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# Time-dependent secular evolution in galaxies

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## ABSTRACT

Lynden-Bell & Kalnajs (1972, LBK) presented a useful formula for computing the long-range torque between spiral arms and the disk at large. The derivation uses second-order perturbation theory and assumes that the perturbation slowly grows over a very long time (the *time-asymptotic limit*). This formula has been widely used to predict the angular momentum transport between spiral arms and stellar bars between disks and dark-matter halos. However, this paper shows that the LBK time-asymptotic limit is *not* appropriate because the characteristic evolution time for galaxies is too close to the relevant dynamical times. We demonstrate that transients, not present in the time-asymptotic formula, can play a major role in the evolution for realistic astronomical time scales. A generalisation for arbitrary time dependence is presented and illustrated by the bar–halo and satellite–halo interaction. The natural time dependence in bar-driven halo evolution causes quantitative differences in the overall torque and qualitative differences in the physical- and phase-space location of angular momentum transfer. The time-dependent theory predicts that four principal resonances dominate the torque at different times and accurately predicts the results of an N-body simulation. In addition, we show that the Inner Lindblad Resonance (ILR) is responsible for the peak angular momentum exchange but, due to the time dependence, the changes occur over a broad range of energies, radii and frequencies. We describe the implication of these findings for the satellite–halo interaction using a simple model and end with a discussion of possible impact on other aspects secular galaxy evolution.

**Key words:** stellar dynamics — methods: n-body simulations — galaxies: evolution — galaxies: kinematics and dynamics — dark matter — Galaxy: structure

## 1 INTRODUCTION

A near-equilibrium galaxy only evolves through the excitation of some non-axisymmetric structure such as spiral arms, bars, infalling satellites, etc. The resulting “waves” are likely actors in triggering star formation, mediating inward gas flow thereby fueling AGN, heating the stellar disk, among other observable phenomena. In addition, the overall evolution of near equilibrium galaxies is caused by the collective transport of energy and angular momentum from these same structures. An accurate theory of secular evolution is essential for developing our understanding of these interrelated dynamics.

The first step is predicting the evolution of the non-axisymmetric distortions. The underlying dynamical mechanism for the stellar and dark matter components is understood. At the level of individual orbits, an excitation such as a bar, a spiral pattern, or orbiting satellite produces a periodic distortion on all orbits, much like a driven harmonic oscillator. Although the torque distorts a typical orbit, this distortion averages to zero over many orbital periods for most orbits. However, for nearly closed orbits<sup>1</sup>, the net torque does not vanish. An adiabatic invariant is broken by the vanishing precession frequency of the closed orbit; clearly, there is no adiabatic invariant for a degree of freedom with no motion. The perturbation then couples to the oscillatory distortion for these orbits and net torque is transferred to or from these orbits. This coupling between the forced oscillation and the perturbation will be second order in the perturbation strength. In time, the perturbation itself and

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<sup>1</sup> A closed orbit is one with integral commensurabilities between the orbital frequencies and the pattern frequency.

the underlying equilibrium is slowly changed by this torque. This leads to a finite measure of orbits that transfer angular momentum to and from the pattern (for additional discussion, see Weinberg & Katz 2004, hereafter Paper 2).

Although the first-order perturbation may dominate the instantaneous changes to phase space, the second-order changes describe the net changes that will persist after many dynamical times. This process was described mathematically by Lynden-Bell & Kalnajs (1972, hereafter LBK) in the limit that the change in the perturbation is very slow but still fast enough that the orbital perturbations remain linear. LBK derived a formula describing the exchange of angular momentum between a spiral pattern and the rest of the disk assuming that the perturbation began infinitely long in the past, the time-asymptotic limit. In this limit, any vestigial response from the formation of the pattern is gone. The LBK approach has been widely applied to estimate secular evolution (e.g. Goldreich & Tremaine 1979; Weinberg 1985; Zhang 1998; Athanassoula 2003). Carlberg & Sellwood (1985) took the first step towards a time-dependent generalisation of the LBK formalism for studying secular disk evolution. They included time-dependent transients but evaluated the secular changes in the infinite time limit. This is appropriate for studying the cumulative effect of short-lived transients. However, the perturbation theory may be fully generalised to treat arbitrary time-dependent perturbations over arbitrary intervals of time. This allows us to treat the long-lived bar and satellite interactions over the full age of a galaxy with some surprising results.

In attempting a detailed description of the angular momentum exchange between a rotating bar and a dark halo in an N-body simulation (see Paper 2), I found a significant discrepancy between the predictions of the second-order LBK perturbation theory and the N-body simulation. This discrepancy persisted for perturbations over a wide range of bar amplitudes and scaled with time and perturbation amplitude as expected from second-order linear theory, implying that its source is not a break down in linearity. Rather, the problem is due to the assumption of an infinitely slow growth of the perturbation. Not only is the number of characteristic dynamical times in a galactic age modest, the growth of a bar, arms and of course satellites is most likely a small fraction of the galactic age. Because patterns often appear over several orbital periods and the total number of orbital periods available are small, transients may be significant and the time-asymptotic limit does not apply. I will show in this paper that a finite-time-limit generalisation of the LBK formula gives quantitatively and qualitatively different results. Quantitatively, we will see that the overall torque is smaller than that computed from the LBK formula for a rotating bar. Qualitatively, different commensurabilities are important at different epochs of the evolution. Because the location of the resonances in physical space are important for the long-term evolution of galaxy, including the finite-time response is important for understanding this evolution. In this limit, the time dependence is more than simply a transient but embodies the galaxy's evolutionary history.

I will present a new, generalised secular evolution formula and describe the important details of the dynamics in §2. The new torque formula replaces the delta function in the LBK formula by an integral over the time-dependent perturbation and is similarly straightforward to apply. Two specific applications are presented in §3: bar–dark halo coupling and satellite–dark halo coupling. N-body simulations will demonstrate the importance of the finite-time response to the net torque and illustrate the discrepancy with the LBK formula. In §3.1, we will see that the ILR dominates the bar evolution and use this new formalism to explicitly predict the location of the angular momentum deposited. The same dynamics is then applied to sinking satellites in §3.2. We will conclude with §4.

## 2 BASIC PRINCIPLES

The total torque on an ensemble of orbits perturbed from equilibrium is

$$\left\langle \frac{dL_z}{dt} \right\rangle = \int_{\Gamma} d^3x d^3v f(\mathbf{x}, \mathbf{v}, t) \frac{dL_z}{dt}(\mathbf{x}, \mathbf{v}) \quad (1)$$

where  $\Gamma$  is the phase-space domain and  $f(\mathbf{x}, \mathbf{v}, t)$  is the phase-space distribution function,  $\dot{L}_z \equiv dL_z/dt$  is the torque per orbit, and  $\langle \cdot \rangle$  denotes the phase-space average. The angular momentum  $L_z$  is a conserved quantity in an equilibrium axisymmetric system and therefore  $\dot{L}_z$  is first-order in the perturbation amplitude. The distribution function has, of course, non-vanishing zeroth- and first-order terms. Over a long period of time, the first-order contribution to  $\langle \dot{L}_z \rangle$  averages to zero. At the next order, the oscillating components from the first-order dependence in  $\langle \dot{L}_z \rangle$  and the first-order induced change in the distribution function,  $f_1(\mathbf{x}, \mathbf{v}, t)$ , coherently reinforce each other, leading to a non-vanishing contribution as  $t \rightarrow \infty$ . However, a closed-form solution to this problem does not demand the  $t \rightarrow \infty$  limit. We will derive the first-order changes in these two quantities in §2.1 and §2.2 and put these together §2.3 to derive the torque for a finite duration perturbation.

### 2.1 Change in angular momentum for a single orbit

Hamilton's equations determine the rate of change in angular momentum for each orbit as follows:

$$\frac{dL_z}{dt} = \frac{\partial L_z}{\partial t} + [H, L_z] \quad (2)$$

where  $[\cdot, \cdot]$  are the Poisson brackets. Because  $L_z$  is a conserved quantity in the absence of any perturbation, this equation becomes

$$\frac{dL_z}{dt} = \frac{\partial H}{\partial \mathbf{I}} \cdot \frac{\partial L_z}{\partial \mathbf{w}} - \frac{\partial H}{\partial \mathbf{w}} \cdot \frac{\partial L_z}{\partial \mathbf{I}} = -\frac{\partial H_1}{\partial \mathbf{w}} \cdot \frac{\partial L_z}{\partial \mathbf{I}} + \mathcal{O}(A^2) \quad (3)$$

using action-angle variables  $\mathbf{I}$ ,  $\mathbf{w}$ ,  $H_1$  is the first-order, perturbed Hamiltonian and  $A$  is the perturbation amplitude. The subscripts ‘0’ and ‘1’ will indicate zeroth- and first-order quantities, respectively.

The orbits in an axisymmetric equilibrium galaxy are purely quasi-periodic in the angles  $\mathbf{w}$  and the actions  $\mathbf{I}$  are invariant. The advantage of action-angle variables is that any phase-space quantity may be expanded in a Fourier series in  $\mathbf{w}$ . In particular, we now expand  $H_1$  in action-angle variables:

$$H_1(\mathbf{I}, \mathbf{w}, t) = \sum_{\mathbf{l}} H_{1\mathbf{l}}(\mathbf{I}, t) e^{i\mathbf{l} \cdot \mathbf{w}} \quad (4)$$

where

$$H_{1\mathbf{l}}(\mathbf{I}, t) = \frac{1}{(2\pi)^3} \int_0^{2\pi} d\mathbf{w} e^{-i\mathbf{l} \cdot \mathbf{w}} H_1(\mathbf{I}, \mathbf{w}, t) \quad (5)$$

and  $\mathbf{l} = (l_1, l_2, l_3)$  is a vector of integers, one for every degree of freedom. Applying the angle transform to equation yields (3)

$$\frac{dL_z}{dt} = -\sum_{\mathbf{l}} i l_3 H_{1\mathbf{l}}(\mathbf{I}, t) e^{i\mathbf{l} \cdot \mathbf{w}}. \quad (6)$$

Physically, equations (3) and (6) describe the first-order evolution of a particular orbit’s  $z$  angular momentum component.

## 2.2 Changes in the distribution function

The development begins with the first-order solution of the linearised collisionless Boltzmann equation (CBE):

$$\frac{\partial f_1}{\partial t} + \frac{\partial H_0}{\partial \mathbf{I}} \cdot \frac{\partial f}{\partial \mathbf{w}} - \frac{\partial H_1}{\partial \mathbf{w}} \cdot \frac{\partial f_0}{\partial \mathbf{I}} = 0. \quad (7)$$

The quantities  $f_0$  and  $H_0$  are the unperturbed phase-space distribution function and the unperturbed Hamiltonian, etc. We will solve for the phase-space dependence by an action-angle expansion (as in eq. 4) and for the time evolution by Laplace transform of the CBE. We denote a Laplace transformed variable by a tilde and an action-angle transformed variable by subscript  $\mathbf{l}$ . The Fourier-Laplace transform of the CBE is

$$s \tilde{f}_1 + i\mathbf{l} \cdot \boldsymbol{\Omega} \tilde{f}_1 - i\mathbf{l} \cdot \frac{\partial f_0}{\partial \mathbf{I}} \tilde{H}_{1\mathbf{l}} = 0 \quad (8)$$

or, upon solving for  $\tilde{f}_1$ ,

$$\tilde{f}_1(\mathbf{l}, s) = \frac{i\mathbf{l} \cdot \frac{\partial f_0}{\partial \mathbf{I}}}{s + i\mathbf{l} \cdot \boldsymbol{\Omega}} e^{i\mathbf{l} \cdot \mathbf{w}} \tilde{H}_{1\mathbf{l}}(\mathbf{l}, s). \quad (9)$$

where  $\boldsymbol{\Omega} \equiv \partial H_0 / \partial \mathbf{I}$ . We will now show that the Laplace transform rather than inclusion of a term  $\exp(\epsilon t)$  the limit  $\epsilon \rightarrow 0$  to impose the arrow of time allows one to solve for an arbitrary time dependence. The total perturbing potential  $H_1$  is the response of the galaxy combined with the external perturbation. The quantity  $\tilde{H}_{1\mathbf{l}}(\mathbf{l}, s)$  is the Laplace transform of the time-dependent coefficient of the action-angle expansion describing the perturbation:

$$\tilde{H}_{1\mathbf{l}}(\mathbf{l}, s) \equiv \int_0^\infty dt e^{-st} H_{1\mathbf{l}}(\mathbf{l}, t). \quad (10)$$

In cases where the perturbation is conveniently described in spherical harmonics, intrinsic symmetries allow  $\tilde{H}_{1\mathbf{l}}$  to be further simplified (see Tremaine & Weinberg 1984).

The desired solution is the inverse Laplace transform of the series:

$$f_1(s) = \sum_{l_1, l_2, l_3 = -\infty}^{\infty} \tilde{f}_1(\mathbf{l}, s) e^{i(l_1 w_1 + l_2 w_2 + l_3 w_3)} \quad (11)$$

where each coefficient  $\tilde{f}_1$  is given by equation (9). The inverse Fourier-Laplace transform of equation (11) is straightforward albeit cumbersome. Recall that the product of the first-order distribution function and the first-order torque per orbit gives the second-order torque. To proceed we first need to transform equation (9) back to the time domain. Combining equations (11) and (10), the inverse Laplace transform of equation (9) requires evaluation of the following expression:

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds \frac{e^{st}}{s + i\mathbf{l} \cdot \boldsymbol{\Omega}} \int_0^\infty dt' e^{-st'} H_{1\mathbf{l}}(\mathbf{l}, t') = \int_0^\infty dt' H_{1\mathbf{l}}(\mathbf{l}, t') \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds \frac{e^{s(t-t')}}{s + i\mathbf{l} \cdot \boldsymbol{\Omega}}$$

$$\begin{aligned}
&= \int_0^\infty dt' H_{11}(\mathbf{I}, t') \begin{cases} e^{-i\mathbf{l} \cdot \boldsymbol{\Omega}(t-t')} & \text{if } t > t' \\ 0 & \text{otherwise} \end{cases} \\
&= \int_0^t dt' H_{11}(\mathbf{I}, t') e^{i\mathbf{l} \cdot \boldsymbol{\Omega}(t'-t)}
\end{aligned} \tag{12}$$

where  $c$  is more positive than the any exponent of divergence in  $H_{11}(\mathbf{I}, t)$ . This is not restrictive for any realistic astronomical perturbation. The second equality follows by exchanging the order of integration and using Cauchy's theorem. Putting this together, we have the following time-dependent solution for the first-order distribution function

$$f_1(\mathbf{w}, \mathbf{I}, t) = \sum_{l_1, l_2, l_3 = -\infty}^{\infty} i\mathbf{l} \cdot \frac{\partial f_o}{\partial \mathbf{I}} e^{i\mathbf{l} \cdot \mathbf{w}} \left\{ \int_0^t dt' H_{11}(\mathbf{I}, t') e^{i\mathbf{l} \cdot \boldsymbol{\Omega}(t'-t)} \right\} \tag{13}$$

### 2.3 Second-order torque

To get the second-order torque as a function of actions (or other conserved quantities following Jeans' theorem), we multiply by equation (6) by equation (13) and integrate over all of phase space:

$$\left\langle \frac{dL_z}{dt} \right\rangle \equiv \int d\mathbf{I} \int d\mathbf{w} \frac{dL_z}{dt} [f_o(\mathbf{I}) + f_1(\mathbf{I}, \mathbf{w}) + \dots] = \int d\mathbf{I} \int d\mathbf{w} \frac{dL_z}{dt} f_1(\mathbf{I}, \mathbf{w}) + \mathcal{O}(A^3).$$

The first term is oscillatory in  $\mathbf{w}$  and vanishes. The product of the two first-order solutions, denoted here by subscripts  $\mathbf{l}$  and  $\mathbf{l}'$ , has an angle dependence of the form

$$\exp(-i\mathbf{l}' \cdot \mathbf{w}) \exp(i\mathbf{l} \cdot \mathbf{w}) = \exp(i[\mathbf{l} - \mathbf{l}'] \cdot \mathbf{w})$$

and therefore the angle integral is only non-zero when  $\mathbf{l}' = \mathbf{l}$ . Integrating this product over the remaining phase-space variables (actions) gives an expression for the total torque on the phase space from the perturbation:

$$\left\langle \frac{dL_z}{dt} \right\rangle = -(2\pi)^3 \sum_{l_1, l_2, l_3 = -\infty}^{\infty} l_3 \int d\mathbf{I} \mathbf{l} \cdot \frac{\partial f_o}{\partial \mathbf{I}} \left\{ \int_0^t dt' H_{11}(\mathbf{I}, t') e^{i\mathbf{l} \cdot \boldsymbol{\Omega}(t'-t)} \right\} H_{1-1}(\mathbf{I}, t). \tag{14}$$

Since physical quantities are real, equation (4) implies  $H_{11}^*(\mathbf{I}, t) = H_{1-1}(\mathbf{I}, t)$ . We may now change variables from actions to energy and angular momentum and the third action  $L_z$ , the  $z$ -component of the angular momentum, in terms of its ratio to the total angular momentum  $J$  as follows:  $\cos \beta \equiv L_z/J$ . The angle  $\beta$  describes the colatitude of the orbital plane. Equation (14) becomes

$$\left\langle \frac{dL_z}{dt} \right\rangle = -(2\pi)^3 \sum_{l_1, l_2, l_3 = -\infty}^{\infty} l_3 \int \int \frac{dE dJ J d(\cos \beta)}{\Omega_1(E, J)} \mathbf{l} \cdot \frac{\partial f_o}{\partial \mathbf{I}} \left\{ \int_0^t dt' e^{i\mathbf{l} \cdot \boldsymbol{\Omega}(t'-t)} H_{11}(\mathbf{I}, t') \right\} H_{11}^*(\mathbf{I}, t). \tag{15}$$

For a fixed gravitational potential profile whose only time dependence is through a rotating pattern, the time dependence in the perturbation separates as  $H_1(\mathbf{I}, \mathbf{w}, t) = a(t)H_1(\mathbf{I}, \mathbf{w})$  and therefore  $H_{11}(\mathbf{I}, t) = a(t)H_{11}(\mathbf{I})$ . Furthermore for a constant pattern speed, the time-dependent coefficients have the form:  $a(t) = a(0) \exp(im\Omega_p t)$ . The time-dependent terms in equation (15) then take the simple form

$$\begin{aligned}
\left\{ \int_0^t dt' e^{i\mathbf{l} \cdot \boldsymbol{\Omega}(t'-t)} a(t') \right\} a^*(t) &= e^{-i\mathbf{l} \cdot \boldsymbol{\Omega} t} \left\{ \int_0^t dt' e^{i\mathbf{l} \cdot \boldsymbol{\Omega} t'} a(t') \right\} a^*(t) \\
&= a(0) a^*(0) \frac{\sin[(\mathbf{l} \cdot \boldsymbol{\Omega} - m\Omega_p)t]}{\mathbf{l} \cdot \boldsymbol{\Omega} - m\Omega_p} \\
&\longrightarrow a(0) a^*(0) \pi \delta(\mathbf{l} \cdot \boldsymbol{\Omega} - m\Omega_p)
\end{aligned} \tag{16}$$

where the final expression obtains in the limit  $t \rightarrow \infty$ . Upon substituting into equation (15), we recover the LBK formula. The mathematics of this convergence also indicates a possible numerical pitfall in applying equation (14): at large times  $T$ , equation (16) will oscillate rapidly with  $E$  and  $J$  or  $\mathbf{I}$  and may require special treatment.

Equation (15) can be integrated to find the run of angular momentum  $L_z$  with time for some arbitrary time-dependent perturbation. At each time step, one must expand  $H_1(\mathbf{I}, t)$  for the desired range of  $\mathbf{I}$ , integrate over the temporal history, and then integrate over phase space. The solution for a general time dependence is computationally intensive, therefore. In the slowing-bar example (§3.1), the problem has separable time dependence of the form  $H_1(\mathbf{I}, \mathbf{w}, t) = a(t)H_1(\mathbf{I}, \mathbf{w})$ . In this equation (15) may be written

$$\left\langle \frac{dL_z}{dt} \right\rangle = -(2\pi)^3 \sum_{l_1, l_2, l_3 = -\infty}^{\infty} l_3 \int \int \int \frac{dE dJ J d(\cos \beta)}{\Omega_1(E, J)} \mathbf{l} \cdot \frac{\partial f_o}{\partial \mathbf{I}} \times$$

$$\left\{ \int_0^t dt' e^{i\mathbf{l} \cdot \boldsymbol{\Omega}(t'-t)} a(t') \right\} a^*(t) H_{11}(\mathbf{I}) H_{1-1}^*(\mathbf{I}). \quad (17)$$

In this case, a grid of  $H_{11}(\mathbf{I})$  along with the other phase-space dependent terms may be determined once and tabled for all time, simplifying the calculation. The integral  $\int_0^t dt' \exp[i\mathbf{l} \cdot \boldsymbol{\Omega}(t'-t)] H_{11}(\mathbf{I}, t')$  must be computed with sufficient small time steps to resolve the oscillations obtained with variations in  $\mathbf{I}$ . Finally, the right- and left-hand sides of equation (14) are generally coupled. For a bar–halo interaction with a fixed bar shape for example, the pattern speed is  $\Omega_p = L_z/I_z$  where  $I_z$  is the bar’s moment of inertia. Stability of the solution is then a concern. I used Gear’s method Gear (1969) to maintain stability for the numerical solutions presented here. The problem is numerically more stable when the right hand side is independent of  $L_z$  as is the case for a satellite–halo interaction whose orbital decay is computed from local Chandrasekhar dynamical friction (§3.2).

### 3 EXAMPLES

#### 3.1 Slowing bar

Following Paper 2, we compare the results of N-body simulations and linear theory for an ellipsoidal bar in an NFW dark-matter halo profile (Navarro et al. 1997):  $\rho(r) \propto r^{-1}(r+r_s)^{-2}$ . The fiducial model has a bar of length (semi-major axis)  $r_s$  with and axis ratios  $b/a = 0.2, c/a = 0.05$  and bar mass equal to one-half of the enclosed dark-halo mass. The total mass of the bar is 0.1 of the halo mass inside the virial radius for a concentration  $c = 15$ . A realistic present-day stellar bar would have a length roughly 25% of  $r_s$ . However, this length greatly increases the number of particles required in the N-body simulation. Because the NFW profile is scale free inside of  $r_s$ , the behaviour should scale to smaller bars. This is verified in Paper 2.

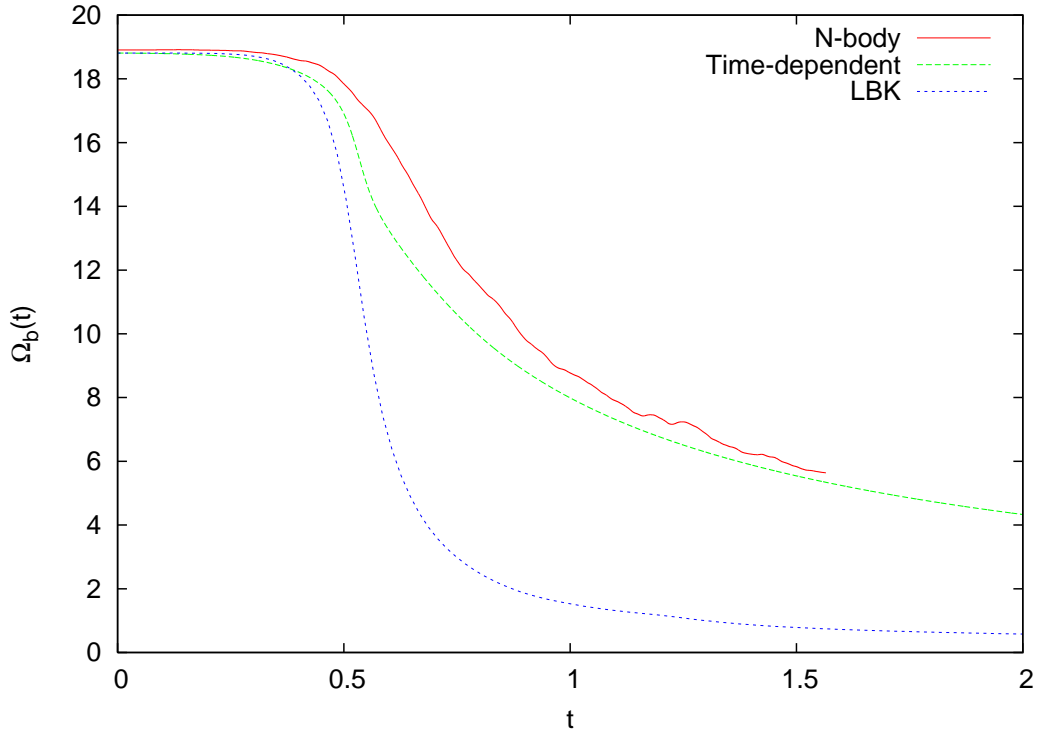
The pattern speed evolves while conserving the total angular momentum of the initial bar and dark halo system. The bar perturbation is slowly turned on according to  $f(t) = (1 + \operatorname{erf}[(t - T_0)/\Delta])/2$  with  $T_0 = 1/2$  Gyr and  $\Delta = 1/4$ , which corresponds to a full-width turn-on time of 1 Gyr scaled to the Milky Way. Bar formation in simulations occurs over a small number of dynamical timescales (e.g. within 200 Myr scaled to the Milky Way). An assumption of  $T_0 = 1/2$  and  $\Delta = 1/4$  then is an upper bound to a realistic formation time scale; we are choosing the slowest time dependence possible, accounting for bar length scale. Paper 2 describes the additional details of the simulation procedure.

Figure 1 compares the resulting evolution of the N-body simulation, the linear theory presented in §2 and the LBK formula. Because the bar figure is constant in time, the torque takes the form of equation (17). The N-body simulation closely follows the time-dependent linear theory prediction. The evolution is significantly slower than predicted by the LBK formula! Clearly, the time-dependence changes the torque. The second panel of Figure 1 repeats the simulation and linear theory prediction for a bar in a King model profile (King 1966) chosen to match the cumulative mass profile of the NFW profile inside of the virial radius ( $W_0 = 6, r_T = 2r_{vir}$ ) demonstrating that the differences due to time dependence are not a consequence of the NFW profile.

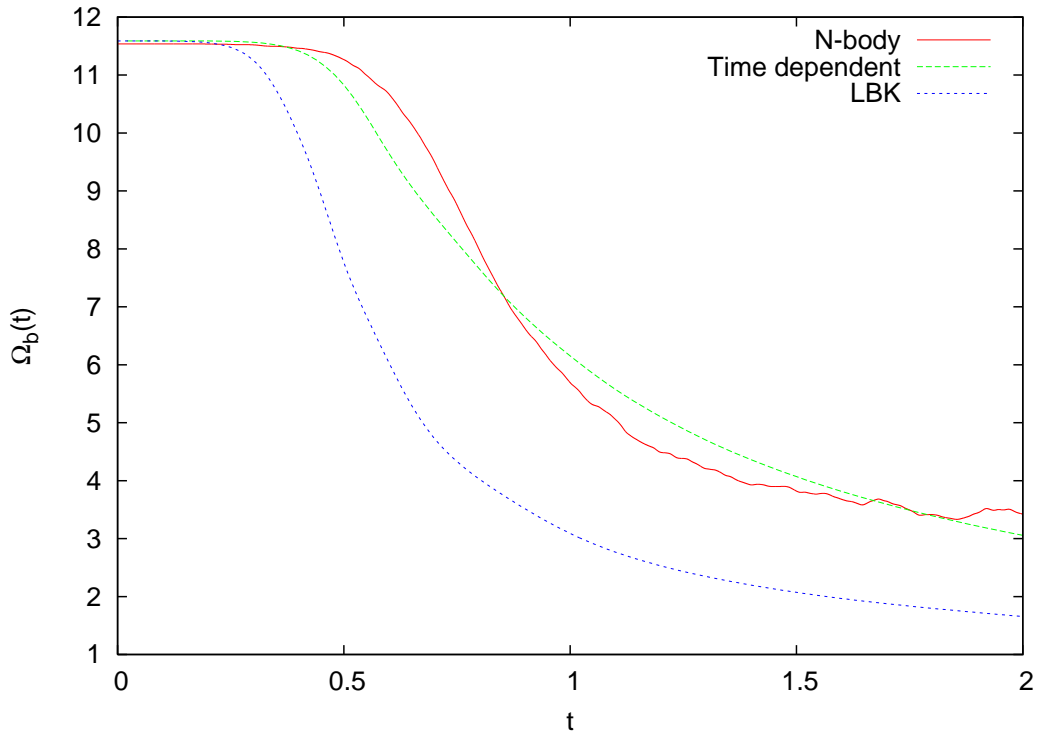
Equation (15) or (17) allows us to determine the contributions from each resonance separately (see Fig. 2) and determine the origin of the differences between the time-dependent theory and LBK. The qualitative differences between the case with explicit time-dependence and the time-asymptotic LBK regime are evident. For the LBK case, the torque is dominated by the ILR-like  $\mathbf{l} = (-1, 2, 2)$  resonance. The contribution peaks as the bar begins to slow and decreases when the pattern frequency becomes too small to easily couple to low-order resonances in the vicinity of the bar. For the time-dependent case, the corotation resonance  $(0, 2, 2)$  dominates at early times. During the rapid slowing phase,  $0.4 \lesssim t \lesssim 1$ , the ILR  $(-1, 2, 2)$  dominates. The ILR is weaker in the time-dependent theory and the simulation. This accounts for a large fraction of the difference in Figure 1. At later times, a near equilibrium obtains between the various commensurabilities, some without true resonances. These all converge to the LBK limit as  $t \rightarrow \infty$ .

In Figure 3, I show the torque from the ILR summed over all angular momentum plotted as a function of energy  $E$  (top axis) and radius  $r$  of the circular orbit with the same energy  $E$  and time  $t$  (bottom axis). The peak angular momentum deposition occurs inside the bar radius (due to the ILR) and corresponds with the radial location of the profile change described in Paper 2. Comparison of Figures 2 and 3 confirms that the peak torque occurs during the rapid pattern speed evolution which also accounts for the broad region of this contribution in energy for  $0.4 \lesssim t \lesssim 1$ . For  $t \gtrsim 1$ , the ILR oscillates at lower amplitude as it approaches the constant  $t \rightarrow \infty$  LBK value.

We now take a closer look at the contributions from all terms ( $l_1, l_2, l_3 = m$ ) in equation (17). The contribution to the torque for each term and the total is shown in Figure 5 for a variety of different bar lengths and turn-on parameters  $T_0, \Delta$ . The mass of the bar has been decreased by a factor of five to  $M_b = 0.01$  to lengthen the slow down period and help isolate the contribution of each resonance. For clarity, only the top four contributing resonances are drawn. The first panel of the describes the torque for a bar with length  $r_s$  with initial period  $1/3, T_0 = 1/2$  and  $\Delta = 1/4$ ; the bar grows to full strength over several bar periods. At  $t < 1/2$ , early in the bar growth, the corotation  $(0, 2, 2)$  and the  $(1, 0, 2)$  resonances dominate the torque (see also Fig. 2). At the peak of the pattern-speed decrease,  $t \approx 0.7$ , the ILR dominates the torque. Moreover, the ILR

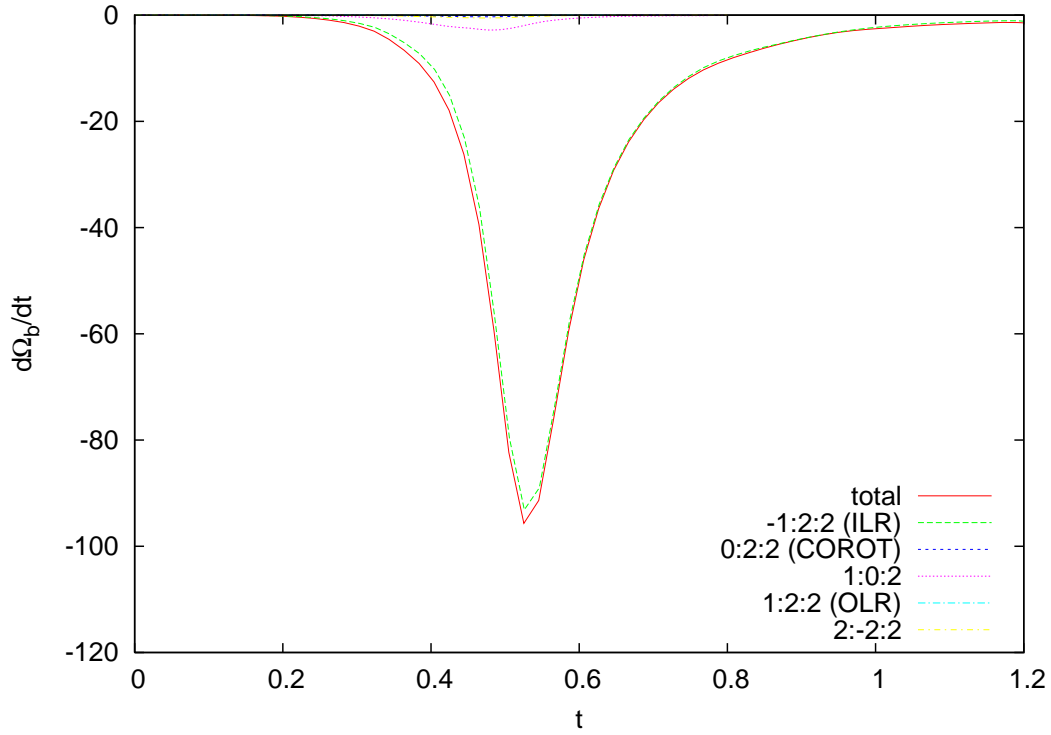


(a) NFW

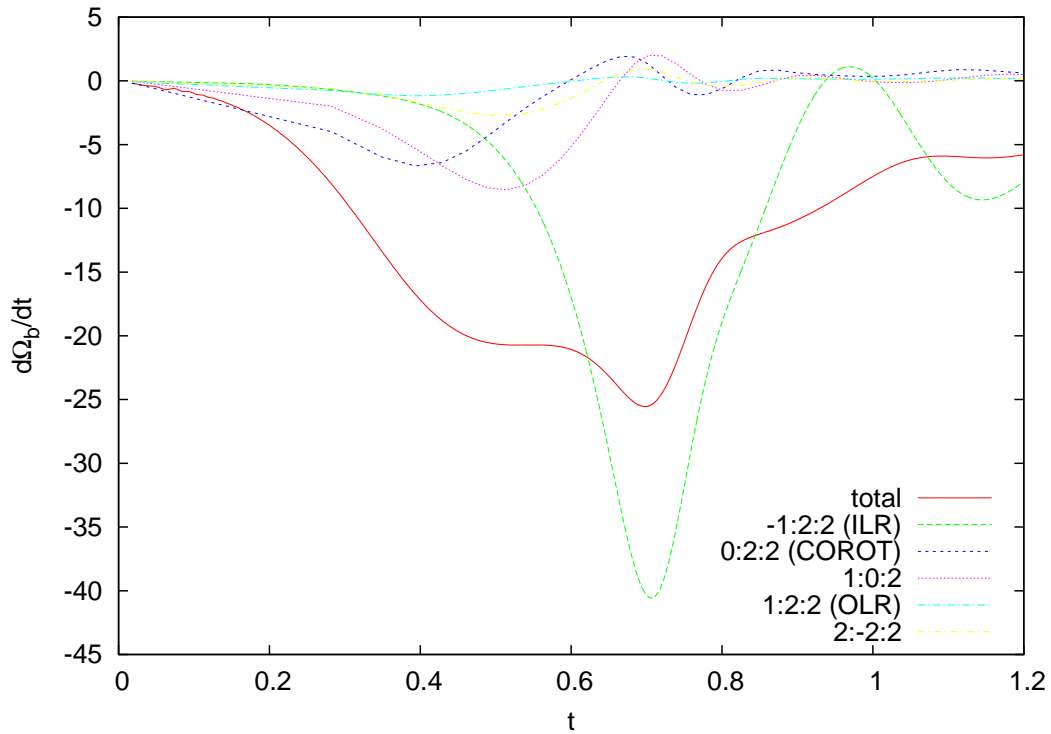


(b) King

**Figure 1.** The pattern speed evolution of a bar in a NFW (a) and King model (b) dark-matter halo for an N-body simulation and computed from the second-order perturbation theory in the time-dependent and LBK limits. The time-dependent theory accurately follows the pattern speed evolution while the time-asymptotic LBK theory does not.



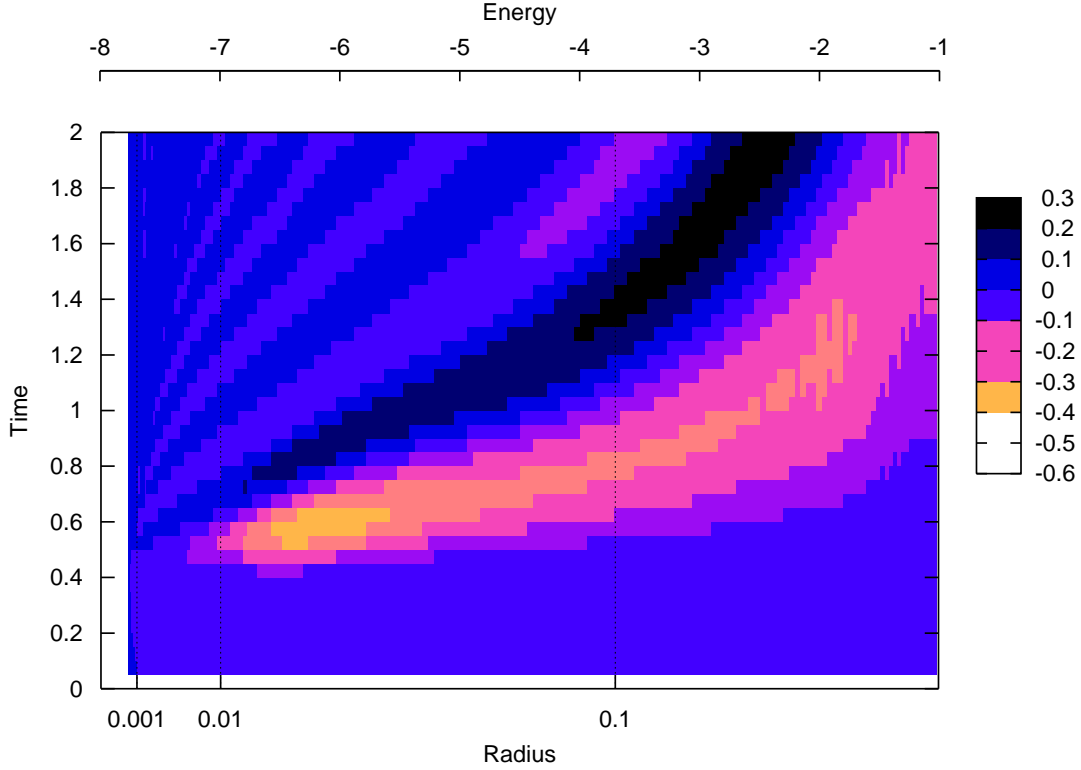
(a) LBK



(b) time-dependent

**Figure 2.** Relative contributions to the total torque  $\dot{L}_z = I_z \dot{\Omega}_z$  from each resonance for time-asymptotic (LBK) regime (a) and with explicit time-dependence (b). Each curve in the figure shows a single 1 term from equation (15) as labelled. The run of  $\dot{\Omega}_z$  derived from the total torque is also shown.





**Figure 3.** ILR contribution to the torque ( $\dot{\Omega}_z = \dot{L}_z/I_z$ ) as a function of energy (top axis) and corresponding radius of the corresponding circular orbit (bottom axis) as a function of time as the pattern speed evolves. This figure is obtained by solving equation (15) and plotting the energy contribution to the phase-space integral as a function of time for the same example shown in Fig. 2.

dominates to the integrated torque for the entire slowdown event. Note that terms  $l$  without pattern-speed commensurabilities,  $l_1\Omega_1 + l_2\Omega_2 \neq m\Omega_p$ , through the phase-space distribution also contribute transiently but significantly to the total torque. For example, the  $(1, -2, 2)$  term, an “negative” ILR contributes to the torque but would only be commensurate for a negative pattern speed. This transient echo decays after the rapid slow down phase since it is purely oscillating without a net second-order contribution. Nonetheless, it diminishes the torque during peak of the slow down. The ILR and the other commensurate terms, on the other hand, approaches the finite non-zero time-asymptotic value predicted by the LBK formula.

The relative importance of the torque to the inner halo is nicely visualised by computing the net change in angular momentum inside of some energy  $E$  from equation (15) as a function of time. That is, let  $\dot{L}(< E)$  denote equation (15) with the energy integral from  $[E_{min}, E]$ . Then, define the cumulative change of angular momentum in time as

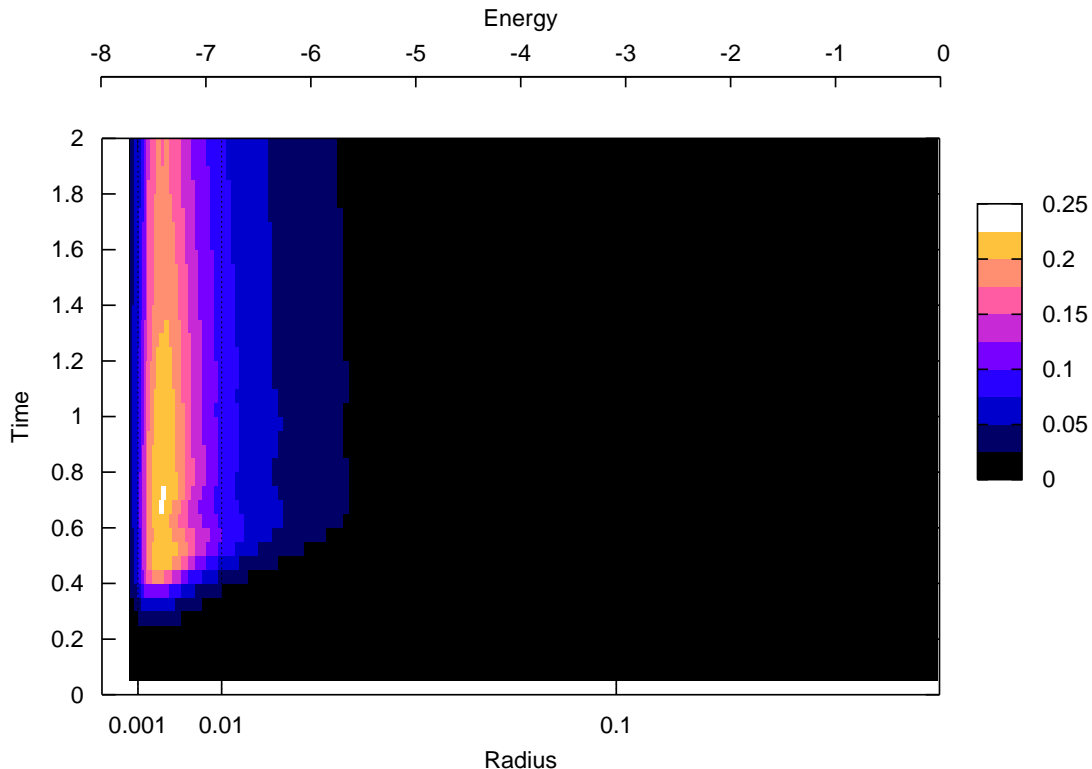
$$\Delta L(< E, < t) = \int_{t_o}^t dt \dot{L}(< E) \quad (18)$$

where  $t_o$  is the initial time. The total orbital angular momentum in the initial phase space with energy less than  $E$  is

$$\mathcal{E}(< E) = (2\pi)^3 \int \int \int \frac{dE dJ d(\cos \beta)}{\Omega_1(E, J)} f(E, J) J. \quad (19)$$

Then, the ratio of these two quantities,  $\Delta L(< E, < t)/\mathcal{E}(< E)$  indicates the importance of the resonant torque to the structure of the inner halo profile. This ratio is shown in Figure 4 for the bar–halo interaction depicted in Figure 3 for the ILR resonance. The contour scale shows the fractional change of the angular momentum in the halo inside of a given energy or radius. A large fractional change indicates a strong possibility of significant structural change. The peak *absolute* torque from the ILR occurs near  $r \approx 0.02$  and  $t \approx 0.6$  (Fig. 3) while the peak *relative* torque occurs near  $r \approx 0.003$  (Fig. 4). In particular, N-body simulations from Paper 2 show that this bar–halo interaction will flatten the NFW cusp for  $0.001 < r < 0.01$  as predicted by Figure 4. Hence, this comparison suggests that a 15%-20% fractional change is sufficient to drive inner-halo evolution.

The sequence in Panels (a)–(c) in Figure 5 shows the same bar model with different turn on parameters  $T_0$  and  $\Delta$ . The corresponding distribution of torque deposited as a function of time and energy (radius) is similar to Figure 3. For longer intervals of bar growth, the resonances are more localised in energy and radius. The steep ILR peak during the rapid evolution



**Figure 4.** Ratio of the cumulative angular momentum contributed by the ILR inside of energy  $E$  and radius  $r$  of the corresponding circular orbit at time  $t$  to the total angular momentum of the halo inside of energy  $E$ . In other words, the contour scale shows the fractional change of the angular momentum in the halo from the ILR resonance inside of a given energy or radius.

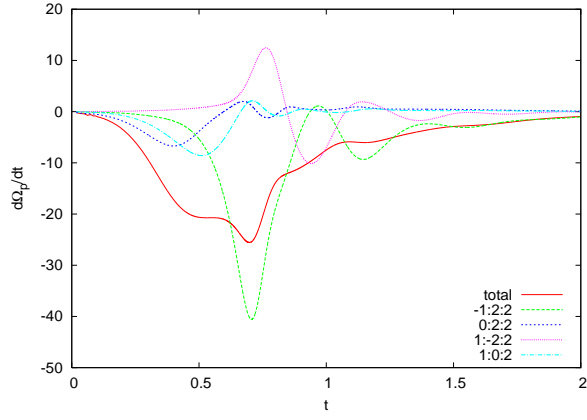
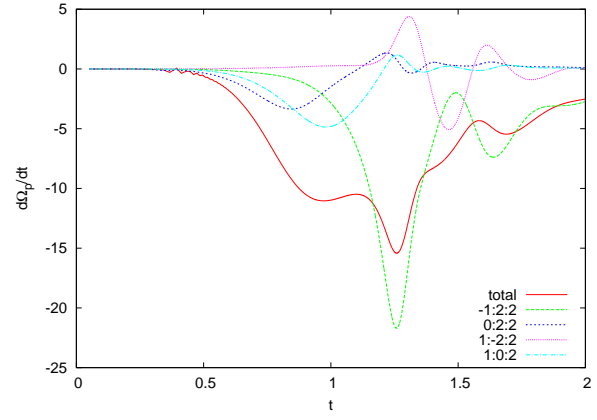
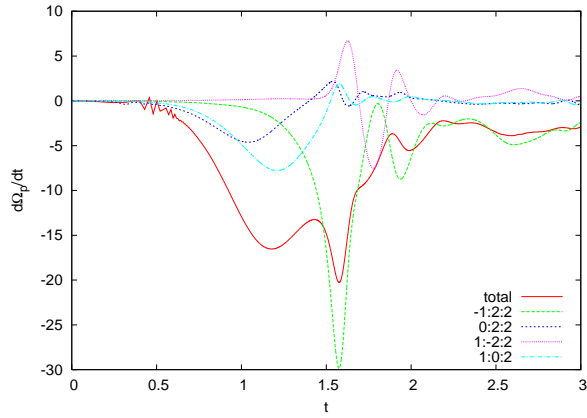
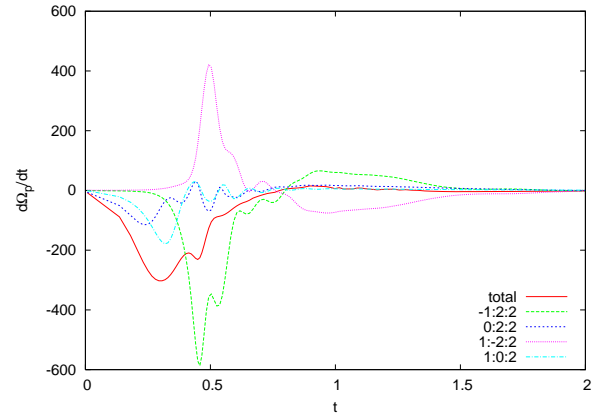
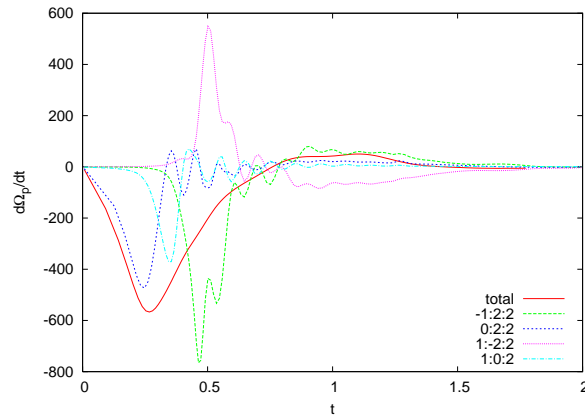
phase covers a broad swath in energy and radius and leads to profile evolution. At later times as the system approaches the time-asymptotic phase, the torque contributions become smaller and more localised. Transients remain important for the unrealistically slow turn on  $\Delta = 1/2$  (full width of 2 Gyr scaled to the Milky Way). The profiles for smaller, more realistic size bars are qualitatively similar to those for the large bar (Panels d & e in Fig. 5); ILR continues to dominate the total torque with corotation and  $(1, 0, 2)$  important at early times. For these smaller bars, the ILR changes sign after the main peak ( $T \approx 1$ ) as the initial response laps the slowing bar before settling to the time-asymptotic LBK value. The torque is deposited as smaller radii and energy as expected but still over a broad region inside of the bar radius. Paper 2 presents estimates of the particle number required to simulate the resonant interaction in these examples.

## 3.2 Decaying satellite

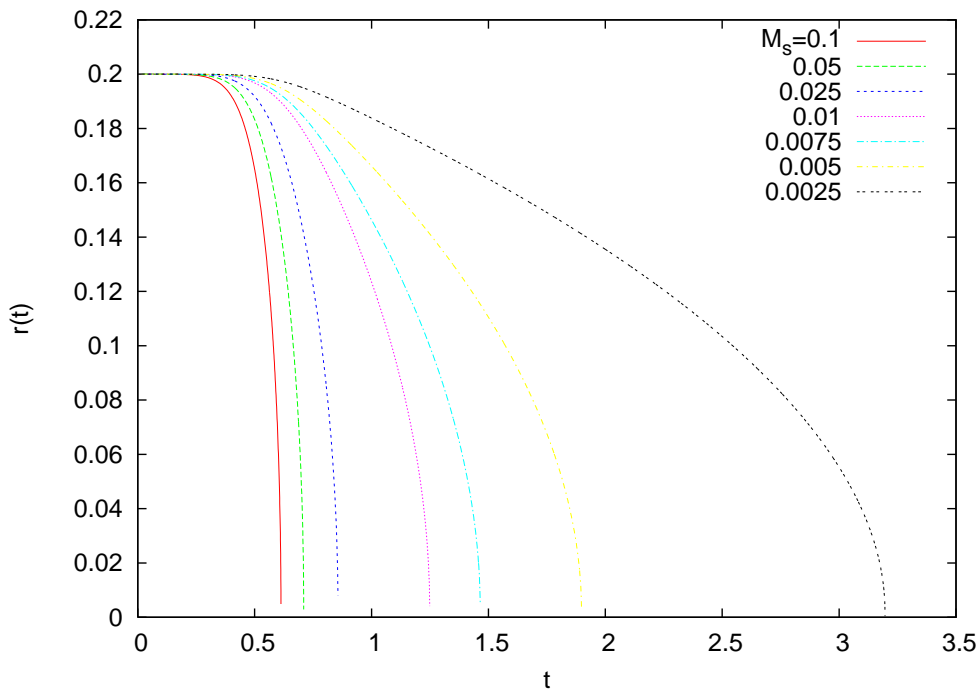
### 3.2.1 Circular orbits

Orbiting satellites couple to a *live halo* through resonances in much the same way as does a rotating bar. However, because the satellite size is generally much smaller than the halo scale, the total torque will have significant contributions from harmonic orders<sup>2</sup> larger than quadrupole. Just as in the case of the bar, the  $(-1, 2, 2)$  resonance can couple to very small radii although the satellite perturbation will have a different frequency spectrum and amplitude than the bar perturbation. Higher-order contributions have energies and response locations similar to the satellite. To investigate this situation, I assume that the satellite's orbit decays according to the local Chandrasekhar formula while remaining on a circular orbit. This, then determines the radius of the satellite orbit and the instantaneous orbital frequency,  $\Omega_s$ . The low-order response is then evaluated using the theory from §2. It is possible to compute the torque self-consistently by including a larger number of  $l$  terms (eq. 14). However, in this example, we only want to illustrate the contribution of the lower-order terms for comparison with the bar example and therefore the additional complexity and time to compute the self-consistent evolution is not justified. Because the satellite orbital radius changes in time, the torque takes the form of equation (15): the time-dependence in the

<sup>2</sup> The contribution scales as  $(\ln l)^{-1}$  where  $l$  is the harmonic order (Weinberg 1986).


 (a)  $a = 0.067$ ,  $M_b = 0.01$ ,  $T_o = 1/2$ ,  $\Delta = 1/4$ 

 (b)  $a = 0.067$ ,  $M_b = 0.01$ ,  $T_o = 1$ ,  $\Delta = 1/3$ 

 (c)  $a = 0.067$ ,  $M_b = 0.01$ ,  $T_o = 3/2$ ,  $\Delta = 1/2$ 

 (d)  $a = 0.02$ ,  $M_b = 0.015$ ,  $T_o = 1/2$ ,  $\Delta = 1/4$ 

 (e)  $a = 0.01$ ,  $M_b = 0.011$ ,  $T_o = 1/2$ ,  $\Delta = 1/4$ 

**Figure 5.** Relative contributions to the total torque  $\dot{L}_z = I_z \dot{\Omega}_z$  as in Fig. 2 for a range of bar lengths ( $a$ ), bar masses ( $M_B$ ) and growth times ( $T_0$  and  $\Delta$ ). Only the top four contributors are shown individually. The total torque is the sum over all contributing terms for  $l = m = 2$ .



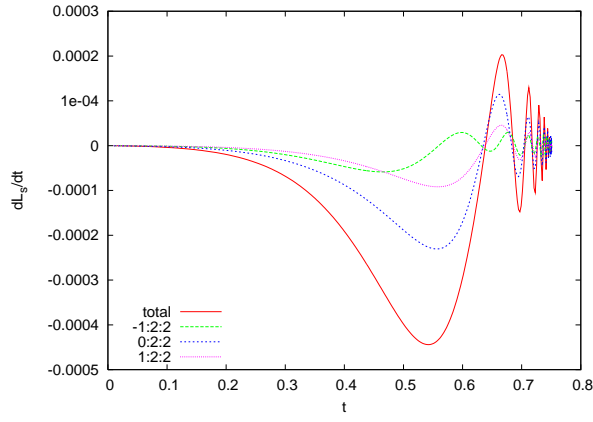
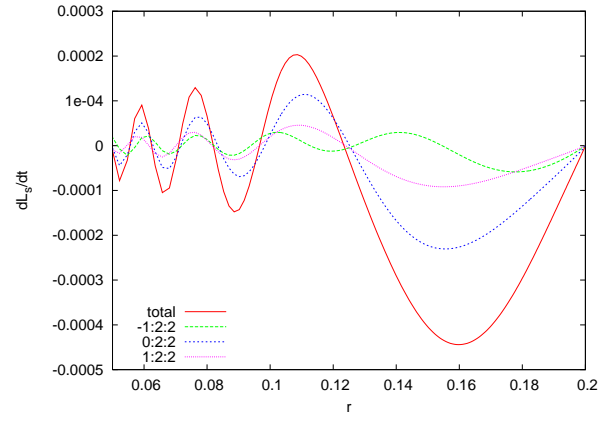
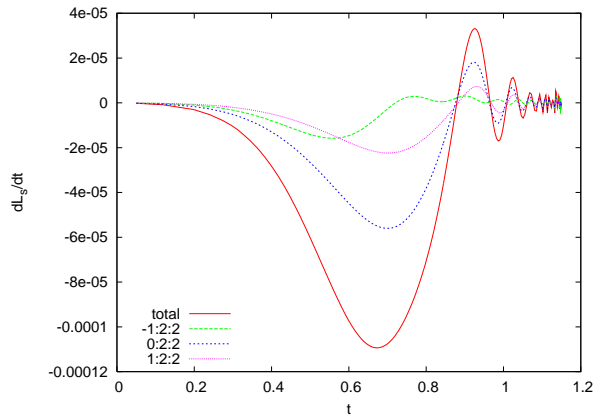
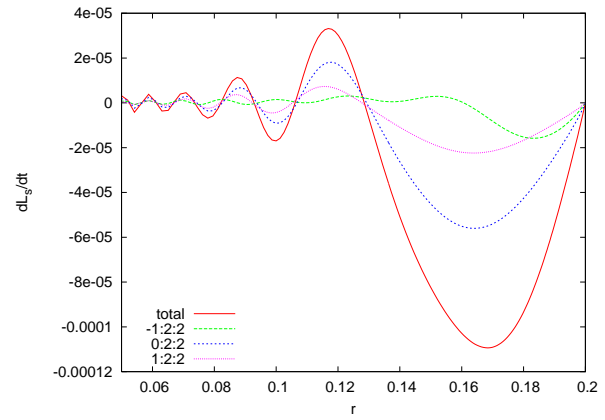
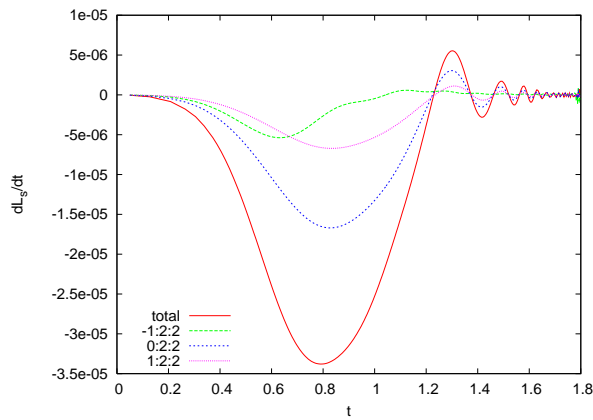
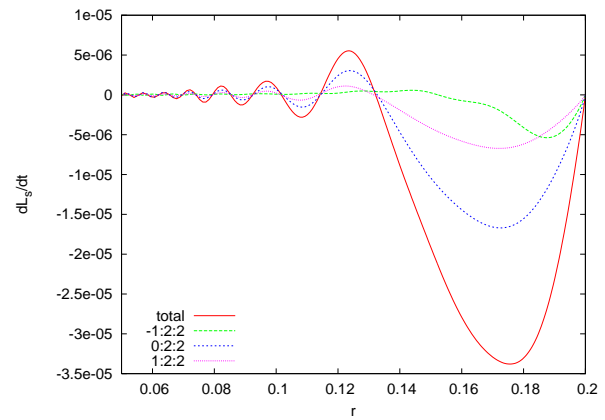
**Figure 6.** Change in radii with time due to orbital decay according to the local Chandrasekhar dynamical friction formula for nearly circular orbits for a variety of satellite masses  $M_s$  (in units of the halo virial mass).

rotation is not separable from the time-dependence of the changing Fourier coefficients in equation (5). The torque is then straightforwardly computed for low-order  $l$  and  $m$  using equation (17) although it is more computationally intensive than the bar example due to the non-separable time dependence than the bar evolution case.

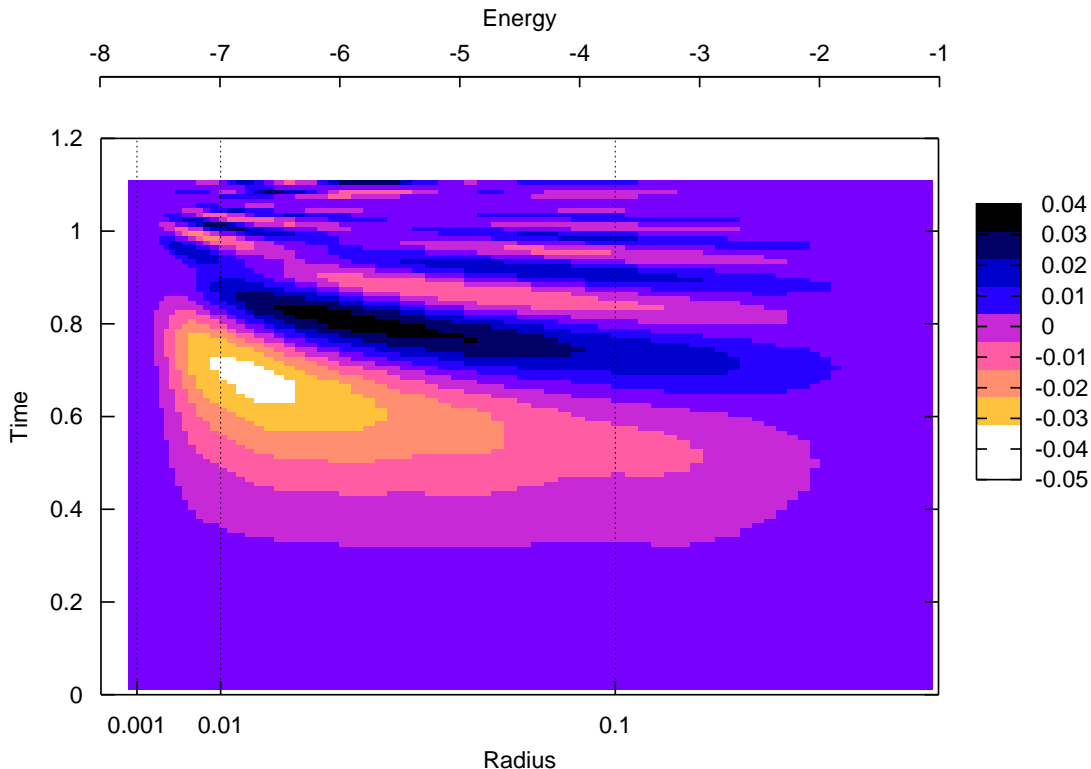
The initial orbital radius is  $r = 0.2$  (20% of the virial radius or approximately 60 kpc scaled to the Milky Way). Figure 6 shows the radius as a function of time for a variety of satellite masses  $M_s$  from 0.0025 to 0.1 virial mass units for interpreting the torque plots that follow. As in the case of the bar, the mass is slowly increased using  $f(t)$  (from §3.2) with  $T_0 = 1/2$  and  $\Delta = 1/4$ . The smallest mass satellite requires approximately 6 Gyr to decay from  $r = 0.2$  and the most massive satellite decays in less than 100 Myr.

Figure 7 shows the torque,  $dL_s/dt$ , as a function of time and orbital radius for the quadrupole  $l = m = 2$  spherical harmonic contribution. The torque for all values of  $M_s$  share some common trends: 1) the  $l = m = 2$  torque is dominated by corotation (0:2:2). At early times, there is a moderate ILR transient (-1:2:2) which changes sign and oscillates at later times and the satellite position laps the transient wake, alternatively shifting the sign of the torque. We can remove the turn-on and extrinsic amplitude dependence by dividing  $dL_s/dt$  by  $M_s^2$  and multiplying the time by  $M_s \int_0^t dt' f(t')$ . Each value of time, then, corresponds to the same satellite radius in each panel. The initial transient in  $dL_z/dt$  between radii 0.12 and 0.2 (Panels b, d, and f of Fig. 7) has a nearly self-similar form. The differences are due to *intrinsic* time dependence from transients. For example, as  $M_s$  decreases, the ILR becomes more localised in time with a larger peak amplitude.

Figure 8 illustrates the distribution of torque from the ILR as a function of time and energy and radius similar to Figure 3. Torque from the initial transient for the satellite orbiting at  $r \approx 0.15 \approx 2r_s$  is deposited at radii between 0.005 and 0.02 or between 7% and 30% of  $r_s$ . Figure 9 plots the ratio of the cumulative change in angular momentum in energy and time,  $\Delta L(< E, < t)/\mathcal{E}(< E)$ , for the satellite decay shown in Figure 8. As in the bar-halo interaction, the peak total torque occurs at much larger radii than the peak relative torque. Comparing to Figure 4, we see that the change in angular momentum is in the same region of phase space and larger than the 15%-20% fractional change found in the bar example. Since the halo evolves for the bar simulation, we anticipate evolution for the halo driven by this satellite. The N-body simulation is more difficult to perform in this case because of the disparate scales and will be the subject of a later paper (Choi et al. 2004). Since the decay time is proportional to  $M_s^{-1}$  and the torque is proportional to  $M_s^2$ , the total torque deposited in the transient phase is proportional to satellite mass. Therefore, as long as  $M_s$  is sufficiently large that the orbital decay takes place in a galaxy age, the angular momentum transport from a few larger merger events may affect the halo profile as described in Paper 2.

(a)  $M_s = 0.025$ (b)  $M_s = 0.025$ (c)  $M_s = 0.01$ (d)  $M_s = 0.01$ (e)  $M_s = 0.005$ (f)  $M_s = 0.005$ 

**Figure 7.** Time-dependent torque for various resonances for a decaying circular satellite orbit with  $\log \Lambda = 4$ ,  $T_o = 1/2$ ,  $\Delta = 1/4$ , and initial radius  $R_0 = 0.2$ . Only the top three contributing terms are shown individually.



**Figure 8.** ILR contribution to the torque as in Fig. 3 but for the satellite decay with  $M_s = 0.025$ .

### 3.2.2 Eccentric orbits

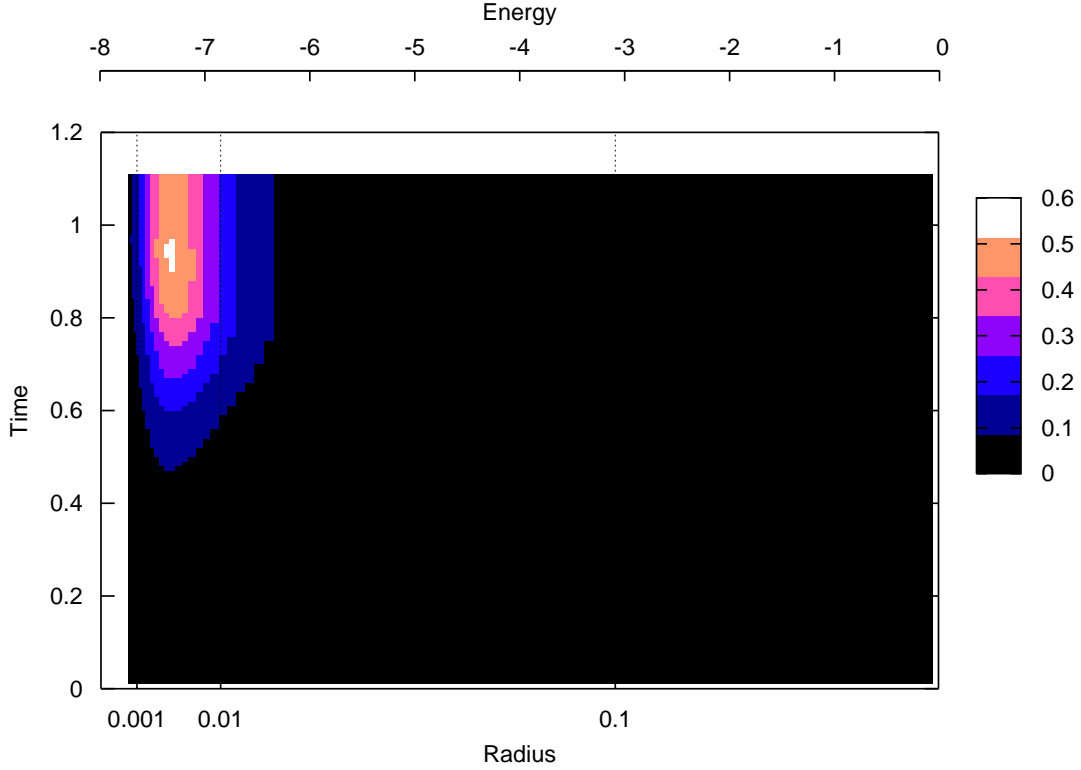
Eccentric orbits may be treated similarly using equation (17). Rather than enforce a circular orbit, one may solve the equations of motion for an orbit decaying according to the Chandrasekhar formula and use the solution, simultaneously, as the input to equation (17). In practice, it is convenient to solve the entire problem with a standard ODE solver. Unfortunately, there is a wide range of time scales in the problem: the decay time and the orbital time scales throughout phase space. In practice, then, accurate solutions require small time steps.

The additional satellite orbital frequency complicates the spectrum and, because the orbital radius now oscillates as well as decays, complicates the representation of the interaction. However, qualitatively, the behaviour is similar to the circular case: low-order resonances such as the ILR provide strong positive torque on the halo at early times and then become oscillatory. Unlike the circular case, an eccentric orbit will deposit its orbital angular momentum at lower energies because the amplitude of the perturbation will be larger at smaller radii. This response of the halo may be a wake that influences the subsequent evolution of the disk and the halo. We will describe our detailed findings for the satellite-halo interaction for eccentric orbits in a later paper.

## 4 SUMMARY

This paper shows the Lynden-Bell & Kalnajs (1972, LBK) secular torque formula is not a quantitatively accurate description of secular evolution in galaxies. The LBK formula assumes that the growth time of a perturbation driving secular evolution is infinitely long compared the characteristic dynamical time. However, not only is the number of characteristic dynamical times in a galactic age modest, the growth of a spiral arms, a bar or the decay of a large dwarf satellite has evolutionary time scales of order 100 Myr to 1 Gyr, uncomfortably close to the equilibrium galaxy lifetime of 10 Gyr or smaller. We have seen that the finite time effects yield a different distribution of angular momentum in the halo and a different total torque than for a time-asymptotic system assumed by LBK. Similarly, it is unlikely that time-dependent effects will affect time-asymptotic perturbation theory predictions in planetary systems Goldreich & Tremaine (1979); Goldreich & Tremaine (1980) where the secular evolution time scale is much larger than the dynamical time scale.

In §2, I presented generalised the LBK formula for perturbations of finite duration. The new formula shares many of the



**Figure 9.** Ratio of cumulative ILR angular momentum transfer to total angular momentum as in Fig. 4 but for the satellite decay with  $M_s = 0.025$ .

features of the LBK formula; the main difference is that the Dirac delta function  $\delta(\mathbf{l} \cdot \boldsymbol{\Omega} - m\Omega_p)$  is replaced with a time integral (see eq. 15). Over astronomically-relevant time scales, the finite-time generalisation shows that transients from the formation history of the system play an important role in the overall dynamical evolution. For example, Figures 1 and 2 illustrate the total torque and differential contributions of the principal resonances during slow down of a bar in dark matter halo. The correct result is the net contribution for approximately 8 resonances (not all shown in Fig. 2) and therefore these resonances must be accurately represented in an N-body simulation to obtain the correct result (see Paper 2).

The most important general finding is that the history of galaxy evolution can not be ignored in understanding and predicting a particular galaxy’s evolution. These dynamics are illustrated in two cases: the bar–halo interaction and the satellite–halo interaction. The details of the bar–halo interaction have been recently described in Paper 2 and explored in a self-consistent simulation in Paper 3 (Kelly Holley-Bockelmann & Katz 2004). Both papers demonstrate that bar–halo coupling through the ILR agrees with the predictions presented here. Both the LBK formula and the time-dependent formula predict that the ILR resonance dominates the overall angular momentum transfer. However, the time-dependent evolution spreads the resonant interaction over a broader range of lower energies that are more populated in phase space and therefore have a larger overall evolutionary consequence. The time-asymptotic ILR is located at smaller energies and radii and would be inaccessible to most N-body simulations. The time-asymptotic approximation is especially poor for the ILR resonance in a cuspy halo where the time-dependent spread in frequency corresponds to a large region in radius and energy. This is strong encouragement to attempt an understanding of the dynamical interactions prior to simulation. One expects a similar halo response in the satellite–halo interaction because the quadrupole perturbation from a rotating bar shares much in common with the quadrupole perturbation from an orbiting satellite. Indeed, we show that this is the case. However, because the symmetries, the radial profile, and the distribution of frequencies for the two perturbations are different, the ILR no longer dominates in the satellite interaction. Nonetheless, the ILR does play a role during the early stages of orbital evolution and this has been verified in N-body simulation (see Choi et al. 2004). The aggregate effect of this mechanism from a population of substructure will be the subject of a later investigation.

The wakes excited by interactions between satellites, disks and dark matter halos give rise to a variety of observable consequences. For example, large-scale transport of angular momentum throughout the disk and to the halo may will tend to decrease the disk scale length. The response of the galaxy halo to the surrounding group environment can propagate features to

the inner galaxy, exciting disk waves and subsequent star formation. In addition, the interaction between the satellite and disk will drive bending modes and the energy in these modes will heat the disk. Detailed simulations for these processes of disparate scales are difficult. Time-asymptotic secular perturbation theory has often been used to estimate evolution and compare with simulation. However, we have found that the difference in amplitude and significance between the time-dependent and time-asymptotic responses for the bar–halo interaction (Fig. 2) is dramatic. If these differences are representative of the difference in these other processes, our current picture of secular evolution must be revisited by considering the time-dependent evolution of these “waves” in the context of a galaxy’s environmental history.

## ACKNOWLEDGMENTS

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