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Product Einstein Manifolds, Zeta-Function Regularization and the Multiplicative Anomaly

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Abstract: The global additive and multiplicative properties of Laplace type operators acting on irreducible rank 1 symmetric spaces are considered. The explicit form of the zeta function on product spaces and of the multiplicative anomaly is derived.

1 Introduction

Among the most important geometric structures on manifolds almost-product structures play an essential role. Structures of this kind appear in a natural way from a variational principle based on a general class of Lagrangians depending on the Ricci square invariant constructed out of a metric and a symmetric connection. Geometric properties of (pseudo-) Riemannian almost-product structures have been studied in Refs.

In this class of structures an almost-product structure on Einstein manifold is associated with an Einstein metric (i.e. holds, where is the Ricci tensor and is a constant).

It is well known that gravitational Lagrangians which are non-linear in the scalar curvature of a metric give rise to equations with higher derivatives or to appearance of additional matter fields. An important example of a non-linear Lagrangian leading to equations with higher derivatives is given by Calabi’s variational principle. The non-linear gravitational Lagrangians which still generate Einstein equations are particularly important since they provide a general approach to governing topology in low dimension models and can be adapted in string theory.

In higher derivative field theories (for example, in higher derivative quantum gravity) as a consequence one has to deal with the product of two (or more) elliptic differential operators. In some cases an elliptic (pseudo-) differential operator has a well-defined zeta-regularized determinant. It is natural to investigate multiplicative properties of the determinants of differential operators, in particular the so-called multiplicative anomaly; for the definition of the anomaly see Sect. 3 below. The multiplicative anomaly can be expressed by means of the non-commutative residue associated with a classical pseudo-differential operator, the Wodzicki residue.
Recently the important role of this residue has been recognized in physics. The Wodzicki residue, which is the unique extension of the Dixmier trace to the wider class of pseudo-differential operators (PDOs) \[17, 18\], has been considered within the non-commutative geometrical approach to the standard model of the electroweak interactions \[19, 20, 21, 22, 23, 24\]. This residue is also used to write down the Yang-Mills action functional. Other recent papers along these lines can be found in Refs. \[25, 26, 27\]. The residue formulae have been used also for dealing with the singularity structure of zeta functions \[28\] and the commutator anomalies of current algebras \[29\].

The purpose of the present paper is to investigate the global additive and multiplicative properties of invertible elliptic operators of Laplace type acting on Einstein manifolds, especially those containing a summary of a meromorphic continuation of the zeta functions \[29\] and the corresponding multiplicative anomaly (for the spaces mentioned above) are studied. The explicit calculation is given for essentially all rank 1 symmetric spaces. We end with some conclusions in Sect. 4. Finally the Appendices A and B contain a summary of a meromorphic continuation of the zeta functions \(\zeta(s)\) and the commutator anomalies of current algebras \[29\].

2 Determinant Regularization and Product of Einstein Manifolds

In this section we consider the problem of the global existence of zeta functions on (pseudo-) Riemannian product manifolds, a product of two Einstein manifolds \(\mathbb{M}, \mathbb{N}\)

\[(\mathbb{M}, \mathbb{N}, \mathcal{P}) = (\mathbb{M}_1, \mathbb{G}_1, \mathcal{P}_1) \otimes (\mathbb{M}_2, \mathbb{G}_2, \mathcal{P}_2),\]

where \(\mathbb{G} = \mathbb{G}_1 \otimes \mathbb{G}_2\) and the metric \(\mathbb{G}\) separates the variables, i.e.

\[ds^2 = \mathbb{G}_{\alpha\beta}(x)dx^\alpha \otimes dx^\beta + \mathbb{G}_{\mu\nu}(y)dy^\mu \otimes dy^\nu.\]

The tangent bundle splits as \(TM = TM_1 \oplus TM_2\) and \(\mathcal{P} = \mathcal{P}_1 + \mathcal{P}_2\), where \(\mathcal{P}_p\) \((p = 1, 2)\) are the corresponding projections on \(TM_p\),

\[\text{Ric}(\mathbb{G}) = \gamma \mathbb{G},\]

\[\mathcal{P}^2 = \text{Id}, \quad \mathbb{G}(\mathcal{P}\mathcal{X}, \mathcal{P}\mathcal{Y}) = \mathbb{G}(\mathcal{X}, \mathcal{Y}), \quad \mathcal{X}, \mathcal{Y} \in \chi(M),\]

and \(\chi(M)\) being the Lie algebra of vector fields \(\mathcal{X}\) and \(\mathcal{Y}\) on \(M\). The trivial examples of an almost-product structure are given by the choices \(\mathcal{P} = \pm \text{Id} (\pm \text{identity}).\)

We recall some facts about Einstein manifolds necessary for the next considerations. We start with an almost-product (pseudo-) Riemannian structure \((\mathbb{G}, \mathcal{P})\) which is integrable iff \(\triangle_\mathbb{G}\mathcal{P} = 0\) for the Levi-Civita connection \(\triangle_\mathbb{G}\) of \(\mathbb{G}\). The two integrable complementary subbundles, i.e. both foliations, are totally geodesic \[1, 30\]. Let \(M\) be a pseudo-Kähler manifold; the metric \(\mathbb{G}\) splits as \(\mathbb{G} = \mathbb{G}_1 \otimes \mathbb{G}_2\). Such a manifold is an Einstein manifold iff in any adapted coordinates \((x^\alpha, y^\beta)\) both metrics \(\mathbb{G}_1\) and \(\mathbb{G}_2\) are Einstein metrics for the same constant \(\gamma\) \[1, 30, 31\], i.e.

\[R_{\alpha\beta\gamma\delta} = \gamma g_{\alpha\beta}g_{\gamma\delta}, \quad (p = 1, 2).\]

Our consideration will be restricted to only locally decomposable manifolds. A wide class of (pseudo-) Riemannian manifolds includes non-locally decomposable manifolds as well, which are given by warped product space-times \[32, 33, 34\]. Note that many exact solutions of Einstein equations (associated with Schwarzschild, Robertson-Walker, Reissner- Nordström, de Sitter space-times) and \(p\)-brane solutions \[35, 5\] are, in fact, warped product space-times \[31, 34\].
2.1 The Spectral Zeta Function

We shall be working with irreducible rank 1 symmetric spaces $\mathcal{M}_p \equiv X_p = G_p/K_p$ of non-compact type. Thus $G_p$ will be connected non-compact simple split rank 1 Lie groups with finite center and $K_p \subset G_p$ will be maximal compact subgroups. Let $\Gamma_p \subset G_p$ be discrete, co-compact, torsion free subgroups.

Let $L_p : C^\infty(V(M_p)) \to C^\infty(V(M_p))$ be partial differential operators acting on smooth sections of vector bundles $V(M_p)$. Let $\chi_p$ be a finite-dimensional unitary representation of $\Gamma_p$, let $\{\lambda_i(p)\}_{i=0}^\infty$ be the set of eigenvalues of the second-order operator of Laplace type $L_p = -\Delta_{\Gamma_p}$ acting on smooth sections of the vector bundle over $\Gamma_p \backslash X_p$ induced by $\chi_p$, and let $n_l(\chi_p)$ denote the multiplicity of $\lambda_l(p)$.

We shall need further a suitable regularization of the determinant of an elliptic differential operator, since the naive definition of the product of eigenvalues gives rise to a badly divergent quantity. We shall make the choice of zeta-function regularization (see (3.2)). The zeta function associated with the operators $\mathcal{L}_p \equiv L_p + b_p$ have the form

$$\zeta(s|\mathcal{L}_p) = \sum_l n_l(\chi_p)\{\lambda_l(p) + b_p\}^{-s},$$

(2.6)

here $b_p$ are arbitrary constants (endomorphisms of the vector bundle $V(M_p)$), called in the physical literature the potential terms. $\zeta(s|\mathcal{L}_p)$ is a well-defined analytic function for $\Re s > \dim(M_p)/2$, and can be analytically continued to a meromorphic function on the complex plane $\mathbb{C}$, regular at $s = 0$. One can define the heat kernel of the elliptic operator $\mathcal{L}_p$ by

$$\omega_{T_p}(t) \equiv \text{Tr} \left( e^{-t\mathcal{L}_p} \right) = \frac{-1}{2\pi i} \text{Tr} \int_{\mathcal{C}_0} e^{-z(t - \mathcal{L}_p)^{-1}} dz,$$

(2.7)

where $\mathcal{C}_0$ is an arc in the complex plane $\mathbb{C}$. By standard results in operator theory there exist $\epsilon, \delta > 0$ such that for $0 < t < \delta$ the heat kernel expansion holds

$$\omega_{T_p}(t) = \sum_{l=0}^{\infty} n_l(\chi_p) e^{-(\lambda_0(p) + b_p)t} = \sum_{0 \leq l \leq l_0} a_l(\mathcal{L}_p)t^{-l} + O(t^\epsilon).$$

(2.8)

Eventually we would like also to take $b_p = 0$, but for now we consider only non-zero modes: $b_p + \lambda_l(p) > 0$, $\forall l : \lambda_0(p) = 0$, $b_p > 0$.

2.2 The Explicit Form of the Zeta Function

The following representations of $X_p$ up to local isomorphism can be chosen

$$X_p = \begin{bmatrix} SO_1(n, 1)/SO(n) & (I) \\ SU(n, 1)/U(n) & (II) \\ SP(n, 1)/(SP(n) \otimes SP(1)) & (III) \\ F_{4(-20)}/Spin(9) & (IV) \end{bmatrix},$$

(2.9)

where $n \geq 2$, and $F_{4(-20)}$ is the unique real form of $F_4$ (with Dynkin diagram $\circ - \circ = \circ - \circ$) for which the character $(\dim X - \dim K)$ assumes the value $(-20)$. We assume that if $G_1$ or $G_2 = SO(m, 1)$ or $SU(q, 1)$ then $m$ is even and $q$ is odd.

We study the zeta function

$$\zeta(s) \bigoplus_{p} \mathcal{L}_p = \zeta_{\Gamma_1 \backslash X_1 \otimes \Gamma_2 \backslash X_2}(s; b_1 b_2) = \frac{1}{\Gamma(s)} \int_0^\infty \prod_p \omega_{T_p}(t) t^{s-1} dt, \quad \Re s > \frac{d_1 + d_2}{2},$$

(2.10)
where \( d_p = \dim X_p \). The suitable Harish-Chandra-Plancherel measure is given as follows:

\[
|C_p(r)|^{-2} = C_{G_p} \pi r P_p(r) \tanh (a(G_p)r) = C_{G_p} \pi \sum_{l=0}^{d_p-1} a_{2l}(p)r^{2l+1} \tanh (a(G_p)r), \tag{2.11}
\]

where

\[
a(G_p) = \begin{cases} 
\pi & \text{for } G_p = SO(1, 2n, 1) \\
\pi/2 & \text{for } G_p = SU(q, 1), \quad q \text{ odd} \\
& \text{or } G_p = SP(m, 1), \quad F_4(-20) 
\end{cases}
\tag{2.12}
\]

while \( C_{G_p} \) is some constant depending on \( G \), and where the \( P_p(r) \) are even polynomials (with suitable coefficients \( a_{2l}(p) \)) of degree \( d - 2 \) for \( G \neq SO(2n + 1, 1) \), and of degree \( d - 1 = 2n \) for \( G = SO(2n + 1, 1) \) \([37, 38]\).

The explicit construction during the proof of Eq. (A.14) (of the Appendix A) gives a little more, namely

**Theorem 1** The function \( \zeta(s \| \bigoplus \mathcal{L}_p) \) admits an explicit meromorphic continuation to \( \mathbb{C} \) with at most a simple pole at \( s = 1, 2, \ldots, \frac{d_1 + d_2}{2} \). In particular on the domain \( \Re s < 1 \),

\[
\zeta(s \| \bigoplus \mathcal{L}_p) = \frac{\pi^2}{2} \prod_k V_k a(G_k) C_{G_k} \sum_{j=0}^{d_1-1} \sum_{l=0}^{d_2-1} \sum_{\mu=0}^{\mu} \sum_{\nu=0}^{\nu} a_{2j}(1) a_{2\mu}(2)(j-l)! (\mu-\nu)! \\
\times \int_0^{\infty} r^{2(j-l)} \frac{\sinh^2(a(G_1)r)K_{\mu-\nu}(s-l-\nu-1; r^2 + B, a(G_1)) + \text{d.r.}}{(s-1)(s-2)(s-(l+1))(s-(l+2))(s-(l+1+\nu+1))} \\
+ C_{G_1} V_1 \sin (\pi s) \sum_{j=0}^{d_2-1} a_{2j}(2) \int_{\mathbb{R}} dr r^{2j+1} \tanh(a(G_1)r) \\
\times \left[ \int_0^{\infty} d\rho \psi_2 \left( \rho_0(2) + t + \sqrt{r^2 + B}; \chi_2 \right) \left( 2t \sqrt{r^2 + B} + t^2 \right)^{-s} \right] \\
+ C_{G_2} V_2 \sin (\pi s) \sum_{j=0}^{d_1-1} a_{2j}(1) \int_{\mathbb{R}} dr r^{2j+1} \tanh(a(G_2)r) \\
\times \left[ \int_0^{\infty} d\rho \psi_1 \left( \rho_0(1) + t + \sqrt{r^2 + B}; \chi_1 \right) \left( 2t \sqrt{r^2 + B} + t^2 \right)^{-s} \right] \\
+ \frac{1}{2\pi^2 i \Gamma(s)} \int_{\Re z = \epsilon} dz [\sin \pi(z + \frac{s}{2})] [\sin \pi(z)] [\Gamma(z + \frac{s}{2}) \Gamma(z - \frac{s}{2})] \\
\times \frac{\int_{\mathbb{R}} d\rho \psi_1 \left( \rho_0(k) + t + B^2 \chi_1 \right) \left( 2t B^2 + t^2 \right)^{-y_k(s,z)}}{k} \right], \tag{2.13}
\]

for any \(-\frac{1}{2} \leq \epsilon \leq \frac{1}{2} \). All of the integrals are entire functions of \( s \).

The simplest case is, for example, \( G_1 = G_2 = G = SO_4(2, l) \simeq SL(2, R) \); besides \( X_1 = X_2 = H^2 \) is a two-dimensional real hyperbolic space. Then we have \( \rho_0^2 = \rho_0^2(1) + \rho_0^2(2) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \), \( a_{20} = 1 \), \( C_G = 1 \), \( a(G) = \pi \), \( \Gamma_1 = \Gamma_2 = \Gamma \), and finally \( |C(r)|^{-2} = \pi r \tanh(\pi r) \). Using the Eq. (2.13) of the Theorem 1 for \( \Re s < 1 \) we have

\[
\zeta(s \| \bigoplus \mathcal{L}_p) = \frac{\pi^3 V_1^2}{2(s-1)(s-2)} \int_0^{\infty} dr \text{sech}^2(\pi r)K_0 \left( s-2; r^2 + B, \pi \right)
\]
where we assume a $\zeta$-form approximation can be given in terms of the multiplicative anomaly [39, 13, 14], related with one-loop approximation in quantum field theory. This statement follows from the general theory of Laplace type operators (see, for example, Ref. [15]).

In this section the product of the second-order operators of Laplace type $\bigotimes \mathcal{L}_p$, $p = 1, 2$ will be considered. We are interested in multiplicative properties of determinants, the multiplicative anomaly [39, 13, 14], related with one-loop approximation in quantum field theory. This approximation can be given in terms of the multiplicative anomaly $F(\mathcal{L}_1, \mathcal{L}_2)$, which has the form

$$F(\mathcal{L}_1, \mathcal{L}_2) = \det_\zeta(\bigotimes_{p} \mathcal{L}_p)[\det_\zeta(\mathcal{L}_1)\det_\zeta(\mathcal{L}_2)]^{-1},$$

(3.1)

where we assume a $\zeta$-regularization of determinants, i.e.

$$\det_\zeta(\mathcal{L}_p) \overset{\text{def}}{=} \exp\left(-\frac{\partial}{\partial s}\zeta(s = 0|\mathcal{L}_p)\right).$$

(3.2)

Generally speaking, if a multiplicative anomaly related to elliptic operators is nonvanishing then the relation $\log\det(\bigotimes \mathcal{L}_p) = \text{Trlog}(\bigotimes \mathcal{L}_p)$ does not hold.

The operator product $\bigotimes \mathcal{L}_p$ can arise in higher-derivative field theories. Note also that the partition function $Z$, associated with the product of two elliptic differential operators, for the simplest $O(2)$ invariant model of self-interacting charged fields in $\mathbb{R}^4$ has been analyzed recently in Ref. [15]:

$$\log Z \propto -\log\det (\bigotimes \mathcal{L}_p).$$

(3.3)

Theorem 2 Given the notation and results of Eqs. (B.11), (B.17), (B.18) and Theorem 4 of Appendix B the explicit formula for the multiplicative anomaly is

$$\mathcal{A}(\mathcal{L}_1, \mathcal{L}_2) = A \sum_{j=0}^{d-1} \frac{a_2 j (-1)^j}{2} \left\{ \frac{j}{2} (B_1 - B_2)^2 B_2^{j-1} + \frac{j(j-1)}{4} (B_1 - B_2)^3 B_2^{j-2} \right\}$$

$$+ \sum_{p=3}^{j} \frac{j!}{(p+1)p!(j-p)!} \left[ \frac{1}{p} + \frac{1}{p-1} + \sum_{q=1}^{p-2} \frac{1}{p-q-1} \right] (B_1 - B_2)^{p+1} B_2^{j-p}. \quad (3.4)$$

In a special case, namely for $d = 2$, Theorem 2 gives $\mathcal{A}(\mathcal{L}_1, \mathcal{L}_2) = 0$. Finally for any odd $d$ the multiplicative anomaly is zero. This statement follows from the general theory of Laplace type operators (see, for example, Ref. [15]).
4 The Residue Formula and the Multiplicative Anomaly

The value of $F(L_1, L_2)$ can be expressed by means of the non-commutative Wodzicki residue \[13\]. Let $\mathcal{O}_p$, $p = 1, 2$ be invertible elliptic PDOs of real non-zero orders $\alpha$ and $\beta$ such that $\alpha + \beta \neq 0$.

Even if the zeta functions for operators $\mathcal{O}_1, \mathcal{O}_2$ and $\mathcal{O}_1 \otimes \mathcal{O}_2$ are well defined and if their principal symbols obey the Agmon-Nirenberg condition (with appropriate spectra cuts) one has in general that $F(\mathcal{O}_1, \mathcal{O}_2) \neq 1$. For such invertible elliptic operators the formula for the anomaly of commuting operators holds \[15\]:

$$A(\mathcal{O}_1, \mathcal{O}_2) = \zeta'(0|\mathcal{O}_1 \mathcal{O}_2) - \zeta'(0|\mathcal{O}_1) - \zeta'(0|\mathcal{O}_2),$$

$$A(\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3) = \zeta'(0|\bigotimes_j \mathcal{O}_j) - \sum_j \zeta'(0|\mathcal{O}_j) - A(\mathcal{O}_1, \mathcal{O}_2),$$

$$\ldots \ldots \ldots \ldots$$

$$A(\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_n) = \zeta'(0|\bigotimes_j \mathcal{O}_j) - \sum_j \zeta'(0|\mathcal{O}_j) - A(\mathcal{O}_1, \mathcal{O}_2) - A(\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3) - \ldots - A(\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_{n-1}).$$

More general formulae have been derived in Refs. \[13, 14\]. Furthermore the anomaly can be iterated consistently. Indeed, using Eqs. (3.2) and (4.1) we have

$$A(\mathcal{O}_1, \mathcal{O}_2) = \log(F(\mathcal{O}_1, \mathcal{O}_2)) = (2\alpha\beta(\alpha + \beta))^{-1} \text{res} \left\{ \log(\mathcal{O}_1^{\beta} \otimes \mathcal{O}_2^{-\alpha}) \right\}^2.$$

In particular, for $n = 2$ and $\mathcal{O}_p \equiv L_p$ the explicit form of anomaly is given by the Eqs. (B.18) (of the Appendix B) and (3.4).

We note that for the four-dimensional space with $G = SO(4, 1)$, one derives from Theorem 2 the result

$$A(L_1, L_2) = -A_G(b_1 - b_2)^2, \quad d = 4,$$

which also follows from Wodzicki’s formula (4.1), where $A_G = \frac{1}{4} A_{a_2}$.

5 Conclusions

In this paper the global additive and multiplicative properties of Laplace type operators and related zeta functions has been studied. We have considered product structures on Einstein manifolds, especially on an irreducible rank 1 symmetric spaces.

In fact the explicit form of the zeta function on product spaces (Theorem 1) is derived. As an example the zeta function associated with the Kr"{o}necker sum of Laplacians acting on two-dimensional real hyperbolic spaces is calculated.

We have obtained also the explicit formula for the multiplicative anomaly, in the main theorem, Theorem 2. It has been shown that the anomaly is equal to zero for $d = 2$ and for the odd dimensional cases; this result is in agreement with the calculation given in Ref. \[15\]. We have preferred to limit ourselves here to discuss in detail various particular cases and emphasize the general picture. It seems to us that the explicit results for the anomaly (3.4), (4.3) are not only interesting as mathematical results but are of physical interest, in view of future applications to concrete problems in field theory and gravity, both at classical and quantum level. Note also that spectral properties of products of differential operators related to higher spin fields might differ in principal from the properties considered in this paper; we hope return to this problem elsewhere.
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A Zeta Functions on Product of Rank 1 Symmetric Spaces

In this Appendix we consider the trace formula for the partition function and zeta function associated with the product of symmetric spaces. Let the data \((G, K, \Gamma)\) be as in Sect. 2, therefore \(G\) being one of the four groups of Eq. (2.9). The trace formula holds

\[
\omega_{\Gamma}(t; b, \chi) = V \int_{\mathbb{R}} d\gamma e^{-(t^2 + b + \rho_0^2)t}|C(\gamma)|^{-2} + \theta_{\Gamma}(t; b, \chi),
\]

where, by definition,

\[
V \overset{\text{def}}{=} \frac{1}{4\pi} \chi(1) \text{vol}(\Gamma \setminus G),
\]

where \(\chi\) is a finite-dimensional unitary representation (or a character) of \(\Gamma\), and the number \(\rho_0\) is associated with the positive restricted (real) roots of \(G\) (with multiplicity) with respect to a nilpotent factor \(N\) of \(G\) in an Iwasawa decomposition \(G = KAN\). One has \(\rho_0 = (n-1)/2, n, 2n+1, 11\) in the cases \((I)\) – \((IV)\) respectively in Eq. (2.9). Finally the function \(\theta_{\Gamma}(t; b, \chi)\) is defined as follows

\[
\theta_{\Gamma}(t; b, \chi) \overset{\text{def}}{=} \frac{1}{\sqrt{4\pi t}} \sum_{\gamma \in \Gamma \setminus \{1\}} \chi(\gamma) \partial_{\gamma} \frac{1}{j(\gamma)} C(\gamma) e^{-(t^2 + \rho_0^2 t + t^2/4t)},
\]

for a function \(C(\gamma), \gamma \in \Gamma\), defined on \(\Gamma \setminus \{1\}\) by

\[
C(\gamma) \overset{\text{def}}{=} e^{-\rho_0^2 t} |\det_{n_0} \left( \text{Ad}(m_\gamma e^{t\gamma} H_0)^{-1} - 1 \right)|^{-1}.
\]

The notation used in Eqs. (A.3) and (A.4) is the following. Let \(a_0, n_0\) denote the Lie algebras of \(A, N\). Since the rank of \(G\) is 1, \(\dim a_0 = 1\) by definition, say \(a_0 = RH_0\) for a suitable basis vector \(H_0\). One can normalize the choice of \(H_0\) by \(\beta(H_0) = 1\), where \(\beta : a_0 \rightarrow \mathbb{R}\) is the positive root which defines \(n_0\); for more detail see Ref. [38]. Since \(\Gamma\) is torsion free, each \(\gamma \in \Gamma \setminus \{1\}\) can be represented uniquely as some power of a primitive element \(\delta : \gamma = \delta^j(\gamma)\) where \(j(\gamma) \geq 1\) is an integer and \(\delta\) cannot be written as \(\gamma^j_1\) for \(\gamma_1 \in \Gamma, j > 1\) an integer. Taking \(\gamma \in \Gamma, \gamma \neq 1\), one can find \(t_\gamma > 0\) and \(m_\gamma \in K\) satisfying \(m_\gamma a = am_\gamma\) for every \(a \in A\) such that \(\gamma\) is \(G\) conjugate to \(m_\gamma \exp(t_\gamma H_0)\), namely for some \(g \in G, g\gamma g^{-1} = m_\gamma \exp(t_\gamma H_0)\). For \(\text{Ad}\) denoting the adjoint representation of \(G\) on its complexified Lie algebra, one can compute \(t_\gamma\) as follows

\[
e^{t_\gamma} = \max\{|c| | \text{c is an eigenvalue of } \text{Ad}(\gamma)\},
\]

in case \(G = SO_1(m, 1)\), with \(|c|\) replaced by \(|c|^{1/2}\) in the other cases of Eq. (2.9).

A.1 Zero Modes

Let us start with the zero modes case, i.e. \(b = 0\). It can be shown that the Mellin transform of \(\theta_{\Gamma}(t; b, \chi)\) at \(b = 0\),

\[
\hat{\theta}_{\Gamma}(s; 0, \chi) \overset{\text{def}}{=} \int_{0}^{\infty} dt \theta_{\Gamma}(t; 0, \chi)t^{s-1},
\]
is a holomorphic function on the domain $\text{Re} s < 0$. Then using the result of Refs. [37, 38] one can obtain on $\text{Re} s < 0$,

$$
\hat{\theta}_T(s; 0, \chi) = \sum_{\gamma \in \mathbb{C}_1 \{1\}} \chi(\gamma) t_\gamma j(\gamma)^{-1} C(\gamma) \int_0^\infty dt \frac{e^{-(\rho_0^2 t^2/(4t))}}{\sqrt{4\pi t}} t^{s-1}
\tag{A.7}
$$

where $K_\nu(s)$ is the modified Bessel function, and finally

$$
\hat{\theta}_T(s; 0, \chi) = \frac{\sin(\pi s)}{\pi} \Gamma(s) \int_0^\infty dt \psi_T(t + 2\rho_0; \chi)(2\rho_0 t + t^2)^{-s}. \tag{A.8}
$$

Here $\psi_T(s; \chi) \equiv d(\log Z_T(s; \chi))/ds$, and $Z_T(s; \chi)$ is a meromorphic suitably normalized Selberg zeta function [44, 45, 46, 47, 48, 49, 37].

### A.2 A Meromorphic Continuation

For $B_p \overset{\text{def}}{=} b_p + \rho_0^2(p) \ (p = 1, 2), \ y_p(s; z) \overset{\text{def}}{=} \frac{s}{2} + (-1)^{p-1}z$, the following proposition holds:

**Proposition 1** The integral of the product of the functions $\hat{\theta}_T(p)(y_p(s; z), b_p, \chi_p)$ is an entire function of $s$ and allows the form

$$
\int_0^\infty dt \prod_p \theta_p(t; b_p, \chi_p) t^{s-1} = \frac{1}{2\pi i} \int_{\text{Re} z = c} dz \prod_p \hat{\theta}_T(p)(y_p(s; z); b_p, \chi_p)
\tag{A.9}
$$

$$
= \frac{1}{2\pi i} \int_{\text{Re} z = c} dz \prod_l \Gamma(y_l(s; z))[\sin \pi(y_l(s; z))]
\times \prod_p \int_0^\infty dt \psi_T(p_0(p) + t + B_p^4; \chi_p)(2B_p^4 t + t^2)^{-y_p(s; z)},
$$

for $c \in \mathbb{R}$ with $\text{Re} \frac{s}{2} < c < -\text{Re} \frac{s}{2}$, and $b_p \geq 0$ with at least $b_1 > 0$ or $b_2 > 0$.

The proposition is a consequence of the first integral transformation of Eq. (A.9) as an integral in the complex plane with the help of the Mellin-Parseval identity

$$
\int_0^\infty dt h(t) g(t) = \frac{1}{2\pi i} \int_{\text{Re} z = c} dz \hat{h}(z)\hat{g}(1 - z), \quad c \in \mathbb{R}. \tag{A.10}
$$

To this end one can choose $h(t) = \theta_1(t; b_1, \chi_1)t^{s/2}, \ g(t) = \theta_1(t; b_2, \chi_2)t^{s/2-1}$.

We shall also need the $B \overset{\text{def}}{=} \rho_0^2(1) + \rho_0^2(2) + b_1 + b_2$, the function

$$
K_n(s; \delta, a) \overset{\text{def}}{=} \int_{\mathbb{R}} \frac{dr r^{2n} \text{sech}^2(a r)}{(\delta + r^2)^s}, \tag{A.11}
$$

defined for $a, \delta > 0, \ n \in \mathbb{N} \ s \in \mathbb{C}$, and the functions

$$
f_l(t; a, G) \overset{\text{def}}{=} \int_0^\infty dr r^{2l+1} e^{-(r^2 + a)t} \tanh(a(G)r), \tag{A.12}
$$

$$
H^{(0)}_\Gamma(s; a, b, \chi, G) \overset{\text{def}}{=} \int_0^\infty dt f_l(t; a, G)\theta_1(t; b, \chi)t^{s-1}. \tag{A.13}
$$

All the functions (A.11) - (A.13) are entire functions of variable $s$. 

Proposition 2 Suppose $G_p \neq SO_1(m,1)$, $SU(q,1)$ with $m$ odd and $q$ even. Then for $\text{Res} > \frac{d_1}{2} + \frac{d_2}{2}$,

$$
\frac{1}{\Gamma(s)} \int_0^\infty dt \prod_p \omega_{\Gamma_p}(t, b_p, \chi_p) t^{s-1} = \frac{\pi^2}{2} \prod_k V_k a(G_k) C_{G_k} \sum_{j=0}^{d_1-1} \sum_{\mu=0}^{d_1-1} \sum_{\nu=0}^{d_2-1} \frac{a_{2j}(1) a_{2\mu}(2) j! \mu!}{(j-l)! (\mu-\nu)!} \\
\times \int_0^\infty dr (2l-1) sech^2(a(G_1)r) K_{\mu-\nu}(s+l-\nu-1; r^2 + B, a(G_1)) dr \\
\times \frac{(s-1)(s-2)...(s-(l+1))(s-(l+2))...(s-(l+1+\nu+1))}{(s-l-\nu-1; r^2 + B, a(G_1)) dr}
$$

\begin{align}
\frac{2\pi}{\Gamma(s)} \left\{ \sum_{j=0}^{d_1-1} a_{2j}(1) H_{t_1}^{(j)}(s; b_1 + B_1, b_2, \chi_2, G_1) \\
+ V_2 C_{G_2} \sum_{j=0}^{d_2-1} a_{2j}(2) H_{t_2}^{(j)}(s; b_2 + B_2, b_1, \chi_1, G_2) \right\} \\
+ \frac{1}{\Gamma(s)} \int_0^\infty dt \prod_k \theta_{\Gamma_k}(t, b_k, \chi_k) t^{s-1}.
\end{align}

(A.14)

The formulae (A.9), (A.14) give the meromorphic continuation of the zeta function (2.10).

B The Zeta Function of the Product of Laplace-Type Operators

The spectral zeta function associated with the product $\bigotimes \mathcal{L}_p$ has the form

$$
\zeta(s|\bigotimes \mathcal{L}_p) = \sum_{j \geq 0} n_j \prod_p (\lambda_j + b_p)^{-s}.
$$

(B.1)

We shall always assume that $b_1 \neq b_2$, say $b_1 > b_2$. If $b_1 = b_2$ then $\zeta(s|\bigotimes \mathcal{L}_p) = \zeta(2s|\mathcal{L})$ is a well-known function. For $b_1, b_2 \in \mathbb{R}$, set $b_+ \overset{def}{=} (b_1 + b_2)/2$, $b_- \overset{def}{=} (b_1 - b_2)/2$, thus $b_1 = b_+ - b_-$. and $b_2 = b_+ - b_-$.  

Remark 1 The spectral zeta function can be written as follows

$$
\zeta(s|\bigotimes \mathcal{L}_p) = (2b_-)^{-s} \frac{\sqrt{\pi}}{\Gamma(s)} \int_0^\infty dt \omega_{\Gamma}(t, b_+) I_{s-\frac{1}{2}}(b_- t),
$$

(B.2)

where the integral converges absolutely for $\text{Res} > \frac{d}{4}$ ($d = \text{dim}(G/K)$).

This formula is a main starting point to study the zeta function. It expresses $\zeta(s|\bigotimes \mathcal{L}_p)$ in terms of the Bessel function $I_{s-\frac{1}{2}}(b_- t)$ and $\omega_{\Gamma}(t, b_+)$, where the trace formula applies to $\omega_{\Gamma}(t, b_+)$. Thus we can study the zeta function by the trace formula. One can use the trace formula for $\omega_{\Gamma}(t, b_+)$ (A.1); as a result

$$
\zeta(s|\bigotimes \mathcal{L}_p) = (2b_-)^{-s} \frac{\sqrt{\pi}}{\Gamma(s)} \int_0^\infty \frac{\chi(1) \text{vol}(\Gamma \setminus G)}{4\pi} \int_{\mathbb{R}} dr e^{-(r^2 + b_+ + b_-) t} |C(r)|^{-2} \theta_{\Gamma}(t) I_{s-\frac{1}{2}}(r^{s-\frac{1}{2}}).
$$

(B.3)

Then for $\text{Re} s > 0$ Fubini’s theorem gives

$$
\int_0^\infty \frac{\chi(1) \text{vol}(\Gamma \setminus G)}{4\pi} \int_{\mathbb{R}} dr e^{-(r^2 + b_+ + b_-) t} |C(r)|^{-2} I_{s-\frac{1}{2}}(r^{s-\frac{1}{2}}) =
$$
\[ (2b_-)^{\frac{1}{2}-s} \frac{\chi(1)\text{vol}(\Gamma\setminus G)\Gamma(s)}{4\pi^{s/2}} \int_{\mathbb{R}} dr|C(r)|^{-2} \prod_{p}(r^2 + B_p)^{-s}. \]  

(B.4)

In order to analyze the last integral in Eq. (B.4) (for the possibility of a meromorphic continuation) it is useful to rewrite the function \(|C(r)|^{-2}\) (see Eq. (2.11)), using the identity \(\tanh(ar) \equiv 1 - 2(1 + e^{2ar})^{-1}\). Then one can calculate a suitable integral in terms of the hypergeometric function \(F(\alpha, \beta; \gamma; z)\), namely

\[
\int_{0}^{\infty} dr r^{2j+1} \prod_{p}(r^2 + B_p)^{-s} = \frac{\sqrt{\pi}(2s - j - 1)!}{2^{2s}\Gamma(s + \frac{1}{2})} B_1^{-s} B_2^{j+1-s} \left(\frac{2B_1}{B_1 + B_2}\right)^{j+1} 
\times \binom{\alpha}{\gamma} \binom{\beta - \gamma}{\gamma - 1} \binom{B_1 - B_2}{B_1 + B_2}^2 \right),
\]

(B.5)

which is holomorphic function on \(\text{Re} \ s > (j + 1)/2\) and admits a meromorphic continuation to \(\mathbb{C}\) with only simple poles at points \(s = (j + 1 - n)/2, \ n \in \mathbb{N}\). Let us introduce the function

\[ E_j(s) \overset{\text{def}}{=} 2 \int_{0}^{\infty} \frac{dr r^{2j+1}}{(1 + e^{2a(r)r})} \prod_{p}(r^2 + B_p)^{-s}, \]

(B.6)

which is an entire function of \(s\). Then a consequence of Eqs. (B.2) - (B.6) is

Theorem 3 For \(\text{Re} \ s > \frac{d}{4}\) the explicit meromorphic continuation holds:

\[ \zeta(s) \bigotimes_{p} L_p = \mathcal{F}(s) - 2A \sum_{j=0}^{\frac{d}{2}-1} a_{2j} E_j(s) + \mathcal{I}(s), \]

(B.7)

where

\[ A \overset{\text{def}}{=} \frac{1}{4} \chi(1)\text{vol}(\Gamma\setminus G)C_G, \]

(B.8)

\[ \mathcal{F}(s) \overset{\text{def}}{=} A(B_1 B_2)^{-s} \sum_{j=0}^{\frac{d}{2}-1} a_{2j} j! \frac{\binom{2B_1 B_2}{B_1 + B_2}^{j+1} \binom{j+1}{2} \binom{j+2}{2} \gamma \left(\frac{B_1 - B_2}{B_1 + B_2} \right)^2}{(2s - 1)(2s - 2)\ldots(2s - (j + 1))}, \]

(B.9)

\[ \mathcal{I}(s) \overset{\text{def}}{=} (2b_-)^{\frac{1}{2}-s} \frac{\sqrt{\pi}}{\Gamma(s)} \int_{0}^{\infty} dt \theta(t, b_+) I_{s-\frac{1}{2}}(b_- t)^{s-\frac{1}{2}}. \]

(B.10)

The goal now is to compute the derivative of the zeta function at \(s = 0\). Thus we have

\[ \zeta'(0) \bigotimes_{p} L_p = A \sum_{j=0}^{\frac{d}{2}-1} a_{2j} \sum_{l=1}^{4} \mathcal{E}_l, \]

(B.11)

where

\[ \mathcal{E}_1 = j!(B_1^{j+1} + B_2^{j+1}) \sum_{k=0}^{j} \frac{(-1)^{k+1}}{k!(j-k)!(j+1-k)!}, \]

(B.12)

\[ \mathcal{E}_2 = B_2^{j+1} \left(\frac{B_1 - B_2}{2B_1}\right) \frac{(-1)^j}{(j+1)!} \sum_{k=1}^{\infty} \frac{(j+k+1)!}{(k+1)!} \sigma_k \left(\frac{B_1 - B_2}{B_1}\right)^k, \]

(B.13)

\[ \mathcal{E}_3 = \log(B_1 B_2) \frac{(-1)^j}{2(j+1)} \left(\frac{B_1^{j+1} + B_2^{j+1}}{2}\right) - 4 \int_{0}^{\infty} \frac{dr r^{2j+1} \log \left(\frac{r^2 + B_1}{r^2 + B_2}\right)}{1 + e^{2a(r)r}}, \]

(B.14)

\[ \mathcal{E}_4 \equiv \mathcal{I}'(s = 0) = T_{\Gamma}(0, b_1, \chi_1) + T_{\Gamma}(0, b_2, \chi_2), \]

(B.15)
and
\[
T_\Gamma(0, b_p, \chi_p) \overset{\text{def}}{=} \int_0^\infty dt \theta_\Gamma(t, b_p) t^{-1}, \quad \sigma_k \overset{\text{def}}{=} \sum_{k=1}^{n} \frac{1}{k}.
\] (B.16)

Similarly using results from Ref. [43] one can show
\[
\zeta'(0|L_p) = A \sum_{j=0}^{\frac{d}{2} - 1} a_{2j} B_{j+1}^2 \left[ j! B_p^{j+1} \frac{(-1)^{j+1}}{k!(j-k)!(j+1-k)^2} + \frac{(-1)^j}{(j+1)!} \log B_p \right]
\]
\[
+ 4 \int_0^\infty \frac{dr r^{2j+1} \log(r^2 + B_p)}{1 + e^{2a(G)p}} \right] + T_\Gamma(0, b_p, \chi_p).
\] (B.17)

Then we can put these results together to compute the anomaly
\[
A(L_1, L_2) \overset{\text{def}}{=} \zeta'(0|\bigotimes L_p) - \zeta'(0|L_1) - \zeta'(0|\bigotimes L_2).
\] (B.18)

**Proposition 3** A preliminary form of the multiplicative anomaly is
\[
A(L_1, L_2) = A \sum_{j=0}^{\frac{d}{2} - 1} a_{2j} B_{j+1}^2 \left[ \frac{B_1 - B_2}{2B_1} \right] \frac{(-1)^{j+1}}{(j+1)!} \sum_{k=1}^\infty \sigma_k \left( \frac{B_1 - B_2}{B_1} \right)^k (j+k+1)! \frac{k (j+k+1)!}{(k+1)!}
\]
\[
+ A \sum_p \log(B_p) \sum_{j=0}^{\frac{d}{2} - 1} a_{2j} \frac{(-1)^{j+p}}{2(j+1)} \left( B_{j+1}^{2j+1} - B_{j+1}^{j+1} \right).
\] (B.19)

To obtain the final explicit form of the anomaly we must compute the infinite series appearing in the statement of Proposition 3. For \(0 < x < 1, j = 0, 1, 2, \ldots\) let
\[
S_j(x) = \sum_{k=1}^{\infty} \frac{(j+k+1)!}{(k+1)!} \sigma_k x^k.
\] (B.20)

Thus in Proposition 3 we choose \(x = (B_1 - B_2)/B_1\). After a tedious calculation one obtains the following result.

**Theorem 4** The function \(S_j(x)\) admits the representation
\[
(1 - x)^{j+1} S_j(x) = P_j(x) + Q_j(x) \log(1 - x),
\] (B.21)

where \(P_j(x)\) and \(Q_j(x)\) are polynomials of degree \(j\) given by
\[
\frac{P_j(x)}{(j+1)!} = \frac{jx}{2} (1 - x)^{j-1} + \frac{j(j-1)}{4} x^2 (1 - x)^{j-2} + \sum_{p=3}^{j} \frac{j!}{(p+1)!}(j-p)!
\]
\[
\times \left[ \frac{1}{p} + \frac{1}{p-1} + \sum_{q=1}^{p-2} \frac{1}{p-q-1} \right] x^p (1 - x)^{j-p},
\] (B.22)
\[
\frac{xQ_j(x)}{(j+1)!} = -\frac{1}{j+1} \left[ 1 - (1 - x)^{j+1} \right].
\] (B.23)
References


[38] F.L. Williams, JMP 38, 796 (1997).


