	0.917	1.058	0.971	0.891	0.875	0.943	0.661
Initial Funds under Current Method	1.326	1.055	1.334	0.809	0.423	0.321	1.049
Additional Funds	0	0	0	0	0	0	0.600

Note: "Eq" refers to equity-ensuring method, and "Ef" refers to efficiency-focused method. These results are also depicted pictorially in Figure 4.2 in Section 4.3.

B.4 Extensions

B.4.1 Uncertainty in Outreach Investment Return Gets Resolved

In our base model, we consider an uncertainty in the rate of investment return of outreach (captured by ρ; see Section 4.1). To derive our main analytical results in Section 4.2, we also consider that this outreach investment related uncertainty remains unresolved when the additional funds become available. Given the temporal gap between the initial allocation of funds and the availability of additional funds, it is

possible that the funding agency and the service agencies obtain a better understanding on the rate of return of outreach investments. Accordingly, for completeness, we extend our model to analyze how resolution of the uncertainty in outreach investment return rate affects the funding agency's optimal funds allocation decisions.

Since we consider that all entities will know whether outreach investment return is either β_H or β_L when additional funds become available for service area 2, we need to modify our model to include investment decisions (when distributing f) by the service agency 2 under two different possibilities. Specifically, we denote by χ_2^H and ψ_2^H (χ_2^L and ψ_2^L) the optimal levels of investment that the service agency 2 makes toward quality improvement and outreach activities when the outreach investment return rate is β_H (β_L), respectively, on top of the initial respective investments X_2 and Ψ_2. Further, we denote by $v_2^H(\cdot)$ and $v_2^L(\cdot)$ the overall social impact generated by service agency 2 using the initial and additional funds under β_H and β_L, respectively. Then, we have: $v_2^H\left(X_2, \Psi_2, \chi_2^H, \psi_2^H\right) \doteq q_2\sqrt{\alpha\left(X_2+\chi_2^H\right)} + (1-q_2)\sqrt{\beta_H\left(\Psi_2+\psi_2^H\right)}$, and $v_2^L\left(X_2, \Psi_2, \chi_2^L, \psi_2^L\right) \doteq q_2\sqrt{\alpha\left(X_2+\chi_2^L\right)} + (1-q_2)\sqrt{\beta_L\left(\Psi_2+\psi_2^L\right)}$.

Following similar steps as described in Section 4.1, the optimal levels of investment χ_2^{H*}, ψ_2^{H*}, χ_2^{L*}, and ψ_2^{L*} can be obtained by solving the following two optimization problems: $\chi_2^{H*}, \psi_2^{H*} \in \arg\max_{\left\{\chi_2^{lH}, \psi_2^{lH}\right\}} \Big\{ v_2^H\left(X_2, \Psi_2, \chi_2^H, \psi_2^H\right) : \chi_2^{lH} + \psi_2^{lH} \leq f, \chi_2^{lH}, \psi_2^{lH} \geq 0 \Big\}$ and $\chi_2^{L*}, \psi_2^{L*} \in \arg\max_{\left\{\chi_2^{lL}, \psi_2^{lL}\right\}} \Big\{ v_2^L\left(X_2, \Psi_2, \chi_2^L, \psi_2^L\right) : \chi_2^{lL} + \psi_2^{lL} \leq f, \chi_2^{lL}, \psi_2^{lL} \geq 0 \Big\}$.

Next, the funding agency's objective function, the total expected social impact, will be revised as: $TSI = V_1\left(X_1, \Psi_1\right) + (1-p)\, V_2\left(X_2, \Psi_2\right) + p\Big(\rho v_2^H\left(X_2, \Psi_2, \chi_2^{H*}, \psi_2^{H*}\right) + (1-\rho)\, v_2^L\left(X_2, \Psi_2, \chi_2^{L*}, \psi_2^{L*}\right)\Big)$. The modification can be noted in the last part of this expression, which represents the expectation of the overall social impact generated by service agency 2 using initial and additional funds over the realization of outreach investment return rate. Incorporating these changed elements of the funding agency and service agency 2's investment problems, we solve the modified optimization prob-

lem to characterize the optimal levels of initial funds allocated to each service agency (i.e., B_1^* and B_2^*) in Lemma B.5 below.

Lemma B.5 *Consider that uncertainty in investment return rate of outreach gets resolved for the funding and service agencies when additional funds become available for service area 2. Then, the funding agency's optimal levels of initial funds to service agencies 1 and 2* (B_1^*, B_2^*) *are:*
$B_1^* = \frac{m_2 m_4(m_1F-2f)+F^2N_{b1}^2\left(N_{b1}^4\left(N_{b2}^2+1\right)+\left(N_{b2}^2-1\right)^2\left(2m_3-N_{b2}^2-1\right)\right)+f(m_1F-f)\left(m_2^2-4N_{b1}^2N_{b2}^2\right)}{\left(m_3^2-4N_{b2}^2\right)\left(Fm_3N_{b1}^2-m_4-fm_2\right)}$ *and* $B_2^* = F - B_1^*$, *respectively, where* $N_{b1} \doteq \frac{2\theta_2\sqrt{\alpha q_1^2+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2(1-q_1)^2}}{\rho\sqrt{\alpha q_2^2+\beta_L(1-q_2)^2}+(1-\rho)\sqrt{\alpha q_2^2+\beta_H(1-q_2)^2}} > 0$, $N_{b2} \doteq \frac{\sqrt{\alpha q_2^2+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2(1-q_2)^2}}{\rho\sqrt{\alpha q_2^2+\beta_L(1-q_2)^2}+(1-\rho)\sqrt{\alpha q_2^2+\beta_H(1-q_2)^2}} > 0$, $m_1 \doteq N_{b1}^2+N_{b2}^2-1$, $m_2 \doteq N_{b1}^2-N_{b2}^2+1$, $m_3 \doteq N_{b1}^2 + N_{b2}^2 + 1$, *and* $m_4 \doteq 2N_{b1}N_{b2} - \sqrt{N_{b1}^2F^2 + fm_1F - f^2}$.

Using the optimal budget decisions presented in the lemma above, we conduct extensive numerical studies and find that in such a setting, the funding agency should allocate more initial budget to area 1 (B_1^*) as compared to our base setting where the uncertainty remains unresolved. This is because if the additional funds arrive, the funding agency will know for certain whether the outreach investment return is β_H or β_L, and thus, it will know for certain the level of social impact generated in area 2 using the initially allocated budget as well as using the additional funds. As a result, the need to allocate a higher initial budget to area 2 to hedge against an uncertain outreach investment is less salient, which in turn implies that B_1^* is higher.

We also compare inequity outcomes and total expected social impact under the formula-based versus equity-ensuring methods and find that insights presented in Section 4.2 (i.e., based on Propositions 4.3 and 4.4) continue to hold under this model extension. Further, we find that our proposed optimization-based equity-ensuring method is more valuable for the funding agency, that is, $I^{Fo} - I^{Eq}$ increases and TSI^{Eq} is significantly higher than TSI^{Fo}, when the uncertainty in outreach invest-

ment return gets resolved and the disparity in the mix of service providers across different areas is not too large.

Finally, we also use this extension to consider another form of information asymmetry between the funding agency and service agencies. Although our review of the related literature of subsidy welfare programs and our interviews with practitioners reveal that the funding and service agencies have same knowledge about contextual parameters, it is possible that the service agencies could gain more clarity on the investment return rate of community outreach by the time additional funds arrive (as the service agencies are closer to their local communities). Thus, we take our model extension one step further by considering that when additional funds are available to service agency 2 (if at all), its uncertainty about outreach investment return gets resolved, whereas the funding agency gets no such resolution. Under such a setting, although the funding agency should allocate the same amount of funds to the service agency 2 (as in the setting without the information asymmetry on outreach investment return), the resulting level of inequity in the social impact between areas 1 and 2 increases as outreach investment is more likely to yield a higher investment return rate, or service area 2 has a skewed mix of service providers.

B.4.2 Time Value of Social Impact

In this section, we extend our model and analysis to consider situations with time value of the generated social impact. Given that there is a temporal time gap (even if it may be small) between investments using initial funds and additional funds, it is possible that under certain subsidy welfare programs (say, healthcare related), it is beneficial to create the social impact sooner than later. Accordingly, we use a discount factor, denoted by δ with $0 \leq \delta \leq 1$, to capture the time value of the generated social impact. In particular, we consider that the funding agency discounts the social impact generated using additional funds by δ in the total expected

social impact and expected inequity formulations. That is, we modify the expressions outlined in Section 4.1.1 by replacing the term $v_2(\cdot)$ with $V_2(\cdot) + \delta(v_2(\cdot) - V_2(\cdot))$, such that $TSI = V_1(\cdot) + (1-p)V_2(\cdot) + p[V_2(\cdot) + \delta(v_2(\cdot) - V_2(\cdot))]$ and $I = \left|\frac{V_1(\cdot)}{\theta_1} - \frac{(1-p)V_2(\cdot)+p[V_2(\cdot)+\delta(v_2(\cdot)-V_2(\cdot))]}{\theta_2}\right|$. Then, if and when additional funds become available to service agency 2 in the future, the service agency 2 decides the optimal level of investment by solving the same optimization model as presented in equation (4.1) in Section 4.1. Lemma B.6 characterizes the optimal levels of initial funds allocated to each service agency (i.e., B_1^* and B_2^*).

Lemma B.6 *In the presence of time value of the generated social impact, the funding agency's optimal levels of initial funds to service agencies 1 and 2 (B_1^*, B_2^*) are: $B_1^* = \frac{\sqrt{(\delta_2-2)^2\delta^2\theta_2^2N_1\left(4FN_1(F+f)\theta_2^2+4(1-\delta)fF-\delta^2f^2\right)}+f\delta^3+(2F+f)\theta_2^2\delta^2N_1+4(1-\delta)\left(\theta_2^2N_1+1-\delta\right)F-\delta^2f}{4\left(\theta_2^2N_1+1\right)\left(\theta_2^2N_1+(1-\delta)^2\right)}$ and $B_2^* = F - B_1^*$, respectively, where N_1 is as characterized in the statement of Lemma 4.1.*

Combining both analytical and numerical analyses, we find that the key insights presented in Section 4.2 are robust to the inclusion of a discount factor in our model setup. As expected, we find that the optimal level of budget allocated by the funding agency to service agency 1 from its initial pool of funds (B_1^*) decreases when the time value of the social impact is quite large (i.e., when δ is low). This is because, at low values of δ, the funding agency puts relatively less emphasis on the social impact that can be generated from the expected additional funds versus the initial pool of funds. Therefore, the need to hedge against such additional funding becoming available by increasing B_1^* is less pronounced when δ is low. Further, this decrement is the largest when the mix of subsidy-accepting and non-accepting service providers in area 1 is more balanced.

We also numerically verify that the insights on comparisons of outcomes under different funding methods continue to hold in the presence of time value of social impact. Further, consider the area not receiving any additional funds has a relatively skewed mix of service providers. Then, when the time value of social impact

is relatively low, we find that the equity-ensuring method (versus the formula-based method) can generate relatively more social impact while eliminating inequity. Overall, we can conclude that our proposed optimization-based equity-ensuring method is more valuable for the funding agency when the time value of social impact is low, which as we remarked earlier is the case in most subsidy programs under study.

B.4.3 Maximum Allowed Inequity Deviation

In this extension, we consider that instead of ensuring perfect equity (i.e., $I = 0$) the funding agency allows a maximum $K \geq 0$ amount of inequity deviation between the social impact in the two service areas. Since we consider that only service agency 2 is expected to receive additional funds, if any, we only need to update the inequity related constraint (given by equation (4.3) in the base model) under this extension. Note that there are two solution regimes under this optimization setup: specifically, the inequity constraint, given by $\left| \frac{V_1(X_1,\Psi_1)}{\theta_1} - \frac{(1-p)V_2(X_2,\Psi_2)+pv_2\left(X_2,\Psi_2,\chi_2^*,\psi_2^*\right)}{\theta_2} \right| \leq K$, either binds or does not not bind. In the next lemma, we characterize the optimal levels of funds allocated to each service agency under each of these two regimes.

Lemma B.7 *Denote $K \geq 0$ as the maximum amount of inequity deviation. (i) When the inequity constraint binds, the funding agency's optimal levels of initial funds to service agencies 1 and 2 (B_1^*, B_2^*) are: $B_1^* = \frac{\sqrt{N_1\left(4FN_1(F+f)\theta_2^2-f^2\right)}+(2F+f)\theta_2 N_1}{4\theta_2 N_1\left(\theta_2^2 N_1+1\right)}$ and $B_2^* = F - B_1^*$. (ii) When the inequity constraint does not bind, B_1^* is the unique solution the equation $B_1\left(\frac{1}{\sqrt{F-B_1}} + \frac{1}{\sqrt{F+f-B_1}}\right)^2 = N_1$ and $B_2^* = F - B_1^*$, where N_1 is as characterized in the statement of Lemma 4.1 presented in Section 4.2.*

We use the characterizations outlined in Lemma B.7 to conduct extensive numerical experiments that help us to generate the following insights. When the allowed inequity deviation is smaller, the funding agency allocates budgets from its initial pool of funds to ensure that the social impact generated in the two areas deviates exactly by K. However, when the allowed deviation in inequity is above a threshold $\bar{K}$, the

funding agency focuses on efficiency (i.e, maximizing the total expected social impact across both areas) in its funds allocation strategy as it is able to satisfy the equity consideration (since K is large). While this is an expected finding, it is interesting to note that this situation is more likely to arise when the mix of service providers in area 1 (which is not expected to receive any additional funds) is more balanced. The explanation is as follows: Consider that the allowable inequity constraint binds (i.e., $I = K$). Then, when q_1 moves away from either 0 or 1, the funding agency should increase the budget B_1^* provided to the service agency 1 at the expense of the budget B_2^* provided to the service agency 2 (as shown in Proposition 4.1 in Section 4.2). However, when q_1 moves farther away from either 0 or 1 (i.e., the mix of service providers in area 1 becomes more balanced), any additional increase in B_1^* will lead to a less pronounced increment in the total expected social impact due to the salient effect of diminishing returns at higher values of B_1^*. As a result, in such situations, the funding agency finds it more valuable to increase the budget B_2^* at the expense of the budget B_1^*, as it can achieve a greater total expected social impact without violating the allowable inequity constraint $I \leq K$.

Next, we highlight a few selected findings based on our numerical comparison of outcomes (levels of inequity and total social impact) under formula-based versus equity-ensuring methods. When K is sufficiently large, the equity-ensuring method generates greater social impact than the formula-based always. This is because, under a relatively high level of allowed inequity deviation, the equity-ensuring funding agency will focus on an efficiency objective (instead of its equity consideration). On the other hand, when K is not too large, the equity-ensuring method generates greater social impact than the formula-based only under certain conditions (same as the conditions in Proposition 4.4 in Section 4.2). Thus, we can say that our proposed optimization-based equity-ensuring method is more valuable for the funding agency

when the maximum allowed inequity deviation is not too low and the disparity in the size of the pool of beneficiaries across service areas is not too large.

B.4.4 No Information Asymmetry Between the Funding and Service Agencies

Our main model and analysis consider an information asymmetry (about the availability of such funds) between the entities at the two different hierarchical levels of the subsidy welfare programs. Recall that, we consider that the funding agency considers that $f > 0$ amount of additional funds will be approved by the legislative body for investments in area 2 at a future instance during the planned horizon of the program with a probability $p \geq 0$; whereas, the service agencies make their investment decisions without incorporating any such likelihood of additional funds becoming available.

In this section, as a benchmark setup, we model and analyze the funds allocation problem when there is no information asymmetry between the funding and service agencies. That is, both the funding and service agencies have the same information on the likelihood and amount of additional funds that may become available for area 2 in the future. Accordingly, we consider that service agency 2 incorporates such information by making allocation decisions between the two provider-facing activities at the beginning of the planning horizon based on the total expected funds (given by, $B_2 + pf$). Denote by X_2^f and Ψ_2^f the optimal levels of investment service agency 2 makes toward quality improvement and outreach activities, respectively. They can be obtained by solving the following optimization problem: $X_2^f, \Psi_2^f \in \arg\max_{\left\{X_2^{lf}, \Psi_2^{lf}\right\}} \left\{V_2\left(X_2^f, \Psi_2^f\right) : X_2^{lf} + \Psi_2^{lf} \leq B_2 + pf, \quad X_2^{lf}, \Psi_2^{lf} \geq 0\right\}$. Incorporating these modified model elements, Lemma B.8 presents the optimal levels of initial funds allocated by the funding agency to service agencies 1 and 2. (A footnote in Section 4.2 explains the insights based on the results below.)

Lemma B.8 *Consider there is no information asymmetry between the funding and service agencies on the future additional funds. Then, the funding agency's optimal levels of initial funds to service agencies 1 and 2* (B_1^*, B_2^*) *are:* $B_1^* = \frac{2F+f}{2\left(\theta_2^2 N_1+1\right)}$ *and* $B_2^* = F - B_1^*$*, respectively, where* N_1 *is as characterized in the statement of Lemma 4.1 presented in Section 4.2. Further,* B_1^* *(*B_2^**) is higher (lower) when there is no information asymmetry versus when there is an information asymmetry between the funding and service agencies.*

Further, our comparative analysis of outcomes (levels of inequity and total social impact) under the formula-based method and equity-ensuring method reveal that the insights presented in Section 4.2 continue to hold in the absence of the information asymmetry on the future additional funds between the funding and service agencies. Naturally, we also find that our proposed optimization-based equity-ensuring method is more valuable for the funding agency when there is no information asymmetry on future additional funds between entities at two different levels (because of the forward-looking feature in the setup).

B.4.5 Regional Asymmetry within Service Areas

A few practitioner reports highlight disparities in the number of service providers between different regions, say predominantly rural and predominantly urban regions, within a service area (Department of Health and Human Services 2023). In this section, we extend our model and analysis to consider the regional differences between the areas in addition to the activity- and area-specific differences considered in our base model (e.g., as captured by α, $\beta_{\{H,L\}}$, ρ, θ_i, and q_i). This can be done by including different values of the scaling factors associated with the social impact of investment in a specific type of activity across different regions within areas. We consider two types of regions, denoted by r and u, within each area. Therefore, each service agency makes four activity- and region-specific decisions, in-

cluding quality improvement and outreach related investment in regions r and u. In particular, we denote by X_{ir}, Ψ_{ir}, X_{iu}, and Ψ_{iu} the levels of investment service agency $i \in \{1,2\}$ makes toward quality improvement in region r, outreach in region r, quality improvement in region u, and outreach in region u, respectively, using its initially allocated funds. As an example, the social impact generated in area i using the budget provided from the initial funds will be modified as: $V_i\left(X_{ir}, \Psi_{ir}, X_{iu}, \Psi_{iu}\right) = q_i r_i \sqrt{\alpha X_{ir}} + (1-q_i)\, r_i \left(\rho\sqrt{\beta_H \Psi_{ir}} + (1-\rho)\sqrt{\beta_L \Psi_{ir}}\right) + q_i\,(1-r_i)\sqrt{\alpha X_{iu}} + (1-q_i)\,(1-r_i)\left(\rho\sqrt{\beta_H \Psi_{iu}} + (1-\rho)\sqrt{\beta_L \Psi_{iu}}\right)$, where the parameter $r_i \in (0,1)$ adjusts the scaling factors q_i and $1-q_i$ to account for the region-specific differences within area i. For example, the more service providers operating in region r as compared to those in region u within area i, the higher the value of r_i.

If and when additional funds become available in the future to service agency 2, we denote by χ_{2r}, ψ_{2r}, χ_{2u}, and ψ_{2u} the four investment decisions that service agency 2 makes using the additional funds on top of the initial respective investments X_{2r}, Ψ_{2r}, X_{2u}, and Ψ_{2u}. Then, the overall social impact generated by service agency 2 using the initial and additional funds, $v_2(\cdot)$, can be revised as:

$$
\begin{aligned}
&v_2\left(X_{2r}, \Psi_{2r}, X_{2u}, \Psi_{2u}, \chi_{2r}, \psi_{2r}, \chi_{2u}, \psi_{2u}\right) = \\
&\qquad q_2 r_2 \sqrt{\alpha\left(X_{2r} + \chi_{2r}\right)} + q_2\,(1-r_2)\sqrt{\alpha\left(X_{2u} + \chi_{2u}\right)} \\
&\quad + (1-q_2)\, r_2 \left(\rho\sqrt{\beta_H\left(\Psi_{2r} + \psi_{2r}\right)} + (1-\rho)\sqrt{\beta_L\left(\Psi_{2r} + \psi_{2r}\right)}\right) \\
&\quad + (1-q_2)\,(1-r_2)\left(\rho\sqrt{\beta_H\left(\Psi_{2u} + \psi_{2u}\right)} + (1-\rho)\sqrt{\beta_L\left(\Psi_{2u} + \psi_{2u}\right)}\right).
\end{aligned}
$$

Following a similar flow of logic as in Section 4.1, we solve the optimization problem outlined above to characterize the optimal levels of initial funds allocated to each service agency (i.e., B_1^* and B_2^*) in Lemma B.9 below.

Lemma B.9 *Consider there is a regional asymmetry within each service area. Then, the funding agency's optimal levels of initial funds to service agencies 1 and 2* (B_1^*, B_2^*)

are: $B_1^* = \frac{\sqrt{N_r\left(4FN_r(F+f)\theta_2^2 - f^2\right)} + (2F+f)\theta_2 N_r}{4\theta_2 N_r\left(\theta_2^2 N_r + 1\right)}$ *and* $B_2^* = F - B_1^*$*, where* $N_r \doteq \left(\frac{r_1^2+(1-r_2)^2}{r_2^2+(1-r_2)^2}\right)$ $\left(\frac{\alpha q_1^2 + \left(\rho\sqrt{\beta_H} + (1-\rho)\sqrt{\beta_L}\right)^2 (1-q_1)^2}{\alpha q_2^2 + \left(\rho\sqrt{\beta_H} + (1-\rho)\sqrt{\beta_L}\right)^2 (1-q_2)^2}\right) > 0$

Next, Lemma B.10 presents effects of several contextual factors, including r_i, q_i, ρ, and α, on the optimal level of initial funds allocated to service agency $i \in \{1, 2\}$ under the efficiency-focused funding method.

Lemma B.10 *Consider* B_1^* *and* B_2^* *as presented in Lemma B.9. Then, for* $i \in \{1, 2\}$*:*
(i) With Respect To Area-specific factors. B_1^* *increases in* r_i *and* B_{3-i}^* *decreases in* r_i *if* $r_i < \frac{1}{2}$*, and* B_1^* *decreases in* q_i *and* B_{3-i}^* *increases in* r_i *otherwise. Further, denote threshold* $\dot{q}$ *as characterized in the proof. Then,* B_1^* *increases in* q_i *and* B_{3-i}^* *decreases in* q_i *if* $q_i < \dot{q}$*, and* B_1^* *decreases in* q_i *and* B_{3-i}^* *increases in* q_i *otherwise.*
(ii) With Respect To Activity-specific factors. B_1^* *decreases in* ρ *and* B_{3-i}^* *increases in* ρ *if* $q_i < q_{3-i}$*, and* B_1^* *increases in* ρ *and* B_{3-i}^* *decreases in* ρ *otherwise. Further,* B_i^* *increases in* α *and* B_{3-i}^* *decreases in* α *if* $q_i < q_{3-i}$*, and* B_i^* *decreases* α *and* B_{3-i}^* *increases* α *otherwise.*

Our analysis reveals that the insights presented in Section 4.2 (i.e., based on Proposition 4.1) continue to hold when there are region-specific differences in the mix of service providers within service areas. Further, we find that the effect of the scaling factor r_i is similar to the effect of the scaling factor q_i on the funding agency's optimal funds allocation decisions (which we formally show in Lemma B.10). Taken together, these two scaling factors–representing, the distribution of subsidy-accepting and non-accepting providers and the distribution of service providers across different regions–have a complementary effect on the optimal allocation strategy of the funding agency. Specifically, the funding agency should allocate more initial budget to the area with a relatively more balanced activity- and region-based service providers (i.e.,

when q_i and r_i are away from 0 or 1). The explanation is similar to that for the result stated in Proposition 4.1 in Section 4.2.

This model extension also allows us to conduct additional analysis on the impact of possibility of additional funds being sanctioned for a particular region within a particular service area (say, the rural region within the Western Massachusetts area under the child care subsidy voucher program in Massachusetts). Considering the setting where the additional funds f are expected to be offered only to area 2 for quality improvement and outreach in its region r, our analysis reveals the following interesting result. In this setting, the effect of the scaling factor r_i (region-specific) is not always similar to the effect of the scaling factor q_i (area-specific) on the funding agency's optimal funds allocation decisions. The different effect is observed when the amount of the expected additional funds is sufficiently large, in which the funding agency should decrease the optimal level of budget B_2^* to area 2 (from its initial pool of funds) when the region-specific scaling factor r_2 increases. This is driven by the joint effect of the following two forces: First, when r_2 is farther away from 0 and 1, the funding agency should increase B_2^* to provide ample amount of funds to the service agency 2 for distribution between a more balanced regional mix of service providers. Second, since the level of social impact generated in service area 2 using additional funds increases in r_2 (because the additional funds are restricted for use in the region r only), the funding agency should decrease B_2^* to provide more funds to the service agency 1 in order to ensure equity in the social impact generated across both areas. At high values of f, the latter force plays a dominant role and hence, B_2^* decreases in r_2; whereas, at low values of f, B_2^* first increases and then decreases in r_2.

Further, with regard to comparative analysis of levels of inequity and total expected social impact under different funding methods, we find that all the insights presented in Section 4.2.2 continue to hold in the presence of regional asymmetry within service areas.

B.4.6 Asymmetry In Quality Improvement Investment Return Rate

In this extension, we allow for the efficiency of investment in the activities across the two service areas to be different. In particular, we consider different investment return rates of quality improvement across service areas 1 and 2, which we denote by α_1 and α_2, respectively. The revised optimization problem under this setting can be obtained by changing α in the functions $V_i(\cdot)$ and $v_2(\cdot)$ in the funding agency's optimization problem (4.2)-(4.6) presented in Section 4.1 into $\alpha_{i\in\{1,2\}}$ accordingly. We solve this optimization problem to characterize the optimal levels of initial funds allocated to each service agency (i.e., B_1^* and B_2^*) in Lemma B.11 below.

Lemma B.11 *Consider asymmetry in the investment return rate of quality improvement. Then, the funding agency's optimal levels of initial funds to service agencies 1 and 2* (B_1^*, B_2^*) *are:* $B_1^* = \frac{\sqrt{N_q\left(4FN_q(F+f)\theta_2^2-f^2\right)}+(2F+f)\theta_2 N_q}{4\theta_2 N_q\left(\theta_2^2 N_q+1\right)}$ *and* $B_2^* = F - B_1^*$*, where* $N_q \doteq \frac{\alpha_1 q_1^2+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2(1-q_1)^2}{\alpha_2 q_2^2+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2(1-q_2)^2} > 0$.

Based on our analytical and numerical analyses, we report that all the main insights presented in Section 4.2 continue to hold in the presence of asymmetry in the investment return rate of quality improvement. We next present a few selected additional insights. First, consider an area with relatively more non-accepting service providers that are targeted by outreach activities. As the chance that outreach investment yields high return (ρ) increases, by Proposition 4.1(ii) presented in Section 4.2, the funding agency should decrease the optimal budget level to this area in order to balance the social impact generated between the two areas. However, we find that this may not hold true in the presence of asymmetry in the investment return rate of quality improvement. In particular, we find that when the quality improvement investment yields a relatively high return in this area, it may still be optimal for the funding agency to allocate more initial funds to this area even when ρ increases. This can be explained by a salient effect of diminishing returns at higher levels of investment in both quality improvement and outreach activities in this area.

Next, we find that as the investment return rate of quality improvement in areas 1 and 2 (i.e., α_1 and α_2) become quite different from each other, the difference in inequity outcomes under the formula-based method versus equity-ensuring method expands. Further, consider the area receiving additional funds has relatively more subsidy-accepting providers (i.e., q_2 is closer to 1). We find that $TSI^{Fo} - TSI^{Eq}$ decreases when $\frac{\alpha_2}{\alpha_1}$ takes a value closer to 1. This is because, when there is a low disparity in the investment return rate of quality improvement activities in the two service areas, the equity-ensuring funding agency is less likely to focus on balancing equity in the social impact generated across different areas; see similar explanations after Proposition 4.4 in Section 4.2. Alternatively, consider the area receiving additional funds has relatively few subsidy-accepting service providers (i.e., q_2 is closer to 0). We find that $TSI^{Fo} - TSI^{Eq}$ decreases when $\frac{\alpha_2}{\alpha_1}$ takes a low value. Overall, we conclude our proposed equity-ensuring method becomes less valuable when there is a large asymmetry in quality improvement investment return rate across different service areas, which is not the most likely situation in most subsidy welfare programs as the service agencies in different areas follow governments' standardized guideline to conduct quality improvement activities.

B.5 Proofs of Analytical Results

We denote the optimal level of initial funds that the funding agency allocates to service agency $i \in \{1, 2\}$ using its initial pool of funds (F) with equity and without equity considerations as B_i^* and B_i^{Ef}, respectively. For outlining proofs, without loss of generality, we normalize the volume adjustment factor for service area 1 as 1, i.e., $\theta_1 = 1$. Additional technical lemmas, which help us outline proofs of the propositions, are presented and proved in Appendix B.6.

Proof of Lemma 4.1: This proof uses Lemma B.12 presented in Appendix B.6.

(i) We solve the funding agency's optimization problem (4.2)-(4.6) presented in Sec-

tion 4.1 by backward induction. That is, the additional funds allocation problem (4.1) in Section 4.1 is solved first (for given X_2 and Ψ_2) followed by the funding and service agencies' initial funds allocation problems.

(i-a). Additional funds allocation problem: Since $v_2(\cdot)$ is a sum of a concave increasing function in χ_2 (which is independent of ψ_2) and a concave increasing function in ψ_2 (which is independent of χ_2), we have that $v_2(\cdot)$ is jointly concave in χ_2 and ψ_2. For a given χ_2, it can be seen that $v_2(\cdot)$ is increasing in ψ_2, which implies that the constraint in equation (4.1) binds. Substituting $\psi_2 = f - \chi_2$ in equation (4.1) and applying the first-order conditions with respect to χ_2 and ψ_2, we obtain χ_2^* and ψ_2^* as functions of X_2 and Ψ_2 as follows: $\chi_2^*(X_2, \Psi_2) = \frac{\alpha q_2^2(\Psi_2+f)-\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2(1-q_2)^2 X_2}{\alpha q_2^2+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2(1-q_2)^2}$ and $\psi_2^*(X_2, \Psi_2) = \frac{\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2(1-q_2)^2(f+X_2)-\alpha q_2^2\Psi_2}{\alpha q_2^2+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2(1-q_2)^2}$.

(i-b). Initial funds allocation problem: Under area-only additional funds, service agency i makes initial funds allocation decisions based on its initial budget B_i. Since $V_i(\cdot)$ is a sum of a concave increasing function in X_i (which is independent of Ψ_i) and a concave increasing function Ψ_i (which is independent of X_i), we have that $V_i(\cdot)$ is jointly concave in X_i and Ψ_i. For a given X_i, it can be seen that $V_i(\cdot)$ is increasing in Ψ_i, which implies that the constraint in equation (4.4) binds. Substituting $\Psi_i = B_i - X_i$ in equation (4.4) and applying the first-order conditions with respect to X_i and Ψ_i, we obtain the following expressions of X_i^* and Ψ_i^* as functions of B_i: $X_i^*(B_i) = \frac{\alpha q_i^2 B_i}{\alpha q_i^2+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2(1-q_i)^2}$ and $\Psi_i^*(B_i) = \frac{\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2(1-q_i)^2 B_i}{\alpha q_i^2+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2(1-q_i)^2}$.

Next, substituting X_1^* and Ψ_1^* in service agency 1's expected social impact function $V_1(\cdot)$, we have $V_1\left(X_1^*(B_1), \Psi_1^*(B_1)\right) = \sqrt{\alpha q_1^2 + \left(\rho\sqrt{\beta_H} + (1-\rho)\sqrt{\beta_L}\right)^2 (1-q_1)^2}\sqrt{B_1}$. After substituting $X_2^*(B_2)$ and $\Psi_2^*(B_2)$ in equations $\chi_2^*(X_2, \Psi_2)$ and $\psi_2^*(X_2, \Psi_2)$, we have: $\chi_2^* = \frac{\alpha q_2^2 f}{\alpha q_2^2+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2(1-q_2)^2}$ and $\Psi_2^* = \frac{\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2(1-q_2)^2 f}{\alpha q_2^2+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2(1-q_2)^2}$.

Substituting $p = \frac{1}{2}$, $X_2^*(B_2)$, $\Psi_2^*(B_2)$, χ_2^*, and ψ_2^* (using equations from above) in service agency 2's expected social impact function $(1-p)V_2(\cdot) + pv_2(\cdot)$, we have

$\frac{1}{2}\left(V_2\left(X_2^*(B_2),\Psi_2^*(B_2)\right)+v_2\left(X_2^*(B_2),\Psi_2^*(B_2),\chi_2^*,\psi_2^*\right)\right)$
$=\frac{1}{2}\sqrt{\alpha q_2^2+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2(1-q_2)^2}\left(\sqrt{B_2}+\sqrt{f+B_2}\right)$.

Since $V_1(B_1)$ and $\frac{1}{2}\left(V_2(B_2)+v_2(B_2)\right)$ are concave increasing functions in B_1 and B_2, respectively, and the funding agency aims to maximize the sum of $V_1(B_1)$ and $\frac{1}{2}\left(V_2(B_2)+v_2(B_2)\right)$, we can conclude that the constraint in equation (4.5) binds, i.e, $B_2=F-B_1$. Then, substituting $B_2=F-B_1$, $\theta_1=1$, and the expressions of $V_1\left(X_2^*(B_1),\Psi_2^*(B_1)\right)$, $V_2\left(X_2^*(B_1),\Psi_2^*(B_1)\right)$, and $v_2\left(X_2^*(B_1),\Psi_2^*(B_1),\chi_2^*,\psi_2^*\right)$ in $\frac{V_1(B_1)}{\theta_1}-\frac{\frac{1}{2}(V_2(B_1)+v_2(B_1))}{\theta_2}$, we have that a unique solution exists to the equation $\frac{V_1(B_1)}{\theta_1}-\frac{\frac{1}{2}(V_2(B_1)+v_2(B_1))}{\theta_2}=0$ if and only if $f<\dot{f}$ (by Lemma B.12).

Then, considering $f<\dot{f}$, we obtain that $\hat{B}_1\doteq\frac{\sqrt{N_1\left(4FN_1(F+f)\theta_2^2-f^2\right)}+(2F+f)\theta_2N_1}{4\theta_2N_1\left(\theta_2^2N_1+1\right)}$ is that unique solution, where $N_1\doteq\frac{\alpha q_1^2+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2(1-q_1)^2}{\alpha q_2^2+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2(1-q_2)^2}>0$. Denote $\hat{B}_2\doteq F-\hat{B}_1$. Since the obtained vector $\left\{\hat{B}_1,\hat{B}_2\right\}$ is the unique solution, it implies that $B_1^*=\hat{B}_1$ and $B_2^*=\hat{B}_2$ are the unique solutions of the optimization problem in equations (4.2)-(4.6).

(ii) Differentiating B_1^* as characterized in the statement of Lemma 4.1 in Section 4.2 with respect to f, we obtain $\frac{\partial}{\partial f}B_1^*(f)=\frac{H_1(f)}{4\left(\theta_2^2N_1+1\right)\sqrt{\theta_2^2N_1\left(4F(F+f)\theta_2^2N_1-f^2\right)}}$, where $H_1(f)\doteq 2F\theta_2^2N_1-f+\sqrt{\theta_2^2N_1\left(4F\left(F+f\right)\theta_2^2N_1-f^2\right)}$. Since $\theta_2>0$ and $N_1>0$, we have that the sign of $\frac{\partial}{\partial f}B_1^*(f)$ depends on the sign of $H_1(f)$. Differentiating $H_1(f)$ with respect to f, it follows that $H_1(f)$ decreases in f if and only if $f<4\theta_2^2N_1F$. Given that $f<4\theta_2^2N_1F$, we have that $H_1(f)$ decreases in f. Further, since $H_1(f)=4\theta_2^2N_1F$ when $f=0$ and $H_1(f)=0$ when $f=4\theta_2^2N_1F$, we have that $H_1(f)>0$ for any $0\le f<4\theta_2^2N_1F$. Hence, we have $\frac{\partial}{\partial f}B_1^*(f)>0$. □

Proof of Proposition 4.1: This proof uses Lemma 4.1 in Section 4.2 and Lemma B.12 presented in Appendix B.6. Per Lemma B.12, we consider $0\le f<4\theta_2^2N_1F$ to ensure perfect equity, where $N_1=\frac{\alpha q_1^2+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2(1-q_1)^2}{\alpha q_2^2+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2(1-q_2)^2}>0$.
(i) We outline the proof in two steps as follows. First, we show $B_1^*(N_1)$ decreases in

N_1. Differentiating B_1^* as characterized in the statement of Lemma B.12 with respect to N_1, we have $\frac{\partial}{\partial N_1} B_1^*(N_1) = \frac{H_2(N_1)}{8N_1\left(N_1\theta_2^2+1\right)^2\sqrt{N_1\theta_2^2\left(4FN_1(F+f)\theta_2^2-f^2\right)}}$, where $H_2(N_1) \doteq -2(2F+f)N_1\theta_2^2\sqrt{N_1\theta_2^2(4FN_1(F+f)\theta_2^2-f^2)} + (N_1\theta_2^2+1)\,f^2 - 2N_1\theta_2^2(4FN_1(F+f)\theta_2^2-f^2)$. Given that $\theta_2, F, N_1 > 0$ and $f \geq 0$, the sign of $\frac{\partial}{\partial N_1} B_1^*(N_1)$ depends on the sign of $H_2(N_1)$. Differentiating $H_2(N_1)$ with respect to N_1, we have $\frac{\partial}{\partial N_1} H_2(N_1) = -\frac{\theta_2^2\left(f^2+4\left(4FN_1(F+f)\theta_2^2-f^2\right)\right)\left((2F+f)N_1\theta_2^2+\sqrt{N_1\theta_2^2\left(4FN_1(F+f)\theta_2^2-f^2\right)}\right)}{\sqrt{N_1\theta_2^2\left(4FN_1(F+f)\theta_2^2-f^2\right)}} < 0$, where the inequality holds because $F, f, N_1, \theta_2 > 0$, and $(4F\,(F+f)\,\theta_2^2 N_1 - f^2) > 0$. Further, since $f < 4\theta_2^2 N_1 F$, we must have $N_1 > \frac{f}{4F\theta_2^2}$. Substituting $N_1 = \frac{f}{4F\theta_2^2}$ in $H_2(N_1)$, we have $H_2(N_1) = 0$. This implies that $H_2(N_1) < 0$ for any $N_1 > \frac{f}{4F\theta_2^2}$. Therefore, we have $\frac{\partial}{\partial N_1} B_1^*(N_1) < 0$.

Second, we show that there is a unique threshold $\hat{q}$, such that N_1 decreases in q_1 when $q_1 \leq \hat{q}$ and N_1 increases in q_1 otherwise. Differentiating N_1 with respect to q_1, we have $\frac{\partial}{\partial q_1} N_1(q_1) = 2\left(\frac{\alpha+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2}{\alpha q_2^2+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2(1-q_2)^2}\right)\left(q_1 - \frac{\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2}{\alpha+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2}\right)$. Since $\alpha, \beta_L, \beta_H > 0$, $0 \leq \rho \leq 1$, $0 < q_1 < 1$, and $0 < q_2 < 1$, we have that the sign of $\frac{\partial}{\partial q_1} N_1(q_1)$ depends on the sign of $\left(q_1 - \frac{\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2}{\alpha+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2}\right)$. Denoting $\hat{q} \doteq \frac{\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2}{\alpha+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2}$, we have that $\frac{\partial}{\partial q_1} N_1(q_1) < 0$ when $q_1 < \hat{q}$ and $\frac{\partial}{\partial q_1} N_1(q_1) \geq 0$ otherwise.

Combining the aforementioned results–specifically, $B_1^*(N_1)$ decreases in N_1 and N_1 decreases in q_1 when $0 < q_1 < \hat{q}$–we can conclude that B_1^* increases in q_1 when $0 < q_1 < \hat{q}$. Also, we have that B_1^* decreases in q_1 when $\hat{q} \leq q_1 < 1$. Further, since $B_2^* = F - B_1^*$ (because the constraint given by equation (4.5) in Section 4.1 binds; see Lemma 4.1), we have that B_2^* decreases in q_1 when $0 < q_1 < \hat{q}$ and increases in q_1 when $\hat{q} \leq q_1 < 1$. Proof of how B_1^* and B_2^* change with respect to q_2 follows similar steps, and therefore, we omit it for brevity.

(ii) Following similar steps as in the proof of Proposition 4.1(i), we have that $B_1^*(N_1)$ decreases in N_1, where $N_1 = \frac{\alpha q_1^2+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2(1-q_1)^2}{\alpha q_2^2+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2(1-q_2)^2} > 0$. Next, we show that N_1 increases in ρ when $q_1 < q_2$ and N_1 decreases in ρ otherwise. Differentiating

N_1 with respect to ρ, we have $\frac{\partial}{\partial\rho}N_1(\rho) = \frac{2\alpha\eta_4\left(\sqrt{\beta_H}-\sqrt{\beta_L}\right)\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)}{\left(\alpha q_2^2+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2(1-q_2)^2\right)^2}$, where $\eta_4 \doteq (q_1-q_2)(2q_1q_2-q_1-q_2)$. Since $\alpha > 0$, $0 \le \rho \le 1$, and $\beta_H \ge \beta_L > 0$, we have that the sign of $\frac{\partial}{\partial\rho}N_1(\rho)$ depends on the sign of η_4. Differentiating η_4 with respect to q_1, we have $\frac{\partial}{\partial q_1}\eta_4(q_1) = -2(q_1-q_2)^2 - 2q_1(1-q_1) < 0$, where the inequality holds because $0 < q_1 < 1$. Solving $\eta_4(q_1) = 0$, we have $q_1 = q_2$. Therefore, we can conclude that $\eta_4(q_1) > 0$ when $q_1 < q_2$ and $\eta_4(q_1) \le 0$ otherwise. This implies that $\frac{\partial}{\partial\rho}N_1(\rho) > 0$ when $0 < q_1 < q_2$ and $\frac{\partial}{\partial\rho}N_1(\rho) \le 0$ otherwise.

Combining the aforementioned results–specifically, $B_1^*(N_1)$ decreases in N_1 and N_1 increases in ρ when $q_1 < q_2$–we can conclude that B_1^* increases in ρ when $q_1 < q_2$. In similar vein, we can also conclude that B_1^* decreases in ρ when $q_1 \ge q_2$. Further, since $B_2^* = F - B_1^*$ (because the constraint given by equation (4.5) binds; see Lemma 4.1), we have that B_2^* increases in ρ when $q_1 < q_2$ and decreases in ρ otherwise.

(iii) Differentiating N_1 with respect to α, we have $\frac{\partial}{\partial\alpha}N_1(\alpha) = -\frac{\eta_4\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2}{\left(\alpha q_2^2+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2(1-q_2)^2\right)^2}$, where η_4 is defined in (ii). Following similar steps as (ii), we can prove for how B_i^* changes with respect to α. Therefore, we omit it for brevity. □

Proof of Proposition 4.2: This proof uses Lemma B.3 presented in Appendix B.2. First, we show $I^{Ef} - I^{Eq}$ decreases in θ_2 if $\theta_2 < \hat{\theta}_2$ and increases in θ_2 otherwise. Differentiating $\frac{V_1^{Ef}(\theta_2)}{\theta_1} - \frac{\frac{1}{2}\left(V_2^{Ef}(\theta_2)+v_2^{Ef}(\theta_2)\right)}{\theta_2}$ (as characterized in the proof of Lemma B.3) with respect to θ_2, we have $\frac{\partial}{\partial\theta_2}\left(\frac{V_1^{Ef}(\theta_2)}{\theta_1} - \frac{\frac{1}{2}\left(V_2^{Ef}(\theta_2)+v_2^{Ef}(\theta_2)\right)}{\theta_2}\right) = \frac{1}{\theta_2^2}\sqrt{\frac{F}{N_1+1}}\sqrt{\alpha q_2^2 + \left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2(1-q_2)^2} > 0$, where the inequality holds because $F, N_1, \beta_L, \beta_H, \theta_2 > 0$, $0 \le \rho \le 1$, and $0 < q_2 < 1$.

Further, we have $\frac{\partial}{\partial\theta_2}\left(\frac{V_1^{Ef}(\theta_2)}{\theta_1} - \frac{\frac{1}{2}\left(V_2^{Ef}(\theta_2)+v_2^{Ef}(\theta_2)\right)}{\theta_2}\right) = -\infty$ when $\theta_2 = 0$ and $\frac{\partial}{\partial\theta_2}\left(\frac{V_1^{Ef}(\theta_2)}{\theta_1} - \frac{\frac{1}{2}\left(V_2^{Ef}(\theta_2)+v_2^{Ef}(\theta_2)\right)}{\theta_2}\right) = \sqrt{\frac{N_1F\left(\alpha q_1^2+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2(1-q_1)^2\right)}{N_1+1}} > 0$ when $\theta_2 = +\infty$. Thus, there exists a unique threshold of θ_2 denoted by $\hat{\theta}_2$, such that $\frac{\partial}{\partial\theta_2}\left(\frac{V_1^{Ef}(\theta_2)}{\theta_1} -\right.$

$\frac{\frac{1}{2}\left(V_2^{Ef}(\theta_2)+v_2^{Ef}(\theta_2)\right)}{\theta_2}\Bigg) > 0$ when $\theta_2 < \hat{\theta}_2$ and $\frac{\partial}{\partial\theta_2}\left(\frac{V_1^{Ef}(\theta_2)}{\theta_1} - \frac{\frac{1}{2}\left(V_2^{Ef}(\theta_2)+v_2^{Ef}(\theta_2)\right)}{\theta_2}\right) \leq 0$ otherwise.

Solving $\frac{\partial}{\partial\theta_2}\left(\frac{V_1^{Ef}(\theta_2)}{\theta_1} - \frac{\frac{1}{2}\left(V_2^{Ef}(\theta_2)+v_2^{Ef}(\theta_2)\right)}{\theta_2}\right) = 0$, we have $\hat{\theta}_2 = \frac{1}{N_1}$, where N_1 is characterized in the statement of Lemma 4.1 in Section 4.2. Above all, we conclude that $I^{Ef} = \left|\frac{V_1^{Ef}(\theta_2)}{\theta_1} - \frac{\frac{1}{2}\left(V_2^{Ef}(\theta_2)+v_2^{Ef}(\theta_2)\right)}{\theta_2}\right| > 0$ decreases in θ_2 if $\theta_2 < \hat{\theta}_2$ and increases in θ_2 otherwise. Given that $I^{Eq} = 0$ (by equation (4.3) in Section 4.1), we have $I^{Ef} - I^{Eq} > 0$ decreases in θ_2 if $\theta_2 < \hat{\theta}_2$ and increases in θ_2 otherwise.

Second, we show $TSI^{Ef} - TSI^{Eq}$ decreases in θ_2 if $\theta_2 < \hat{\theta}_2$ and increases in θ_2 otherwise. Differentiating $TSI^{Ef} - TSI^{Eq} > 0$ (as characterized in the statement of Lemma B.3) with respect to θ_2, we have $\frac{\partial}{\partial\theta_2}\left(TSI^{Ef}(\theta_2) - TSI^{Eq}(\theta_2)\right) = \frac{2FN_1(\theta_2+1)(N_1\theta_2-1)}{\left(N_1\theta_2^2+1\right)^2}$, where N_1 is characterized in the statement of Lemma 4.1. Given that $F, N_1, \theta_2 > 0$, the sign of $\frac{\partial}{\partial\theta_2}\left(TSI^{Ef}(\theta_2) - TSI^{Eq}(\theta_2)\right)$ depends on the sign of $N_1\theta_2 - 1$. It follows $\frac{\partial}{\partial\theta_2}\left(TSI^{Ef}(\theta_2) - TSI^{Eq}(\theta_2)\right) < 0$ if $\theta_2 < \hat{\theta}_2 = \frac{1}{N_1}$ and $\frac{\partial}{\partial\theta_2}\left(TSI^{Ef}(\theta_2) - TSI^{Eq}(\theta_2)\right) \geq 0$ otherwise.

Last, in the proof of Proposition 4.1, we show that $\frac{\partial}{\partial q_1}N_1(q_1) < 0$ when $q_1 < \hat{q} = \frac{\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2}{\alpha+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2}$ and $\frac{\partial}{\partial q_1}N_1(q_1) \geq 0$ otherwise. Therefore, $\frac{\partial\hat{\theta}_2}{\partial q_1} = \frac{\partial}{\partial q_1}\left(\frac{1}{N_1(q_1)}\right) > 0$ when $q_1 < \hat{q}$ and $\frac{\partial\hat{\theta}_2}{\partial q_1} = \frac{\partial}{\partial q_1}\left(\frac{1}{N_1(q_1)}\right) \leq 0$ otherwise. Proof of how $\hat{\theta}_2$ changes with respect to q_2 follows similar steps, and therefore, we omit it for brevity. □

Proof of Lemma 4.2: This proof uses Lemma B.3 in Appendix B.2 and Lemmas B.13 and B.14 in Appendix B.6.

First, we show when θ_2 is sufficiently small (large) compared with θ_1, $I^{Ef} - I^{Eq} > 0$ first decreases (increases) and then increases (decreases) in q_2. Differentiating $\frac{V_1^{Ef}(q_2)}{\theta_1} - \frac{\frac{1}{2}\left(V_2^{Ef}(q_2)+v_2^{Ef}(q_2)\right)}{\theta_2}$ (as characterized in the proof of Lemma B.3) with respect to q_2, we have $\frac{\partial}{\partial q_2}\left(\frac{V_1^{Ef}(q_2)}{\theta_1} - \frac{\frac{1}{2}\left(V_2^{Ef}(q_2)+v_2^{Ef}(q_2)\right)}{\theta_2}\right) = \frac{\sqrt{F}\left(\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2(1-q_2)-\alpha q_2\right)}{\sqrt{(1+N_1)\left(\alpha q_2^2+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2(1-q_2)^2\right)}}$, where N_1 is characterized in the statement of Lemma 4.1. Since $F, N_1, \alpha, \beta_L, \beta_H > 0$,

$0 \leq \rho \leq 1$, and $0 < q_2 < 1$, we conclude the sign of $\frac{\partial}{\partial q_2}\left(\frac{V_1^{Ef}(q_2)}{\theta_1} - \frac{\frac{1}{2}\left(V_2^{Ef}(q_2)+v_2^{Ef}(q_2)\right)}{\theta_2}\right)$ depends on the sign of $\left(\rho\sqrt{\beta_H} + (1-\rho)\sqrt{\beta_L}\right)^2(1-q_2) - \alpha q_2$.

It follows $(\rho\sqrt{\beta_H} + (1-\rho)\sqrt{\beta_L})^2(1-q_2) - \alpha q_2 \geq 0$ when $q_2 \leq \frac{\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2}{\alpha+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2}$ and $\left(\rho\sqrt{\beta_H} + (1-\rho)\sqrt{\beta_L}\right)^2(1-q_2) - \alpha q_2 < 0$ otherwise. Therefore, we conclude $\frac{V_1^{Ef}(q_2)}{\theta_1} - \frac{\frac{1}{2}\left(V_2^{Ef}(q_2)+v_2^{Ef}(q_2)\right)}{\theta_2}$ increases in q_2 when $q_2 \leq \frac{\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2}{\alpha+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2}$ and decreases in q_2 otherwise. Next, we consider the following two cases (I-a and I-b) based on the magnitude of θ_2.

Case I-a: Suppose $\theta_2 < \underline{\theta}_2$, where $\underline{\theta}_2$ is characterized in the proof of Lemma B.13 in Appendix B.6. By Lemma B.13, we have $\frac{V_1^{Ef}(q_2)}{\theta_1} - \frac{\frac{1}{2}\left(V_2^{Ef}(q_2)+v_2^{Ef}(q_2)\right)}{\theta_2} < 0$ for $0 < q_2 < 1$. Given that $I^{Eq} = 0$ (by equation (4.3) in Section 4.1), we have $I^{Ef}(q_2) - I^{Eq}(q_2) = \left|\frac{V_1^{Ef}(q_2)}{\theta_1} - \frac{\frac{1}{2}\left(V_2^{Ef}(q_2)+v_2^{Ef}(q_2)\right)}{\theta_2}\right| > 0$ first decreases and then increases in q_2.

Case I-b: Suppose $\theta_2 > \dddot{\theta}_2$, where $\dddot{\theta}_2$ is characterized in the proof of Lemma B.13 in Appendix B.6. By Lemma B.13, we have $\frac{V_1^{Ef}(q_2)}{\theta_1} - \frac{\frac{1}{2}\left(V_2^{Ef}(q_2)+v_2^{Ef}(q_2)\right)}{\theta_2} > 0$ for $0 < q_2 < 1$. Given $I^{Eq} = 0$ (by equation (4.3)), we have $I^{Ef}(q_2) - I^{Eq}(q_2) = \left|\frac{V_1^{Ef}(q_2)}{\theta_1} - \frac{\frac{1}{2}\left(V_2^{Ef}(q_2)+v_2^{Ef}(q_2)\right)}{\theta_2}\right| > 0$ first increases and then decreases in q_2.

The proof for how $I^{Ef} - I^{Eq}$ with respect to q_1 follow similar steps, and therefore, we omit it for brevity.

Second, we show when θ_2 is sufficiently small (large) compared with θ_1, $TSI^{Ef} - TSI^{Eq} > 0$ first decreases (increases) and then increases (decreases) in q_2. Differentiating $TSI^{Ef} - TSI^{Eq}$ (as characterized in the statement of Lemma B.3) with respect to q_2, we have $\frac{\partial}{\partial q_2}\left(TSI^{Ef}(q_2) - TSI^{Eq}(q_2)\right) = \frac{F\left(\alpha q_2 - \left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2(1-q_2)\right)}{\sqrt{\alpha q_2^2 + \left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2(1-q_2)^2}}\left(\frac{1}{N_1+1} - \frac{N_1^2\theta_2^2(\theta_2+1)}{\sqrt{N_1}\left(N_1\theta_2^2+1\right)^{\frac{3}{2}}}\right)$, where N_1 is characterized in the statement of Lemma 4.1 presented in Section 4.2.

Since $F, N_1, \alpha, \beta_L, \beta_H > 0$, $0 \leq \rho \leq 1$, and $0 < q_2 < 1$, the sign of $\frac{\partial}{\partial q_2}\left(TSI^{Ef}(q_2) - TSI^{Eq}(q_2)\right)$ depends on the sign of $\left(\alpha q_2 - \left(\rho\sqrt{\beta_H} + (1-\rho)\sqrt{\beta_L}\right)^2(1-q_2)\right)\left(\frac{1}{N_1+1} -\right.$

$\frac{N_1^2\theta_2^2(\theta_2+1)}{\sqrt{N_1}\left(N_1\theta_2^2+1\right)^{\frac{3}{2}}}\Bigg)$. The sign of $\alpha q_2 - \left(\rho\sqrt{\beta_H} + (1-\rho)\sqrt{\beta_L}\right)^2(1-q_2)$ depends on the value of q_2, which follows $\alpha q_2 - \left(\rho\sqrt{\beta_H} + (1-\rho)\sqrt{\beta_L}\right)^2(1-q_2) \leq 0$ when $q_2 \leq \frac{\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2}{\alpha+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2}$, and $\alpha q_2 - \left(\rho\sqrt{\beta_H} + (1-\rho)\sqrt{\beta_L}\right)^2(1-q_2) > 0$ otherwise. Next, we consider the following two cases (II-a and II-b) based on the magnitude of θ_2.

Case II-a: Suppose $\theta_2 \leq \bar{\bar{\bar{\theta}}}_2$, where the existence of $\bar{\bar{\bar{\theta}}}_2$ is shown in the proof of Lemma B.14 in Appendix B.6. By Lemma B.14, we have $\frac{1}{N_1+1} - \frac{N_1^2\theta_2^2(\theta_2+1)}{\sqrt{N_1}\left(N_1\theta_2^2+1\right)^{\frac{3}{2}}} \geq 0$. Given $TSI^{Ef}(q_2) - TSI^{Eq}(q_2) > 0$ (as explained in the proof of Lemma B.3) and $\alpha q_2 - \left(\rho\sqrt{\beta_H} + (1-\rho)\sqrt{\beta_L}\right)^2(1-q_2)$ first decreases and then increases in q_2, we can conclude $TSI^{Ef}(q_2) - TSI^{Eq}(q_2) > 0$ first decreases and then increases in q_2.

Case II-b: Suppose $\theta_2 > \bar{\bar{\bar{\theta}}}_2$. Following similar steps as in the case II-a, we can conclude $TSI^{Ef}(q_2) - TSI^{Eq}(q_2) > 0$ first increases and then decreases in q_2.

The proof for how $TSI^{Ef} - TSI^{Eq}$ with respect to q_1 follow similar steps. Thus, we omit it for brevity. □

Proof of Proposition 4.3: This proof uses Lemma B.4 in Appendix B.2 and Lemmas B.15 and B.16 in Appendix B.6. First, we show $\frac{V_1^{Fo}(q_1)}{\theta_1} - \frac{\frac{1}{2}\left(V_2^{Fo}(q_1)+v_2^{Fo}(q_1)\right)}{\theta_2}$ first decreases in q_1 and then increases in q_1. Substituting $f = 0$ in the expression of $\frac{V_1^{Fo}(\cdot)}{\theta_1} - \frac{\frac{1}{2}\left(V_2^{Fo}(\cdot)+v_2^{Fo}(\cdot)\right)}{\theta_2}$ characterized in the proof of Lemma B.4 (i) and differentiating it with respect to q_1, we have $\frac{\partial}{\partial q_1}\left(\frac{V_1^{Fo}(q_1)}{\theta_1} - \frac{\frac{1}{2}\left(V_2^{Fo}(q_1)+v_2^{Fo}(q_1)\right)}{\theta_2}\right) = \frac{3}{2}\sqrt{\frac{F}{1+\theta_2}}\left(\sqrt{\alpha}\sqrt{q_1} - \left(\rho\sqrt{\beta_H} + (1-\rho)\sqrt{\beta_L}\right)\sqrt{1-q_1}\right)$, which sign depends on the sign of $\sqrt{\alpha}\sqrt{q_1} - \left(\rho\sqrt{\beta_H} + (1-\rho)\sqrt{\beta_L}\right)\sqrt{1-q_1}$.
Since $\frac{\partial}{\partial q_1}\left(\sqrt{\alpha}\sqrt{q_1} - \left(\rho\sqrt{\beta_H} + (1-\rho)\sqrt{\beta_L}\right)\sqrt{1-q_1}\right) = \frac{\sqrt{\alpha}\sqrt{1-q_1}+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)\sqrt{q_1}}{2\sqrt{q_1(1-q_1)}} > 0$, $\sqrt{\alpha}\sqrt{q_1} - \left(\rho\sqrt{\beta_H} + (1-\rho)\sqrt{\beta_L}\right)\sqrt{1-q_1} = -\left(\rho\sqrt{\beta_H} + (1-\rho)\sqrt{\beta_L}\right) < 0$ when $q_1 = 0$, and $\sqrt{\alpha}\sqrt{q_1} - \left(\rho\sqrt{\beta_H} + (1-\rho)\sqrt{\beta_L}\right)\sqrt{1-q_1} = \sqrt{\alpha} > 0$ when $q_1 = 1$, we conclude there exists a unique threshold of q_1, denoted by $\bar{q}_1$, such that $\frac{\partial}{\partial q_1}\Bigg(\frac{V_1^{Fo}(q_1)}{\theta_1} -$

$\frac{\frac{1}{2}\left(V_2^{Fo}(q_1)+v_2^{Fo}(q_1)\right)}{\theta_2}\Bigg) \leq 0$ when $0 < q_1 \leq \bar{q}_1$ and $\frac{\partial}{\partial q_1}\left(\frac{V_1^{Fo}(q_1)}{\theta_1} - \frac{\frac{1}{2}\left(V_2^{Fo}(q_1)+v_2^{Fo}(q_1)\right)}{\theta_2}\right) > 0$ otherwise.

Solving $\frac{\partial}{\partial q_1}\left(\frac{V_1^{Fo}(q_1)}{\theta_1} - \frac{\frac{1}{2}\left(V_2^{Fo}(q_1)+v_2^{Fo}(q_1)\right)}{\theta_2}\right) = 0$, we have $\bar{q}_1 = \frac{\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2}{\alpha+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2}$. Next, consider the following two cases (cases I and II) based on the magnitude of α.

Case I: Suppose $\alpha > \left(\rho\sqrt{\beta_H} + (1-\rho)\sqrt{\beta_L}\right)^2$. By Lemma B.16, we have $\tilde{\tilde{\theta}}_2 < \tilde{\theta}_2 < \dot{\theta}_2$. We consider the following three sub-cases (cases I-a, I-b, and I-c) based on the magnitude of θ_2.

(I-a) Suppose $\theta_2 \leq \tilde{\tilde{\theta}}_2$. By Lemma B.15, we have $\frac{V_1^{Fo}(q_1)}{\theta_1} - \frac{\frac{1}{2}\left(V_2^{Fo}(q_1)+v_2^{Fo}(q_1)\right)}{\theta_2} \leq 0$ when $q_1 = 0$, $q_1 = \frac{\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2}{\alpha+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2}$, and $q_1 = 1$. Hence, $\frac{V_1^{Fo}(q_1)}{\theta_1} - \frac{\frac{1}{2}\left(V_2^{Fo}(q_1)+v_2^{Fo}(q_1)\right)}{\theta_2} < 0$ for any $0 < q_1 < 1$. This implies $I^{Fo}(q_1) = \left|\frac{V_1^{Fo}(q_1)}{\theta_1} - \frac{\frac{1}{2}\left(V_2^{Fo}(q_1)+v_2^{Fo}(q_1)\right)}{\theta_2}\right| \geq 0$ increases in q_1 if $q_1 \leq \bar{q}_1$ and decreases in q_1 otherwise.

(I-b) Suppose $\theta_2 \geq \dot{\theta}_2$. By Lemma B.15, we have $\frac{V_1^{Fo}(q_1)}{\theta_1} - \frac{\frac{1}{2}\left(V_2^{Fo}(q_1)+v_2^{Fo}(q_1)\right)}{\theta_2} \geq 0$ when $q_1 = 0$, $q_1 = \bar{q}_1$, and $q_1 = 1$. Similar to case (I-a), we can conclude $I^{Fo}(q_1) \geq 0$ decreases in q_1 if $q_1 \leq \bar{q}_1$ and increases in q_1 otherwise.

(I-c) Suppose $\tilde{\tilde{\theta}}_2 < \theta_2 < \dot{\theta}_2$. By Lemma B.15, we have: (i) $\frac{V_1^{Fo}(q_1)}{\theta_1} - \frac{\frac{1}{2}\left(V_2^{Fo}(q_1)+v_2(q_1)\right)}{\theta_2} \leq 0$ if $\tilde{\tilde{\theta}}_2 < \theta_2 \leq \tilde{\theta}_2$, and $\frac{V_1^{Fo}(q_1)}{\theta_1} - \frac{\frac{1}{2}\left(V_2^{Fo}(q_1)+v_2^{Fo}(q_1)\right)}{\theta_2} > 0$ if $\tilde{\theta}_2 < \theta_2 < \dot{\theta}_2$ when $q_1 = 0$; (ii) $\frac{V_1^{Fo}(q_1)}{\theta_1} - \frac{\frac{1}{2}\left(V_2^{Fo}(q_1)+v_2^{Fo}(q_1)\right)}{\theta_2} < 0$ when $q_1 = \bar{q}_1$; and (iii) $\frac{V_1^{Fo}(q_1)}{\theta_1} - \frac{\frac{1}{2}\left(V_2^{Fo}(q_1)+v_2^{Fo}(q_1)\right)}{\theta_2} > 0$ when $q_1 = 1$. Solving $\frac{V_1^{Fo}(q_1)}{\theta_1} - \frac{\frac{1}{2}\left(V_2^{Fo}(q_1)+v_2^{Fo}(q_1)\right)}{\theta_2} = 0$, we have two solutions of q_1, denoted by $\check{q}_1$ and $\check{\check{q}}_1$, where $\check{q}_1 < \check{\check{q}}_1$. We conclude $\frac{V_1^{Fo}(q_1)}{\theta_1} - \frac{\frac{1}{2}\left(V_2^{Fo}(q_1)+v_2^{Fo}(q_1)\right)}{\theta_2} > 0$ decreases in q_1 when $0 < q_1 < \max\{0, \check{q}_1\}$; $\frac{V_1^{Fo}(q_1)}{\theta_1} - \frac{\frac{1}{2}\left(V_2^{Fo}(q_1)+v_2^{Fo}(q_1)\right)}{\theta_2} \leq 0$ increases in q_1 when $\max\{0, \check{q}_1\} \leq q_1 \leq \bar{q}_1$; $\frac{V_1^{Fo}(q_1)}{\theta_1} - \frac{\frac{1}{2}\left(V_2^{Fo}(q_1)+v_2^{Fo}(q_1)\right)}{\theta_2} \leq 0$ decreases in q_1 when $\bar{q}_1 < q_1 \leq \check{\check{q}}_1$; and $\frac{V_1^{Fo}(q_1)}{\theta_1} - \frac{\frac{1}{2}\left(V_2^{Fo}(q_1)+v_2^{Fo}(q_1)\right)}{\theta_2} > 0$ increases in q_1 when $\check{\check{q}}_1 < q_1 < 1$. This implies that: $I^{Fo}(q_1) = \left|\frac{V_1^{Fo}(q_1)}{\theta_1} - \frac{\frac{1}{2}\left(V_2^{Fo}(q_1)+v_2^{Fo}(q_1)\right)}{\theta_2}\right| \geq 0$ decreases in q_1 when $0 < q_1 < \max\{0, \check{q}_1\}$; I^{Fo} increases in q_1 when $\max\{0, \check{q}_1\} \leq q_1 \leq \bar{q}_1$; I^{Fo} decreases in q_1 when $\bar{q}_1 < q_1 \leq \check{\check{q}}_1$; and I^{Fo} increases in q_1 when $\check{\check{q}}_1 < q_1 < 1$.

Case II: Suppose $\alpha \leq \left(\rho\sqrt{\beta_H} + (1-\rho)\sqrt{\beta_L}\right)^2$. By Lemma B.16, we have $\tilde{\theta}_2 \leq \tilde{\tilde{\theta}}_2 < \dot{\theta}_2$. Similar to Case I, we next consider the following three sub-cases based on

the magnitude of θ_2. (II-a) Suppose $\theta_2 \leq \tilde{\tilde{\theta}}_2$. The results are similar to case (I-a) above. We omit for brevity.

(II-b) Suppose $\theta_2 \geq \dot{\theta}_2$. The results are similar to case (I-b) above. We omit for brevity.

(II-c) Suppose $\tilde{\tilde{\theta}}_2 < \theta_2 < \dot{\theta}_2$. By Lemma B.15, we have: (i) $\frac{V_1^{Fo}(q_1)}{\theta_1} - \frac{\frac{1}{2}\left(V_2^{Fo}(q_1)+v_2^{Fo}(q_1)\right)}{\theta_2} > 0$ when $q_1 = 0$; (ii) $\frac{V_1^{Fo}(q_1)}{\theta_1} - \frac{\frac{1}{2}\left(V_2^{Fo}(q_1)+v_2^{Fo}(q_1)\right)}{\theta_2} < 0$ when $q_1 = \bar{q}_1$; and (iii) $\frac{V_1^{Fo}(q_1)}{\theta_1} - \frac{\frac{1}{2}\left(V_2^{Fo}(q_1)+v_2^{Fo}(q_1)\right)}{\theta_2} < 0$ if $\tilde{\tilde{\theta}}_2 < \theta_2 < \tilde{\tilde{\tilde{\theta}}}_2$, and $\frac{V_1^{Fo}(q_1)}{\theta_1} - \frac{\frac{1}{2}\left(V_2^{Fo}(q_1)+v_2^{Fo}(q_1)\right)}{\theta_2} \geq 0$ if $\tilde{\tilde{\tilde{\theta}}}_2 \leq \theta_2 < \dot{\theta}_2$ when $q_1 = 1$. Similar to part (I-c), we can conclude: $I^{Fo}(q_1) \geq 0$ decreases in q_1 when $0 < q_1 < \check{q}_1$; I^{Fo} increases in q_1 when $\check{q}_1 \leq q_1 \leq \bar{q}_1$; $I^{Fo}(q_1)$ decreases in q_1 when $\bar{q}_1 < q_1 \leq \min\{\check{\check{q}}_1, 1\}$; and $I^{Fo}(q_1)$ increases in q_1 when $\min\{\check{\check{q}}_1, 1\} < q_1 < 1$.

Given that $\theta_1 = 1$ (because of normalization) and $I^{Eq} = 0$ (by equation (4.3) in Section 4.1) and defining thresholds $\underline{\theta} \doteq \dot{\theta}_2^{-1}$, $\bar{\theta} \doteq \max\{\tilde{\tilde{\theta}}_2^{-1}, \tilde{\tilde{\tilde{\theta}}}_2^{-1}\}$, $\underline{q}_1 \doteq \max\{0, \check{q}_1\}$, and $\dot{q}_1 \doteq \min\{\check{\check{q}}_1, 1\}$, we have: (i) When $\frac{\theta_1}{\theta_2} \leq \underline{\theta}$, $I^{Fo}(q_1) - I^{Eq}(q_1) \geq 0$ first decreases and then increases in q_1 (based on cases I-b and II-b above); (ii) When $\underline{\theta} < \frac{\theta_1}{\theta_2} < \bar{\theta}$, $I^{Fo}(q_1) - I^{Eq}(q_1) \geq 0$ increases in q_1 if $q_1 \in \left(\underline{q}_1, \bar{q}_1\right) \cup (\dot{q}_1, 1)$ and decreases in q_1 otherwise (based on cases I-c and II-c above); (iii) When $\frac{\theta_1}{\theta_2} \geq \bar{\theta}$, $I^{Fo}(q_1) - I^{Eq}(q_1) \geq 0$ first increases and then decreases in q_1 (based on cases I-a and II-a above). Proof of how $I^{Fo} - I^{Eq}$ changes with respect to q_2 follows similar steps, and therefore, we omit it for brevity. □

Proof of Proposition 4.4: This proof uses Lemma B.4 in Appendix B.2 and Lemmas B.19, B.20, and B.21 in Appendix B.6.

(i) Substituting $f = 0$, $q_2 = \frac{1}{2}$, and $\sqrt{\alpha} = \rho\sqrt{\beta_H} + (1-\rho)\sqrt{\beta_L}$ in the expression of $TSI^{Fo} - TSI^{Eq}$ characterized in the statement of Lemma B.4, we have $TSI^{Fo} - TSI^{Eq} = \sqrt{\frac{\alpha F}{1+\theta_2}}\left(q_1^{\frac{3}{2}} + (1-q_1)^{\frac{3}{2}} + \sqrt{\frac{\theta_2}{2}} - (1+\theta_2)^{\frac{3}{2}}\sqrt{\frac{2q_1^2-2q_1+1}{1+2\theta_2^2\left(2q_1^2-2q_1+1\right)}}\right)$.

Differentiating $TSI^{Fo} - TSI^{Eq}$ with respect to θ_2, we have $\frac{\partial}{\partial\theta_2}\big(TSI^{Fo}(\theta_2) - TSI^{Eq}(\theta_2)\big) = \frac{\sqrt{\alpha F}}{2(1+\theta_2)^{\frac{3}{2}}}\left(H_3(\theta_2) - q_1^{\frac{3}{2}} - (1-q_1)^{\frac{3}{2}}\right)$, where $H_3(\theta_2) \doteq \frac{(1+\theta_2)^{\frac{3}{2}}\sqrt{2q_1^2-2q_1+1}\big(1+\theta_2^2\big(4q_1^2-4q_1+2\big)\big)-2\theta_2(1+\theta_2)^{\frac{5}{2}}\big(2q_1^2-2q_1+1\big)^{\frac{3}{2}}}{-\frac{1}{2}\big(1+\theta_2^2\big(4q_1^2-4q_1+2\big)\big)^{\frac{3}{2}}} + \frac{1}{\sqrt{2\theta_2}}$.

Since $\alpha, F, \theta_2 > 0$, the sign of $\frac{\partial}{\partial\theta_2}\big(TSI^{Fo}(\theta_2) - TSI^{Eq}(\theta_2)\big)$ depends on the sign of $H_3(\theta_2) - q_1^{\frac{3}{2}} - (1-q_1)^{\frac{3}{2}}$. Given that $\theta_1 = 1$, by Lemma B.19 and defining the threshold $\check{\theta} = \hat{\hat{\theta}}_2$, where the existence of $\hat{\hat{\theta}}_2$ is explained in the proof of Lemma B.19, we complete the proof of the result in Proposition 4.4(i).

(ii). Differentiating $TSI^{Fo} - TSI^{Eq}$ with respect to q_1, we have $\frac{\partial}{\partial q_1}\big(TSI^{Fo}(q_1) - TSI^{Eq}(q_1)\big) = (1-2q_1)\sqrt{\frac{aF}{(1+\theta_2)\big(2q_1^2-2q_1+1\big)}}\varrho_1$, where $\varrho_1 \doteq -\frac{3\sqrt{2q_1^2-2q_1+1}\big(\sqrt{1-q_1}-\sqrt{q_1}\big)}{2(1-2q_1)} + \frac{(1+\theta_2)^{\frac{3}{2}}}{\big(1+2\big(2q_1^2-2q_1+1\big)\theta_2^2\big)^{\frac{3}{2}}}$. Since $q_1 \leq \frac{1}{2}$, the sign of $\frac{\partial}{\partial q_1}\big(TSI^{Fo}(q_1) - TSI^{Eq}(q_1)\big)$ depends on the sign of ϱ_1. By Lemma B.20, we have $\frac{(1+\theta_2)^{\frac{3}{2}}}{\big(1+2\big(2q_1^2-2q_1+1\big)\theta_2^2\big)^{\frac{3}{2}}}$ increases in q_1 and $\frac{3\sqrt{2q_1^2-2q_1+1}\big(\sqrt{1-q_1}-\sqrt{q_1}\big)}{2(1-2q_1)}$ decreases in $0 < q_1 \leq \frac{1}{2}$. Next, we consider the following two cases (cases I and II) based on the magnitude of θ_2.

Case I: Suppose $\theta_2 \leq \ddot{\theta}_2$, which existence is shown in the proof of Lemma B.20 (and as shown in B.20, $\ddot{\theta}_2 > 1$). By Lemma B.20, we conclude there exists a threshold of q_1, denoted by $\ddot{q}_1$, such that $\frac{(1+\theta_2)^{\frac{3}{2}}}{\big(1+2\big(2q_1^2-2q_1+1\big)\theta_2^2\big)^{\frac{3}{2}}} \leq \frac{3\sqrt{2q_1^2-2q_1+1}\big(\sqrt{1-q_1}-\sqrt{q_1}\big)}{2(1-2q_1)}$ when $q_1 \leq \ddot{q}_1$ and $\frac{(1+\theta_2)^{\frac{3}{2}}}{\big(1+2\big(2q_1^2-2q_1+1\big)\theta_2^2\big)^{\frac{3}{2}}} > \frac{3\sqrt{2q_1^2-2q_1+1}\big(\sqrt{1-q_1}-\sqrt{q_1}\big)}{2(1-2q_1)}$ otherwise. This implies $\frac{\partial}{\partial q_1}\big(TSI^{Fo}(q_1) - TSI^{Eq}(q_1)\big) \leq 0$ when $q_1 \leq \ddot{q}_1$ and $\frac{\partial}{\partial q_1}\big(TSI^{Fo}(q_1) - TSI^{Eq}(q_1)\big) > 0$ otherwise.

Case II: Suppose $\theta_2 > \ddot{\theta}_2$. By Lemma B.20, we have $\frac{\partial}{\partial\theta_2}\big(TSI^{Fo}(q_1) - TSI^{Eq}(q_1)\big) \geq 0$ for any $0 < q_1 < \frac{1}{2}$.

By Lemma B.21 and Proposition 4.4(i), we can conclude there exists a threshold of $\theta_2 \geq \frac{1}{2}$, denoted by $\ddot{\ddot{\theta}}_2$. (i) When $\theta_2 \leq \ddot{\ddot{\theta}}_2$, there exists two thresholds of q_1, denoted by $\tilde{q}_1$ and $\tilde{\tilde{q}}_1$, such that: $TSI^{Fo}(q_1) - TSI^{Eq}(q_1) > 0$ when $0 < q_1 < \tilde{q}_1$; $TSI^{Fo}(q_1) - TSI^{Eq}(q_1) \leq 0$ when $\tilde{q}_1 \leq q_1 \leq \tilde{\tilde{q}}_1$; and $TSI^{Fo}(q_1) - TSI^{Eq}(q_1) > 0$ when $\tilde{\tilde{q}}_1 < q_1 \leq \frac{1}{2}$. (ii) When $\theta_2 > \ddot{\ddot{\theta}}_2$, $TSI^{Fo}(q_1) - TSI^{Eq}(q_1) > 0$ for any $0 < q_1 \leq \frac{1}{2}$.

Given that $\theta_1 = 1$ and defining the threshold $\dot{\theta} = \max\{\check{\theta}_2, \ddot{\theta}_2\}$, we complete the proof of the result in Proposition 4.4(ii). $\square$

B.6 Proofs of Lemmas Referred in Appendix B.5

We present lemmas (and their proofs), which help us outline proofs of propositions (as outlined in Appendix B.5). We denote the optimal level of initial funds that the funding agency allocates to service agency $i \in \{1, 2\}$ using its initial pool of funds (F) with equity and without equity considerations as B_i^* and B_i^{Ef}, respectively. Further, recall that for outlining proofs in Appendix B.5, without loss of generality, we normalize the volume adjustment factor for service area 1 as 1, i.e., $\theta_1 = 1$.

Lemma B.12 *The optimal decisions $B_1^* \in (0, F)$ and $B_2^* \in (0, F)$, as characterized in Lemma 4.1 presented in Section 4.2, ensure perfect equity (i.e., $I = 0$) if and only if $f < \dot{f}$, where $\dot{f} \doteq 4\theta_2^2 N_1 F > 0$.*

Proof of Lemma B.12: As shown within the proof for Lemma 4.1, service agencies 1 and 2's expected social impact functions can be expressed as functions of B_1 as follows: $V_1(B_1) = \sqrt{\alpha q_1^2 + \left(\rho\sqrt{\beta_H} + (1-\rho)\sqrt{\beta_L}\right)^2 (1-q_1)^2}\sqrt{B_1}$, and $\frac{V_2(B_1)+v_2(B_1)}{2\theta_2} = \frac{1}{2\theta_2}\sqrt{\alpha q_2^2 + \left(\rho\sqrt{\beta_H} + (1-\rho)\sqrt{\beta_L}\right)^2 (1-q_2)^2}\left(\sqrt{F-B_1} + \sqrt{F+f-B_1}\right)$ (where, $p = \frac{1}{2}$ and $\theta_1 = 1$). Differentiating $V_1(B_1)$ and $\frac{V_2(B_1)+v_2(B_1)}{2\theta_2}$ with respect to B_1, we have $\frac{\partial}{\partial B_1}V_1(B_1) = \frac{1}{2}\sqrt{\frac{\alpha q_1^2 + \left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2(1-q_1)^2}{B_1}} > 0$ and $\frac{\partial}{\partial B_1}\left(\frac{V_2(B_1)+v_2(B_1)}{2\theta_2}\right) = \frac{-1}{4\theta_2}\left(\frac{1}{\sqrt{F-B_1}} + \frac{1}{\sqrt{F+f-B_1}}\right)\sqrt{\alpha q_2^2 + \left(\rho\sqrt{\beta_H} + (1-\rho)\sqrt{\beta_L}\right)^2 (1-q_2)^2} < 0$, where inequalities hold because $\alpha > 0$, $0 < q_2 < 1$, $0 < B_1 < F$, $\theta_2 > 0$, and $f \geq 0$. Further, when $B_1 = 0$, we have $V_1(B_1) = 0$ and $\frac{V_2(B_1)+v_2(B_1)}{2\theta_2} = \frac{\left(\sqrt{F}+\sqrt{F+f}\right)\sqrt{\alpha q_2^2 + \left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2(1-q_2)^2}}{2\theta_2} > 0$, which imply that $V_1(B_1) < \frac{V_2(B_1)+v_2(B_1)}{2\theta_2}$ at $B_1 = 0$. Then, since $V_1(B_1)$ increases in B_1, $\frac{V_2(B_1)+v_2(B_1)}{2\theta_2}$ decreases in B_1, and $V_1(B_1) < \frac{V_2(B_1)+v_2(B_1)}{2\theta_2}$ at $B_1 = 0$, for the equity constraint (4.3) in Section 4.1 to hold when $B_1 = B_1^* \in (0, F)$, we must have $V_1(B_1) > \frac{V_2(B_1)+v_2(B_1)}{2\theta_2}$ at $B_1 = F$. That is, $\sqrt{\alpha q_1^2 + \left(\rho\sqrt{\beta_H} + (1-\rho)\sqrt{\beta_L}\right)^2 (1-q_1)^2}\sqrt{F} >$

$\frac{\sqrt{f}}{2\theta_2}\sqrt{\alpha q_2^2+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2(1-q_2)^2}$. Upon taking square on both sides of the aforementioned inequality and re-arranging terms, we have that $V_1(B_1) > \frac{V_2(B_1)+v_2(B_1)}{2\theta_2}$ at $B_1 = F$ holds if and only if $f < 4\theta_2^2 N_1 F$, where N_1 is characterized in the statement of Lemma 4.1. This completes the proof. □

Lemma B.13 *Consider the difference of the normalized total expected social impact between areas 1 and 2 under the efficiency-focused method,* $\frac{V_1^{Ef}(\cdot)}{\theta_1} - \frac{\frac{1}{2}\left(V_2^{Ef}(\cdot)+v_2^{Ef}(\cdot)\right)}{\theta_2}$, *as characterized in the proof of Lemma B.3. Then:*
(i) If $\theta_2 < \underline{\theta}_2$, $\frac{V_1^{Ef}(\cdot)}{\theta_1} - \frac{\frac{1}{2}\left(V_2^{Ef}(\cdot)+v_2^{Ef}(\cdot)\right)}{\theta_2} \leq 0$ *when* $q_2 = 0$, $q_2 = \frac{\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2}{\alpha+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2}$, *and* $q_2 = 1$;
(ii) If $\theta_2 > \ddot{\theta}_2$, $\frac{V_1^{Ef}(\cdot)}{\theta_1} - \frac{\frac{1}{2}\left(V_2^{Ef}(\cdot)+v_2^{Ef}(\cdot)\right)}{\theta_2} \geq 0$ *when* $q_2 = 0$, $q_2 = \frac{\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2}{\alpha+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2}$, *and* $q_2 = 1$, *where* $\underline{\theta}_2$ *and* $\ddot{\theta}_2$ *are characterized in the proof of Lemma B.13.*

Proof of Lemma B.13: We examine the sign of $\frac{V_1^{Ef}(\cdot)}{\theta_1} - \frac{\frac{1}{2}\left(V_2^{Ef}(\cdot)+v_2^{Ef}(\cdot)\right)}{\theta_2}$ based on when $q_2 = 0$, $q_2 = \frac{\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2}{\alpha+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2}$, and $q_2 = 1$. First, when $q_2 = 0$, we have $\frac{V_1^{Ef}(\cdot)}{\theta_1} - \frac{\frac{1}{2}\left(V_2^{Ef}(\cdot)+v_2^{Ef}(\cdot)\right)}{\theta_2} = \varrho_2\left(\theta_2 - \varrho_3\right)$, where $\varrho_2 \doteq \frac{\sqrt{F}\left(\alpha q_1^2+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2(1-q_1)^2\right)}{\theta_2\sqrt{\alpha q_1^2+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2\left(q_1^2-2q_1+2\right)}} > 0$ and $\varrho_3 \doteq \frac{\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2}{\alpha q_1^2+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2(1-q_1)^2}$. Since $\alpha > 0$, $\theta_2 > 0$, and $0 \leq \rho \leq 1$, the sign of $\frac{V_1^{Ef}(\cdot)}{\theta_1} - \frac{\frac{1}{2}\left(V_2^{Ef}(\cdot)+v_2^{Ef}(\cdot)\right)}{\theta_2}$ depends on the sign of $\theta_2 - \varrho_3$. It follows $\frac{V_1^{Ef}(\cdot)}{\theta_1} - \frac{\frac{1}{2}\left(V_2^{Ef}(\cdot)+v_2^{Ef}(\cdot)\right)}{\theta_2} \leq 0$ when $\theta_2 \leq \varrho_3$ and $\frac{V_1^{Ef}(\cdot)}{\theta_1} - \frac{\frac{1}{2}\left(V_2^{Ef}(\cdot)+v_2^{Ef}(\cdot)\right)}{\theta_2} > 0$ otherwise.

When $q_2 = \frac{\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2}{\alpha+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2}$, we have $\frac{V_1^{Ef}(\cdot)}{\theta_1} - \frac{\frac{1}{2}\left(V_2^{Ef}(\cdot)+v_2^{Ef}(\cdot)\right)}{\theta_2} = \eta_7\eta_8\left(\theta_2 - \varrho_4\right)$, where $\varrho_4 \doteq \frac{\frac{\alpha\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2}{\alpha+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2}}{\alpha q_1^2+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2(1-q_1)^2}$, $\eta_7 \doteq \frac{\alpha q_1^2+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2(1-q_1)^2}{\theta_2} > 0$, and $\eta_8 \doteq \frac{\sqrt{F\left(\alpha+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2\right)}}{\sqrt{\alpha q_1^2+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2(1-q_1)^2+\frac{\alpha\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2}{\alpha+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2}}} > 0$. Since $\theta_2, \eta_7, \eta_8 > 0$, the sign of $\frac{V_1^{Ef}(\cdot)}{\theta_1} - \frac{\frac{1}{2}\left(V_2^{Ef}(\cdot)+v_2^{Ef}(\cdot)\right)}{\theta_2}$ depends on the sign of $\theta_2 - \varrho_4$. It follows $\frac{V_1^{Ef}(\cdot)}{\theta_1} - \frac{\frac{1}{2}\left(V_2^{Ef}(\cdot)+v_2^{Ef}(\cdot)\right)}{\theta_2} \leq 0$ when $\theta_2 \leq \varrho_4$ and $\frac{V_1^{Ef}(\cdot)}{\theta_1} - \frac{\frac{1}{2}\left(V_2^{Ef}(\cdot)+v_2^{Ef}(\cdot)\right)}{\theta_2} > 0$ otherwise.

Then, when $q_2 = 1$, we have $\frac{V_1^{Ef}(\cdot)}{\theta_1} - \frac{\frac{1}{2}\left(V_2^{Ef}(\cdot)+v_2^{Ef}(\cdot)\right)}{\theta_2} = \eta_9\left(\theta_2 - \varrho_5\right)$, where $\eta_9 \doteq \frac{\sqrt{F}\left(\alpha q_1^2+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2(1-q_1)^2\right)}{\theta_2\sqrt{\alpha\left(1+q_1^2\right)+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2(1-q_1)^2}} > 0$ and $\varrho_5 \doteq \frac{\alpha}{\alpha q_1^2+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2(1-q_1)^2}$. The sign of $\frac{V_1^{Ef}(\cdot)}{\theta_1} - \frac{\frac{1}{2}\left(V_2^{Ef}(\cdot)+v_2^{Ef}(\cdot)\right)}{\theta_2}$ depends on the sign of $\theta_2 - \varrho_5$. It follows $\frac{V_1^{Ef}(\cdot)}{\theta_1} - \frac{\frac{1}{2}\left(V_2^{Ef}(\cdot)+v_2^{Ef}(\cdot)\right)}{\theta_2} \leq 0$ when $\theta_2 \leq \varrho_5$ and $\frac{V_1^{Ef}(\cdot)}{\theta_1} - \frac{\frac{1}{2}\left(V_2^{Ef}(\cdot)+v_2^{Ef}(\cdot)\right)}{\theta_2} > 0$ otherwise.

The results in Lemma B.13 follow by combining the results above, and by denoting two thresholds $\underline{\theta}_2$ and $\dddot{\theta}_2$ as $\underline{\theta}_2 \doteq \min\left\{\varrho_3, \varrho_4, \varrho_5\right\}$ and $\dddot{\theta}_2 \doteq \max\left\{\varrho_3, \varrho_4, \varrho_5\right\}$. □

Lemma B.14 *Consider* $\frac{1}{N_1+1} - \frac{N_1^2\theta_2^2(\theta_2+1)}{\sqrt{N_1}\left(N_1\theta_2^2+1\right)^{\frac{3}{2}}}$, *where* $N_1 > 0$ *and* $\theta_2 > 0$. *Denote a threshold* $\dddot{\theta}_2$. *Then,* $\frac{1}{N_1+1} - \frac{N_1^2\theta_2^2(\theta_2+1)}{\sqrt{N_1}\left(N_1\theta_2^2+1\right)^{\frac{3}{2}}} \geq 0$ *if* $\theta_2 \leq \dddot{\theta}_2$, *and* $\frac{1}{N_1+1} - \frac{N_1^2\theta_2^2(\theta_2+1)}{\sqrt{N_1}\left(N_1\theta_2^2+1\right)^{\frac{3}{2}}} < 0$ *otherwise, where the existence of* $\dddot{\theta}_2$ *is explained in the proof of Lemma B.14.*

Proof of Lemma B.14: Differentiating $\frac{1}{N_1+1} - \frac{N_1^2\theta_2^2(\theta_2+1)}{\sqrt{N_1}\left(N_1\theta_2^2+1\right)^{\frac{3}{2}}}$ with respect to θ_2, we have $\frac{\partial}{\partial\theta_2}\left(\frac{1}{N_1+1} - \frac{N_1^2\theta_2^2(\theta_2+1)}{\sqrt{N_1}\left(N_1\theta_2^2+1\right)^{\frac{3}{2}}}\right) = \frac{\theta_2 N_1^{\frac{3}{2}}(N_1\theta_2-3\theta_2-2)}{\left(N_1\theta_2+1\right)^2}$. Since $\theta_2 > 0$ and $N_1 > 0$, the sign of $\frac{\partial}{\partial\theta_2}\left(\frac{1}{N_1+1} - \frac{N_1^2\theta_2^2(\theta_2+1)}{\sqrt{N_1}\left(N_1\theta_2^2+1\right)^{\frac{3}{2}}}\right)$ depends on the sign of $N_1\theta_2-3\theta_2-2$. Applying the first-order condition on $N_1\theta_2 - 3\theta_2 - 2$ with respect to θ_2, we have $\theta_2 = \frac{3+\sqrt{9+8N_1}}{2N_1} > 0$ is the unique minima of $N_1\theta_2 - 3\theta_2 - 2$. Therefore, there exists a unique threshold of $\theta_2 > 0$, $N_1\theta_2-3\theta_2-2 \leq 0$ when $\theta_2 \leq \frac{3+\sqrt{9+8N_1}}{2N_1}$ and $N_1\theta_2-3\theta_2-2 > 0$ otherwise. This implies that $\frac{1}{N_1+1} - \frac{N_1^2\theta_2^2(\theta_2+1)}{\sqrt{N_1}\left(N_1\theta_2^2+1\right)^{\frac{3}{2}}}$ decreases in θ_2 when $\theta_2 \leq \frac{3+\sqrt{9+8N_1}}{2N_1}$ and increases in θ_2 otherwise.

When $\theta_2 = 0$, we have $\frac{1}{N_1+1} - \frac{N_1^2\theta_2^2(\theta_2+1)}{\sqrt{N_1}\left(N_1\theta_2^2+1\right)^{\frac{3}{2}}} = \frac{1}{1+N_1} > 0$. When $\theta_2 = +\infty$, applying L'Hôpital's rule, we have $\frac{1}{N_1+1} - \frac{N_1^2\theta_2^2(\theta_2+1)}{\sqrt{N_1}\left(N_1\theta_2^2+1\right)^{\frac{3}{2}}} = -1 < 0$. Therefore, there exists a threshold of θ_2, denoted by $\dddot{\theta}_2$, $\frac{1}{N_1+1} - \frac{N_1^2\theta_2^2(\theta_2+1)}{\sqrt{N_1}\left(N_1\theta_2^2+1\right)^{\frac{3}{2}}} \geq 0$ if $\theta_2 \leq \dddot{\theta}_2$, and $\frac{1}{N_1+1} - \frac{N_1^2\theta_2^2(\theta_2+1)}{\sqrt{N_1}\left(N_1\theta_2^2+1\right)^{\frac{3}{2}}} < 0$ otherwise. □

Lemma B.15 *Consider $f=0$ and $\frac{V_1^{Fo}(\cdot)}{\theta_1}-\frac{\frac{1}{2}\left(V_2^{Fo}(\cdot)+v_2^{Fo}(\cdot)\right)}{\theta_2}$ characterized in the proof of Lemma B.4. Then: (i) When $q_1=0$, then $\frac{V_1^{Fo}(\cdot)}{\theta_1}-\frac{\frac{1}{2}\left(V_2^{Fo}(\cdot)+v_2^{Fo}(\cdot)\right)}{\theta_2}\leq 0$ if $\theta_2\leq\tilde{\tilde{\theta}}_2$ and $\frac{V_1^{Fo}(\cdot)}{\theta_1}-\frac{\frac{1}{2}\left(V_2^{Fo}(\cdot)+v_2^{Fo}(\cdot)\right)}{\theta_2}>0$ otherwise; (ii) When $q_1=\frac{\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2}{\alpha+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2}$, then $\frac{V_1^{Fo}(\cdot)}{\theta_1}-\frac{\frac{1}{2}\left(V_2^{Fo}(\cdot)+v_2^{Fo}(\cdot)\right)}{\theta_2}\leq 0$ if $\theta_2\leq\dot{\theta}_2$ and $\frac{V_1^{Fo}(\cdot)}{\theta_1}-\frac{\frac{1}{2}\left(V_2^{Fo}(\cdot)+v_2^{Fo}(\cdot)\right)}{\theta_2}>0$ otherwise; (iii) When $q_1=1$, then $\frac{V_1^{Fo}(\cdot)}{\theta_1}-\frac{\frac{1}{2}\left(V_2^{Fo}(\cdot)+v_2^{Fo}(\cdot)\right)}{\theta_2}\leq 0$ if $\theta_2\leq\tilde{\tilde{\tilde{\theta}}}_2$ and $\frac{V_1^{Fo}(\cdot)}{\theta_1}-\frac{\frac{1}{2}\left(V_2^{Fo}(\cdot)+v_2^{Fo}(\cdot)\right)}{\theta_2}>0$ otherwise, where $\dot{\theta}_2$, $\tilde{\tilde{\theta}}_2$, and $\tilde{\tilde{\tilde{\theta}}}_2$ are characterized in the proof of Lemma B.15.*

Proof of Lemma B.15: (i) Substituting $f=0$ and $q_1=0$ in $\frac{V_1^{Fo}}{\theta_1}-\frac{\frac{1}{2}\left(V_2^{Fo}+v_2^{Fo}\right)}{\theta_2}$ and denoting it by M_1, we have $M_1=\sqrt{\frac{F}{\theta_2(1+\theta_2)}}\Big(\sqrt{\theta_2}\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)-\sqrt{\alpha}q_2^{\frac{3}{2}}-\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)(1-q_2)^{\frac{3}{2}}\Big)$. Differentiating M_1 with respect to θ_2, we have $\frac{\partial}{\partial\theta_2}M_1(\theta_2)=\frac{\sqrt{F}G_1(\theta_2)}{2\theta_2^{\frac{3}{2}}(1+\theta_2)^{\frac{3}{2}}}$, where $G_1(\theta_2)\doteq(1+2\theta_2)\Big(\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)(1-q_2)^{\frac{3}{2}}+\sqrt{\alpha}q_2^{\frac{3}{2}}\Big)$. The sign of $\frac{\partial}{\partial\theta_2}M_1(\theta_2)$ depends on the sign of $G_1(\theta_2)$. Differentiating $G_1(\theta_2)$ with respect to θ_2, we have $\frac{\partial}{\partial\theta_2}G_1(\theta_2)=-\frac{3\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)}{4\sqrt{\theta_2}}<0$, where inequality holds because $0<\rho<1$. Since $G_1(\theta_2)=\sqrt{\alpha}q_2^{\frac{3}{2}}+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)(1-q_2)^{\frac{3}{2}}>0$ when $\theta_2=0$ and $G_1(\theta_2)=-\infty$ when $\theta_2=+\infty$, there exists a threshold of θ_2, denoted by $\underline{\underline{\theta_2}}$, such that $G_1(\theta_2)\geq 0$ when $\theta_2\leq\underline{\underline{\theta_2}}$ and $G_1(\theta_2)<0$ otherwise. Therefore, we conclude $M_1(\theta_2)$ increases in θ_2 when $\theta_2\leq\underline{\underline{\theta_2}}$ and decreases in θ_2 otherwise. Further, since $M_1(\theta_2)=-\infty$ when $\theta_2=0$ and $M_1(\theta_2)=0$ when $\theta_2=+\infty$, there exists a threshold of θ_2, denoted by $\tilde{\tilde{\theta}}_2$, such that $M_1\leq 0$ if $\theta_2\leq\tilde{\tilde{\theta}}_2$ and $M_1>0$ otherwise. Solving $M_1(\theta_2)=0$, we have $\tilde{\tilde{\theta}}_2=\left(\frac{\sqrt{\alpha}q_2^{\frac{3}{2}}+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)(1-q_2)^{\frac{3}{2}}}{\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}}\right)^2$.

(ii) Substitute $f=0$ and $q_1=\frac{\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2}{\alpha+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2}$ in the expression of $\frac{V_1^{Fo}}{\theta_1}-\frac{\frac{1}{2}\left(V_2^{Fo}+v_2^{Fo}\right)}{\theta_2}$ and denote it by M_2. Following similar steps as in (i) above, we can conclude there exists a threshold of θ_2, denoted by $\dot{\theta}_2$, such that $M_2\leq 0$ if $\theta_2\leq\dot{\theta}_2$ and $M_2>0$ otherwise. (For brevity, we omit details.) Solving $M_2(\theta_2)=0$, we have $\dot{\theta}_2=\frac{\left(\sqrt{\alpha}q_2^{\frac{3}{2}}+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)(1-q_2)^{\frac{3}{2}}\right)^2}{\frac{\alpha\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2}{\alpha+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2}}$

(iii) Substitute $f = 0$ and $q_1 = 1$ in the expression of $\frac{V_1^{Fo}}{\theta_1} - \frac{\frac{1}{2}\left(V_2^{Fo}+v_2^{Fo}\right)}{\theta_2}$ and denote it by M_3. Similar to (i), we show there exists a threshold of θ_2, denoted by $\dot{\theta}_2$, such that $M_3 \leq 0$ if $\theta_2 \leq \tilde{\tilde{\theta}}_2$ and $M_3 > 0$ otherwise. (For brevity, we omit details.) Solving $M_3(\theta_2) = 0$, we have $\tilde{\tilde{\theta}}_2 = \left(\frac{\sqrt{\alpha}q_2^{\frac{3}{2}}+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)(1-q_2)^{\frac{3}{2}}}{\alpha}\right)^2$. □

Lemma B.16 *Consider $\dot{\theta}_2$, $\tilde{\tilde{\theta}}_2$, and $\tilde{\tilde{\tilde{\theta}}}_2$ as characterized in the proof of Lemma B.15. Then, $\tilde{\tilde{\tilde{\theta}}}_2 < \tilde{\tilde{\theta}}_2 < \dot{\theta}_2$ when $\alpha > \left(\rho\sqrt{\beta_H} + (1-\rho)\sqrt{\beta_L}\right)^2$ and $\tilde{\tilde{\theta}}_2 \leq \tilde{\tilde{\tilde{\theta}}}_2 < \dot{\theta}_2$ otherwise.*

Proof of Lemma B.15: First, we show $\tilde{\tilde{\theta}}_2 < \dot{\theta}_2$ and $\tilde{\tilde{\tilde{\theta}}}_2 < \dot{\theta}_2$. Observing expressions of $\tilde{\tilde{\theta}}_2$ and $\dot{\theta}_2$, we have their comparison is equivalent to the comparison between 1 with $\frac{\alpha+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2}{\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2}$. Since $\alpha > 0$, we can conclude that $1 < \frac{\alpha+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2}{\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2}$, or $\tilde{\tilde{\theta}}_2 < \dot{\theta}_2$. Following similar steps, we can conclude that $\tilde{\tilde{\tilde{\theta}}}_2 < \dot{\theta}_2$.

Next, we compare $\tilde{\tilde{\theta}}_2$ with $\tilde{\tilde{\tilde{\theta}}}_2$. Observing their expressions, we have their comparison is equivalent to the comparison between α with $\rho\sqrt{\beta_H} + (1-\rho)\sqrt{\beta_L}$. It follows $\tilde{\tilde{\theta}}_2 > \tilde{\tilde{\tilde{\theta}}}_2$ when $\alpha > \left(\rho\sqrt{\beta_H} + (1-\rho)\sqrt{\beta_L}\right)$ and $\tilde{\tilde{\theta}}_2 \leq \tilde{\tilde{\tilde{\theta}}}_2$ otherwise. Combining the results above, we complete the proof. □

Lemma B.17 *Consider $h_1(\theta_2) = -\left(\frac{3}{4} + (\theta_2+4)\,\theta_2^2 u^2 - \left(\frac{3}{2}\theta_2^2 + \frac{11}{2}\theta_2 + 1\right)u\right)$ and $h_2(\theta_2) = \frac{\sqrt{4u\theta_2^2+2}\left(u\theta_2^2+\frac{1}{2}\right)^2}{4\sqrt{u(1+\theta_2)}\theta_2^{\frac{3}{2}}}$, where $\theta_2 > 0$ and $\frac{1}{2} \leq u \leq 1$. Then, there exist thresholds of $\underline{\underline{\theta_2}}$ and $\dot{\dot{\theta}}_2$, where $0 < \underline{\underline{\theta_2}} < \frac{1}{2}$ and $\dot{\dot{\theta}}_2 > 1$, such that: (i) $h_1(\theta_2) \leq h_2(\theta_2)$ if $0 < \theta_2 \leq \underline{\underline{\theta_2}}$; (ii) $h_1(\theta_2) > h_2(\theta_2)$ if $\underline{\underline{\theta_2}} < \theta_2 < \dot{\dot{\theta}}_2$; and (iii) $h_1(\theta_2) \leq h_2(\theta_2)$ if $\theta_2 \geq \dot{\dot{\theta}}_2$.*

Proof of Lemma B.17: First, we show $h_1(\theta_2)$ is a concave function in θ_2. Twice differentiating $h_1(\theta_2)$ with respect to θ_2, we have $\frac{\partial^2}{\partial\theta_2^2}h_1(\theta_2) = 3u - (6\theta_2+8)\,u^2$. Twice differentiating $3u - (6\theta_2+8)\,u^2$ with respect to u, we have $\frac{\partial^2}{\partial u^2}\left(3u - (6\theta_2+8)\,u^2\right) = -12\theta_2 - 16 < 0$, where inequity holds because $\theta_2 > 0$. Hence, $3u - (6\theta_2+8)\,u^2$ is a concave function in u. Applying the first-order condition on $3u - (6\theta_2+8)\,u^2$, we

have $u = \frac{3}{12\theta_2+16} \leq \frac{1}{2}$, where inequity holds because $\theta_2 > 0$. Recall $\frac{1}{2} \leq u \leq 1$, we can conclude $3u - (6\theta_2 + 8)\, u^2$ decreases in u for any $\frac{1}{2} \leq u \leq 1$. Since $3u - (6\theta_2 + 8)\, u^2 = -\frac{1}{2}\,(3\theta_2 + 1) < 0$ when $u = \frac{1}{2}$, therefore, $3u - (6\theta_2 + 8)\, u^2 < 0$. This implies that $\frac{\partial^2}{\partial \theta_2^2} h_1(\theta_2) > 0$.

Next, we show $h_2(\theta_2)$ is a convex in θ_2. Twice differentiating $h_2(\theta_2)$ with respect to θ_2, we have $\frac{\partial^2}{\partial \theta_2^2} h_2(\theta_2) = \frac{\sqrt{4u\theta_2^2+2}M_4(u)}{64\sqrt{u}\theta_2^{\frac{7}{2}}(1+\theta_2)^{\frac{5}{2}}}$, where $M_4(u) \doteq (96\theta_2^6 + 224\theta_2^5 + 140\theta_2^4)\, u^2 - (24\theta_2^4 + 56\theta_2^3 + 20\theta_2^2)\, u + 24\theta_2^2 + 36\theta_2 + 15$. The sign of $\frac{\partial^2}{\partial \theta_2^2} h_2(\theta_2)$ depends on the sign of $M_4(u)$. Twice differentiating $M_4(u)$ with respect to u, we have $\frac{\partial^2}{\partial u^2} M_4(u) = 196\theta_2^6 + 448\theta_2^5 + 280\theta_2^4 > 0$, where inequity holds because $\theta_2 > 0$. Therefore, $M_4(u)$ is a convex function in u. Applying the first-order condition on $M_4(u)$, we have $u = \frac{6\theta_2^2+14\theta_2+5}{48\theta_2^4+112\theta_2^3+70\theta_2^2}$. Substituting $u = \frac{6\theta_2^2+14\theta_2+5}{48\theta_2^4+112\theta_2^3+70\theta_2^2}$ in $M_4(u)$, we obtain $M_4(u) = \frac{20(\theta_2+1)^2\left(27\theta_2^2+48\theta_2+25\right)^2}{24\theta_2^2+56\theta_2+25} > 0$. This implies that $\frac{\partial^2}{\partial \theta_2^2} h_2(\theta_2) > 0$.

Last, compare $h_1(\theta_2)$ with $h_2(\theta_2)$ when $\theta_2 = 0$, $\theta_2 = \frac{1}{2}$, and $\theta_2 = 1$. When $\theta_2 = 0$, we have $h_1(\theta_2) = -\frac{3}{4}$ and $h_2(\theta_2) = +\infty$. Therefore, $h_1(\theta_2) < h_2(\theta_2)$ when $\theta_2 = 0$. When $\theta_2 = \frac{1}{2}$, we have $h_1(\theta_2) = -\frac{9}{8}\left(u - \frac{33}{18}\right)^2 + \frac{97}{32} > 0$ and $h_2(\theta_2) = \frac{\sqrt{12}(u+2)^{\frac{5}{2}}}{96\sqrt{u}} > 0$, where inequality signs hold because $\frac{1}{2} \leq u \leq 1$. Differentiating $\frac{-\frac{9}{8}\left(u-\frac{33}{18}\right)^2+\frac{97}{32}}{\frac{\sqrt{12}(u+2)^{\frac{5}{2}}}{96\sqrt{u}}}$ with respect to u, we have $\frac{\partial}{\partial u}\left(\frac{-\frac{9}{8}\left(u-\frac{33}{18}\right)^2+\frac{97}{32}}{\frac{\sqrt{12}(u+2)^{\frac{5}{2}}}{96\sqrt{u}}}\right) = \frac{\sqrt{3}\left(-156\left(u-\frac{37}{52}\right)^2+\frac{3483}{52}\right)}{\sqrt{u}(u+2)^{\frac{7}{2}}} > 0$, where inequity holds because $\frac{1}{2} \leq u \leq 1$. Therefore, $\frac{-\frac{9}{8}\left(u-\frac{33}{18}\right)^2+\frac{97}{32}}{\frac{\sqrt{12}(u+2)^{\frac{5}{2}}}{96\sqrt{u}}}$ increases in u. Further, since $\frac{-\frac{9}{8}\left(u-\frac{33}{18}\right)^2+\frac{97}{32}}{\frac{\sqrt{12}(u+2)^{\frac{5}{2}}}{96\sqrt{u}}} = \frac{66\sqrt{15}}{125} > 1$ when $u = \frac{1}{2}$, we conclude $\frac{-\frac{9}{8}\left(u-\frac{33}{18}\right)^2+\frac{97}{32}}{\frac{\sqrt{12}(u+2)^{\frac{5}{2}}}{96\sqrt{u}}} > 1$ for any $\frac{1}{2} \leq u \leq 1$. Therefore, $h_1(\theta_2) > h_2(\theta_2)$ when $\theta_2 = \frac{1}{2}$. When $\theta_2 = 1$, we have $h_1(\theta_2) = -5\left(u - \frac{4}{5}\right)^2 + \frac{49}{20} > 0$ and $h_2(\theta_2) = \frac{(2u+1)^{\frac{5}{2}}}{16\sqrt{u}} > 0$, where inequality signs hold because $\frac{1}{2} \leq u \leq 1$. Similarly, we show $h_1(\theta_2) > h_2(\theta_2)$ when $\theta_2 = 1$. Above all, we complete the proof by defining two thresholds of θ_2, denoted by $\underline{\underline{\theta_2}}$ and $\dot{\dot{\theta}}_2$ where $0 < \underline{\underline{\theta_2}} < \frac{1}{2}$ and $\dot{\dot{\theta}}_2 > 1$. $\square$

Lemma B.18 *Consider* $M_5(q_1) = -69120q_1^4 + 138240q_1^3 - 91008q_1^2 + 21888q_1 + 96$. *Then,* $M_5(q_1) > 0\ \forall 0 < q_1 \leq \frac{1}{2}$.

Differentiating $M_5(q_1)$ with respect to q_1, we have $\frac{\partial}{\partial q_1} M_5(q_1) = -276480q_1^3 + 414720q_1^2 - 182016q_1 + 21888$. Differentiating $-276480q_1^3 + 414720q_1^2 - 182016q_1 + 21888$ with respect to q_1, we have $\frac{\partial}{\partial q_1}\left(-276480q_1^3 + 414720q_1^2 - 182016q_1 + 21888\right) = -829440\left(\frac{1}{2} - q_1\right)^2 + 25344$, which increases in q_1 when $0 < q_1 \leq \frac{1}{2}$.

Since $-829440\left(\frac{1}{2} - q_1\right)^2 + 25344 = -182016$ when $q_1 = 0$ and $-829440\left(\frac{1}{2} - q_1\right)^2 + 25344 = 25344$ when $q_1 = 1$, we conclude $-276480q_1^3 + 414720q_1^2 - 182016q_1 + 21888$ first decreases and then increases in q_1 when $0 < q_1 \leq \frac{1}{2}$. Further, since $-276480q_1^3 + 414720q_1^2 - 182016q_1 + 21888 = 21888$ when $q_1 = 0$ and $-276480q_1^3 + 414720q_1^2 - 182016q_1 + 21888 = 0$ when $q_1 = \frac{1}{2}$, we can conclude there exists a threshold of q_1, denoted by $\underline{\underline{q}}_1$, where $0 < \underline{\underline{q}}_1 < \frac{1}{2}$, such that $-276480q_1^3 + 414720q_1^2 - 182016q_1 + 21888 > 0$ if $q_1 < \underline{\underline{q}}_1$ and $-276480q_1^3 + 414720q_1^2 - 182016q_1 + 21888 \leq 0$ otherwise. This implies that $M_5(q_1)$ first increases then decreases in q_1. Give that $M_5(q_1) = 96$ when $q_1 = 0$ and $M_5(q_1) = 1248$ when $q_1 = \frac{1}{2}$, we have $M_5(q_1) \geq 0$ for any $0 < q_1 \leq \frac{1}{2}$. □

Lemma B.19 *Consider* $H_3(\theta_2) - q_1^{\frac{3}{2}} - (1-q_1)^{\frac{3}{2}}$ *as characterized in the proof of Proposition 4.4, where* $0 < q_1 < 1$ *and* $\theta_2 > 0$. *Then, there exists three thresholds of* θ_2, *denoted by* $\hat{\hat{\theta}}_2$, $\hat{\hat{\hat{\theta}}}_2$, *and* $\check{\check{\theta}}_2$, *such that:* $H_3(\theta_2) - q_1^{\frac{3}{2}} - (1-q_1)^{\frac{3}{2}} > 0$ *when* $\theta_2 < \hat{\hat{\theta}}_2$, $H_3(\theta_2) - q_1^{\frac{3}{2}} - (1-q_1)^{\frac{3}{2}} \leq 0$ *when* $\hat{\hat{\theta}}_2 \leq \theta_2 \leq \hat{\hat{\hat{\theta}}}_2$, $H_3(\theta_2) - q_1^{\frac{3}{2}} - (1-q_1)^{\frac{3}{2}} > 0$ *when* $\hat{\hat{\hat{\theta}}}_2 < \theta_2 < \check{\check{\theta}}_2$, *and* $H_3(\theta_2) - q_1^{\frac{3}{2}} - (1-q_1)^{\frac{3}{2}} \leq 0$ *when* $\theta_2 \geq \check{\check{\theta}}_2$.

Proof of Lemma B.19: This proof uses Lemmas B.17 and B.18 in Appendix B.6. Differentiating $H_3(\theta_2) - q_1^{\frac{3}{2}} - (1-q_1)^{\frac{3}{2}}$ with respect to θ_2, we have $\frac{\partial}{\partial \theta_2}\Big(H_3(\theta_2) - q_1^{\frac{3}{2}} - (1-q_1)^{\frac{3}{2}}\Big) = \frac{4\sqrt{(1+\theta_2)u}}{\left(2\theta_2^2 u + 1\right)^{\frac{5}{2}}}\left(h_1(\theta_2) - h_2(\theta_2)\right)$, where $u \doteq 2q_1^2 - 2q_1 + 1$, $h_1(\theta_2) \doteq -\left(\frac{3}{4} + (\theta_2 + 4)\,\theta_2^2 u^2 - \left(\frac{3}{2}\theta_2^2 + \frac{11}{2}\theta_2 + 1\right)u\right)$ and $h_2(\theta_2) \doteq \frac{\sqrt{4u\theta_2^2 + 2}\left(u\theta_2^2 + \frac{1}{2}\right)^2}{4\sqrt{u(1+\theta_2)}\,\theta_2^{\frac{3}{2}}}$. Since $u =$

$2\left(q_1-\frac{1}{2}\right)^2+\frac{1}{2}$ and $0<q_1\leq\frac{1}{2}$, we have $\frac{1}{2}\leq u<1$. The sign of $\frac{\partial}{\partial\theta_2}\Big(H_3(\theta_2)-q_1^{\frac{3}{2}}-(1-q_1)^{\frac{3}{2}}\Big)$ depends on the sign of $h_1(\theta_2)-h_2(\theta_2)$.

By Lemma B.17, we have there exist two thresholds $\underline{\underline{\theta_2}}$ and $\dot{\dot{\theta}}_2$, where $0<\underline{\underline{\theta_2}}<\frac{1}{2}$ and $\dot{\dot{\theta}}_2>1$: (i) $\frac{\partial}{\partial\theta_2}\left(H_3(\theta_2)-q_1^{\frac{3}{2}}-(1-q_1)^{\frac{3}{2}}\right)\leq 0$ if $0<\theta_2\leq\underline{\underline{\theta_2}}$; (ii) $\frac{\partial}{\partial\theta_2}\Big(H_3(\theta_2)-q_1^{\frac{3}{2}}-(1-q_1)^{\frac{3}{2}}\Big)>0$ if $\underline{\underline{\theta_2}}<\theta_2<\dot{\dot{\theta}}_2$; and (iii) $\frac{\partial}{\partial\theta_2}\left(H_3(\theta_2)-q_1^{\frac{3}{2}}-(1-q_1)^{\frac{3}{2}}\right)\leq 0$ if $\theta_2\geq\dot{\dot{\theta}}_2$. Next, we compare $H_3(\theta_2)$ with $q_1^{\frac{3}{2}}+(1-q_1)^{\frac{3}{2}}$ in the following four cases (cases I, II, III, and VI) in which $\theta_2=0$, $\theta_2=\frac{1}{2}$, $\theta_2=1$, and $\theta_2=+\infty$.

Case I: When $\theta_2=0$, we have $H_3=+\infty$. Since $0<q_1\leq\frac{1}{2}$, therefore, $H_3>q_1^{\frac{3}{2}}+(1-q_1)^{\frac{3}{2}}$ when $\theta_2=0$.

Case II: When $\theta_2=\frac{1}{2}$, we have $H_3=\frac{\sqrt{6}M_6(q_1)}{\sqrt{2}(2q_1^2-2q_1+3)^{\frac{7}{2}}(2q_1^2-2q_1+1)^{\frac{3}{2}}}$, where $M_6(q_1)\doteq -2304q_1^6+6912q_1^5-7584q_1^4+3648q_1^3+48q_1^2-720q_1+216$. We can conclude that the sign of $\frac{\sqrt{6}M_6(q_1)}{\sqrt{2}(2q_1^2-2q_1+3)^{\frac{7}{2}}(2q_1^2-2q_1+1)^{\frac{3}{2}}}$ depends on the sign of $M_6(q_1)$. Twice differentiating $M_6(q_1)$ with respect to q_1, we have $\frac{\partial^2}{\partial q_1^2}M_6(q_1)=M_5(q_1)$, where $M_5(q_1)\doteq -69120q_1^4+138240q_1^3-91008q_1^2+21888q_1+96$. By Lemma B.18, we have $M_5(q_1)>0$ for any $0<q_1\leq\frac{1}{2}$. Therefore, we conclude $\frac{\sqrt{6}M_6(q_1)}{\sqrt{2}(2q_1^2-2q_1+3)^{\frac{7}{2}}(2q_1^2-2q_1+1)^{\frac{3}{2}}}$ is a convex function in $0<q_1\leq\frac{1}{2}$. Applying the first-order condition on $\frac{\sqrt{6}M_6(q_1)}{\sqrt{2}(2q_1^2-2q_1+3)^{\frac{7}{2}}(2q_1^2-2q_1+1)^{\frac{3}{2}}}$ we have $q_1=\frac{1}{2}$. Twice differentiating $q_1^{\frac{3}{2}}+(1-q_1)^{\frac{3}{2}}$ with respect to q_1, we have $\frac{\partial^2}{\partial q_1^2}\left(q_1^{\frac{3}{2}}+(1-q_1)^{\frac{3}{2}}\right)=\frac{3\left(\sqrt{q_1}+\sqrt{1-q_1}\right)}{4\sqrt{q_1}\sqrt{1-q_1}}>0$. Therefore, $q_1^{\frac{3}{2}}+(1-q_1)^{\frac{3}{2}}$ is a convex function in q_1. Applying the first-order condition on $q_1^{\frac{3}{2}}+(1-q_1)^{\frac{3}{2}}$, we have $q_1=\frac{1}{2}$. Therefore, we conclude both $\frac{\sqrt{6}M_6(q_1)}{\sqrt{2}(2q_1^2-2q_1+3)^{\frac{7}{2}}(2q_1^2-2q_1+1)^{\frac{3}{2}}}$ and $q_1^{\frac{3}{2}}+(1-q_1)^{\frac{3}{2}}$ convex decrease in q_1 when $0<q_1\leq\frac{1}{2}$. Further, since $\frac{\sqrt{6}M_6(q_1)}{\sqrt{2}(2q_1^2-2q_1+3)^{\frac{7}{2}}(2q_1^2-2q_1+1)^{\frac{3}{2}}}=q_1^{\frac{3}{2}}+(1-q_1)^{\frac{3}{2}}=1$ when $q_1=0$, and $\frac{\sqrt{6}M_6(q_1)}{\sqrt{2}(2q_1^2-2q_1+3)^{\frac{7}{2}}(2q_1^2-2q_1+1)^{\frac{3}{2}}}=1-\frac{6\sqrt{15}}{25}<q_1^{\frac{3}{2}}+(1-q_1)^{\frac{3}{2}}=\frac{\sqrt{2}}{2}$ when $q_1=\frac{1}{2}$, therefore, $H_3<q_1^{\frac{3}{2}}+(1-q_1)^{\frac{3}{2}}$ when $\theta_2=\frac{1}{2}$.

Case III: When $\theta_2=1$, following similar steps as in case II, we have $H_3\geq q_1^{\frac{3}{2}}+(1-q_1)^{\frac{3}{2}}$ when $\theta_2=1$. For brevity, we omit it.

Case IV: When $\theta_2=+\infty$, we have $H_3=0$. Since $0<q_1\leq\frac{1}{2}$, therefore, $H_3<q_1^{\frac{3}{2}}+(1-q_1)^{\frac{3}{2}}$ when $\theta_2=0$.

Above all, we complete the proof by combining the results in four cases above, and by defining three thresholds of θ_2, $\hat{\hat{\theta}}_2$, $\hat{\hat{\hat{\theta}}}_2$, and $\check{\check{\theta}}_2$, where $0 < \hat{\hat{\theta}}_2 < \frac{1}{2} < \hat{\hat{\hat{\theta}}}_2 < 1 < \check{\check{\theta}}_2$.

□

Lemma B.20 *Consider* $\frac{(1+\theta_2)^{\frac{3}{2}}}{\left(1+2\left(2q_1^2-2q_1+1\right)\theta_2^2\right)^{\frac{3}{2}}}$ *and* $\frac{3\sqrt{2q_1^2-2q_1+1}\left(\sqrt{1-q_1}-\sqrt{q_1}\right)}{2(1-2q_1)}$, *where* $\theta_2 > 0$ *and* $0 < q_1 \leq \frac{1}{2}$. *Let* $\ddot{\theta}_2$ *denote threshold on* θ_2. *Then (i). When* $\theta_2 \leq \ddot{\theta}_2$. $\frac{(1+\theta_2)^{\frac{3}{2}}}{\left(1+2\left(2q_1^2-2q_1+1\right)\theta_2^2\right)^{\frac{3}{2}}} < \frac{3\sqrt{2q_1^2-2q_1+1}\left(\sqrt{1-q_1}-\sqrt{q_1}\right)}{2(1-2q_1)}$ *when* $q_1 = 0$ *and* $\frac{(1+\theta_2)^{\frac{3}{2}}}{\left(1+2\left(2q_1^2-2q_1+1\right)\theta_2^2\right)^{\frac{3}{2}}} \geq \frac{3\sqrt{2q_1^2-2q_1+1}\left(\sqrt{1-q_1}-\sqrt{q_1}\right)}{2(1-2q_1)}$ *when* $q_1 = \frac{1}{2}$.
(ii). When $\theta_2 > \ddot{\theta}_2$. $\frac{(1+\theta_2)^{\frac{3}{2}}}{\left(1+2\left(2q_1^2-2q_1+1\right)\theta_2^2\right)^{\frac{3}{2}}} < \frac{3\sqrt{2q_1^2-2q_1+1}\left(\sqrt{1-q_1}-\sqrt{q_1}\right)}{2(1-2q_1)}$ *for any* $0 < q_1 < \frac{1}{2}$.
The existence of $\ddot{\theta}_2$ *is explained in the proof of Lemma B.20.*

Proof of Lemma B.20: First, we show $\frac{(1+\theta_2)^{\frac{3}{2}}}{\left(1+2\left(2q_1^2-2q_1+1\right)\theta_2^2\right)^{\frac{3}{2}}}$ increases in q_1 if $0 < q_1 \leq \frac{1}{2}$. Since $\frac{\partial}{\partial q_1}\left(\frac{(1+\theta_2)^{\frac{3}{2}}}{\left(1+2\left(2q_1^2-2q_1+1\right)\theta_2^2\right)^{\frac{3}{2}}}\right) = \frac{(6-12q_1)\theta_2^2(1+\theta_2)^{\frac{3}{2}}}{\left(1+2\left(2q_1^2-2q_1+1\right)\theta_2^2\right)^{\frac{5}{2}}}$, we conclude that the sign of $\frac{\partial}{\partial q_1}\left(\frac{(1+\theta_2)^{\frac{3}{2}}}{\left(1+2\left(2q_1^2-2q_1+1\right)\theta_2^2\right)^{\frac{3}{2}}}\right)$ depends on the sign of $6-12q_1$. It follows $\frac{\partial}{\partial q_1}\left(\frac{(1+\theta_2)^{\frac{3}{2}}}{\left(1+2\left(2q_1^2-2q_1+1\right)\theta_2^2\right)^{\frac{3}{2}}}\right) > 0$ if $0 < q_1 < \frac{1}{2}$.

Second, we show $\frac{3\sqrt{2q_1^2-2q_1+1}\left(\sqrt{1-q_1}-\sqrt{q_1}\right)}{2(1-2q_1)}$ decreases in q_1 if $0 < q_1 \leq \frac{1}{2}$. Differentiating $\frac{\sqrt{1-q_1}-\sqrt{q_1}}{1-2q_1}$ with respect to q_1, we have $\frac{\partial}{\partial q_1}\left(\frac{\sqrt{1-q_1}-\sqrt{q_1}}{1-2q_1}\right) = \frac{\left(1-2\sqrt{q_1}\sqrt{1-q_1}\right)\left(\sqrt{q_1}-\sqrt{1-q_1}\right)}{2\sqrt{q_1}\sqrt{1-q_1}(1-2q_1)^2} < 0$, where inequality holds because $0 < q_1 \leq \frac{1}{2}$. Differentiating $\sqrt{2q_1^2-2q_1+1}$ with respect to q_1, we have $\frac{\partial}{\partial q_1}\left(\sqrt{2q_1^2-2q_1+1}\right) = \frac{2q_1-1}{\sqrt{2q_1^2-2q_1+1}} < 0$, where inequality holds because $0 < q_1 \leq \frac{1}{2}$. Combining $\frac{\sqrt{1-q_1}-\sqrt{q_1}}{1-2q_1}$ with $\sqrt{2q_1^2-2q_1+1}$, we can conclude that $\frac{3\sqrt{2q_1^2-2q_1+1}\left(\sqrt{1-q_1}-\sqrt{q_1}\right)}{2(1-2q_1)}$ decreases in q_1 when $0 < q_1 \leq \frac{1}{2}$. Next, we compare $\frac{(1+\theta_2)^{\frac{3}{2}}}{\left(1+2\left(2q_1^2-2q_1+1\right)\theta_2^2\right)^{\frac{3}{2}}}$ with $\frac{3\sqrt{2q_1^2-2q_1+1}\left(\sqrt{1-q_1}-\sqrt{q_1}\right)}{2(1-2q_1)}$ in the following two cases (cases I and II) in which $q_1 = 0$ and $q_1 = \frac{1}{2}$.

Case I: When $q_1 = 0$, $\frac{(1+\theta_2)^{\frac{3}{2}}}{\left(1+2\left(2q_1^2-2q_1+1\right)\theta_2^2\right)^{\frac{3}{2}}} = \left(\frac{1+\theta_2}{1+2\theta_2^2}\right)^{\frac{3}{2}}$ and $\frac{3\sqrt{2q_1^2-2q_1+1}\left(\sqrt{1-q_1}-\sqrt{q_1}\right)}{2(1-2q_1)} = \frac{3}{2}$. Differentiating $\left(\frac{1+\theta_2}{1+2\theta_2^2}\right)^{\frac{3}{2}}$ with respect to θ_2, we have $\frac{\partial}{\partial q_1}\left(\left(\frac{1+\theta_2}{1+2\theta_2^2}\right)^{\frac{3}{2}}\right) = \frac{-2(\theta_2+1)^2+3}{\left(2\theta_2^2+1\right)^2}$,

which follows $\frac{\partial}{\partial q_1}\left(\left(\frac{1+\theta_2}{1+2\theta_2^2}\right)^{\frac{3}{2}}\right) > 0$ when $\theta_2 = \frac{2-\sqrt{6}}{2}$ and $\frac{\partial}{\partial q_1}\left(\left(\frac{1+\theta_2}{1+2\theta_2^2}\right)^{\frac{3}{2}}\right) \leq 0$ otherwise. Substituting $\theta_2 = \frac{2-\sqrt{6}}{2}$ in $\left(\frac{1+\theta_2}{1+2\theta_2^2}\right)^{\frac{3}{2}}$, we have $\left(\frac{1+\theta_2}{1+2\theta_2^2}\right)^{\frac{3}{2}} \approx 1.17$, which is the maximum value of $\left(\frac{1+\theta_2}{1+2\theta_2^2}\right)^{\frac{3}{2}}$. Hence, $\frac{(1+\theta_2)^{\frac{3}{2}}}{\left(1+2\left(2q_1^2-2q_1+1\right)\theta_2^2\right)^{\frac{3}{2}}} < \frac{3\sqrt{2q_1^2-2q_1+1}\left(\sqrt{1-q_1}-\sqrt{q_1}\right)}{2(1-2q_1)}$ when $q_1 = 0$.

Case II: When $q_1 = \frac{1}{2}$, $\frac{(1+\theta_2)^{\frac{3}{2}}}{\left(1+2\left(2q_1^2-2q_1+1\right)\theta_2^2\right)^{\frac{3}{2}}} = \left(\frac{1+\theta_2}{1+\theta_2^2}\right)^{\frac{3}{2}}$ and $\frac{3\sqrt{2q_1^2-2q_1+1}\left(\sqrt{1-q_1}-\sqrt{q_1}\right)}{2(1-2q_1)} = \frac{3}{4}$. Following similar steps as in Case I, we can show $\left(\frac{1+\theta_2}{1+\theta_2^2}\right)^{\frac{3}{2}}$ increases in θ_2 when $\theta_2 < \sqrt{2}-1$ and decreases in θ_2 otherwise. Further, since $\left(\frac{1+\theta_2}{1+\theta_2^2}\right)^{\frac{3}{2}} = 1$ when $\theta_2 = 0$, we conclude there exists a threshold of θ_2, denoted by $\ddot{\theta}_2$, such that $\left(\frac{1+\theta_2}{1+\theta_2^2}\right)^{\frac{3}{2}} \geq \frac{3}{4}$ when $\theta_2 \leq \ddot{\theta}_2$ and $\left(\frac{1+\theta_2}{1+\theta_2^2}\right)^{\frac{3}{2}} < \frac{3}{4}$ otherwise. Since $\left(\frac{1+\theta_2}{1+\theta_2^2}\right)^{\frac{3}{2}} = 1$ when $\theta_2 = 1$, we conclude $\ddot{\theta}_2 > 1$. Thus, $\frac{(1+\theta_2)^{\frac{3}{2}}}{\left(1+2\left(2q_1^2-2q_1+1\right)\theta_2^2\right)^{\frac{3}{2}}} \geq \frac{3\sqrt{2q_1^2-2q_1+1}\left(\sqrt{1-q_1}-\sqrt{q_1}\right)}{2(1-2q_1)}$ when $\theta_2 \leq \ddot{\theta}_2$ and $\frac{(1+\theta_2)^{\frac{3}{2}}}{\left(1+2\left(2q_1^2-2q_1+1\right)\theta_2^2\right)^{\frac{3}{2}}} < \frac{3\sqrt{2q_1^2-2q_1+1}\left(\sqrt{1-q_1}-\sqrt{q_1}\right)}{2(1-2q_1)}$ otherwise.

We complete the proof by combining the results in two cases above. □

Lemma B.21 *Consider $TSI^{Fo}-TSI^{Eq}$, as characterized in the statement of Lemma B.4, and $q_2 = \frac{1}{2}$ and $\sqrt{\alpha} = \rho\sqrt{\beta_H} + (1-\rho)\sqrt{\beta_L}$, where $\theta_2 > 0$ and $q_1 \in \left(0, \frac{1}{2}\right]$. Then, (i) $TSI^{Fo} - TSI^{Eq} \geq 0$ when $q_1 = 0$ $\forall\theta_2 > 0$, and $TSI^{Fo} - TSI^{Eq} = 0$ when $q_1 = 0$ and $\theta_2 = \frac{1}{2}$; (ii) $TSI^{Fo} - TSI^{Eq} \geq 0$ when $q_1 = \frac{1}{2}$ $\forall\theta_2 > 0$. Further, if $\theta_2 = +\infty$, $TSI^{Fo}(q_1) - TSI^{Eq}(q_1) = 0$ $\forall q_1 \in \left(0, \frac{1}{2}\right]$.*

Proof of Lemma B.21: First, we show, when $q_1 = 0$, $TSI^{Fo} - TSI^{Eq} \geq 0$ for any $\theta_2 > 0$ and $TSI^{Fo} - TSI^{Eq} = 0$ when $\theta_2 = \frac{1}{2}$. Substituting $f = 0$, $\sqrt{\alpha} = \rho\sqrt{\beta_H} + (1-\rho)\sqrt{\beta_L}$, $q_1 = 0$, and $q_2 = \frac{1}{2}$ in $TSI^{Fo} - TSI^{Eq}$, we have $TSI^{Fo} - TSI^{Eq} = \sqrt{\frac{\alpha F}{2}}\left(\left(\sqrt{\theta_2}+\sqrt{2}\right)\sqrt{\frac{1}{\theta_2+1}} - (1+\theta_2)\sqrt{\frac{2}{2\theta_2^2+1}}\right)$. The sign of $TSI^{Fo} - TSI^{Eq}$ depends on the sign of $\left(\sqrt{\theta_2}+\sqrt{2}\right)\sqrt{\frac{1}{\theta_2+1}} - (1+\theta_2)\sqrt{\frac{2}{2\theta_2^2+1}}$. Since $\theta_2 > 0$, the sign of $\left(\sqrt{\theta_2}+\sqrt{2}\right)\sqrt{\frac{1}{\theta_2+1}} - (1+\theta_2)\sqrt{\frac{2}{2\theta_2^2+1}}$ is equivalent to that of $\left(\left(\sqrt{\theta_2}+\sqrt{2}\right)\sqrt{\frac{1}{\theta_2+1}}\right)^2$

$-\left((1+\theta_2)\sqrt{\frac{2}{2\theta_2^2+1}}\right)^2 = \frac{\sqrt{\theta_2}\left(\left(4\sqrt{2}\theta_2^2+2\sqrt{2}\right)-\left(2\theta_2^{\frac{3}{2}}+5\sqrt{\theta_2}\right)\right)}{(1+\theta_2)\left(2\theta_2^2+1\right)}$, which sign depends on the sign of $\left(4\sqrt{2}\theta_2^2+2\sqrt{2}\right)-\left(2\theta_2^{\frac{3}{2}}+5\sqrt{\theta_2}\right)$. The sign of $\left(4\sqrt{2}\theta_2^2+2\sqrt{2}\right)-\left(2\theta_2^{\frac{3}{2}}+5\sqrt{\theta_2}\right)$ is equivalent to the sign of $\left(4\sqrt{2}\theta_2^2+2\sqrt{2}\right)^2-\left(2\theta_2^{\frac{3}{2}}+5\sqrt{\theta_2}\right)^2 = 32\theta_2^4-4\theta_2^3+12\theta_2^2-25\theta_2+8$. Twice differentiating $32\theta_2^4-4\theta_2^3+12\theta_2^2-25\theta_2+8$ with respect to θ_2, we have $\frac{\partial^2}{\partial\theta_2^2}\left(32\theta_2^4-4\theta_2^3+12\theta_2^2-25\theta_2+8\right) = 384\left(\theta_2-\frac{1}{32}\right)^2+\frac{189}{8} \geq 0$, where inequality holds because $\theta_2 > 0$. Therefore, $32\theta_2^4-4\theta_2^3+12\theta_2^2-25\theta_2+8$ is a convex function in θ_2. Applying the first-order condition on $32\theta_2^4-4\theta_2^3+12\theta_2^2-25\theta_2+8$, we have $\theta_2 = \frac{1}{2}$. Substituting $\theta_2 = \frac{1}{2}$ in $32\theta_2^4-4\theta_2^3+12\theta_2^2-25\theta_2+8$, we have $32\theta_2^4-4\theta_2^3+12\theta_2^2-25\theta_2+8 = 0$. Hence, we can conclude $32\theta_2^4-4\theta_2^3+12\theta_2^2-25\theta_2+8 \geq 0$ for any $\theta_2 > 0$. This implies that $\left(\sqrt{\theta_2}+\sqrt{2}\right)\sqrt{\frac{1}{\theta_2+1}}-(1+\theta_2)\sqrt{\frac{2}{2\theta_2^2+1}} \geq 0$ for any $\theta_2 > 0$. Further, substituting $\theta_2 = 1$ in $TSI^{Fo}-TSI^{Eq}$, we have $TSI^{Fo}-TSI^{Eq} = 0$.

Second, we show, when $q_1 = \frac{1}{2}$, $TSI^{Fo}-TSI^{Eq} \geq 0$ for any $\theta_2 > 0$. Substituting $f = 0$, $\sqrt{\alpha} = \rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}$, $q_1 = \frac{1}{2}$, and $q_2 = \frac{1}{2}$ in $TSI^{Fo}-TSI^{Eq}$, we have $TSI^{Fo}-TSI^{Eq} = \sqrt{\frac{\alpha F}{2}}\left(\left(\sqrt{\theta_2}+1\right)\sqrt{\frac{1}{\theta_2+1}}-(1+\theta_2)\sqrt{\frac{1}{\theta_2^2+1}}\right)$. The sign of $TSI^{Fo}-TSI^{Eq}$ depends on the sign of $\left(\sqrt{\theta_2}+1\right)\sqrt{\frac{1}{\theta_2+1}}-(1+\theta_2)\sqrt{\frac{1}{\theta_2^2+1}}$. Following similar steps as above, we can show $\left(\sqrt{\theta_2}+1\right)\sqrt{\frac{1}{\theta_2+1}}-(1+\theta_2)\sqrt{\frac{1}{\theta_2^2+1}} \geq 0$ for any $\theta_2 > 0$. We omit details for brevity.

Last, by applying L'Hôpital's rule, we have $TSI^{Fo}(q_1)-TSI^{Eq}(q_1) = 0$ when $\theta_2 = +\infty$ for any $q_1 \in \left(0, \frac{1}{2}\right]$. □

B.7 Proofs of Lemmas Referred in Section B.2

Proof of Lemma B.1: As per the proof of Lemma 4.1, we have: $V_1(B_1) = \sqrt{\alpha q_1^2+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2(1-q_1)^2}\sqrt{B_1}$ and $\frac{1}{2}\left(V_2(B_1)+v_2(B_1)\right) = \frac{\sqrt{F-B_1}+\sqrt{F+f-B_1}}{2}\sqrt{\alpha q_2^2+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2(1-q_2)^2}$. Let B_1^{Ef} denote the maxima of $TSI = V_1(B_1)+\frac{1}{2}\left(V_2(B_1)+v_2(B_1)\right)$. Twice differentiating TSI with respect to B_1, we have $\frac{\partial^2}{\partial B_1^2}TSI(B_1) = -\frac{1}{4}\sqrt{\alpha q_1^2+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2(1-q_1)^2}B_1^{-\frac{3}{2}}-$

$\frac{1}{8(F-B_1)^{-\frac{3}{2}}+(F+f-B_1)^{-\frac{3}{2}}}\sqrt{\alpha q_2^2+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2(1-q_2)^2}<0$, where inequality holds because $\alpha>0$, $0<q_1<1$, $0<q_2<1$, $0<B_1<F$, and $f\geq 0$. Thus, we have that TSI is concave in B_1, which implies exists a unique $B_1^{Ef}\in(0,F)$, such that TSI obtains the maximum value when $B_1=B_1^{Ef}$. Applying the first-order condition on TSI with respect to B_1, we have the B_1^{Ef} is the unique solution to the equation stated in the statement of lemma. □

Proof of Lemma B.2: This proof uses Lemma B.1 presented in Appendix B.2.1. (i) Per Lemma B.1, B_1^{Ef} is the unique solution to $B_1\left(\frac{1}{\sqrt{F-B_1}}+\frac{1}{\sqrt{F+f-B_1}}\right)^2=N_1$, where N_1 is characterized in the statement of lemma 4.1. Applying Implicit Function Theorem to this equation, we obtain $\frac{\partial}{\partial q_1}B_1^{Ef}(q_1)=\frac{\eta_1\eta_2}{\eta_3}\left(q_1-\frac{\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2}{\alpha+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2}\right)$, where $\eta_1\doteq\frac{\sqrt{F-B_1^{Ef}}+\sqrt{F+f-B_1^{Ef}}}{\sqrt{F-B_1^{Ef}}\sqrt{F+f-B_1^{Ef}}}$, $\eta_2\doteq\frac{2\left(\alpha+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2\right)}{N_1\left(\alpha q_2^2+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2(1-q_2)^2\right)}$, and $\eta_3\doteq\frac{2\sqrt{N_1}}{B_1^{Ef\,\frac{3}{2}}}+\frac{1}{(F-B_1^{Ef})^{\frac{3}{2}}}+\frac{1}{\left(F+f-B_1^{Ef}\right)^{\frac{3}{2}}}$. Given that $\eta_1,\eta_2,\eta_3>0$, the sign of $\frac{\partial}{\partial q_1}B_1^{Ef}(q_1)$ depends on the sign of $\left(q_1-\frac{\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2}{\alpha+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2}\right)$, which follows $\frac{\partial}{\partial q_1}B_1^{Ef}(q_1)<0$ if $0<q_1<\hat{q}$ and $\frac{\partial}{\partial q_1}B_1^{Ef}(q_1)\geq 0$ otherwise, where $\hat{q}$ is defined in Proposition 4.1. Further, since $B_2^{Ef}=F-B_1^{Ef}$ (because the budget constraint binds), we have $\frac{\partial}{\partial q_1}B_2^{Ef}(q_1)>0$ if $0<q_1<\hat{q}$ and $\frac{\partial}{\partial q_1}B_2^{Ef}(q_1)\leq 0$ otherwise. Proof for how $B_1^{Ef}(q_2)$ and $B_2^{Ef}(q_2)$ change with respect to q_2 follows similar steps. Therefore, we omit it for brevity.

(ii) Following similar steps as in part (i), we obtain $\frac{\partial}{\partial\rho}B_1^{Ef}(\rho)=\frac{N^{-\frac{3}{2}}\eta_4\left(\frac{\partial}{\partial\rho}N_1(\rho)\right)}{(B_1^{Ef})^{-\frac{3}{2}}+N^{-\frac{1}{2}}\eta_5}$, where $\eta_4\doteq\frac{1}{2\sqrt{F+f-B_1^{Ef}}}+\frac{1}{2\sqrt{F-B_1^{Ef}}}>0$ and $\eta_5\doteq\frac{1}{2\left(F+f-B_1^{Ef}\right)^{\frac{3}{2}}}+\frac{1}{2\left(F-B_1^{Ef}\right)^{\frac{3}{2}}}>0$. Since $\eta_4,\eta_5>0$, and $N_1>0$, the sign of $\frac{\partial}{\partial\rho}B_1^{Ef}(\rho)$ depends on the sign of $\frac{\partial}{\partial\rho}N_1(\rho)$. By proof of Proposition 4.1, we have N_1 increases in ρ when $q_1<q_2$ and N_1 decreases in ρ otherwise. Therefore, we conclude $\frac{\partial}{\partial\rho}B_1^{Ef}(\rho)>0$ when $q_1<q_2$ and $\frac{\partial}{\partial\rho}B_1^{Ef}(\rho)\leq 0$ otherwise. Further, since $B_2^*=F-B_1^*$ (because the constraint given by equation (4.5)

binds; see Lemma 4.1), we have that B_2^* decreases in ρ when $q_1 < q_2$ and increases in ρ otherwise.

Following similar steps, we can prove how B_i^{Ef} changes with respect to α. For brevity, we omit it. $\square$

Proof of Lemma B.3: Substituting $f = 0$ in the expression of B_1^{Ef} characterized in the statement of Lemma B.1, we have $B_1^{Ef} = \frac{N_1 F}{N_1+1}$, where N_1 is characterized in the statement of Lemma 4.1. Denote the expected social impact in area 1 under the efficiency-focused method by $V_1^{Ef}(\cdot)$. Denote the expected social impact in area 2 if no additional funds become available and if additional funds become available under the efficiency-focused method by $V_2^{Ef}(\cdot)$ and $v_2^{Ef}(\cdot)$, respectively.

(i) Substituting B_1^{Ef} in the expression of $V_1(\cdot)$ and $\frac{1}{2}(V_2(\cdot) + v_2(\cdot))$ as characterized in the proof of Lemma 4.1, we have the difference of normalized total expected social impact between areas 1 and 2 under the efficiency-focused method, $\frac{V_1^{Ef}(\cdot)}{\theta_1} - \frac{\frac{1}{2}\left(V_2^{Ef}(\cdot)+v_2^{Ef}(\cdot)\right)}{\theta_2} = \left(N_1 - \frac{1}{\theta_2}\right)\sqrt{\frac{F\left(\alpha q_2^2+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2(1-q_2)^2\right)}{N+1}}$. Hence, the resulting inequity outcomes under the efficiency-focused allocation method is $I^{Ef} = \left|\frac{V_1^{Ef}(\cdot)}{\theta_1} - \frac{\frac{1}{2}\left(V_2^{Ef}(\cdot)+v_2^{Ef}(\cdot)\right)}{\theta_2}\right| = \left|\left(N_1 - \frac{1}{\theta_2}\right)\sqrt{\frac{F\left(\alpha q_2^2+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2(1-q_2)^2\right)}{N+1}}\right|$. Since the inequity outcomes under the equity-ensuring method is zero, i.e., $I^{Eq} = 0$ (by equation (4.3)), the difference in the inequity outcomes under the efficiency-focused method versus the equity-ensuring method is: $I^{Ef} - I^{Eq} = I^{Ef}$.

(ii) Using the expression of B_1^{Ef}, we have $TSI^{Ef} = V_1^{Ef}(\cdot) + \frac{1}{2}(V_2^{Ef}(\cdot) + v_2^{Ef}(\cdot)) = \sqrt{F(N_1+1)\left(\alpha q_2^2 + \left(\rho\sqrt{\beta_H} + (1-\rho)\sqrt{\beta_L}\right)^2(1-q_2)^2\right)}$. Substituting $f = 0$ in the expression of B_i^* characterized in the statement of Lemma B.1, we have $TSI^{Eq} = V_1(\cdot) + \frac{1}{2}(V_2(\cdot) + v_2(\cdot)) = (1+\theta_2 N_1)\sqrt{\frac{\alpha q_2^2+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2(1-q_1)^2}{\theta_2^2 N_1+1}}$. Hence, the difference of total expected social impact under the efficiency-focused method versus equity-ensuring method, $TSI^{Ef} - TSI^{Eq}$, is characterized in the statement of Lemma B.3(ii). Further, since $F > 0$, the sign of $TSI^{Ef} - TSI^{Eq}$ depends on

the sign of $\sqrt{(N_1+1)(N_1\theta_2^2+1)} - (\theta_2+1)\sqrt{N_1}$. Since $\left(\sqrt{(N_1+1)(N_1\theta_2^2+1)}\right)^2 - \left((\theta_2+1)\sqrt{N_1}\right)^2 = (N_1\theta_2-1)^2 > 0$ and $\theta_2 > 0$, we conclude that $TSI^{Ef} - TSI^{Eq} > 0$. □

Proof of Lemma B.4: This proof uses Lemma B.3 presented in Appendix B.2.1. The funding agency allocates initial funds F to service agency i based on the relative size of the pool of the families who need subsidy assistance in service area i. Denote B_i^{Fo} the level of initial funds received by service agency i under this formula-based allocation method. Therefore, $B_i^{Fo} = \left(\frac{\theta_i}{\theta_1+\theta_2}\right)F$. After receiving initial funds, each service agency distributes them between quality improvement and outreach activities based on the proportion of subsidy-accepting and non-accepting service providers in its area, denoted by X_i^{Fo} and Ψ_i^{Fo}, respectively. Hence, $X_i^{Fo} = q_i\left(\frac{\theta_i}{\theta_1+\theta_2}\right)F$ and $\Psi_i^{Fo} = (1-q_i)\left(\frac{\theta_i}{\theta_1+\theta_2}\right)F$. Similarly, when additional funds f become available to service agency 2 in the future, service agency 2 distributes it between two provider-facing activities within its area proportionally, denoted by χ_2^{Fo} and ψ_2^{Fo}, such that $\psi_2^{Fo} = q_2 f$ and $\chi_2^{Fo} = (1-q_2)f$.

(i) Denote the expected social impact in area 1 under the formula-based method by $V_1^{Fo}(\cdot)$. Denote the expected social impact in area 2 if no additional funds become available and if additional funds become available under the formula-based method by $V_2^{Fo}(\cdot)$ and $v_2^{Fo}(\cdot)$, respectively. Substituting B_1^{Fo} in the expression of $V_1(\cdot)$ and $\frac{1}{2}(V_2(\cdot)+v_2(\cdot))$ as characterized in the proof of Lemma 4.1, we have the difference in the normalized total expected social impact between areas 1 and 2 under the formula-based method, $\frac{V_1^{Fo}(\cdot)}{\theta_1} - \frac{\frac{1}{2}\left(V_2^{Fo}(\cdot)+v_2^{Fo}(\cdot)\right)}{\theta_2} = \frac{2\theta_2 n_1\sqrt{F} - n_2\left(\sqrt{\theta_2 F}+\sqrt{\theta_2 F+(1+\theta_2)f}\right)}{2\theta_2\sqrt{1+\theta_2}}$, where $n_1 \doteq \left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)(1-q_1)^{\frac{3}{2}} + \sqrt{a}q_1^{\frac{3}{2}}$, $n_2 \doteq \left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)(1-q_2)^{\frac{3}{2}} + \sqrt{a}q_2^{\frac{3}{2}}$. Given that $I^{Fo} = \left|\frac{V_1^{Fo}(\cdot)}{\theta_1} - \frac{\frac{1}{2}\left(V_2^{Fo}(\cdot)+v_2^{Fo}(\cdot)\right)}{\theta_2}\right|$ and the inequity outcomes under the equity-ensuring method is zero (i.e., $I^{Eq} = 0$, which is by equation (4.3)), the difference in the inequity outcomes under the formula-based versus the equity-

ensuring methods is: $I^{Fo} - I^{Eq} = I^{Fo}$, which is characterized in the statement of Lemma B.4(i).

(ii) Following similar steps as in part (i), we have $TSI^{Fo} = V_1^{Fo}(\cdot) + \frac{V_2^{Fo}(\cdot)+v_2^{Fo}(\cdot)}{2} = \frac{2n_1\sqrt{F}+\left(\sqrt{\theta_2 F}+\sqrt{\theta_2 F+n_2(1+\theta_2)f}\right)}{2\sqrt{1+\theta_2}}$. Using the expression of TSI^{Eq} expressed in the proof of Lemma B.3, we have the difference in the total expected social impact under the formula-based method versus equity-ensuring method, $TSI^{Fo} - TSI^{Eq}$, as characterized in the statement of Lemma B.4(ii), where $n_3 \doteq \sqrt{4F(F+f)N_1\theta_2^2 - f^2}$, $n_4 \doteq \frac{1}{4}\sqrt{\frac{\alpha q_2^2+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2(1-q_2)^2}{\sqrt{N_1}\theta_2\left(N_1\theta_2^2+1\right)}}$, $n_5 \doteq \frac{1}{2\sqrt{1+\theta_2}}\left(\sqrt{\alpha}q_2^{\frac{3}{2}}+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)(1-q_2)^{\frac{3}{2}}\right)\left(\sqrt{F}\left(2+\sqrt{\theta_2}\right)+\sqrt{\theta_2 F+(1+\theta_2)f}\right)$, $n_6 \doteq 4(F+f)N_1^{\frac{3}{2}}\theta_2^3 + \sqrt{N_1}\theta_2\left(2F+3f\right)$, and $n_7 \doteq 4FN_1^{\frac{3}{2}}\theta_2^3 + \sqrt{N_1}\theta_2\left(2F-f\right)$. □

B.8 Proofs of Lemmas Referred in Section B.4

Proof of Lemma B.5: We follow similar steps as in the proof of Lemma 4.1 to solve the funding agency's optimization problem using backward induction.

I. Additional funds allocation problem: Since $v_2^H(\cdot)$ is a sum of a concave increasing function in χ_2^H (independent of ψ_2^H) and a concave increasing function in ψ_2^H (independent of χ_2^H), we can conclude that $v_2^H(\cdot)$ is jointly concave in χ_2^H and ψ_2^H. After substituting $\psi_2^H = f - \chi_2^H$ (because service agency 2's budget constraint binds) in the expression of $v_2^H(\cdot)$ and applying the first-order conditions, we have expressions of χ_2^{H*} and ψ_2^{H*} as functions of X_2 and Ψ_2 as follows: $\chi_2^{H*}(X_2,\Psi_2) = \frac{-(1-q_2)^2\beta_H X_2+\alpha q_2^2\Psi_2+\alpha q_2^2 f}{\alpha q_2^2+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2(1-q_2)^2}$ and $\psi_2^{L*}(X_2,\Psi_2) = \frac{(1-q_2)^2\beta_L(f+X_2)-\alpha q_2^2\Psi_2}{\alpha q_2^2+\left(\rho\sqrt{\beta_L}+(1-\rho)\sqrt{\beta_L}\right)^2(1-q_2)^2}$. Next, considering $v_2^L(\cdot)$ and following the similar steps as above, we obtain the expressions of χ_2^{L*} and ψ_2^{L*} as functions of X_2 and Ψ_2 as follows: $\chi_2^{L*}(X_2,\Psi_2) = \frac{-(1-q_2)^2\beta_L X_2+\alpha q_2^2\Psi_2+\alpha q_2^2 f}{\alpha q_2^2+\left(\rho\sqrt{\beta_L}+(1-\rho)\sqrt{\beta_L}\right)^2(1-q_2)^2}$ and $\psi_2^{L*}(X_2,\Psi_2) = \frac{(1-q_2)^2\beta_L(f+X_2)-\alpha q_2^2\Psi_2}{\alpha q_2^2+\left(\rho\sqrt{\beta_L}+(1-\rho)\sqrt{\beta_L}\right)^2(1-q_2)^2}$.

II. Initial funds allocation problem: Substituting $X_1^*(B_1)$ and $\Psi_1^*(B_1)$ (characterized in the proof of Lemma 4.1) in $V_1(\cdot)$, we have $V_1(B_1) = \sqrt{\alpha q_1^2 + \left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2(1-q_1)^2}\sqrt{B_1}$. Substituting χ_2^{H*}, ψ_2^{H*},

χ_2^{L*}, ψ_2^{L*}, $B_2 = F - B_1$, $X_2^*(B_2)$, $\Psi_2^*(B_2)$ (as characterized in the proof of Lemma 4.1), and $p = \frac{1}{2}$ in service agency 2's expected social impact function, we have $(1-p)\,V_2\left(X_2, \Psi_2\right) + p\left(\rho v_2^H\left(X_2, \Psi_2, \chi_2^{H*}, \psi_2^{H*}\right) + (1-\rho)\,v_2^L\left(X_2, \Psi_2, \chi_2^{L*}, \psi_2^{L*}\right)\right) =$

$$\frac{1}{2}\sqrt{\alpha q_2^2 + \left(\rho\sqrt{\beta_H} + (1-\rho)\sqrt{\beta_L}\right)^2 (1-q_2)^2}\sqrt{F - B_1} + \\ \frac{1}{2}\sqrt{F + f - B_1}\left(\rho\sqrt{\alpha q_2^2 + \beta_H (1-q_2)^2} + (1-\rho)\sqrt{\alpha q_2^2 + \beta_L (1-q_2)^2}\right).$$

Now, following similar steps as in the proof of Lemma 4.1, it follows that B_1^* and B_2^*, as shown in the statement of lemma, constitute the unique vector that solves the optimization problem under consideration. □

Proof of Lemma B.6: Substituting X_1^* and Ψ_1^* characterized in the proof of Lemma 4.1 in $V_1(\cdot)$, we obtain $V_1\left(B_1\right) = \sqrt{\alpha q_1^2 + \left(\rho\sqrt{\beta_H} + (1-\rho)\sqrt{\beta_L}\right)^2 (1-q_1)^2}\sqrt{B_1}$. Substituting $X_2^*(B_2)$, $\Psi_2^*(B_2)$, χ_2^*, and ψ_2^* (as characterized in the proof of Lemma 4.1), $B_2 = F - B_1$ (because the constraint binds), and $p = \frac{1}{2}$ in service agency 2's expected social impact function, we have $(1-p)\,V_2\left(\cdot\right) + p\left(V_2\left(\cdot\right) + \delta\left(V_2\left(\cdot\right) - v_2\left(\cdot\right)\right)\right) = \frac{\sqrt{\alpha q_2^2 + \left(\rho\sqrt{\beta_H} + (1-\rho)\sqrt{\beta_L}\right)^2 (1-q_2)^2}}{2}\left((1-\delta)\sqrt{F - B_1} + \delta\sqrt{F + f - B_1}\right)$. Following similar steps as in the proof of Lemma 4.1, it follows that B_1^* and B_2^*, as shown in the statement of lemma, constitute the unique vector that solves the optimization problem under consideration. □

Proof of Lemma B.7: (i) When the maximum allowed inequity deviation constraint binds (i.e., $I = K$), the optimization problem is equivalent to the optimization problem outlined in (4.2)-(4.6). Then, using Lemma 4.1, the expressions of B_1^* and B_2^* are as expressed in Lemma B.7(i).

(ii) When the maximum allowed inequity deviation constraint does not bind (i.e., $I < K$), the optimization problem is equivalent to the optimization problem in which

the funding agency has efficiency focus, that is, without the equity constraint. Then, using Lemma B.1, the optimal decisions B_1^* and B_2^* are as expressed in Lemma B.7(ii). $\square$

Proof of Lemma B.8: We follow similar steps as in the proof of Lemma 4.1 to solve the funding agency's optimization problem. Under this benchmark setup, service agencies 1 and 2 only need to solve initial funds allocation problem.

Using the proof of Lemma 4.1, for a given B_1, we have two allocation decisions made by service agency 1: $X_1^*(B_1) = \frac{\alpha q_1^2 B_1}{\alpha q_1^2 + \left(\rho\sqrt{\beta_H} + (1-\rho)\sqrt{\beta_L}\right)^2 (1-q_1)^2}$ and $\Psi_1^*(B_1) = \frac{\left(\rho\sqrt{\beta_H} + (1-\rho)\sqrt{\beta_L}\right)^2 (1-q_1)^2 B_1}{\alpha q_1^2 + \left(\rho\sqrt{\beta_H} + (1-\rho)\sqrt{\beta_L}\right)^2 (1-q_1)^2}$. Since service agency 2 has $B_2 + pf$ amount of total expected funds, similarly, we have $X_2^f(B_2) = \frac{\alpha q_1^2 (B_2 + pf)}{\alpha q_2^2 + \left(\rho\sqrt{\beta_H} + (1-\rho)\sqrt{\beta_L}\right)^2 (1-q_2)^2}$ and $\Psi_2^f(B_2) = \frac{\left(\rho\sqrt{\beta_H} + (1-\rho)\sqrt{\beta_L}\right)^2 (1-q_2)^2 (B_2 + pf)}{\alpha q_2^2 + \left(\rho\sqrt{\beta_H} + (1-\rho)\sqrt{\beta_L}\right)^2 (1-q_2)^2}$. Using $B_2 = F - B_1$ (because the budget constraint binds) and $p = \frac{1}{2}$ in these aforementioned expressions and then replacing them in the expressions of $V_1(\cdot)$ and $\frac{1}{2}\left(V_2(\cdot) + v_2(\cdot)\right)$, we obtain:

$V_1(B_1) = \sqrt{\alpha q_1^2 + \left(\rho\sqrt{\beta_H} + (1-\rho)\sqrt{\beta_L}\right)^2 (1-q_1)^2}\sqrt{B_1}$ and

$V_2(B_2) = \sqrt{\alpha q_2^2 + \left(\rho\sqrt{\beta_H} + (1-\rho)\sqrt{\beta_L}\right)^2 (1-q_2)^2}\sqrt{B_2 + \frac{f}{2}}$.

Following similar steps as in the proof of Lemma 4.1, it follows that B_1^* and B_2^*, as shown in the statement of Lemma B.11, constitute the unique vector that solves the optimization problem under consideration.

Compare the expressions of B_1^* presented in this lemma versus that presented in Lemma 4.1, we have their difference, which we denote by $B_1^{info} = \frac{1}{4\left(\theta_2^2 N_1 + 1\right)\theta_2^2 N_1}\Big((2F + f)N_1\theta_2^2 - 2\sqrt{N_1\theta_2^2\left(FN_1(F+f)\theta_2^2 - \frac{f^2}{4}\right)}\Big)$, where N_1 is characterized in the statement of Lemma 4.1. The sign of B_1^{info} depends on the sign of $(2F + f)N_1\theta_2^2 - 2\sqrt{N_1\theta_2^2\left(FN_1(F+f)\theta_2^2 - \frac{f^2}{4}\right)}$. Since $F > 0$ and $f \geq 0$, we conclude the sign of $(2F + f)N_1\theta_2^2 - 2\sqrt{N_1\theta_2^2\left(FN_1(F+f)\theta_2^2 - \frac{f^2}{4}\right)}$ is equivalent to the sign of $((2F + f)N_1\theta_2^2) -$

$\left(2\sqrt{N_1\theta_2^2\left(FN_1(F+f)\theta_2^2-\frac{f^2}{4}\right)}\right) = N_1 f^2\theta_2^2(\theta_2^2 N_1+1) > 0$. Therefore, we have $B_1^{info} > 0$. □

Proof of Lemma B.9: We follow similar steps as in the proof of Lemma 4.1 to solve the funding agency's optimization problem using backward induction.

I. Additional funds allocation problem. Substituting $\psi_{2u} = f - \chi_{2r} - \psi_{2r} - \chi_{2u}$ in $v_2\left(X_{2r}, \Psi_{2r}, X_{2u}, \Psi_{2u}, \chi_{2r}, \psi_{2r}, \chi_{2u}, \psi_{2u}\right)$ and applying the first-order conditions, we obtain the following expressions of $\chi_{2r}^*\left(X_{2r}, \Psi_{2r}, X_{2u}, \Psi_{2u}\right)$, $\chi_{2u}^*\left(X_{2r}, \Psi_{2r}, X_{2u}, \Psi_{2u}\right)$, $\psi_{2r}^*\left(X_{2r}, \Psi_{2r}, X_{2u}, \Psi_{2u}\right)$, and $\psi_{2u}^*\left(X_{2r}, \Psi_{2r}, X_{2u}, \Psi_{2u}\right)$ as functions of X_{2r}, Ψ_{2r}, X_{2u}, and Ψ_{2u}:

$$\chi_{2r}^*(\cdot) = \frac{-\left(E_1\left(2r_2^2-2r_2+1\right)(1-q_2)^2+\alpha(1-r_2)^2q_2^2\right)X_{2r}+\alpha q_2^2r_2^2(f+X_{2u}+\Psi_{2r}+\Psi_{2u})}{\left(\alpha q_2^2+E_1(1-q_2)^2\right)\left(2r_2^2-2r_2+1\right)},$$

$$\chi_{2u}^*(\cdot) = \frac{-\left(E_1\left(2r_2^2-2r_2+1\right)(1-q_2)^2+\alpha r_2^2q_2^2\right)X_{2u}+\alpha q_2^2(1-r_2)^2(f+X_{2r}+\Psi_{2r}+\Psi_{2u})}{\left(\alpha q_2^2+E_1(1-q_2)^2\right)\left(2r_2^2-2r_2+1\right)},$$

$$\psi_{2r}^*(\cdot) = \frac{-\left(E_1(1-r_2)^2(1-q_2)^2+\alpha\left(2r_2^2-2r_2+1\right)q_2^2\right)\Psi_{2r}+E_1(1-q_2)^2r_2^2(f+X_{2r}+X_{2u}+\Psi_{2u})}{\left(\alpha q_2^2+E_1(1-q_2)^2\right)\left(2r_2^2-2r_2+1\right)},$$

$\psi_{2u}^*(\cdot) = f - \chi_{2r}^* - \psi_{2r}^* - \chi_{2u}^*$, where $E_1 \doteq \left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2$.

II. Initial funds allocation problem. For a given B_i, substituting $\Psi_{iu} = B_i - X_{ir} - \Psi_{ir} - X_{iu}$ in $V_i\left(X_{ir}, \Psi_{ir}, X_{iu}, \Psi_{iu}\right)$ and applying the first-order conditions, the obtained expressions of X_{ir}^*, Ψ_{ir}^*, X_{iu}^*, and Ψ_{iu}^* are as follows:

$$X_{ir}^*(B_i) = \frac{\alpha q_i^2r_i^2B_i}{\left(\alpha q_i^2+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2(1-q_i)^2\right)\left(2r_i^2-2r_i+1\right)},$$

$$X_{iu}^*(B_i) = \frac{\alpha q_i^2(1-r_i)^2B_i}{\left(\alpha q_i^2+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2(1-q_i)^2\right)\left(2r_i^2-2r_i+1\right)},$$

$$\Psi_{ir}^*(B_i) = \frac{\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2(1-q_i)^2r_i^2B_i}{\left(\alpha q_i^2+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2(1-q_i)^2\right)\left(2r_i^2-2r_i+1\right)},$$

$$\Psi_{iu}^*(B_i) = \frac{\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2(1-q_i)^2(1-r_i)^2B_i}{\left(\alpha q_i^2+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2(1-q_i)^2\right)\left(2r_i^2-2r_i+1\right)}.$$

Using $B_2 = F - B_1$ in expressions above and replacing them in the expressions of $V_1(\cdot)$ and $\frac{1}{2}\left(V_2(\cdot)+v_2(\cdot)\right)$, we have:

$V_1(B_1) = \sqrt{\left(\alpha q_1^2+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2(1-q_1)^2\right)(r_1^2+(1-r_1)^2)}\sqrt{B_1}$, and

$$\frac{1}{2}\left(V_2(B_1)+v_2(B_1)\right) = \frac{\sqrt{\left(\alpha q_2^2+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2(1-q_2)^2\right)\left(r_2^2+(1-r_2)^2\right)}\left(\sqrt{F-B_1}+\sqrt{F+f-B_1}\right)}{2}.$$

Following similar steps as in the proof of Lemma 4.1, it follows that B_1^* and B_2^*, as shown in the statement of Lemma B.9, constitute the unique vector that solves the optimization problem under consideration. $\square$

Proof of Lemma B.10: Consider the optimal decisions B_1^* and B_2^* as characterized in Lemma B.9.

(i) Following similar steps as in the proof of Proposition 4.1, we can show that $B_1^*(N_r)$ decreases in N_r. Differentiating $N_r(r_1)$ with respect to r_1, we have $\frac{\partial}{\partial r_1} N_r(r_1) = \frac{4\left(\alpha q_1^2 + \left(\rho\sqrt{\beta_H} + (1-\rho)\sqrt{\beta_L}\right)^2 (1-q_1)^2\right)\left(r_1 - \frac{1}{2}\right)}{\left(\alpha q_2^2 + \left(\rho\sqrt{\beta_H} + (1-\rho)\sqrt{\beta_L}\right)^2 (1-q_2)^2\right)\left(r_2^2 + (1-r_2)^2\right)}$. Since $\alpha, \beta_L, \beta_H > 0$, $0 \leq \rho \leq 1$, $0 < q_1 < 1$, $0 < q_2 < 1$, $0 < r_1 < 1$, and $0 < r_2 < 1$, we have that the sign of $\frac{\partial}{\partial r_1} N_r(r_1)$ depends on the sign of $\left(r_1 - \frac{1}{2}\right)$. This implies that $\frac{\partial}{\partial r_1} N_r(r_1) < 0$ when $r_1 < \frac{1}{2}$ and $\frac{\partial}{\partial r_1} N_r(r_1) \geq 0$ otherwise. Thus, we can conclude $B_1^*(r_1)$ increases in r_1 when $0 < r_1 < \frac{1}{2}$ and decreases in r_1 otherwise. Proof of how $B_1^*(r_2)$ and $B_2^*(r_2)$ change with respect to r_2 follows similar steps, and we therefore omit it for brevity.

Differentiating $N_r(q_1)$ with respect to q_1, we have $\frac{\partial}{\partial q_1} N_r(q_1) = \frac{2\left(\alpha q_1 - \left(\rho\sqrt{\beta_H} + (1-\rho)\sqrt{\beta_L}\right)^2 (1-q_1)\right)\left(\left(r_1 - \frac{1}{2}\right)^2 + \frac{1}{4}\right)}{\left(\alpha q_2^2 + \left(\rho\sqrt{\beta_H} + (1-\rho)\sqrt{\beta_L}\right)^2 (1-q_2)^2\right)\left(\left(r_2 - \frac{1}{2}\right)^2 + \frac{1}{4}\right)}$. Since $\alpha, \beta_L, \beta_H > 0$, $0 \leq \rho \leq 1$, $0 < q_1 < 1$, $0 < q_2 < 1$, $0 < r_1 < 1$, and $0 < r_2 < 1$, we have that the sign of $\frac{\partial}{\partial q_1} N_r(q_1)$ depends on the sign of $\alpha q_1 - \left(\rho\sqrt{\beta_H} + (1-\rho)\sqrt{\beta_L}\right)^2 (1-q_1)$. Following similar steps in the proof of Proposition 4.1(i), we can conclude $\frac{\partial}{\partial q_1} N_r(q_1) < 0$ when $q_1 < \hat{q}_1$ and $\frac{\partial}{\partial q_1} N_r(q_1) \geq 0$ otherwise, where $\hat{q}_1$ is characterized in Proposition 4.1. The rest of the proof is similar to the proof of Proposition 4.1(i). For brevity, we omit details.

(ii) The proof for part *(ii)* follows similar steps as the proof of Proposition 4.1 in Appendix B.5. $\square$

Proof of Lemma B.11: We follow similar steps as in the proof of Lemma 4.1 to solve the funding agency's optimization problem using backward induction.

I. Additional funds allocation problem. Substituting $\psi_2 = f - \chi_2$ in $v_2\left(X_2, \Psi_2\right)$ and applying the first-order conditions, we obtain the following expressions of χ_2^* and ψ_2^* as functions of X_2 and Ψ_2: $\chi_2^*(X_2, \Psi_2) = \frac{-\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2(1-q_2)^2 X_2+\alpha_2 q_2^2\Psi_2+\alpha_2 q_2^2 f}{\alpha_2 q_2^2+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2(1-q_2)^2}$ and $\psi_2^*(X_2, \Psi_2) = \frac{\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2(1-q_2)^2(f+X_2)-\alpha_2 q_2^2\Psi_2}{\alpha_2 q_2^2+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2(1-q_2)^2}$.

II. Initial funds allocation problem. For a given B_i, substituting $\Psi_i = B_i - X_i$ in $V_i\left(X_i, \Psi_i\right)$ and applying the first-order conditions, the obtained expressions of X_i^* and Ψ_i^* are as follows: $X_i^*(B_i) = \frac{\alpha_i q_i^2 B_i}{\alpha_i q_i^2+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2(1-q_i)^2}$ and $\Psi_i^*(B_i) = \frac{\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2(1-q_i)^2 B_i}{\alpha_i q_i^2+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2(1-q_i)^2}$.

Using $B_2 = F - B_1$ (because the budget constraint binds) in these aforementioned expressions and then replacing them in the expressions of $V_1(\cdot)$ and $\frac{1}{2}\left(V_2\left(\cdot\right) + v_2\left(\cdot\right)\right)$, we have: $V_1\left(B_1\right) = \sqrt{\alpha_1 q_1^2 + \left(\rho\sqrt{\beta_H} + (1-\rho)\sqrt{\beta_L}\right)^2(1-q_1)^2}\sqrt{B_1}$, and $\frac{1}{2}\big(V_2\left(B_1\right) + v_2\left(B_1\right)\big) = \frac{\sqrt{\alpha_2 q_2^2+\left(\rho\sqrt{\beta_H}+(1-\rho)\sqrt{\beta_L}\right)^2(1-q_2)^2}\left(\sqrt{F-B_1}+\sqrt{F+f-B_1}\right)}{2}$.

Following similar steps as in the proof of Lemma 4.1, we have B_1^* and B_2^* (as shown in the statement of Lemma B.11) constitute the unique vector that solves the optimization problem under consideration. $\square$

APPENDIX C

APPENDIX FOR CHAPTER 5: SUBSIDIZING SOCIAL WELFARE PROGRAMS: CONTRACTED SLOTS OR VOUCHERS?

C.1 Table

Table C.1. Description of Model Notation

Notation	
$K_{i\in\{L,H\}}$	Capacity of each low(high)-quality provider
$d_{i\in\{L,H\}}$	Demand for each low(high)-quality provider in the private market
$\underline{D}_L$	Minimum demand for each low-quality provider in the private market
$\bar{D}_L$	Maximum demand for each low-quality provider in the private market
δ	Relative demand for a high-quality provider to a low-quality provider
$p_{i\in\{L,H\}}$	Market price per private-pay client at a low(high)-quality provider
$s^v_{i\in\{L,H\}}$	Reimbursement rate for a low(high)-quality provider in the voucher program
$\rho_{i\in\{L,H\}}$	Voucher fill-out rate at a low(high)-quality provider
$d^c_{i\in\{L,H\}}$	Demand for each low(high)-quality provider in the private market in the contracted slot program
$d_{i\in\{lo,ho\}}$	Demand adjustment factor for each low(high)-quality provider in the contracted slot program
c_v	Service provider's fixed cost of managing the voucher program
c_c	Service provider's fixed cost of managing the contracted slot program
$c_{improve}$	Cost of improving quality from low-quality to high-quality for a low-quality provider
s^c_H	Reimbursement rate for the high-quality provider in the contracted slot program
Decision Variables and Functions	
f	Proportion of low-quality providers improve their quality levels
$x^c_{i\in\{L,H\}}$	Number of slots low(high)-quality provider reserves for the contracted slot program
$\Pi^{v0}_{i\in\{L,H\}}$	Low(High)-quality provider's expected payoff under no participation in the voucher program
$\Pi^{v1}_{i\in\{L,H\}}$	Low(High)-quality provider's expected payoff under participation in the voucher program
$\Pi^{c0}_{i\in\{L,H\}}$	Low(High)-quality provider's expected payoff under no participation in the contracted slot program
Π^{c1}_L	Low-to-high-quality provider's expected payoff under participation in the contracted slot program
Π^{c1}_H	High-quality provider's expected payoff under participation in the contracted slot program

C.2 Additional Details of the Case Study

We use data from the child care literature, Massachusetts Department of Early Education and Care (MEEC), and our interviews with managers in several daycare providers in Boston to estimate the values of the model parameters needed to solve the above-mentioned game-theoretical model setup under study.

Type, Number, Quality Level, and Capacity of Service Providers. Based on various criteria, such as staff qualifications and professional development, curriculum and learning activities, and administration and business practices, governments have used

a quality rating and improvement system (QRIS), a systemic approach to assess the level of quality in child care domain (Department of Health & Human Services 2023). Specifically, there are four quality levels, from 1 to 4. 1 represents the lowest quality level while 4 represents the highest quality level. For simplicity, governments and child care literature categorize these providers into two main types: those who have a QRIS rating of 2 or below are considered low-quality service providers; those who have a QRIS rating of 3 or above are considered high-quality service providers (Campbell and Patil 2019). To align with practice, we consider there are two types of service providers in the private market–low-quality providers and high-quality providers. Assuming the average QRIS rating for low-quality service providers to be 2 and high-quality service providers to be 3 (based on our interviews), we consider $Q_H = {}^3\!/_2 \times Q_L$, which implies that $Q_H = 1.5$ when Q_L is normalized to 1.

To estimate numbers and capacities of service providers in Boston, Massachusetts, we combine information on the (i) MEEC's service providers database, (ii) child care literature, and (iii) our interviews with several service providers operating in the Boston area (Campbell and Patil 2019, Massachusetts Department of Early Education and Care 2023). Based on these information, we estimate the number of low- and high-quality service providers in the Boston area as $N_L = 725$ and $N_H = 97$. The capacity of low- and high-quality service providers in the Boston area is estimated as $K_L = 60$ and $K_H = 100$.

Market Price for Service Providers. Using child care reports, we have the average market price at a low-quality service provider and a high-quality service provider as \$88 and \$120 daily, respectively (Center for Early Learning Funding Equity 2023). Since service providers usually plan for their operations based on a one-year horizon, we estimate the related price and cost based on a year in this case study. Consider each service provider operates only on weekdays, hence, 260 days. Therefore, the

market price for low-quality and high-quality service providers in a year is $p_L = \$88 \times 260 = \$22,880$ and $p_H = \$120 \times 260 = \$31,200$, respectively.

Reimbursement Rate for Service Providers in Subsidy Voucher Program and Contracted Slot Program. Using data provided by MEEC, we have the daily reimbursement rate for a low-quality service provider and a high-quality service provider as \$88 and \$94, respectively, in the subsidy voucher program; we also have the daily reimbursement rate for a provider in the contracted slot program as \$113 (Massachusetts Department of Early Education and Care 2022). Consider each service provider operates only on weekdays, hence, 260 days. Therefore, the reimbursement rate for low-quality and high-quality service providers in the subsidy voucher program in a year is $s_L^v = \$88 \times 260 = \$22,880$ and $s_H^v = \$94 \times 260 = \$24,440$, respectively; the reimbursement rate for service providers in the contracted slot program in a year is $s_H^c = \$113 \times 260 = \$29,380$.

Voucher Fill-Out Rate in Subsidy Voucher Program. Using studies on the child care market in the Boston area, we have the average fill-out rate for voucher-accepting providers in the Boston area across both low- and high-quality service providers are 45% (Campbell and Patil 2019). Further, child care experts and reports point out that, on average, the fill-out rate at a high-quality service provider is much higher than that at a low-quality service provider (Ryan et al. 2011, Krafft et al. 2017). Therefore, we reasonably estimate the fill-out rate at a low-quality provider and a high-quality provider is $\rho_L = 35\%$ and $\rho_H = 55\%$, respectively.

Service Provider's Cost of Managing Subsidy Voucher Program. In Massachusetts, a voucher-accepting provider is required to: (i) ensure that its license is active and up-to-date, (ii) participate in the QRIS in the state, and (iii) track subsidy recipients' attendance, and (iv) submit their records of children's attendance and billing to the state for reimbursement, among others (Schneider et al. 2017). Therefore, using data from child care accreditation association and child care reports, we estimate the

cost of managing the subsidy voucher program in Boston in a year is $c_v = \$50,000$ (Schneider et al. 2017, The Association for Early Learning Leaders 2023, The National Association for the Education of Young Children 2023).

Service Provider's Cost of Managing Contracted Slot Program. A provider in the contracted slot program must not only meet the requirements that a voucher-accepting provider follows, it must also meet the additional requirements. These requirements include using the state's centralized waitlist to identify eligible families to fill their contracted slots, committing employee time to manage contracts, and conducting eligibility authorizations and re-authorizations for beneficiaries using their contracted slots (Schneider et al. 2017). Therefore, the cost of managing a contracted slot program can be much higher than managing a subsidy voucher program. In this case study, we consider it uniformly distributed from $1 \times c_v$ to $4 \times c_v$. Hence, the cost of managing the contracted slot program in Boston in a year $c_c \sim U[\$50,000, \$200,000]$.

Service Provider's Cost of Improving Quality Level. As described earlier, the average quality QRIS rating for low-quality and high-quality providers in Boston is $Q_L = 2$ and $Q_H = 3$, respectively. In order to participate in the contracted slot program, a low-quality provider needs to invest in quality improvement activities and improve the quality level from QRIS Level 2 to Level 3 (Recall that a high-quality provider is expected to have a QRIS level of 3 or above). Using child care studies, we estimate such a investment in quality improvement activities is estimated to be approximately $\$900,000$ (Workman 2021, Center for Early Learning Funding Equity 2023). This investment is used for employees' professional training, designing appropriate curricula that are tailored to the age of the children, maintaining safe and hygienic child care facilities, and expanding the outdoor space for children to move and engage with the natural world (Workman and Ullrich 2017, Department of Health & Human Services 2023). In this case study, we consider this improvement

cost to be uniformly distributed. Accordingly, we have a low-quality provider's cost of improving its quality level $c_{improve} \sim U[\$800,000, \$1,000,000]$.

Demand for Service Provider in Private Market. Through our interviews with experts and survey of the related reports, we learn that the demand for both low-quality and high-quality service providers is not fixed. That is, service providers sometimes can fill all of their slots with clients from the private market, and sometimes they cannot. Therefore, we consider the demand for a service provider in the private market to be uniformly distributed around its capacity, and we use low-quality providers as a benchmark. Recall $K_L = 60$; hence, we estimate the demand for a low-quality provider is uniformly distributed between 40 and 70, i.e., $\underline{D}_L = 40$ and $\bar{D}_L = 70$. Studies show that providing access to high-quality child care can help improve the long-term outcomes of children, including educational attainment, earning capacity, and reduced anxiety and depression (Barnett and Masse 2007, Herbst 2017). As such, managers in child care centers who we interview point out that the demand for high-quality providers is higher than for low-quality providers in the private market. Accordingly, we estimate that the demand for a high-quality provider is 1.6 times the demand for a low-quality provider. That is, $\delta = 1.6$.

Recall that, as explained earlier, in the contracted slot program, with f proportion of low-quality providers improving their quality levels and becoming high-quality providers, the demand for low- and high-quality service providers should be revised accordingly. We use the following procedures to estimate demand adjustment factors d_{lo} and d_{ho}. Consider 99% of low-quality providers improving their quality level. Then, the demand for remaining low-quality providers reaches its maximum level, $\bar{D}_L = 70$. As we described in Section 5.2.1, with f proportion of low-quality providers improving their quality levels, the demand for a low-quality provider is between $\underline{D}_L + d_{lo}f$ and $\bar{D}_L + d_{lo}f$; hence, its average demand is $\frac{1}{2}\left(\underline{D}_L + \bar{D}_L\right) + d_{lo}f$. Therefore, using $\frac{1}{2}\left(\underline{D}_L + \bar{D}_L\right) + d_{lo}f = \frac{1}{2}(40 + 70) + d_{lo}99\% = 70$, we have $d_{lo} = 15.2$. In this

case study, we allow the demand adjustment factor for a high-quality provider, d_{ho}, to be as same as d_{lo}. So, we have $d_{ho} = 15.2$.

C.3 Proofs of Analytical Results

Additional technical lemmas, which help us outline proofs below, are available in Appendix C.4.

Proof of Proposition 5.1. First, consider low-quality service providers' participation decisions. For each low-quality service provider, to decide whether participate in a subsidy voucher program or not, it compares its expected payoff if it participates in the voucher program (i.e., $\Pi_L^{v1}(\cdot)$ expressed in Section 5.1.1) versus if it does not participate in the voucher program (i.e., $\Pi_L^{v0}(\cdot)$ expressed in Section 5.1.1). Hence, we have $\Pi_L^{v1}(\cdot) - \Pi_L^{v0}(\cdot) = \frac{s_L^v \rho_L (K_L - \underline{D}_L)^2}{2\left(\bar{D}_L - \underline{D}_L\right)} - c_v$. It follows that $\Pi_L^{v1}(\cdot) - \Pi_L^{v0}(\cdot) > 0$ when $c_v < \frac{s_L^v \rho_L (K_L - \underline{D}_L)^2}{2\left(\bar{D}_L - \underline{D}_L\right)}$ and $\Pi_L^{v1}(\cdot) - \Pi_L^{v0}(\cdot) \leq 0$ otherwise. Since ρ_L, s_L^v, $\bar{D}_L$, and $\underline{D}_L$ are same for each low-quality service provider, we conclude each low-quality service provide makes the same participation decision.

Consider high-quality service providers' participation decisions. Similarly, comparing each high-quality service provider's expected payoff if it participate in the subsidy voucher program versus if it does not participates in the voucher program, we have $\Pi_H^{v1}(\cdot) - \Pi_H^{v0}(\cdot) = \frac{s_H^v \rho_H (K_H - \delta \underline{D}_L)^2}{2\delta\left(\bar{D}_L - \underline{D}_L\right)} - c_v$. It follows that $\Pi_H^{v1}(\cdot) - \Pi_H^{v0}(\cdot) > 0$ when $c_v < \frac{s_H^v \rho_H (K_H - \delta \underline{D}_L)^2}{2\delta\left(\bar{D}_L - \underline{D}_L\right)}$ and $\Pi_H^{v1}(\cdot) - \Pi_H^{v0}(\cdot) \leq 0$ otherwise. Similarly, we conclude each high-quality service provide makes the same participation decision.

Next, we compare $\frac{s_L^v \rho_L (K_L - \underline{D}_L)^2}{2\left(\bar{D}_L - \underline{D}_L\right)}$ with $\frac{s_H^v \rho_H (K_H - \delta \underline{D}_L)^2}{2\delta\left(\bar{D}_L - \underline{D}_L\right)}$. Since $0 < \rho_L < 1$, $0 < \rho_H < 1$, $s_L^v > 0$, $s_H^v > 0$, $\bar{D}_L > \underline{D}_L$, and $\delta > 0$, the comparison of $\frac{s_L^v \rho_L (K_L - \underline{D}_L)^2}{2\left(\bar{D}_L - \underline{D}_L\right)}$ and $\frac{s_H^v \rho_H (K_H - \delta \underline{D}_L)^2}{2\delta\left(\bar{D}_L - \underline{D}_L\right)}$ is equivalent to the comparison of $\delta \left(\frac{\rho_L}{\rho_H}\right) \left(\frac{K_L - \underline{D}_L}{K_H - \delta \underline{D}_L}\right)^2$ and s_H^v. Therefore, we have $\frac{s_L^v \rho_L (K_L - \underline{D}_L)^2}{2\left(\bar{D}_L - \underline{D}_L\right)} > \frac{s_H^v \rho_H (K_H - \delta \underline{D}_L)^2}{2\delta\left(\bar{D}_L - \underline{D}_L\right)}$ when $s_H^v < \delta \left(\frac{\rho_L}{\rho_H}\right) \left(\frac{K_L - \underline{D}_L}{K_H - \delta \underline{D}_L}\right)^2$ and $\frac{s_L^v \rho_L (K_L - \underline{D}_L)^2}{2\left(\bar{D}_L - \underline{D}_L\right)} \leq \frac{s_H^v \rho_H (K_H - \delta \underline{D}_L)^2}{2\delta\left(\bar{D}_L - \underline{D}_L\right)}$ otherwise. Denote $\dot{s} \doteq \delta \left(\frac{\rho_L}{\rho_H}\right) \left(\frac{K_L - \underline{D}_L}{K_H - \delta \underline{D}_L}\right)^2$. Then, we consider the following two cases (I and II) based on the magnitude of s_H^v.

Case I: Suppose $s_H^v < \dot{s}$. From above, we have $\frac{s_L^v \rho_L (K_L - \underline{D}_L)^2}{2\left(\bar{D}_L - \underline{D}_L\right)} > \frac{s_H^v \rho_H (K_H - \delta \underline{D}_L)^2}{2\delta\left(\bar{D}_L - \underline{D}_L\right)}$. We next consider the following three sub-cases based on the magnitude of c_v

(I-a) Suppose $c_v \leq \frac{s_H^v \rho_H (K_H - \delta \underline{D}_L)^2}{2\delta\left(\bar{D}_L - \underline{D}_L\right)}$, then $\Pi_L^{v1}(\cdot) - \Pi_L^{v0}(\cdot) \geq 0$ and $\Pi_H^{v1}(\cdot) - \Pi_H^{v0}(\cdot) \geq 0$. This implies that both low- and high-quality service providers participate in the subsidy voucher program.

(I-b) Suppose $\frac{s_H^v \rho_H (K_H - \delta \underline{D}_L)^2}{2\delta\left(\bar{D}_L - \underline{D}_L\right)} < c_v < \frac{s_L^v \rho_L (K_L - \underline{D}_L)^2}{2\left(\bar{D}_L - \underline{D}_L\right)}$, then $\Pi_L^{v1}(\cdot) - \Pi_L^{v0}(\cdot) > 0$ and $\Pi_H^{v1}(\cdot) - \Pi_H^{v0}(\cdot) < 0$. This implies that all low-quality service providers participate in and none of high-quality service providers participate in the subsidy voucher program.

(I-c) Suppose $c_v \geq \frac{s_L^v \rho_L (K_L - \underline{D}_L)^2}{2\left(\bar{D}_L - \underline{D}_L\right)}$, then $\Pi_L^{v1}(\cdot) - \Pi_L^{v0}(\cdot) \leq 0$ and $\Pi_H^{v1}(\cdot) - \Pi_H^{v0}(\cdot) \leq 0$. This implies that none of the low- and high-quality service providers participate in the subsidy voucher program.

Case II: Suppose $s_H^v \geq \dot{s}$. From above, we have $\frac{s_L^v \rho_L (K_L - \underline{D}_L)^2}{2\left(\bar{D}_L - \underline{D}_L\right)} \leq \frac{s_H^v \rho_H (K_H - \delta \underline{D}_L)^2}{2\delta\left(\bar{D}_L - \underline{D}_L\right)}$. We next consider the following three sub-cases based on the magnitude of c_v

(II-a) Suppose $c_v \leq \frac{s_L^v \rho_L (K_L - \underline{D}_L)^2}{2\left(\bar{D}_L - \underline{D}_L\right)}$, then $\Pi_L^{v1}(\cdot) - \Pi_L^{v0}(\cdot) \geq 0$ and $\Pi_H^{v1}(\cdot) - \Pi_H^{v0}(\cdot) \geq 0$. This implies that both low- and high-quality service providers participate in the subsidy voucher program.

(II-b) Suppose $\frac{s_L^v \rho_L (K_L - \underline{D}_L)^2}{2\left(\bar{D}_L - \underline{D}_L\right)} < c_v < \frac{s_H^v \rho_H (K_H - \delta \underline{D}_L)^2}{2\delta\left(\bar{D}_L - \underline{D}_L\right)}$, then $\Pi_L^{v1}(\cdot) - \Pi_L^{v0}(\cdot) < 0$ and $\Pi_H^{v1}(\cdot) - \Pi_H^{v0}(\cdot) > 0$. This implies that none of low-quality service providers participate in and all high-quality service providers participate in the subsidy voucher program.

(II-c) Suppose $c_v \geq \frac{s_H^v \rho_H (K_H - \delta \underline{D}_L)^2}{2\delta\left(\bar{D}_L - \underline{D}_L\right)}$, then $\Pi_L^{v1}(\cdot) - \Pi_L^{v0}(\cdot) \leq 0$ and $\Pi_H^{v1}(\cdot) - \Pi_H^{v0}(\cdot) \leq 0$. This implies that none of the low- and high-quality service providers participate in the subsidy voucher program.

The results in the proposition follow by combining the results in cases I and II above and by defining thresholds $\underline{c}_v \doteq \min\left\{\frac{s_H^v \rho_H (K_H - \delta \underline{D}_L)^2}{2\delta\left(\bar{D}_L - \underline{D}_L\right)}, \frac{s_L^v \rho_L (K_L - \underline{D}_L)^2}{2\delta\left(\bar{D}_L - \underline{D}_L\right)}\right\}$ and $\bar{c}_v \doteq \max\left\{\frac{s_H^v \rho_H (K_H - \delta \underline{D}_L)^2}{2\delta\left(\bar{D}_L - \underline{D}_L\right)}, \frac{s_L^v \rho_L (K_L - \underline{D}_L)^2}{2\delta\left(\bar{D}_L - \underline{D}_L\right)}\right\}$. $\square$

Proof of Lemma 5.1. To decide whether participate in a contracted slot program or not, each high-quality service provider compares its expected payoff if it participates in the contracted slot program (i.e., $\Pi_H^{c1}(\cdot)$ expressed in Section 5.2.1) versus if it does not participate in contracted slot programs (i.e., $\Pi_H^{c0}(\cdot)$ expressed in Section 5.2.1). To compute its expected payoff in the contracted slot program, it needs to decide the optimal number of slots that it should reserve for the program. Twice differentiating $\Pi_H^{c1}(\cdot)$ with respect to x_H, we have $\frac{\partial^2}{\partial x_H^2}\Pi_H^{c1}(x_H) = -\frac{p_H}{\delta\left(\bar{D}_L-\underline{D}_L\right)} < 0$, which implies that $\Pi_H^{c1}(x_H)$ is a concave function in x_H. Applying the first-order condition on $\Pi_H^{c1}(x_H)$, we have $x_H^* = \frac{\delta s_H^c\left(\bar{D}_L-\underline{D}_L\right)+p_H\left(K_H+d_{ho}f-\delta\bar{D}_L\right)}{p_H}$. Substituting x_H^* to $\Pi_H^{c1}(x_H)$, we have $\Pi_H^{c1*}(\cdot) - \Pi_H^{c0}(\cdot) = m_1 f^2 + m_2 f + m_3$, where
$m_1 \doteq \frac{d_{ho}^2 p_H}{2\delta\left(\bar{D}_L-\underline{D}_L\right)} > 0$, $m_2 \doteq \frac{d_{ho}\left(K_H p_H-\delta\left(\left(p_H-s_H^c\right)\bar{D}_L+s_H^c\underline{D}_L\right)\right)}{\delta\left(\bar{D}_L-\underline{D}_L\right)}$, and
$m_3 \doteq \frac{\left(p_H\bar{D}_L+s_H^c\left(\bar{D}_L-\underline{D}_L\right)\right)^2\delta^2-2p_H\left(\bar{D}_L K_H p_H-\left(K_H s_H^c-c_c\right)\left(\bar{D}_L-\underline{D}_L\right)\right)\delta+K_H^2 p_H^2}{2p_H\delta\left(\bar{D}_L-\underline{D}_L\right)}$.
Given that m_1, m_2, and m_3 are same for every high-quality service providers, we conclude every high-quality provider makes the same participation decision.

Since $p_H, \delta, d_{ho} > 0$ and $\bar{D}_L - \underline{D}_L > 0$, we conclude the sign of m_2 depends on the sign of $K_H p_H - \delta\left(\left(p_H - s_H^c\right)\bar{D}_L + s_H^c\underline{D}_L\right) = \delta\left(\bar{D}_L - \underline{D}_L\right)\left(s_H^c - \frac{p_H\left(\bar{D}_L-K_H\right)}{\delta\left(\bar{D}_L-\underline{D}_L\right)}\right)$. Since $\delta > 0$ and $\bar{D}_L - \underline{D}_L > 0$, we have the sign of $K_H p_H - \delta\left(\left(p_H - s_H^c\right)\bar{D}_L + s_H^c\underline{D}_L\right)$ depends on the sign of $s_H^c - \frac{p_H\left(\bar{D}_L-K_H\right)}{\delta\left(\bar{D}_L-\underline{D}_L\right)}$, which follows $s_H^c - \frac{p_H\left(\bar{D}_L-K_H\right)}{\delta\left(\bar{D}_L-\underline{D}_L\right)} > 0$ when $s_H^c > \frac{p_H\left(\bar{D}_L-K_H\right)}{\delta\left(\bar{D}_L-\underline{D}_L\right)}$. To better match with the observations from practice that s_H^c is usually relatively high and for analytical tractability, we focus on the case when $s_H^c > \frac{p_H\left(\bar{D}_L-K_H\right)}{\delta\left(\bar{D}_L-\underline{D}_L\right)}$. Our numerical analysis shows that the main insights still hold under a large range of s_H^c. Therefore, we conclude $m_2 > 0$.

Differentiating m_3 with respect to c_c, we have $\frac{\partial}{\partial c_c}m_3(c_c) = -1$. Therefore, m_3 decreases in c_c. Next, we examine the sign of m_3 when $c_c = 0$. Denote m_4 by the expression of m_3 when $c_c = 0$. Hence, $m_4 = \frac{m_5}{2p_H\delta\left(\bar{D}_L-\underline{D}_L\right)}$, where $m_5 \doteq K_H^2 p_H^2 + \left(p_H\bar{D}_L + s_H^c\left(\bar{D}_L - \underline{D}_L\right)\right)^2\delta^2 - 2p_H\left(\bar{D}_L K_H p_H - \left(K_H s_H^c\right)\left(\bar{D}_L - \underline{D}_L\right)\right)\delta$. Since $p_H > 0$, $\delta > 0$, and $\bar{D}_L - \underline{D}_L > 0$, the sign of m_4 depends on the sign of m_5. Ap-

plying the first-order condition with respect to δ on m_5, we have its minima, $\delta = \frac{p_H K_H}{\underline{D}_L s_H^c + \bar{D}_L p_H - \bar{D}_L s_H^c}$. Substituting $\delta = \frac{p_H K_H}{\underline{D}_L s_H^c + \bar{D}_L p_H - \bar{D}_L s_H^c}$ to m_5, we have its minimum value 0. This implies that $m_5 > 0$, or $m_3 > 0$, when $c_c = 0$. Therefore, we conclude there exists a threshold of c_c, denoted by $\dot{c}_c$, such that $m_3 \geq 0$ when $c \leq \dot{c}_c$ and $m_3 < 0$ otherwise. Solving $m_3 = 0$, we have $\dot{c}_c = \frac{\left(\delta\left(p_H \bar{D}_L - s_H^c\left(\bar{D}_L - \underline{D}_L\right)\right) - K_H p_H\right)^2}{2\delta p_H\left(\bar{D}_L - \underline{D}_L\right)}$. Then, we consider the following two cases (I and II) based on the magnitude of c_c.

Case I: Suppose $c \leq \dot{c}_c$, then $m_3 \geq 0$. Since $m_1 > 0$, $m_2 > 0$, and $m_3 \geq 0$, we conclude $\Pi_H^{c1*}(\cdot) - \Pi_H^{c0}(\cdot) \geq 0$ for any $f > 0$. This implies that all high-quality service provider participate in the contracted slot program.

Case II: Suppose $c > \dot{c}_c$, then $m_3 < 0$. Since $m_1 > 0$, $m_2 > 0$, and $m_3 < 0$, we conclude, there exists a threshold of f, denoted by $\bar{f}$, such that $\Pi_H^{c1*}(\cdot) - \Pi_H^{c0}(\cdot) \leq 0$ when $f \leq \bar{f}$ and $\Pi_H^{c1*}(\cdot) - \Pi_H^{c0}(\cdot) > 0$ otherwise. Solving $\Pi_H^{c1*}(\cdot) - \Pi_H^{c0}(\cdot) = 0$, we have $\bar{f} = \frac{\sqrt{2\delta c_c p_H\left(\bar{D}_L - \underline{D}_L\right)} + \left(\delta \bar{D}_L - K_H\right) p_H - \delta s_H^c\left(\bar{D}_L - \underline{D}_L\right)}{p_H d_{ho}}$, which increases in c_c. By proof of Proposition 5.2, we have when $c_c > \ddot{c}_c$, $f^* = 0$ or $f^* = f_2$ that are characterized in the proof of Proposition 5.2. Align with the practice wherein c_c is usually high. Next, we focus on this scenario. Differentiating f_2 with respect to c_c, we have $\frac{\partial}{\partial c_c} f_2(c_c) = -\sqrt{\frac{p_H\left(\bar{D}_L - \underline{D}_L\right)}{h1\underline{D}_L + h2\bar{D}_L + h3}} < 0$, where $h_1 \doteq \left(\delta\left(p_H^2 - \left(s_H^c\right)^2\right) d_{lo} + 2h_4 p_H\right) d_{lo} p_L + \left(h_4 + C\right)^2 p_H$, $h_2 \doteq \delta p_L d_{lo}^2 (p_H - s_H^c)^2 - p_H h_4^2$, $h_3 \doteq -2p_L p_H d_{lo}\left(p_L d_{lo} k_L + \left(-K_L s_H^c + c_c\right) d_{lo} + K_L h_4\right)$, and $h_4 \doteq d_{ho}\left(p_H - s_H^c\right) + C$. Therefore, these exists a threshold of c_c, denoted by $\tilde{c}_c$, such that $\bar{f} < f^*$ when $c_c < \tilde{c}_c$ and $\bar{f} \geq f^*$ otherwise. Next, we consider the following three sub-cases based on the magnitude of c_c.

(II-a) Suppose $\tilde{c}_c < \dot{c}_c$, then $\bar{f} > f^*$ for any $c > \dot{c}_c$. This implies that $\Pi_H^{c1*}(\cdot) - \Pi_H^{c0}(\cdot) < 0$ for $f = f^*$, i.e., none of the high-quality service providers participate in the contracted slot program.

(II-b) Suppose $\tilde{c}_c \geq \dot{c}_c$, then $\bar{f} \leq f^*$ when $\dot{c}_c \leq c_c < \tilde{c}_c$ and $\bar{f} > f^*$ otherwise. This implies that $\Pi_H^{c1*}(\cdot) - \Pi_H^{c0}(\cdot) \geq 0$ for $f = f^*$ when $\dot{c}_c \leq c_c < \tilde{c}_c$ and $\Pi_H^{c1*}(\cdot) - \Pi_H^{c0}(\cdot) < 0$ for $f = f^*$ otherwise. Therefore, all of the high-quality service providers participate

in the contracted slot program when $\dot{c}_c \leq c_c < \tilde{c}_c$ and none of them participate in the contracted slot program otherwise.

The results in the Lemma follow by combining the results in cases I and II above and by defining a threshold $\hat{c}_c \doteq \max\{\dot{c}_c, \tilde{c}_c\}$. $\square$

Proof of Proposition 5.2. This proof uses Lemmas C.4.1, C.4.2, and C.4.3 (presented in Appendix C.4). To decide whether participate in a contracted slot program or not, each low-quality service provider compares its expected payoff if it participates in the contracted slot program (i.e., $\Pi_L^{c1}(\cdot)$ expressed in Section 5.2.1) versus if it does not participate in contracted slot programs (i.e., $\Pi_L^{c0}(\cdot)$ expressed in Section 5.2.1). To compute its expected payoff in the contracted slot program, it needs to decide the optimal number of slots that it should reserve for the program. Twice differentiating $\Pi_L^{c1}(\cdot)$ with respect to x_L^c, we have $\frac{\partial^2}{\partial (x_L^c)^2}\Pi_L^{c1}(x_L^c) = -\frac{p_H}{\delta\left(\bar{D}_L - \underline{D}_L\right)} < 0$, which implies that $\Pi_L^{c1}(x_L^c)$ is a concave function in x_L^c. Applying the first-order condition on $\Pi_L^{c1}(x_L^c)$, we have $x_L^{c*} = \frac{\delta s_H^c\left(\bar{D}_L - \underline{D}_L\right) + p_H\left(K_L + d_{ho}f - \delta\bar{D}_L\right)}{p_H}$.

Since we consider low-quality providers are heterogeneous in their costs of improving their quality, and allow $c_{improve}$ to be uniformly distributed between 0 and C, and, therefore, a low-quality provider will improve its quality if and only if $\Pi_L^{c0}(f) < \Pi_L^{c1}(x_L^{c*}, f)$, where f denotes the proportion of low-quality providers that improve their quality (out of all low-quality providers). Denote f^* proportion of low-quality providers improve their quality levels in equilibrium. Substituting $x_L^{c*} = \frac{\delta s_H^c\left(\bar{D}_L - \underline{D}_L\right) + p_H\left(K_L + d_{ho}f - \delta\bar{D}_L\right)}{p_H}$ and $c_{improve} = fC$ to $\Pi_L^{c1}(x_L^{c*}, f)$, we have:

$\Pi_L^{c1*}(f) - \Pi_L^{c0}(f) = \frac{n_1 f^2 + n_2 f + n_3}{2p_H\left(\bar{D}_L - \underline{D}_L\right)}$, where

$n_1 \doteq d_{lo}^2 p_H p_L > 0$,

$n_2 \doteq 2d_{ho}p_H\left(\bar{D}_L - \underline{D}_L\right)\left(s_H^c - \frac{p_L d_{lo}(K_L - \underline{D}_L) + (p_H d_{ho} + C)\left(\bar{D}_L - \underline{D}_L\right)}{d_{ho}\left(\bar{D}_L - \underline{D}_L\right)}\right)$, and

$n_3 \doteq \delta p_H^2\left(\bar{D}_L^2 - \underline{D}_L^2\right) + \left(-2\delta s_H^c \bar{D}_L^2 + 2s_H^c\left((K_L + \delta\underline{D}_L) - 2K_L p_L - 2c_c\right)\bar{D}_L + p_L\underline{D}_L^2 - 2\underline{D}_L K_L s_H^c + K_L^2 p_L + 2\underline{D}_L c_c\right)p_H + \delta\left(s_H^c\left(\bar{D}_L - \underline{D}_L\right)\right)^2$.

By Lemma C.4.1, we have $\Pi_L^{c1*}(f) - \Pi_L^{c0}(f) = 0$ has two real roots if and only if $c_c > \underline{\underline{c}}_c$, where $\underline{\underline{c}}_c$ is characterized in the proof of Lemma C.4.1. Since in practice, the cost of managing the contracted slot programs is relatively high, Here, we focus on the scenario wherein $\Pi_L^{c1*}(f) - \Pi_L^{c0}(f) = 0$ has real root, i.e, $c_c > \underline{\underline{c}}_c$. Solving $\Pi_L^{c1*}(f) - \Pi_L^{c0}(f) = 0$, we have two solutions of f, denoted by f_1 and f_2.
$f_1 \doteq \frac{p_H\left(\bar{D}_L-\underline{D}_L\right)\left(d_{ho}\left(p_H-s_H^c\right)+C\right)+p_Lp_Hd_{lo}(K_L-\underline{D}_L)+\sqrt{p_H\left(\bar{D}_L-\underline{D}_L\right)\left(h1\underline{D}_L+h2\bar{D}_L+h3\right)}}{p_Lp_Hd_{lo}^2}$ and $f_2 \doteq \frac{p_H\left(\bar{D}_L-\underline{D}_L\right)\left(d_{ho}\left(p_H-s_H^c\right)+C\right)+p_Lp_Hd_{lo}(K_L-\underline{D}_L)-\sqrt{p_H\left(\bar{D}_L-\underline{D}_L\right)\left(h1\underline{D}_L+h2\bar{D}_L+h3\right)}}{p_Lp_Hd_{lo}^2}$, where
$h_1 \doteq \left(\delta\left(p_H^2-(s_H^c)^2\right)d_{lo}+2h_4p_H\right)d_{lo}p_L+(h_4+C)^2p_H$, $h_2 \doteq \delta p_Ld_{lo}^2(p_H-s_H^c)^2-p_Hh_4^2$, $h_3 \doteq -2p_Lp_Hd_{lo}\left(p_Ld_{lo}k_L+\left(-K_Ls_H^c+c_c\right)d_{lo}+K_Lh_4\right)$, and $h_4 \doteq d_{ho}\left(p_H-s_H^c\right)+C$. Since $f_1-f_2=\frac{2\sqrt{p_H\left(\bar{D}_L-\underline{D}_L\right)\left(h1\underline{D}_L+h2\bar{D}_L+h3\right)}}{p_Lp_Hd_{lo}^2}>0$, we conclude $f_1>f_2$.

n_3 can be rewritten as $n_3 = 2p_H\left(\bar{D}_L-\underline{D}_L\right)\left(\frac{\sigma_1}{2p_H\left(\bar{D}_L-\underline{D}_L\right)}-c_c\right)$, where $\sigma_1 \doteq \delta\left(\bar{D}_L^2-\underline{D}_L^2\right)p_H^2+\delta(s_H^c)^2\left(\bar{D}_L-\underline{D}_L\right)^2+(-2\delta s_H^c\bar{D}_L^2+2((K_L+\underline{D}_L\delta)s_H^c-K_Lp_L)\bar{D}_L+p_L\underline{D}_L^2-2\underline{D}_LK_Ls_H^c+K_L^2p_L)p_H$. Since $p_H>0$ and $\left(\bar{D}_L-\underline{D}_L\right)>0$, the sign of n_3 depends on the sign of $\frac{\sigma_1}{2p_H\left(\bar{D}_L-\underline{D}_L\right)}-c_c$. Denote $\tilde{c}_c \doteq \frac{\sigma_1}{2p_H\left(\bar{D}_L-\underline{D}_L\right)}$. Therefore, we have $n_3>0$ when $c_c \le \tilde{c}_c$ and $n_3>0$ otherwise.

Since $d_{ho}, p_H, \bar{D}_L-\underline{D}_L>0$, the sign of n_2 depends on the sign of $s_H^c-\frac{p_Ld_{lo}(K_L-\underline{D}_L)+(p_Hd_{ho}+C)\left(\bar{D}_L-\underline{D}_L\right)}{d_{ho}\left(\bar{D}_L-\underline{D}_L\right)}$. Denote $\hat{s}_H^c \doteq \frac{p_Ld_{lo}(K_L-\underline{D}_L)+(p_Hd_{ho}+C)\left(\bar{D}_L-\underline{D}_L\right)}{d_{ho}\left(\bar{D}_L-\underline{D}_L\right)}$. It follows $n_2<0$ when $s_H^c<\hat{s}_H^c$ and $n_2 \ge 0$ otherwise. Then, we consider the following two cases (I and II) based on the magnitude of s_H^c.

Case I: Suppose $s_H^c > \hat{s}_H^c$, then $n_2>0$. Since $n_1>0$, we conclude $f_2<0$. Next, we consider the following two sub-cases based on the magnitude of c_c.

(I-a) Suppose $c_c \le \ddot{c}_c$. Since $2n_1+n_2>0$, as per Lemmas C.4.1 and C.4.2, we have $f_1 \le 1$. Further, we have: (i) $f_1 \le 0$ when $c_c \le \tilde{c}_c$ (because $n_3 \ge 0$). This implies that $\Pi_L^{c1*}(f)-\Pi_L^{c0}(f) \ge 0$ for any $0 \le f \le 1$. Therefore, $f^*=1$; (ii) $f_1>0$ when $c_c>\tilde{c}_c$ (because $n_3<0$). This implies that $0<f_1<1$. Therefore, $f^*=f_1$.

(I-b) Suppose $c_c > \ddot{c}_c$. Since $2n_1 + n_2 > 0$, as per Lemmas C.4.1 and C.4.2, $f_1 > 1$. Similar to (I-a), we have: (i) $f_1 \leq 0$ when $c_c \leq \tilde{c}_c$, which contradict with $f_1 > 1$. (ii) $f_1 > 0$ when $c_c > \tilde{c}_c$. This implies that $f_1 \geq 1$. Therefore, $f^* = 0$.

Combining two sub-cases above (i.e., (I-a) and (I-b)), we have $f^* = 1$ when $c_c \leq \min\{\ddot{c}_c, \tilde{c}_c\}$; $f^* = f_1 \in (0, 1)$ when $\min\{\ddot{c}_c, \tilde{c}_c\} < c_c < \max\{\ddot{c}_c, \tilde{c}_c\}$; and $f^* = 0$ when $c_c \geq \max\{\ddot{c}_c, \tilde{c}_c\}$.

Case II: Suppose $s_H^c > \hat{s}_H^c$, then $n_2 < 0$. As per Lemmas C.4.1 and C.4.2, when $s_H^c > \underline{s}_H^c$, then $f_2 < 1$. Align with the practice that s_H^c is relatively high, we focus on this scenario. Next, we consider the following two sub-cases based on the magnitude of c_c.

(II-a) Suppose $c_c \leq \ddot{c}_c$. As per Lemmas C.4.1 and C.4.2, $f_1 < 1$. Since $n_1 > 0$ and $n_2 < 0$, we conclude $0 < f_1 < 1$. Using Lemma C.4.3, we conclude $f^* = f_1$.

(II-b) Suppose $c_c > \ddot{c}_c$. As per Lemmas C.4.1 and C.4.2, $f_1 \geq 1$. Similar to case (I-a), we have: (i) $f_2 > 0$ when $c_c \leq \tilde{c}_c$. This implies that $0 < f_2 < 1$. Therefore, $f^* = f_2$. (ii) $f_2 < 0$ when $c_c > \tilde{c}_c$. This implies that $\Pi_L^{c1*}(f) - \Pi_L^{c0}(f) < 0$ for any $0 \leq f \leq 1$. Therefore, $f^* = 0$;

Combining two sub-cases above (i.e., (II-a) and (II-b)), we have $f^* = f_1 \in (0, 1)$ when $c_c \leq \min\{\ddot{c}_c, \tilde{c}_c\}$; $f^* = f_1 \in (0, 1)$ or $f^* = f_2 \in (0, 1)$ when $\min\{\ddot{c}_c, \tilde{c}_c\} < c_c < \max\{\ddot{c}_c, \tilde{c}_c\}$; and $f^* = 0$ when $c_c \geq \max\{\ddot{c}_c, \tilde{c}_c\}$.

The results in the Lemma follow by combining the results in cases I and II above and by defining two thresholds $\underline{c}_c \doteq \max\left\{\min\{\ddot{c}_c, \tilde{c}_c\}, \underline{\underline{c}}_c\right\}$ and $\bar{c}_c \doteq \max\{\ddot{c}_c, \tilde{c}_c\}$. $\square$

C.4 Proofs of Lemmas Referred in Appendix C.3

We present lemmas (and their proofs), which help us outline proofs of propositions and lemma (as outlined in Appendix C.3).

Lemma C.4.1 *Consider $G_1(g) = n_1 g^2 + n_2 g + n_3$, where $n_1 > 0$, n_2, and n_3 are characterized in the proof of Proposition 5.2. Then, $G_1(g)$ has two real roots if and*

only if $c_c > \underline{\underline{c}}_c$*, where* $\underline{\underline{c}}_c$ *is characterized in the proof. Further, denote two real roots of* $G_1(g)$ *by* $g_1 \doteq \frac{-n_2+\sqrt{n_2^2-4n_1n_3}}{2n_1}$ *and* $g_2 \doteq \frac{-n_2-\sqrt{n_2^2-4n_1n_3}}{2n_1}$*. Then,*

(i) $g_1 < 1$ *when* $2n_1 + n_2 > 0$ *and* $n_1 + n_2 + n_3 > 0$*;* $g_1 \geq 1$ *when* $2n_1 + n_2 \leq 0$ *or when* $2n_1 + n_2 > 0$ *and* $n_1 + n_2 + n_3 \leq 0$*.*

(ii) $g_2 < 1$ *when* $2n_1 + n_2 > 0$ *or when* $2n_1 + n_2 \leq 0$ *and* $n_1 + n_2 + n_3 < 0$*;* $g_2 \geq 1$ *when* $2n_1 + n_2 \leq 0$ *and* $n_1 + n_2 + n_3 \geq 0$*.*

Proof of Lemma C.4.1: $G_1(g)$ has two real roots if and only if its discriminant is non-negative. That is, $n_2^2 - 4n_1n_3 \geq 0$. $n_2^2 - 4n_1n_3 = 8d_{lo}^2 p_L p_H^2 \left(\bar{D}_L - \underline{D}_L\right)\left(c_c - \frac{\sigma_2}{2d_{lo}^2 p_L p_H^2 \left(\bar{D}_L - \underline{D}_L\right)}\right)$, where $\sigma_2 \doteq d_{lo}^2 p_L p_H \Big(\delta\left(\bar{D}_L^2 - \underline{D}_L^2\right) p_H^2 + \big(-2\delta s_H^c \bar{D}_L^2 + 2((\delta \underline{D}_L + K_L)s_H^c - K_L p_L)\bar{D}_L + p_L \bar{D}_L^2 - 2\underline{D}_L K_L s_H^c + K_L^2 p_L\big)p_H + \delta (s_H^c)^2 \left(\bar{D}_L - \underline{D}_L\right)^2\Big) - p_H^2\Big((s_H - p_H)\left(\bar{D}_L - \underline{D}_L\right) d_{ho} - d_{lo} p_L (K_L - \underline{D}_L) - C\left(\bar{D}_L - \underline{D}_L\right)^2\Big)^2$. Since $d_{lo} > 0$, $p_L > 0$, $p_H > 0$, and $\bar{D}_L - \underline{D}_L > 0$, the sign of $n_2^2 - 4n_1n_3 > 0$ depends on the sign of $c_c - \frac{\sigma_2}{2d_{lo}^2 p_L p_H^2 \left(\bar{D}_L - \underline{D}_L\right)}$. Denote $\underline{\underline{c}}_c \doteq \frac{\sigma_2}{2d_{lo}^2 p_L p_H^2 \left(\bar{D}_L - \underline{D}_L\right)}$. Therefore, we have $n_2^2 - 4n_1n_3 \geq 0$ if and only if $c_c > \underline{\underline{c}}_c$.

(i) Since $n_1 > 0$, the comparison between g_1 and 1 is equivalent to the comparison between $-n_2 + \sqrt{n_2^2 - 4n_1n_3}$ and $2n_1$, or the comparison between $\sqrt{n_2^2 - 4n_1n_3}$ and $2n_1 + n_2$. Next, consider the following two cases based on the magnitude of $2n_1 + n_2$.

Case I: Suppose $2n_1 + n_2 \leq 0$, then $\sqrt{n_2^2 - 4n_1n_3} \geq 2n_1 + n_2$. This implies that $g_1 \geq 1$.

Case II: Suppose $2n_1 + n_2 > 0$. Then, the comparison between $\sqrt{n_2^2 - 4n_1n_3}$ and $2n_1 + n_2$ is equivalent to the comparison between $\left(\sqrt{n_2^2 - 4n_1n_3}\right)^2$ and $(2n_1 + n_2)^2$. $\left(\sqrt{n_2^2 - 4n_1n_3}\right)^2 - (2n_1 + n_2)^2 = -4n_1(n_1 + n_2 + n_3)$, which sign depends on the sign of $-(n_1 + n_2 + n_3)$. Therefore, we conclude $g_1 < 1$ when $n_1 + n_2 + n_3 > 0$ and $g_1 \geq 1$ when $n_1 + n_2 + n_3 \leq 0$.

Combining Case I and II, we complete the proof of part (i) of this lemma. Following similar steps, we can prove part (ii) of this lemma. Thus, we omit details for brevity. □

Lemma C.4.2 *Consider $n_1 > 0$, n_2, and n_3 as characterized in the proof of Proposition 5.2. Then, there exists a threshold of s_H^c, denoted by $\underline{s}_H^c$, $2n_1 + n_2 \leq 0$ when $s_H^c < \underline{s}_H^c$ and $2n_1 + n_2 > 0$ otherwise. Further, there exists a threshold of c_c, denoted by $\ddot{c}_c$, $n_1 + n_2 + n_3 \leq 0$ when $c \geq \ddot{c}_c$ and $n_1 + n_2 + n_3 > 0$ otherwise. $\underline{s}_H^c$ and $\ddot{c}_c$ are characterized in the proof of lemma.*

Proof of Lemma C.4.2: Using the expressions of n_1 and n_2, we have $2n_1 + n_2 = 2d_{ho}p_H\left(\bar{D}_L - \underline{D}_L\right)\left(s_H^c - \frac{(p_H d_{ho}+C)\left(\bar{D}_L-\underline{D}_L\right)+d_{lo}p_L(K_L-\underline{D}_L-d_{lo})}{d_{ho}\left(\bar{D}_L-\underline{D}_L\right)}\right)$. Since $d_{ho} > 0$, $p_H > 0$, and $\left(\bar{D}_L - \underline{D}_L\right) > 0$, the sign of $2n_1 + n_2$ depends on the sign of $s_H^c - \frac{(p_H d_{ho}+C)\left(\bar{D}_L-\underline{D}_L\right)+d_{lo}p_L(K_L-\underline{D}_L-d_{lo})}{d_{ho}\left(\bar{D}_L-\underline{D}_L\right)}$. Denote $\underline{s}_H^c \doteq \frac{(p_H d_{ho}+C)\left(\bar{D}_L-\underline{D}_L\right)+d_{lo}p_L(K_L-\underline{D}_L-d_{lo})}{d_{ho}\left(\bar{D}_L-\underline{D}_L\right)}$. Therefore, we have $2n_1 + n_2 \leq 0$ when $s_H^c < \underline{s}_H^c$ and $2n_1 + n_2 > 0$ otherwise.

Using the expressions of n_1 and n_2, we have $n_1+n_2+n_3 = 2p_H\left(\bar{D}_L - \underline{D}_L\right)\left(\frac{\sigma_3}{2p_H\left(\bar{D}_L-\underline{D}_L\right)} - c_c\right)$, where $\sigma_3 \doteq \delta\left(\bar{D}_L - \underline{D}_L\right)^2(s_H^c)^2 - 2p_H\left(\bar{D}_L - \underline{D}_L\right)\left(\underline{D}_L\delta - d_{ho} - K_L\right)s_H^c + \left(\bar{D}_L - \underline{D}_L\right)\left(\delta\left(\bar{D}_L + \underline{D}_L\right) - 2d_{ho}\right)p_H^2 + (((K_L - d_{lo})^2 + 2\underline{D}_L d_{lo} + \underline{D}_L^2 - 2K_L\bar{D}_L)p_L - 2C\left(\bar{D}_L - \underline{D}_L\right))p_H$. Since $p_H > 0$ and $\left(\bar{D}_L - \underline{D}_L\right) > 0$, the sign of $n_1 + n_2 + n_3$ depends on the sign of $\frac{\sigma}{2p_H\left(\bar{D}_L-\underline{D}_L\right)} - c_c$. Denote $\ddot{c}_c \doteq \frac{\sigma_3}{2p_H\left(\bar{D}_L-\underline{D}_L\right)}$. Therefore, we have $n_1 + n_2 + n_3 \leq 0$ when $c \geq \ddot{c}_c$ and $n_1 + n_2 + n_3 > 0$ otherwise. □

Lemma C.4.3 *Consider f_1 and f_2 as characterized in the proof of Proposition 5.2. $\Pi_L^{c1}\left(x_L^{c*}, f_1\right) - \Pi_L^{c1}\left(x_L^{c*}, f_2\right) > 0$ for any $s_H^c > \underline{s}_H^c$, where $\underline{s}_H^c$ is characterized in the proof of Lemma C.4.2.*

Proof of Lemma C.4.1: Substituting f_1 and f_2 to $\Pi_L^{c1}\left(x_L^{c*}, f\right)$, we have $\Pi_L^{c1}\left(x_L^{c*}, f_1\right) - \Pi_L^{c1}\left(x_L^{c*}, f_2\right) = \frac{2d_{ho}\left(s_H^c-p_H\right)\sqrt{p_H\left(\bar{D}_L-\underline{D}_L\right)\left(h1\underline{D}_L+h2\bar{D}_L+h3\right)}}{p_L p_H d_{lo}^2}$. Since $p_L, p_H, d_{lo}, d_{ho} > 0$, the sign of $\Pi_L^{c1}\left(x_L^{c*}, f_1\right) - \Pi_L^{c1}\left(x_L^{c*}, f_2\right)$ depends on the sign of $s_H^c - p_H$.

Rewriting $\underline{s}_H^c$, we have $\underline{s}_H^c = p_H + \frac{C}{d_{ho}} + \frac{p_L d_{lo}(K_L - (\underline{D}_L + d_{lo}))}{d_{ho}\left(\bar{D}_L - \underline{D}_L\right)}$. Since $C > 0$, $d_{ho} > 0$, $d_{lo} > 0$, $p_L > 0$ and $K_L > \underline{D}_L + d_{lo}$, we conclude $\underline{s}_H^c > p_H$. This implies that $s_H^c - p_H$ for any $s_H^c > \underline{s}_H^c$, or $\Pi_L^{c1}\left(x_L^{c*}, f_1\right) - \Pi_L^{c1}\left(x_L^{c*}, f_2\right) > 0$. $\square$

BIBLIOGRAPHY

Adams, G., Ewen, D. and Luetmer, G. (2021). Using contracts to support the child care workforce. https://www.urban.org/research/publication/using-contracts-support-child-care-workforce. Accessed July 2, 2023.

Adams, G., Snyder, K. and Tout, K. (2003). Child care subsidy policies and practices: Implications for child care providers. *Series A, No. 57, Assessing the New Federalism, The Urban Institute*.

Alizamir, S., Iravani, F. and Mamani, H. (2019). An analysis of price vs. revenue protection: Government subsidies in the agriculture industry, *Management Science* **65**(1): 32–49.

Anderson, S. G., Ramsburg, D. M. and Scott, J. (2003). Illinois study of license-exempt child care: Interim report. https://researchconnections.org/sites/default/files/pdf/rc2968.pdf. Accessed July 2, 2023.

Andritsos, D. A. and Tang, C. S. (2018). Incentive programs for reducing readmissions when patient care is co-produced, *Production and Operations Management* **27**(6): 999–1020.

Arnosti, N. and Shi, P. (2020). Design of lotteries and wait-lists for affordable housing allocation, *Management Science* **66**(6): 2291–2799.

Arora, P., Rahmani, M. and Ramachandran, K. (2022). Doing less to do more? Optimal service portfolio of non-profits that serve distressed individuals, *Manufacturing & Service Operations Management* **24**(2): 883–901.

Arora, P. and Subramanian, R. (2019). Improving societal outcomes in the organ donation value chain, *Production and Operations Management* **28**(8): 2110–2131.

Arya, A. and Mittendorf, B. (2016). Donor reliance on accounting and its consequences for the charitable distribution channel, *Production and Operations Management* **25**(8): 1319–1331.

Ata, B., Skaro, A. and Tayur, S. (2016). Organjet: Overcoming geographical disparities in access to deceased donor kidneys in the United States, *Management Science* **63**(9): 2776–2794.

Atasu, A., Toktay, B., Yeo, W. M. and Zhang, C. (2017). Effective medical surplus recovery, *Production and Operations Management* **26**(6): 1142–1162.

Austermuhle, M. (2015). D.C. is one of the most expensive places in the U.S. for child care. But why?. https://tinyurl.com/ycjpj6js. Accessed July 2, 2023.

Banerjee, M. M., Friedline, T. and Phipps, B. J. (2017). Financial capability of parents of kindergarteners, *Children and Youth Services Review* **81**: 178–187.

Banful, A. B. (2011). Do formula-based intergovernmental transfer mechanisms eliminate politically motivated targeting? Evidence from Ghana, *Journal of Development Economics* **96**(2): 380–390.

Banghart, P., Guerra, G. and Daily, S. (2021). Strategies to guide the equitable allocation of COVID-19 relief funding for early care and education. https://cms.childtrends.org/wp-content/uploads/2021/12/ABC-brief_ChildTrends_Dec2021.pdf. Accessed July 2, 2023.

Barnett, W. S. and Masse, L. N. (2007). Comparative benefit–cost analysis of the Abecedarian program and its policy implications, *Economics of Education Review* **26**(1): 113–125.

Bartels, L. M. (2018). Partisanship in the Trump era, *The Journal of Politics* **80**(4): 1483–1494.

Berenguer, G. and Shen, Z.-J. (2020). OM forum—Challenges and strategies in managing nonprofit operations: An operations management perspective, *Manufacturing & Service Operations Management* **22**(5): 888–905.

Bertsimas, D., Delarue, A. and Martin, S. (2019). Optimizing schools' start time and bus routes, *Proceedings of the National Academy of Sciences* **116**(13): 5943–5948.

Bertsimas, D., Farias, V. F. and Trichakis, N. (2011). The price of fairness, *Operations Research* **59**(1): 17–31.

Bertsimas, D., Farias, V. F. and Trichakis, N. (2013). Fairness, efficiency, and flexibility in organ allocation for kidney transplantation, *Operations Research* **61**(1): 73–87.

Besiou, M., Pedraza-Martinez, A. J. and Van Wassenhove, L. N. (2014). Vehicle supply chains in humanitarian operations: Decentralization, operational mix, and earmarked funding, *Production and Operations Management* **23**(11): 1950–1965.

Besiou, M. and Van Wassenhove, L. N. (2015). Addressing the challenge of modeling for decision-making in socially responsible operations, *Production and Operations Management* **24**(9): 1390–1401.

Besiou, M. and Van Wassenhove, L. N. (2020). Humanitarian operations: A world of opportunity for relevant and impactful research, *Manufacturing & Service Operations Management* **22**(1): 135–145.

Besley, T. and Kanbur, R. (1988). Food subsidies and poverty alleviation, *The Economic Journal* **98**(392): 701–719.

Bierhoff, H.-W. (2005). *Prosocial Behaviour*, Psychology Press, Abingdon, UK.

Bipartisan Policy Center (2021). Payment practices to stabilize child care. `https://bipartisanpolicy.org/explainer/payment-practices-to-stabilize-child-care/`. Accessed July 2, 2023.

Brest, P. and Harvey, H. (2008). *Money Well Spent: A strategic plan for smart philanthropy*, Bloomberg Press, New York.

Burkart, C., Besiou, M. and Wakolbinger, T. (2016). The funding–Humanitarian supply chain interface, *Surveys in Operations Research and Management Science* **21**(2): 31–45.

Cameron, M. E., Keller, K. E. and Whorton, S. C. (2005). Policy and budget analysis of Virginia's child care subsidy program (CCDF). `https://www.wm.edu/as/publicpolicy/documents/prs/voices.pdf`. Accessed July 2, 2023.

Campbell, F. and Patil, P. (2019). State of early education and care in Boston: Supply, demand, affordability and quality. `https://files.eric.ed.gov/fulltext/ED606431.pdf`. Accessed July 2, 2023.

CCAoA (2018). The US and the high cost of child care: A review of prices and proposed solutions for a broken system. `https://tinyurl.com/y6fawhrq`. Accessed July 2, 2023.

Center for Early Learning Funding Equity (2023). Massachusetts 2022 market rate survey and narrow cost analysis final report. `https://www.mass.gov/doc/massachusetts-2022-market-rate-survey-and-narrow-cost-analysis-final-report/download`. Accessed July 2, 2023.

Center on Budget and Policy Priorities (2009). Introduction to the housing voucher program. `https://www.cbpp.org/research/introduction-to-the-housing-voucher-program`. Accessed July 6, 2023.

Chaudry, A., Pedroza, J. M., Sandstrom, H., Danzinger, A., Grosz, M., Scott, M. and Ting, S. (2011). Child care choices of low-income working families. `https://tinyurl.com/y78znu73`. Accessed July 2, 2023.

Chinchilla, M., Gabrielian, S., Horan, W. P., Glasmeier, A., Hellemann, G., Wynn, J. K., Capone-Newton, P. and Green, M. F. (2019). Comparing tenant and neighborhood characteristics of the VA's project-vs. tenant-based supportive housing program in Los Angeles county, *Journal of Health Care for the Poor and Underserved* **30**(4): 1373–1393.

Cohen, M. C., Lobel, R. and Perakis, G. (2016). The impact of demand uncertainty on consumer subsidies for green technology adoption, *Management Science* **62**(5): 1235–1258.

Congressional Research Service (2015). Mandatory spending since 1962. https://fas.org/sgp/crs/misc/RL33074.pdf. Accessed July 2, 2023.

Connecticut Department of Housing (2019). Administrative plan for the rental assistance payments program "RAP". https://portal.ct.gov/-/media/DOH/2019-RAP-Admin-Plan.pdf. Accessed July 2, 2023.

de Véricourt, F. and Lobo, M. S. (2009). Resource and revenue management in nonprofit operations, *Operations Research* **57**(5): 1114–1128.

Department of Health & Human Services (2019). FY 2019 preliminary data table 2 - Percent of children served by payment method. https://www.acf.hhs.gov/occ/data/fy-2019-preliminary-data-table-2. Accessed July 2, 2023.

Department of Health & Human Services (2023). QRIS resource guide. https://ecquality.acf.hhs.gov/. Accessed July 2, 2023.

Department of Health and Human Services (2016). Child care and development fund (CCDF) program rules and regulations 2016. https://www.govinfo.gov/content/pkg/FR-2016-09-30/pdf/2016-22986.pdf. Accessed July 2, 2023.

Department of Health and Human Services (2019). Addressing the decreasing number of family child care providers in the United States. https://tinyurl.com/73kz29pm. Accessed July 2, 2023.

Department of Health and Human Services (2021). HUD announces additional assistance to keep foster youth housed. https://www.usich.gov/news/hud-announces-additional-assistance-to-keep-foster-youth-housed/. Accessed July 2, 2023.

Department of Health and Human Services (2023). Fiscal year 2022 budget in brief strengthening health and opportunity for all Americans. https://tinyurl.com/4zp8d9eu. Accessed July 2, 2023.

Department of Housing and Urban Development (2017). Housing project-based rental assistance: 2017 summary statement and initiatives. https://www.hud.gov/sites/documents/24-PROJ.BASED_RENT_ASSIST.PDF. Accessed July 2, 2023.

Department of Housing and Urban Development (2019). Fiscal year 2020 annual performance plan. https://tinyurl.com/336865an. Accessed July 2, 2023.

Department of Housing and Urban Development (2021). Federal account: 086-0303. https://www.usaspending.gov/federal_account/086-0303. Accessed July 2, 2023.

Department of Housing and Urban Development (2023a). HOME and CDBG guidebook. https://tinyurl.com/zv3t58ef. Accessed July 2, 2023.

Department of Housing and Urban Development (2023b). HOME tenant-based rental assistance. https://www.hudexchange.info/programs/home/topics/tbra/#policy-guidance-and-faqs. Accessed July 2, 2023.

Department of Housing and Urban Development (2023c). HUD announces the availability of $30.3 billion for housing authorities to provide affordable housing. https://www.hud.gov/press/press_releases_media_advisories/hud_no_23_076. Accessed July 2, 2023.

Department of Housing and Urban Development (2023d). HUD announces the availability of $30.3 billion for housing authorities to provide affordable housing. https://www.hud.gov/press/press_releases_media_advisories/hud_no_23_076#:~:text=Additionally%2C%20the%20President's%202024%20Budget,support%20youth%20leaving%20foster%20care. Accessed July 6, 2023.

Devalkar, S. K., Sohoni, M. G. and Arora, P. (2017). Ex-post funding: How should a resource-constrained non-profit organization allocate its funds?, *Production and Operations Management* **26**(6): 1035–1055.

Dilger, R. J. and Boyd, E. (2014). *Block grants: Perspectives and controversies*, Congressional Research Service Washington, DC.

Dorn, C. (2020). Infant and toddler contracted slots pilot program: Evaluation report. https://s35729.pcdn.co/wp-content/uploads/2020/11/IT-Pilot-Evaluation-Report_PA_Final.V2.pdf. Accessed July 2, 2023.

Education Above All (2016). Education and the SDGs: Educate a child, *Occasional Paper Number 2*. https://educationaboveall.org/uploads/library/file/2a8e15847d.pdf. Accessed July 2, 2023.

Epstein, D., Chalabi, Z., Claxton, K. and Sculpher, M. (2005). Mathematical programming for the optimal allocation of health care resources, *Brookings Monographs* .

Feng, Q. and Shanthikumar, J. G. (2016). Not-for-profit operations management.

Gandjour, A. (2010). Investment in quality improvement: How to maximize the return, *Health Economics* **19**(1): 31–42.

Garboden, P. M., Rosen, E., DeLuca, S. and Edin, K. (2018). Taking stock: What drives landlord participation in the housing choice voucher program, *Housing Policy Debate* **28**(6): 979–1003.

Garboden, P., Rosen, E., Greif, M., DeLuca, S. and Edin, K. (2018). Urban landlords and the housing choice voucher program: A research report. https://tinyurl.com/7578esm9. Accessed July 2, 2023.

Government Accountability Office (2019a). Child care and early education: Most states offer preschool programs and rely on multiple funding sources. https://www.gao.gov/assets/700/698961.pdf. Accessed July 2, 2023.

Government Accountability Office (2019b). Child care: States report child care and development funds benefit all children in care. https://tinyurl.com/5epy4vv8. Accessed July 2, 2023.

Greenberg, E., Adams, G. and Michie, M. (2016). Barriers to preschool participation for low-income children of immigrants in Silicon Valley. https://urbn.is/2HqA3Uc. Accessed July 2, 2023.

Grunewald, R. and Stepick, L. (2022). What drives the cost of high-quality early care and education? https://www.minneapolisfed.org/article/2022/what-drives-the-cost-of-high-quality-early-care-and-education. Accessed July 2, 2023.

Guthrie, J. W. (2006). "Modern" education finance: How it differs from the "old" and the analytic and data collection changes it implies, *Education Finance and Policy* **1**(1): 3–16.

Hamm, K. (2014). Cuts to the child care subsidy system force parents and providers to make difficult choices. https://tinyurl.com/y6xo8co6. Accessed July 2, 2023.

Hardy, E., Joshi, P., Ha, Y. and Schneider, K. G. (2018). Subsidized child care in Massachusetts: Exploring geography, access, and equity. https://tinyurl.com/2p9y52jd. Accessed July 2, 2023.

Hartmann, H., Spalter-Roth, R. and Sills, M. (2003). Survival at the bottom: The income packages of low-income families with children. https://tinyurl.com/y3s238g8. Accessed July 2, 2023.

Hatt, L., Nguyen, H., Sloan, N., Miner, S., Magvanjav, O., Sharma, A., Chowdhury, J., Chowdhury, R., Paul, D., Islam, M., and Wang, H. (2010). Economic evaluation of demand-side financing (DSF) for maternal health in Bangladesh. https://tinyurl.com/yawltpay. Accessed July 2, 2023.

Heckman, J. J. (2012). Invest in early childhood development: Reduce deficits, strengthen the economy, *The Heckman Equation* **7**: 1–2.

Herbst, C. M. (2017). Universal child care, maternal employment, and children's long-run outcomes: Evidence from the U.S. Lanham Act of 1940, *Journal of Labor Economics* **35**(2): 519–564.

Hotchkiss, M. and Phelan, J. (2017). Uses of Census Bureau data in federal funds distribution. https://tinyurl.com/yy5nd5xa. Accessed July 2, 2023.

Howden-Chapman, P., Bennett, J., Edwards, R., Jacobs, D., Nathan, K. and Ormandy, D. (2023). Review of the impact of housing quality on inequalities in health and well-being, *Annual review of public health* **44**: 233–254.

Illinois Department of Human Services (2017). FY17 Illinois child care assistance program (CCAP) report. https://bit.ly/2KaBCeu. Accessed July 2, 2023.

Illinois Housing Development Authority (2017). The Illinois rental housing support program (RHS): Program guide. https://tinyurl.com/IHDA2017. Accessed July 2, 2023.

Ir, P., Horemans, D., Souk, N. and Van Damme, W. (2010). Using targeted vouchers and health equity funds to improve access to skilled birth attendants for poor women: A case study in three rural health districts in Cambodia, *BMC Pregnancy and Childbirth* **10**(1): 1–11.

Isaacs, J. B., Katz, M. and Amin, R. (2016). Improving the efficiency of benefit delivery. https://tinyurl.com/ybsnfnyb. Accessed July 2, 2023.

Isaacs, J. B., Katz, M., Minton, S. and Michie, M. (2015). Review of child care needs of eligible families in Massachusetts. https://urbn.is/2LXszhm, last accessed on July 2, 2023.

Johnson-Staub, C. (2020). COVID-19 and state child care assistance programs: Immediate considerations for state CCDF lead agencies. https://tinyurl.com/y9y2qjep. Accessed July 2, 2023.

Jones, L., Quigley, J. and Foster, G. (2006). Vouchers revisited–Can small enterprises save government programmes?, *Small Enterprise Development* **17**(4): 43–51.

Karaer, Ö., Kraft, T. and Khawam, J. (2017). Buyer and nonprofit levers to improve supplier environmental performance, *Production and Operations Management* **26**(6): 1163–1190.

Katz, M. and Adams, G. (2015). Balancing quality early education and parents' workforce success. Urban Institute. https://tinyurl.com/st8p9byp. Accessed July 2, 2023.

Keith, T. (2019). White House and Ivanka Trump propose new spending on child care. https://n.pr/2TGmycX. Accessed July 2, 2023.

Korpeoglu, C., Tang, C. and Yu, J. (2023). Market entry of a non-profit food cooperative and deterrence pricing strategy of a for-profit retailer, *Working Paper*.

Kotler, P. and Armstrong, G. (2017). *Principles of Marketing*, 17th edn, Pearson Education, London, UK.

Kotsi, T., Aflaki, A., Aydin, G. and Pedraza Martinez, A. (2020). Allocation of nonprofit funds among program, fundraising, and administration, *Fundraising, and Administration (December 24, 2020)* .

Kotsi, T. O., Wu, O. Q. and Pedraza-Martinez, A. J. (2022). Donations for refugee crises: In-kind vs. cash assistance, *Manufacturing & Service Operations Management* **24**(6): 3001–3018.

Krafft, C., Davis, E. E. and Tout, K. (2017). Child care subsidies and the stability and quality of child care arrangements, *Early Childhood Research Quarterly* **39**: 14–34.

Kraft, T., Zheng, Y. and Erhun, F. (2013). The NGO's dilemma: How to influence firms to replace a potentially hazardous substance, *Manufacturing & Service Operations Management* **15**(4): 649–669.

Kretschmer, A., Spinler, S. and Van Wassenhove, L. N. (2014). A school feeding supply chain framework: Critical factors for sustainable program design, *Production and Operations Management* **23**(6): 990–1001.

Lee, H. L. and Tang, C. S. (2017). Socially and environmentally responsible value chain innovations: New operations management research opportunities, *Management Science* **64**(3): 983–996.

Leventhal, G. S. (1980a). What should be done with equity theory?, Springer, Boston, pp. 27–55.

Leventhal, G. S. (1980b). What should be done with equity theory?, *Social Exchange*, Springer, pp. 27–55.

Levi, R., Perakis, G. and Romero, G. (2017). On the effectiveness of uniform subsidies in increasing market consumption, *Management Science* **63**(1): 40–57.

Lien, R. W., Iravani, S. M. and Smilowitz, K. R. (2014). Sequential resource allocation for nonprofit operations, *Operations Research* **62**(2): 301–317.

Lynch, S. (2020). Massachusetts public housing authorities awarded nearly $5 million in funding to provide rental assistance for people with disabilities. https://tinyurl.com/4u8f4zvu. Accessed July 2, 2023.

Maine Department of Health and Human Services (2023). Early childhood education improvement. https://tinyurl.com/shvst5dj. Accessed July 2, 2023.

Malik, R., Hamm, K., Schochet, L., Novoa, C., Workman, S. and Jessen-Howard, S. (2018). America's child care deserts in 2018. Accessed July 2, 2023.

Marshall, N. L., Robeson, W. W., Tracy, A. J., Frye, A. and Roberts, J. (2013). Subsidized child care, maternal employment and access to quality, affordable child care, *Early Childhood Research Quarterly* **28**(4): 808–819.

Martin, P., Gupta, D. and Natarajan, K. V. (2020). Vaccine procurement contracts for developing countries, *Production and Operations Management* **29**(11): 2601–2620.

Massachusetts Department of Early Education and Care (2005). CCR&R and CPC delivery system assessment final report. https://archives.lib.state.ma.us/bitstream/handle/2452/392582/ocn945988195.pdf?sequence=1&isAllowed=y. Accessed July 2, 2023.

Massachusetts Department of Early Education and Care (2017). Bid solicitation: BD-17-1037-1CEN0-C0000-14687. https://tinyurl.com/119hw85p. Accessed July 2, 2023.

Massachusetts Department of Early Education and Care (2019). Department of Early Education and Care 2018 annual report. https://malegislature.gov/Bills/191/SD2340.pdf. Accessed July 2, 2023.

Massachusetts Department of Early Education and Care (2022). Daily reimbursement rates - Fiscal year 2023. https://www.mass.gov/doc/fy23-daily-reimbursement-rates/download. Accessed July 2, 2023.

Massachusetts Department of Early Education and Care (2023). Find a licensed family, group or school age child care program. https://www.mass.gov/guides/find-a-licensed-family-group-or-school-age-child-care-program#-find-a-licensed-child-care-program-. Accessed July 2, 2023.

Matthews, H. and Schumacher, R. (2008). Ensuring quality care for low-income babies: Contracting directly with providers to expand and improve infant and toddler care. https://www.clasp.org/sites/default/files/public/resources-and-publications/files/0422.pdf. Accessed July 2, 2023.

McCarty, M. (2014). Introduction to public housing. https://sgp.fas.org/crs/misc/R41654.pdf. Accessed July 2, 2023.

McCoy, J. H. and Lee, H. L. (2014). Using fairness models to improve equity in health delivery fleet management, *Production and Operations Management* **23**(6): 965–977.

McKeown, M. P. (1996). State funding formulas for public four-year institutions, Denver, CO: State Higher Education Executive Officers.

Mehrotra, M. and Natarajan, K. V. (2020). Value of combining patient and provider incentives in humanitarian health care service programs, *Production and Operations Management* **29**(3): 571–594.

Metropolitan Area Planning Council (2017). Fair housing and equity assessment for metropolitan Boston. https://tinyurl.com/2x68j8eb. Accessed July 2, 2023.

Mo, J. and Walrand, J. (2000). Fair end-to-end window-based congestion control, *IEEE/ACM Transactions on Networking* **8**(5): 556–567.

Morrisey, T. and Workman, S. (2020). Grants and contracts: A strategy for building the supply of subsidized infant and toddler child care, *Washington: Center for American Progress* .

Nagurney, A., Daniele, P. and Cappello, G. (2021). Human migration networks and policy interventions: Bringing population distributions in line with system optimization, *International Transactions in Operational Research* **28**(1): 5–26.

Nagurney, A., Daniele, P., Flores, E. A. and Caruso, V. (2018). A variational equilibrium network framework for humanitarian organizations in disaster relief: Effective product delivery under competition for financial funds, *Dynamics of Disasters: Algorithmic Approaches and Applications 3* pp. 109–133.

Nagurney, A., Flores, E. A. and Soylu, C. (2016). A generalized nash equilibrium network model for post-disaster humanitarian relief, *Transportation Research Part E: Logistics and Transportation Review* **95**: 1–18.

Natarajan, K. V. and Swaminathan, J. M. (2017). Multi-treatment inventory allocation in humanitarian health settings under funding constraints, *Production and Operations Management* **26**(6): 1015–1034.

National Housing Law Project (2023). Project-based rental assistance. `https://www.nhlp.org/resource-center/project-based-rental-assistance/`. Accessed July 2, 2023.

National League of Cities (2017). Equitable early care and education an alignment framework. `https://tinyurl.com/yybgsfuv`. Accessed July 2, 2023.

National Research Council (2001). *Choosing the Right Formula: Initial Report*, The National Academies Press, Washington, DC.

New England Farm Workers Council (2017). New England Farm Workers Council budget narrative. `https://tinyurl.com/fakhb9m2`. Accessed July 2, 2023.

Nguyen, T. and Vohra, R. (2019). Stable matching with proportionality constraints, *Operations Research* **67**(6): 1503–1519.

Ohsawa, Y., Ozaki, N. and Plastria, F. (2008). Equity-efficiency bicriteria location with squared euclidean distances, *Operations Research* **56**(1): 79–87.

Olsder, W., Martagan, T. and Tang, C. S. (2022). Improving access to rare disease treatments: Subsidy, pricing, and payment schemes, *Management Science* .

Orgut, I. S., Ivy, J. S., Uzsoy, R. and Hale, C. (2018). Robust optimization approaches for the equitable and effective distribution of donated food, *European Journal of Operational Research* **269**(2): 516–531.

Pedraza-Martinez, A. J., Hasija, S. and Van Wassenhove, L. N. (2020). Fleet coordination in decentralized humanitarian operations funded by earmarked donations, *Operations Research* **68**(4): 984–999.

Pilarz, A. R. (2018). Child care subsidy programs and child care choices: Effects on the number and type of arrangements, *Children and Youth Services Review* **95**: 160–173.

Pilarz, A. R., Claessens, A. and Gelatt, J. (2016). Patterns of child care subsidy use and stability of subsidized care arrangements: Evidence from Illinois and New York, *Children and Youth Services Review* **65**: 231–243.

Privett, N. and Erhun, F. (2011). Efficient funding: Auditing in the nonprofit sector, *Manufacturing & Service Operations Management* **13**(4): 471–488.

Putnam, R. D. (2001). *Bowling alone: The collapse and revival of American community*, Simon and Schuster, New York, NY.

Rachidi, A. (2017). Child care assistance in the United States, *in* R. Doar (ed.), *A Safety Net That Works: Improving Federal Programs for Low-Income Americans*, Rowman & Littlefield, Washington DC, pp. 160–178.

Ratcliffe, M., Burd, C., Holder, K. and Fields, A. (2016). Defining rural at the U.S. Census Bureau: American community survey and geography brief.

Rice, D. (2016). Chart book: Cuts in federal assistance have exacerbated families' struggles to afford housing. `https://tinyurl.com/y6tfe637`. Accessed July 2, 2023.

Richter, L. M., Daelmans, B., Lombardi, J., Heymann, J., Boo, F. L., Behrman, J. R., Lu, C., Lucas, J. E., Perez-Escamilla, R., Dua, T. et al. (2017). Investing in the foundation of sustainable development: Pathways to scale up for early childhood development, *The Lancet* **389**(10064): 103–118.

Roels, G., Karmarkar, U. S. and Carr, S. (2010). Contracting for collaborative services, *Management Science* **56**(5): 849–863.

Rohacek, M. and Adams, G. (2017). Providers in the child care subsidy system: Insights into factors shaping participation, financial well-being, and quality. `https://www.urban.org/sites/default/files/publication/95221/providers-and-subsidies.pdf`. Accessed July 2, 2023.

Ryan, R. M., Johnson, A., Rigby, E. and Brooks-Gunn, J. (2011). The impact of child care subsidy use on child care quality, *Early Childhood Research Quarterly* **26**(3): 320–331.

Sargrad, S., Partelow, L., Yin, J. and Harris, K. M. (2020). Public education opportunity grants. `https://tinyurl.com/2nkzy779`. Accessed July 2, 2023.

Schneider, A. and Gibbs, H. (2022). Disparities in housing, health care, child care, and economic security affect babies for life. `https://tinyurl.com/3aty4dc7`. Accessed July 2, 2023.

Schneider, K. G., Joshi, P. and Ha, Y. (2021). An examination of child care provider participation in state subsidy contract systems, *Children and Youth Services Review* **127**: 106099.

Schneider, K. G., Warfield, M. E., Joshi, P., Ha, Y. and Hodgkin, D. (2017). Insights into the black box of child care supply: Predictors of provider participation in the massachusetts child care subsidy system, *Children and Youth Services Review* **79**: 148–159.

Schochet, L. and Malik, R. (2017). 2 million parents forced to make career sacrifices due to problems with child care, *Center for American Progress* **13**.

Schumacher, R., Irish, K. and Greenberg, M. H. (2003). Untapped potential? How states contract directly with providers to shore up child care choices, *Child Care and Early Education Series, April 2003* pp. 1–8.

SDG (2017). SDG Indicator 4.5.1. `https://sdg-tracker.org/quality-education`. Accessed July 2, 2023.

Sharma, N., Devalkar, S. K. and Sohoni, M. G. (2021). Payment for results: Funding non-profit operations, *Production and Operations Management* **30**(6): 1686–1702.

Slaugh, V. W., Akan, M., Kesten, O. and Ünver, M. U. (2016). The Pennsylvania adoption exchange improves its matching process, *Interfaces* **46**(2): 133–153.

Smith, K. and Adams, N. (2013). Child care subsidies critical for low-income families amid rising child care expenses, *Policy Brief No. 20, Carsey Institute, University of New Hampshire*.

Solak, S., Bayram, A., Gumus, M. and Zhuo, Y. (2018). Optimizing foreclosed housing acquisitions in societal response to foreclosure, *Operations Research* **67**(4): 950–964.

Speirs, K. E., Vesely, C. K. and Roy, K. (2015). Is stability always a good thing? Low-income mothers' experiences with child care transitions, *Children and Youth Services Review* **53**: 147–156.

Su, X. and Zenios, S. A. (2006). Recipient choice can address the efficiency-equity trade-off in kidney transplantation: A mechanism design model, *Management Science* **52**(11): 1647–1660.

Swenson, K. (2008). Child care arrangements in urban and rural areas. `https://aspe.hhs.gov/sites/default/files/migrated_legacy_files//42806/report.pdf`. Accessed July 2, 2023.

Taylor, T. A. and Xiao, W. (2014). Subsidizing the distribution channel: Donor funding to improve the availability of malaria drugs, *Management Science* **60**(10): 2461–2477.

The Association for Early Learning Leaders (2023). Accreditation fee schedule. `https://www.earlylearningleaders.org/accreditation/accreditation-fees/`. Accessed July 2, 2023.

The National Association for the Education of Young Children (2023). Accreditation fees. `https://www.naeyc.org/accreditation/early-learning/fees`. Accessed July 2, 2023.

Toyasaki, F. and Wakolbinger, T. (2014). Impacts of earmarked private donations for disaster fundraising, *Annals of Operations Research* **221**(1): 427–447.

Trisi, D. and Saenz, M. (2019). Economic security programs cut poverty nearly in half over last 50 years. `https://tinyurl.com/yyetcfza`. Accessed July 2, 2023.

Turner, M. A. (2003). Strengths and weaknesses of the Housing Voucher Program. https://tinyurl.com/ychdmnds. Accessed July 2, 2023.

Ullrich, R., Schmit, S. and Cosse, R. (2019). Inequitable access to child care subsidies. https://tinyurl.com/uk9v6tp. Accessed July 2, 2023.

UNICEF (2015). The investment case for education and equity. https://www.unicef.org/reports/investment-case-education-and-equity. Accessed July 2, 2023.

Urban Institute (2020). State and local finance initiative. https://tinyurl.com/y55efnph. Accessed July 2, 2023.

Urrea, G., Pedraza-Martinez, A. J. and Besiou, M. (2019). Volunteer management in charity storehouses: Experience, congestion and operational performance, *Production and Operations Management* **28**(10): 2653–2671.

U.S. Census Bureau (2015). 21.3 percent of U.S. population participates in government assistance programs each month. https://www.census.gov/newsroom/archives/2015-pr/cb15-97.html. Accessed July 2, 2023.

Walensky, R. P., Freedberg, K. A., Weinstein, M. C. and Paltiel, A. D. (2007). Cost-effectiveness of HIV testing and treatment in the United States, *Clinical Infectious Diseases* **45**(Supplement_4): S248–S254.

Walker, S. K. and Reschke, K. L. (2004). Child care use by low-income families in rural areas: A contemporary look at the influence of women's work and partner availability, *Journal of Children and Poverty* **10**(2): 149–167.

Weber, R. and Grobe, D. (2014). Director's survey for contracted slots pilot program: At the end of Year 1. https://health.oregonstate.edu/sites/health.oregonstate.edu/files/occrp/pdf/directors-survey-report-post-12-16-2014-to-committee.pdf. Accessed July 2, 2023.

WHO (2023). Health equity. https://www.who.int/health-topics/health-equity#tab=tab_1. Accessed July 2, 2023.

Williams, S. (2020). Santa Barbara: Need for day care in rural areas must be addressed. https://tinyurl.com/y9kca7fn. Accessed July 2, 2023.

Workman, S. (2021). The true cost of high-quality child care across the United States. https://www.americanprogress.org/article/true-cost-high-quality-child-care-across-united-states/. Accessed July 2, 2023.

Workman, S. and Ullrich, R. (2017). Quality 101: Identifying the core components of a high-quality early childhood program. https://tinyurl.com/293r6v4y. Accessed July 2, 2023.

Wykstra, S. (2014). Vouchers can help the poor find homes. But landlords often won't accept them. https://tinyurl.com/wgem3xv. Accessed July 2, 2023.

Yu, J. J., Tang, C. S. and Shen, Z.-J. M. (2018). Improving consumer welfare and manufacturer profit via government subsidy programs: Subsidizing consumers or manufacturers?, *Manufacturing & Service Operations Management* **20**(4): 752–766.

Zenios, S. A., Chertow, G. M. and Wein, L. M. (2000). Dynamic allocation of kidneys to candidates on the transplant waiting list, *Operations Research* **48**(4): 549–569.

Zhang, C., Atasu, A., Ayer, T. and Toktay, L. B. (2020). Truthful mechanisms for medical surplus product allocation, *Manufacturing & Service Operations Management* **22**(4): 735–753.

Zhu, L., Gong, Y., Xu, Y. and Gu, J. (2019). Emergency relief routing models for injured victims considering equity and priority, *Annals of Operations Research* **283**(1): 1573–1606.