A Two Dimensional Analysis Of Vertical Axis Windmills

E. S. Van Dusen

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A Two Dimensional Analysis
of
Vertical Axis Windmills

Technical Report
by
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Amherst, Mass. 01003

June 1978

Prepared for the United States Department of Energy
and Rockwell International, Rocky Flats Plant
Under Contract Number PF67025F
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CHAPTER I

INTRODUCTION

1.1 Historical Development

This study develops the theory for determining the flow field and performance of a momentum exchange rotor, which turns primarily due to aerodynamic lift forces, and is oriented so that the flow is perpendicular to the axis of rotation. When operating as windmills, these devices are termed vertical axis machines to distinguish them from the axial flow or propeller type. The analytical development of cross flow rotors has lagged other types, because there is not any reference frame that can be devised to make the flow appear steady. Since the orientation of the rotor to the flow is always changing, a series of solutions at differing angles of rotation are needed to adequately describe the performance. The unsteady and more complex fluid dynamics make consideration of this problem impractical without current computational methods.

Cross flow rotors can be separated into two groups depending on their tip speed ratio, \( \frac{\Omega R}{U_0} \), where \( \Omega \) is the rotational speed, \( R \) the maximum rotor radius, and \( U_0 \) the speed of the incoming flow. When the tip speed ratio is approximately one or less, a foil on the side of the rotor traveling into the flow may provide torque by acting as an aerofoil, while a foil traveling down stream may provide torque by acting as a drag device. This group, of which the Savonius rotor is the most successful, is characterized by large curved foils and by the fact that the flow
across a foil may reverse direction during a revolution. The other group has a tip speed ratio significantly greater than unity and slender blades resembling a cycloidal propeller. Due to its high rotational speeds, the flow across the foils is always in the same direction; although the incidence angle and the speed of the flow relative to the blades varies throughout a revolution. Generally, tip speed ratios of six or more are chosen to prevent the foils from stalling and the resulting power loss. This type of machine is often referred to as a Darrieus rotor. (see Figure 1.)

In 1924 the design of the Savonius rotor developed from Sigurd Savonius' interest in the Flettner rotor ship. Flettner propelled a ship in a cross wind using the Magnus effect on circular cylinders mounted vertically and mechanically rotated. Savonius saw the opportunity to dispense with the mechanical drive by replacing the cylinders with a rotor in the shape of an "S", which would rotate by itself. He then demonstrated the validity of this design by propelling a small boat and constructing some interesting windmills. This success motivated him to test over 30 variations of the "S" rotor and 20 models of more conventional wind turbines which he used to verify his results. Overlapping the half cylinders of the rotor to provide a slot produced more than three times the power of the simple "S" shape. Many variations were tried including various numbers of blades, different slot sizes, and various shapes. The most successful was a two-bladed design with a slot about one quarter of the diameter and discs capping each end.
FIGURE 1 - Savonius and Darrieus Rotors
Savonius reported a power coefficient of .31 at a tip speed ratio of .85, with the power falling off two or three percent at .65 and 1.1, and a maximum tip speed ratio of 1.74. Good starting torque was also reported. Unfortunately, no details of the wind tunnel are presented so that corrections could be made for performance in an unbounded flow. Savonius reports an improvement in performance in a natural wind, claiming a power coefficient of .37 at tip speed ratios of .92 to 1.0, which he attributes to both the effect of the wind tunnel and the improved response of the rotor in gusts and changes in wind direction over horizontal axis machines.\[^{1,2}\]

The Savonius rotor became popular for use in pumping water where it exhibits additional advantages to its easy starting and good performance in gusts. Its shape is simple and inexpensive to build. Mechanical complexity is considerably reduced, not only in the rotor itself, but in easy mounting on a tower and the lack of gearing aloft. Flaps on the foils acting as wind brakes actuated by centrifugal force were reported by Savonius to provide smooth and reliable control in high winds and gusts. Varying the slot size was also found effective in controlling the rotor. Additional applications are to turn a building ventilator, a water wheel in rivers, harnessing wave power when mounted horizontally just below the surface, and to measure oceanographic currents.

Less has been reported about the origin of the Darrieus rotor; although a United States patent was filed in 1926 by Georges Darrieus of
Paris for a vertical axis windmill. Maximum power coefficients in the range of .34 to .38 have been recorded by several investigators.[3] The high power coefficient, fairly simple shape, and small blade area, hence material requirement, make this design attractive. The Darrieus rotor is reluctant to start by itself; however, its high tip speed ratio is an advantage when connected to a high speed electrical generator. If straight blades are used, the centrifugal forces cause high stresses in the blades, but pitch control is readily accomplished allowing performance optimization and speed control. In an effort to reduce blade stresses, a curved shape has been tried, so that the forces are only tensile. The curve resulting is similar to a parabola and is given the name troposkien. Dynamic forces, principally occurring when the downstream blade must pass through the wake from an upstream blade, can be of considerable magnitude and to date have kept these rotors relatively small.

Figure 2 shows the power produced by different types of wind machines as a function of rotational speed. The Darrieus rotor has a performance slightly below the best high speed horizontal axis machines. The Savonius rotor is similar to the fan type horizontal axis machine. Unfortunately, there is a wide variation in the measured performance of a Savonius rotor.[1,4,5,6,7,8] Frequently, when wind tunnel tests are performed, no correction is made for the tunnel effects; furthermore, details of tunnel geometry and whether if open or closed test sections are omitted so that it is impossible to estimate the direction of the
FIGURE 2 - Typical Windmill Performance

POWER COEFFICIENT, $C_p = \frac{1}{2} \omega^2 

SPEED RATIO = TIP PERIPHERAL SPEED/STREAM SPEED

Ideal Efficiency for Propeller Type Windmills

High Speed Two Blade Type

American Multi-Blade Type

Savonius Rotor

Dutch Four Arm Type

Darrieus Rotor
correction, not to mention its magnitude. When field tests are performed, the response of the wind measuring device to gusts is not well documented, so that the additional energy available due to varying wind strengths is not taken into account. Model dissimilarities can produce additional differences. Various length to diameter ratios have been used, allowing 3-D effects to cloud the results. Different camber shapes, foil thickness, shape of the leading and trailing edges, and flow turbulence level can have a major effect on performance.

1.2 Simple Operational Analysis

A brief discussion of one-dimensional flow analysis is useful as an overview of rotor performance. Momentum theory compares the energy withdrawn from a stream tube by some device with the fluid momentum entering and leaving the stream tube. The development uses quantities integrated across the flow so that transverse variations are not considered; for simplicity, uniform flows will be used here. The thrust, $T$, extracted from the flow at (2), Figure 3, is equal to the rate of change of momentum in the stream tube between (1) and (3):

$$T = \frac{d}{dt}(mv) = \rho A(U_0 - v)(U_0 - U_w)$$

(1.1)

The power produced is the rate of change of kinetic energy;

$$P = \frac{d}{dt}\left(\frac{1}{2}mV^2\right) = \frac{1}{2} \rho A(U_0 - v)(U_0^2 - U_w^2)$$

(1.2)
and also the product of the thrust and flow rate:

\[ P = T(U_0 - v). \]  

(1.3)

Equating both expressions for power gives:

\[ v = \frac{U_0 - U_w}{2} \]  

(1.4)

\[ U_0 - v = \frac{U_0 + U_w}{2} \]  

(1.5)

Which can be solved to find the maximum power coefficient giving:

\[ \frac{U_w}{U_0} = \frac{1}{3} \]  

(1.6)

\[ v = \frac{1}{3} U_0 \]  

(1.7)

\[ C_{p\text{max}} = \frac{P_{\text{max}}}{\frac{1}{2} \rho U_0^3} = \frac{16}{27} = .593 \]  

(1.8)

Hence, the maximum power that can be removed from the flow by any means is \( \frac{16}{27} \) of its kinetic energy, assuming the most ideal conditions - uniform wake and frictionless flow.\(^9,10\)

If energy were removed from the flow solely by fluid dynamic drag carrying an object downstream, the power produced would be the product of the drag force, \( D \), and the translational velocity, \( v \)\(^{11} \) (Refer to Figure 4).

\[ P = Dv = \frac{1}{2} \rho (U_0 - v)^2 C_D A v \]  

(1.9)

Solving for maximum power:

\[ v = \frac{1}{3} U_0 \]  

(1.10)
FIGURE 3 - Stream Tube for Momentum Theory

FIGURE 4 - Drag on Body Traveling Slowly Downstream
Drag coefficients are likely to be approximately 1., so for a rotating device relying only on drag forces, the power coefficient should be considerably less than .15, since there will be losses in returning the drag device to an upstream position. Also, \(v\) must be less than \(U_o\), since the drag force is zero when \(v\) equals \(U_o\). The fact that a Savonius rotor can reach a tip speed ratio significantly above one and that the maximum power occurs just below unity, where the drag force tending to rotate the foils is small, leads one to conclude that the predominant power must be due to aerodynamic lift. The high power coefficients found by some investigators add further support to this assertion.

A simple three-dimensional model of a cross flow rotor may be constructed to provide insight into the rotational forces and wake development. For simplicity consider the high tip speed ratio type with long thin aerofoils.

The relative velocity vector diagram, lift force and circulation are shown in Figure 5. Starting at the point where the foil is travelling downstream and there is no circulation, position (a), Figures 5 and 6, the foil develops lift and circulation as it travels along its path. As circulation develops, an equal and opposite starting vortex is shed from the trailing edge. Since vortex lines cannot end within a fluid, the bound and starting vortices are connected by trailing vortices from the ends of the foil. Half a revolution later the bound vorticity again approaches zero and is shed into the wake, forming a vortex ring which
FIGURE 5 - Relative Velocity Diagram for a Rotor

FIGURE 6 - Rotor Wake Formation
will be carried downstream by the fluid.

The rotation of the foils keeps adding these vortex rings to the wake which produce regions of high vorticity and a velocity component that retards the flow through the rotor. Due to manufacturing simplicity, both Savonius and Darrieus rotors are most frequently constructed with constant chord untwisted foil sections; this produces a constant circulation along the span of the blade. An elliptical spanwise circulation distribution is needed to produce a constant downwash velocity in the wake, which minimizes the kinetic energy lost to the wake. Hence, some loss in performance from the ideal case must be expected unless an effort is made to control the spanwise load distribution as is done in the more efficient axial flow machines. Curving the foils radially to the troposkien shape further compromises performance, since the ends of the foil act at a lower speed ratio and experience a wider variation of incidence angle, increasing the likelihood of stall.

Inspecting Figure 5 leads one to expect the Darrieus rotor to produce large thrust forces; however, the side forces and bound circulation averaged over one revolution should have relatively small values. This is not the case with the Savonius rotor, which produces a side force of the same order of magnitude as its thrust. The large foils operating in a less balanced fashion, lifting and stalled, create large vorticity in the fluid flowing through the rotor. In order to produce large forces, conservation of momentum requires that the flow be deflected sideways. Hence, the wake behind the Savonius rotor is a jet of more slowly moving fluid with regions of high vorticity that travels through the outer flow at an angle to the velocity at infinity.
1.3 Previous Analysis

Very little analytical work has been published about the Savonius rotor. Wilson, Lissaman, and Walker\cite{3} present an interesting series of photographs of smoke streak lines for a very lightly loaded Savonius rotor spinning at high speed ratio. Although these are quite different from streamlines that would be seen in a reference frame fixed in the rotor, they are able to identify the more salient flow trends. Wilson, et al proceed with an analysis of an "S" shaped rotor which they hope will model the more significant performance characteristics of a two bladed Savonius rotor. They assume that the wake can be represented as a constant stream tube flowing at an angle to the outer flow in a manner used to study powered wakes from V/STOL aircraft and match this solution to the flow through a single rotating "S" vane. Difficulty is encountered in determining the circulation produced by the rotor. Normally the Kutta condition would be applied to the trailing edge of a foil; however, the complex flow pattern makes this specification difficult at times. They decide to specify a circulation that minimizes the flow around both ends of the foil. The results of a few model tests are presented and discussion is made of the importance of wind tunnel corrections. This theoretical analysis predicts power coefficients that are typical for a Savonius rotor; however, tests performed by Savonius demonstrated that the "S" shaped rotor produced only 1/3 the power. This fact together with the lack of a clear physical interpretation of the circulation specification suggest that the details of this solution need further scrutiny.
Due to the quasi-steady flow regime, current analysis of the Darrieus rotor has modeled most of the important phenomena and agrees fairly well with empirical results. Flow models published by Wilson, et al., Shankar, Strickland, Templin, and Holme attempt to determine how much the flow at the rotor is slowed by the wake and then to find the aerodynamic forces that the rotating aerofoils would experience in such a flow. Templin assumes that flow through the rotor and in the wake is uniform. Wilson and Lissaman, Shankar, and Strickland consider the wake to be a series of stream tubes across the rotor and equate the time averaged blade forces in front of each stream tube to the mean change in momentum flux through the tube. This creates a non-uniform flow across the rotor and deformation of the wake, but neglects mutual interference between foils and the effects of blades passing through shed vortex sheets. Holme's analysis is the most sophisticated and includes the effects of bound and wake vorticity; however, it considers only 2-D rotors and linear aerodynamics. Holme's results show that the foils encounter higher angles of attack on the upstream side of the rotor; whereas, the other analyses predict no difference. Wilson, et al., predict a maximum power coefficient of .554 under the following conditions: no drag, straight blades, \( C_L = 2\pi \sin \alpha \) (\( \alpha \) is the angle of incidence), and the chord positioned tangentially on the rotor. The difference between this and the power coefficient predicted by momentum theory is due to the non-uniform wake considered by their model. Then their analysis is extended to consider empirical lift coefficients, curved blades, and foil drag.
1.4 Current Analysis

The purpose of this study is to determine the flow field and performance of any arbitrary cross flow rotor by means of a detailed application of aerodynamic theory. The force and pressure distribution on each foil will be determined by solution of the entire flow including the complete structure of the wake. Such an approach avoids using a simplified model that loses the structure of the flow, and must rely heavily on the capabilities of a modern digital computer. This analysis will be limited to two dimensional inviscid flow, which should provide considerable insight into the operation of the rotors and a representative evaluation of performance. Should a future analysis need to consider viscous effects, this solution would form the logical basis upon which a boundary layer analysis could be performed.

After developing this model, the performance of typical vertical axis machines will be predicted and compared with published empirical values. The geometry and tip speed ratio of a Savonius Rotor will be varied through a wide range to identify trends and plots of the streamlines presented to aid in visualizing the flow patterns. It is hoped that these results will enable more thorough experimental measurement to be performed, which are unfortunately beyond the scope of this study and would be desirable before engineering applications are considered.
CHAPTER II

2-D ANALYSIS

2.1 Problem Definition

A 2-D flow solution is needed for two foils rotating about an axis in a uniform stream which is flowing perpendicular to the axis. The shape of each foil is arbitrary, although most frequently one will be symmetrical with respect to the other about the center of rotation. An inertial Cartesian axis system is positioned so that the foils rotate about the origin at the rate $\Omega$, positive in the counterclockwise sense. A uniform stream with velocity $U_o$ flows parallel to the x axis from left to right, as shown in Figure 7. Since the geometry, $\Omega$, and $U_o$ are arbitrary the solution is general for any class of momentum exchange device with two foils and an axis of rotation across the flow.

Considering a fluid that is inviscid and incompressible, the conditions of continuity and irrotationality must be satisfied throughout the flow. Letting $U$ be the fluid velocity at a point, these conditions may be written:

Continuity: \[ \nabla \cdot U = 0, \] \[ (2.1) \]

Irrotationality: \[ \nabla \times U = 0. \] \[ (2.2) \]

At this time a velocity potential function, $\phi$, may be defined so that $\frac{\partial \phi}{\partial x} = u$ and $\frac{\partial \phi}{\partial y} = v$. Continuity requires $\nabla^2 \phi = 0$. Likewise, a stream function, $\psi$, so that $\frac{\partial \psi}{\partial y} = u$ and $\frac{\partial \psi}{\partial x} = -v$, and irrotational flow produce $\nabla^2 \psi = 0$, Laplace's equation.
FIGURE 7 - Axis System for a 2-D Rotor

FIGURE 8 - Rotating Axis System
The first boundary condition is that the flow disturbance due to the rotor must approach zero at large distances from the origin; hence,

\[ \mathbf{U} \text{ at infinity} = \mathbf{U}_0. \]  

The other boundary condition is that foil surfaces are impermeable to the fluid so that

\[ \mathbf{U} \cdot \mathbf{n} = 0 \text{ on the foil}, \]  

where \( \mathbf{n} \) is a vector normal to the surface of the foil.

Since the object of the rotor is to produce forces, it is assumed that the ends of the rotor where the flow leaves the surfaces are sufficiently sharp so that they behave like the trailing edges of aerofoils in order that the Kutta-Joukowskii law may be used to determine the circulation on each foil. Clearly, it is possible that the angle of attack of a foil could become so large that the flow separates and the above condition no longer applies to that foil. In such a case the total circulation about that foil is required to be zero. Hence, the magnitude of the circulation for each foil will increase with increasing angle of attack until a value is reached that is physically impossible for a real fluid and then will be set to zero. These maximum and minimum values of circulation must be supplied to the analysis and depend upon the properties of the fluid and the particular geometry of the foils such as the sharpness of the leading edge.

Normally a solution of Laplace's equation would be independent of time; however, a wake develops behind the foil which makes the solution vary in time. At a particular instant the orientation of each foil to the
flow requires a certain value of circulation to satisfy the Kutta condition. A short time later the rotor will be in a slightly different position requiring different circulation. Conservation of vorticity causes a vortex to be shed into the wake of each foil, equal and opposite to the change in bound vorticity. This process of vortex shedding from the trailing edge is assumed to occur simultaneously with the change in bound vorticity. As time progresses these shed vortices are moved along by the flow and form a large wake downstream, which has a significant effect on the performance of the rotor. In order to predict the performance of the foils, a start-up period must be determined until steady state conditions are approached. Thus a series of instantaneous solutions separated by a short time interval must be found as the rotor turns. Overall performance quantities are determined by time averaging these solutions over one revolution. An understanding of the particular flow mechanisms occurring may be obtained by viewing a series of instantaneous solution streamlines separated by a small time interval. This complete process must be repeated for each change in rotor geometry, wind velocity and rotational speed.

2.2 Methods of Solution

Two classical methods of solution are suggested: conformal transformation of simple flows and thin aerofoil theory. Both will be developed more fully later in this writing, but are outlined briefly here. The former method seeks a simple related flow solution using flow singularities, which in this case might be two source doublets with circulation in a uniform stream. Next, the body surfaces are determined by the stagnation
streamlines and a conformal transformation is found which will map these surfaces into the desired geometry. Then the transform is used to produce the desired solution from the basic flow.

Thin aerofoil theory is based on the premise that when bodies are long and thin in the flow direction, the effect of body thickness may be found separately from the effects of body curvature and orientation and the two solutions superposed with negligible error. Since the performance of many thickness profiles is well documented, and they generally effect local velocities and not foil forces, the main problem for consideration becomes that of solving for the flow about two thin foils defined by the centerlines of the original bodies. Such a solution is achieved by placing a particular vortex distribution on these mean lines so that they become streamlines. This method produces most of the interesting performance information; however, a quantity depending on real fluid effects such as maximum lift coefficient would have to be determined experimentally or by combining the mean line and thickness solutions and performing a boundary layer analysis.[17]

If one has the good fortune to find a satisfactory transformation, the method of simple flow transformation can offer several advantages. First, the complete solution including body thickness is found directly, following a series of mathematical steps without the need for human judgement. In thin aerofoil theory one is invariably faced with the decision of singularity size and location in a discrete model or of series truncation in a continuous model. Second, the solution of the
basic flow is much more simple, saving time due to the large number of solutions required. Once the transformation is found, it may be used repeatedly to map to the desired geometry.

2.3 Reference Axes

Before proceeding with the solution details, it will be useful to review the axis systems and their effect on the velocity field. Although the entire solution can be obtained using an inertial axis system, there are distinct advantages to choosing a reference frame that rotates with the rotor and has the origin placed at the center of rotation. This slightly simplifies the boundary conditions; because the foils do not move with respect to the axes, hence are no longer functions of time. The rotating stagnation streamlines follow the body outlines and the other streamlines flow past in the conventional manner. Any other axis system will have streamlines passing through the foils, thereby making interpretation much more difficult. The possible complexities of unsteady flow and a developing wake in this problem make expression of the solution in rotating axes desirable.

Consider velocity fields $\mathbf{U}$ and $\mathbf{U}'$ in both axis systems, where the prime denotes the non-inertial frame, rotating at rate $\Omega$. $\mathbf{R}$ is the position vector locating the origin of the moving system, while $\mathbf{r}'$ locates the position of a point with respect to the rotating axes (Figure 8). The inertial velocity $\mathbf{U}$ may be found from the moving frame velocity $\mathbf{U}'$ by:

$$\mathbf{U} = \mathbf{U}' + \frac{d\mathbf{R}}{dt} + \Omega \times \mathbf{r}'$$  \hspace{1cm} (2.5)
For this problem it is convenient to have the same origin for both axes so $\dot{\mathbf{R}} = 0$ and:

$$\dot{\mathbf{U}} = \dot{\mathbf{U}}' + \dot{\mathbf{x}} \dot{\mathbf{r}}'.$$

(2.6)

Restricting $\mathbf{r}'$ to two dimensions and using $\mathbf{i}, \mathbf{j},$ and $\mathbf{k}$ as orthogonal unit vectors, then $\dot{\mathbf{r}}' = x'i + y'j$ and $\dot{\mathbf{U}} = \dot{\mathbf{r}} k$, positive counterclockwise, hence:

$$\dot{\mathbf{U}} = \dot{\mathbf{U}}' + \dot{\mathbf{r}} k (x'i + y'j)$$

$$= \dot{\mathbf{U}}' + \omega y'i - \omega x'j$$

or,

$$\dot{\mathbf{U}}' = \dot{\mathbf{U}} - \omega y'i + \omega x'j.$$  

(2.7)

Hence, when finding the velocity in the rotating axis system, the velocity components $-\omega y'$ and $\omega x'$ must be added to the inertial velocity. The transformation relating positions in the two frames is:

$$x = x'\cos\theta - y'\sin\theta$$

$$y = y'\cos\theta + x'\sin\theta$$

(2.8)

where $\theta$ is the angle from the inertial to the rotating frame.

In two dimensional inviscid flow it is often convenient to work in a complex plane, $z = x + yi$, and to define a complex potential, $W(z) = \phi + iv$, so that $\frac{dW}{dz} = V$, where $V$ is the complex velocity, the conjugate of the real velocity. The complex velocity in the rotating frame becomes:

$$V' = V - \omega \overline{z}'i = \frac{dW}{dz} - \omega \overline{z}'i$$

(2.9)

where $\overline{z}'$ is the complex conjugate of $z'$. Since $W(z)$ must be an analytic function and $\overline{z}$ is not analytic, a complex potential can not exist in the rotating system. The last term represents solid body rotation as seen by
the rotating frame, which is clearly a violation of irrotational flow, \( \nabla^2 \psi = 0 \).

Consider the function \( \psi_r \) such that \( \frac{\partial \psi_r}{\partial y} = \Omega y \) and \( \frac{\partial \psi_r}{\partial x} = \Omega x \). These form exact differentials so that \( d\psi = \frac{\partial \psi_r}{\partial x} dx + \frac{\partial \psi_r}{\partial y} dy \), if \( \frac{\partial^2 \psi_r}{\partial y \partial x} = \frac{\partial^2 \psi_r}{\partial x \partial y} \), so that one may find \( \psi_r = \Omega \frac{x^2}{2} + y^2 \). The requirement that continuity be satisfied, \( \nabla \cdot U = 0 \), is

\[
\frac{\partial \psi}{\partial y} = \frac{2u}{\partial x}
\]

which is consistent with exact differential theory when \( \frac{\partial \psi}{\partial x} = -v \) and \( \frac{\partial \psi}{\partial y} = u \), also consistent with our former definition of the stream function. This demonstrates that \( \psi_r \)

satisfies continuity when \( \frac{\partial \psi_r}{\partial y} = \Omega y = u_r \) and \( \frac{\partial \psi_r}{\partial x} = -\Omega x = v_r \), \( u_r \) and \( v_r \)

being the components of \( \Omega x \) derived above.

The velocity in the rotating frame may be considered the sum of a rotational and an irrotational velocity field, namely,

\[
\mathbf{U}' = \mathbf{U}_i + \mathbf{U}_r
\]

where \( \mathbf{U}_i \) satisfies \( \nabla \cdot \mathbf{U}_i = 0 \) and \( \nabla \times \mathbf{U}_i = 0 \)

\( \mathbf{U}_r \) satisfies \( \nabla \cdot \mathbf{U}_r = 0 \) and \( \nabla \times \mathbf{U}_r = \omega \),

\( \omega \) specifying the vorticity everywhere in the flow. \( \mathbf{U}_i \) is the irrotational solution for inertial axes and \( \mathbf{U}_r = \Omega x \mathbf{r} \), which from above has the stream function \( \psi_r \); hence, the stream function for rotating axes may be written as:

\[
\psi' = \psi_i + \psi_r
\]

\[
= \psi_i + \frac{\Omega}{2}(x^2 + y^2)
\]

This relation, not emphasized by many texts on fluid dynamics, permits an inertial solution to be readily transformed to rotating axes.
Using this stream function transformation to view a simple uniform stream as seen from both axes systems will aid interpretation of the streamlines when altered by the presence of the rotor. (See Figure 9 for typical streamlines and equations.)

In the rotating frame all streamlines are concentric circles whose center is offset perpendicular to the uniform stream to where the velocity of rotation equals $U_0$. Since the streaming flow is always parallel to the x axis in the inertial axes, it will appear to be rotating at $-\Omega$ in the frame rotating at $\Omega$. 

FIGURE 9 - Streamlines and Equations

Inertial Axes

a) No flow
$\psi = 0$

b) Uniform Stream
$\psi = U_0 y$

Rotating Axes

$\psi' = \frac{\Omega(x'^2 + y'^2)}{2}$

$\psi' = U_0(y'\cos\omega t + x'\sin\omega t) + \frac{\Omega}{2}(x'^2 + y'^2)$

$\Omega = \Omega_0$

$\tilde{U} = U_0$

$t = 0$

$\tilde{\Omega} = \Omega_0$

$\tilde{U} = U_0$

$t = \frac{\pi}{2\Omega}$
CHAPTER III

SIMPLE FLOW WITH TRANSFORMATION

3.1 Solution Planes

The method pursued here is similar to Theodorsen's analysis of aerofoils except that it has been expanded to consider two bodies in unsteady flow. First, a general solution is found in an inertial axis system for two source-sink doublets with circulation rotating about the origin at \( \omega \) and placed in a uniform stream, \( U_0 \). This solution plane will be referred to as the \( z \) plane. Next, the solution is transformed to a reference frame rotating with the rotor, the \( z' \) plane, so that the stagnation streamlines will define the body shapes. The body contours will be ovals that approach circles as the distance between them increases. The actual geometry desired is represented in the \( \zeta \) plane and a Joukowski type transformation is used to map the arbitrary foil shapes in the \( \zeta \) plane to nearly circular bodies in the \( z'' \) plane. A second transformation is required to map the contours in the \( z' \) plane to the \( z'' \) plane. This sequence is illustrated in Figure 10.

A transformation must be an analytic function throughout the flow field, say \( \zeta = f(z'') \). If the velocity, \( V_{z''} \), is known in the \( z'' \) plane, the velocity in the \( \zeta \) plane, \( V_\zeta \), may be found by

\[
V_\zeta = \frac{V_{z''}}{d\zeta/dz''}
\]

(3.1)
FIGURE 10 - Solution Transformations

Transformation 2

Transformation 1
Once the transformations are known, the velocity field in the solution plane can be easily transferred to the real plane. Vortices shed from the bodies in the $z$ plane can similarly be mapped to the $\zeta$ plane forming a wake.

3.2 Simple Flow

The flow field in the $z$ plane as represented by the complex potential, $W(z)$, may be formed from the superposition of basic flows.[23] A uniform stream paralleled to the $x$ axis is

$$W_1(z) = U_0 z.$$  \hspace{1cm} (3.2)

The complex potential for a source doublet is

$$W_2(z) = \frac{Va^2}{z-z_0}$$

where: $z_0 = R_0 e^{i\omega t}$ is the position of a doublet rotating about the origin at $\omega$.

$$V = U_0 - iR_0 e^{i\omega t}$$ is the fluid velocity relative to the doublet.

$a$ is the nominal radius of the body formed by the doublet.

Hence,

$$W_2(z) = (U_0 - iR_0 e^{i\omega t}) \frac{a^2}{z-z_0}$$ \hspace{1cm} (3.3)

Circulation, $K$, may be added by placing a point vortex at $z_0$ giving

$$W_3(z) = -\frac{iK}{2\pi} \log\left(\frac{z-z_0}{a}\right)$$ \hspace{1cm} (3.4)

Combining the above potentials for two rotating cylinders at $z_0$ and $-z_0$, one obtains:
The complex velocity is found by taking \( \frac{dW}{dz'} \).

Transforming to the rotating axes follows directly from the discussion about axis systems in Section 2.3, where it was shown that a velocity potential did not exist. Since \( W(z) = \phi + i\psi \), the stream function in the rotating frame becomes

\[
\psi' = \text{Im}(W(z)) + \frac{\Omega}{2}(x^2 + y^2) \quad \text{or} \quad \psi' = U_0(y'\cos\Omega t + x'\sin\Omega t)
\]

\[
\begin{align*}
&= U_0\sin(\Omega t)(\frac{a_2^2(x' - R_0)}{(x' - R_0)^2 + y'^2} + \frac{a_2^2(x' + R_0)}{(x' + R_0)^2 + y'^2}) \\
&\quad - U_0\cos(\Omega t)(\frac{a_1^2x'}{(x' - R_0)^2 + y'^2} + \frac{a_2^2y'}{(x' + R_0)^2 + y'^2}) \\
&\quad = \frac{\Omega R_0}{2}(\frac{a_1^2(x' - R_0)}{(x' - R_0)^2 + y'^2} - \frac{a_2^2(x' + R_0)}{(x' + R_0)^2 + y'^2}) \\
&\quad - \frac{K_1}{2}\log(x' - R_0)^2 + y'^2 - \frac{K_2}{2}\log (x' + R_0)^2 + y'^2 \\
&\quad + \frac{\Omega}{2}(x'^2 + y'^2)
\end{align*}
\]

where: \( z = z'e^{i\Omega t} \) and \( z' = x' + iy' \)

The velocity components are:
\[
\begin{align*}
\mathbf{u}' = \frac{\partial \psi'}{\partial y'} & \quad \text{and} \quad \mathbf{v}' = -\frac{\partial \psi'}{\partial x'}
\end{align*}
\] (3.7)

\(R_0, U_0, n, \) and \(t\) will be specified for a particular condition and the four unknowns \(a_1, a_2, K_1,\) and \(K_2\) will have to be determined. The transformations can be used to locate the points in the solution plane which correspond to the trailing edges in the real plane. Requiring these points to be stagnation points produces four equations, namely, that both velocity components at each trailing edge be zero. Should the flow separate so that \(K\) equals zero, \(a\) may be determined by requiring the points to lie on the body surface, i.e., the stagnation streamline.

3.3 Transformation 1

For the first transformation a relation \(\zeta = f(z'')\) is needed that will map semi-circular arcs or aerofoil shapes in the \(\zeta\) plane into closed curves in the \(z''\) plane and have no effect as \(z''\) approaches infinity. A cusp will appear in the \(\zeta\) plane where \(d\zeta/dz''\) equals zero.\[12\]

Consider poles located within the body in the \(z''\) plane at \(B\) and \(-B\) and zeros at \(B \pm C\) and \(-B \pm C\), Figure 11. Hence, an attempt to find a transformation could proceed as follows:

\[
\frac{d\zeta}{dz''} = \frac{(z'' - B - C)(z'' - B + C)(z'' + B - C)(z'' + B + C)}{(z'' - B)^2(z'' + B)^2}
\] (3.8)

Evaluating \(\zeta = \int d\zeta/dz'' dz''\):

\[
\zeta = z'' + \frac{C^2}{z'' + B} + \frac{C^2}{z'' - B} + \frac{C^4}{2B(z'' - b)(z'' + B)} - \frac{C^4}{2B^2(z'' - b)}
+ \frac{C^4}{4B^3} \log\left(\frac{z'' + B}{z'' - B}\right)
\] (3.9)
B and C must be determined so that the cusps occur at \( z_1 \) and \( z_2 \) and then the curve in the \( \zeta \) plane mapped to find the curve in the \( z'' \) plane. Substituting \( z'' = B + C \) and \( z'' = B - C \) into the transform gives

\[
\begin{align*}
B + C &= B + C + \frac{B + C}{C(C + 2B)} (2C^2 - \frac{C^4}{2B^2}) + \frac{C^4}{4B^3} \log\left(\frac{C + 2B}{C}\right) \\
B - C &= B - C + \frac{B - C}{C(C - 2B)} (2C^2 - \frac{C^4}{2B^2}) + \frac{C^4}{4B^3} \log\left(\frac{C - 2B}{C}\right)
\end{align*}
\tag{3.10}
\]

When separated into real and imaginary parts, these four equations may be solved for the real and imaginary parts of B and C. The Newton-Ralphson iterative method of solution may be applied if the equations are expressed as follows:

\[
\begin{align*}
F_1 &= f_1(B_r, B_i, C_r, C_i) - \zeta_{1r} = 0 \\
F_2 &= f_2(B_r, B_i, C_r, C_i) - \zeta_{2r} = 0 \\
F_3 &= f_3(B_r, B_i, C_r, C_i) - \zeta_{1i} = 0 \\
F_4 &= f_4(B_r, B_i, C_r, C_i) - \zeta_{2i} = 0
\end{align*}
\tag{3.11}
\]

Where the subscripts denote the real and imaginary parts.

The method expands these equations about the first estimated values, using a Taylor series of which second derivative and higher terms are omitted, and solves for the incremental change for the next estimate.

\[
F_1 + B_r \frac{\partial F_1}{\partial B_r} + B_i \frac{\partial F_1}{\partial B_i} + C_r \frac{\partial F_1}{\partial C_r} + C_i \frac{\partial F_1}{\partial C_i} = 0
\tag{3.12}
\]

etc. \( F_2, \ldots, F_4 \)
The above set of linear equations may be solved using Cramer's rule

\[
\begin{vmatrix}
F_1 & F_{1B_1} & F_{1C_r} & F_{1C_i} \\
F_2 & F_{2B_1} & F_{2C_r} & F_{2C_i} \\
F_3 & F_{3B_1} & F_{3C_r} & F_{3C_i} \\
F_4 & F_{4B_1} & F_{4C_r} & F_{4C_i}
\end{vmatrix}
\]

\[
\Delta B_r = \frac{a(F_1, F_2, F_3, F_4)}{a(B_r, B_i, C_r, C_i)}
\]

etc. \(\Delta B_1 \ldots \Delta C_i\)

Provided \(\frac{a(...)}{a(\ldots)} \neq 0\)

Where \(\frac{a(...)}{a(\ldots)}\) is the Jacobian coefficient determinant and the subscripts imply partial differentiation.

The new estimate is

\[
B_{r \text{ new}} = B_{r \text{ old}} + \Delta B_r
\]

etc. \(B_i \ldots C_i\)

The process is repeated until convergence.

After finding \(B\) and \(C\) by iteration it will be desirable to transform a set of offsets, \(\zeta_n\), describing the body shape in the real plane, to their corresponding points, \(z''_n\), in the \(z''\) plane. Since such a complicated transformation is not readily inverted it will be useful to use the Newton-Ralphson method again. The transformation is represented in the form
FIGURE 11 - Transformation 1

FIGURE 12 - Sample Transformation
\[ G_1 + iG_2 = 0 \]  

where  
\[ G_1 = g_1(z''_r, z''_i) - \zeta_{nr} = 0 \]
\[ G_2 = g_2(z''_r, z''_i) - \zeta_{nr} = 0 \]

The new estimates are

\[ z''_{nr \text{ new}} = z''_{nr \text{ old}} - \frac{\partial (G_1, G_2)}{\partial (z''_r, z''_i)} \]

\[ G_1 z''_r \quad G_1 \\
G_2 z''_r \quad G_2 \]

\[ z''_{ni \text{ new}} = z''_{ni \text{ old}} - \frac{\partial (G_1, G_2)}{\partial (z''_r, z''_i)} \]

This method converges quite nicely, but is tedious due to the number of partial derivatives which must be evaluated.

Using the above technique and a typical Savonius rotor shape the curves in Figure 12 are produced. The computer program to solve for the constants B and C was 300 lines long in Fortran '76 and compiled and executed in five seconds at Kiewit Computation Center, Dartmouth College. The inverse mapping program was likewise 200 lines long and ran in under ten seconds. The presence of the complex logarithm in the transformation chosen for this example should make the results no surprise. Since the log is multivaluated, traveling around B or \(-B\) adds \(2\pi\) making it impossible for the curves in the \(z''\) plane to be closed and omitting the shaded region in the \(\zeta\) plane from
the transformation. The mapping could be completed by passing through a cut between $-B$ and $B$ in the $z''$ plane onto another plane to complete the flow region in the $\zeta$ plane; however, finding a transformation to return to the solution plane seems unlikely. Hence, a new transformation without multi-valued functions must be used.

A transform such as the following is a likely choice.

$$\zeta = z'' + \sum_{1}^{n} A(n)(\frac{1}{z'' - B(n)} + \frac{1}{z'' + B(n)})$$  \hspace{1cm} (3.17)

where \(\frac{\delta \zeta}{\delta z''} = 0 = 1 - \sum_{1}^{n} A(n)(\frac{1}{(z'' - B(n))^2} + \frac{1}{(z'' + B(n))^2})\)

at $\zeta_1$ and $\zeta_2$.

As the number of poles reaches six or more, closed curves resembling those in the $z''$ plane of Figure 12 are possible.

3.4 Transformation 2

Finally a transformation mapping the closed curves in the $z'$ plane to the $z''$ plane is required. The calculations will be simplified if the centers of the bodies and the stagnation points remain unchanged, since they are used to find a solution in the $z$ plane and can be derived from the $\zeta$ plane. In Figure 13 $P_1$ and $P_2$ are the position of the doublets in the $z'$ plane and the center of the bodies in the $z''$ plane. $S_1$ and $S_2$ are the trailing edge stagnation points. A transformation is required which is an analytic function in the flow region with the following properties:

$$z'' = f(z')$$ such that
FIGURE 13 - Transformation 2
\[ f(z') = z' \text{ at infinity, at } z' = S_1 \text{ and } S_2, \text{ and} \]
\[ \text{at } z' = P_1 \text{ and } P_2 \]
\[ f(C') = C'' \text{ mapping the curves from one plane to the other.} \]

Consider the function
\[ z'' = (z' - P)F_1 + (z' + P)F_2 \]  
(3.18)
where \( F_1 \) and \( F_2 \) are real functions of \( z' \)
\( (z' - P) \) and \( (z' + P) \) are position vectors from \( P = P_1 \) and \( -P = P_2 \).
\( F_1 \) and \( F_2 \) may then be chosen as scaling functions to locally distort the boundary to the desired shape. Such a function may be formed by considering the Fourier series so that
\[ F_1 = A_1^0 + \sum_{n} \alpha_1^n \frac{a_1^n}{r_1^n} \left( A_1^n \cos n\phi_1 + B_1^n \sin n\phi_1 \right) \]
\[ + A_3^0 + \sum_{n} \alpha_2^n \frac{a_2^n}{r_2^n} \left( A_3^n \cos n\phi_2 + B_3^n \sin n\phi_2 \right) \]
\[ F_2 = A_2^0 + \sum_{n} \alpha_2^n \frac{a_2^n}{r_2^n} \left( A_2^n \cos n\phi_2 + B_2^n \sin n\phi_2 \right) \]
\[ \quad + A_4^0 + \sum_{n} \alpha_1^n \frac{a_1^n}{r_1^n} \left( A_4^n \cos n\phi_1 + B_4^n \sin n\phi_1 \right) \]  
(3.19)
where
\[ r_1 = z' - P \quad r_2 = z' + P \]
\[ = \arg(z' - P) \quad = \arg(z' + P) \]
\[ a_1 = \frac{1}{2\pi} \int_{c_1} r_1 d\phi_1 \quad a_2 = \frac{1}{2\pi} \int_{c_2} r_2 d\phi_2 \]
requiring that
\[ (z' - P)F_1 = C_1'' \quad \text{on } z' = C_1' \]
\[ = 0 \quad \text{on } z' = C_2' \]
After determining the transformations, equation (3.1) can be used to determine the velocity field in the real plane from the flow in the solution plane (equation (3.5)).

3.5 Solution

In concluding the description of this method some operational observations are in order. Although all steps in the solution tie together quite well, the added complexity of going from one to two bodies made the transformation cumbersome. For every change in the transform three computer programs were written to evaluate its performance; one to determine the various transformation constants, one to plot curves, and another to invert the transformation. These programs were fairly lengthy and required a large amount of algebra and partial differentiation before writing. This made feedback from new ideas disappointingly slow. The principal difficulty came in mapping the body contours with the inverted transformation. A computer program incrementally followed the body contour using previous points to determine the next estimate and then iterated to the correct mapping. Although the transformation is single-valued in the flow field, each pole adds a multiple value within the body. As poles were added to improve the transformation, the inversion program would be more likely to skip to a point within the body and follow some internal curve. This required the operator to continuously monitor
the results and redirect the mapping to the body contour, making the process quite time consuming. Rather than proceeding to develop the computational complexity to surmount this difficulty, a look at thin aerofoil theory was considered desirable to see if the solution followed more easily.
CHAPTER IV

THIN AEROFOIL THEORY

4.1 Method of Solution

The basis of thin aerofoil theory is that the flow over the camber line of a foil and over an uncambered thickness profile may be determined separately and superposed to produce the total flow pattern. The camber line solution determines the predominant forces, while the thickness profile locally modifies velocity and pressure gradients controlling the development of the boundary layer. The following analysis will concentrate on the camber line solution.

The performance of a single foil has been determined by placing a vortex sheet of continuously varying strength along the mean line in a uniform stream. [12,23] The boundary condition of zero normal velocity on the mean line is used to determine the vortex strength distribution. The mathematical complexity of considering two foils is such that using a discrete model to approximate the continuous case is more compatible with current numerical methods. Two models have been developed; the first places a series of point vortices along the camber line and in the wake. The second is a refinement of the first, which models the camber line and wake as a string of vortex sheets of linearly varying strength. The streamlines produced by a series of point vortices and a vortex sheet are shown in Figure 14. The local flow is quite different both in magnitude and
FIGURE 14 - Flow Near Vortices

FIGURE 15 - Notation for Point Vortex
direction, but the two flows rapidly approach each other as the
distance from the foil surface increases. When the spacing between
the vortices is reduced the local distortion can be contained to a
region very near the foil.

4.2 Point Vortex Model

Due to its relative simplicity, the point vortex model will
be developed first. The foil shape is described by a series of
coordinates and the point vortices are placed in between. The
specification of no normal velocity at each foil offset is not
adequate to prevent flow through the foil. Requiring that all
the coordinates of a foil lie on a streamline will essentially
prevent flow through the mean line provided that their spacing is
close enough so that the streamline follows the camber line. By
keeping both above specifications, namely, no normal flow and constant
value of the stream function on the foil, two vortices may be
specified between each pair of offsets, thereby doubling the
number of vortices for a given set of coordinates. The vortices
are evenly spaced along a straight line connecting each pair of
offsets. See Figure 15.

The stream function in inertial axes may be created by the super-
position of a uniform stream, \( U_0 \), flowing parallel to the \( x \) axis
and a series of point vortices. This is the imaginary part of equations
(3.2) and (3.4) summed over all point vortices in the flow field.

\[
\psi = U_0 y - \sum_{n} \frac{K_n}{2\pi} \log R_n
\]
where: $K_n$ is the strength of the $n^{th}$ vortex

$$R_n = \sqrt{(x - x_n \cos \alpha + y_n \sin \alpha)^2 + (y - y_n \cos \alpha - x_n \sin \alpha)^2}$$

the distance from a point at $x,y$ to the vortex at $x_n,y_n$ when $t = 0$.

Transforming to rotating axes the stream function becomes (see Section 2.3):

$$\psi' = U_0 (y' \cos \alpha + x' \sin \alpha) - \sum \frac{K_n}{2\pi} \log \left( \frac{R_n}{2\pi} \right) log(r')^2 + (y' - y_n')^2$$

$$+ \frac{\Omega}{2} (x'^2 + y'^2)$$ (4.1)

The velocity components are

$$u' = \frac{\partial \psi'}{\partial y'} = U_0 \cos \alpha - \sum \frac{K_n}{2\pi} \frac{y' - y_n'}{(x' - x_n')^2 + (y' - y_n')^2} + \Omega y'$$ (4.2)

$$v' = -\frac{\partial \psi'}{\partial x'} = U_0 \sin \alpha + \sum \frac{K_n}{2\pi} \frac{x' - x_n'}{(x' - x_n')^2 + (y' - y_n')^2} - \Omega x'$$ (4.3)

Now equations specifying the boundary conditions may be written.

The requirement of zero normal velocity applies to all coordinates $(x'_i,y'_i)$ except at the leading and trailing edges. When adjoining segments as specified by the coordinates meet at an angle the normal is considered to be the bisector of the angle. The normal velocity is the scalar product of the normal $N_i$ and the velocity, hence;

$$N_i \cdot U'_i = 0 \text{ at (}x'_i,y'_i\text{)} \quad i = 2 \ldots \text{npt - 1}$$ (4.4)

where npt is the number of camber line coordinates per foil

$$U'_i = u'_i + v'_i$$ is the velocity at $(x'_i,y'_i)$ from the preceeding equations summed over all vortices
Requiring all of the offsets on a foil to be on a streamline may be written as the stream function equaling a constant

\[ \psi'_1 = \psi'_1 \quad \text{at} \ (x'_1, y'_1) \quad i = 1 \ldots \text{npt on foil 1} \]
\[ \psi'_1 = \psi'_2 \quad \text{at} \ (x'_1, y'_1) \quad i = 1 \ldots \text{npt on foil 2} \]  

(4.5)

When the flow along a foil is attached, the Kutta condition may be used to determine the total circulation about that foil. Non-infinite velocity at the trailing edge can only be achieved when the normal velocity is zero, hence;

\[ \mathbf{N}' \cdot \mathbf{U}' = 0 \quad \text{at the trailing edge} \]  

(4.6)

Should this specification produce a physically impossible circulation on a foil, the flow is considered to have separated and the total circulation on that foil will be zero.

\[ \sum_{n=1}^{\text{npt}} K_n = 0 \quad \text{on stalled foil} \]  

(4.7)

For npt coordinates on each foil there will be npt - 1 connecting segments and 2(npt - 1) vortices. The number of unknowns on both foils is; 4(npt - 1) vortex strengths, K, and the stagnation stream function values \( \psi'_1 \), \( \psi'_2 \), totaling 4npt - 2. Checking the number of equations, there are 2(npt - 2) for normal velocity, 2npt for the stream function, and 2 for the Kutta condition or total circulation whichever is applicable, making 4npt - 2. Hence there are a sufficient number of equations to determine all unknowns.

Due to the rotation of the foils, their bound vorticity will be continually changing, requiring a series of vortices to be shed
from the trailing edges. For each increment of rotation a new
pair of vortices must be shed and the remaining wake carried
downstream one step. This process adds two equations and unknowns,
namely;

\[ \sum_{n} K_{n \text{ new}} + K_{w \text{l}} = \sum_{n} K_{n \text{ old}} \text{ for each foil} \]  

(4.8)

where: \( K_{n} \) is the bound vortex strength

\( K_{w \text{l}} \) is the vortex being shed into the wake

The strengths of the remaining wake vortices do not change so their
position may be determined by calculating the fluid velocity at
each wake vortex location and finding how far each point moves in
the time increment between solutions. Due to the curvature of
the streamlines, a predictor-corrector method will produce significantly
better results; hence, the velocity is calculated at the new estimated
position and averaged with the first velocity to find the final
position.

The complete set of equations for the point vortex model are;

a) \( N_{i} \cdot U_{i} = 0 \) at \((x_{i}^{1}, y_{i}^{1})\) \( i = 2 \ldots \text{npt} - 1 \) on each foil

b) \( u_{i}^{1} = u_{j}^{1} \) at \((x_{i}^{1}, y_{i}^{1})\) \( i = 1 \ldots \text{npt} \) on each foil

c) \( N_{i} \cdot U_{i} = 0 \) at the trailing edge if flow attached or

\( K_{n} = 0 \) if stalled

d) \( \sum_{n} K_{n \text{ new}} + K_{w \text{l}} = \sum_{n} K_{n \text{ old}} \) on each foil

(4.9)

These form a set of 4npt linear equations which may be solved employing
the usual matrix techniques.
In matrix form the equations for the point vortex model may be written as:

\[(A) \cdot (X) = (C)\]  \hspace{1cm} (4.10)

where: the coefficient matrix is

\[
(A) = \begin{pmatrix}
A_{11} & A_{12} & A_{13} & \cdots & 0 & 0 & A_{1n-1} & A_{1n} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
B_{1} & B_{12} & B_{13} & \cdots & 1 & 0 & B_{1n-1} & B_{1n} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
1 & 1 & \cdots & 0 & \cdots & 0 & 0 & 1 & 0 \\
0 & 0 & \cdots & 1 & \cdots & 0 & 0 & 0 & 1
\end{pmatrix}
\]

the unknown matrix is

\[(X) = (K_1 \ K_2 \ \cdots \ \psi_1 \ \psi_2 \ \psi_{\text{shed}} \ 1 \ \psi_{\text{shed}} \ 2)\]

and the constant matrix is

\[
(C) = \begin{pmatrix}
C_1 \\
\vdots \\
C_2 \\
\vdots \\
K_1 \ \text{old} \\
K_2 \ \text{old}
\end{pmatrix}
\]

which contains those terms in the equations of motion that do not contain unknowns including contributions from the wake.

4.3 **Vortex Sheet Model**

The corresponding development for the linear vortex sheet model will be outlined as follows. The same set of coordinates specifying the mean line is used. A vortex sheet is placed on
straight line segments connecting the points instead of two vortices. The analysis is similar, although the calculations are more complicated because the segments are connected and the orientation of the vortex sheet to the point where the velocity is calculated is important. If the vortex strength were constant over each segment, there would be steps in the strength at the end of each segment which would produce infinite normal velocities at these points. By requiring adjacent segments to have the same strength at their junction as shown in Figure 16 the infinite velocity is avoided. The velocity at the junction that is produced by a small section of the vortex sheet an infinitesimal distance to one side of the joint is opposed by the corresponding section on the other side of the joint.

The stream function for a vortex sheet may be found by distributing a point vortex along a line segment and integrating over the segment (imaginary part of equation 3.4), and the velocity components by partial differentiation.

\[ \psi_V = -\frac{1}{2\pi} \int_S \gamma \log \sqrt{(\xi - x)^2 + (\eta - y)^2} \, ds \]  
\[ u_V = -\frac{1}{2\pi} \int_S \gamma \frac{\eta - y}{(\xi - x)^2 + (\eta - y)^2} \, ds \]  
\[ v_V = \frac{1}{2\pi} \int_S \gamma \frac{\xi - x}{(\xi - x)^2 + (\eta - y)^2} \, ds \]

where \((x,y)\) defines the locus of the vortex sheet \(S\)

\(\gamma\) is the vortex strength per unit length so that the total circulation is \(\Gamma = \int_S \gamma \, ds\)
FIGURE 16 - Vortex Sheet Notation

FIGURE 17 - Velocity Calculation at End of Vortex Segment
\((\xi, \eta)\) is the point of interest in the flow

Representing the vortex sheet as a straight line segment with linearly varying strength gives:

\[
y = a_n x + b_n
\]

\[
a_n = \frac{y_{n+1} - y_n}{x_{n+1} - x_n}
\]

\[
b_n = y_n - a_n
\]

\[
ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + a^2} \, dx
\]

\[
\Delta \gamma_n = \frac{y_{n+1} - y_n}{x_{n+1} - x_n}
\]

\[
\gamma = \gamma_n + \Delta \gamma_n (x - x_n)
\]

The stream function and velocity become;

\[
\psi_v = \frac{1}{2\pi} \int_{x_n}^{x_{n+1}} (\gamma_n + \Delta \gamma_n(x - x_n)) \log \sqrt{(\xi - x)^2 + (\eta - y)^2} \left(1 + \frac{a^2}{1 + a^2}\right) \, dx
\]

\[
u_v = -\frac{1}{2\pi} \int_{x_n}^{x_{n+1}} (\gamma_n + \Delta \gamma_n(x - x_n)) \frac{n - a_n x - b_n}{(\xi - x)^2 + (\eta - y)^2} \left(1 + \frac{a^2}{1 + a^2}\right) \, dx
\]

\[
v_v = \frac{1}{2\pi} \int_{x_n}^{x_{n+1}} (\gamma_n + \Delta \gamma_n(x - x_n)) \frac{\xi - x}{(\xi - x)^2 + (\eta - y)^2} \left(1 + \frac{a^2}{1 + a^2}\right) \, dx
\]

Arranging terms for ease in evaluating the integrals produces;

\[
\psi_v = -\frac{\sqrt{\xi}}{2\pi} \left[(\gamma_n - \Delta \gamma_n x_n) \int_{x_n}^{x_{n+1}} \log \sqrt{x^2} \, dx + \Delta \gamma_n \int_{x_n}^{x_{n+1}} X \log X \, dx\right]
\]
where: \[ A = \xi^2 + \eta^2 + b_n^2 - 2n b_n \]

\[ B = 2a_n b_n - 2n a_n - 2\xi \]

\[ C = a_n^2 + 1 \]

\[ X = Cx^2 + Bx + A \]

In evaluating the integrals the discriminant, \( Q = 4AC - B^2 \), will be needed. Expressing in terms of the above relations

\[ Q = 4(a_n \xi - (\eta - b_n))^2 \]

which being all real ensures that the discriminant is always greater than or equal to zero. When \( Q \) equals zero the point \((\xi, \eta)\) lies on the line \( y = a_n x + b_n \). It is necessary to evaluate the integrals for \( Q = 0 \) and \( Q > 0 \) in Appendix A. Because the line \( y = a_n x + b_n \) is undefined when the slope becomes infinite, which is the case when any vortex sheet segment curves parallel to the \( y \) axis, the above analysis must be repeated with the variable of integration switched from \( x \) to \( y \).

The complete stream function and velocity component equations for rotating axes are formed from the sum of uniform stream; vortex sheet, and rotational components, analogous to the formation of equations 4.1, 4.2, and 4.3 for the point vortex model.

\[ \tau' = U_0 (y' \cos \Omega t + x' \sin \Omega t) + \frac{\alpha}{2} (x'^2 + y'^2) \]

\[ - \frac{\pi}{n} \sqrt{\frac{\alpha^2}{2\pi}} \int_{x_n}^{x'_n} (\gamma_n + \Delta \gamma_n (x' - x'_n)) \log \sqrt{x'} \, dx' \]
The summation of the vortex segments includes the camber line, the shed vortex, and the wake, which have to be separated into knowns and unknowns to solve for the vortex strengths.

The final system of equations applied to each foil are:

a) All points on the foil are a streamline;
\[ u' = U_0 \cos \theta + \omega y' \]
\[ -\frac{1}{2\pi} \sum_{n=1}^{N} \int_{x_n'}^{x_{n+1}} (\gamma_n + \Delta \gamma_n (x' - x_n')) \frac{n - a_n x' - b_n}{x} \, dx' \]  
\[ (4.21) \]

b) Conservation of vorticity where \( \Gamma = \int_S \gamma \, ds \);
\[ \Gamma_{\text{bound new}} + \Gamma_{\text{shed}} = \Gamma_{\text{bound old}} \]  
\[ (4.24) \]

c) Lifting surface with a finite velocity at the trailing edge;
Without wake
\[ \gamma = 0 \]  
\[ \text{at trailing edge} \]  
With wake and shed vortex
\[ \gamma_{\text{bound}} = \gamma_{\text{shed}} \]  
\[ \text{at trailing edge} \]  
\[ (4.25) \]
\[ (4.26) \]

d) Stalled foil;
\[ \Gamma_{\text{bound}} = 0 \]  
\[ (4.27) \]

In matrix form the equations for the vortex sheet model may be written as:
\[ (A)(X) = (C) \]  
\[ (4.28) \]

where the coefficient matrix is
the unknown matrix is

\[(X) = (G_1 G_2 G_3 \cdots \psi_1 \psi_2 G_{\text{shed}}^1 G_{\text{shed}}^2)\]

and the constant matrix is

\[(C) = \begin{bmatrix} C_1 \\ \vdots \\ \gamma_{\text{bound old}} \\ \vdots \\ 0 \end{bmatrix}\]

Which contains those terms in the equations of motion that do not contain unknowns, including contributions from the wake.

Additional care is required with the vortex sheet wake. As the wake segments travel downstream they are assumed to remain straight line segments, but are allowed to change in length. The circulation per unit length, \(\gamma\), must be changed at each new time so that the total circulation of each segment, \(\Gamma = \int_S \gamma \, ds\), remains constant. Should the flow reverse direction along a foil, the string of wake segments must be terminated and a new string started from the other end of the foil. Also, should the bound vorticity remain constant as is the case when stalled, not only is the total shed circulation zero, but \(\gamma\) must equal zero. These wake manipulations can be handled by the addition of zero length or zero strength segments.
to the string of wake vortices.

4.4 Determination of Forces

After the flow solution has been determined, the rotor forces and torque at each time step are of particular interest. The velocity along the surface of the foils could be calculated and the Bernoulli theorem used to determine a pressure differential across each foil segment. These pressures may be summed over the foil to find the total force and torque. Since the leading edge of the mean line is sharp, these values will be incorrect, unless the leading edge singularity is accounted for, or the foil happens to be at the ideal angle of attack. This can be readily illustrated by considering a flat plate inclined to a uniform stream. A standard textbook calculation is to apply the Kutta condition to the trailing edge, find the velocity, and show that the integral of the pressure differential across the plate is $\rho V \ell$. Since a pressure force can only act normal to a surface, the force must be normal to the plate, clearly contradictory to both empirical and theoretical fluid dynamics, which requires the lift force to be normal to the incoming flow. An ideal fluid will have an infinite velocity at the sharp leading edge, and an infinitely negative pressure at that point. The leading edge produces a force along the plate which makes the lift force normal to the flow and must be considered when this method of force determination is used.

An alternate approach is to use the Blasius theorem and find the total force and moment by simply summing the residues. Since the
flow solution is not readily expressed as a Laurent series about a point, this method is not useful directly. Lagally's theorem,[25,23] which states that the force produced on the fluid by singularities, hence on the singularities by the fluid, is the sum of the forces produced by each singularity placed in a flow produced by all the other singularities except the particular one in question. This can be illustrated using the Blasius theorem and the theory of residues. Consider a complex potential

\[ W = f - \frac{i \Gamma}{2\pi} \log(z - a) \]  

(4.29)

where the last term is the potential for a vortex of strength \( \Gamma \) at \( a \) and \( f \) is the potential for all other singularities. Using the Blasius theorem,

\[ X - iY = i \frac{\partial}{\partial z} \int_S \frac{(dW)^2}{dz} ds \]  

(4.30)

on a contour, \( S \), around the vortex requires

\[ \frac{dW}{dz} = f' - \frac{i \Gamma}{2\pi} \frac{1}{z - a} \]  

(4.31)

When evaluated by the theory of residues the force is

\[ X - iY = i \rho |f'| \]  

(4.32)

where \( f' \) is the complex velocity due to all other singularities. Hence, the forces and the rotor may be determined by summing the forces produced by each singularity on the rotor as determined by that singularity and the flow at that point due to all other causes.

4.5 Arrangement for Programming

Due to the analytical complexity of the rotor in a stream, developing the previous solution in the form that will be programmed
for a digital computer should be an asset in following the details. The basic steps in solving the problem are:

1) Input of geometry, flow information, and initializations.
2) Calculating the wake position for the next time increment.
3) Setting up and solving the system of equations.
4) Checking that the solution is reasonable.
5) Calculating the forces and torque.

The following paragraphs trace the analysis for the linearly varying vortex sheet model. The point vortex model was developed in a similar manner, and has only minor differences in notation and file handling.

Necessary input information is; the stream velocity, \( U_0 \), the rotational speed, \( \Omega \), the time, \( T \), the number of coordinates used to describe the camber line, \( NPT \), and the number of coordinates locating the ends of the wake vortex segments, \( NW \). Files for these \( x \) and \( y \) coordinates and the vortex strength per unit length at each coordinate are created as \( ZR_{i,j} \), and \( \gamma_{i,j} \) respectively, where \( i \) indicates foil 1 or 2 and \( j \) is the number of the point starting at the leading edge of the foil and continuing through the wake for a total of \( NPT + NW \). \( ZR_{i,j} \) and \( ZI_{i,j} \) are considered to be the real and imaginary parts of the point \( Z_{i,j} \) in the complex plane. Since the vortex strength may be different on each side of the trailing edge, the file is set up so that \( Z_{i,NPT} = Z_{i,NPT + 1} \) and \( \gamma_{i,NPT} \) is the bound strength while \( \gamma_{i,NPT + 1} \) is the wake strength at the trailing edge. In other words, a zero width segment is placed at the trailing edge to simplify computations. For convenience,
the foils are assumed to be symmetrical, so the second foil is created within the program to be the image of the first. This could easily be changed to the more general case if desired.

In order to find the next solution the wake position must be updated to the new time, \( T = T_{\text{old}} + \Delta T \). The velocity is calculated at the end of each vortex segment in the wake, \( Z_{m,n} \ m = 1,2 \ n = \text{NPT} + 1 \ldots \text{NPT} + \text{NW} \) giving;

\[
U_{m,n} = U_0 \cos \omega T + \omega Z_{m,n} + \sum_{i=1}^{2} \sum_{j=1}^{\text{NPT}+\text{NW}-1} U_v i,j
\]

\[
V_{m,n} = -U_0 \sin \omega T - \omega Z_{m,n} + \sum_{i=1}^{2} \sum_{j=1}^{\text{NPT}+\text{NW}-1} V_v i,j
\]

where,

\[
U_v i,j = -\frac{\sqrt{C_{\mu}}}{2\pi} - \Delta \gamma_{m,n} a_{m,n} \int_{Z_{i,j}}^{Z_{i,j+1}} \frac{x^2}{X} + D \int_{Z_{i,j}}^{Z_{i,j+1}} \frac{dx}{X}
\]

\[
+ E \int_{Z_{i,j}}^{Z_{i,j+1}} \frac{dx}{X}
\]

\[
V_v i,j = \frac{\sqrt{C_{\mu}}}{2\pi} - \Delta \gamma_{m,n} \int_{Z_{i,j}}^{Z_{i,j+1}} \frac{x^2}{X} + F \int_{Z_{i,j}}^{Z_{i,j+1}} \frac{xdx}{X}
\]

\[
+ H \int_{Z_{i,j}}^{Z_{i,j+1}} \frac{dx}{X}
\]

from equations 4.18 and 4.19, letting \((\xi,\eta) = (Z_{R_{m,n}}, Z_{I_{m,n}})\). When the point where the velocity is being calculated falls on the end of the vortex sheet, then \( m = i \) and \( n = j \) or \( j + 1 \), and infinite velocities are avoided by noting that the strength of the adjacent segment is the same as the one being calculated at their common point. Hence, a uniform strength vortex may be removed from both segments, giving zero strength.
at the point, as illustrated in Figure 17.

The first estimated position at the new time is determined for each point:

\[
\begin{align*}
Z_{ER_{m,n}} &= ZR_{m,n} + \Delta T U_{I_{m,n}} \\
Z_{EI_{m,n}} &= ZI_{m,n} + \Delta T V_{I_{m,n}}
\end{align*}
\]  

The length of the segments will be different so that new vortex strengths must be calculated to give the same total circulation.

\[
\gamma_{m,n \text{ new}} = \gamma_{m,n \text{ old}} \frac{|Z_{m,n} - Z_{m,n-1}| + |Z_{m,n+1} - Z_{m,n}|}{|ZE_{m,n} - ZE_{m,n-1}| + |ZE_{m,n+1} - ZE_{m,n}|} \quad (4.35)
\]

Next, the velocity is calculated at each estimated position, \(ZE_{m,n}\) \(m = 1,2\), \(n = NPT + 1 \ldots NPT + NW\), using \(\gamma_{m,n \text{ new}}\), producing \(U_{2_{m,n}}\) and \(V_{2_{m,n}}\). The final position for the wake at the new time is determined by averaging both velocities; hence,

\[
\begin{align*}
Z_{R_{m,n \text{ new}}} &= ZR_{m,n \text{ old}} + \Delta T (U_{1_{m,n}} + U_{2_{m,n}})/2 \\
Z_{I_{m,n \text{ new}}} &= ZI_{m,n \text{ old}} + \Delta T (V_{1_{m,n}} + U_{2_{m,n}})/2
\end{align*}
\]  

and the vortex strengths adjusted as before.

Now the system of equations may be set up. In preparation for solution using matrix methods, the unknowns are listed in an array; the bound vortex strengths, \(\gamma_{1,n} n = 1 \ldots NPT\), \(\gamma_{2,n} n = 1 \ldots NPT\), the stream function value on each foil, \(\psi_{1}, \psi_{2}\), and the shed vortex strength, \(\gamma_{1,NPT + 1}, \gamma_{2,NPT + 1}\). The equations become:

a) The camberline of each foil must be a streamline; \(\psi' = \text{constant}\).
For each point on foil 1.

\[
\sum_{m=1,2} \sum_{n=1,NPT} \hat{\psi}_m, n \gamma_{m,n} + \psi_1 + \hat{\psi}_1, NPT+1 \gamma_{1,NPT+1} + \\
\hat{\psi}_2, NPT+1 \gamma_{2,NPT+1} = -U_0 Z_{1,i,j} \cos \omega T - U_0 Z_{2,i,j} \sin \omega T - \\
- \frac{a}{2} (Z_{R_{1,i,j}}^2 + Z_{I_{1,i,j}}^2)
\]  
(4.37)

For each point on foil 2.

\[
\sum_{m=1,2} \sum_{n=1,NPT} \hat{\psi}_m, n \gamma_{m,n} + \psi_2 + \hat{\psi}_1, NPT+1 \gamma_{1,NPT+1} + \\
\hat{\psi}_2, NPT+1 \gamma_{2,NPT+1} = -U_0 Z_{1,i,j} \cos \omega T - U_0 Z_{2,i,j} \sin \omega T - \\
- \frac{a}{2} (Z_{R_{1,i,j}}^2 + Z_{I_{1,i,j}}^2)
\]  
(4.38)

Where \( \hat{\psi}_{m,n} \) is the contribution to the stream function at \((\xi, \eta) = Z_{i,j}\) of two adjacent segments with unit strength at \(Z_{m,n}\) and zero strength at \(Z_{m,n-1}\) and \(Z_{m,n+1}\), equation 5.1.

b) Conservation of vorticity; \( \Gamma_{\text{bound new}} + \Gamma_{\text{shed}} = \Gamma_{\text{bound old}} \).

\[
\sum_{n=1,NPT} C_{1,1,n} \gamma_{1,n} + \frac{1}{2} |Z_{1,NPT+1} - Z_{1,NPT+2}| \gamma_{1,NPT+1} = \Gamma_{1 \text{ bound old}}
\]  
(4.39)

\[
\sum_{n=1,NPT} C_{2,2,n} \gamma_{2,n} + \frac{1}{2} |Z_{2,NPT+1} - Z_{2,NPT+2}| \gamma_{2,NPT+1} = \Gamma_{2 \text{ bound old}}
\]  
(4.40)

where \( C_{m,n} = \frac{1}{2} (|Z_{m,n-1} - Z_{m,n}| + |Z_{m,n} - Z_{m,n+1}|) \).

c) Kutta condition for each foil

1) Without wake

\( \gamma_{m,NPT} = 0 \)  
(4.41)
2) With wake and shed vortex

\[ \gamma_{m,NPT} = \gamma_{m,NPT+1} \quad (4.42) \]

3) Stalled foil

\[ \sum_{n=1}^{NPT} C_{m,n} \gamma_{m,n} = 0 \quad (4.43) \]

This system of equations is expressed in matrix form \((A)(X) = (C)\), where \((A)\) is a square matrix formed from the coefficients of the unknown terms, \((X)\) is the list of unknowns, and \((C)\) is formed from the constant terms on the right hand side of the equations. The Gauss eliminator method is used to solve for the unknowns \[26\].

Each solution must be checked to see if it is physically reasonable. The first question to ask is whether the point chosen to apply the Kutta condition is really the trailing edge. The direction of flow along a foil may be determined as follows. A chord vector, \(\vec{CV}\), is defined from the originally assumed leading edge to the trailing edge of the foil and the velocity, \(\vec{V}\), calculated at the middle of the camber line. If the scalar product, \(\overrightarrow{CV} \cdot \vec{V}\), is greater than zero, the Kutta condition should have been applied to the original trailing edge, and if less than zero to the original leading edge. If this is not the case, the Kutta condition must be applied to the other end of the foil. This is accomplished by reversing the order of the files \(ZR_{i,j}\), \(ZI_{i,j}\), and \(\gamma_{i,j}\) along the foil only, advancing the wake one step, and adding a zero strength segment next to the foil to terminate the wake at the old trailing edge so that a new wake segment may start from the other end. The system of equations must be solved again for the new condition.

After determining that the trailing edge condition is correct, the bound circulation must be checked to see if it is within the range possible...
for aerofoils. A simple model is chosen here so as not to prolong the execution time, though a more sophisticated model relating to the boundary layer development may prove desirable. Presently, it is assumed that the lift coefficient for each foil must lie between empirically determined maximum and minimum values for the flow to remain attached. Otherwise, the flow is assumed to have separated and the bound vorticity set to zero. No attempt has been made to model the separated wake region in the hopes that the ideal flow solution will not grossly misrepresent the performance of a real fluid. The concept of a lift coefficient, \( C_L = \frac{1}{2} \rho V^2 A \), on a foil in a varying velocity flow is confusing, since it is unclear what value to use for the velocity; however, it is intuitively meaningful. For this simple model, the average velocity, \( V_{av} \), is taken to be the sum of the free stream velocity and that produced by the rotation of the mid-point of the camber line, ignoring the effect of both foils. The lift coefficient per unit span is

\[
C_L = -2 \Gamma \frac{V_{av} \cdot CV}{|V_{av} \cdot CV|^2} \quad (4.44)
\]

Care must be taken with the signs so that the lift coefficient is positive when the lift force is predominately outward from the origin and in a positive torque sense, when the chord vector is defined to go from the predominately leading to trailing edges of the foil. Should the lift coefficient be out of the selected range, the appropriate changes are made in the system of equations and a new solution found.

This catastrophic stall model was found to produce alternating attached and separated solutions at small time increments, giving large oscillations in rotor torque values. When a small \( \Delta T \) is used, the shed vortex segment is short and yet must contain the entire bound vortex.
FIGURE 18

Abrupt Stall Model

Partial Stall Model
Such a high concentration of vorticity at the trailing edge when just past the stall point alters the flow enough so that the flow may be attached in the next solution. Real aerofoils exhibit an early linear increase at lift vs. angle of attack up to a maximum value. When stall occurs the lift typically drops suddenly to 1/2 to 3/4 of the maximum value and then gradually reduces to zero at an angle of attack of 90 degrees. The more realistic model used in the program is illustrated in Figure 18. At an angle of attack greater than stall the lift drops immediately to 60% of the maximum and then tapers to zero over twice the change in angle of attack as when the flow was attached. Since the actual angle of attack is undefined in this flow field, a similar behavior may be deduced by noting how much the attached flow lift coefficient produced in the initial solution exceeds the maximum lift coefficient, assuming a nearly linear relationship between the attached lift coefficients and the angle of attack. For example, when the attached lift coefficient is three times $C_{L \text{ max}}$ the lift is zero, when twice, is 30% $C_{L \text{ max}}$, etc. This new value of the lift coefficient is specified for the foil and a new solution obtained for the flow field.

The determination of the foil forces and torque follow Lagally's theorem. The velocity, $V_{i,j}$, is calculated at the middle of each vortex segment on the foils due to all singularities except that segment. The force produced by the segment is normal to $V_{i,j}$:

$$F_{i,j} = \rho|V_{i,j}||Z_{i,j} - Z_{i,j+1}| \left( \frac{\gamma_{i,j} + \gamma_{i,j+1}}{2} \right)$$

(4.45)

Here is it assumed that the segment size is small enough to avoid distributing the vorticity and velocity over the segment, so both are
considered to act at the middle. The force components, X and Y, are determined from the sum of $F_{ij}$ over each foil and the torque from the moment of these forces about the center of rotation.
5.1 Scope of Analysis

The results presented here fall into two categories; those used for model development and those for analyzing rotor performance. The purpose of the model development results is to determine the suitability of the approach and its sensitivity to various parameters. A typical Savonius rotor with semicircular foils and a .20 diameter slot is used for these tests. The cases studied here are as follows:

1. The point vortex model is used to test the program logic in determining the stall point, proper trailing edge, wake development, and accuracy of the analysis. A wide range of tip speed ratios are investigated to see if the model predicts observed trends in experimental data.

2. The vortex sheet model is developed because it more closely represents the actual fluid flow. It is used to investigate the sensitivity of the results to the time interval between solutions at a tip speed ratio near maximum power. Time steps corresponding to 3.75 degrees up to 30 degrees of rotor rotation are used.

3. Alterations in the determination of foil circulation after stall are made which better represent real aerofoils.

Those cases for analyzing rotor performance were chosen to be typical so that they may be used for comparison with other theories and test results. They include:

1. A Savonius rotor with a .20 diameter gap is operated throughout its speed range. This example is carefully presented with many streamline
plots of the flow field, torque curves, and printed output giving
details of the wake and flow information along each foil to give a
thorough insight into the operation of the rotor.

2. The sensitivity of the Savonius rotor to geometry variations
is investigated by changing the slot to .1 and .33 diameters and also
the camberline to a "J" shape at a tip speed ratio near maximum power.

3. A typical two bladed Darrieus rotor with uncambered tangential
foils having a solidity of .2 is tested at a tip speed ratio of six to
determine the performance and observe unsteady forces that may effect
structural dynamics.

4. The Darrieus example above is run with a simple model of the
far field wake to reduce the time required to achieve steady state
conditions.

Due to the unsteady nature of cross flow rotors and the number of
cases studied, a vast amount of numerical information is created during
solution. In the interests of clarity, most of the salient features
are presented graphically, de-emphasizing many of the finer details
considered by this method. It must be remembered that this is a complete
inviscid solution of the entire flow field, so that specific information
such as velocity, pressure, etc. may be found at any point that is of
interest.

5.2 Annotated Examples of Output

Selected examples of the output will be explained here to avoid
excessive repetition in the following material and to acquaint the reader
with the methods used to present the results. All examples studied are
for rotors with a diameter of ten feet placed in a uniform thirty feet
per second flow of air. The program can readily handle non-dimensional terms, but it was felt that customary engineering units would be easier to visualize.

Figure 19a shows the printed output from the Thin Aerofoil Solution program at a selected time increment. The case chosen here is a semi-circular vaned Savonius rotor.

Lines 1 and 2

UO = Free stream velocity (feet per second)
OMEGA = Rotational speed (radians per second)
T = Time (seconds)
R*OM/UO = Tip speed ratio

Lines 3 to 14

X and y coordinates of camber line for foil 1 (feet)

Line 15

Bound vortex strength from the previous solution for foils 1 and 2 (ft²/sec)

Lines 16 to 22

X and y coordinates and vortex strength per length (ft/sec) for the updated end points of the wake vortex segments as used for solution. Note that the vortex strengths of the wake at the trailing edges appear as zeros because they are known at this time. These values appear on line 28 after solution is found.

Line 23

Maximum and minimum lift coefficient for attached flow used by the stall model.

Lines 24 to 26

List of pseudo lift coefficient, location of the trailing edge
(A 0 refers to the first and 1 the last of the foil coordinates on Lines 3 to 14), and an indicator of stalled flow (A 0 indicates attached flow and 1 separated) for each foil is repeated until an acceptable solution is found.

Lines 27 and 28
The vortex strength of the wake at the trailing edge is listed for each foil. (ft/sec)

Line 29
The value of the stream function (ft^2/sec) on each foil is listed.

Line 30
Bound vortex strength found by the solution for each foil. (ft^2/sec)

Lines 31 to 37
List of vortex strengths at the end of each vortex segment on the foils solved for by the solution. They are listed in order across the page from the current leading to trailing edges of foil 1 and then foil 2. (ft/sec)

Line 38
Check for round off error by substituting the solution into the stream function equation, to be compared to the solution value for foil 1 on line 29.

Lines 39 to 50 and 52 to 62
List in order for each vortex segment on each foil of the relative velocity (speed (ft/sec) and angle from horizontal in radians), the horizontal and vertical forces (lbs), and the torque (lbs-ft).

Lines 51, 63, and 64
The horizontal and vertical forces and the torque for each foil and the total for the rotor.
Some of this information is presented graphically in Figure 19b. In addition to instantaneous printed output for the flow field and performance, files are created containing all the flow information for use at the next time increment.

In order to aid visualization of the flow, two additional programs are employed. One reads the files created in the above solution and finds the locus of selected streamlines for the solutions at each specified time. When the list of vortex segments in the wake is large, the run time may exceed several minutes. This limits its utility to the first few revolutions, which is still very valuable for interpreting the numerical results and understanding the complexities of the flow. The second program presents the output files of the first to a plotting device.

Figure 20a shows a typical streamline plot corresponding to Figure 19. The information listed along the bottom identifies the run, i.e. the tip speed ratio, time, rotational speed, and vector direction of the incoming free stream in degrees counterclockwise from the x axis. Since a rotating frame, fixed with respect to the rotor, is used, the reader may wish to refer to Section 2.3 for a description of the axis system. At the center of the plot are two semicircular lines representing the camber lines of a Savonius rotor. The solid lines ending at the numbers are the streamlines, which are evenly spaced so as to have equal discharges between them. The numbers indicate the value of the stream function. The dotted and dashed lines represent the continuous string of vortex sheets shed from each foil, thus forming the wake. The dotted lines refer to sheets of zero vortex strength, having no effect on the flow and merely connecting the other vortex sheets together.
All examples illustrated here have a rotational speed that is positive (counterclockwise). Since the axes are fixed relative to the rotor, the foils will remain in the same position on each plot and the free stream will move clockwise from plot to plot as time increases. Generally, flow along the streamlines follows a clockwise direction. A complete picture of the flow field throughout a revolution can be obtained by viewing a series of these plots separated by a small time interval.

After the rotor has operated for a few revolutions, a considerable number of vortex sheets will build up in the wake. Figure 20b shows their location. Here, the wind angle is 180 degrees, indicating a flow in the negative x direction. At the center of the plot the two curved foils of the Savonius are visible. Although the wake is too complex to easily trace the path of the vortex sheets, the more general information is useful. There are definite regions with high and low concentrations of vorticity, a spreading of the wake width leaving the rotor, and the extent of the wake may be useful in considering how closely steady state conditions are approached. Note also that the wake is deflected upwards from the horizontal x axis. This represents conservation of momentum of the flow through the machine, since the Savonius rotor generates considerable side force.

Figure 20c summarized the performance of the rotor over one half revolution. The information along the bottom of the plot gives the tip speed ratio, the half revolution since startup, the wake size indicating that all wake vortices are retained or limited to a certain number, and the size of the step between solutions, given in degrees of rotation. The abscissa shows the rotation and the ordinate, the torque in pound-feet.
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<th>Figure 19a</th>
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Figure 19a (continued)

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**FOIL X, Y, T**

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**TOTAL**

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FIGURE 19b - Savonius Rotor Showing Force, Torque, and Vortex Strength

Vortex Strength - Plotted 200 ft/sec = 1"
Force - Vectors 10 lbs/ft = 1"
Torque - Numbers by segment lb-ft/ft

\[ \Sigma T_1 = 49.6 \]
\[ \Sigma T_2 = 18.4 \]

OM \cdot R/U = 0.83 \quad \theta = 0.16 \quad \text{OMEGA} = 5.0
FIGURE 20c
The solid line represents the total torque produced by the machine. The dotted line always refers to the torque generated by foil 1 and the dashed line, foil 2. Comparison of the streamline and the torque plots permits easy identification of factors influencing performance.

5.3 **Point Vortex Model**

The point vortex model was the first to be completed in this study. In order to assess the validity of the approach, a wide range of cases were studied before proceeding with the vortex sheet model and other refinements. Some of the results are presented here in graphical form to illustrate the success and shortcomings of this model. A solution was obtained every 7.5 degrees of rotation and only eight shed vortex points were retained from each foil. Hence, the character of the very near field was determined, but not enough points were considered to materially alter the flow velocity as would occur in a fully developed wake at high power coefficients, since the cumulative effect of the eight points has much less retarding effect than a full wake.

The case presented is a range of tip speed ratios for a semicircular Savonius rotor having a 1/5 diameter slot. The simplest stall model is used (Figure 18); fully attached flow with lift coefficients between -.5 and 2.0, which is within the range of high lift foils [24], and zero circulation otherwise. The foils are represented by 20 point vortices each and marked by an "x" in the streamline plots. Wind velocity is 30 feet per second and the rotor diameter 10 feet, so a rotational speed of 6 radians per second corresponds to \( \frac{\varpi R}{U_0} = 1.0 \). Figure 21 shows the power coefficient vs. tip speed ratio, having a maximum of .3y at \( \varpi = 5 \) radians per second. This is just below the maximum value in the literature.
of .37 reported by Savonius, and should drop somewhat as the wake develops. The power peaks at approximately the same tip speed ratio as reported in nearly all empirical results, and follows closely the typical trends at lower and higher speeds. Figure 22 shows the torque vs. angle of rotation for all the tip speed ratios. Except for the stationary case, a sudden increase in torque occurs at approximately 60 degrees and is due to the transition from stall to lifting flow on the downstream foil, as determined by reviewing the printed output.

The reader may wish to look over the sequence of streamline plots to gain a feeling for the flow structure at various angles of rotation and to see how the character changes at different speed ratios. Figures 23 to 28 correspond to a speed ratio of 0, .33, .66, .83, 1.33, and 1.67 respectively. Each vortex on the camberline and in the wake is marked with an "x". The plots are presented every 30 degrees of rotation so several solutions occur in between. Occasionally, a streamline makes an abrupt change in direction around one of the point vortices, usually in the wake. Although the general representation of the flow field is good, such local distortion is characteristic of the point vortex model when a streamline passes near a strong vortex. If this were to occur very near a high lift area on a foil, as could happen when many wake vortices fill up the flow field, its effect may dominate the performance predictions and produce entirely unrealistic values at random times in the analysis. The better local representation created by the vortex sheet model should significantly ameliorate this condition, because the vortex sheet model more closely approximates natural conditions.
FIGURE 22 - Savonius Rotor Torque vs. Angle of Rotation Point Vortex Model

\[
\frac{\Omega R}{U} = 0  
\]

Wind Speed = 30 fps
Rotor Diameter = 10 feet

Torque - pound-feet per foot length

**Legend:**
- .33 ——-
- .67 ——-
- .83 ——
- 1.0 ——
- 1.33 ——
- 1.67 ——

Angles (degrees):
- 0° - 90°
- 90° - 180°
5.4 Vortex Sheet Model

5.4.1 Time Step between Solutions and Stall Model

As soon as the vortex sheet model was operational, the sensitivity of the analysis to the time interval between solutions was studied. The semicircular \( \frac{1}{5} \) diameter slot Savonius used earlier was chosen as a typical example. The tip speed ratio was .83 and the camberlines were each represented by ten vortex segments. Also used was the very simple stall model of fully attached flow within the lift coefficient range and zero circulation otherwise, as shown in Figure 18.

During early runs with the point vortex and this model, occasionally very large torque values appeared as in Figure 33f; these occurred predominately during the zero circulation condition outside the permissable range of lift coefficients when most of the foil vorticity is concentrated at the ends. When a shed vortex in the wake alters the flow direction near one of these end points, the lift force, although of proper magnitude, is oriented so as to produce a higher contribution to the torque. Since a real fluid would be likely to separate from the ends under such conditions, it was felt that at this stage in the analysis the most expedient way to proceed would be to require that under these special conditions only, the force on the end segments be normal to the vortex sheet. This modification was discontinued in later cases because it was found to predict increasingly higher power coefficients at higher tip speed ratios. For this reason and subsequent improvements in the stall model, the results presented in Table 1 will not agree with those appearing later. The wake starts to develop at time \( T = 0 \) and continues increasing throughout the run; hence, the execution time for the small step sizes preclude their
running out more than the first half revolution, exceeding 45 minutes CPU time, Kiewit Computation Center, Dartmouth College.

Figure 29 presents plots of the streamlines and wake one half revolution after starting for step sizes ranging from 3.75 to 30 degrees. The general pattern of the streamlines is similar for all plots, with both the foils and the wake having significant influence on the otherwise circular shape. The 3.75 degree step in Figure 29a shows several regions of high wake vorticity with the vortex sheets rolled up quite tightly. Remember that the dotted lines represent zero vortex strength and may be ignored. The wake in the 7.5 and 10 degree steps closely resemble the 3.75 degree case with some loss of vortex sheet bunching due to the fewer segments. The 15 and 22.5 degree cases lose much of the detailed character of the previous examples, yet have regions of high vorticity in nearly the same places. The size of the wake segments is just too large in the 30 degree plot to capture the details. Hence, one may deduce from the streamline plots that, although the finer steps produce a detailed picture of the wake, the step size may be increased to 15 or 22.5 degrees and still retain the general character of the flow quite well.

Figure 30 shows the torque plots by half revolution for all the above examples. Examining Figure 30a for the 3.75 degree step size, one can see that foil 1 produces most of the torque until approximately 100 degrees at which point it stalls and a negative torque is produced. Then instead of returning gradually to zero, the torque from this foil jumps drastically around. Meanwhile, the torque from foil 2 begins to increase gradually as it approaches the position that foil 1 occupied originally. The behavior of foil 1 after stall may be attributed to the catastrophic
loss of circulation required by the stall model. When stall occurs, all the bound vorticity must be dumped into the wake. When the step size, hence wake segment length, is small, the resulting vortex strength is very large. This has such a large influence on the flow that in the next step the predicted lift coefficients are reasonable so that stall is not required. Then as the strong shed vortex moves away from the foil, stall must occur again causing the torque to oscillate drastically. This is more precisely defined in the printed output omitted here in the interests of space. Such behavior is unlikely for a real fluid and has not been observed on Savonius rotors.

Studying the other torque diagrams in the first half revolution shows that the behavior of foil 2 is quite similar throughout the entire range of step sizes and likewise with foil 1 up to stall at approximately 100 degrees. Thereafter, the intensity of the oscillations reduces with increasing step size as the effect of the larger step is to move further into the stalled region and to disperse the intensity of the shed vortex. The areas under the torque curves for foil 1 up to stall for all cases are similar. This behavior explains the lack of any identifiable trends in the power coefficients presented in Table 1.

Figure 31 gives the results of rerunning the 10 and 15 degree step cases with the improved stall model illustrated in Figure 18. At stall the circulation suddenly drops to 60 percent of the maximum value and then tapers to zero at increasing angles of attack. The oscillations after stall are no longer present. The power coefficients integrated over the first half revolution are .39 for both examples, a vast improvement in agreement over the former stall model. The lift forces for the foils
are found by solution of the equations of motion, so that the purpose of the stall model is only to determine when a real fluid would stall and does not effect the magnitude of the lift with attached flow. However, the way the circulation is handled after stall has a profound effect on the predicted performance and must be controlled in a realistic way for the results to be valuable.

5.4.2 Savonius Rotor at Various Tip Speed Ratios

The results for the semicircular Savonius rotor are summarized in Table 2, which presents power coefficients integrated over each half revolution. The range of lift coefficients for attached flow is -0.5 to 1.5 and the improved stall model is used. The standard rotor diameter of ten feet, slot size of 0.20 the diameter, and a uniform stream of air at 30 feet per second are the same as before. The torque values are pound-feet per foot along the rotor axis. The computational time for two revolutions at any speed ratio is about 25 minutes CPU time at Kiewit Computation Center, Dartmouth College.

Figure 32 shows the torque and streamlines for a stationary rotor at several orientations. The stall model sets the circulation to zero for all foils except the downstream one in the 45 degree case. In all cases other than 45 degrees the forces produced by the foils are opposite so that the net force on the rotor is zero, in agreement with d'Alembert's Paradox for an inviscid fluid. However, a positive torque is generated in all the examples tested.

Figures 33 to 37 present the results for tip speed ratios of 0.5, 0.83, 1.0, 1.33, and 1.67 which correspond to rotational speeds of 3, 5, 6, 8, and 10 radians per second respectively. A step size of 22.5 degrees
is used from 0 to 1.5 revolutions and 15 degrees, on the last half revolution. In each case a plot of the streamlines after one half revolution is presented for comparative purposes. Figure 34, for $\frac{OR}{U} = .83$, also includes these plots for every 22.5 degrees through the first half revolution and the corresponding printed output, clearly showing the development of the wake from time $T = 0$. Also, Figure 19b plots torque, force, and vortex strength along the foils taken from this output at 45 degrees. The next plot shows the extent and structure of the wake after 2.0 revolutions. Then plots of the torque are presented throughout the first two revolutions.

Figure 33b shows two shed vortex sheets intersecting the foils after two revolutions. Their presence accounts for the unrealistically large torque in Figure 33f just before reaching that position. Further sophistication of the stall model or a different time step may be needed to avoid such occurrences. Figures 34c and 34g to give considerable insight into the operation of a Savonius rotor near its maximum power output. The numerical output, Figure 34p, is partially summarized in Figure 34q for easy reference, where the angle of rotation corresponds to foil 2; foil 1 differs from the other plots by 180 degrees. These substantiate that the flow is attached until 45 degrees ($T = .16$) on foil 1 which makes a large positive contribution to the torque. Foil 1 is stalled from 45 to 135 degrees ($T = .47$) and the torque drops sharply to negative values. At 135 degrees the flow reverses direction on foil 1 and the torque increases. Foil 2, on the other hand, remains stalled from 45 to 90 degrees ($T = .31$) and produces little torque, since it is in a low velocity region, indicated by the wide spacing of the streamlines.
After the flow on foil 2 reverses direction and attaches at 90 degrees, the torque steadily increases to a maximum at 180 to 202.5 degrees; then stall occurs and the torque drops off. At 180 degrees the streamlines are tightly bunched near foil 2, indicating a high velocity region. Here, the pseudo lift coefficient used in the stall model reaches a maximum of 1.4, and the magnitude of the bound vortex strength is much larger than elsewhere in the cycle.

Hence, the method of operation of the Savonius rotor is due to the large positive torque occurring when the downstream foil traverses the flow in a high velocity region. Very little torque is provided by the foils catching the wind as they travel downstream, since the relative velocity is low. The torque plots are quite similar considering the range of tip speed ratios studied, indicating that changes in the flow occur in roughly the same position. The trend with increasing rotational speed is for a slight lessening of the torque provided by the traversing of the downstream foil and an increase in the drag as the foil reaches the upstream position. Above a tip speed ratio of 1.0 the average torque over one revolution diminishes faster than the rotational speed increases, causing a reduction in power production. The average torque goes negative above a tip speed ratio of 1.67, indicating that power must be supplied to the rotor to achieve higher speeds.

Wilson et al. [3] present smoke streak lines for a Savonius spinning freely at a tip speed ratio of 1.64, from which they make several observations about the flow. It is interesting to compare these with the results of the 1.67 speed ratio case, also at nearly zero power, due to their similarity, even though the appearance of streaklines and stream-
lines for unsteady flow is radically different. They report:

1) Distinct vortices are shed from vane tips at approximately 90 degrees.

2) These vortices are counter rotating with the one from the retreating vane rotating in the same sense as the rotor.

3) Shed vortices move downstream at free stream speed.

4) There appears to be attached flow on both sides of the retreating vane.

5) Separation appears to occur on the advancing blade at 90 degrees.

6) This separated flow is shed as a bubble with shed vortex.

7) Advancing blade vortex contains a region of low energy flow extending toward the wake centerline.

Upon inspection of the smoke streak photographs, the shed vortices appear slightly before 90 degrees on the advancing foil and after 90 degrees on the retreating foil.

Figure 37 plots important flow information taken from the printed output. A large amount of positive circulation is shed into the wake after 90 degrees from the retreating foil as its lift, hence bound vorticity, increases (positive rotor rotation); and negative circulation is shed from the advancing foil before 270 degrees as the bound vorticity decreases. This is consistent with observations 1) and 2) above. Point 3) is implicit in the model insofar as the vortices are carried downstream by the local flow, which must be nearly the free stream velocity since the power is essentially zero. Flow is attached to the retreating blade except momentarily at 45 degrees just after the flow reverses direction.
TABLE 1 - Power Coefficient vs. Step Size for a Savonius Rotor

at $\frac{R}{U} = .83$

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FIGURE 29b - Step Size = 7.5°

\text{OM*R/U} = 0.63 \quad T = 0.63 \quad \text{OMEGA} = 5.0 \quad \text{WIND} = 180°
DM * R/U = .833 REVOLUTION = 0 - .5 WAKE SIZE = 0 - 8 STEP = 22.5 DEG
FIGURE 301

DM*R/U = .833 REVOLUTION = 1.5-2 WAKE SIZE = 24-32 STEP = 22.5°
FIGURE 31a - Step Size = 10°
FIGURE 31c - Step Size = 15°
FIGURE 32e - Savonius Rotor $\frac{OR}{U} = 0$
FIGURE 34d

\[ \text{TORQUE} \]

\[ \text{OMAUE} = 0.83 \text{ REV } = 0.51, \text{ WAKE } = \text{ FULL STEP } = 22.5 \text{ DEG PS} \]
FIGURE 34g
FIGURE 34h

\[ \Omega = \frac{R}{U} = 0.83 \quad \theta = 0.08 \quad \Omega = 5.0 \quad \gamma = 3.7 \]
FIGURE 341

$D/R/U = 0.83 \quad T = 0.16 \quad \Omega = 5.0 \quad \text{WIN} = 319.$
FIGURE 34k

\[ \text{\textit{Figure details go here}} \]
FIGURE 34n

$OM\times R/U = 0.83 \quad T = 0.55 \quad \Omega = 5.0 \quad \text{WIND} = 202.$
FIGURE 340

$OM+R/U = 0.83 \quad T = 0.63 \quad OMEGA = 5.0 \quad WIND = 180$. 
### FIGURE 34p

**U0, OMEGA+T*R*OM/U0**  
0.3000000E+02  0.5000000E+01  0.0000000E-38  0.8333333E+00

**FOIL OFFSETS**

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**SOLUTIONS: CL, TE**  
0 = 1, 1 = LAST PT; STALL = 1

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**G1**  
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**G**  
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### FORCES AND TORQUE

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**TOTAL X,Y,T**  
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**U0,OMEGA,T,R*OM/U0**  
.3000000E+02  .5000000E+01  .7854000E-01  .8333333E+00

**FOIL OFFSETS**  
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**OLD BOUND VORTICITY**  
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**WAKE LOCATION AND CIRCULATION**  
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**CLMAX,CLMIN**  
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**SOLUTIONS:**  
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**PSI**  
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**G1**  
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**G**  
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**FOIL X, Y, T**

-1508180E+02, -6400829E+01, -4801602E+02

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**U0, OMEGA, T, R*Q*M/U0**

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**FOIL OFFSETS**

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| .3200636E+01 | .7292748E+01 | .9788111E+01 |
| .3799866E+01 | .8513535E+01 | .0000000E-38 |
| .1000000E+01 | .0000000E-38 | .0000000E-38 |
| .7776110E+00 | .1628279E+01 | .0000000E-38 |
| .7776110E+00 | .1628279E+01 | .4306927E+02 |
| .3001747E+01 | .3308945E+01 | .3616500E+02 |
| .2867591E+01 | .4451392E+01 | .0000000E-38 |

| CLMAX, CLMIN | .1500000E+01 | -.5000000E+00 |

| SOLUTIONS: CL; TE 0=1, 1=LAST PT; STALL=1 |            |            |            |
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| -1.09 | 1 0 | 0.00 | 0 1 |
| -0.12 | 1 1 | 0.00 | 0 1 |

<p>| SHED VORTICES |            |            |            |
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| PSI | .9874646E+02 | -.1557052E+01 |            |            |
| G | .1691230E+02 | .82 - .1823753E-03 |            |            |
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OLD BOUND VORTICITY .1691230E+02 -.1823753E-03
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PSI .8530938E+02 .5564772E+01
G1 .1989305E-05 G2 -.2956336E+02
G
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<td>2927052E+01</td>
</tr>
<tr>
<td>3763356E+01</td>
</tr>
<tr>
<td>Wake Location and Circulation</td>
</tr>
<tr>
<td>-------------------------------</td>
</tr>
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<td>-1.000000E+01</td>
</tr>
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</table>

Solutions: CL; TE 0=1, 1=LAST PT; STALL=1

-shed vortices
-3.412231E+02
 PSI 9.666386E+01

G 1 -3.728635E+02
g -4.022195E+03

CHECK PSI 9.666390E+01

Forces and Torque

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</tr>
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<td>3</td>
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<tr>
<td>1</td>
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<td>4.1212</td>
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<tr>
<td>1</td>
<td>4.0827</td>
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</table>

| FOIL X, Y, T | 1.2152327E+02 | -.5624969E+01 | -.5516535E+02 |
| FOIL 2     |               |               |               |

|   | 15. | 3.5862 | -0.93 | 1.96 | 1.38 |   |   |
| 2  | 18. | 4.1433 | 0.58 | -0.24 | 0.36 |   |   |
| 2  | 28. | 3.8346 | 0.93 | -1.12 | 1.84 |   |   |
| 2  | 35. | 3.5713 | 1.24 | -2.71 | 5.06 |   |   |
| 2  | 37. | 3.2834 | 0.56 | -3.92 | 7.66 |   |   |
| 2  | 40. | 2.9922 | -0.62 | -4.12 | 8.34 |   |   |
| 2  | 30. | 2.7003 | -1.53 | -3.24 | 6.79 |   |   |
| 2  | 35. | 2.4109 | -1.62 | -1.80 | 4.00 |   |   |
| 2  | 30. | 2.1019 | -1.15 | -0.67 | 1.58 |   |   |
| 2  | 30. | 1.8783 | 0.44 | 0.14 | -0.49 |   |   |

| FOIL X, Y, T | -.2288431E+01 | -.1573064E+02 | .3652775E+02 |
| TOTAL X, Y, T | .9863842E+01 | -.2155561E+02 | -.1663760E+02 |


<table>
<thead>
<tr>
<th>U0, OMEGA, T, R*X/0U0</th>
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<tbody>
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<td>.3000000E+02</td>
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</table>

<table>
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<tr>
<td>-.4270510E+00</td>
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<td>-.2366440E+00</td>
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<td>-.1072949E+01</td>
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<td>.2000000E+01</td>
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</tr>
<tr>
<td>.4853170E+01</td>
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<td>.5000000E+01</td>
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</table>

| OLD BOUND VORTICITY | -.3728635E+02 | -.1761239E+03 |

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<th>WAKE LOCATION AND CIRCULATION</th>
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<tbody>
<tr>
<td>-.1000000E+01</td>
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<tr>
<td>-.1711514E+01</td>
</tr>
<tr>
<td>-.2415801E+01</td>
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<td>-.9429233E+01</td>
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<tr>
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<tr>
<td>2</td>
</tr>
</tbody>
</table>

FOIL $X, Y, T = -1.28556E+02, -1.963474E+02, 5.424539E+02$

TOTAL $X, Y, T = -7.306900E+01, -2.309855E+02, 2.039119E+02$
FIGURE 34q - Savonius Performance Analysis at $\frac{\Omega R}{U} = .83$

- Torque (lb-ft/ft)
- Lift
- Stall
- Flow Direction (Outward, Inward)
- Bound Vortex Strength (ft$^2$/sec)
- Circulation Shed into Wake (ft$^2$/sec)

- Retreating
- 180
- Advancing
- 360
FIGURE 35d

TORQUE

-70 - OM&R/U = 1, REV = 0.5-1, WAKE = FULL STEP = 22.5 DEG PS
FIGURE 35e

TORQUE

-70 - CH1/R1 = 1. REV = 1-1.5 WAKE = FULL STEP = 22.5 DEG PS
FIGURE 37g - Savonius Performance Analysis at $\frac{OR}{U} = 1.67$. 

- Foil Torque (lb-ft/ft)
- Lift
- Stall
- Flow Direction Along Foil
- Outward
- Inward
- Bound Vortex Strength (ft$^2$/sec)
- Circulation Shed into Wake (ft$^2$/sec)
along the foil, in agreement with 4). Observation 5) locates advancing blade stall about 90 degrees; the model has stalled by the next step. A separated flow bubble as observed in 6) has not been considered by this analysis; hence its significance is unknown. Point 7) is consistent with all streamline plots since the concave side of the advancing foil has widely spaced streamlines indicating low velocity flow. Hence, this model appears to predict the observed flow field remarkably well.

5.4.3 Savonius Rotor Geometry Variations

Several different geometries were analyzed near the maximum power coefficient at a tip speed ratio of .83. All other variables are the same as the former example. The range of attached flow for the stall model is unchanged since no information was available which could be used to discriminate between foil shapes. A time step equivalent to 22.5 degrees of rotation was used throughout the two revolutions. Figure 38 presents the results for a semicircular Savonius rotor with a diameter of ten feet and the curvature of the foils adjusted so that the slot is 1/3 the diameter. Figure 39 has the foils changed to provide a gap of 1/10 the diameter. Figure 40 presents a more "J" shaped camberline similar to that which some investigators [1] report to produce more power and a higher rotational speed. The results of these examples appear similar to those discussed formerly; however, upon integrating the power over one revolution, see Table 3, they all appear to have a loss in power, producing about 2/3 that of the 1/5 diameter gap. This indicates considerable performance sensitivity to the geometry. Unfortunately, time did not permit investigation over several tip speed ratios.
FIGURE 38b

-70 DM*RU = .83  REV = .5-1.  WAKE = FULL  STEP = 22.5  GAP = .33
5.4.4 Darrieus Rotor

Figure 41 presents the graphical results and Table 4, the average power for a two bladed Darrieus rotor ten feet in diameter, having uncambered foils with a one foot long chord normal to the diameter. The rotational speed is 36 radians per second, giving a tip speed ratio of 6.0. The simple stall model of catastrophic stall outside a lift coefficient range of -1.0 to 1.0 is employed. Figures 41a, b, and c present streamline plots at .5, 1.0, and 1.5 revolutions after starting ($T = .09, .17, \text{ and } .26$ respectively). The foils appear as solid horizontal lines above and below the middle of the page, which the dotted wake vortex lines follow. Figures 41d and e are wake traces at 2.0 and 4.0 revolutions ($T = .35 \text{ and } .70$).

The remaining torque plots show that the upstream foil produces more torque than the downstream one. Just before reaching 90 degrees the upstream foil stalls, since the relative velocity incidence angle is too great for attached flow. Shortly thereafter the incidence angle reduces and the flow reattaches, producing a torque peak smaller than the first. The downstream foil appears to be operating at a smaller angle of attack and does not stall; hence, the torque peaks at 90 degrees. The stalling of the upstream foil can be avoided by operating at higher rotational speeds which reduce the maximum incidence angle, employing a foil shape that will operate at higher lift coefficients, or changing the orientation of the foil.

In the torque plots the effect of the developing wake at higher revolutions can be seen as a general reduction in torque as the wake effectively slows the flow through the rotor. This retarding effect is
more predominant at the sides of the wake than the center and further reduces the torque near 0 and 180 degrees, thereby narrowing the torque peak around 90 degrees. These effects are consistent with observations and are considered by other sophisticated theoretical models\textsuperscript{(3)}.

Referring to Table 4, the power coefficient shows a trend of dropping from 1.8 to 1.1 over four revolutions. Unfortunately, the lengthy computational time prevented continued running of the program to see if the power coefficient would continue to drop down into agreement with empirical results. Although Figure 41e shows that the wake is quite well developed in the region near the rotor, the effect of a larger wake would be to slow the flow even more, which is necessary to reduce the power coefficient. The primary effect of the added length would be to more tightly bunch the vortices leaving the rotor, thereby increasing their effectiveness in retarding the flow at the rotor.

In order to avoid the long time required for the wake to more fully develop, a case was tried where the wake at large distances from the rotor was considered to be a steady jet as predicted by momentum theory. This was modeled by point vortex pairs on each side of the wake at a few diameters downstream, and the program run to create the normal vortex sheet wake within 1.5 diameters of the rotor. See Appendix B for details. The far field vortex strengths were adjusted so that the velocity in their middle was the proper value as predicted by momentum theory for the power coefficient calculated by the program. Hence, this is merely matching a simple far field wake to a detailed near field solution. The resulting power coefficient for this case is .42, quite typical of experimental results. Although this wake matching method may prove very useful for
engineering applications, it is felt that, due to the sensitivity of the results to the wake structure, evaluation of the fluid dynamic model would be more difficult at this time. It is hoped that initially operating the program at large time steps could fill up the wake reasonably quickly and then a finer spacing could be used for the final analysis. When one considers a Savonius rotor with varying side force hence, transverse deflection of the wake, the difficulties in determining the proper far field wake become apparent. Therefore, the near and far field wake matching technique will not be pursued further here.

In addition to verifying the predicted power coefficient integrated over a revolution, an instantaneous force calculation was performed as an additional check on the validity of the method. The instant the Darrieus rotor used in the above example is started there is no wake; hence, there are only two flat plates in a flow with circular streamlines. The force on these flat plates in the curved flow may be approximated by considering a Joukowski aerofoil placed in a uniform stream with zero thickness and a circular camberline matching the curvature of the streamlines in the example. Joukowski finds that for small angles the lift coefficient is; \[ C_L = 2\pi(\alpha + \beta) \]

where

\[ \alpha = \text{the angle of attack} \]
\[ \beta = 2 \frac{\text{max camber}}{\text{chord}} \]
Figure 410 is the printed output for such a case with the foils represented by only four vortex segments. The center of the circular streamlines is offset toward foil 2 by $\frac{U_0}{\Omega} = .83$ feet; hence, the radius of curvature at foil 1 is 5.83 feet and foil 2, 4.17 feet. Using trigonometry $\beta$ and $C_L$ are found to be .042 and .264 for foil 1 and .060 and .377 for foil 2. The density of air is .00237 slugs/ft$^3$ and the velocity at the foils 210 and 150 feet per second respectively.

The lift force is then:

Foil 1  \[ L = 0.5(0.00237)(0.264)(210)^2 = 13.8\text{#} \]

Foil 2  \[ L = 0.5(0.00237)(0.377)(150)^2 = 10.0\text{#} \]

as compared to the model predictions of 12.17 and 8.67 pounds, a difference of 15%. As the number of vortex segments is increased, the agreement with Joukowski improves.
FIGURE 41h

TORQUE

-20 - 0M*R/U = 6.  REV = 1.1-1.5  WAKE = FULL  STEP = 22.5 DEG

180

90
FIGURE 410

\[ U_0, \Omega, T \]
\[ 30000000E+02 \quad 36000000E+02 \quad 06000000E-38 \]

FOIL OFFSETS
\[ -50000000E+00 \quad 49750000E+01 \]
\[ -25000000E+00 \quad 49750000E+01 \]
\[ 00000000E-38 \quad 49750000E+01 \]
\[ 25000000E+00 \quad 49750000E+01 \]
\[ 50000000E+00 \quad 49750000E+01 \]

OLD BOUND VORTICITY \[ 00000000E-21 \quad 00000000E-21 \]

WAKE LOCATION AND CIRCULATION
CLMAX, CLMIN \[ 10000000E+01 \quad -10000000E+01 \]

SOLUTIONS: CL, TE 0=1, 1=LAST PT;
-0.24 1 -0.33 1

G1 \[ 2459412E+02 \]
G2 \[ 2459413E+02 \]

\[ 0.1378059E-03 \quad 3204935E+02 \quad 3427773E+02 \quad 3204932E+02 \]
\[ 0.0000000E-38 \quad 1459122E-03 \quad 3204923E+02 \quad 3427790E+02 \]
\[ 3204932E+02 \quad 00000000E-38 \]

PSI \[ 5935290E+03 \]

FORCES AND TORQUE

FOIL 1
1 209. 0.0197 0.04 -1.98 0.80
2 209. 0.0027 0.01 -4.10 0.97
3 209. -0.0027 -0.01 -4.10 0.03
4 209. -0.0197 -0.04 -1.98 -0.30

FOIL X, Y, T \[ 6081536E-06 \quad -1216513E+02 \quad 1520442E+01 \]

FOIL 2
1 149. 3.1692 -0.04 1.41 0.51
2 149. 3.1453 -0.01 2.92 0.68
3 149. 3.1379 0.01 2.92 0.05
4 149. 3.1139 0.04 1.41 -0.16

FOIL X, Y, T \[ 2275339E-04 \quad 8667948E+01 \quad 1083596E+01 \]

TOTAL X, Y, T \[ 2356153E-04 \quad 3497279E+01 \quad 2804238E+01 \]
CHAPTER VI

SUMMARY OF RESULTS

6.1 Savonius Rotor at Various Speed Ratios

The performance of a Savonius rotor with semicircular vanes and a 1/5 diameter slot is shown in Figure 42, derived from Table 2. Experimental results are included with notes so that the reader may make allowances for the difference between the two-dimensional analysis and three-dimensional tests under varying conditions.

An analytical model is proposed by Wilson et al. [3] which is limited to one "S" shaped vane without any slot, but goes on to deal with the complex problems of three-dimensional and viscous effects. Their closest example to this study is for a semi-circular "S" rotor with no gap and an aspect ratio of 100, using a nonuniform wake model. They predict a power coefficient of .44 at a speed ratio of .7 and a maximum speed ratio of 1.4. The differences between their model and the one considered here prevent further comparison.

6.2 Savonius Rotor Geometry Variation

Semicircular vaned rotors with 1/10 and 1/3 diameter slots and another with a more "J" shaped camberline were investigated near the optimum power speed ratio. In these cases, summarized in Table 3, the power coefficients fell to about two-thirds that of the 1/5 diameter slot case at a tip speed ratio of .83.

6.3 Darrieus Rotor

A two bladed Darrieus rotor with uncambered tangential foils of
FIGURE 42 - Savonius Rotor Power Coefficient vs. $\frac{\Omega R}{U}$ Vortex Sheet Model
FIGURE 42 (continued)

**Summary of Savonius Rotor Test Results**

<table>
<thead>
<tr>
<th>Reference</th>
<th>( C_{p} )</th>
<th>( C_{L} )</th>
<th>( L/D )</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Carver &amp; MacPherson (5)</td>
<td>.33</td>
<td>.90</td>
<td>1.7</td>
<td>Open section wind tunnel L/D = 2, Slot = 33%</td>
</tr>
<tr>
<td>2. Mercier (7)</td>
<td>.15</td>
<td>.8</td>
<td>1.4</td>
<td>Well documented test in large water basin, very low L/D = .57, Slot = 14.3%. 25% increase in L increases ( C_{p} ) 14% showing models tested are off optimum geometry.</td>
</tr>
<tr>
<td>3. Newman (27)</td>
<td>.24</td>
<td>1.0</td>
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<tr>
<td>4. NYU (6)</td>
<td>.27</td>
<td>.91</td>
<td></td>
<td>Wind tunnel, L/D = 6, Slot = 20%</td>
</tr>
<tr>
<td>5. Robinson (28)</td>
<td>.125</td>
<td>.6</td>
<td>.75</td>
<td>Water tunnel, L/D = 1.83, Slot = 16.7% with central shaft.</td>
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<tr>
<td>6. Savonius (1)</td>
<td>.31</td>
<td>.85</td>
<td>1.74</td>
<td>Wind tunnel</td>
</tr>
<tr>
<td>7. Simonds &amp; Bodek (4)</td>
<td>.14</td>
<td>.7</td>
<td></td>
<td>Open site test of 44 gallon oil drum. Slot = 20%, variable wind and experimental scatter.</td>
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TABLE 2 - Power Coefficients for Semicircular Savonius Rotor with
1/5 Diameter Slot

<table>
<thead>
<tr>
<th>$\frac{\Omega R}{U}$</th>
<th>Revolution</th>
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<tr>
<td>0</td>
<td>0</td>
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<td></td>
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<td>.50</td>
<td>.09</td>
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<td>.83</td>
<td>.12</td>
<td>.18</td>
<td>.18</td>
<td>.17</td>
<td>.18</td>
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<td>1.00</td>
<td>.21</td>
<td>.27</td>
<td>.23</td>
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<td>1.33</td>
<td>.17</td>
<td>.14</td>
<td>.12</td>
<td>.29</td>
<td>.26</td>
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<tr>
<td>1.67</td>
<td>.03</td>
<td>-.02</td>
<td>-.06</td>
<td>-.01</td>
<td>-.04</td>
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TABLE 3 - Power Coefficients for Savonius Rotor at $\frac{\Omega R}{U} = .83$ with Various Foil Geometries

<table>
<thead>
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<th>Description</th>
<th>Revolution</th>
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<tr>
<td>Semicircular</td>
<td>0-.5</td>
<td>.15-1.0</td>
<td>.09</td>
<td>.16</td>
<td>.13</td>
</tr>
<tr>
<td>1/3 slot</td>
<td></td>
<td></td>
<td></td>
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</tr>
<tr>
<td>Semicircular</td>
<td>.03</td>
<td>.11</td>
<td>.11</td>
<td>.11</td>
<td>.11</td>
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<tr>
<td>1/10 slot</td>
<td></td>
<td></td>
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<td></td>
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<tr>
<td>&quot;J&quot; Camber</td>
<td>.08</td>
<td>.06</td>
<td>.23</td>
<td>0.0</td>
<td>.12</td>
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</table>

TABLE 4 - Power Coefficients for Darrieus Rotor at $\frac{\Omega R}{U} = 6$

<table>
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</thead>
<tbody>
<tr>
<td>0-.5</td>
<td>1.82</td>
<td>1.31</td>
<td>1.17</td>
<td>1.33</td>
<td>1.08</td>
<td>1.29</td>
</tr>
<tr>
<td>1.5-2.0</td>
<td>1.11</td>
<td>1.16</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.5-3.0</td>
<td></td>
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<tr>
<td>3.5-4.0</td>
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<td></td>
<td>1.16</td>
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<tr>
<td>4.5-5.0</td>
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<td></td>
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</tbody>
</table>
solidity .2 was tested at a speed ratio of 6. Table 4 shows the power coefficients dropping from 1.8 to 1.1 in the first four revolutions after starting. Excessive computational time prevented further investigation to see if the $C_p$ dropped to reasonable values.

A second analysis using the vortex sheet model to create a near field solution to match with a simple far field wake as predicted by momentum theory produced a power coefficient of .42.
CHAPTER VII

CONCLUSIONS

7.1 Discussion of Results

This analysis provides a useful method for evaluating the performance and optimizing the design of cross flow rotors. Exact determination of the accuracy of the thin aerofoil vortex sheet model is made difficult by the wide variation of test results and the absence of detailed experimental information other than average power. The performance predictions must be considered in good agreement with empirical values since for the Savonius rotor the predicted power is close to the maximum experimental results at all tip speed ratios, and for the Darrieus rotor observed phenomena are modeled as the wake starts to approach steady conditions. The following factors are likely to adjust the results closer to the experimental mean if considered:

a) Steady state conditions may not be fully achieved
b) Viscous drag is not considered
c) Three Dimensional effects are ignored
d) A very simple stall model is employed which has representative values but lacks empirical justification.

When fairly large time intervals between solutions are used, the basic character of the flow is still well represented, which enables a sizeable wake to be developed and then finer spacing used for greater detail. Present limitations of the theory appear to be having sufficient computational time to reach steady conditions and the suitability of the stall model in describing the particular foils used. The point vortex
model occasionally encounters difficulty with unrealistic performance predictions due to the local distortion around point vortices in the wake; when this effect is reduced by a finer step size, the advantage of a faster run time is lost. The alternate approach of a simple flow with transformation, although theoretically feasible, appears to produce a numerical solution that is cumbersome enough to preclude its use.

7.2 Recommendations

The stall model should be tailored to suit the particular purpose of the analysis. For general performance predictions and dynamic loads for structural design a model based on typical foil performance such as the one used here should be sufficient. Detailed experimental results of specific aerofoil sections could be used to study the Darrieus rotor. For a careful optimization of a Savonius rotor, a model that considers important factors in boundary layer development is desirable in order to determine accurately the stall point for slightly different camber shapes and thickness profiles. This analysis could be very complex due to the varying location of the stagnation points and the unsteady flow. Hopefully, the principles of real fluid flow and empirical results could be combined into one subprogram which uses pressure distribution information from the solution and from thickness forms.

The improved insight into the mechanisms of operation of cross flow rotors provided by this analysis enables interpretation of more sophisticated experimental results. The instantaneous streamline plots, torque, and pressure distribution need experimental documentation. It must be demonstrated that the flow pattern with a stalled foil and its separated wake is similar to that predicted by the inviscid theory.
REFERENCES


APPENDIX A
EVALUATION OF INTEGRALS

Integrals appearing in expressions for the stream function and velocity of a vortex sheet with linearly varying strength need to be evaluated when the discriminant, \( Q = 4AC - B^2 \), equals zero and is greater than zero.

1) \[ \int \log x \, dx = \frac{1}{2} \int \log x \, dx \]

\[ = \frac{1}{2} \left( x + \frac{B}{2C} \right) \log x - 2x + \frac{\sqrt{C}}{C} \tan^{-1} \left( \frac{2Cx + B}{\sqrt{Q}} \right) \quad Q > 0 \]

\[ = \frac{1}{2} (x_n + \frac{B}{2C}) \log x - x_n \quad Q = 0 \]

since

\[ \lim_{Q \to 0} \tan^{-1} \left( \frac{2Cx + B}{\sqrt{Q}} \right) = 0 \cdot \frac{\pi}{2} = 0 \]

\[ = -x_n \quad \text{when } Q = 0 \text{ and } \xi = x, \eta = y \]

Since \( \lim_{x \to 0} (x_n + \frac{B}{2C}) \log x = \lim_{x \to 0} (x - \xi) \log x = \lim_{x \to 0} k x \log x \)

by l'Hôpital's Rule

\[ \lim_{x \to 0} \frac{\log x}{1/x} + k \frac{1/x}{-1/x^2} = -kx + 0 \]

\[ \int x \log \sqrt{x} \, dx = \frac{1}{4C} (x \log x - x) - \frac{B}{2C} \int \log \sqrt{x} \, dx \]

2) \[ \int \frac{dx}{x} = 2 \sqrt{\frac{C}{Q}} \tan^{-1} \left( \frac{2Cx + B}{\sqrt{Q}} \right) \quad Q > 0 \]

When \( Q = 0, x = (\pm \sqrt{A} \pm \sqrt{C} x_n)^2 \)

\[ \int \frac{dx}{x} = -\frac{1}{\sqrt{C} - \sqrt{A} + \sqrt{C} x_n} \quad Q = 0 \]
Where the signs are chosen so that \( u = \infty \) when \( \xi = x_n \), noting that
\[
A = x^2 + (y - b_n)^2 = Cx^2,
\]
and the sense of the circulation is positive counterclockwise.

3) \[
\int \frac{x \, dx}{X} = \frac{1}{2C} \log X - \frac{B}{2C} \int \frac{dx}{X} \quad Q \neq 0
\]
\[
= \frac{1}{C} \log(-\sqrt{A} + \sqrt{C} \, x) + \frac{A}{C} \int \frac{dx}{X} \quad Q = 0
\]

4) \[
\int \frac{x^2 \, dx}{X} = \frac{x}{C} - \frac{B}{2C} \log X + \frac{B^2 - 2AC}{2C^2} \int \frac{dx}{X} \quad Q \neq 0
\]
\[
= \frac{1}{C \sqrt{C}} (-\sqrt{A} + \sqrt{C} \, x) + 2\sqrt{A} \log(-\sqrt{A} + \sqrt{C} \, x) + \frac{A}{C} \int \frac{dx}{X} \quad Q = 0
\]
Figure 42 shows the control volume for momentum theory with far field point vortices as positioned several diameters downstream. Equating the power removed from the stream tube as calculated by a change in momentum to that representing the change in kinetic energy produces the following equations.

\[ P = A(V_0 - v)(V_0 - V_w)(V_0 - v) = \frac{1}{2} A(V_0 - v)(V_o^2 - V_w^2) \]

giving

\[ v = \frac{1}{2}(V_0 - V_w) \]

Hence, the power may be written:

\[ P = \frac{1}{2} V_o^3 C_p = \frac{1}{4} A(V_0 + V_w)(V_o^2 - V_w^2) \]

Letting \( R_w = \frac{V_w}{V_0} \), this may be written:

\[ R_w^3 + R_w^2 - R_w - 1 + 2C_p = 0 \] (1)

This equation may be solved to find the wake velocity downstream given the value of the power coefficient. The width of the wake is:

\[ \frac{A_w}{A} = \frac{V}{V_w} = \frac{R_w + 1}{2R_w} \]

For this example far downstream conditions are assumed to occur at 2.5 rotor diameters downstream from the center of the rotor. Equal strength point vortices are placed on each side of the wake at 1.75, 2.25, 2.75, and 3.25 diameters downstream and their strength adjusted so that the velocity along the center of the wake at 2.5 D is that calculated for the downstream condition by momentum theory for the estimated \( C_p \). The vortex sheet model is then run to determine the detailed wake structure.
within 1.5 diameters of the rotor. If the power produced is not the same as that estimated then its value must be changed and the program run until agreement is achieved.

The following example illustrates the procedure as used in the Darrieus rotor case. The flow conditions are:

\[ D = 10 \text{ feet} \]
\[ V_o = 30 \text{ feet per second} \]
\[ \frac{\omega R}{U} = 6 \]

1. Estimate \( C_p = 0.42 \)

Solving equation (1) by trial and error: \( R_w = 0.714107 \)
giving: \( V_w = 21.42 \text{ fps} \) and \( \frac{A_w}{A} = 1.200175 \). Hence the wake will expand to a width of 12.00 feet and the velocity downstream will be 21.42 fps. Point vortices are located at \( x = 17.5, 22.5, 27.5, \) and 32.5 and \( y = \pm 6.00 \text{ ft} \).

2. The horizontal velocity produced by a point vortex at \((x_n, y_n)\) is

\[ u = \frac{\Gamma}{2\pi} \frac{y - y_n}{(x-x_n)^2 + (y-y_n)^2} \]  

Adding up the contribution for all vortices for the velocity decrement in the wake, \( \Sigma u = V_o - V_w \), at 2.5 \( D \) and solving for the vortex strength gives \( \Gamma = \pm 65.07 \text{ ft}^2/\text{sec} \).

3. Solve the vortex sheet model with the addition of the far field wake points as determined above until the wake fills up the area to 1.5 \( D \). In this case the average \( C_p \) over a revolution is 0.42, the same as the estimate.
It is interesting to use equation (2) to determine the effect of the far field vortex wake at the center of the rotor. For this example $u = -0.863$ fps; hence, the flow through the rotor is only slowed down by 2.9% by the far field wake.
APPENDIX C

PROGRAM TASD - THIN AEROFOIL SOLUTION FOR VORTEX SHEET

Fortran '76 Program TASD represents the cross flow rotor flow field as a series of vortex sheets with linearly varying strength along the camber lines of the foils and the wake. The resulting system of linear equations is solved for the strength of each vortex. A complete outline of the logic is too lengthy to include here; however, a brief description of the main program and the many comment statements within the program should enable the steps to be followed.

Summary of Main Program

I. Input coordinates of foils and flow information.

II. Initializations

III. Call WAKE

A. Advance wake from previous solution to position at next time step.

B. Adjust vortex strength to allow for vortex stretching.

IV. Call COEF

A. Set up linear equations for new vortex strengths on foils.

V. Call DGELD

A. Solve equations

VI. Find bound vorticity and pseudo lift coefficient.

VII. Call CKFLO

A. Correct flow direction on foil.

B. Bound vorticity within limits set by stall model.

C. If not satisfactory return to set up new set of equations.
VIII. Stabilize wake after stall by inserting a zero strength segment.

IX. Check solution for roundoff error by substitution.

X. Call FORCE
   A. Determine force and torque on rotor.

XI. Update output files for solution at next time step.
PROGRAM TASD
TASD SOLVES FOR VORTEX SHEETS WITH LINEARLY VARYING STRENGTH ON CAMBER LINES OF TWO IDENTICAL ROTATING FOILS WITH VORTEX SHEETS SHED INTO WAKE

KEY

UO = VELOCITY AT INFINITY
OM = OMEGA RATE OF ROTATION
T = TIME LAPSE
Q = CIRCULATION/LENGTH OF EACH SEGMENT
ZR, ZI = COORDINATES OF MEAN LINE LIST NOSE TO TAIL AND VORTEX SHEET SEGMENTS INCLUDING WAKE
NPT = NUMBER OF POINTS ON MEAN LINE
NW = NO. PTS. IN WAKE - SEGMENTS + 1
N(1) = NPT + NW FOR EACH FOIL
IS = 0 KUTTA CONDITION AT FIRST POINT ON CAMBER LINE
IR = 1 BOUND VORTICITY = 0
ST = STEADY STATE NO WAKE SOLUTION - NW=0
START UP PROCEDURE
FIRST RUN: TASIN - NW=0, CLEAR TASOUT FOLLOWING RUNS: TASIN - NW=FINAL NUMBER, TASOUT - SORT COMMON ZR(2,100), ZI(2,100), Q(2,100), SL(2,100), N(2)
& NPT, NW, UO, OM, T COMMON /C/ VAV1R, VAV1I, VAV2R, VAV2I, CL1, CL2, CVR, CVI,
& JS1, JS2, N, CLMIN, CLMAX, TOL DIMENSION A(676), B(26), GOLD(2)
DOUBLE PRECISION A, B OPENFILE 1, "TASIN"
OPENFILE 2, "TASOUT"
READ(1,101) UO, OM, T READ(1,102) NPT READ(1,102) NPT, NW READ(1,101) CLMIN, CLMAX RR=0.
DO 1 I=1,NPT
READ(1,101) ZR(1,I), ZI(1,I)
R=SQR(ZR(1,I)**2+ZI(1,I)**2)
IF(R.GT.RR) RR=R
ZR(2,I)=-ZR(1,I)
IZ(2,I)=-IZ(1,I)
ORDU=OM*RR/UO
PRINT, "UO, OMEGA, T, R*OM/UO"
PRINT, "UO, OM, T, ORDU"
520 IF(NPRT.LT.2) GO TO 9
530 PRINT, "FOIL OFFSETS"
540 DO 20 I=1,NPT
550 20 PRINT, ZR(I,1),ZI(I,1)
560 NTIMES = 0
570 IS1 = 1
580 IS2 = 1
590 JS1 = 0
600 JS2 = 0
610 IR1 = 0
620 IR2 = 0
630 NU = 2*(NPT+2)
640 * NOSE TO TAIL CHORD VECTOR
650 CVR = ZR(1,NPT)-ZR(1,1)
660 CVI = ZI(1,NPT)-ZI(1,1)
670 CHORD = SQRT(CVR+CVR+CVI*CVI)
680 * MEAN FOIL VELOCITY
690 N=(NPT+1)/2
700 COT = COS(OM*T)
710 SOT = SIN(OM*T)
720 VAV1R = U0*COT+OM*ZI(1,N)
730 VAV1I = U0*SOT+OM*ZR(1,N)
740 VAV1 = SQRT(VAV1R**2+VAV1I**2)
750 VAV2R = U0*COT+OM*ZI(2,N)
760 VAV2I = U0*SOT+OM*ZR(2,N)
770 VAV2 = SQRT(VAV2R**2+VAV2I**2)
780 TOL = 0.00025*CHORD*(VAV1+VAV2)
790 * FIND NEW WAKE POSITION
810 IF(NW.EQ.0) GO TO 22
820 CALL WAKE(GOLD,IS1,IS2)
830 GO TO 16
840 22 NV(1)=NPT
850 NV(2)=NPT
860 NU = NU - 2
870 CALL SLEN(ZR, ZI, NV, SL)
880 16 IF(NPRT.LT.2) GO TO 3
890 PRINT, "OLD BOUND VORTICITY ", GOLD(1), GOLD(2)
900 PRINT, "WAKE LOCATION AND CIRCULATION"
910 DO 23 I=1,2
920 DO 23 J=1,NPT+1,NV(I)
930 23 PRINT, ZR(I,J),ZI(I,J),G(I,J)
940 PRINT, "CLMAX,CLMIN",CLMAX,CLMIN
950 3 PRINT, "SOLUTIONS: CL; TE 0=1.1=LAST PT; STALL=1"
960 * SET UP EQUATIONS
970 6 CALL COEF(A,B,GOLD,IR1,IR2,G1S,G2S)
980 NTIMES=NTIMES+1
990 IF(NTIMES.GT.8) GO TO 2
1000 CALL DGEL(D,B,A,NU,1,1E-15,IER)
1010 DO 30 I=1,2
1020 DO 30 J=1,NPT
1030 30 G(I,J)=B(J+NPT*(I-1))
1040 C(I,NPT+1)=B(NU)
1050 G(2,NPT+1)=B(NU)
1060 IF(IER.EQ.-1) PRINT, "SINGULAR SOLUTION"
PARTIAL STALL MODEL

IF(CL1.LT.CLMAX) GO TO 10
G1S=60*(3.*CLMAX-CL1)*G1/(2.*CL1)

IF(CL1.GT.3.*CLMAX) G1S=0.

10 IF(CL2.LT.CLMAX) GO TO 11
G2S=60*(3.*CLMAX-CL2)*G2/(2.*CL2)

IF(CL2.GT.3.*CLMAX) G2S=0.

11 IF(CL1.LT.CLMIN) GO TO 15
G1S=60*(3.*CLMIN-CL1)*G1/(2.*CL1)

12 IF(CL1.LT.CLMIN) G1S=0.

15 IF(CL2.LT.CLMIN) GO TO 17
G2S=60*(3.*CLMIN-CL2)*G2/(2.*CL2)

17 CONTINUE

END STALL MODEL

CALL CKFLD(IS1,IS2,IR1,IR2,NR)

IF(NR.NE.1) GO TO 7

IF(NV(1).EQ.NPT+2) GO TO 5

GO TO 6

7 IF(NW.EQ.0) GO TO 19

PRINT,"SHED VORTICES"

19 PRINT,"PSI",B(2*NPT+1),B(2*NPT+2)

* STABILIZE WAKE AFTER STALL

IF(IR1.NE.1) GO TO 12

IF(ABS(GOLD(1)).LT.TOL.OR.ABS(G(1,NPT+1)).LT.TOL) GO TO 12

CALL ADV(1,0)

12 IF(IR2.NE.1) GO TO 13

IF(ABS(GOLD(2)).LT.TOL.OR.ABS(G(2,NPT+1)).LT.TOL) GO TO 13

CALL ADV(0,1)

13 PRINT,"G1","G2";

IF(NPRT.LT.2) GO TO 4

PRINT,"G1",G1,"G2",G2

4 PRINT, "*"

140 PRINT,"(G(I,J),I=1,NPT),J=1,2)

1480 * CHECK PSI

1490 PSIC=UC*(COT*ZI(I,1)+SOT*ZR(I,1))strom(ZR(1,1)**2+ZI(1,1))

& +/**2/2.

1500 DD 31 I=1,2

1520 DD 31 J=1,NV(1)-1

1530 CALL PSI(ZR(1,J),ZI(I,J),ZR(I,J+1),ZI(I,J+1),

& ZR(1,1),ZI(I,1)),G(I,J),G(J,I),PSI

1550 31 PSIC=PSIC+PSI

1560 PRINT, "CHECK PSI",PSIC

1570 * OUTPUT TO FILE TASSOUT

1580 4 WRITE(2,201) 110,UC,DM,T,ORDU

1590 WRITE(2,202) 120,NPT,NV(1),NV(2),IS1,IS2

1600 WRITE(2,201) 125,G1,G2

1610 2 DD 14 I=1,2
1620 DO 14 J=1,NV(I)
1630 WRITE(2,201) 129+J+NV(I)*(I-1), ZR(I,J), ZI(I,J), Q(I,J)
1640 IF(NV(I).EQ.0) GO TO 1
1650 CALL FORCENPRT, IR1, IR2)
1660 101 FORMAT(4X, 4F10.2)
1670 102 FORMAT(4X, I3, 4E15.7)
1680 103 FORMAT(4X, 4E15.7)
1690 201 FORMAT(I3, 4X, 4E15.7)
1700 202 FORMAT(I3, 5(X, I3))
1710 END

SUBROUTINE ADV(N1, N2)

ADVANCE WAKE ONE STEP IF NS .NE. 0

1740 COMMON ZR(2,100), ZI(2,100), G(2,100), SL(2,100), NV(2), NPT, NW, U0, QM, T
1750 & DIMENSION NS(2)
1760 NS(1)=N1
1770 NS(2)=NW
1780 NM=NPT+NW
1790 DO 1 I=1,2
1800 IF(NV(I).EQ.0) GO TO 1
1810 NV(I)=NV(I)+1
1820 IF(NV(I).EQ.NM) NV(I)=NM
1830 N=NV(I)-NPT
1840 DO 2 J=1,N
1850 ZR(I,J)=ZR(I,J-1)
1860 ZI(I,J)=ZI(I,J-1)
1870 IR=NV(I)-J
1880 ZR(I,IR+1)=ZR(I,IR)
1890 ZI(I,IR+1)=ZI(I,IR)
1900 2 CONTINUE
1910 DO 3 J=1, N-1
1920 IR=NV(I)-J
1930 G(I,IR)=Q(I,IR)
1940 SL(I,IR)=SL(I,IR-1)
1950 3 CONTINUE
1960 SL(I,NPT+1)=0.
1970 Q(I,NPT+1)=0.
1980 1 CONTINUE
1990 RETURN
2000 END

SUBROUTINE WAKE(GOLD, IS1, IS2)

2030 COMMON ZR(2,100), ZI(2,100), G(2,100), SL(2,100), NV(2), NPT, NW, U0, QM, T
2040 & DIMENSION GOLD(2), US(2,100), VS(2,100)
2050 COMMON /B/ NV(2), ZER(2,100), ZERI(2,100), GE(2,100), SLE(2,100)
2060 OPENFILE 2, "TASOUT"
2070 READ(2,202) TOLD
2080 READ(2,203) GOLD(1), GOLD(2)
2090 DT=T-TOLD
2100 DO 1 I=1,2
2110 DO 1 J=1,NV(I)
2120 READ(2,204) NV(1), NV(2), IS1, IS2
2130 READ(2,203) ZR(I,J), ZI(I,J), Q(I,J)
2140 1 CONTINUE
2150 1 CONTINUE
2160 FIND NW POSITION
CALL SLEN(ZR, ZI, NV, SL)

IF(NV(I)-NPT, NE, O) GO TO 14

* PROVIDE FIRST WAKE SEGMENT

DO 15 I=1,2
  ZR(I, NPT+1)=ZR(I, NPT)
  ZI(I, NPT+1)=ZI(I, NPT)
15 CONTINUE

IF(NV(1)-NPT, NE, O) GO TO TC 14

PROVIDE FIRST WAKE SEGMENT

DO 15 I=1,2
  ZR(I, NPT+1)=ZR(I, NPT)
  ZI(I, NPT+1)=ZI(I, NPT)
15 CONTINUE

SL(I, NPT+1)=SQRT((ZR(I, NPT+2)-ZR(I, NPT+1))**2+(ZI(I, NPT+1))**2)

CALL SLEN(ZR, ZI, NV, SL)

NS=NPT+1

DO 6 I=1,2
  S1=0.
  S2=0.
  SN1=0.
  SN2=0.
6 CONTINUE

IF(J.EQ. NS) GO TO 7

S1=SL(I, J-1)

SN1=SLE(I, J-1)

IF(J.EQ. NV(I). AND. J. NE. NS) GO TO 6
S2 = SL(I, J)
SN2 = SLE(I, J)
S = S1 * S2
SN = SN1 + SN2

IF (SN. EQ. 0.) GO TO 16
GE(I, J) = G(I, J) + S/SN
GO TO 6

GE(I, J) = G(I, J)
CONTINUE

UCOT = U0 * COS(OM * T)
USOT = U0 * SIN(OM * T)

DO 3 I = 1, 2
DO 3 J = NPT + I, NV(I)

UT = UCOT + OM * ZER(I, J)
VT = USOT - OM * ZER(I, J)

DO 5 IK = 1, 2
DO 5 JK = 1, NV(1K) - 1
CALL VELD(IK, JK, ZER(I, J), U, V)

UT = UT + U
VT = VT + V

END

AVERAGE VELOCITY AT OLD AND ESTIMATED POSITION
US(I, J) = 5 * (US(I, J) + UT)
VS(I, J) = 5 * (VS(I, J) + VT)
ZR(I, J) = ZR(I, J) + DT * US(I, J)
ZI(I, J) = ZI(I, J) + DT * VS(I, J)

ADJUST WAKE STRENGTH FOR SEGMENT STRENGTH
CALL SLEN(ZR, ZI, NV, SLE)

IF(J. EQ. NS) GO TO 10
SN1 = SLE(I, J - 1)
SN2 = SLE(I, J)
S = S1 + S2
SN = SN1 + SN2
IF(SN, NE. 0.) GO TO 9
SN = 1.
S = 1.
G(I, J) = G(I, J) * S/SN
SL(1, J) = SLE(1, J)
CONTINUE

CALL ADV(1, 1)
RETURN

FORMAT(34X, E15.7)
FORMAT(4X, E15.7)
FORMAT(4X, E15.7)
FORMAT(4X, 5(I3, X))
END
SUBROUTINE SLEN(ZR, ZI, NV, SL)
DIMENSION ZR(2, 100), ZI(2, 100), NV(2), SL(2, 100)
DO 1 I = 1, 2
DO 1 J = 1, NV(I) - 1
1 SL(I, J) = SQRT((ZR(I, J) - ZR(I, J-1))**2 + (ZI(I, J) - ZI(I, J+1))**2)
RETURN
END

SUBROUTINE DGELD(On, An, fln, EPS, IER)
DIMENSION A(676), B(26)
DOUBLE PRECISION 3, An, PIv, TB, TOL, PIVI
IF (fl) 23.23.
IER = 0
PIV = 0.0
n = n * n
NM = N * M
DO 3 L = 1, NM
TB = DABS(A(L))
IF (TB - PIV) 3, 2
PIV = TB
3 CONTINUE
TOL = EPS + PIV
LST = 1
DO 17 K = 1, M
3 L = L + I
TB = PIV * B(LL)
S(LL) = B(L)
B(LL) = TB
IF (K - M) 9, 18, 18
LST = L + M - K
IF (J) 12, 12, 10
II = J * M
DO 11 L = LST, LEND
TB = A(L)
LL = L + II
A(LL) = A(L)
11 A(LL) = TB
9 LEND = LST + M - K
10 II = J * M
11 L = LST, LEND
SUBROUTINE CDEF(A, B, GOLD, IR1, IR2, G1, G2)

* UNKNOWNS: G1(NPT), G2(NPT), PSI1, PSI2, G1SHED, G2SHED
* EQUATIONS: PSI1(NPT), PSI2(NPT)

INTEGER LL, L, M, LST, J, I, K, NPT, IR1, IR2, M1, M2
REAL A(LST), B(LST), TLL, TB, TLLT, TLLT, G1, G2, G1SHED, G2SHED

A(LST) = J
PIV = 0
LST = LST + 1
J = 0
DO 16 I = LST, LEND
PIV = - A(I)
IST = II + M
J = J + 1
DO 15 II = 15, 15, 14

TB = DABS(A(LL))
IF(TB - PIV) 15, 15, 14

CONTINUE

DO 16 L = K, NM, PI
LST = LST + 1
16 B(LL) = B(LL) + PIV * B(L)

18 IF(M - 1) 23, 22, 19

IST = MM + M
LST = M + 1
DO 21 I = 2, M
19 II = LST - I
IST = IST - LST
L = IST - M
L = A(L) + 5
DO 20 J = 1, M, NM, PI
20 TB = TB - A(K) * B(LL)
K = J + L
21 B(J) = B(K)
22 TB = TB + A(J) * B(LL)
K = J + L
23 IER = -1
RETURN

END

SUBROUTINE CDEF(A, B, GOLD, IR1, IR2, G1, G2)
BOUND G + SHED G = OLD BOUND G
KUTTA CONDITION - G AT TE OR
BOUND G = 0
COMMON ZR(2,100), ZI(2,100), G(2,100), SL(2,100), NV(2).
DIMENSION A(676), B(26), GOLD(2)
DOUBLE PRECISION A, B
NU=2*(NPT+2)
IF(NW.EQ.0) NU=NU-2
UCOT=UO*COS(OM*T)
USOT=UO*SIN(OM*T)
G(1,NPT+1)=0.
G(2,NPT+1)=0.
PSI EQUATIONS
DO 1 I=1,2
DO 1 J=1,NPT
G(I,J)=O.
G(I,NPT+1)=O.
BASIC FLOW
B(I,J)=UCOT*ZI(I,J)-USOT*ZR(I,J)-.5+OM+(ZR(I,J)**2 & ZI(I,J)**2)
DO 2 L=1,2
DO 2 K=NPT+1, NV(L)-1
CALL PSID(ZR(L,K), ZI(L,K), ZR(L,K+1), ZI(L,K+1), ZR(I,J), & ZI(I,J), G(L,K), G(L,K+1), PSI)
B(I,J)=B(I,J)-PSI
BOUND VORTEX COEF
DO 3 L=1,2
DO 3 K=1,NPT
PSI1=0.
PSI2=0.
IF(K.EQ.1) GO TO 4
CALL PSID(ZR(L,K-1), ZI(L,K-1), ZR(L,K), ZI(L,K), ZR(I,J), & ZI(I,J), 0., 1.0., PSI1)
4 IF(K.EQ.NPT) GO TO 3
CALL PSID(ZR(L,K), ZI(L,K), ZR(L,K+1), ZI(L,K+1), ZR(I,J), & ZI(I,J), 1., 0., PSI2)
A(I,J+NU*((K-1)+NPT*(L-1)))=PSI1+PSI2
SHED VORTEX COEF
DO 5 K=1,2
IF(NW.EQ.0) GO TO 12
CALL PSID(ZR(K,NPT+1), ZI(K,NPT+1), ZR(K,NPT+2), ZI(K,NPT+2), & ZR(I,J), ZI(I,J), 1., 0., PSI)
A(I,J+NU*(NU-3+K))=PSI
BOUND VORTEX COEF
CONSERVATION OF VORTICITY EQN - BOUND VORTEX COEF
IF(NW.EQ.0) GO TO 10
C1=0.
C2=0.
IF(J.EQ.1) GO TO 6
C1=SL(1,J-1)
IF(J.EQ.NPT) GO TO 7
C2=SL(I,J)
7 C=(C1+C2)/2.
A(2*NPT+I+NU*(J-1+NPT*(I-1)))=C
A(2*NPT+I+NU*(J-1+NPT*(2-I)))=0.
KUTTA CONDITION BOUND VORTEX CDEF
10 A(NU-2+I+NU*(J-1))=C
1 CONTINUE

CONSERVATION OF VORTICITY EGN AND KUTTA COND
DO 8 I=1,2
B(I+NU-I-2)=0.
DO 8 J=1,2
IF(NW.EQ.0) GO TO 8
A(NU-I-4+NU*(NU-J-5))=0.
8 A(NU-I-2+NU*(NU-J-5))=C.
C=0.
IF(NW.EQ.0) GO TO 11
A(NU-3+NU*(NU-2))=SL(1,NPT+1)/2.
A(NU-3+NU*(NU-1))=0.
A(NU-2+NU*(NU-2))=0.
A(NU-2+NU*(NU-1))=SL(2,NPT+1)/2.
B(NU-3)=GOLD(1)-Q(1,NPT+2)*SL(1,NPT+1)/2.
B(NU-2)=GOLD(2)-Q(2,NPT+2)*SL(2,NPT+1)/2.
C=-1.

KUTTA COND EGN
11 A(NU-I-NU*(NU-2))=C
A(NU-I-NU*(NU-1))=0.
A(NU+NU-I-I))=0.
5200 A(NU+NU)=C
5210 A(NU-I-NU*(NU-1))=I.
5220 A(NU+NU*(NU+NU*(NU-1)))=I.

BOUND VORTICITY = 0 WHEN CIRCULATION OUTSIDE LIMITS
IF(IR1.NE.1) GO TO 15
IR=NU-1
DO 14 I=1,NPT
S=0.
IF(I.NE.1) S=S+SL(I-1,1-I-1)/2.
A(I-1)*NU+IR)=S
14 A(I-1+NPT)*NU+IR)=0.
IF(NW.EQ.0) GO TO 15
A(IR+NU*(NU-2))=0.
B(NU-1)=G1S
15 IF(IR2.NE.1) GO TO 16
16 RETURN
END
SUBROUTINE FORCE(NPRT, I1, I2)

THIS PROGRAM CALCULATES THE FORCE, MOMENT AND TORQUE FOR PROGRAM TAS BY CALCULATING
RHO*G*V AT EACH VORTEX (LAGALLY'S THEOREM).

COMMON ZR(2, 100), ZI(2, 100), G(2, 100), SL(2, 100), NV(2),
& NPRT, NU, UO, OM, T

DIMENSION VEL(2, 10), TH(2, 100), TQ(2), IR(2)

COMMON /B/ NV(2), ZER(2, 100), ZEI(2, 100), GE(2, 100), SLE(2, 100)

ARCTAN(U)=-.5*AIMAG(CLOG(CMPLX(CNPLX(1., 0.), CNPLX(1., 0.))))

OPENFILE 3, "TORQ"

RHO= 0.00237 SLUGS/FT**3

PD2= 3.14159/2.

IR(1)=1
IR(2)=12

COT=COS(OM+T)
SOT=SIN(OM+T)

FIND VELOCITY AT EACH POINT

DO 2 I=1, 2
DO 2 J=1, NV(I)
ZER(I, J)=ZR(I, J)
ZEI(I, J)=ZI(I, J)
SLE(I, J)=SL(I, J)

2 St£(I, J)=SL(I, J)

DO 16 J=1, NPT-1
16 TH(I, J)=TH(I, J)+3.14159
14 TH(I, J)=PD2
16 CONTINUE

CALCULATE FORCES

XT=0.
YT=0.
TT=0.

PRINT, "FORCES AND TORQUE"
SUBROUTINE PSID(X1, Y1, X2, Y2, X, Y, G1, G2, PSI)

VALUE OF STREAM FUNCTION AT X, Y FROM VORTEX SHEET WITH LINEARLY VARYING STRENGTH

IF (X1.EQ. X2_ AND. Y1.EQ. Y2) GO TO 2

XX = X2 - X1
YY = Y2 - Y1

A = XX/YY
BB = Y1 - AA*X1

A = X*X + Y*Y + BB*BB - 2.*BB*Y
B = 2.*(AA*BB - AA*Y - X)
CC = SQRT(C) / 6.28318

DG = (G2 - G1) / XX
CALL INTLN(X1, X2, A, B, C, SLNRX, SXLNRX)

PSI = CC + (G1 - DG*X1)*SLNRX + DG*SXLNRX

RETURN

END

CALL INTLN(Y1, Y2, A, B, C, SLNRX, SXLRX)
PSI=-CC*((G1-DG*Y1)*SLNRX+DG*SXLNX)
RETURN
2 PSI=O.
RETURN
END

SUBROUTINE VELD(IF, IV, X, Y, U, V)

COMMON /B/ NV(2), ZR(2, 100), ZI(2, 100), G(2, 100), SL(2, 100)

X1=ZR(IF, IV)
Y1=ZI(IF, IV)
X2=ZR(IF, IV+1)
Y2=ZI(IF, IV+1)

GL=GG(IF, IV)

XX=X2-X1
YY=Y2-Y1

IF(XX.EQ.0. AND. YY.EQ.0.) GO TO 9

IF(XX.EQ.0.) GO TO 1

AA=YY/XX

IF(ABS(AA).GT.1.60 TO 1

RZ=t*XZ+X2+B*X2+A

IF(R2.LE.0.) GO TO 2

V=-CC*(-AA*DC*SX2DX+D*SXDX+E*SDX)

V=-CC+DG*XX/C

IF(R1.LE.0.) GO TO 2

IF(R1.EQ.0. OR. X1.EQ. X) GO TO 4

IF(IV.EQ.NV(IF)+1) GO TO 3

IF(SL(IF, IV+1).EQ.0.) IS=2

IF(SL(IF, IV+2).EQ.0. AND. IS.EQ.2) IS=3
10 IF(SL(IF, IV). LE. SL(IF, IV+IS)) GO TO 3
110 GO=G2
1140 S1=X2-SL(IF, IV)/SC
1150 S2=X2-SL(IF, IV+IS)/SC
1160 GO TO 5
1170 4 IF(IV. EQ. 1) GO TO 3
1180 IF(SL(IF, IV-1). EQ. 0.) IS=2
1190 IF(IV. LE. 2) GO TO 11
1200 IF(SL(IF, IV-2). EQ. 0. AND. IS. EQ. 2) IS=3
1210 GO=G2
1220 S1=X1-SL(IF, IV+IS)/SC
1230 S2=X1-SL(IF, IV)/SC
1240 CALL INTX(S1, S2, X, A, B, C, SDX, SXDX, SX2DX)
1250 U=U-CC*(-AA*GG*SDX+GG*(Y-BB)*SDX)
1260 V=V+CC*(-GG*SDX*GG*X*SDX)
1270 RETURN
1280 Y IS VARIABLE OF INTEGRATION
1290 1 AA=XX/YY
1300 BB=X1-AA*Y1
1310 A=X*Y+Y+BB+BB/2+BB*X
1320 B=2*(AA*BB-AA*X-Y)
1330 C=AA*AA+1
1340 SC=SQRT(C)
1350 CALL INTX(Y1, Y2, X, A, B, C, SDX, SXDX, SX2DX)
1360 D=-Gl+DG*(Y+Y1)
1370 E=(Gl-DG*Y1)*Y
1380 F=-AA*G1-DG*(AA*Y1+X-BB)
1390 H=(G1-DG*Y1)*(X-BB)
1400 CALL INTX(Y1, Y2, X, A, B, C, SDX, SXDX, SX2DX)
1410 U=U-CC*(-DG*SDX+DG*SDX+H*SDX)
1420 V=V+CC*(-GG*SDX*GG*X*SDX)
1430 RETURN
1440 VELOCITY AT END OF SEGMENT
1450 6 U=CC*DG*YY/C
1460 IF(Y2. LT. Y1) U=U
1470 V=AA*U
1480 IS=1
1490 IF(R1. EQ. 1) GO TO 7
1500 IF(SL(IF, 1). EQ. 0. AND. IS. EQ. 2) IS=2
1510 IF(SL(IF, 2). EQ. 0. AND. IS. EQ. 2) IS=3
1520 GO=G2
1530 S1=Y2-SL(IF, IV)/SC
1540 S2=Y2-SL(IF, IV+IS)/SC
1550 GO TO 8
7670. 7 IF (IV.EQ.1) RETURN
7680  IF (SL(IF, IV-1).EQ. 0.) IS=2
7690  IF (IV.LE.2) GO TO 13
7700  IF (SL(IF, IV-2).EQ.0. AND. IS.EQ.2) IS=3
7710 13 IF (SL(IF, IV).LE.SL(IF, IV-1)) GO TO 3
7720  G0=Q1
7730  S1=Y1+SL(IF, IV-1)/5C
7740  S2=Y1+SL(IF, IV)/SC
7750  8 CALL INTX(S1, S2, Y, A, B, C, SDX, SXDX, SX2DX)
7760  U=U-CC*(-GG*SDX+A*Y*SDX)
7770  V=U+CC*(-AA*GG*SDX+GG*(X-BB)*SDX)
7780  RETURN
7790 9 U=0.
7800  V=0.
7810  RETURN
7820  END
7830 *
7840  SUBROUTINE INTLNX(X1, X2, A, B, C, SlnRX, SxlnRX)
7850  PROVIDES INTEGRALS OF LOG(R) AND X*LOG(R)
7860  ARCTAN(U)=-.S*AIMAG(CLOG(CMPLX(1.,-U)/CMPLX(1.,U)))
7870  SN=1.
7880  IF(X2.LT.X1) SN=-1.
7890  Q=4*A+B*B
7900  B2=B/(2.*C)
7910  R1=C*X1*X1+B*X1+A
7920  R2=C*X2*X2+B*X2+A
7930  IF (G.LE.0.) GO TO 2
7940  T1=2*C*X1+B
7950  T2=2*C*X2+B
7960  SQ=SQR(T(Q)
7970  ATN=SQ*(ARCTAN(T2/SQ)-ARCTAN(T1/SQ))/C
7980  GO TO 3
7990 2 ATN=0.
8000  IF(R1.LE.0.) GO TO 4
8010  AL1=ALOG(R1)
8020  GO TO 5
8030 4 AL1=0.
8040  IF(R2.LE.0.) GO TO 6
8050  AL2=ALOG(R2)
8060  GO TO 7
8070 6 AL2=0.
8080 7 SlnRX=SN*(.5*(X2+B2)AL2-(X1+B1)AL1+ATN)-X2+X1)
8090  SxlnRX=SN*(.25*(R2*(AL2-1)-R1*(AL1-1))/C-B2*SlnRX
8100  RETURN
8110  END
8120 *
8130  SUBROUTINE INTX(X1, X2, A, B, C, Sdx, SxDX, SX2DX)
8140  PROVIDES INTEGRALS OF DX/R, DX/R, X**2DX/R
8150  ARCTAN(U)=-.S*AIMAG(CLOG(CMPLX(1.,-U)/CMPLX(1.,U)))
8160  SN=1.
8170  IF(X2.LT.X1) SN=-1.
8180  R1=C*X1*X1+B*X1+A
8190  R2=C*X2*X2+B*X2+A
8200  Q=4*A+C-B*B
8210  IF (G.LE.0.) GO TO 2
SQ=SQR(T)
T2=(2.*C*X2+B)/SQ
T1=(2.*C*X1+B)/SQ
SDX=SN*ALN/(ARCTAN(T2)-ARCTAN(T1))/SQ
ALN=ALOG(R2)-ALOG(R1)
SDX=SN*ALN/(2*C)-B*SDX/(2*C)
SX2DX=(SN+(X2-X1)-B*SDX)/(2*C)
RETURN.

POINT ON SEGMENT LINE
2 IF(A.LT.0.) A=0.
SA=SRT(A)*SIGN(1.,X)
SC=SRT(C)
T2=SA+SC*X2
T1=SA+SC*X1
POINT WITHIN SEGMENT
IF(SIGN(T1,T2).EQ.T1) GO TO 1
SDX=0.
SXDX=0.
SX2DX=0.
RETURN
1 ALN=ALOG(T2/T1)
SDX=SN*(T1-T2)/SC
SXDX=SA+SDX/SC+SN*ALN/C
SX2DX=SA+SDX/C+SN*(T2-T1+2.*SA*ALN)/(C*SC)
RETURN
END
SUBROUTINE CKFLO(IS1, IS2, IR1, IR2, NR)
COMMON ZR(2,100), ZI(2,100), G(2,100), SL(2,100), NV(2),
& N., CLMIN, CLMAX, TOL
NR=1
CHECK THAT KUTTA CONDITION IS AT TRAILING EDGE
I=IS1
J2=IS2
JR1=IR1
JR2=IR2
DO 9 I=1,2
NVE(I)=NV(I)
DO 9 J=1,NV(I)
ZER(I,J)=ZR(I,J)
ZI(I,J)=ZI(I,J)
GE(I,J)=G(I,J)
SLE(I,J)=SL(I,J)
UI=UA1
V=VA1
UI=UA2
V=VA2
DO 21 I=1,2
DO 21 J=1,NV(I)-1
CALL VELD(I,J,ZR(I,N),ZI(I,N),U,V)
UI=UI+U
V=V+V
CALL VELD(I, J, ZR(2, N), ZI(2, N), U, V)
U2=U2+U
V2=V2+V
CONTINUE
PRINT 300, CL1, IS1, IR1, CL2, IS2, IR2
IF(JS1. EQ. 2) GO TO 7
S=U1*CVR+V1*CVI
IF(IS1. EQ. 0. AND. S. LE. 0.) GO TO 2
IF(IS1. EQ. 1. AND. S. GE. 0.) GO TO 2
IS1=IS1+1
JS1=JS1+1
IF(IS1. GE. 2) IS1=0

REVERSE FOIL 1 POINTS IF TE NOT LAST
Q(1, NPT+1)=0.
ZR(1, NPT+1)=ZR(1, NPT+2)
IZ(1, NPT+1)=IZ(1, NPT+2)
IF(ABS(Q(1, NPT+2)). GT. TOL) CALL ADV(1, 0)
CALL ADV(1, 0)
DO 15 J=1, NPT
J1=NPT+1-J
ZR(1, J)=ZER(1, J1)
15 ZI(1, J)=ZER(1, J1)
ZR(1, NPT+1)=ZR(1, NPT)
IZ(1, NPT+1)=IZ(1, NPT)
ZR(1, NPT+2)=2.*ZR(1, NPT)-ZR(1, NPT-1)
IZ(1, NPT+2)=2.*IZ(1, NPT)-IZ(1, NPT-1)
CALL SLEN(ZR, ZI, NV, SL)
IF(JS2. EQ. 2) GO TO 7
S=U2*CVR-V2*CVI
IF(IS2. EQ. 0. AND. S. LE. 0.) GO TO 5
IF(IS2. EQ. 1. AND. S. GE. 0.) GO TO 5
IS2=IS2+1
JS2=JS2+1
IF(IS2. EQ. 2) IS2=0

REVERSE FOIL 2 POINTS IF TE NOT LAST
Q(2, NPT+1)=0.
ZR(2, NPT+1)=ZR(2, NPT+2)
IZ(2, NPT+1)=IZ(2, NPT+2)
IF(ABS(Q(2, NPT+2)). GT. TOL) CALL ADV(0, 1)
CALL ADV(0, 1)
DO 17 J=1, NPT
J2=NPT+1-J
ZR(2, J)=ZER(2, J2)
17 ZI(2, J)=ZER(2, J2)
ZR(2, NPT+2)=2.*ZR(2, NPT)-ZR(2, NPT-1)
IZ(2, NPT+2)=2.*IZ(2, NPT)-IZ(2, NPT-1)
CALL SLEN(ZR, ZI, NV, SL)
IF(IS1. EQ. 11. OR. IS2. EQ. 12) RETURN

IF FOIL CIRCULATION TOO LARGE SET TO ZERO
CLM=(CLMAX+CLMIN)/2.
IF(CL1. GT. CLMIN. AND. CL1. LT. CLMAX) GO TO 10
IF(ABS(CL1-CLM). LT. ABS(CL2-CLM)) GO TO 10
JS1=1
9320  JS2=1
9330  IR1=1
9340  10 IF(CL2 .GT. CLMIN. AND. CL2 .LT. CLMAX) GO TO 11
9350  IF(ABS(CL1-CLM) .GE. ABS(CL2-CLM)) GO TO 11
9360  JS2=1
9370  JS1=1
9380  IR2=1
9390  11 IF(JR1. NE. IR1. OR. JR2. NE. IR2) RETURN
9400  NR=0
9410  RETURN
9420  300 FORMAT(2(F6.2,2(I4,2X),3X))
9430  END
Fortran '76 program STR finds the locus of NPSI equally spaced streamlines between specified values PSI1 and PSI2 within a circle of radius RL. Function PSIC(X,Y) is defined to take on the value of the stream function at the point X,Y. The program finds each streamline by searching for the proper value of PSIC(X,Y) along the circumference of the circle of radius RL and then along the x and y axes. After each streamline is located, its path is determined by projecting out DS in the direction of the previous two points and interpolating on each side of the projection until the proper value of PSIC(X,Y) is found. A provision is included for reducing the step size DS near the rotor where the streamlines change direction more abruptly, and for plotting the wake vorticity with a different symbol when its strength is very small. A file is created for use by a plotting program which contains a coordinate list for each streamline and the foils, labeling, and title information.
100 * PROGRAM STR
110 * THIS PROGRAM FINDS THE LOCUS OF STREAMLINES OF
120 * FUNCTION PSIC(X,Y)
130 COMMON UO, OMEGA, T, VR(312), VI(312), G(312), P12, NV(2)
140 DIMENSION X(251), Y(251), P12(2)
150 OPEN (UNIT=1, NAME="STRLIN")
160 OPEN (UNIT=2, NAME="STRLOUT")
170 P12=6.28318
180 * PLOT INFORMATION
190 * NPSI=NUMBER OF STREAMLINES
200 * RL=PLLOT RADIUS
210 * PSI1, PSI2=FIRST AND LAST VALUES OF PSI
220 * DS=STEP SIZE ALONG STREAMLINE
230 READ(1, 101) NPSI, RL, PSI1, PSI2, DS
240 PRINT, "NPSI, RL, PSI1, PSI2, DS"
250 WRITE(2, 203) RL
260 DSS=DS
270 * FLOW INFORMATION
280 * UO=VELOCITY AT INFINITY
290 * OMEGA=SPEED OF ROTATION
300 * T=TIME
310 * VR, VI=LOCATION OF VORTICES
320 * G=VORTEX STRENGTH
330 * NV=NUMBER OF VORTICES
340 * READ(1, 102) UO, OMEGA, T, ORDU
350 READ(1, 103) NPT, NV(1), NV(2)
360 WRITE(2, 205) NPT, NV(1), NV(2)
370 * DUMMY READ
380 READ(1, 102) VR(1), VI(1)
390 WANG=OMEGA*T+360. / P12
400 30 IF(WANG, GE. 0., AND. WANG, LE. 360.) GO TO 29
410 IF(WANG, LT. 0.) WANG=WANG+360.
420 IF(WANG, GT. 360.) WANG=WANG-360.
430 GO TO 30
440 29 WRITE(2, 202) ORDU, T, OMEGA, WANG
450 NVW=NV(1)-NV(2)
460 WRITE(2, 205) NPT, NV(1), NV(2)
470 * SYMBOL CHANGE WHEN WAKE CIRCULATION LESS THAN
480 * GMIN
490 GMIN=.10
500 DO B I=1, NVW
510 READ(1, 102) VR(I), VI(I), G(I)
IF(I.L.E. NPT) GO TO 28
IF(I.GT. NV(I). AND. I.LE. NV(1)+NPT) GO TO 28
ISY=1
IF(ABS(Q(I)).LT. GMIN) ISY=0
GO TO 8
28 ISY=0
WRITE(2,204) I, VR(I), VI(I), ISY
PRINT, "U0, OMEGA, T, NPT"
PRINT, U0, OMEGA, T, NPT
SET UP LIMITS
DPSI=ABS((PSI2-PSI1)/FLOAT(NPSI-1))
WRITE(2,201) NPSI
MEND=IFIX(15*RL/DS)
IF(MEND. GT. 250) MEND=250
CHORD=SQRT(((VR(1)-VR(NPT/2))**2+(VI(1)-VI(NPT/2))**2)
DE=RADIUS NEAR ENDS FOR REDUCING STEP SIZE
DE=CHORD/3
FIND INITIAL VALUE ALONG STREAMLINE
DO 20 L=1,NPSI
PSI=PSI1+(PSI2-PSI1)*FLOAT(L-1)/FLOAT(NPSI-1)
DTH=PI/24.
THT=--DTH
NTIMES=0
SEARCH AROUND CIRCUMFERENCE
P2=PSI-PSIC(RL,0.)
3 THT=THT+DTH
NTIMES=NTIMES+1
IF(NTIMES. GT. 25) GO TO 6
P1=P2
THT=THT+DTH
P2=PSI-PSIC(RL*COS(THT1),RL*SIN(THT1))
IF(SIGN(P1,P2). EQ. P1) GO TO 3
FIRST INTERPOLATION
NTIMES=0
PP=P1-P2
P2=PSI-P2
5 THT2=THT1-DTH*ABS((PSI-P2)/PP)
X(1)=RL*COS(THT2)
Y(1)=RL*SIN(THT2)
P3=PSIC(X(1),Y(1))
IF(ABS(P3-P3). LT. DPSI. 01) GO TO 4
DTH=2*DTH=(PSI-P3)/PP
THT1=THT2-DTH
X2=RL*COS(THT1)
Y2=RL*SIN(THT1)
P2=PSIC(X2,Y2)
PP=P2-P3
NTIMES=NTIMES+1
IF(NTIMES. GT. 10) GO TO 6
GO TO 5
SEARCH ALONG Y AXIS
DO 20 L=1,20
1070  Y(2)=-RL*DTH*FLOAT(I)
1080  P1=P2
1090  P2=PSI-PSIC(0.,Y(2))
1100  IF(SIGN(P1,P2).NE.P1) GO TO 21
1110  2 CONTINUE
1120  GO TO 27
1130  21 THS2=1.5708
1140  PP=P1-P2
1150  P2=PSI-P2
1160  X(1)=0.
1170  DO 22 I=1,10
1180  Y(1)=Y(2)-DTH*ABS((PSI-P2)/PP)
1190  P3=PSIC(0.,Y(1))
1200  IF(ABS(PSI-P3).LT.DPSI*01) GO TO 4
1210  DTH=-2*DTH*(PSI-P3)/PP
1220  Y(2)=Y(1)+DTH
1230  P2=PSIC(0.,Y(2))
1240  22 PP=P2-P3
1250  SEARCH ALONG X AXIS
1260  27 DTH=RL/10.
1270  P2=PSI-PSIC(-RL,0.)
1280  DO 24 I=1,20
1290  X(2)=-RL*DTH*FLOAT(I)
1300  P1=P2
1310  P2=PSI-PSIC(X(2),0.)
1320  IF(SIGN(P1,P2).NE.P1) GO TO 25
1330  24 CONTINUE
1340  GO TO 23
1350  25 PP=P1-P2
1360  P2=PSI-P2
1370  Y(1)=0.
1380  DO 26 I=1,10
1390  X(1)=X(2)-DTH*ABS(PSI-P2)/PP
1400  P3=PSIC(X(1),0.)
1410  IF(ABS(PSI-P3).LT.DPSI*01) GO TO 4
1420  DTH=-2*DTH*(PSI-P3)/PP
1430  X(2)=X(1)+DTH
1440  P2=PSIC(X(2),0.)
1450  26 PP=P2-P3
1460  23 PRINT, "NO INITIAL PSI", PSI
1470  WRITE(2,201) 0
1480  WRITE(2,201) 1,X(1),Y(1),PSI
1490  GO TO 1
1500  4 CONTINUE
1510  PRINT, "INIT PSIC,X,Y",P3,X(1),Y(1)
1520  * FIND LOCUS OF STREAMLINE
1530  * FIND DIRECTION OF NEXT STEP
1540  IF(THS2.GT.3.14159) THS2=THS2-5.28318
1550  THS=THS2+3.14159
1560  14 DO 7 M=1,10
1570  TH1=THS
1580  DS=DSS
1590  REDUCE STEP SIZE NEAR ENDS OF BODY
1600  A11=ABS(ABS(Y(M))-ABS(VI(1)))
1610  A12=ABS(ABS(Y(M))-ABS(VI(NPT)))
1620 IF(A11.GE.DE.AND.A12.GT.DE) GO TO 13
1630 AR1=ABS(ABS(X(M))-ABS(VR(1)))
1640 AR2=ABS(ABS(X(M))-ABS(VR(NPT)))
1650 IF(AR1.LT.DE.OR.AR2.LT.DE) DS=DSS/3.
1660 13 X1=X(M)+DS*COS(TH1)
1670 Y1=Y(M)+DS*SIN(TH1)
1680 + SEARCH EACH SIDE OF STEP FOR PSI
1690 9 ITER=0
1700 TH=0.
1710 DTH=.2
1720 FPI(I)=PSI-PSIC(X1,Y1)
1730 PPI(2)=PPI(I)
1740 19 TH=TH+DTH
1750 DO 18 I=1,2
1760 TH2=TH1+TH
1770 IF(I.EQ.2) TH2=TH1-TH
1780 X2=X(M)+DS*COS(TH2)
1790 Y2=Y(M)+DS*SIN(TH2)
1800 P2=PSI-PSIC(X2,Y2)
1810 IF(SIGP=PPI(I),P2),NE.PPI(I)) GO TO 10
1820 PPI(I)=P2
1830 18 CONTINUE
1840 IF(TH.GT.3) GO TO 15
1850 GO TO 19
1860 + FIRST INTERPOLATION
1870 10 IF(I.EQ.2) DTH=DTH
1880 PP=PPI(I)-P2
1890 P2=PSIC(X2,Y2)
1900 11 THS=TH2=DTH*ABS((PSI-P2)/PP)
1910 X3=X(M)+DS*COS(THS)
1920 Y3=Y(M)+DS*SIN(THS)
1930 P3=PSIC(X3,Y3)
1940 IF(ABS(PSI-P3),LT.DPSI*.01) GO TO 12
1950 DTH=2*DTH+(PSI-P3)/PP
1960 TH2=THS+DTH
1970 X2=X(M)+DS*COS(TH2)
1980 Y2=Y(M)+DS*SIN(TH2)
1990 P2=PSIC(X2,Y2)
2000 PP=P2-P3
2010 ITER=ITER+1
2020 IF(ITER.GT.6) GO TO 15
2030 GO TO 11
2040 12 X(M+1)=X3
2050 Y(M+1)=Y3
2060 + DO NOT REPEAT LINE
2070 IF(ABS((X1-X2)**2+(Y1-Y2)**2),LT.,6*DS) GO TO 15
2080 MF=M+1
2090 IF(ABS(X2).GT.RL.OR.ABS(Y2).GT.RL) GO TO 15
2100 7 CONTINUE
2110 + OUTPUT LOCUS OF STREAMLINE
2120 15 WRITE(2,201)MF
2130 WRITE(2,201)1,X(1),Y(1),PSI
2140 DO 16 I=1,MF
2150 WRITE(2,201)I,X(I),Y(I)
2160 16 CONTINUE
FUNCTION PSIC(X, Y)

THE VC0-UE OF PSI IS CALCULATED FOR NV VORTICES LOCATED AT VR, VI ROTATING AT OMEGA IN FREE STREAM U0

COMMON UO, OMEGA, T, VR(312), VI(312), G(312), PI2, NV(2)

COMMON /A/ X, Y

XP=X
YP=Y

CALL PSID(VR(I), VI(I), VR(I+1), VI(I+1), G(I), G(I+1), PSI)

CALL PSID(VR(I+1), VI(I+1), VR(I+2), VI(I+2), G(I+1), G(I+2), PSI)

RETURN
END

SUBROUTINE PSID(X1, Y1, X2, Y2, G1, G2, PSI)

VALUE OF STREAM FUNCTION AT X, Y FROM VORTEX SHEET WITH LINEARLY VARYING STRENGTH

COMMON /A/ X, Y

IF(X1.EQ.X2.AND.Y1.EQ.Y2) GO TO 2

XX=X2-X1
YY=Y2-Y1

IF(XX.EQ.0.) GO TO 1

AA=YY/XX

BB=YY/XX

C=AA+BB*XX

CC=SGRT(C)/6. 28318

DG=(G2-G1)/XX

CALL INTLN(X1, X2, A, B, C, SLNRX, SXLNX)

PSI=C-(G1-DG*X1)*SLNRX+DG*SXLNX

RETURN

AA=XX/YY

BB=X1-AA*YY

A=XX+B*YY-BB*BB-2.*BB*Y

B=2*(AA*BB-AA*YY-X)

C=AA+AA+1

CC=SGRT(C)/6. 28318

DG=(G2-G1)/XX

CALL INTLN(X1, X2, A, B, C, SLNRX, SXLNX)

PSI=C-(G1-DG*X1)*SLNRX+DG*SXLNX

RETURN

AA=XX/YY

BB=X1-AA*YY

A=XX+Y*Y+BB*BB+2.*BB*Y

B=2*(AA*BB-AA*YY-X)

C=AA+AA+1
2720 \( CC = \text{SGRT}(C)/6.28318 \)
2730 \( DG = (G2-G1)/YY \)
2740 CALL INTLN(Y1, Y2, A, B, C, SLNRX, SXLNRX)
2750 PSI = -CC*((G1-DG*Y1)*SLNRX+DG*SXLNRX)
2760 RETURN
2770 2 PSI = 0.
2780 RETURN
2790 END
2800 *
2810 * SUBROUTINE INTLN(X1, X2, A, B, C, SLNRX, SXLNRX)
2820 * PROVIDES INTEGRALS OF LOG(R) AND X*LOG(R)
2830 ARCTAN(U) = -.5*AIMAG(CLOG(CMPLX(1.,-U)/CMPLX(1.,U)))
2840 SN = 1.
2850 IF (X2 LT. X1) SN = -1.
2860 SN = 4*A*C-B*8
2870 B2 = B/(2.*C)
2880 R1 = C*X1**X1+B*X1+A
2890 R2 = C*X2**X2+B*X2+A
2900 IF (Q LE. 0.) GO TO 2
2910 T1 = 2*C*X1+B
2920 T2 = 2*C*X2+B
2930 SQ = SGRT(Q)
2940 ATN = SQ*(ARCTAN(T2/SQ)-ARCTAN(T1/SQ))/C
2950 GO TO 3
2960 2 ATN = 0.
2970 3 IF (R1 LE. 0.) GO TO 4
2980 AL1 = ALOG(R1)
2990 GO TO 5
3000 4 AL1 = 0.
3010 5 IF (R2 LE. 0.) GO TO 6
3020 AL2 = ALOG(R2)
3030 GO TO 7
3040 6 AL2 = 0.
3050 7 SLNRX = SN*.5*((X2+B2)*AL2-(X1+B2)*AL1+ATN)-X2+X1)
3060 SXLNRX = SN*.25*(R2*(AL2-1.0)-R1*(AL1-1.0))/C-B2*SLNRX
3070 RETURN
3080 END
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