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The Influence of Dimension Eight Operators on Weak Matrix Elements

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I describe recent work with V. Cirigliano and E. Golowich on the effect of dimension eight operators on weak nonleptonic amplitudes. The basic message is that there is an inconsistency in the way that many calculations are traditionally performed. If one calculates matrix elements involving only physics below some scale μ, then one needs dimension eight operators explicitly in the weak OPE. On the other hand if one wants to use dimensional regularization throughout, then one needs all scales to be included within the matrix element, and this results in the same net effect. A numerical estimate indicates that this is important below $\mu = 2$ GeV, and this calls into question many of the models that have been used to predict ϵ'/ϵ.

1. Introduction

In a way, the message of this talk is somewhat surprising as it implies that we have often not been proceeding correctly in calculating weak nonleptonic matrix elements over the last 25 years. There needs to be a significant modification to standard practice in most models. This is less of an issue for lattice evaluations but may be fatal for many models that work only at low scales.

We normally describe the weak Hamiltonian using the Operator Product Expansion (OPE)[1]. In words, we describe this as separating the physics into that involving energies above a scale μ and that below. The physics at high energies is represented by a series of local operators and one calculates the coefficients of these operators using perturbation theory. The physics below the scale μ goes into the hadronic matrix elements of the operators. The OPE for weak matrix elements then reads

$$\langle \mathcal{H}_W^{(\Delta S=1)} \rangle \;=\; \frac{G_F}{\sqrt{2}}\, V_{us} V_{ud}^* \sum_d \sum_i \, \mathcal{C}_i^{(d)}(\mu)\, \langle \mathcal{Q}_i^{(d)} \rangle_\mu \;. \quad (1)$$

where $\{\mathcal{Q}_i^{(d)}\}$ represents a complete basis set of operators of increasing dimension d, and $\mathcal{C}_i^{(d)}(\mu)$ are the coefficient functions. Dimension-six operators enter the OPE with a dimensionless coefficient, and dimension-eight would have a coefficient which scales as $1/(\text{mass})^2$.

Present practice uses dimensional regularization in the calculation of the coefficient functions[2–4]. In this case, one has the dimension eight coefficients of order $1/M_W^2$ and/or $1/m_c^2$ [5,6]. For the matrix elements one employs a framework which describes physics up to the scale μ. It is this combination that we will show is inconsistent. It must be modified in one of the following ways:

- If one calculates the matrix elements including physics only up to the scale μ, one must include dimension-eight operators in the OPE with coefficient of order $1/\mu^2$.

- If instead one wants to use dimensional regularization throughout, one must include physics of all scales in the matrix elements. The high energy portion of the matrix element above the scale μ is the effect of dimension-eight (and higher) operators.

2. Demonstration in a calculable framework

In our paper[7], we explicitly calculate the dimension eight operators within the Standard Model, and we could phrase the argument entirely within this theory. However, for pedagogical reasons we prefer to present the understanding of

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dimension eight effects using a slightly different interaction. The reason that this is useful is that it builds on the theory of QCD sum rules [8,9] in a simple way that lets us concretely demonstrate each of the key points. The hamiltonian which we use has the feature that hadronic matrix elements can be rigorously related to vacuum polarization functions, which are well understood.

This hamiltonian[10,11] contains one left-handed and one right-handed current instead of the usual Standard Model hamiltonian in which both currents are left-handed. Specifically we define[2]

$$\begin{aligned}
\mathcal{H}_{\rm LR} &\equiv \frac{g_2^2}{8}\int d^4x\ \mathcal{D}_{\mu\nu}(x,M_W^2)\ J^{\mu\nu}(x) \quad , \\
J^{\mu\nu}(x) &\equiv \frac{1}{2}T\left[\bar{d}(x)\Gamma^\mu_L u(x)\ \bar{u}(0)\Gamma^\nu_R s(0)\right] \\
&= \frac{1}{2}T\left[\left(V^\mu_{1-i2}(x)+A^\mu_{1-i2}(x)\right)\right. \\
&\qquad \left.\left(V^\nu_{4+i5}(0)-A^\nu_{4+i5}(0)\right)\right] \quad , \qquad (2)
\end{aligned}$$

where $\mathcal{D}_{\mu\nu}$ is the W-boson propagator and V^μ_a, A^μ_a $(a=1,\dots 8)$ are the flavor-octet vector, axialvector currents.

In the chiral limit, the K-to-pi matrix element is given in the chiral limit of zero momentum and vanishing light-quark masses by the vacuum matrix element

$$\begin{aligned}
\mathcal{M} &\equiv \lim_{p=0}\mathcal{M}(p) = \frac{g_2^2}{16F_\pi^2}\int d^4x\ \mathcal{D}(x,M_W^2) \\
&\quad \langle 0|T\left(V^\mu_3(x)V_{\mu,3}(0)-A^\mu_3(x)A_{\mu,3}(0)\right)|0\rangle \qquad (3)
\end{aligned}$$

Using properties of the vacuum polarization function, this can be transformed into momentum space

$$\begin{aligned}
\mathcal{M} &= \frac{3G_F M_W^2}{32\sqrt{2}\pi^2 F_\pi^2}\int_0^\infty dQ^2\ \frac{Q^4}{Q^2+M_W^2} \\
&\qquad \left[\Pi_{V,3}(Q^2)-\Pi_{A,3}(Q^2)\right] \quad . \qquad (4)
\end{aligned}$$

Here $\Pi_{V,A}$ are the vector and axial vector vacuum polarization functions We will use this simple expression in our analysis. The reader is interested in more details of how this result is obtained is referred to the original papers[10,11]. However the only essential point for the present argument is that this matrix element involves an integration over all scales of some hadronic quantity.

The usual OPE of this matrix element is expressed in terms of two LR dimension-six operators. Although it it is easy to perform the renormalization group summation, for our purpose here I just display the OPE to first order in α_s,

$$\mathcal{M} \simeq \frac{G_F}{2\sqrt{2}F_\pi^2}\left[\langle\mathcal{O}_1^{(6)}\rangle_\mu + \frac{3}{8\pi}\ln\left(\frac{M_W^2}{\mu^2}\right)\langle\alpha_s\mathcal{O}_8^{(6)}\rangle_\mu\right] . \qquad (5)$$

where

$$\begin{aligned}
\mathcal{O}_1^{(6)} &\equiv \bar{q}\gamma_\mu\frac{\tau_3}{2}q\ \bar{q}\gamma^\mu\frac{\tau_3}{2}q - \bar{q}\gamma_\mu\gamma_5\frac{\tau_3}{2}q\ \bar{q}\gamma^\mu\gamma_5\frac{\tau_3}{2}q \quad , \\
\mathcal{O}_8^{(6)} &\equiv \bar{q}\gamma_\mu\lambda^a\frac{\tau_3}{2}q\ \bar{q}\gamma^\mu\lambda^a\frac{\tau_3}{2}q \\
&\quad -\bar{q}\gamma_\mu\gamma_5\lambda^a\frac{\tau_3}{2}q\ \bar{q}\gamma^\mu\gamma_5\lambda^a\frac{\tau_3}{2}q \quad . \qquad (6)
\end{aligned}$$

In the above, $q = u,d,s$, τ_3 is a Pauli (flavor) matrix, $\{\lambda^a\}$ are the Gell Mann color matrices and the subscripts on $\mathcal{O}_1^{(6)}$, $\mathcal{O}_8^{(6)}$ refer to the color carried by their currents.

Now lets look at the direct calculation of the matrix element using the vacuum polarization functions. First we will consider the case where we separate the physics above and below some value of $Q^2=\mu^2$, with μ large enough that we can use perturbation theory for the high energy portion. Most interesting for our purposes here is the high energy contribution. The asymptotic behavior of the vacuum polarization operator is described by the operator product expansion, involving a series of local operators ordered by increasing dimension. In the chiral limit the leading contribution to the difference of vector and axial-vector correlators is a four-quark operator of dimension six[8,12], followed by a series of higher dimensional operators,

$$(\Pi_{V,3}-\Pi_{A,3})(Q^2) \sim \frac{2\pi\langle\alpha_s\mathcal{O}_8^{(6)}\rangle_\mu}{Q^6}+\frac{\mathcal{E}_\mu^{(8)}}{Q^8}+\dots \qquad (7)$$

Here $\mathcal{E}_\mu^{(8)}$ represents the combination of local operators carrying dimension eight. These have been discussed and partially calculated by Broadhurst and Generalis [13]. For our purposes, it is not necessary to know their specific form, but

[2] We define the chiral matrices $\Gamma^\mu_{\substack{L\\R}} \equiv \gamma^\mu(1\pm\gamma_5)$.

only the fact of their existence. Upon performing the integration over Q^2 at high energies, we find

$$\mathcal{M}_>(\mu) = \frac{3G_F}{32\sqrt{2}\pi^2F_\pi^2}\left[\ln\left(\frac{M_W^2}{\mu^2}\right)2\pi\langle\alpha_s\mathcal{O}_8^{(6)}\rangle_\mu + \frac{\mathcal{E}_\mu^{(8)}}{\mu^2}+\ldots\right] \quad . \tag{8}$$

Here is the first indication of the dimension-eight effect. In this matrix element it clearly appears as a contribution to the final answer, it is scaled with $1/\mu^2$, and it arises from the high energy portion of the calculation, above μ.

The low energy portion of the integral goes all into the matrix element of $\mathcal{O}_1^{(6)}$. This is more or less clear, but will be shown explicitly below. The full amplitude is then becomes

$$\mathcal{M} \simeq \frac{G_F}{2\sqrt{2}F_\pi^2}\left[\langle\mathcal{O}_1^{(6)}\rangle_\mu^{(\mathrm{c.o.})} + \frac{3}{8\pi}\ln\left(\frac{M_W^2}{\mu^2}\right)\langle\alpha_s\mathcal{O}_8^{(6)}\rangle_\mu + \frac{3}{16\pi^2}\frac{\mathcal{E}_\mu^{(8)}}{\mu^2}+\ldots\right] \quad . \tag{9}$$

Comparison of this result with the usual OPE shows that the dimension-eight term was not properly accounted for in the OPE. This is an illustration of the first itemized point in the introduction - if one performs a strict separation of scales, one needs dimension-eight terms in the OPE, scaled by $1/\mu^2$.

However, we can never get $1/\mu^2$ effects in dimensional regularization because the scale that enters there (which I will call $\mu_{\mathrm{d.r.}}$ from now on) can only appear in logs in 4 dimensions. The point is that dimensional regularization is not a separation of scales. We can see this explicitly in the calculation of the local operator matrix element. We can obtain this amplitude by taking the $x \to 0$ limit of the vacuum polarization functions as defined in d dimensions, which results in

$$\begin{aligned}&\langle\mathcal{O}_1^{(6)}\rangle_{\mu_{\mathrm{d.r.}}}^{(\mathrm{d.r.})} \\ &\quad\equiv \langle 0|T\left(V_3^\mu(0)V_{\mu,3}(0)-A_3^\mu(0)A_{\mu,3}(0)\right)|0\rangle \qquad (10)\\ &= \frac{(d-1)\mu_{\mathrm{d.r.}}^{4-d}}{(4\pi)^{d/2}\Gamma(d/2)}\int_0^\infty dQ^2\;\; Q^d\left(\Pi_{V,3}-\Pi_{A,3}\right)(Q^2)\end{aligned}$$

When $d<4$, this expression is finite. A key point is that the integral continues to run over all Q^2. Even without evaluating the integral we can see that to know its value we must include physics from above the scale $\mu_{\mathrm{d.r.}}$, since there is no separation of scales.

Let us look at the relation of these two schemes To this end, we split again the Q^2 integral into regions below and above $Q^2=\mu^2$. For the part of the integration below separation scale μ^2 the integral is finite for all dimensions, and we can take the limit $d\to 4$. This portion of the integration then reproduces exactly the cutoff version of the matrix element. The difference between the cutoff and dimensional regularization comes entirely from the high energy region and again can be calculated

$$\langle\mathcal{O}_1^{(6)}\rangle_{\mu_{\mathrm{d.r.}}}^{\overline{(\mathrm{MS})}} = \langle\mathcal{O}_1^{(6)}\rangle_\mu^{(\mathrm{c.o.})} + \frac{3\alpha_s}{8\pi}\left[\ln\left(\frac{\mu_{\mathrm{d.r.}}^2}{\mu^2}\right)-\frac{1}{6}\right]\langle\mathcal{O}_8^{(6)}\rangle_\mu + \frac{3}{16\pi^2}\frac{\mathcal{E}_\mu^{(8)}}{\mu^2} \quad . \tag{11}$$

The mixing with the dimension six operator is expected. (This result is obtained in the $\overline{\mathrm{MS}}-NDR$ scheme - see [10] for details.) For our purposes, the main point here is that the high energy region does contribute to the matrix element, through the dimension eight effect. Comparison with Eq. (9) shows that *all* of the dimension-eight operator is shifted into the $\overline{\mathrm{MS}}$ definition of the dimension-six operator. This is seen to be consistent:

1. When one performs a separation of scales, one has the need for dimension-eight operators in the OPE scaled by $1/\mu^2$.

2. When one defines instead the OPE using dimensional regularization, one cannot get effects proportional to $1/\mu_{\mathrm{d.r.}}^2$, but the same effect appears contained within the dimension-six operator matrix element. Overall one obtains the same total matrix element in either case.[3].

[3] Since we are treating the dimension-six coefficients at leading-log order, we can ignore the nonlogarithmic

Another way to phrase this result is to note that, within dimensional regularization, the OPE is employed as a method for regularizing operators, not for separating scales. Regularizaton accounts for the most singular aspects of short distance physics, but not all of it. The dimension eight effect represents the leading component of the residual short distance physics. The two ideas of the OPE and the separation of scales were both put forward by Wilson at about the same time and they have tended to be combined together in our thinking. It is interesting that in dimensional regularization of nonleptonic operators we employ one of Wilson's ideas (the OPE, used a regularization tool) but not the other.

Within this calculation, we can also calculate reliably the magnitude of the dimension eight effect. This is because the vacuum polarization functions satisfy dispersion relations with the input being given by data on e^+e^- reactions and τ decays. The details are found in [7,10]. The relevant comparison is between the first and last terms of Eq. (11), i.e. by how much does the dimension eight effect shift the matrix element . In units of 10^{-7} we find

$$\mathcal{M} = \begin{cases} -0.12 \ + \ 0.64 + \dots & (\mu = 1 \text{ GeV}) \\ -0.28 \ + \ 0.30 + \dots & (\mu = 1.5 \text{ GeV}) \\ -0.44 \ + \ 0.17 + \dots & (\mu = 2 \text{ GeV}) \\ -0.89 \ + \ 0.04 + \dots & (\mu = 4 \text{ GeV}) \ . \end{cases} \quad (12)$$

where the first entry is the cut-off matrix element and the second is the dimension eight effect. We see that the effect is of order 100% for $\mu = 1.5$ GeV. At this scale and below, we would also need to consider yet higher dimension effects, and the whole calculation is out of control. At higher values of μ, the dimension-eight effect can be treated as a perturbation on the usual calculation, and it is still significant at $\mu = 2$ GeV. Certainly at $\mu = 4$ GeV, the effect is small enough to be neglected.

3. The problem with standard practice

The conflict with present practice comes because we mix these two methods. We use dimensional regularization for the calculation of the coefficient functions, but we generally calculate matrix elements using a framework that only included physics up to the scale μ. This is inconsistent and, one way or the other, some physics is missing.

The present methods for calculating matrix elements involve either lattice gauge theory or low energy models. In lattice gauge theory[14], there is a cutoff of the physics at a momentum scale $p \sim 1/a$, where a is the lattice spacing. Most weak matrix element calculations are evaluated at $\mu \sim 1/a$, and so are missing the physics beyond this scale. In the lattice community there already exists the recognition that one needs to add back in the physics from higher energies. One example of this insight is embodied in the Syzmansik improvement program[15,16], which does involve adding higher dimension operators. However, it is my understanding that this is not yet implemented in weak matrix element calculations.

Low energy models[17] use quark models or models based on low-energy effective field theories. These involve physics which is valid at low energies only. Therefore these techniques most often involve a cut-off that separates low-energy from high-energy physics, and includes only low energy physics in the matrix elements. As we have shown, this is inconsistent when used in connection with the usual OPE. Very often the cut-off is taken to be very low, $\mu \sim 0.7 \to 1.0$ GeV. This leads to an enormous uncertainty due to dimension eight effects. The uncertainty of such methods is then far greater than was previously estimated.

Most of the estimates of ϵ'/ϵ involve low energy models for the estimate of the gluonic penguin operator matrix element, often labeled B_6. This matrix element has proven difficult to calculate on the lattice, and therefore in most of the reviews one obtains B_6 from low energy models of some sort. However, these are all suspect in light of their use of a low cut-off and their lack of the dimension eight effects. At this stage, I certainly cannot claim that this explains why the theoretical predictions are below the experimental value, as we do not even know the sign of the missing dimension-eight effects. However, I know of no

dimension-six portion of Eq. (11). To include it only requires an inclusion of the nonlogarithmic terms in the coefficient function.

complete evaluation of ϵ'/ϵ that is fully consistent theoretically.

4. What can be done?

Let me assume from now on that we agree to calculate using dimensional regularization. This is the most convenient framework for perturbative calculations. In this case, there are no dimension eight terms in the OPE and we can use all of the past results for the coefficients without change. However, in this case the matrix elements must be evaluated using a method that includes all scales. How can this be done?

In the case of lattice methods, there are several options. One is to simply push the scale μ high enough that the residual uncertainty from dimension-eight operators is small. By our estimates, $\mu \sim 4$ GeV should be sufficient. Alternatively one may use a range of lattice spacings and use the data to extrapolate to the continuum limit at fixed μ. This has been done successfully for the kaon B parameter[18]. A third option would be to use the Syzmansik improvement program also for weak matrix elements. This will involve adding in dimension-eight operators.

It will be more difficult to improve many of the quark-model and effective field theory models. Sometimes these models can be formulated in a way that includes physics over all scales, even if the short-distance physics differs from perturbative QCD (for example, see [19]). These "all scale" models can be used to estimate dimensionally regularized matrix elements if treated properly. However it is more common in such quark models to use a cut-off, as one recognizes that the model is not valid at higher energies. In this case, we would only be able to add back in the physics at high energies through the use of dimension-eight operators. This may be a difficult task.

The dispersive approach[10,20] of my collaborators and myself is a special case. Because our approach to two special matrix elements uses data as input, at least in the chiral limit, it is not a model. It can be applied over all energy scales and hence can readily provide dimensionally regularized matrix elements. In our previous work the impact of dimension eight operators was seen but not fully understood. It was treated as an uncertainty in the error bars which were quoted. Now that we have a better understanding of this effect, we will provide an update to the previous work which clarifies the proper treatment.

In our paper[7], we have displayed the dimension eight operators relevant for the Standard Model at one loop, in a particular method for the separation of scales. Perhaps these will be useful in estimating the magnitude of such effects, or correcting existing calculations.

REFERENCES

1. K. G. Wilson, "Nonlagrangian models of current algebra," Phys. Rev. **179**, 1499 (1969).
2. M. Ciuchini, E. Franco, G. Martinelli and L. Reina, "The ΔS = 1 effective Hamiltonian including next-to-leading order QCD and QED corrections," Nucl. Phys. **B415**, 403 (1994) [hep-ph/9304257].
3. A. J. Buras, M. Jamin, M. E. Lautenbacher and P. H. Weisz, "Two loop anomalous dimension matrix for ΔS = 1 weak nonleptonic decays. 1. O(α_s^2)," Nucl. Phys. **B400**, 37 (1993) [hep-ph/9211304].
4. For reviews *etc* see, A. J. Buras, "Weak hamiltonian, CP violation and rare decays," hep-ph/9806471.
   A. J. Buras, "Operator product expansion, renormalization group and weak decays," hep-ph/9901409.
   J. F. Donoghue, E. Golowich and B. R. Holstein, *Dynamics of the Standard Model*, (Cambridge Univ. Press, 1992).
   G. Buchalla, A. J. Buras and M. E. Lautenbacher, "Weak decays beyond leading logarithms," Rev. Mod. Phys. **68**, 1125 (1996) [hep-ph/9512380].
5. A. A. Penin and A. A. Pivovarov, "$\Delta S = 1$ effective nonleptonic Hamiltonian in the standard model: Dimension-eight operators," Phys. Rev. **D49**, 265 (1994).
6. J. Melsom, "The effect of higher-dimensional operators in $K \to \pi\pi$ decays," Z. Phys. **C75**, 471 (1997).
7. V. Cirigliano, J. F. Donoghue and E. Golowich, "Dimension-eight operators in

the weak OPE," hep-ph/0007196.

8. M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, "QCD and resonance physics – sum rules," Nucl. Phys. **B147**, 385 (1979).
9. L. J. Reinders, H. Rubinstein and S. Yazaki, "Hadron properties from QCD sum rules," Phys. Rept. **127**, 1 (1985).
   S. Narison, *QCD Spectral Sum Rules* (World Scientific, Singapore, 1989)
10. J. F. Donoghue and E. Golowich, "Dispersive calculation of $B_7^{(3/2)}$ and $B_8^{(3/2)}$ in the chiral limit," Phys. Lett. **B478**, 172 (2000) [hep-ph/9911309].
11. Other relevant work using this basic technique is found in: M. Knecht, S. Peris and E. de Rafael, "Matrix elements of electroweak penguin operators in the $1/N_c$ expansion," Phys. Lett. **B457**, 227 (1999) [hep-ph/9812471].
    S. Narison, "New QCD-estimate of the kaon penguin matrix elements and ϵ'/ϵ," hep-ph/0004247.
    J. F. Donoghue and E. Golowich, "Anatomy of a weak matrix element," Phys. Lett. **B315**, 406 (1993) [hep-ph/9307263].
12. L. V. Lanin, V. P. Spiridonov and K. G. Chetyrkin, "Contribution of four quark condensates to sum rules for rho and a_1 mesons", Sov. J. Nucl. Phys. **44**, 892 (1986).
13. D.J. Broadhurst and S.C. Generalis, "Dimension eight contributions to light quark QCD sum rules," Phys.Lett.**B165** (1985), 175.
14. A. Donini, V. Gimenez, L. Giusti and G. Martinelli, "Renormalization group invariant matrix elements of ΔS = 2 and ΔI = 3/2 four-fermion operators without quark masses," Phys. Lett. **B470**, 233 (1999) [hep-lat/9910017].
    L. Lellouch and C. J. Lin [(UKQCD Collaboration)], "ΔS = 2 and ΔI = 3/2 matrix elements in quenched QCD," Nucl. Phys. Proc. Suppl. **73**, 312 (1999) [hep-lat/9809142].
    D. Pekurovsky and G. Kilcup, "Lattice calculation of matrix elements relevant for ΔI = 1/2 rule and ϵ'/ϵ," hep-lat/9903025.
    D. Pekurovsky and G. Kilcup, "ΔI = 1/2 rule from staggered fermions," Nucl. Phys. Proc. Suppl. **73**, 309 (1999) [hep-lat/9809115].
    C. Dawson, G. Martinelli, G. C. Rossi, C. T. Sachrajda, S. Sharpe, M. Talevi and M. Testa, "New lattice approaches to the ΔI = 1/2 rule," Nucl. Phys. **B514**, 313 (1998) [hep-lat/9707009].
    R. Gupta, T. Bhattacharya and S. Sharpe, "Matrix elements of 4-fermion operators with quenched Wilson fermions," Phys. Rev. **D55**, 4036 (1997) [hep-lat/9611023].
    S. R. Sharpe, "Phenomenology from the lattice," hep-ph/9412243.
15. K. Symanzik, "Continuum Limit And Improved Action In Lattice Theories. 1. Principles And ϕ^4 Theory," Nucl. Phys. **B226**, 187 (1983).
16. I thank Chris Sachrajda for pointing out the relevance of the lattice improvement program and for discussions on these points.
17. J. Bijnens and J. Prades, "ϵ'_K/ϵ_K in the chiral limit," hep-ph/0005189.
    J. Bijnens and J. Prades, "The ΔI = 1/2 rule in the chiral limit," JHEP **9901**, 023 (1999) [hep-ph/9811472].
    S. Bertolini, J. O. Eeg and M. Fabbrichesi, "An updated analysis of ϵ'/ϵ in the standard model with hadronic matrix elements from the chiral quark model," hep-ph/0002234.
    S. Bertolini, J. O. Eeg, M. Fabbrichesi and E. I. Lashin, "The ΔI = 1/2 rule and $\hat{B}_K$ at O(p^4) in the chiral expansion," Nucl. Phys. **B514**, 63 (1998) [hep-ph/9705244].
    W. A. Bardeen, A. J. Buras and J. M. Gerard, "The ΔI = 1/2 rule in the large N limit," Phys. Lett. **B180**, 133 (1986).
    A. Buras, M. Jamin and M. E. Lautenbacher, "The anatomy of ϵ'/ϵ beyond leading logarithms with improved hadronic matrix elements," Nucl. Phys. **B408**, 209 (1993) [hep-ph/9303284].
    T. Hambye, G. O. Kohler, E. A. Paschos, P. H. Soldan and W. A. Bardeen, "$1/N_c$ corrections to the hadronic matrix elements of Q_6 and Q_8 in $K \to \pi\pi$ decays," Phys. Rev. **D58**, 014017 (1998) [hep-ph/9802300].
    T. Hambye, G. O. Kohler and P. H. Soldan, "New analysis of the ΔI = 1/2 rule in kaon decays and the $\hat{B}_K$ parameter," Eur. Phys. J.

C10, 271 (1999) [hep-ph/9902334].
18. S. R. Sharpe, "B_K Using staggered fermions: An update," Nucl. Phys. Proc. Suppl. **34**, 403 (1994) [hep-lat/9312009].
19. S. Peris and E. de Rafael, "$K^0 - \bar{K}^0$ mixing in the $1/N_c$ expansion," hep-ph/0006146.
J. Bijnens and J. Prades, "ϵ'_K/ϵ_K at next-to-leading in $1/N_c$ and to lowest order CHPT," hep-ph/0009156.
20. V. Cirigliano and E. Golowich, "Analysis of O(p^2) corrections to $< \pi\pi|\mathcal{Q}_{7,8}|K >$" Phys. Lett. **B475**, 351 (2000)