SEMI-INFINITE FLAGS AND ZASTAVA SPACES

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SEMI-INFINITE FLAGS AND ZASTAVA SPACES

A Dissertation Presented

By

ANDREAS HAYASH

Submitted to the Graduate School of the University of Massachusetts Amherst in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

September 2023

Department of Mathematics and Statistics
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ACKNOWLEDGMENTS

I wish to thank my advisor Ivan Mirković for his continued support and encouragement, as well as for introducing me to many beautiful ideas in mathematics. I would also like to thank Tom Braden, Justin Campbell, Chris Elliott, Michael Finkelberg, Dennis Gaitsgory, and Owen Gwilliam for many useful and interesting discussions, some of which led to the ideas developed in this paper.

Special thanks are due to Joakim Faergeman and Sam Raskin, both of whom contributed crucial pieces of technical help in the paper in addition to many other interesting discussions. In particular, Joakim offered a simpler argument for conservativity in Lemma 7.1.0.1 than the one I originally had in mind, and Sam taught me the proof of Lemma 7.1.0.2.

I am glad to thank my friends and colleagues at the UMass Math Department, including Fil Dul, Bela Nelson, Chunlin Shao, Arthur Wang, and Chujiao Zhang for many good conversations, as well as for helping to foster a sense of community in the department.

Last but not least, I would also like to thank my family and friends for their continued love and support throughout these years. Without them none of this would have been possible.
We give an interpretation of Dennis Gaitsgory’s semi-infinite intersection cohomology sheaf associated to a semisimple simply-connected algebraic group in terms of finite-dimensional geometry. Specifically, we construct machinery to build factorization spaces over the Ran space from factorization spaces over the configuration space, and show that under this procedure the compactified Zastava space is sent to the support of the semi-infinite intersection cohomology sheaf in the Beilinson-Drinfeld Grassmannian. We also construct a partial resolution of singularities of the compactified Zastava space and show that the Zastava version of the semi-infinite intersection cohomology sheaf is pulled back to the ordinary (perverse) intersection cohomology sheaf of the partial resolution. Lastly, we show that there is a monad acting on sheaves over the resolution whose category of modules embeds fully faithfully in sheaves on the affine Grassmannian.
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1.1 Notation.

Fix an algebraically closed field $k$ of characteristic zero, and let $X$ be a connected smooth projective curve over $k$ of genus $g$. We will also fix a semisimple and simply-connected algebraic group $G$ over $k$ with Lie algebra $\mathfrak{g}$, together with a choice of Borel subgroup $B \subseteq G$. Choose a splitting $B = TN$ of $B$ into a maximal torus $T$ and unipotent radical $N$, and let $B^-$ be the Borel opposite to $B$ containing $T$. By $\Lambda$ we will denote the coweight lattice of the pair $(G, T)$, and our choice of Borel determines a subset $\{\alpha_i\} \subseteq \Lambda$ of simple coroots indexed by a set $\mathcal{I}_G$. Moreover, by $\Lambda^-$ we mean those coweights $\lambda$ which can be expressed as a sum $\sum_{i \in \mathcal{I}_G} (-n_i)\alpha_i$ with each $n_i$ a nonnegative integer. Dually, we will denote by $\check{\Lambda}$ the weight lattice of $G$, and by $\check{\Lambda}^+$ the subset of dominant weights.

1.2 Semi-infinite flags and Zastava spaces.

The purpose of this text is to establish a new connection between the semi-infinite flag variety of Feigin-Frenkel and the Zastava spaces of Drinfeld-Finkelberg-Mirković.

The semi-infinite flag variety $\mathcal{F}^{\infty}_\ell$ at a point $x \in X$ is an object of great interest

\footnote{More generally, one can work with a reductive group with simply-connected derived subgroup. However, for simplicity we will only consider semisimple groups.}
in geometric representation theory. First defined in (9), \( \mathcal{F} \ell \tilde{\mathcal{Z}} \) is given as the quotient

\[
\mathcal{F} \ell \tilde{\mathcal{Z}} := \mathcal{L}_x G / \mathcal{L}_x N \mathcal{L}_x^+ T
\]

where for an algebraic group \( H \) we denote by \( \mathcal{L}_x H \) its loop group at \( x \) and by \( \mathcal{L}_x^+ H \) its group of arcs\(^2\) Its category \( \mathcal{D}(\mathcal{F} \ell \tilde{\mathcal{Z}}) \) of D-modules is expected to be the target of a localization theory for modules over the Kac-Moody algebra \( \hat{\mathfrak{g}} \) (15)(16), similar in spirit to the celebrated result of Beilinson-Bernstein (3). For related reasons, \( \mathcal{D}(\mathcal{F} \ell \tilde{\mathcal{Z}}) \) is also expected to be the \( \mathcal{L}_x G \)-category associated to the trivial \( \bar{G} \)-local system on \( \hat{D}_x \) under the conjectural local geometric Langlands correspondence (10). Here \( \hat{D}_x := D_x \setminus x \) is the punctured formal disc about \( x \).

Early constructions of the category \( \mathcal{D}(\mathcal{F} \ell \tilde{\mathcal{Z}})\mathcal{L}_x^+ G \) of \( \mathcal{L}_x^+ G \)-invariants in \( D(\mathcal{F} \ell \tilde{\mathcal{Z}}) \) were proposed in (11), and then later in (1). Both use in an essential way the Zastava spaces \( \mathcal{Z} \), also first defined in (11). The Zastava space \( \mathcal{Z} \) associated to \( G \) is a finite-dimensional local model for \( \mathcal{F} \ell \tilde{\mathcal{Z}} \) and is defined as the moduli space of \( G \)-bundles together with a generalized \( N \)-reduction and a generically transverse \( B^- \)-reduction. Here by "generalized" we mean that the reduction is allowed to have points of degeneracy. More precisely, the Zastava space organizes transverse slices to \( \mathcal{L}_x^+ G \)-orbits in \( \mathcal{F} \ell \tilde{\mathcal{Z}} \) into a single space.

A crucial feature of \( \mathcal{Z} \) is that it factorizes over the configuration space \( \text{Conf} \) of points in \( X \) with coefficients in \( \Lambda^- \). Roughly, this means that the fiber of \( \mathcal{Z} \) over a disjoint union of divisors can be written as the product of fibers over each component divisor. Moreover, \( \mathcal{Z} \) contains a smooth and factorizable dense open subscheme, and as a result the intersection cohomology sheaf \( \text{IC}_\mathcal{Z} \) has a canonical structure of a factorization algebra. Intuitively, the factorization structure on \( \mathcal{Z} \) should be thought of

\(^2\)Strictly speaking, there are in the literature various inequivalent, though related definitions of the semi-infinite flag variety. For example, the approach taken in (22) is expected to yield a different object than the approach taken in the present text. Although the two are closely related, we will refrain from a discussion of the latter.
as making \( Z \) into a kind of monoid\(^3\), and \( \text{IC}_Z \) should be thought of as a multiplicative sheaf with respect to this monoid structure. For any factorization algebra there is also an associated category of factorization modules at the point \( x \). In (11) and (1), the category \( \mathcal{D}(\mathcal{F}(\overline{Z}))^{\text{op}}_G \) is essentially defined to be the category of factorization modules for the intersection cohomology sheaf \( \text{IC}_Z \) at \( x \).

### 1.3 Semi-infinite flags and \( \text{IC}_{\text{Ran}}^\infty \)

More recently, another construction of \( \mathcal{D}(\mathcal{F}(\overline{Z}))^{\text{op}}_G \) was proposed in (15) and (16) using the affine Grassmannian \( \text{Gr}_{G,\text{Ran}} \) over the moduli \( \text{Ran} \) of finite subsets of \( X \). There is a natural action of the loop group \( \mathfrak{L}_{\text{Ran}}N \) of \( N \) relative to the \( \text{Ran} \) space on \( \text{Gr}_{G,\text{Ran}} \) whose orbits are indexed by the coweight lattice \( \Lambda \) and denoted \( S^\lambda_{\text{Ran}} \) for \( \lambda \in \Lambda \). In (16) Gaitsgory constructs a factorization algebra \( \text{IC}_{\text{Ran}}^\infty \) in \( D \)-modules over the Beilinson-Drinfeld Grassmannian supported on the closure \( \overline{S^0_{\text{Ran}}} \) of the semi-infinite orbit \( S^0_{\text{Ran}} \) whose category of factorization modules is expected to possess the properties desired of \( \mathcal{D}(\mathcal{F}(\overline{Z}))^{\text{op}}_G \) (15).

By definition, \( \text{IC}_{\text{Ran}}^\infty \) is the intermediate extension of the dualizing sheaf \( \omega_{S^0_{\text{Ran}}} \) relative to a semi-infinite \( t \)-structure defined on the category of \( \mathfrak{L}_{\text{Ran}}N \)-equivariant \( D \)-modules on \( \overline{S^0_{\text{Ran}}} \). The latter \( t \)-structure can loosely be thought of as a perverse \( t \)-structure on \( D \)-modules over \( \overline{S^0_{\text{Ran}}} \) which are “constructible” with respect to the semi-infinite stratification. Note that this context is quite different from the traditional one since the strata \( S^\lambda_{\text{Ran}} \) are of infinite dimension and of infinite codimension. In particular, in the semi-infinite \( t \)-structure \( \omega_{S^0_{\text{Ran}}} \) lies in the heart whereas in the classical perverse \( t \)-structure it lies in cohomological degree minus infinity.

---

\(^3\)In fact, this is more than an analogy since a factorization structure is a monoid (or commutative algebra) structure in a category of correspondences.
1.3.1 The obvious question one might ask is if the early constructions from (11) and (1) can be compared to the later one in (16). A first step towards doing so is to reconstruct the semi-infinite intersection cohomology sheaf $\text{IC}_{\text{Ran}}^\infty$ from the geometry of the Zastava spaces. In the present paper such a reconstruction is performed (see Theorems 6.0.0.1 and 7.1.0.3). In slightly more detail, we produce a sheaf $\text{IC}_{\mathcal{Z}}^\infty$ on a compactification $\mathcal{Z}$ of $\mathcal{Z}$ which is in a precise sense equivalent to $\text{IC}_{\text{Ran}}^\infty$ and which becomes the ordinary intersection cohomology after pullback to a partial resolution $\mathcal{Z}_K$ of $\mathcal{Z}$. The compactification $\mathcal{Z}$ is obtained by allowing generalized $B^-$-reductions of $G$-bundles in addition to considering generalized $N$-reductions in the definition of $\mathcal{Z}$. Its resolution $\mathcal{Z}_K$ is slightly harder to define and uses a resolution of singularities by stable maps of Drinfeld’s compactification constructed in (6).

Although a comparison of factorization modules for $\text{IC}_{\mathcal{Z}}^\infty$ and $\text{IC}_{\text{Ran}}^\infty$ is beyond the scope of the paper and will be carried out in a future work, we can nevertheless say a few words. It is fairly clear that a suitably defined category of factorization modules with extra structure for $\text{IC}_{\mathcal{Z}}^\infty$ is equivalent to factorization modules for $\text{IC}_{\text{Ran}}^\infty$. Additionally, calculations carried out by the author indicate that this category is not equivalent to factorization modules for $\text{IC}_{\mathcal{Z}}$, but that it is indeed an infinity categorical enhancement of the category defined in (1). As such, it does indeed seem that $\mathcal{Z}$ on its own is insufficient to describe $\mathcal{F}\ell^\infty$.

1.4 The reconstruction principle, aka accumulation of dust.

The origin of the idea for this paper can be found in (19). In loc. cit. Mirković observes that many interesting spaces $\mathcal{Y}_x$, usually ind-schemes, associated to the formal disc $D_x$ around a point $x \in X$ can be “reconstructed” as a colimit from a family of objects, usually schemes, living over the $n$-th infinitesimal neighborhoods
$D^\circ_x \cong \text{Spec} \left( k[z]/(z^n) \right)$ of $x$.

More precisely, such a $\mathcal{Y}_x$ can often be endowed with an assignment of a space $\mathcal{Y}_x^n$ to each $D_x^n$, associative maps $\mathcal{Y}_x^n \to \mathcal{Y}_x^m$ whenever $n \leq m$, and an isomorphism

$$\text{colim}_n \mathcal{Y}_x^n \xrightarrow{\sim} \mathcal{Y}_x$$

as objects associated to $D_x$. As a toy example, note that the $D$-module $\mathcal{O}_{D_x}^\vee$ of distributions on $D_x$ can be written as a colimit

$$\text{colim}_n \mathcal{O}_{D_x}^\vee \xrightarrow{\sim} \mathcal{O}_{D_x}^\vee$$

of distributions on the finite schemes $D_x^n$.

It is often the case that such reconstructions also occur “factorizably,” i.e. where we allow the point $x$ to move and collide with other points of $X$. Although this idea is not made precise in (19), part of the motivation of the present work is to give such a construction. To do so, we employ a correspondence

$$\text{Conf} \leftrightarrow \text{Conf}_{\text{corr}} \leftrightarrow \text{Ran}$$

(1.4.0.1)

between the Hilbert space $\text{Conf}$ of points on $X$ with coefficients in the negative cone $\Lambda^-$ and the moduli $\text{Ran}$ of finite subsets of $X$.

Whenever the fibers of a space $\mathcal{Y} \to \text{Conf}$ vary functorially$^4$ under inclusions of divisors, we explain in Section 6.1 how to obtain a space $\mathcal{Y}_{\text{Ran}}$ over $\text{Ran}$ from $\mathcal{Y}$ using a pull-push procedure along the diagram (1.4.0.1). Explicitly, the fiber of $\mathcal{Y}_{\text{Ran}}$ at a point $x$ should think of $\mathcal{Y}$ as a sort of coCartesian fibration lying over $\text{Conf}$. To make this precise, we use the language of lax prestacks (see Section 2).
finite subset \( x_I \subseteq X \) is given by the colimit

\[
\colim_{D \subseteq x_I} \mathcal{Y}_D \sim (\mathcal{Y}_{\text{Ran}})_{x_I}
\]

over the moduli of divisors \( D \) which are set-theoretically supported on the formal completion in \( X \) of the union of points contained in \( x_I \). It is useful to think of the isomorphisms (1.4.0.2) as endowing the space \( \mathcal{Y}_{\text{Ran}} \) with a sort of filtration relative to \( \text{Ran} \) indexed by the cone \( \Lambda^- \).

### 1.5 Reconstruction and Zastava spaces.

The central example of a space equipped with a notion of functoriality with respect to inclusions of divisors which will be considered in the present text is the compactified Zastava \( \overline{Z} \). The scheme \( \text{Conf} \) is a monoid under addition of divisors, and one can equip \( \overline{Z} \) with the structure of an augmented module for \( \text{Conf} \). Such an action endows \( \overline{Z} \) with the necessary functoriality to plug it into the machinery discussed above.

The fiber of \( \overline{Z} \) over a point \( \lambda x \in \text{Conf} \) is given by the intersection

\[
\overline{Z}_{\lambda x} := \overline{S}_x^0 \cap \overline{S}^{-,\lambda}_x
\]

where \( S_x^0 \) is the orbit of \( L_x N \) in the fiber \( \text{Gr}_{G,x} \) of the Beilinson-Drinfeld Grassmannian at \( x \), and \( S_x^{-,\lambda} \) is the orbit of the loop group at \( x \) of the opposite unipotent \( N^- \). Whenever \( \lambda \leq \mu \) we have an inclusion \( \overline{S}^{-,\lambda}_x \hookrightarrow \overline{S}^{-,\mu}_x \) induced by the closure relations of semi-infinite orbits, and it follows that we have an isomorphism

\[
\colim_{\lambda} \overline{Z}_{\lambda x} \sim \overline{S}_x^0.
\]
These isomorphisms assemble into an equivalence

\[ \overline{Z}_{\text{Ran}} \sim \overline{S}_{\text{Ran}}^0 \]

which respects factorization structures on both sides. As a result, the geometry of \( \overline{S}_{\text{Ran}}^0 \) can be analyzed in terms of the (finite-dimensional) geometry of \( \overline{Z} \).

### 1.6 Zastava spaces and \( \text{IC}^\infty_{\overline{Z}_{\text{Ran}}} \)

A feature of the assignment \( \mathcal{Y} \mapsto \mathcal{Y}_{\text{Ran}} \) taking a space over \( \text{Conf} \) to a space over \( \text{Ran} \) is that it allows us to calculate the category \( \mathcal{D}(\mathcal{Y}_{\text{Ran}}) \) of \( D \)-modules on \( \mathcal{Y}_{\text{Ran}} \) in terms of the category \( \mathcal{D}(\mathcal{Y}) \) (see Corollary 6.1.0.1.1). An advantage of this is that \( \mathcal{D}(\mathcal{Y}) \) is often "easier" to work with than \( \mathcal{D}(\mathcal{Y}_{\text{Ran}}) \), especially in the case \( \mathcal{Y} \) is finite-dimensional.

More explicitly, let \( \mathcal{F} \) be a \( D \)-module on \( \mathcal{Y}_{\text{Ran}} \). Using the isomorphism (1.4.0.2), the pullback \( \mathcal{F}_{x_I} \) of \( \mathcal{F} \) to the fiber of \( \mathcal{Y}_{\text{Ran}} \) over \( x_I \) can be written as a system \( (\mathcal{F}_{x_I}^D)_{D \subseteq x_I} \) of \( D \)-modules on the spaces \( \mathcal{Y}_D \) together with homotopy associative isomorphisms

\[ f^{E,D}_{E,D}(\mathcal{F}_{x_I}^D) \sim \mathcal{F}_{x_I}^E, \]

where \( f_{E,D} : \mathcal{Y}_E \to \mathcal{Y}_D \) is the transition map associated to an inclusion \( E \subseteq D \) of divisors. Working backwards, to construct a \( D \)-module \( \mathcal{F} \) on \( \mathcal{Y}_{\text{Ran}} \), it is therefore enough to construct the \( D \)-modules \( \mathcal{F}_{x_I}^D \) and the isomorphisms \( f^{E,D}_{E,D} \). We call such systems effective sheaves on \( \mathcal{Y} \).

As an application of the above discussion to the semi-infinite IC sheaf \( \text{IC}^\infty_{\overline{Z}_{\text{Ran}}} \), we construct such a system of \( D \)-modules on \( \overline{Z} \) via descent along a factorizable partial resolution \( \overline{Z}_K \to \overline{Z} \) of singularities. The space \( \overline{Z}_K \) is a Deligne-Mumford stack which is introduced in Section 7.0.1.1. Its (ordinary) intersection cohomology sheaf can be endowed with descent data which produces an effective factorization algebra \( \text{IC}^\infty_{\overline{Z}} \) on
that corresponds to $\text{IC}_{\overline{\text{Ran}}}^\infty$ under the reconstruction principle.

The reason for introducing $\overline{Z}_K$ can be thought of heuristically as follows. Although the Zastava space $Z$ models transverse slices to $\mathcal{L}^+G$-orbits in $\mathcal{F}(\xi)$, unlike $Z$ it does not contain enough redundancy to apply the reconstruction principle. Indeed if $Z$ can be thought of as a filtration of $\overline{S}_0\text{Ran}$, the Zastava $Z$ is akin to its associated graded. Unfortunately, in compactifying $Z$ we introduce certain singularities that are not detected by $\text{IC}_{\overline{\text{Ran}}}^\infty$. Fortunately, only these “bad” singularities are resolved by $\overline{Z}_K$. Working with $\overline{Z}_K$ and its intersection cohomology sheaf therefore has the effect of retaining the redundancy of $Z$ but keeping only the singularities coming from $Z$.

1.7 Possible applications to Langlands duality for Kac-Moody algebras.

A notable triumph of (19) together with (20) is a generalization of the compactified Zastava (and therefore of the Beilinson-Drinfeld Grassmannian) which uses only a bilinear form and a line bundle on $\text{Conf}$. This is a possible first step in a generalization of the geometric Langlands program to Kac-Moody algebras. One can therefore view the present work as being partially motivated by such an extension of the geometry associated to Langlands duality.

Since the construction of $\overline{Z}_K$ involves geometric data associated to the group $G$, it is not obvious at the moment how to define a semi-infinite intersection cohomology sheaf for a Kac-Moody algebra using the techniques of the present paper. Nevertheless, it is highly plausible that an alternative definition of $\overline{Z}_K$ exists which only depends on the Killing form.
1.8 The plan and statement of results.

We will now give a brief overview of the paper, as well as precise statements of some of the central results.

In Section 2 we review the theory of lax prestacks and their sheaves. Roughly, a lax prestack is a functor from commutative algebras to categories. The notion of lax prestack is a convenient framework in which to unify the theory of unital factorization structures, as well as their configuration space analogues effective factorization structures, under a single banner. A review of the latter will occupy Section 3.

In Section 4 we will review Gaitsgory’s construction of the Ran semi-infinite intersection cohomology sheaf $\text{IC}_\text{Ran}^{\infty}$. Specifically, we will sketch the construction of the $t$-structure inside whose heart $\text{IC}_\text{Ran}^{\infty}$ belongs, as well as the factorization structure on $\text{IC}_\text{Ran}^{\infty}$. Following this, we will review in Section 5 the theory of Zastava spaces, and introduce the Zastava semi-infinite intersection cohomology sheaf as Definition 5.2.0.1.

Section 6 will contain our first main result. Namely, we will identify a category $\mathcal{D}_{\text{eff}}(\mathcal{Z})$ of sheaves on $\mathcal{Z}$ with extra structure to which $\text{IC}_\mathcal{Z}^{\infty}$ will belong. The following will be proved as Theorem 6.0.0.1.

**Theorem 1.8.0.1.** There is an equivalence of categories

$$\mathcal{D}_{\text{eff}}(\mathcal{Z}) \sim \mathcal{D}_{\text{untl}}(\mathcal{S}_\text{Ran}^{0})$$

which sends $\text{IC}_\mathcal{Z}^{\infty}$ to $\text{IC}_\text{Ran}^{\infty}$.

In the statement of the theorem, the category $\mathcal{D}_{\text{untl}}(\mathcal{S}_\text{Ran}^{0})$ is the category of unital $D$-modules on $\mathcal{S}_\text{Ran}^{0}$, i.e. $D$-modules which are equivariant for the action of Ran on $\mathcal{S}_\text{Ran}^{0}$. The proof will proceed geometrically; we will identify the quotient $\mathcal{Z}/\text{Conf}$ with the quotient $\mathcal{S}_\text{Ran}^{0}/\text{Ran}$ (the latter is also referred to as the generic Zastava
As a corollary, we will show that the category of D-modules on Drinfeld’s compactification \( \overline{\text{Bun}}_N \) embeds fully faithfully in \( \mathcal{D}_{\text{eff}}(\mathcal{Z}) \). Here \( \overline{\text{Bun}}_N \) denotes the moduli of \( G \)-bundles together with a generalized \( N \)-reduction.

Section 7.0.1.1 may be viewed as the heart of the paper. We will introduce the Kontsevich Zastava space \( \mathcal{Z}_K \) in 7.0.1, a Deligne-Mumford factorizable stack equipped with a map \( \mathfrak{r}_Z : \mathcal{Z}_K \to \mathcal{Z} \) which is built out of a resolution of singularities for Drinfeld’s compactification. In fact, \( \mathcal{Z}_K \) is equivalent to \( \overline{\text{Bun}}_N \) locally in the smooth topology. As a result, \( \mathcal{Z}_K \) is a factorizable model for \( \overline{\text{Bun}}_N \) on which \( \text{IC}_Z^\infty \) “lives” as an ordinary IC sheaf. Our main result (Theorem 7.0.3.1) will be the following.

**Theorem 1.8.0.2.** There is a canonical isomorphism

\[
\mathfrak{r}_Z^!(\text{IC}_Z^\infty) \sim \text{IC}_Z^\infty
\]

of factorization algebras. Here \( \text{IC}_Z^\infty \) is a factorization algebra equivalent to \( \text{IC}_Z \) up to a cohomological shift.

In other words, although the sheaf itself \( \text{IC}_Z^\infty \) may not be perverse\(^5\), it restricts to a perverse sheaf after smoothing the singularities in \( \overline{\text{Bun}}_{B^{-}} \). Moreover, we will show in Theorem 7.1.0.3 that the sheaf \( \text{IC}_Z^\infty \), and by extension the sheaf \( \text{IC}_\text{Ran}^\infty \), can be recovered from \( \text{IC}_Z \). More precisely, we will construct a monad \( \mathcal{M}_K \) acting on the category \( \mathcal{D}(\mathcal{Z}_K) \) together with an equivalence

\[
\mathcal{M}_K \text{- mod}(\mathcal{D}(\mathcal{Z}_K)) \sim \mathcal{D}_{\text{eff}}(\mathcal{Z})
\]

which sends (a shift of) \( \text{IC}_{\mathcal{Z}_K} \) to the Zastava semi-infinite IC sheaf.

---

\(^5\)Strictly speaking, whether or not \( \text{IC}_Z^\infty \) is perverse is still an open question.
CHAPTER 2

LAX PRESTACKS AND THEIR
SHEAVES

In this chapter we will review the concept of lax prestack. For a general survey on lax prestacks see (21). Let AffSch denote the category of (non-derived) affine schemes over $k$, and let Cat denote the $\infty$-category of $\infty$-categories.

2.1 Lax prestacks.

A lax prestack is a functor $\text{AffSch}^{\text{op}} \rightarrow \text{Cat}$. The category $\text{Fun}(\text{AffSch}^{\text{op}}, \text{Cat})$ of lax prestacks will be denoted $\text{LaxPrStk}$.

Remark 2.1.0.1. In this paper most functors $\text{AffSch}^{\text{op}} \rightarrow \text{Cat}$ we consider will take values in (nerves of) ordinary categories. Nevertheless, it is still desirable to fit these objects into the broader framework of higher category theory, in particular to have a well-behaved theory of sheaves.

Let $\text{Spc}$ denote the $\infty$-category of $\infty$-groupoids. Then there is a natural inclusion $\text{Spc} \rightarrow \text{Cat}$ and so every prestack $\text{AffSch}^{\text{op}} \rightarrow \text{Spc}$ may be thought of as a particular kind of lax prestack. We will denote the left adjoint to the inclusion $\text{PrStk} \rightarrow \text{LaxPrStk}$ by $\text{str} : \text{LaxPrStk} \rightarrow \text{PrStk}$. Roughly, for an affine scheme $S$, the space $\mathcal{Y}_{\text{str}}(S) := \text{str}(\mathcal{Y})(S)$ is the universal groupoid admitting a map from $\mathcal{Y}(S)$.

Here is an equivalent formulation of $\mathcal{Y}_{\text{str}}$. There is a fully faithful functor $\text{Cat} \rightarrow$
Fun(Δ^{op}, Spc), whose essential image consists of the complete Segal spaces. Hence to a lax prestack \( \mathcal{Y} \) we may associate a simplicial prestack \( s\mathcal{Y} : \text{AffSch}^{op} \to \text{Fun}(\Delta^{op}, \text{Spc}) \) satisfying the Segal conditions. The following lemma gives a more concrete construction of \( \mathcal{Y}_{\text{str}} \).

**Lemma 2.1.0.2.** Let \( \mathcal{Y} \) be a lax prestack and let \( s\mathcal{Y} \) be the associated simplicial prestack. Then there is a canonical equivalence of prestacks

\[
\mathcal{Y}_{\text{str}} \longrightarrow |s\mathcal{Y}|,
\]

where \( |s\mathcal{Y}| \) is the prestack obtained from \( s\mathcal{Y} \) by composing with the geometric realization functor \( \text{Fun}(\Delta^{op}, \text{Spc}) \to \text{Spc} \).

**Proof.** This follows from the fact that the straightening functor \( \text{Cat} \to \text{Spc} \) is equivalent to the composition

\[
\text{Cat} \longrightarrow \text{Fun}(\Delta^{op}, \text{Spc}) \longrightarrow \text{Spc},
\]

where the first arrow is the functor taking a category to its associated complete Segal space, and the second arrow is geometric realization. \( \square \)

Recall that a category \( \mathcal{C} \) is weakly contractible if \( \mathcal{C}_{\text{str}} \) is equivalent to a point in \( \text{Spc} \). We will say that a lax prestack \( \mathcal{Y} \) is weakly contractible if for all affine schemes \( S \), the category \( \mathcal{Y}(S) \) is weakly contractible.

**Remark 2.1.0.3.** The straightening functor does not preserve homotopy groups in general. This means that even if \( \mathcal{Y} \) is a lax prestack taking values in ordinary categories, the straightening \( \mathcal{Y}_{\text{str}} \) may take values in higher groupoids.

Consider the following example of a lax prestack, which will be important throughout the text. Let \( \mathcal{M} \) be a presheaf of commutative monoids on \( \text{AffSch}_k \). That is, for
every affine scheme $S$ the groupoid $\mathcal{M}(S)$ is a set, and $\mathcal{M}$ has a factorization through the forgetful functor from commutative monoids to sets.

The functor $\mathcal{M}$ can be upgraded to a lax prestack as follows. Define $\mathcal{M}^{-}(S)$ to be the category whose objects are given by the set $\mathcal{M}(S)$ and such that there is a unique arrow $x \to y$ for every $m$ such that $mx = y$. For every morphism $S \to T$ of affine schemes, the morphism $\mathcal{M}(T) \to \mathcal{M}(S)$ of monoids given by restriction can be upgraded to a functor

$$\mathcal{M}^{-}(T) \to \mathcal{M}^{-}(S)$$

in the obvious way. It is easy to see that the associated simplicial prestack is given by the simplicial object

$$\ldots \xrightarrow{\cdot} \mathcal{M} \times \mathcal{M} \xrightarrow{\cdot} \mathcal{M} \xrightarrow{\cdot} \mathcal{M}$$

induced by multiplication in $\mathcal{M}$.

**Remark 2.1.0.4.** Note that $\mathcal{M}^{-}$ is weakly contractible if $\mathcal{M}$ is a presheaf of cancellative monoids. Explicitly, $\mathcal{M}$ is cancellative if for all $x, y, z \in \mathcal{M}(S)$ if $xy = xz$ then $y = z$. In this case, $\mathcal{M}^{-}$ is a functor of filtered categories, and all such are known to be weakly contractible.

### 2.2 Sheaves on lax prestacks.

We will now briefly review sheaves on lax prestacks (21). Let us assume we are given a sheaf theory $\text{Shv} : \text{AffSch}^{op} \to \text{Cat}$. For all intents and purposes $\text{Shv}(S)$ will be the category of quasicoherent sheaves, ind-coherent sheaves, or $D$-modules (see (13) for a survey of ind-coherent sheaves and (17) for a survey of $D$-modules). For a map $f : S \to T$ of affine schemes, denote the resulting functor between categories of sheaves by $f^!$. We can then define the category of sheaves on an arbitrary lax
Definition 2.2.0.1. Let \( \mathcal{Y} \) be a lax prestack. Then the category \( \text{Shv}(\mathcal{Y}) \) of sheaves on \( \mathcal{Y} \) is the category of natural transformations \( \mathcal{Y} \to \text{Shv} \). In particular, the assignment \( \mathcal{Y} \mapsto \text{Shv}(\mathcal{Y}) \) takes arbitrary colimits to limits.

Unwinding the definition of \( \text{Shv}(\mathcal{Y}) \), we see that a sheaf \( F \) on \( \mathcal{Y} \) is the following data.

1. For every \( S \)-point \( f : S \to \mathcal{Y} \) of \( \mathcal{Y} \) we have a sheaf \( F_{S,f} \in \text{Shv}(S) \).
2. For every map \( \varphi : (S, f) \to (T, g) \) of affine schemes over \( \mathcal{Y} \), we have an isomorphisms \( \varphi^! F_{T,g} \simeq F_{S,f} \) with homotopy coherent compatibilities.
3. For every morphism \( f \to g \) of \( S \)-points, we have morphisms \( F_{S,f} \to F_{S,g} \) with homotopy coherent compatibilities.

By definition of \( \mathcal{Y}_{\text{str}} \), it follows that the category \( \text{Shv}(\mathcal{Y}_{\text{str}}) \) is equivalent to the full subcategory \( \text{Shv}_{\text{str}}(\mathcal{Y}) \) of sheaves on \( \mathcal{Y} \) where all the morphisms \( F_{S,f} \to F_{S,g} \) in 3 are isomorphisms. Denote the resulting full subcategory of \( \text{Shv}(\mathcal{Y}) \) by \( \text{Shv}_{\text{str}}(\mathcal{Y}) \). There is the following equivalent simplicial perspective on \( \text{Shv}_{\text{str}}(\mathcal{Y}) \).

Lemma 2.2.0.2. Let \( \mathcal{Y} \) be a lax prestack, and let \( s\mathcal{Y} \) be the associated simplicial prestack. Then there is an equivalence of categories

\[
\text{Shv}_{\text{str}}(\mathcal{Y}) \longrightarrow \text{Tot}(\text{Shv}(s\mathcal{Y})),
\]

where \( \text{Tot}(\text{Shv}(s\mathcal{Y})) \) denotes the totalization of the cosimplicial category \( \text{Shv}(s\mathcal{Y}) \).

Proof. Since \( \text{Shv}_{\text{str}}(\mathcal{Y}) \) is equivalent to \( \text{Shv}(\mathcal{Y}_{\text{str}}) \), it suffices to show the claim with the left hand side replaced by the latter. But by Lemma 2.1.0.2 we have a canonical isomorphism \( \mathcal{Y}_{\text{str}} \simeq |s\mathcal{Y}| \), and so the claim follows from the fact that \( \text{Shv} \) sends arbitrary colimits of prestacks to limits. \( \square \)
2.3 Left fibrations of lax prestacks.

Recall that a functor $F : C \to D$ of $\infty$-categories is a left fibration if it satisfies the right lifting property with respect to all horn inclusions, except possibly the right outer ones. That is, whenever the outer square of

$$\begin{array}{ccc}
\Lambda^k[n] & \rightarrow & C \\
\downarrow & & \downarrow F \\
\Delta[n] & \rightarrow & D
\end{array}$$

commutes, there exists a dotted arrow as above for all $0 \leq k < n$ making the diagram commute. In higher category theory, the left fibrations play a role similar to that of categories cofibered in groupoids in ordinary category theory. In fact, any left fibration is also a coCartesian fibration, and if the base is a Kan complex then it is also a Kan fibration.

2.3.1 We will define here an analogue of left fibrations for lax prestacks. A treatment of the following material, including the notion of multiplicative space, can be found in (5). Consider the lax prestack $\text{Spc}_{/\dash} \dash$ which sends an affine scheme to the slice category $\text{PrStk}/S$. For a lax prestack $Y$, define $\text{Spc}_{/Y}$ to be the category of natural transformations $
abla : Y \rightarrow \text{Spc}_{/\dash}$. For a lax prestack $\mathcal{Y}$, define $\text{Spc}_{/\mathcal{Y}}$ to be the category of natural transformations

$$\Phi : \mathcal{Y} \rightarrow \text{Spc}_{/\dash}$$

One should think of $\Phi$ as assigning to an $S$-point $f : S \rightarrow \mathcal{Y}$ the fiber $\Phi_f$ of the associated left fibration. For any morphism $f \rightarrow g$ of $S$-points, the structure of natural transformation $\mathcal{Y} \rightarrow \text{Spc}_{/\dash}$ gives a morphism

$$\Phi_f \rightarrow \Phi_g$$
as prestacks over $S$. By passing to global sections and using the Grothendieck construction, for each $\Phi$ there is an associated lax prestack $\Phi_{\text{Spc}} \to \mathcal{Y}$ lying over $\mathcal{Y}$. For an affine scheme $S$, the functor $\Phi_{\text{Spc}}(S) \to \mathcal{Y}(S)$ is a left fibration of categories.

We will refer to a natural transformation $\Phi : \mathcal{Y} \to \text{Spc}/-$ as a \textit{left fibration of lax prestacks} or simply as a \textit{left fibration}. Since left fibrations are examples of coCartesian fibrations, we will also sometimes call such functors $\Phi$ \textit{coCartesian fibrations of lax prestacks}. Note that we have the lax prestack $\mathcal{Y}^\text{op}$ taking an affine scheme $S$ to $\mathcal{Y}(S)^\text{op}$. Hence we also have the notion of \textit{right fibration of lax prestacks} or \textit{Cartesian fibrations of lax prestacks}. Additionally, one can define a \underline{pointwise (co)Cartesian fibration of lax prestacks} which is a map $\mathcal{X} \to \mathcal{Y}$ of lax prestacks such that for all affine schemes $S$ the induced functor

$$\mathcal{X}(S) \to \mathcal{Y}(S)$$

is a (co)Cartesian fibration. Although for $\Phi$ as above the map $\Phi_{\text{Spc}} \to \mathcal{Y}$ is a pointwise coCartesian fibration, it is not clear that the converse holds.

An advantage of this perspective is that one can immediately see how to assign to a lax prestack $\mathcal{Y}$ a sheaf of categories, given the existence of such a $\Phi$. Recall that a sheaf of categories is defined to be a lax natural transformation

$$\mathcal{Y} \to \text{ShvCat}/$$

where ShvCat is the lax prestack which assigns to an affine scheme $S$ the category of modules for the symmetric monoidal category $\text{QCoh}(S)$. Note there is a natural transformation

$$\mathcal{D}^{\text{Spc}/-} : \text{Spc}/- \to \text{ShvCat}/-$$

which takes a prestack $\mathcal{X} \to S$ over an affine scheme to $\mathcal{D}(\mathcal{X})$. Hence to any left
fibration $\Phi$ over $\mathcal{Y}$ we may assign a sheaf of categories given as the composition

$$D^{\text{SpCat}}(\Phi) := D^{\text{SpCat}} \circ \Phi.$$  

Note whenever there is a map $f : \mathcal{X} \to \mathcal{Y}$ of lax prestacks, we can define a pullback functor

$$f^{\text{SpCat}} : \text{Spc}/\mathcal{Y} \to \text{Spc}/\mathcal{X}$$

simply by composing a lax natural transformation $\mathcal{Y} \to \text{Spc}/$ with $f$. We therefore obtain a functor

$$\text{LaxPrStk}^{\text{op}} \to \text{Cat}$$

which takes a lax prestack $\mathcal{Y}$ to $\text{Spc}/\mathcal{Y}$.

### 2.4 Correspondences.

In this section we will discuss categories of correspondences. These will be important in defining notions of factorization spaces and their variants (e.g. unital, effective). Of course, the definition of factorization spaces is well-known, and we are not adding anything new to the theory here. However, since we are considering factorization over several different bases, it is convenient to have a unified language to discuss all of them.

Let $\mathcal{C}$ be a category with all finite limits, and let $\text{Corr}(\mathcal{C})$ be the 2-category of correspondences\footnote{We will mostly sweep all 2-categorical complexities under the rug. The ideas can all be made precise, however, using the approach in (21), mutatis mutandis.} in $\mathcal{C}$. Objects in $\text{Corr}(\mathcal{C})$ are just objects of $\mathcal{C}$, but morphisms $\mathcal{Y} \to \mathcal{G}$ are given by correspondences

$$\begin{array}{c}
\mathcal{Z} \\
\mathcal{Y} \leftarrow \mathcal{G}.
\end{array}$$
Composition of morphisms $\mathcal{Y} \to \mathcal{G} \to \mathcal{H}$ is given by the outer legs of the diagram

\[
\begin{array}{ccc}
\mathcal{Y} & \xrightarrow{\mathcal{Z} \times_{\mathcal{G}} \mathcal{Z}'} & \mathcal{H} \\
\downarrow & & \\
\mathcal{Z} & \mathcal{G} & \mathcal{H} \\
\downarrow & \downarrow & \\
\mathcal{Z} & \mathcal{Z}' & \mathcal{H}.
\end{array}
\]

Note that $\text{Corr}(\mathcal{C})$ is naturally symmetric monoidal under the Cartesian monoidal structure. Hence we may consider commutative algebras (aka commutative monoids) in $\text{Corr}(\mathcal{C})$. Roughly, a commutative algebra in $\text{Corr}(\mathcal{C})$ is a pair of correspondences $\text{mult}_\mathcal{Y}$ and $\text{unity}_\mathcal{Y}$ satisfying natural associativity and unitality conditions up to coherent homotopy.

We will apply the above picture to both $\text{PrStk}$ and $\text{LaxPrStk}$ to obtain the categories $\text{Corr(PrStk)}$ and $\text{Corr(LaxPrStk)}$. Note that $\text{Spc}/(-)$ is naturally a commutative algebra in lax prestacks, where multiplication is induced by the functors

\[
\text{Spc}/S \times \text{Spc}/S \to \text{Spc}/S
\]

taking a pair of prestacks $(\mathcal{Y}, \mathcal{X})$ to the fiber product $\mathcal{Y} \times_S \mathcal{X}$. The unit is provided by the inclusion of the identity $S \to S$ as an object of $\text{Spc}/S$.

### 2.5 Multiplicative spaces.

In this section, we’ll review the category of multiplicative spaces over a commutative algebra in correspondences. A general survey of the notion of multiplicative spaces can be found in (5). As we’ll later see, when the commutative algebra is the Ran space...
or the configuration space, multiplicative spaces will coincide with the familiar notion of factorization spaces. Our definition of multiplicative space proceeds by adapting Raskin’s construction of multiplicative sheaves of categories to a geometric context.

Denote by

$$\text{Spc}_{/(-)} : \text{LaxPrStk}^{\text{op}} \rightarrow \text{Cat}$$

the functor taking a lax prestack $\mathcal{Y}$ to its category $\text{Spc}_{/\mathcal{Y}}$ of left fibrations over $\mathcal{Y}$. As in *Chiral categories*, we can construct the category $\text{Groth}(\text{Spc}_{/(-)})$ whose objects are pairs $(\mathcal{Y}, \Phi)$, where $\mathcal{Y}$ is a lax prestack and $\Phi : \mathcal{Y} \rightarrow \text{Spc}_{/(-)}$ is a left fibration over $\mathcal{Y}$. Morphisms $(\mathcal{Y}, \Phi) \rightarrow (\mathcal{X}, \Psi)$ are given by a correspondence

$$
\begin{array}{c}
\mathcal{Y} \\
\downarrow f \\
\mathcal{Z} \\
\downarrow g \\
\mathcal{X}
\end{array}
$$

with a map

$$f^{\text{Spc,*}}(\Phi) \rightarrow g^{\text{Spc,*}}(\Psi)$$

satisfying various homotopy coherent compatibilities.

Since $\text{Spc}_{/(-)}$ is lax symmetric monoidal with respect to the Cartesian monoidal structure on both sides, the category $\text{Groth}(\text{Spc}_{/(-)})$ inherits a symmetric monoidal structure $\otimes$, given pointwise on objects by

$$(\mathcal{Y}, \Phi) \otimes (\mathcal{X}, \Psi) = (\mathcal{Y} \times \mathcal{X}, \Phi \times \Psi).$$

There is a natural forgetful functor

$$\text{oblv}_{\text{Spc}} : \text{Groth}(\text{Spc}_{/(-)}) \rightarrow \text{Corr}(\text{LaxPrStk})$$

which has the structure of a symmetric monoidal functor.
Now let \( \mathcal{Y} \) be a commutative algebra in \( \text{Corr}(\text{LaxPrStk}) \). Define a \textit{weakly multiplicative space} over \( \mathcal{Y} \) to be a commutative algebra object of \( \text{Groth}(\text{Spc}/(\cdot)) \) mapping to \( \mathcal{Y} \) under \( \text{oblv}_{\text{Spc}} \) as a commutative algebra. Note that, among other features, a weakly multiplicative space on \( \mathcal{Y} \) is a left fibration \( \Phi \) over \( \mathcal{Y} \) together with a morphism

\[(\iota_{\mathcal{Y}})^*(\Phi \times \Phi) \to (\mu_{\mathcal{Y}})^*(\Phi) \]  
(2.5.0.1)

where \( \iota_{\mathcal{Y}} \) and \( \mu_{\mathcal{Y}} \) are the left and right legs, respectively, of the multiplication correspondence in (2.4.0.1). We also have analogous maps for all \( n \)-ary operations of \( \mathcal{Y} \).

**Definition 2.5.0.1.** Let \( \mathcal{Y} \) be a commutative algebra in correspondences of lax prestacks. Define a \textit{multiplicative space} over \( \mathcal{Y} \) to be a weakly multiplicative space such that all \( n \)-ary operations are isomorphisms. In particular, the map (2.5.0.1) is an isomorphism. Let \( \text{MultSpc}(\mathcal{Y}) \) denote the category of multiplicative spaces over \( \mathcal{Y} \).

Let us unwind this definition a bit more. By the Grothendieck construction, a left fibration \( \Phi : \mathcal{Y} \to \text{Spc}/ \) corresponds to a particular kind of lax prestack \( \Phi_\tau \) with a map

\[\Phi^{\text{Groth}} : \Phi_\tau \to \mathcal{Y},\]

where here the notation \( \tau \) stands for \textit{total space}, i.e. the total space of the fibration \( \Phi \). Then if \( \Phi \) has the structure of a multiplicative space over \( \mathcal{Y} \) we have isomorphisms

\[(\Phi_\tau \times \Phi_\tau) \times_{\mathcal{Y} \times \mathcal{Y}} \text{mult}_{\mathcal{Y}} \cong \Phi_\tau \times \text{mult}_{\mathcal{Y}} \]

satisfying various commutativity and associativity compatibilities.
2.6 Compatibility of straightening with algebras.

In this section, we will discuss when it is possible to push a multiplicative space forward along the canonical map $\mathcal{Y} \to \mathcal{Y}_{\text{str}}$ for a commutative algebra $\mathcal{Y}$ in $\text{Corr}(\text{LaxPrStk})$.

Note that the straightening functor

$$\text{str} : \text{LaxPrStk} \longrightarrow \text{PrStk}$$

preserves products but not fiber products in general. As such, it is not clear how to upgrade $\text{str}$ to a functor between categories of correspondences. To remedy this, we will restrict the allowable morphisms in $\text{LaxPrStk}$.

Recall that a functor $\mathcal{C} \to \mathcal{D}$ between categories $\mathcal{C}$ and $\mathcal{D}$ is called a realization fibration if for all functors $\mathcal{E} \to \mathcal{D}$ the diagram

$$\begin{array}{ccc}
(\mathcal{C} \times_{\mathcal{D}} \mathcal{E})_{\text{str}} & \longrightarrow & \mathcal{E}_{\text{str}} \\
\downarrow & & \downarrow \\
\mathcal{C}_{\text{str}} & \longrightarrow & \mathcal{D}_{\text{str}}
\end{array}$$

is Cartesian. We will call a map $\mathcal{X} \to \mathcal{Y}$ of lax prestacks a realization fibration if for all affine schemes $S$ the functor

$$\mathcal{X}(S) \longrightarrow \mathcal{Y}(S)$$

is a realization fibration of categories. Denote by $\text{LaxPrStk}_{\text{rf}}$ the subcategory of $\text{LaxPrStk}$ consisting of lax prestacks and realization fibrations as morphisms. Note that, by an application of the pasting lemma, realization fibrations are stable under base change and hence it makes sense to consider the category $\text{Corr}(\text{LaxPrStk}_{\text{rf}})$.

Since the functor $\text{str}$ preserves fiber products when restricted to $\text{LaxPrStk}_{\text{rf}}$, we
may upgrade \((-)_{str}\) to a functor

\[
str^{\text{enh}} : \text{CAlg} \left( \text{Corr}(\text{LaxPrStk}_{str}) \right) \longrightarrow \text{CAlg} \left( \text{Corr}(\text{PrStk}) \right),
\]

where for a category \(\mathcal{C}\) with all finite limits we define \(\text{CAlg}(\mathcal{C})\) to be the category of commutative algebras in \(\mathcal{C}\). In other words, given a commutative algebra \(\mathcal{Y}\) in correspondences of lax prestacks where each leg of the multiplication and unit correspondences is a realization fibration, the straightening \(\mathcal{Y}_{str}\) has a natural structure of a commutative algebra in correspondences of prestacks.

**Proposition 2.6.0.1.** Let \(\mathcal{Y}\) be a commutative algebra in correspondences such that all maps in the diagrams (2.4.0.1) are realization fibrations. Then there is a natural functor

\[
\text{LKE}_{\mathcal{Y} \to \mathcal{Y}_{str}} : \text{MultSpc}(\mathcal{Y}) \longrightarrow \text{MultSpc} \left( \text{str}^{\text{enh}}(\mathcal{Y}) \right)
\]

taking multiplicative spaces over \(\mathcal{Y}\) to multiplicative spaces over the commutative algebra \(\text{str}^{\text{enh}}(\mathcal{Y})\).

**Proof.** Let \(\Phi : \mathcal{Y} \to \text{Spc}_{/-}\) be a multiplicative space over \(\mathcal{Y}\), and denote also by \(\Phi_{\tau}\) the total space of the associated left fibration over \(\mathcal{Y}\). Define \(\text{LKE}_{\mathcal{Y} \to \mathcal{Y}_{str}}(\Phi)\) by left Kan extension as follows:

\[
\begin{array}{c}
\mathcal{Y} \\
\downarrow \\
\mathcal{Y}_{str}
\end{array} \xrightarrow{\Phi} \begin{array}{c}
\text{Spc}_{/-} \\
\downarrow \\
\text{LKE}_{\mathcal{Y} \to \mathcal{Y}_{str}}(\Phi)
\end{array}
\]

where by left Kan extension we mean the functor \(\mathcal{Y}_{str} \to \text{Spc}_/\) which on an affine scheme \(S\) is the left Kan extension of \(\mathcal{Y}(S) \to \text{Spc}_/S\) along \(\mathcal{Y}(S) \to \mathcal{Y}_{str}(S)\). Note that this gives a well-defined morphism of lax prestacks because \(\text{PrStk}\) is locally Cartesian closed.
We need to show that the resulting diagram

\[
\begin{array}{ccc}
\mathcal{Y}_{\text{str}} \times \mathcal{Y}_{\text{str}} & \xrightarrow{\text{mult}_{\mathcal{Y}_{\text{str}}}} & \mathcal{Y}_{\text{str}} \\
\downarrow & & \downarrow \\
\text{Spc}_{/(-)} \times \text{Spc}_{/(-)} & \xrightarrow{\text{LKE}_{\mathcal{Y} \to \mathcal{Y}_{\text{str}}}} & \text{Spc}_{/(-)}
\end{array}
\]

(2.6.0.1)

and all the analogous ones for higher operations commute (and similarly for the unit diagram, which we omit). By the universal property of Kan extensions, we know that (2.6.0.1) commutes up to a possibly non-invertible 2-morphism. We wish to show that this 2-morphism is an equivalence.

Let \((\Phi_{\tau})_{\text{str}} \to \mathcal{Y}_{\text{str}}\) denote the left fibration associated to \(\text{LKE}_{\mathcal{Y} \to \mathcal{Y}_{\text{str}}}\) via the Grothendieck construction. Equivalently, \((\Phi_{\tau})_{\text{str}}\) is the straightening of the lax prestack \(\Phi_{\tau}\). To show that (2.6.0.1) commutes, it suffices to show that the corresponding map

\[
((\Phi_{\tau})_{\text{str}} \times (\Phi_{\tau})_{\text{str}}) \times \mathcal{Y}_{\text{str}} \times \mathcal{Y}_{\text{str}} \xrightarrow{\text{mult}_{\mathcal{Y}_{\text{str}}}} (\Phi_{\tau})_{\text{str}} \times \mathcal{Y}_{\text{str}} \times \mathcal{Y}_{\text{str}} \xrightarrow{\text{mult}_{\mathcal{Y}_{\text{str}}}}
\]

is an equivalence, and similarly for the higher operations. But this follows from the definitions together with our assumption on \(\mathcal{Y}\). □
CHAPTER 3

BASES OF FACTORIZATION

In this chapter we will introduce the bases over which factorization will occur. Namely, we consider the moduli of finite subsets of our curve $X$, as well as a graded version in which we remember multiplicity. We will define what it means to have a factorization structure over both. Later, we will show how to pass from factorization structures over the configuration space to factorization structures over the Ran space. This will allow us to reconstruct sheaves on the affine Grassmannian from sheaves on Zastava spaces.

3.1 The Ran space.

Define a prestack $\text{Ran}$ as follows (see (21) for a more detailed overview). For an affine scheme $S$, a map $S \rightarrow \text{Ran}$ is a nonempty finite subset of $\text{Map}(S, X)$. Equivalently, we can write

$$
\text{Ran} \sim \text{colim}_{\text{FinSet}^{\text{op}}_{\text{surj}}} X^I
$$

where the colimit is over the category opposite to the one whose objects are (possibly empty) finite sets and whose morphisms are surjective functions. Here $X^I$ denotes the scheme of maps $x_I : I \rightarrow X$ from a finite set $I$ to $X$.

For a point $x$ of $X$, define $\Gamma_x$ to be its graph. Accordingly, for a point $x_I = (x_i)_{i \in I}$ of $\text{Ran}$, write $\Gamma_{x_I}$ for the union $\cup_{i \in I} \Gamma_{x_i}$ of graphs. Inside the product $\text{Ran} \times \text{Ran}$ we have the subprestack $[\text{Ran} \times \text{Ran}]_{\text{disj}}$ given by pairs $(x_I, x_J)$ such that $\Gamma_{x_I} \cap \Gamma_{x_J} = \emptyset$. 

Note that Ran is equipped with the structure of a monoid

\[ \text{add}_{\text{Ran}} : \text{Ran} \times \text{Ran} \to \text{Ran} \]

given by union of finite subsets. By abuse of notation, we will denote by \( \text{add}_{\text{Ran}} \) the restriction of the same named map to \( [\text{Ran} \times \text{Ran}]_{\text{disj}} \).

### 3.1.1 An important feature of Ran is that it has a structure of a commutative algebra in correspondences. Namely, there is a correspondence

\[
\begin{array}{ccc}
[\text{Ran} \times \text{Ran}]_{\text{disj}} & \xleftarrow{\text{Ran} \times \text{Ran}} & \text{Ran} \\
\text{Ran} \times \text{Ran} & \xrightarrow{\text{add}_{\text{Ran}}} & \text{Ran}
\end{array}
\]

whose second leg is given \( \text{add}_{\text{Ran}} \). The unit diagram for Ran is just given by the inclusion

\[
\{\emptyset\} \xhookrightarrow{} \text{Ran}
\]

of the empty subset of \( X \).

Define a factorization space over Ran to be a multiplicative space for the correspondence (3.1.1.1). A bit more explicitly, a factorization space is a prestack \( \mathcal{Y} \to \text{Ran} \) together with unital and associative isomorphisms

\[
\text{fact}_{\mathcal{Y}} : (\mathcal{Y} \times \mathcal{Y}) \times_{\text{Ran} \times \text{Ran}} [\text{Ran} \times \text{Ran}]_{\text{disj}} \xrightarrow{\sim} \mathcal{Y} \times_{\text{Ran}} [\text{Ran} \times \text{Ran}]_{\text{disj}}.
\]

We will often denote the left hand side by \( [\mathcal{Y} \times \mathcal{Y}]_{\text{disj}} \) and the right hand side by \( \mathcal{Y}_{\text{disj}} \).
3.2 The unital Ran space.

We will also need an enhancement of Ran as a lax prestack. Define

$$\text{Ran}^{\text{un}} : \text{AffSch}^\text{op}_k \longrightarrow \text{Cat}$$

to be the functor sending $S$ to the following category. Objects in $\text{Ran}^{\text{un}}(S)$ are given by finite subsets $x_I \subseteq \text{Map}(S, X)$ and for any such pair $x_I$ and $x_J$ there is a single morphism $x_I \rightarrow x_J$ precisely if $x_I \subseteq x_J$. Note we are not requiring that $x_I$ be nonempty. We call the lax prestack $\text{Ran}^{\text{un}}$ the unital Ran space.

As in the case of the usual Ran space, there is a correspondence

$$[\text{Ran}^{\text{un}} \times \text{Ran}^{\text{un}}]_{\text{disj}} \quad \text{Ran}^{\text{un}} \times \text{Ran}^{\text{un}} \quad \text{Ran}^{\text{un}}$$

which makes $\text{Ran}^{\text{un}}$ into a commutative algebra in correspondences of lax prestacks. Define a unital factorization space to be a multiplicative space over $\text{Ran}^{\text{un}}$.

More explicitly, we may think of a unital factorization space as a prestack $\mathcal{Y} \rightarrow \text{Ran}$ together with associative and unital maps $\mathcal{Y}_{x_I} \rightarrow \mathcal{Y}_{x_J}$ between fibers whenever we have an inclusion $x_I \subseteq x_J$. By abuse of notation, we will often call $\mathcal{Y}$ a unital factorization space when the associated structure of a left fibration is clear from context.

3.2.1 Let $\mathcal{Y}^{\text{un}} \rightarrow \text{Ran}^{\text{un}}$ be a unital factorization space, and let $\mathcal{Y}$ be the underlying ordinary factorization space. Since $\mathcal{Y}^{\text{un}}$ is a lax prestack, we may consider the category $\mathfrak{D}_{\text{str}}(\mathcal{Y}^{\text{un}})$ of unital $D$-modules on $\mathcal{Y}$. There is a natural forgetful functor $\text{obl}\mathcal{Y} :$
\( \mathcal{D}_{\text{str}}(\mathcal{Y}^{\text{un}}) \to \mathcal{D}(\mathcal{Y}) \) obtained by \(!\)-pull back along the composition

\[
\mathcal{Y} \to \mathcal{Y}^{\text{un}} \to \mathcal{Y}_{\text{str}}
\]

where the first map is the projection \( \mathcal{Y} = \mathcal{Y}^{\text{un}} \times_{\text{Ran}^{\text{un}}} \text{Ran} \to \mathcal{Y}^{\text{un}} \). The following proposition is standard.

**Proposition 3.2.1.1.** The forgetful functor

\[
\text{oblv}_{\mathcal{Y}} : \mathcal{D}_{\text{str}}(\mathcal{Y}^{\text{un}}) \to \mathcal{D}(\mathcal{Y})
\]

is fully faithful.

**Proof.** This follows directly from the homological contractibility of \( \text{Ran} \). \( \square \)

Hence we may call \( \mathcal{D}_{\text{str}}(\mathcal{Y}^{\text{un}}) \) the unital subcategory of \( \mathcal{D}(\mathcal{Y}) \), and often denote it by \( \mathcal{D}_{\text{untl}}(\mathcal{Y}) \).

### 3.3 An alternative description of the unital category.

A unital factorization space \( \mathcal{Y}^{\text{un}} \) is determined by an ordinary factorization space \( \mathcal{Y} \to \text{Ran} \) together with an action

\[
\text{act}_{\mathcal{Y}} : \text{Ran} \times \mathcal{Y} \to \mathcal{Y}
\]

which extends the natural map over the disjoint locus provided by factorization. In particular, we obtain a simplicial prestack

\[
\mathcal{Y}^* := \ldots \xrightarrow{\text{Ran} \times \mathcal{Y}} \xrightarrow{\text{pr}_Y} \mathcal{Y}
\]
whose associated geometric realization \(|\mathcal{Y}^*|\) is canonically isomorphic to \(\mathcal{Y}_{\text{str}}^{\text{un}}\). It follows that there is a canonical equivalence of categories

\[
\text{Tot}(\mathcal{D}(\mathcal{Y}^*)) \overset{\sim}{\longrightarrow} \mathcal{D}_{\text{str}}(\mathcal{Y}^{\text{un}})
\]

which is compatible with factorization. One should think of \(\text{Tot}(\mathcal{D}(\mathcal{Y}^*))\) as a category of Ran-equivariant sheaves on \(\mathcal{Y}\).

Note that by homological contractibility of Ran, the category \(\text{Tot}(\mathcal{D}(\mathcal{Y}^*))\) may be explicitly described as follows. It is the full subcategory of \(\mathcal{D}(\mathcal{Y})\) of sheaves \(\mathcal{F}\) such that there exists a (necessarily unique) isomorphism

\[
\text{pr}_Y^! \mathcal{F} \overset{\sim}{\longrightarrow} \text{act}_Y^! \mathcal{F},
\]

i.e. equivariance is a property rather than an extra structure.

### 3.4 The configuration space.

We will now discuss a “graded” analogue of Ran (see (22)). Let \(\text{Div}^{-\text{eff}}\) denote the moduli of anti-effective divisors on \(X\) and let \(\check{\Lambda}^+\) be the monoid of dominant weights of \(G\). Define the configuration space \(\text{Conf}\) as the space of homomorphisms \(\check{\Lambda}^+ \rightarrow \text{Div}^{-\text{eff}}\). Equivalently, \(\text{Conf}\) is the moduli of sums \(\sum_j \lambda_j x_j\) where \(\lambda_j \in \Lambda^-\) is a negative coweight for each \(j\) and the \(x_j\)'s are pairwise distinct. We will refer to the points of \(\text{Conf}\) as colored divisors on \(X\).

The prestack \(\text{Conf}\) is actually a scheme with \(\pi_0(\text{Conf}) = \Lambda^-\) where each connected component \(\text{Conf}^\lambda\) consists of colored divisors with total degree \(\lambda\). Explicitly, a choice of simple coroots determines an isomorphism

\[
\text{Conf}^\lambda \overset{\sim}{\longrightarrow} \prod_{\alpha_i \in \mathcal{I}_G} X^{(n_i)}
\]
where \( X^{(n_i)} = X^{n_i} / S_{n_i} \) is the GIT quotient of \( X^{n_i} \) by the symmetric group. We will often denote \( \text{Conf}^\lambda \) by \( X^\lambda \) and call it a *partially symmetrized power* of \( X \).

**3.4.1** Consider the incidence divisor in the product \( X \times \text{Conf} \). Given an \( S \)-point \( D : S \to \text{Conf} \), define \( \Gamma_D \) to be the relative effective divisor in \( X \times S \), equal to the pullback along

\[
\text{pr}_X \times D : X \times S \to X \times \text{Conf}
\]

of the incidence divisor. Here \( \text{pr}_X \) is the projection onto \( X \). As in the case of the Ran space, there is a subscheme \([\text{Conf} \times \text{Conf}]_{\text{disj}}\) of \( \text{Conf} \times \text{Conf} \) consisting of pairs \((D, E)\) such that \( \Gamma_D \cap \Gamma_E = \emptyset \). As before, there is a correspondence

\[
\begin{tikzcd}
[\text{Conf} \times \text{Conf}]_{\text{disj}} \ar[dr] \ar[dl] & \\
\text{Conf} \times \text{Conf} \ar[dr] \ar[dl] & \text{Conf}
\end{tikzcd}
\]

which makes \( \text{Conf} \) into a commutative algebra in correspondences. Here the second leg of the correspondence is the restriction of the map \( \text{add}_{\text{Conf}} : \text{Conf} \times \text{Conf} \to \text{Conf} \) given by addition of divisors. Hence we may define a *factorization space over* \( \text{Conf} \) to be a multiplicative space for \( \text{Conf} \).

**3.4.2** Parallel to what happens for the Ran space, the configuration space may be upgraded to a lax prestack

\[
\text{Conf}^- : \text{AffSch}_{\text{k}}^{\text{op}} \longrightarrow \text{Cat}
\]

where there is a unique arrow \( D \to E \) if and only if \( \Gamma_D \subseteq \Gamma_E \). In typical fashion, there is a natural structure of a commutative algebra in correspondences on \( \text{Conf}^- \). As before, we may consider left fibrations over \( \text{Conf}^- \) endowed with a multiplicative structure. To distinguish them from the Ran situation, we will call such objects
effective factorization spaces.

Given an effective factorization space $\mathcal{Y} \rightarrow \text{Conf}$ with underlying factorization space $\mathcal{Y} \rightarrow \text{Conf}$, we may consider the category $\mathcal{D}_{\text{eff}}(\mathcal{Y}) := \mathcal{D}(\mathcal{Y}_{\text{str}})$ of effective sheaves on $\mathcal{Y}$. There is also a description of this category in terms of equivariant sheaves with respect to a corresponding defect action $\text{Conf} \times \mathcal{Y} \rightarrow \mathcal{Y}$. Note however that since $\text{Conf}$ is not contractible, the forgetful functor $\mathcal{D}_{\text{eff}}(\mathcal{Y}) \rightarrow \mathcal{D}(\mathcal{Y})$ is no longer fully-faithful.
CHAPTER 4

THE RAN SEMI-INFINITE IC SHEAF

We will recall here the construction of the *semi-infinite intersection cohomology sheaf* of (15) and (16). Note the definitions of the affine Grassmannian and related structures that will be used in the following may be found in *loc. cit.* and the references therein.

4.0.1 Recall the *affine Grassmannian* $\text{Gr}_{G,\text{Ran}}$ is the factorization space over Ran whose $S$-points are given by triples $(x_I, P_G, \alpha)$, where $x_I$ is an $S$-point of Ran, $P_G$ is a $G$-bundle on $X \times S$, and

$$\alpha : P_G^0|_{X \times S \setminus \Gamma_{x_I}} \sim P_G|_{X \times S \setminus \Gamma_{x_I}} \quad (4.0.1.1)$$

is a trivialization of $P_G$ away from the complement of $\Gamma_{x_I}$. Here $P_G^0$ denotes the trivial bundle.

The factorization structure on $\text{Gr}_{G,\text{Ran}}$ is given by gluing $G$-bundles. Moreover, this upgrades to a unital factorization structure given as follows. Let $x_I \subseteq x_J$ and let $(x_I, P_G, \alpha)$ be a point of $\text{Gr}_{G,\text{Ran}}$ over $x_I$. By restricting $\alpha$ to the complement of $\Gamma_{x_J}$ we obtain a point over $x_J$, and hence the structure of a left fibration over $\text{Ran}^{\text{un}}$. 

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4.0.2 Suppose we have an $S$-point $(x_I, \mathcal{P}_G, \alpha)$ of $\text{Gr}_{G,\text{Ran}}$. Then for each dominant coweight $\hat{\lambda} \in \hat{\Lambda}^+$ the trivialization $\alpha$ induces meromorphic maps

$$O_{X \times S} \rightarrow V_{\mathcal{P}_G}^{\hat{\lambda}} \rightarrow V_{\mathcal{P}_G}^{\hat{\lambda}}$$

where $V^{\hat{\lambda}}$ is the irreducible representation of $G$ with highest weight $\hat{\lambda}$ and $V_{\mathcal{P}_G}^{\hat{\lambda}}$ is the associated vector bundle. Here, the first map in (4.0.2.1) is given by the inclusion of the highest weight line. Define a closed, factorizable sub-indscheme $\mathcal{S}_{\text{Ran}}^0$ of $\text{Gr}_{G,\text{Ran}}$ given by the condition that the compositions (4.0.2.1) extend to regular maps

$$O_{X \times S} \rightarrow V_{\mathcal{P}_G}^{\hat{\lambda}}$$

for all dominant coweights. The prestack $\mathcal{S}_{\text{Ran}}^0$ is naturally closed under the unital structure of $\text{Gr}_{G,\text{Ran}}$.

Let $\overline{\text{Bun}}_N$ denote Drinfeld’s compactification, whose definition will be recalled in detail in Section 5.1. There is a natural map

$$\overline{p}_{\text{Ran}} : \mathcal{S}_{\text{Ran}}^0 \rightarrow \overline{\text{Bun}}_N$$

taking a point of the left hand side to the (possibly degenerate) reduction to $N$ defined by the trivialization. More precisely, the maps (4.0.2.2) satisfy the Plücker relations and hence define a point of $\overline{\text{Bun}}_N$.

4.0.3 For an algebraic group $H$, let $\mathfrak{L}_{\text{Ran}}H$ denote the group prestack over $\text{Ran}$ whose $S$-points are given by an $S$-point $x_I$ of $\text{Ran}$ together with a map $\tilde{D}_{x_I} \rightarrow H$. By bounding the degrees of zeros of the maps (4.0.2.2) we obtain an $\mathfrak{L}_{\text{Ran}}N$-equivariant stratification of $\mathcal{S}_{\text{Ran}}^0$, with strata $S_{\text{Ran}}^{\hat{\lambda}}$ indexed by negative coweights $\lambda \in \Lambda^-$. The
labeled by 0 is given by the *semi-infinite orbit*, an open dense subspace of $\overline{S}^0_{\text{Ran}}$ obtained by the condition that the sections (4.0.2.2) have no zeros. Denote by $p^0_{\text{Ran}}$ the restriction of $p_{\text{Ran}}$ to $S^0_{\text{Ran}}$.

We then consider the category $\mathcal{S}_{\text{Ran}}$ of $\mathcal{L}_{\text{Ran}}N$-equivariant sheaves on the affine Grassmannian. By considering $\mathcal{L}_{\text{Ran}}N$-equivariant sheaves with support on $\overline{S}^0_{\text{Ran}}$ we obtain the category $\mathcal{S}^0_{\text{Ran}}$.

Similarly, there is a full subcategory $\mathcal{S}^\lambda_{\text{Ran}}$ of $\mathcal{D}(S^\lambda_{\text{Ran}})$ for each negative coweight $\lambda$. A feature of these categories is that they are preserved by (! and *) pullback and pushforward between the $S^\lambda_{\text{Ran}}$’s and $\overline{S}^0_{\text{Ran}}$. The following summarizes some of the central results of (16).

**Theorem 4.0.3.1. (Gaitsgory)**

1. For each negative coweight $\lambda$, there is a t-structure on $\mathcal{S}^\lambda_{\text{Ran}}$ characterized by the fact that (up to a cohomological shift) the dualizing sheaf $\omega_{S^\lambda_{\text{Ran}}}$ is in the heart $(\mathcal{S}^\lambda_{\text{Ran}})^\diamondsuit$. In particular, for $\lambda = 0$ the dualizing sheaf $\omega_{S^0_{\text{Ran}}}$ itself is an object of the heart $\mathcal{S}^0_{\text{Ran}}$.

2. The t-structures on the categories $\mathcal{S}^\lambda_{\text{Ran}}$ glue to one on $\mathcal{S}^0_{\text{Ran}}$ and the intermediate extension

$$\text{IC}^\infty_{\text{Ran}} := j^0_! \omega_{S^0_{\text{Ran}}} \in (\mathcal{S}^0_{\text{Ran}})^\diamondsuit$$

possesses a canonical structure of a factorization algebra. This factorization structure is uniquely determined by the requirement that it extends the tautological one on $\omega_{S^0_{\text{Ran}}}$.

**Remark 4.0.3.2.** It should be emphasized that the t-structure on $\mathcal{S}^0_{\text{Ran}}$ is not the perverse t-structure. Indeed, the latter doesn’t make sense with respect the stratification.
of $\overline{S}_0^{\lambda}$ since the strata $S^{\lambda}_{\text{Ran}}$ are both infinite dimensional and of infinite codimension.

The fact that the Gaitsgory t-structure behaves differently from the perverse t-structure is reflected, for example, in the fact that $\omega_{S^{\lambda}_{\text{Ran}}}$ lies in the heart of $\text{SI}_{\text{Ran}}^0$. This is because for an ind-scheme $\mathcal{Y}$ the dualizing sheaf $\omega_{\mathcal{Y}}$ typically lies in cohomological degree $-\infty$ in the perverse t-structure.

Following (16), we call the object $\text{IC}_{\text{Ran}}^{\infty}$ the semi-infinite intersection cohomology sheaf. We will not say much more about the t-structure for which $\text{IC}_{\text{Ran}}^{\infty}$ is an intermediate extension, but it is worth mentioning that the t-structure on $\text{SI}_{\text{Ran}}^\lambda$ is pulled back from the usual one on the scheme of $T$-fixed points of $S^{\lambda}_{\text{Ran}}$. In other words, the scheme of $T$-fixed points of $S^{\lambda}_{\text{Ran}}$ is isomorphic to the partially symmetrized power $X^\lambda$ of the curve and we have an equivalence

$$\text{SI}_{\text{Ran}}^{=\lambda} \sim \mathcal{D}(X^\lambda)$$

which is t-exact up to a cohomological shift (16).

The following is Theorem 3.3.3 in (16) and relates the semi-infinite intersection cohomology sheaf to its global version lying over Drinfeld’s compactification $\overline{\text{Bun}}_N$. The latter is simply given by the ordinary IC sheaf $\text{IC}_{\overline{\text{Bun}}_N}$ up to a cohomological shift.

**Proposition 4.0.3.3.** Let $d = (g-1) \dim(N)$. Then there is a canonical isomorphism

$$\overline{\mathcal{p}}_R^{!*}(\text{IC}_{\overline{\text{Bun}}_N})[d] \sim \text{IC}_{\text{Ran}}^{\infty}$$

extending the tautological isomorphism $(\overline{p}_R^0)^!(\omega_{\text{Bun}}) \simeq \omega_{S^0_{\text{Ran}}}$.
4.1 The configuration space semi-infinite IC sheaf.

We will also need a version of $\text{IC}^{\infty}_{\text{Ran}}$ which lies over the configuration space. To this end, define the configuration space affine Grassmannian $\text{Gr}_{G,\text{Conf}}$ to be the prestack whose $S$-points are given by triples $(D, \mathcal{P}_G, \alpha)$, where

1. $D$ is an $S$-family of colored divisors;
2. $\mathcal{P}_G \to X \times S$ is a $G$-bundle;
3. $\alpha$ is a trivialization of $\mathcal{P}_G$ on $(X \times S) \setminus \Gamma_D$.

As before, we define a closed substack $\overline{S}^0_{\text{Conf}}$ inside $\text{Gr}_{G,\text{Conf}}$ by a positivity condition. More precisely, the trivialization $\alpha$ induces meromorphic maps

$$\mathcal{O}_{X \times S} \longrightarrow V_{\mathcal{P}_G}^{\hat{\lambda}}$$

for every dominant weight $\hat{\lambda}$ and we require that these extend to $X \times S$. The space $\overline{S}^0_{\text{Conf}}$ is also equipped with a canonical map $\overline{\text{Pic}}_{\text{Conf}}$ to $\overline{\text{Bun}}_N$.

We may endow $\text{Gr}_{G,\text{Conf}}$ with the structure of an effective factorization space as follows. Let $D$ be a colored divisor and let $D'$ be another colored divisor such that $\Gamma_D \subseteq \Gamma_{D'}$. Suppose we have a point $(D, \mathcal{P}_G, \alpha)$ of $\text{Gr}_{G,\text{Conf}}$ lying over $D$. The effective structure is given by the map

$$(D, \mathcal{P}_G, \alpha) \mapsto (D', \mathcal{P}_G, \alpha'),$$

where $\alpha'$ is the restriction of $\alpha$ to $X \times S \setminus \Gamma_{D'}$. The factorization structure is again given by gluing $G$-bundles. It is evident that $\overline{S}^0_{\text{Conf}}$ is closed under both structures.

4.1.1 Define the correspondence configuration space $\text{Conf}_{\text{corr}}$ as the subspace of $\text{Conf} \times \text{Ran}$ given by the condition that the divisor is set-theoretically supported.
on the point of Ran.

We have a correspondence

\[ \text{Conf}_{\text{corr}} \]

\[ \text{Conf} \]

\[ \text{Ran} \]

\[ \epsilon_L \]

\[ \epsilon_R \]

defined in the obvious way. For a space \( Y \rightarrow \text{Conf} \) lying over the configuration space, define

\[ Y_{\text{corr}} := Y \times_{\text{Conf}} \text{Conf}_{\text{corr}} \]

as a space over \( \text{Conf}_{\text{corr}} \). Similarly, define

\[ Y_{\text{corr}, R} := Y \times_{\text{Ran}} \text{Conf}_{\text{corr}} \]

for a space \( Y \rightarrow \text{Ran} \).

Note that there is an action of Ran on \( Y_{\text{corr}} \), and hence we may talk about unital objects in \( \mathcal{D}(Y_{\text{corr}}) \). Precisely, given an \( S \)-point \( x_J \) of Ran and an \( S \)-point \( (y, D, x_I) \) of \( Y_{\text{corr}} \), then \( D \) is set-theoretically supported on \( x_J \cup x_I \) giving the formula

\[ (y, D, x_I) \mapsto (y, D, x_I \cup x_J) \]

for the action. Similarly, any unital space \( Y \rightarrow \text{Ran} \) also gives a unital space \( Y_{\text{corr}, R} \) in this sense.

We also have a correspondence analogous to (4.1.1.1) whose left leg is \( \text{Conf}^- \) and whose total space is denoted \( \text{Conf}^{-}_{\text{corr}} \). For a space \( Y \rightarrow \text{Conf}^- \), denote by \( Y^-_{\text{corr}} \) its pullback to \( \text{Conf}^{-}_{\text{corr}} \). Similarly, for a space \( Y \rightarrow \text{Ran} \) we have the space \( Y^-_{\text{corr}, R} \) over \( \text{Conf}^{-}_{\text{corr}} \).
4.1.2 Let us show that $\text{Conf}_{\text{corr}}$ is a commutative algebra in $\text{Corr}(\text{LaxPrStk})$. Define

$$[\text{Conf}_{\text{corr}} \times \text{Conf}_{\text{corr}}]_{\text{disj}} := (\text{Conf}_{\text{corr}} \times \text{Conf}_{\text{corr}}) \times \text{Ran} \times \text{Ran} [\text{Ran} \times \text{Ran}]_{\text{disj}}$$

and note that in addition to the obvious map

$$\text{inc}_{\text{corr}} : [\text{Conf}_{\text{corr}} \times \text{Conf}_{\text{corr}}]_{\text{disj}} \to \text{Conf}_{\text{corr}} \times \text{Conf}_{\text{corr}}$$

we have another map

$$\text{add}_{\text{corr}} : [\text{Conf}_{\text{corr}} \times \text{Conf}_{\text{corr}}]_{\text{disj}} \longrightarrow \text{Conf}_{\text{corr}}$$

which sends a pair $((D, x_I), (E, x_J))$ to $(D + E, x_I \cup x_J)$.

The inclusion of the pair

$$(0, \emptyset) \hookrightarrow \text{Conf}_{\text{corr}}$$

provides the necessary unit for $\text{Conf}_{\text{corr}}$, and it is easy to see that the above maps endow $\text{Conf}_{\text{corr}}$ with the structure of a commutative algebra in $\text{Corr}(\text{LaxPrStk})$. Note that the inclusion of $(\emptyset, \emptyset)$ into $\text{Conf}_{\text{corr}}$ is an inclusion of a connected component.

Now the map $c_L$ upgrades to a lax morphism of commutative algebras, while the second projection $c_R$ is a strict morphism of commutative algebras. It follows that for an effective factorization space $\mathcal{Y} \to \text{Conf}_{\text{corr}}$, the pullback $\mathcal{Y}_{\text{corr}}$ to $\text{Conf}_{\text{corr}}$ obtains a structure of a multiplicative space over $\text{Conf}_{\text{corr}}$. Similarly, if $\mathcal{Y}$ is a factorization space over $\text{Ran}$ then $\mathcal{Y}_{\text{corr}, R}$ obtains a structure of a multiplicative space over $\text{Conf}_{\text{corr}}$. 

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4.1.3 We claim there is a map \( \iota_{\text{corr}} : \mathcal{S}^0_{\text{Conf}, \text{corr}} \to \mathcal{S}^0_{\text{Ran}, \text{corr}, R} \) which makes the triangle commute. Indeed, \( \iota_{\text{corr}}((x_J, D, \mathcal{P}_G, \alpha)) \) is obtained by restricting \( \alpha \) to the complement of \( \Gamma_{x_J} \). In the following proposition, denote by \( t_{\text{Conf}} \) (resp. \( t_{\text{Ran}} \)) the projection \( \mathcal{S}^0_{\text{Conf}, \text{corr}} \to \mathcal{S}^0_{\text{Conf}} \) (resp. \( \mathcal{S}^0_{\text{Ran}, \text{corr}} \to \mathcal{S}^0_{\text{Ran}} \)).

**Proposition 4.1.3.1.** There exists an effective factorization algebra \( \text{IC}^\infty_{\text{Conf}} \) over \( \mathcal{S}^0_{\text{Conf}} \) together with an isomorphism

\[
\iota'_{\text{Conf}}(\text{IC}^\infty_{\text{Conf}}) \sim (t_{\text{Ran}} \circ \iota_{\text{corr}})'(\text{IC}^\infty_{\text{Ran}})
\]

which is compatible with factorization.

**Proof.** It is easy to see that the map \( \iota_{\text{corr}} : \mathcal{S}^0_{\text{Conf}, \text{corr}} \to \mathcal{S}^0_{\text{Ran}, \text{corr}, R} \) is compatible with unital structures on both sides. It follows that the shriek pullback \( \iota'_{\text{corr}} \) restricts to a functor

\[
\iota'_{\text{corr}} : \mathcal{D}_{\text{untl}}(\mathcal{S}^0_{\text{Ran}, \text{corr}, R}) \to \mathcal{D}_{\text{untl}}(\mathcal{S}^0_{\text{Conf}, \text{corr}}).
\]

As in Proposition 4.2.7 of (16), one shows that \( \mathcal{D}_{\text{untl}}(\mathcal{S}^0_{\text{Conf}, \text{corr}}) \) is equivalent to \( \mathcal{D}(\mathcal{S}^0_{\text{Conf}}) \) via \( \iota'_{\text{Conf}} \) using the universal homological contractibility of \( \text{Ran} \). Thus the desired sheaf \( \text{IC}^\infty_{\text{Conf}} \) will be produced as soon as we know that \( \text{IC}^\infty_{\text{Ran}} \) belongs to the unital subcategory of \( \mathcal{D} \)-modules on \( \mathcal{S}^0_{\text{Ran}} \). The latter is the content of Section 4 of (16).

To see that \( \text{IC}^\infty_{\text{Conf}} \) has an effective structure, note that we have constructed a
By Proposition 4.0.3.3, we have an isomorphism

$$\text{IC}_{\text{Ran}}^\infty \xrightarrow{\sim} \overline{\text{p}}_{\text{Ran}}(\text{IC}_{\text{Bun}_N}[d])$$

where $d$ is the dimension of $\text{Bun}_N$. We therefore deduce an isomorphism

$$\text{IC}_{\text{Conf}}^\infty \xrightarrow{\sim} \overline{\text{p}}_{\text{Conf}}(\text{IC}_{\text{Bun}_N}[d]).$$

Since the map $\overline{\text{p}}_{\text{Conf}}$ upgrades to a map

$$\overline{\text{S}}^0_{\text{Conf}} \longrightarrow \text{Bun}_N,$$

we can upgrade $\text{IC}_{\text{Conf}}^\infty$ to an effective sheaf on $\overline{\text{S}}^0_{\text{Conf}}$. 

□
CHAPTER 5

ZASTAVA SPACES AND THE SEMI-INFINITE IC SHEAF

In this section we will review the theory of Drinfeld’s compactifications and Zastava spaces. More information about both objects can be found in (4) and a review of Zastava spaces via the affine Grassmannian can be found in (14).

5.1 Drinfeld’s compactification.

Let $Bun_B$ denote the moduli of $B$-bundles on $X$, and consider the induction morphism $pr_{B,G}: Bun_B \rightarrow Bun_G$. Although the fibers of this map are not proper in general, there is a proper morphism

$$pr_{B,G}: \overline{Bun}_B \rightarrow Bun_G$$

together with an open embedding $j: Bun_B \rightarrow \overline{Bun}_B$ with dense image which commutes with the projection to $Bun_G$. The stack $\overline{Bun}_B$ is known as Drinfeld’s compactification, and is explicitly defined as follows. Let $S$ be an affine scheme. A map $S \rightarrow \overline{Bun}_B$ is the data of a $G$-bundle $\mathcal{P}_G$ and a $T$-bundle $\mathcal{P}_T$ on $X \times S$, together with embeddings of coherent sheaves

$$\lambda(\mathcal{P}_T) \rightarrow V^\lambda_{\mathcal{P}_G}$$
for every \( \check{\lambda} \in \check{\Lambda} \) which satisfy the Plücker relations. There is an evident map \( \overline{\text{Bun}}_B \rightarrow \text{Bun}_T \) whose preimage over \( \text{Bun}^\lambda_T \) will be denoted \( \overline{\text{Bun}}^\lambda_B \). Here \( \text{Bun}^\lambda_T \) is the connected component of \( \text{Bun}_T \) corresponding to \( \lambda \in \Lambda \) consisting of \( T \)-bundles of total degree \( \lambda \).

5.1.1 There is a stratification of \( \overline{\text{Bun}}_B \) by smooth substacks indexed by positive coweights \( \Lambda^+ \). Each stratum \( =_\lambda \overline{\text{Bun}}_B \) is canonically isomorphic to \( X^\lambda \times \text{Bun}^{-1}_B \), and we will write \( \iota_\lambda \) for the inclusion \( =_\lambda \overline{\text{Bun}}_B \rightarrow \overline{\text{Bun}}_B \). The union of strata indexed by coweights which are less than or equal to some fixed coweight \( \mu \) is an open substack of \( \overline{\text{Bun}}_B \) and will be denoted by \( \leq_\mu \overline{\text{Bun}}_B \).

The maps \( \iota_\lambda \) extend to finite morphisms

\[
\tau_\lambda : X^\lambda \times \overline{\text{Bun}}_B \longrightarrow \overline{\text{Bun}}_B
\]

which assemble into an action of the configuration space on \( \overline{\text{Bun}}_B \). Moreover, the diagram

\[
\begin{array}{ccc}
X^\lambda \times \overline{\text{Bun}}_B & \xrightarrow{\tau_\lambda} & \overline{\text{Bun}}_B \\
\downarrow & & \downarrow \\
a_{X^\lambda \times \overline{\text{Bun}}_B} & & \overline{\text{Bun}}_G \\
\overline{\text{Bun}}_G & \xleftarrow{\overline{\text{pr}}_{B,G}} & \overline{\text{pr}}_{B,G}
\end{array}
\]

commutes, where \( a_{X^\lambda} : X^\lambda \rightarrow \text{pt} \) is the unique map. In practice, we will more often use the corresponding structure for \( B^- \).

5.1.2 We will also need a variant \( \overline{\text{Bun}}_N \) of Drinfeld’s compactification for the unipotent radical \( N \) of \( B \). By definition, \( \overline{\text{Bun}}_N \) sits in a pullback square

\[
\begin{array}{ccc}
\overline{\text{Bun}}_N & \longrightarrow & \overline{\text{Bun}}_B \\
\downarrow & & \downarrow \\
\text{Spec}(k) & \longrightarrow & \text{Bun}_T
\end{array}
\]

\(^1\)Whenever convenient we will identify \( X^\lambda \times \text{Bun}_B \) and its image in \( \overline{\text{Bun}}_B \).
where Spec($k$) → Bun$_T$ is the inclusion of the trivial bundle. Since $T$ acts on $N$ by the adjoint action, it also naturally acts on the stack $\overline{\text{Bun}}_N$.

5.1.3 We will denote by $\mathcal{F}_{N,B^-}$ the fiber product

$$\overline{\text{Bun}}_N \times_{\text{Bun}_G} \overline{\text{Bun}}_{B^-}.$$ 

Define the compactified Zastava space $\overline{Z}$ to be the open locus of $\mathcal{F}_{N,B^-}$ where the two generalized reductions are generically transverse. Explicitly, an $S$-point of $\overline{Z}$ consists of a $G$-bundle $\mathcal{P}_G$ and a $T$-bundle $\mathcal{P}_T$ on $X \times S$ together with diagrams

$$\mathcal{O}_{X \times S} \xrightarrow{\kappa^\lambda} V_{\mathcal{P}_G} \xrightarrow{(\kappa^-)^{\lambda}} \lambda(\mathcal{P}_T) \quad (5.1.3.1)$$

for every dominant weight $\lambda$, where the collections $\{\kappa^\lambda\}$ and $\{(\kappa^-)^{\lambda}\}$ satisfy the Plücker equations and each composition $(\kappa^-)^{\lambda} \circ \kappa^\lambda$ is nonzero.

Denote by $\overline{\mathcal{Z}}$ (resp. $\overline{\mathcal{Z}}_N$) the projection $\overline{Z} \to \overline{\text{Bun}}_N$ (resp. $\overline{Z} \to \overline{\text{Bun}}_{B^-}$). Note that $\overline{Z}$ splits into connected components $\overline{Z}^\lambda$ indexed by negative coweights $\lambda$, where $\overline{Z}^\lambda$ is given as the preimage of $\overline{\text{Bun}}_{B^-}^\lambda$ under $\overline{q}$. By taking the zeros of the composition of (5.1.3.1) we obtain a map

$$\pi^\lambda : \overline{Z}^\lambda \to X^\lambda.$$ 

It is well-known that each $\overline{Z}^\lambda$ is a scheme of finite type, and that $\pi^\lambda$ is proper. Moreover, there is a natural factorization structure on $\overline{Z}$ with respect to $\pi$.

By commutativity of the diagram (5.1.1.2) for $B$ replaced by $B^-$, the maps $\tau^\lambda$ give rise to an action of Conf on the fiber product $\mathcal{F}_{N,B^-}$. Moreover, since the condition defining $\overline{Z}$ is generic, the compactified Zastava is preserved by this action. This equips $\overline{Z}$ with the structure of an effective factorization space.
5.1.4 By requiring the maps \((\kappa^-)\hat{\lambda}\) in (5.1.3.1) to be surjective, we obtain the affine Zastava \(\mathcal{Z}\). The space \(\mathcal{Z}\) admits a factorization structure and the dense open embedding

\[ j_\mathcal{Z} : \mathcal{Z} \hookrightarrow \overline{\mathcal{Z}} \]

is compatible with factorization on both sides. Note however that the the factorization structure on \(\mathcal{Z}\) is \textit{not} effective. Similarly, by requiring that the maps \(\kappa^\lambda\) are regular embeddings of vector bundles, we obtain the factorizable open subscheme

\[ j^-_\mathcal{Z} : \mathcal{Z}^- \hookrightarrow \overline{\mathcal{Z}} \]

which we call the \textit{opposite affine Zastava}. The intersection

\[ \hat{\mathcal{Z}} := \mathcal{Z} \cap \mathcal{Z}^- \]

is called the \textit{open Zastava} and is known to be smooth. Note that each variation of the Zastava space carries a \(T\)-action\(^2\).

5.2 Zastava spaces via the affine Grassmannian.

Suppose we have a point of the compactified Zastava space, consisting of two generically transverse generalized \(N\) and \(B^-\) reductions of a \(G\)-bundle \(\mathcal{P}_G\). Over the locus of transversality both reductions are genuine, and intersect in a single point over each fiber of the projection \(\mathcal{P}_G \to X \times S\). It follows that we may equip \(\mathcal{P}_G\) with a section away from the support of a colored divisor \(D\), and hence we obtain a point of \(\text{Gr}_{G,\text{Conf}}\).

\(^2\)In constructions of \(\mathcal{D}(\mathcal{F}t \Xi)\) one imposes \(T\)-equivariance on the categories of sheaves considered, c.f. (1), (11), (16). Although \(T\)-equivariance of, e.g. \(\text{IC}_\Xi\), is clear, it is not necessary to consider for our purposes, and further discussion of this point will be left for future work.
The result is a closed embedding

$$\overline{Z} \hookrightarrow \text{Gr}_{G,\text{Conf}}$$

which we will now describe more explicitly.

Define $\overline{S}_{\text{Conf}}$ to be the subspace of $\text{Gr}_{G,\text{Conf}}$ whose $S$-points consist of triples $(D, \mathcal{P}_G, \alpha)$ such that for each dominant weight $\check{\lambda} \in \check{\Lambda}^+$ the composition

$$V_{\mathcal{P}_G}^{\check{\lambda}} \overset{\alpha}{\longrightarrow} V_{\mathcal{P}_G}^{\check{\lambda}^0} \longrightarrow \mathcal{O}_{X \times S}$$

of meromorphic maps extends to a regular map

$$V_{\mathcal{P}_G}^{\check{\lambda}^0} \longrightarrow \mathcal{O}_{X \times S}(-\check{\lambda}(D)),$$

where the map $V_{\mathcal{P}_G}^{\check{\lambda}} \rightarrow \mathcal{O}_{X \times S}$ corresponds to the unique $N^-$-invariant functional on $V^{\check{\lambda}}$.

The image of the compactified Zastava space in $\text{Gr}_{G,\text{Conf}}$ may be described as the intersection

$$\overline{\mathcal{S}}_{\text{Conf}}^0 \times_{\text{Gr}_{G,\text{Conf}}} \overline{\mathcal{S}}_{\text{Conf}} \hookrightarrow \text{Gr}_{G,\text{Conf}}.$$

As a result of the definitions, we see that the embedding $\overline{Z} \hookrightarrow \text{Gr}_{G,\text{Conf}}$ is compatible with the effective factorization structures on both sides. We are now ready to define our central object of study.

**Definition 5.2.0.1.** Denote by $i_Z : \overline{Z} \hookrightarrow \overline{S}_{\text{Conf}}^0$ the inclusion. Define the *Zastava semi-infinite intersection cohomology sheaf* $\overline{\text{IC}}_{\overline{Z}}$ to be the factorization algebra $i_Z^1 \overline{\text{IC}}_{\text{Conf}}^\infty$ in $\mathcal{D}_{\text{eff}}(\overline{Z})$. Note that $\overline{\text{IC}}_{\overline{Z}}^\infty$ inherits a natural effective structure from that of $\overline{\text{IC}}_{\text{Conf}}^\infty$. 

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CHAPTER 6

THE EFFECTIVE CATEGORY
AND THE GENERIC ZASTAVA

For the remaining chapters we will give a more explicit description of $\text{IC}_{Z}^{\infty}$ and show that it is, in a precise sense, equivalent to $\text{IC}_{\text{Ran}}^{\infty}$. In this section we will establish the following theorem.

**Theorem 6.0.0.1.** There is an equivalence of categories

$$\mathcal{D}_{\text{eff}}(\mathcal{Z}) \xrightarrow{\sim} \mathcal{D}_{\text{untl}}(\mathcal{S}_{\text{Ran}}^0)$$

which is compatible with factorization and sends $\text{IC}_{Z}^{\infty}$ to $\text{IC}_{\text{Ran}}^{\infty}$ up to a cohomological shift.

In fact, we will actually show a slightly stronger statement from which Theorem 6.0.0.1 can be deduced as a corollary. We will begin with some preparations. The following result is an infinity-categorical analogue of Quillen’s theorem A which is a direct corollary of Theorem 4.1.3.1 of (18).

**Lemma 6.0.0.2.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor such that for every object $d \in \mathcal{D}$, the straightening $(\mathcal{C}_d)_{\text{str}}$ of the slice category $\mathcal{C}_d$ of $F$ over $d$ is contractible. Then the induced functor

$$F_{\text{str}} : \mathcal{C}_{\text{str}} \to \mathcal{D}_{\text{str}}$$

is an equivalence.
6.0.1 Let $\varphi : \mathcal{Y} \to \text{Conf}$ be an effective space over $\text{Conf}$ and let $\varphi^{-} : \mathcal{Y}^{-} \to \text{Conf}^{-}$ be the associated lax prestack. Recall the correspondence

$$
\text{Conf}_{\text{corr}} \leftrightarrow \text{Conf}^{-} \leftrightarrow \text{Ran}
$$

and define

$$
\mathcal{Y}^{-}_{\text{corr}} := \mathcal{Y}^{-} \times_{\text{Conf}^{-}} \text{Conf}_{\text{corr}}
$$

to be the lax prestack whose $S$-objects are triples $(y, x_I, D)$, where $y$ is an $S$-point of $\mathcal{Y}$, $D$ is the image of $y$ under $\varphi^{-} : \mathcal{Y}^{-} \to \text{Conf}^{-}$, and $x_I$ is a point of $\text{Ran}$ on which $D$ is set-theoretically supported.

Since the fibers of the map $\mathcal{e}_R : \text{Conf}_{\text{corr}} \to \text{Ran}$ are functors of directed sets, we see by Lemma 6.0.0.2 that the induced map

$$
(\text{Conf}_{\text{corr}}^{-})_{\text{str}} \longrightarrow \text{Ran}
$$

is an equivalence. As a result, the projection $\mathcal{Y}^{-}_{\text{corr}} \to \text{Ran}$ factors through the canonical map $\mathcal{Y}^{-}_{\text{corr}} \to (\mathcal{Y}^{-}_{\text{corr}})_{\text{str}}$. Let $\mathcal{Y}_{\text{Ran}}$ denote the prestack $(\mathcal{Y}^{-}_{\text{corr}})_{\text{str}}$ and write

$$
\varphi_{\text{Ran}} : \mathcal{Y}_{\text{Ran}} \longrightarrow \text{Ran}
$$

for the resulting morphism.

6.1 Factorization over $\text{Conf}$ to factorization over $\text{Ran}$.

In this section we will show that the assignment $\mathcal{Y} \leftrightarrow \mathcal{Y}_{\text{Ran}}$ sends factorization spaces to unital factorization spaces.
Proposition 6.1.0.1. Let $\mathcal{Y} \to \text{Conf}$ be an effective factorization space. Then the prestack $\mathcal{Y}_{\text{Ran}} \to \text{Ran}$ has a natural structure of a unital factorization space.

Proof. By Proposition 2.6.0.1, it suffices to show that the maps $\text{inc}_{\text{corr}}$ and $\text{add}_{\text{corr}}$ (as well as the corresponding maps for the unital diagram) are realization fibrations. By Theorem 2.1 of (23) it is enough to show that $\text{inc}_{\text{corr}}$ and $\text{add}_{\text{corr}}$ are both pointwise Cartesian and coCartesian fibrations. Note that while the right leg of the unital diagram is not a pointwise (co)Cartesian fibration, it’s still a realization fibration since it’s the inclusion of a disjoint base-point.

We will show that $\text{add}_{\text{corr}}$ is a pointwise coCartesian fibration, and the other cases will be left to the reader. Let

$$f : (D + E, x_I \cup x_J) \longrightarrow (D', x_I \cup x_J)$$

be a morphism in $\text{Conf}_{\text{corr}}^\to$ with $((D, x_I), (E, x_J))$ in $[\text{Conf}_{\text{corr}}^\to \times \text{Conf}_{\text{corr}}^\to]_{\text{disj}}$. Since $x_I$ and $x_J$ are disjoint, we can write

$$D' = D'(x_I) + D'(x_J),$$

where $D'(x_I)$ is set-theoretically supported on $x_I$ and $D'(x_J)$ is set-theoretically supported on $x_J$. Then it is easy to see that there exists

$$f' : ((D, x_I), (E, x_J)) \longrightarrow ((D'(x_I), x_I), (D'(x_J), x_J))$$

which is a coCartesian lift of $f$.

To see that $\mathcal{Y}_{\text{Ran}}$ is a unital factorization space, first recall the action of $\text{Ran}$ on $\mathcal{Y}_{\text{corr}}^\to$ constructed in Section 4.1.1.1. Clearly, this action is compatible with the morphisms in $\mathcal{Y}_{\text{corr}}^\to$ and hence descends to an action on $\mathcal{Y}_{\text{Ran}}$. Lastly, it is easy to see that this action is compatible with the factorization structure constructed above. □
Corollary 6.1.0.1.1. There is a natural equivalence of categories

\[ \mathcal{D}_{str}(\mathcal{Y}_{\text{corr}}) \sim \mathcal{D}(\mathcal{Y}_{\text{Ran}}) \]

which sends factorization algebras to factorization algebras.

Proof. The equivalence of categories follows by definition of \( \mathcal{Y}_{\text{Ran}} \), and the fact that it preserves factorization algebras follows from the construction of the factorization structure on \( \mathcal{Y}_{\text{Ran}} \) from Proposition 6.1.0.1. \( \square \)

The next proposition relates the effective category of \( \mathcal{Y}^- \) with the unital subcategory of \( \mathcal{D}(\mathcal{Y}_{\text{Ran}}) \). Note that the natural projection \( \text{pr}_{\mathcal{Y}} : \mathcal{Y}_{\text{corr}} \to \mathcal{Y} \) is a morphism of lax prestacks, and hence we obtain a functor

\[ \text{pr}^!_{\mathcal{Y}_{\text{str}}} : \mathcal{D}_{\text{eff}}(\mathcal{Y}) \to \mathcal{D}(\mathcal{Y}_{\text{Ran}}) \]

from the category of effective sheaves on \( \mathcal{Y} \) and the category of D-modules on \( \mathcal{Y}_{\text{Ran}} \) given by \(!\)-pullback along the induced map of straightenings.

Proposition 6.1.0.2. Let \( \varphi^- : \mathcal{Y}^- \to \text{Conf} \) be an effective factorization space and assume the underlying prestack \( \mathcal{Y} \) takes values in sets. Then the functor

\[ \text{pr}^!_{\mathcal{Y}_{\text{str}}} : \mathcal{D}_{\text{eff}}(\mathcal{Y}) \to \mathcal{D}(\mathcal{Y}_{\text{Ran}}) \]

is fully faithful and its essential image is the unital subcategory of \( \mathcal{D}(\mathcal{Y}_{\text{Ran}}) \).

Proof. By the proof of Proposition 6.1.0.1 we may endow \( \mathcal{Y}_{\text{corr}}^- \) with an action of Ran by functors. This gives us a modified lax prestack \( \mathcal{Y}_{\text{corr}}^- \) whose \( S \)-objects are given by \( \mathcal{Y}_{\text{corr}}^-(S) \) with a morphism

\[ (y, x_I) \to (y', x_I) \]
for every morphism \( y \to y' \) provided \( x_I \) is contained in \( x_J \). By definition of the unital structure on \( \mathcal{Y}_{\text{Ran}} \), the straightening of \( \mathcal{Y}_{\text{corr}} \) is tautologically isomorphic to \((\mathcal{Y}_{\text{Ran}})_{\text{str}}\).

We claim that the obvious map

\[
\mathcal{Y}_{\text{corr}} \to \mathcal{Y}
\]

induces an isomorphism on the level of straightenings. Let \( y_0 \) be an \( S \)-object of \( \mathcal{Y} \).

By Lemma 6.0.0.2, it suffices to show that the category

\[
\mathcal{Y}_{\text{corr}, y_0} := \mathcal{Y}_{\text{corr}}(S) \times_{\mathcal{Y}(S)} \mathcal{Y}_{y_0}/(S)
\]

is weakly contractible.

By our assumption on \( \mathcal{Y} \), the category \( \mathcal{Y}_{\text{corr}, y_0} \) is equivalent to the category whose objects are pairs \((y, x_J)\) as in the definition of \( \mathcal{Y}_{\text{corr}} \) and \( y_0 \) admits a (necessarily unique) map to \( y \). Let \((y, x_I)\) and \((y', x_J)\) be two such objects. Since \( \mathcal{Y} \to \text{Conf} \) is a pointwise coCartesian fibration, we may lift the morphism

\[
\varphi(y) \to \varphi(y) + \varphi(y')
\]

to a coCartesian morphism \( y \to y_1 \). We obtain an analogous morphism \( y' \to y'_1 \).

Let \( \lambda \) be the degree of \( \varphi(y) + \varphi(y') \). Then, as points of \( \mathcal{Y} \), \( y_1 \) and \( y'_1 \) lie in \( \mathcal{Y}^\lambda \) over \( \text{Conf}^\lambda \), and since \( y_1 \) and \( y'_1 \) admit a map from \( y_0 \), we conclude that there is a (necessarily unique) isomorphism \( y_1 \sim y'_1 \). Hence we obtain maps

\[
(y, x_I) \to (y_1, x_I \cup x_J) \to (y', x_J)
\]
and by our hypothesis on $\mathcal{Y}$, we conclude that $\mathcal{Y}_{\text{corr},/y_0}$ is a directed set and is therefore weakly contractible.

To conclude the proof, notice we have constructed a commutative diagram

$$
\begin{array}{ccc}
\mathcal{Y}_{\text{corr}} & \xrightarrow{\sim} & (\mathcal{Y}_{\text{Ran}})_{\text{str}} \\
\mathcal{Y}_{\text{str}} & \xrightarrow{} & \\
\end{array}
$$

where the bottom arrow is an isomorphism. It follows that $\text{pr}_{\mathcal{Y}_{\text{str}}}^1$ is fully faithful and has essential image given by $\mathcal{D}_{\text{untl}}(\mathcal{Y}_{\text{Ran}})$. 

6.1.1 We will now apply the machinery above to the affine Grassmannian. An application of Lemma 6.0.0.2 shows that $(\text{Gr}_{G,\text{Conf}})^{\text{Ran}}$ is canonically isomorphic to $\text{Gr}_{G,\text{Ran}}$. Moreover, the unital structure on $\text{Gr}_{G,\text{Ran}}$ is inherited from the one on $\text{Gr}_{G,\text{Conf,corr}}$ via Proposition 6.1.0.1. Likewise, we have a canonical isomorphism

$$(S^0_{\text{Conf}})^{\text{Ran}} \xrightarrow{\sim} S^0_{\text{Ran}}$$

inducing the unital structure on the right hand side.

The next proposition shows that we can actually obtain all of $\text{Gr}_{G,\text{Ran}}$ from $S^0_{\text{Conf}}$. Denote by $i_{\text{corr}}$ the inclusion $S_{\text{Conf,corr}} \hookrightarrow \text{Gr}_{G,\text{Conf,corr}}$.

**Lemma 6.1.1.1.** The inclusion

$$i_{\text{corr}} : S_{\text{Conf,corr}} \hookrightarrow \text{Gr}_{G,\text{Conf,corr}}$$

induces an equivalence

$$(S_{\text{Conf}})^{\text{Ran}} \xrightarrow{\sim} (\text{Gr}_{G,\text{Conf}})^{\text{Ran}}$$

and hence $(S_{\text{Conf}})^{\text{Ran}}$ is canonically equivalent to $\text{Gr}_{G,\text{Ran}}$.

**Proof.** Let $(D, x_1, \mathcal{P}_G, \alpha)$ be an $S$-point of $\text{Gr}_{G,\text{Conf,corr}}$. We will construct a point
$(D', x_I, \mathcal{P}_G, \alpha')$ of $\overline{S_{\text{Conf}}}$ together with a morphism $(D, x_I, \mathcal{P}_G, \alpha) \to (D', x_I, \mathcal{P}_G, \alpha')$. To this end, let $\omega_i$ be a fundamental weight of $G$, and consider the meromorphic map

$$\beta_i : V_{\mathcal{P}_G}^{\omega_i} \to \mathcal{O}_{X \times S}$$

induced by $\alpha$.

Since $\beta_i$ is defined away from the divisor $\Gamma_D$, there exists an integer $n_i$ large enough such that the composition

$$V_{\mathcal{P}_G}^{\omega_i} \rightarrow \mathcal{O}_{X \times S} \rightarrow \mathcal{O}_{X \times S}(n_i \Gamma_D)$$

is regular. Define the colored divisor

$$D' := D + \sum_{i \in I_G} (-n_i) \alpha_i \Gamma_D$$

and note that there is an obvious map $D \to D'$. Define $\alpha'$ to be the restriction of the map $\alpha$ to the complement of $\Gamma_D'$. Tautologically the tuple $(D', x_I, \mathcal{P}_G, \alpha')$ admits a map from $(D, x_I, \mathcal{P}_G, \alpha)$ and $(D', x_I, \mathcal{P}_G, \alpha')$ is a point of $\overline{S_{\text{Conf}}}$.

Now consider the composition of

$$\overline{S_{\text{Conf,corr}}} \to \overline{\text{Gr}_{G,\text{Conf,corr}}} \to (\overline{\text{Gr}_{G,\text{Conf}}})_{\text{Ran}},$$

where the first arrow is the inclusion and the second arrow is the natural map. By the discussion above, this map is essentially surjective for every affine scheme $S$, and the fact that it induces an equivalence

$$(\overline{S_{\text{Conf}}})_{\text{Ran}} \sim (\overline{\text{Gr}_{G,\text{Conf}}})_{\text{Ran}},$$

is a direct application of Lemma 6.0.0.2. Indeed, the fibers of the functor $\overline{S_{\text{Conf,corr}}}(S) \to$
Gr\(_{G,\text{Ran}}(S)\) are easily seen to be directed sets for every affine scheme \(S\).

We are now ready to prove the theorem.

Proof of Theorem 6.0.0.1. Recall we have a natural embedding \(\mathcal{Z} \to \text{Gr}_{G,\text{Conf}}\) whose essential image is the intersection \(\mathcal{S}_0 \cap \mathcal{S}_{\text{corr}}\). It is easy to see that \(\mathcal{S}_{\text{corr}} \to \text{Gr}_{G,\text{Ran}}\) is both a Cartesian and coCartesian fibration, and hence a realization fibration. By Lemma 6.1.1.1 the inclusion

\[
\mathcal{Z} \to \text{Gr}_{G,\text{Conf}}
\]

induces a canonical isomorphism

\[
\mathcal{Z}_{\text{Ran}} \sim \mathcal{S}_0^0_{\text{Ran}}
\]

and therefore a natural equivalence \(\mathcal{F} : \mathcal{D}_{\text{eff}}(\mathcal{Z}) \sim \mathcal{D}_{\text{untl}}(\mathcal{S}_0^0_{\text{Ran}})\). Moreover, under this equivalence \(\mathcal{Z}_{\text{Ran}}\) is identified with \(\mathcal{S}_0^0_{\text{Ran}}\).

Note that the projection \(\bar{\mathcal{p}} : \mathcal{Z} \to \overline{\text{Bun}}_N\) commutes with the defect action of \(\text{Conf}\) on \(\mathcal{Z}\) and the trivial action on \(\overline{\text{Bun}}_N\), and therefore upgrades to a morphism \(\bar{\mathcal{p}} : \mathcal{Z} \to \overline{\text{Bun}}_N\). Hence we have a commutative diagram

\[
\begin{array}{ccc}
\mathcal{Z}_{\text{corr}} & \longrightarrow & \mathcal{S}_0^0_{\text{Ran}} \\
\downarrow \bar{\mathcal{p}}_{\text{corr}} & & \downarrow \bar{\mathcal{p}}_{\text{Ran}} \\
\overline{\text{Bun}}_N & \longrightarrow & \mathcal{S}_0^0_{\text{Ran}}
\end{array}
\]

where \(\bar{\mathcal{p}}_{\text{corr}}\) is the composition of the projection \(\mathcal{Z}_{\text{corr}} \to \mathcal{Z}\) with \(\bar{\mathcal{p}}\). We conclude that \(\mathcal{F}(\mathcal{IC}_{\mathcal{Z}}^\infty)\) is canonically isomorphic to \(\mathcal{IC}_{\text{Ran}}^\infty\) up to a cohomological shift.

It remains to check that \(\mathcal{IC}_{\text{Ran}}^\infty\) is isomorphic to \(\mathcal{F}(\mathcal{IC}_{\mathcal{Z}}^\infty)\) as a factorization algebra. However, it is easy to check that the isomorphism \(\mathcal{F}(\mathcal{IC}_{\mathcal{Z}}^\infty) \sim \mathcal{IC}_{\text{Ran}}^\infty\) as sheaves extends the tautological isomorphism of factorization algebras over \(\mathcal{S}_0^0_{\text{Ran}}\). Since the
factorization structure on $\text{IC}_{\text{Ran}}^\infty$ is uniquely determined by its restriction to $S^0_{\text{Ran}}$, we’re done. \hfill \square

6.2 The generic Zastava.

The following sections on the generic Zastava will not be used in the rest of the text. Nevertheless, the results contained therein may be of independent interest as they connect the theory of Zastava spaces and the semi-infinite intersection cohomology sheaf to moduli spaces of rationally defined maps and principal bundles (see (2) as well as Appendix A of (16)).

Denote by $\text{Gr}^\text{gen}_G$ the straightening of the lax prestack $\text{Gr}^\infty_{G,\text{Ran}}$ associated to the canonical unital structure. We wish to give a more explicit description of $\text{Gr}^\text{gen}_G$. Let $S$ be an affine scheme, and recall that an open subscheme $U \subseteq X \times S$ is called a domain if it is universally dense with respect to the projection

$$p_S : X \times S \rightarrow S.$$ 

This means that for any closed point $s \in S$, the intersection of the fiber $X_s$ of the projection $p_S$ over $s$ with $U$ is nonempty.

It is well known that $\text{Gr}^\text{gen}_G$ is the moduli space whose $S$-points are triples $(\mathcal{P}_G, U, \alpha)$ where $\mathcal{P}_G \rightarrow X \times S$ is a $G$-bundle, $U \subseteq X \times S$ is a domain, and $\alpha$ is a trivialization of $\mathcal{P}_G$ defined over $U$. An isomorphism of such triples is an isomorphism of $G$-bundles which commutes with the trivializations over some subdomain of the intersection of their domains of definition.

6.2.1 Let us generalize the above construction. For $H \subseteq G$ a subgroup, let $\text{Bun}^\text{gen}_{G,H}$ denote the following moduli space (see Appendix A of (16) for more information).
For an affine scheme $S$, a morphism

$$S \to \text{Bun}^\text{gen}_{G,H}$$

is a triple $(\mathcal{P}_G, U, \kappa)$, where $\mathcal{P}_G \to X \times S$ is a $G$-bundle, $U \subseteq X \times S$ is a domain, and $\kappa$ is a reduction of $\mathcal{P}_G$ to $H$ over $U$. An isomorphism

$$(\mathcal{P}_G, U, \kappa) \sim \to (\mathcal{P}_G', U', \kappa')$$

is an isomorphism $\mathcal{P}_G \sim \to \mathcal{P}_G'$ over $X \times S$ which commutes with $\kappa$ and $\kappa'$ over a domain contained in $U \cap U'$. In this notation, note that we have a tautological isomorphism

$$\text{Gr}^\text{gen}_G \sim \to \text{Bun}^\text{gen}_{G,1},$$

where $1 \subseteq G$ denotes the trivial subgroup.

For any subgroup $H$ of $G$, there is a morphism

$$\text{Gr}^\text{gen}_G \longrightarrow \text{Bun}^\text{gen}_{G,H}$$

which takes a triple $(\mathcal{P}_G, U, \alpha)$ to $(\mathcal{P}_G, U, \alpha_H)$, where $\alpha_H$ is the isomorphism given by the composition

$$\mathcal{P}_G \sim \to \mathcal{P}_G^0 \simeq \text{ind}_1^G(\mathcal{P}_1^0) \sim \to \text{ind}_H^G(\mathcal{P}_H^0)$$

defined over $U$.

6.2.2 When $H = N$, there is also a morphism $\overline{\text{Bun}}_N \to \text{Bun}^\text{gen}_{G,N}$ obtained by restricting a generalized reduction $\kappa$ of a $G$-bundle $\mathcal{P}_G$ to its non-degeneracy domain, i.e. the maximal domain over which the Plücker maps are injective maps of vector
bundles. Now define a prestack $\mathcal{S}^{\text{gen}}$ by the condition that it sits in a Cartesian square

\[
\begin{array}{ccc}
\mathcal{S}^{\text{gen}} & \rightarrow & G_{\mathcal{S}^{\text{gen}}} \\
p^{\text{gen}} \downarrow & & \downarrow \\
\text{Bun}_N & \rightarrow & \text{Bun}_{G,N}^{\text{gen}}.
\end{array}
\]

Explicitly, an $S$-point of $\mathcal{S}^{\text{gen}}$ is given by a $G$-bundle $\mathcal{P}: \mathcal{G} \rightarrow X \times S$, a domain $U \subseteq X \times S$, and a trivialization $\alpha$ of $\mathcal{P}_G$ defined over $U$ such that the following condition holds. For all dominant weights $\lambda \in \hat{\Lambda}^+$, the meromorphic maps

\[
\mathcal{O}_{X \times S} \rightarrow V_{\mathcal{P}_G}^\lambda \xrightarrow{\alpha} V_{\mathcal{G}}^\lambda
\]

induced by $\alpha$ extend to regular maps defined over all of $X \times S$. As usual, the map $\mathcal{O}_{X \times S} \rightarrow V_{\mathcal{P}_G}^\lambda$ is the inclusion of the highest weight line.

**6.2.3** In what follows, denote by $\text{Map}^{\text{gen}}(X, N)$ the group prestack of generically defined maps $X \rightarrow N$ (see (2) for a careful definition). In other words, an $S$-point of $\text{Map}^{\text{gen}}(X, N)$ is a map $U \rightarrow N$, where $U \subseteq X \times S$ is a domain. Two such maps $U \rightarrow N$ and $U' \rightarrow N$ are equivalent if they agree on a subdomain of $U \cap U'$. It is known that $\text{Map}^{\text{gen}}(X, N)$ is homologically contractible. As the next proposition shows, the projection $p^{\text{gen}} : \mathcal{S}^{\text{gen}} \rightarrow \text{Bun}_N$ is very close to being an equivalence. Note the following result is closely related to arguments from Appendix A of (16).

**Proposition 6.2.3.1.** The map $p^{\text{gen}} : \mathcal{S}^{\text{gen}} \rightarrow \text{Bun}_N$ is a torsor for the group prestack $\text{Map}^{\text{gen}}(X, N)$. In particular, $p^{\text{gen}}$ is universally homologically contractible.

**Proof.** For a subgroup $H$ of $G$, define a prestack $\text{Bun}_H^{\text{gen}}$ by declaring a morphism
$S \to \operatorname{Bun}_H^{\text{gen}}$ to be a domain $U \subseteq X \times S$ as well as an $H$-bundle

$$\mathcal{P}_H \to U \subseteq X \times S.$$  

An isomorphism $(U, \mathcal{P}_H) \to (U', \mathcal{P}_H')$ of pairs as above is an isomorphism $\mathcal{P}_H \to \mathcal{P}_H'$ defined over a domain contained in $U \cap U'$. There is an obvious morphism $\operatorname{Bun}_H^{\text{gen}} \to \operatorname{Bun}_G^{\text{gen}}$. Moreover, it is easy to see that there is a canonical isomorphism

$$\psi_H : \operatorname{Bun}_{G,H}^{\text{gen}} \sim \to \operatorname{Bun}_H^{\text{gen}} \times_{\operatorname{Bun}_G^{\text{gen}}} \operatorname{Bun}_G.$$

Now since $N$ is unipotent, for any affine scheme $S$ the étale cohomology group $H^1_{\text{et}}(S, N)$ vanishes, and hence there is an isomorphism

$$\beta : \operatorname{Bun}_N^{\text{gen}} \sim \to B(\operatorname{Map}_{\text{gen}}(X, N))$$

which commutes with the obvious map from $\text{Spec}(k)$ to both sides. Therefore under the equivalences $\psi_1$, $\psi_N$, and $\beta$, the map $\operatorname{Gr}_G^{\text{gen}} \to \operatorname{Bun}_{G,N}^{\text{gen}}$ corresponds to the map

$$(u \times \text{id}_{\operatorname{Bun}_G}) : \text{Spec}(k) \times_{\operatorname{Bun}_G^{\text{gen}}} \operatorname{Bun}_G \to B(\operatorname{Map}_{\text{gen}}(X, N)) \times_{\operatorname{Bun}_G^{\text{gen}}} \operatorname{Bun}_G$$

obtained by base change from

$$u : \text{Spec}(k) \to B(\operatorname{Map}_{\text{gen}}(X, N)).$$

Since $p^{\text{gen}}$ is a further base change of $(u \times \text{id}_{\operatorname{Bun}_G})$, we conclude that it's a torsor for $\operatorname{Map}_{\text{gen}}(X, N)$.

The claim that $p^{\text{gen}}$ is universally homologically contractible follows from the following general observation. Let $\mathcal{Y}$ be a prestack and let $\pi : \mathcal{X} \to \mathcal{Y}$ be a torsor for a homologically contractible group prestack $\mathcal{H}$. Then $\pi$ is universally homologically
contractible (see Lemma 6.2.10 of (2)). □

6.3 Equivalence between $\mathcal{Z}^{\text{gen}}$ and $\mathcal{S}^{\text{gen}}$.

Define the generic Zastava $\mathcal{Z}^{\text{gen}}$ to be the straightening $\mathcal{Z}^{\text{str}}$. Denote by

$$t^\text{gen}_Z : \mathcal{Z} \rightarrow \mathcal{Z}^{\text{gen}}$$

the quotient map.

In this section we will identify $\mathcal{Z}^{\text{gen}}$ with $\mathcal{S}^{\text{gen}}$. First note that there is a canonical equivalence

$$(\text{Gr}^{\text{str}}_{G,\text{Conf}}) \rightarrow \text{Gr}^\text{gen}_G.$$

Indeed, this follows from the fact that for every domain $U \subseteq X \times S$ there exists a Zariski cover $S' \rightarrow S$ such that $U \times_{X \times S} (X \times S')$ is a divisor complement, as well as an application of Lemma 6.0.0.2.

6.4 The prestack $\mathcal{Z}^{\text{gen}}$ in detail.

In this section we will give an alternative description of $\mathcal{Z}^{\text{gen}}$ to justify the “Zastava” part of its name.

Recall the stack $\text{Bun}^{\text{gen}}_{G,B^{-}}$ with its map to $\text{Bun}_G$. Then we can form the fiber product

$$\text{Bun}_N \times_{\text{Bun}_G} \text{Bun}^{\text{gen}}_{G,B^{-}}$$

together with its open locus $\mathcal{Z}^{\text{gen}}_0$ consisting of a generalized $N$-reduction with a generically transverse generically defined $B^{-}$-reduction. Note this is a well-defined
condition. We claim there is an isomorphism

$$Z^\text{gen} \xrightarrow{\sim} Z_0^\text{gen}.$$  (6.4.0.1)

Indeed, $Z^\text{gen}$ is the quotient of $Z$ by the action of the configuration space. But Conf inherits its action from a similar action on $\overline{ \text{Bun}_B}$, and it is easy to see that the quotient of the latter is equivalent to $\text{Bun}^\text{gen}_{G,B}$.

From this perspective, we get another understanding of Proposition 6.2.3.1. The fiber of the map

$$\text{Bun}^\text{gen}_{G,B} \to \text{Bun}_G$$

over a $G$-bundle $\mathcal{P}_G \to X \times S$ is given by the prestack

$$\text{Gsect}_{X \times S}((G/B^-)_{\mathcal{P}_G}) \xrightarrow{\sim} \text{Bun}^\text{gen}_{G,B^-} \times_{\text{Bun}_G} S$$

of generically defined sections of the bundle $(G/B^-)_{\mathcal{P}_G}$ associated to $\mathcal{P}_G$ with fiber $G/B^-$ (see (2) for a definition).

In particular, the fiber over the trivial bundle is tautologically equivalent to the prestack $\text{Map}^\text{gen}(X \times S, G/B^-)$. Inside the latter we have the open subprestack consisting of generically defined maps $X \times S \to G/B^-$ which factor through the open cell $N \hookrightarrow G/B^-$. This is precisely the fiber of the map

$$p^\text{gen} : Z^\text{gen} \to \overline{\text{Bun}_N}$$

over the trivial reduction of $\mathcal{P}^\text{fl}_G$ exhibiting $p^\text{gen}$ as a torsor for $\text{Map}^\text{gen}(X,N)$.

Let us also identify a stratification on $Z^\text{gen}$. First recall that $\text{Bun}^\text{gen}_{G,B^-}$. By the equivalence (6.4.0.1) we see that $Z^\text{gen}$ has a (non-smooth) stratification whose strata
are isomorphic to the connected components of $\mathcal{Z}$. In particular, the composition

$$\mathcal{Z} \longrightarrow \mathcal{Z}^{\text{gen}}$$

of the open embedding $j_Z : \mathcal{Z} \hookrightarrow \mathcal{Z}$ with the quotient map $t^{\text{gen}}_Z : \mathcal{Z} \rightarrow \mathcal{Z}^{\text{gen}}$ is a bijection on $k$-points.
CHAPTER 7

KONTSEVICH ZASTAVA SPACES

In this section we will introduce a new version $\mathcal{Z}_K$ of the compactified Zastava space and establish the existence of a factorization structure on $\mathcal{Z}_K$. We will then show that its ordinary intersection cohomology sheaf is equivalent in a sense which we will make precise to the semi-infinite intersection cohomology sheaf.

7.0.1 Recall the resolution $\overline{\text{Bun}}_B^K \to \overline{\text{Bun}}_B$ of singularities of $\overline{\text{Bun}}_B$ over $\text{Bun}_G$ by stable maps introduced in (6). All results mentioned here about $\overline{\text{Bun}}_B^K$ can be found in loc. cit. An $S$-point of $\overline{\text{Bun}}_B^K$ is a commutative diagram

$$
\begin{array}{c}
C \longrightarrow \text{pt}/B \\
p \downarrow \quad \downarrow \\
X \times S \longrightarrow \mathcal{P}_G \longrightarrow \text{pt}/G
\end{array}
$$

(7.0.1.1)

where $C \to S$ is a flat family of projective nodal curves of arithmetic genus $g = \text{genus}(X)$, $p$ has degree 1, and the induced map $C \to (G/B)_p \mathcal{P}_G$ is stable over each geometric point of $S$. One shows that $\overline{\text{Bun}}_B^K$ is a smooth, locally of finite-type algebraic stack containing $\text{Bun}_B$ as a dense open substack.

The stack $\overline{\text{Bun}}_B^K$ is equipped with a proper morphism

$$r : \overline{\text{Bun}}_B^K \longrightarrow \overline{\text{Bun}}_B$$

which is an isomorphism when restricted to $\text{Bun}_B$. To define $r$, take an $S$-point of
$\overline{\text{Bun}}^K_B$ as in (7.0.1.1). Then the map $C \to \text{pt} / B$ induces nondegenerate Plücker maps

$$p^\lambda_K : \check{\lambda}(\mathcal{P}_T) \longrightarrow V_{p^*\mathcal{P}_G}$$

for every dominant weight $\check{\lambda} \in \check{\Lambda}^+$. Define a point of $\overline{\text{Bun}}_B$ where the Plücker map corresponding to $\check{\lambda}$ is given by

$$\text{det}(p^\lambda_K) : \text{det}(p_*(\check{\lambda}(\mathcal{P}_T))) \longrightarrow V^\lambda_{\mathcal{P}_G}$$

where $\text{det}$ denotes the determinant operation on perfect complexes. Clearly, over the locus where $p$ is an isomorphism, the maps $\text{det}(p^\lambda_K)$ are embeddings of vector bundles.

Let $\mathcal{F}^K_{N,B^-}$ denote the fiber product $\overline{\text{Bun}}_N \times_{\overline{\text{Bun}}_G} \overline{\text{Bun}}^K_B$. Define the Kontsevich compactified Zastava space $\overline{\mathcal{Z}}^\mathcal{K}$ by requiring that it sits in a Cartesian square

$$\begin{array}{ccc}
\overline{\mathcal{Z}}^\mathcal{K} & \longrightarrow & \mathcal{F}^K_{N,B^-} \\
\tau_Z \downarrow & & \downarrow \text{id}_{\overline{\text{Bun}}_N \times r} \\
\mathcal{Z} & \longleftarrow & \frac{\overline{\text{Bun}}_N \times r}{\overline{\text{Bun}}_G}
\end{array}$$

and the bottom arrow is the usual open embedding. There are the analogous versions $\mathcal{Z}^\mathcal{K}$ and $\mathcal{Z}^-\mathcal{K}$ together with open embeddings $j_K$ and $j^-\mathcal{K}$, respectively, into $\overline{\mathcal{Z}}^\mathcal{K}$. Note that $\tau_Z$ is proper and birational, and the open locus on which the map $\tau_Z$ is an isomorphism is given by the affine Zastava $\mathcal{Z} \simeq \mathcal{Z}^\mathcal{K}$.

**Remark 7.0.1.1.** The space $\mathcal{Z}^-\mathcal{K}$ appeared in (8) as a resolution for $\mathcal{Z}^-$ over the affine curve $\mathbb{A}^1$. However, to our knowledge the present paper is the first time the space $\mathcal{Z}^\mathcal{K}$ has been considered in the literature. Proposition 7.0.1.2 also appears to be new, even for $\mathcal{Z}^-\mathcal{K}$.

The next proposition establishes the factorization property of the Kontsevich Zastava spaces, and will be an important ingredient in the remainder of the paper.
Proposition 7.0.1.2. The Kontsevich Zastava space $Z_K$ has a factorization structure which commutes with that of $\overline{Z}$. The same holds for the open subspaces $Z_K$ and $Z_K^-$, and their open embeddings into $\overline{Z}_K$ are compatible with factorization.

Proof. Let $S$ be an affine scheme. For all $\lambda, \mu \in \Lambda^-$ we will only define a map

$$[\mathcal{Z}_K^\lambda \times \mathcal{Z}_K^\mu]_{\text{disj}} \longrightarrow \mathcal{Z}_K^{\lambda+\mu}_{\text{disj}}$$

since by the construction it will be clear how define its inverse. Consider an $S$-point $(z, z')$ of the left hand side. Then $z$ consists of a $G$-bundle $\mathcal{P}_G \rightarrow X \times S$, a generalized $N$-reduction $\{\kappa^\lambda\}_{\lambda \in \Lambda^+}$ of $\mathcal{P}_G$, and a point $z_K$ of $\overline{\text{Bun}}_B^K$ whose underlying generalized $B^-$-reduction of $\mathcal{P}_G$ is generically transverse to $\{\kappa^\lambda\}_{\lambda \in \Lambda^+}$. Write the point $z_K$ of $\overline{\text{Bun}}_B^K$ as a commutative diagram

$$
\begin{array}{ccc}
C & \longrightarrow & \text{pt} / B \\
\downarrow p & & \downarrow \\
X \times S & \longrightarrow & \text{pt} / G
\end{array}
$$

where $C$ is a flat, connected family of nodal curves over $S$, the map $p$ has degree 1, and $C \to (G / B)_{\mathcal{P}_G}$ is stable over every geometric point of $S$. Generic transversality identifies $z_K$ with the trivial diagram over a domain $U_z \subseteq X \times S$, complement to the $S$-family of colored divisors associated to $z$. For the point $z'$ we get a similar curve $p' : C' \to X \times S$ with locus $U_{z'}$ over which $p'$ is an isomorphism.

Let $C_{U_z}$ denote the fiber product $U_{z'} \times_{X \times S} C$. Note that by disjointness, the curve $C_{U_z}$ contains all the nodes of $C$. Likewise, define $C'_{U_z} = U_z \times_{X \times S} C'$ which contains all the nodes of $C'$. Write $C_{U_z \cap U_{z'}}$ (resp. $C'_{U_z \cap U_{z'}}$) for the fiber product $U_z \cap U_{z'} \times_{X \times S} C$ (resp. $U_z \cap U_{z'} \times_{X \times S} C'$). There is a unique isomorphism

$$C_{U_z \cap U_{z'}} \sim C'_{U_z \cap U_{z'}}$$
over $U_z \cap U_{z'}$ since the latter is contained in the locus over which $p$ and $p'$ are both isomorphisms.

By the assumption that the projections of $z$ and $z'$ to the configuration space have disjoint support we obtain a pushout diagram

$$
\begin{array}{ccc}
C_{U_z \cap U_{z'}} & \longrightarrow & C_{U_z'} \\
\downarrow & & \downarrow \\
C'_{U_z} & \longrightarrow & C_{z',z''}
\end{array}
$$

with a map $p_{z,z'} : C_{z,z'} \rightarrow X \times S$. Note that $C_{z,z'}$ is flat over $S$ by construction. Furthermore, since $C'_{U_z} \subseteq C'$ (resp. $C_{U_z'} \subseteq C$) intersects each fiber of $C' \rightarrow S$ (resp. $C \rightarrow S$) nontrivially and in the nonsingular locus, the scheme $C_{z,z'}$ is a proper family of nodal curves relative to $S$.

From the fact that $C \rightarrow \text{pt}/B$ and $C' \rightarrow \text{pt}/B$ factor through the canonical map $\text{pt} \rightarrow \text{pt}/B$ when restricted to $C_U$, we obtain a commutative diagram

$$
\begin{array}{ccc}
C_{z,z'} & \longrightarrow & \text{pt}/B \\
\downarrow^{p'} & & \downarrow \\
X \times S & \longrightarrow & \text{pt}/G
\end{array}
$$

with $C_{z,z'} \rightarrow (G/B)_{\mathfrak{g}_{G'}}$ stable over each geometric point $s \in S$. Indeed, the locus along which we glue $C_{U_z}$ and $C_{U_z'}$ does not contain any nodes. □

We will also record the following lemma for future use.

**Lemma 7.0.1.3.** The Kontsevich Zastava $\overline{Z}_K$ is a Deligne-Mumford stack. In particular, it has a schematic and proper diagonal. Moreover, $\overline{Z}_K$ is a union of connected components $\overline{Z}^\lambda_K$ of dimension $-\langle \lambda, 2\hat{\rho} \rangle$, where $\lambda \in \Lambda^-$.  

**Proof.** It is known that the fiber of the map $\overline{\text{Bun}}_B^K \rightarrow \text{Bun}_G$ over an affine scheme $S$ is a Deligne-Mumford stack. Hence the same is true of the fiber of the resolution $\overline{\text{Bun}}_B^K \rightarrow \overline{\text{Bun}}_{B^-}$, as well as of the map $\overline{\mathcal{F}}_{N, B^-}^K \rightarrow \overline{\mathcal{F}}_{N, B^-}$. Since $\overline{Z}$ is a scheme, we
see that $\mathcal{Z}_K$ is a Deligne-Mumford stack. Lastly, the dimension calculation for $\mathcal{Z}^\lambda_K$ follows from the corresponding well-known fact for $\mathcal{Z}^\lambda$.

**7.0.2** Taking defects of $k$-points of $\overline{\text{Bun}}_{B^-}^K$ gives us a stratification indexed by elements of $\Lambda^-$. Denote by $\leq_\mu \overline{\text{Bun}}_{B^-}^{K,\lambda}$ the open locus of $\overline{\text{Bun}}_{B^-}^{K,\lambda}$ which is a union of strata indexed by coweights greater than or equal to $\mu$. We therefore obtain open subsets $\leq_\mu \mathcal{Z}^\lambda_K$ and $\leq_\mu \mathcal{Z}^-_{K^-}$ of the Kontsevich Zastavas.

**Proposition 7.0.2.1.** The Kontsevich opposite affine Zastava $\mathcal{Z}^-_{K^-}$ is smooth, and therefore the map

$$r^-_Z : \mathcal{Z}^-_{K^-} \longrightarrow \mathcal{Z}^-$$

is a resolution of singularities. The locus over which $r^-_Z$ is an isomorphism is equivalent to the open Zastava $\hat{\mathcal{Z}}$.

**Proof.** First we claim that for $\lambda$ sufficiently dominant (see (6) for a precise bound), the open locus $\leq_\mu \mathcal{Z}^-_{K^-}^{\lambda}$ of $\mathcal{Z}^-_{K^-}^{\lambda}$ is smooth for all $\mu \in \Lambda^-$. Indeed, by the hypothesis on $\lambda$, the projection $\leq_\mu \overline{\text{Bun}}_{B^-}^{K,\lambda} \rightarrow \text{Bun}_G$ is smooth. By base-change, so is the map $\leq_\mu \mathcal{Z}^-_{K^-}^{\lambda} \rightarrow \text{Bun}_N$. But $\text{Bun}_N$ is smooth, and hence so is $\leq_\mu \mathcal{Z}^-_{K^-}^{\lambda}$.

Now let $\lambda$ be arbitrary, and let $\nu$ be large enough so that $\lambda + \nu$ is sufficiently dominant. Factorization provides an étale morphism

$$[\hat{\mathcal{Z}}^\nu \times \mathcal{Z}^-_{K^-}^{\lambda}]_{\text{disj}} \longrightarrow \mathcal{Z}^-_{K^-}^{\nu + \lambda}$$

and since the degeneracy of the $B^-$-reduction is bounded by $\lambda$, this map factors through $\leq_\lambda \mathcal{Z}^-_{K^-}^{\nu + \lambda}$. We can therefore conclude that the left side is smooth. To finish, note the projection

$$[\hat{\mathcal{Z}}^\nu \times \mathcal{Z}^-_{K^-}^{\lambda}]_{\text{disj}} \longrightarrow \mathcal{Z}^-_{K^-}^{\lambda}$$

is smooth and surjective, whence the result. \qed
Remark 7.0.2.2. An argument similar to the proof of Proposition 7.0.2.1 shows that $\mathcal{Z}_K^-$ is dense in $\overline{\mathcal{Z}}_K$. Indeed, the key point is that the preimage of a dense open substack under a smooth (and therefore open) map is dense.

7.0.3 We are ready to prove our main theorem. We will start by fixing some notation. In what follows, denote by

$$\overline{p}_K : \overline{\mathcal{Z}}_K \to \overline{\text{Bun}}_N$$

the projection obtained via base change from the map $\overline{\text{Bun}}^K_B \to \overline{\text{Bun}}_G$.

To make the exposition cleaner, for an algebraic stack $\mathcal{Y}$ with a map $\pi_{cY}$ to $\text{Conf}$ such that each preimage

$$\mathcal{Y}^\lambda := \pi_{cY}^{-1}(\text{Conf}^\lambda)$$

is connected and of finite type, let us denote by $IC^{\text{ren}}_\mathcal{Y}$ the renormalized intersection cohomology sheaf of $\mathcal{Y}$. On a component $\mathcal{Y}^\lambda$ the renormalized IC sheaf $IC^{\text{ren}}_{\mathcal{Y}^\lambda}$ is defined to be $IC_{\mathcal{Y}^\lambda}[-\langle \lambda, 2\hat{\rho} \rangle]$, where $IC_{\mathcal{Y}^\lambda}$ is the usual intersection cohomology sheaf. In particular, when $\mathcal{Y} = \overline{\mathcal{Z}}_K$ we have

$$IC^{\text{ren}}_{\overline{\mathcal{Z}}_K} = IC_{\overline{\mathcal{Z}}_K}[-\langle \lambda, 2\hat{\rho} \rangle],$$

where we recall that $-\langle \lambda, 2\hat{\rho} \rangle$ is a nonnegative integer, and positive whenever $\lambda \neq 0$. Note that $IC^{\text{ren}}_{\overline{\mathcal{Z}}_K}$ is naturally a factorization algebra.

Theorem 7.0.3.1. The pullback $r^*_Z(IC^{\infty}_Z)$ is canonically isomorphic to the renormalized intersection cohomology sheaf $IC^{\text{ren}}_{\overline{\mathcal{Z}}_K}$ as a factorization algebra.

Proof. Fix $\mu \in \Lambda^-$ and let $\lambda$ be sufficiently dominant for any $\nu \leq \mu$. Then by (6) the projection

$$\leq \mu \overline{\text{Bun}}^K_{B^-} \longrightarrow \text{Bun}_G$$

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is smooth, and hence so is the restriction

\[ \leq \underline{\mu} \overline{p}^\lambda_K : \leq \underline{\mu} \overline{Z}^\lambda_K \longrightarrow \text{Bun}_N \]

of \( \overline{p}^\lambda_K \) to \( \leq \underline{\mu} \overline{Z}^\lambda_K \) by base-change. We conclude that \( (\leq \underline{\mu} \overline{p}^\lambda_K)^!(\text{IC}_{\text{Bun}_N}) \) is the IC sheaf up to a cohomological shift. The statement of the theorem is therefore true on the latter space (note the cohomological shift is absorbed into the definition of \( \text{IC}_{\overline{Z}}^\infty \)).

For the general case, let \( \lambda \) be arbitrary and let \( \nu \) be big enough such that \( \lambda + \nu \) is sufficiently dominant for all \( \mu \leq \nu \). Factorization provides a commutative square

\[
\begin{array}{ccc}
[\check{Z}^\nu_K \times \overline{Z}^\lambda_K]_{\text{disj}} & \longrightarrow & \leq \lambda \overline{Z}^{\lambda + \nu}_{K, \text{disj}} \\
\downarrow & & \downarrow \\
[\check{Z}^\nu \times \overline{Z}^\lambda]_{\text{disj}} & \longrightarrow & \leq \lambda \overline{Z}^{\lambda + \nu}_{\text{disj}}
\end{array}
\]

where both horizontal arrows are open immersions (see also the proof of Proposition 7.0.2.1). Hence there is a natural isomorphism

\[ \alpha : \text{IC}^\text{ren}_{\check{Z}^\nu_K} \boxtimes_{\text{disj}} \text{IC}^\text{ren}_{\overline{Z}^\lambda_K} \sim \text{IC}^\text{ren}_{\check{Z}^\nu} \boxtimes_{\text{disj}} \nu^!_Z (\text{IC}_{\overline{Z}}^\infty) \quad (7.0.3.1) \]

where \( (\_) \boxtimes_{\text{disj}} (\_) \) denotes the external product restricted to the disjoint locus.

Since \( \check{Z}^\nu_K \) is smooth, the left hand side (resp. right hand side) of (7.0.3.1) is the \(!\)-pullback of \( \text{IC}^\text{ren}_{\overline{Z}^\lambda_K} \) (resp. \( \nu^!_Z (\text{IC}_{\overline{Z}}^\infty) \)) along the projection

\[ h : [\check{Z}^\nu_K \times \overline{Z}^\lambda_K]_{\text{disj}} \rightarrow \overline{Z}^\lambda_K \]

up to a cohomological shift. Since \( h \) is schematic and surjective, \( h^! \) is conservative and \( t \)-exact up to a cohomological shift. It follows that \( \nu^!_Z (\text{IC}_{\overline{Z}}^\infty) \) is perverse up to a cohomological shift\(^1\). Since the geometric fibers of \( h \) are connected, the functor \( h^! \)

\(^1\)For us "perverse" means a holonomic D-module concentrated in cohomological degree 0.
is fully faithful when restricted to perverse sheaves, and hence the isomorphism \( \alpha \) descends to an isomorphism \( \beta : \text{IC}^{\text{ren}}_{\mathcal{Z}_K} \xrightarrow{\sim} \mathfrak{r}_{\mathcal{Z}}'(\text{IC}^\infty_{\mathcal{Z}}) \) in \( \mathcal{D}(\mathcal{Z}_K) \).

It remains to check that \( \beta \) is an isomorphism of factorization algebras. A priori, we have two factorization structures on \( \text{IC}^{\text{ren}}_{\mathcal{Z}_K} \): one by definition and the other obtained via transport of structure from \( \beta \). However, the equivalence \( \beta \) extends the tautological isomorphism

\[
\omega_{\mathcal{Z}_K} = \text{IC}^{\text{ren}}_{\mathcal{Z}_K} \big|_{\mathcal{Z}_K} \xrightarrow{\sim} \mathfrak{r}_{\mathcal{Z}}'(\text{IC}^\infty_{\mathcal{Z}}) \big|_{\mathcal{Z}_K} = \omega_{\mathcal{Z}_K}
\]

of factorization algebras over \( \mathcal{Z}_K \) obtained by Proposition 7.0.2.1. The latter condition uniquely determines the factorization structure on \( \text{IC}^{\text{ren}}_{\mathcal{Z}_K} \) by Remark 7.0.2.2. \( \square \)

**Remark 7.0.3.2.** Note that in the course of the proof of Theorem 7.0.3.1 we have showed that \( \mathcal{Z}_K \) is locally in the smooth topology isomorphic to \( \overline{\text{Bun}}_N \). From this perspective, it is clear why the intersection cohomology of \( \mathcal{Z}_K \) should model the semi-infinite intersection cohomology sheaf.

### 7.1 Recovering the effective category.

The goal of this section is to show that the effective category \( \mathcal{D}_{\text{eff}}(\mathcal{Z}) \) (and consequently the unital category \( \mathcal{D}_{\text{unit}}(\mathcal{Z}_{\text{Ran}}) \)) can be recovered from \( \mathcal{D}(\mathcal{Z}_K) \) after imposing equivariance with respect to a certain groupoid. Under this equivalence, the renormalized intersection cohomology sheaf \( \text{IC}^{\text{ren}}_{\mathcal{Z}_K} \) will match up with \( \text{IC}^\infty_{\text{Ran}} \).

Recall that a prestack \( \mathcal{Y} \) is QCA if it is quasi-compact with affine stabilizers (7). Moreover, a stack is called safe if the neutral component of any stabilizer of a geometric point is a unipotent algebraic group (7). For example, any Deligne-Mumford stack is safe. We have the following general lemma.

**Lemma 7.1.0.1.** Let \( \varphi : \mathcal{X} \to \mathcal{Y} \) be a proper (not necessarily schematic) morphism of safe QCA algebraic stacks which is surjective on geometric points. Then \( \varphi^! \) is monadic, and in particular admits a left adjoint.
Proof. We wish to show that \( \varphi^! \) satisfies the hypotheses of the Barr-Beck-Lurie theorem. First we'll show that \( \varphi^! \) is conservative. Let \( c \in \mathcal{D}(\mathcal{Y}) \) be an arbitrary nonzero D-module on \( \mathcal{Y} \). By definition, we have an equivalence

\[
\mathcal{D}(\mathcal{Y}) \sim \lim_{S \in \text{AffSch}/\mathcal{Y}} \mathcal{D}(S)
\]

so we may choose an affine scheme \( S \) with a map \( f : S \to \mathcal{Y} \) such that \( f^!(c) \) is not zero.

The fiber product \( \mathcal{X} \times_{\mathcal{Y}} S \) is algebraic, so there exists a smooth cover

\[
g : T \to \mathcal{X} \times_{\mathcal{Y}} S
\]

by a scheme \( T \). Letting \( p_S : \mathcal{X} \times_{\mathcal{Y}} S \to S \) denote the projection, we see it suffices to show that \( (p_S \circ g)^! c \) is nonzero.

By assumption, \( p_S \circ g \) is surjective on geometric points. Then the right vertical arrow in the commutative diagram

\[
\begin{array}{ccc}
\mathcal{D}(S) & \overset{\text{obl}_{T}}{\longrightarrow} & \text{IndCoh}(T) \\
\downarrow_{(p_S \circ g)^!} & & \downarrow_{(p_S \circ g)^! \circ \text{IndCoh}} \\
\mathcal{D}(T) & \overset{\text{obl}_{S}}{\longrightarrow} & \text{IndCoh}(S)
\end{array}
\]

is conservative by Proposition 8.1.2 of (13). Since the forgetful functor from D-modules to indcoherent sheaves is conservative, we see that \( (p_S \circ g)^! c \) is conservative as well.

We'll now verify that \( \varphi^! \) admits a left adjoint. Since \( \mathcal{X} \) and \( \mathcal{Y} \) are QCA, their categories of D-modules are compactly generated. Moreover, since they're both safe we have equalities

\[
\mathcal{D}(\mathcal{X})^c = \mathcal{D}_{\text{coh}}(\mathcal{X}) \quad \text{and} \quad \mathcal{D}(\mathcal{Y})^c = \mathcal{D}_{\text{coh}}(\mathcal{Y})
\]
where for a stack \( \mathcal{Y}' \) we denote by \( \mathcal{D}_{\text{coh}}(\mathcal{Y}') \) the category of D-modules \( c \) such that for every scheme \( S \) with a smooth map \( f : S \to \mathcal{Y}' \) the pullback \( f^!(c) \) has coherent cohomologies.

By Proposition 10.6.2 of (7), the de Rham pushforward functor

\[
\varphi_* : \mathcal{D}(\mathcal{X}) \longrightarrow \mathcal{D}(\mathcal{Y})
\]

preserves the coherent subcategories, and hence preserves compact objects by the discussion above. Now the category of D-modules for any QCA stack is dualizable, and under Verdier duality we have \( \varphi^! \cong (\varphi_*')' \). Hence by (12) the functor \( \varphi^! \) admits a left adjoint \( \varphi_! \). Since \( \varphi^! \) is continuous, it is monadic by the Barr-Beck-Lurie theorem and the associated monad has underlying functor given by the composition \( \varphi_! \varphi_* : \mathcal{D}(\mathcal{X}) \to \mathcal{D}(\mathcal{X}) \).

Note that, unless \( \varphi \) is schematic, in general \( \varphi_! \) will differ from \( \varphi_* \). In fact, the difference is related to the pseudo-identity functor of Gaitsgory. Nevertheless, we can relax the schematicity assumption in the following case. A map \( \varphi : \mathcal{X} \to \mathcal{Y} \) is called Deligne-Mumford if for every affine scheme \( S \) with a map \( S \to \mathcal{Y} \), the fiber product \( \mathcal{X} \times_{\mathcal{Y}} S \) is a Deligne-Mumford stack. We have the following lemma.

**Lemma 7.1.0.2.** Let \( \varphi : \mathcal{X} \to \mathcal{Y} \) be a proper Deligne-Mumford morphism of safe QCA stacks. Then the left adjoint to \( \varphi^! \) is defined and coincides with the de Rham pushforward \( \varphi_* \).

**Proof.** Since the question is local on \( \mathcal{Y} \), we may assume that \( \mathcal{Y} \) is an affine scheme. The fact that the left adjoint \( \varphi_! \) to \( \varphi^! \) is defined follows from Lemma 7.1.0.1. Since the categories of D-modules on \( \mathcal{X} \) and \( \mathcal{Y} \) are dualizable and canonically self-dual, the functor \( \varphi_! \) is given by a kernel \( K \in \mathcal{D}(\mathcal{X} \times \mathcal{Y}) \).

By definition,

\[
K = (\text{id}_{\mathcal{D}(\mathcal{X})} \otimes \varphi_!)((\Delta_\mathcal{X})_* (\omega_\mathcal{X}))
\]
where $\text{id}_D(\mathcal{X}) \otimes \varphi!$ is the functor

$$D(\mathcal{X} \times \mathcal{X}) \simeq D(\mathcal{X}) \otimes D(\mathcal{X}) \longrightarrow D(\mathcal{X}) \otimes D(\mathcal{Y}) \simeq D(\mathcal{X} \times \mathcal{Y})$$

given by the tensor product of the identity with $\varphi!$. Here, $\Delta_X$ denotes the diagonal of $\mathcal{X}$.

Since

$$\text{id}_X \times \varphi : \mathcal{X} \times \mathcal{X} \longrightarrow \mathcal{X} \times \mathcal{Y}$$

is proper, $K$ also identifies with $(\text{id}_X \times \varphi)!((\Delta_X)_*(\omega))$. By assumption, $\Delta_X$ is schematic and proper, and hence $(\Delta_X)! = (\Delta_X)_*$.

We therefore obtain that $K$ is given by $(\gamma_\varphi)_!(\omega_X)$, where $\gamma_\varphi$ denotes the inclusion

$$\Gamma_\varphi \hookrightarrow \mathcal{X} \times \mathcal{Y}$$

of the graph of $\varphi$. But the latter is proper and schematic, and therefore $(\gamma_\varphi)_!$ is equivalent to $(\gamma_\varphi)_*$. Since $(\gamma_\varphi)_*(\omega_X)$ is the kernel of $\varphi_*$, we're done.

Denote by $r^\text{gen}_Z$ the composition

$$\mathcal{Z}_K \xrightarrow{r_Z} \mathcal{Z} \xrightarrow{r^\text{gen}_Z} \mathcal{Z}^\text{gen}$$

We are now ready to prove the following reconstruction theorem.

**Theorem 7.1.0.3.** There is a monad $\mathcal{M}_K$ acting on $D(\mathcal{Z}_K)$ such that

1. $\mathcal{M}_K$ sends factorization algebras to factorization algebras;

2. $\text{IC}^\text{gen}_{\mathcal{Z}_K}$ is a module for $\mathcal{M}_K$.

Moreover, there is an equivalence

$$(r^\text{gen}_Z)^!: D(\mathcal{Z}^\text{gen}) \xrightarrow{\sim} \mathcal{M}_K - \text{mod} (D(\mathcal{Z}_K))$$
such that \((r^\text{gen}_Z)_!^{\text{enh}}(\text{IC}^\infty_Z)\) is canonically isomorphic to \(\text{IC}^\text{gen}_{Z_K}\) as a factorization algebra.

**Proof.** We will begin with the construction of the monad and a proof of the stated equivalence. Recall that the projection \(\text{Bun}_{B^-} \to \text{Bun}^\text{gen}_{G,B^-}\) is proper and schematic, as well as fppf locally surjective. Since these properties are preserved under base change, the same holds for the map

\[
t^\text{gen}_Z : \mathcal{Z} \to \mathcal{Z}^\text{gen}.
\]

In particular, the de Rham pushforward functor \((r^\text{gen}_Z)_*\) is left adjoint to the conservative and continuous functor \((r^\text{gen}_Z)_!\).

Fix a coweight \(\lambda \in \Lambda^-\). Then \(\mathcal{Z}^\lambda\) is quasicompact, and hence \(\mathcal{Z}^\lambda_K\) is a safe QCA stack. Since \(r^\lambda_Z : \mathcal{Z}^\lambda_K \to \mathcal{Z}^\lambda\) is proper and surjective on geometric points for every \(\lambda\), by Lemma 7.1.0.1 the functor

\[
(r^\lambda_Z)_! : \mathcal{D}(\mathcal{Z}^\lambda) \to \mathcal{D}(\mathcal{Z}^\lambda_K)
\]

is monadic with left adjoint \((r^\lambda_Z)_!\).

By Lemma 7.0.1.3, the morphism \(r^\lambda_Z\) is Deligne-Mumford, and hence by Lemma 7.1.0.2 the left adjoint \((r^\lambda_Z)_!\) coincides with \((r^\lambda_Z)_*\). Since \(\mathcal{Z}_K\) is the disjoint union of the connected components \(\mathcal{Z}^\lambda_K\), by the previous paragraph the functor \((r^\text{gen}_Z)_!\) admits a left adjoint given by the de Rham pushforward functor \((r^\text{gen}_Z)_*\).

Let \(\mathcal{M}_K\) denote the monad

\[
(r^\text{gen}_Z)_! \circ (r^\text{gen}_Z)_* : \mathcal{D}(\mathcal{Z}_K) \to \mathcal{D}(\mathcal{Z}_K)
\]

acting on \(\mathcal{D}(\mathcal{Z}_K)\). Since \((r^\text{gen}_Z)_!\) is conservative, by Barr-Beck-Lurie we conclude that
there is an equivalence of categories

$$(r^\text{gen}_Z)^!_{\text{enh}} : \mathcal{D}(Z^\text{gen}) \simto \mathcal{M}_K - \text{mod}(\mathcal{D}(\overline{Z}_K))$$

whose composition with the forgetful functor

$$\text{oblv.}_{\#_K} : \mathcal{M}_K - \text{mod}(\mathcal{D}(\overline{Z}_K)) \longrightarrow \mathcal{D}(\overline{Z}_K)$$

canonically identifies with $(r^\text{gen}_Z)^!$.

Note that since $r_Z$ is a morphism of factorization spaces, the monad $\mathcal{M}_K$ preserves factorization algebras, proving point 1. To show point 2, note that by Theorem 6.0.0.1 the Zastava semi-infinite IC sheaf $\text{IC}^{\infty}_{Z}$ has an effective structure, and hence may be upgraded to an object of $\mathcal{D}(Z^\text{gen})$, which we denote by the same name. By Theorem 7.0.3.1 there is a canonical isomorphism

$$\text{IC}^\text{gen}_{Z_K} \simto (r^\text{gen}_Z)^!_{\text{enh}}(\text{IC}^{\infty}_{Z}),$$

from which we conclude that $\text{IC}^\text{gen}_{Z_K}$ has a natural structure of an $\mathcal{M}_K$-module and is canonically equivalent to $(r^\text{gen}_Z)^!_{\text{enh}}(\text{IC}^{\infty}_{Z})$. 

The monad $\mathcal{M}_K$ has the following concrete description. Consider the Cartesian diagram

$$\begin{array}{ccc}
Z_K \times_{Z^\text{gen}} Z_K & \xrightarrow{pr_2} & Z_K \\
pr_1 \downarrow & & \downarrow r^\text{gen}_Z \\
Z_K & \xrightarrow{r^\text{gen}_Z} & Z^\text{gen}
\end{array}$$

where $pr_i$ for $i \in \{1, 2\}$ denotes the projection onto the $i$th factor. Base change along this diagram identifies the underlying functor of the monad $\mathcal{M}_K$ with the composition of functors $(pr_2)_* \circ (pr_1)^!$. 

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BIBLIOGRAPHY


