9-2010

Geometry of Satake and Toroidal Compactifications

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GEOMETRY OF SATAKE AND TOROIDAL COMPACTIFICATIONS

A Dissertation Presented

by

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Submitted to the Graduate School of the
University of Massachusetts Amherst in partial fulfillment
of the requirements for the degree of

DOCTOR OF PHILOSOPHY

September 2010

Department of Mathematics and Statistics

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Dedication

To the memory of my brother Jeffrey Thomas Boland

ACKNOWLEDGMENTS

I would first like to thank my advisor Paul E. Gunnells. His careful preparation of several topics courses and countless hours of personal attention played a definitive role in my graduate education. I am grateful for his love of mathematics and infinite patience.

There are several other professors to whom I am indebted: Tom Braden, Eduardo Cattani, and David Kastor for their support as members of my thesis committee; Dan Yasaki, Rob Kusner, and Les Saper for helpful conversations concerning my thesis research; Eric Sommers and Farshid Hajir for their inspiring classes and support; Szilárd Révész and Zoltán Füredi for their encouraging words and excellent lectures; Joanna Su, Liam Donohoe, and Jim Tattersall for introducing me to abstract mathematics; and Lola Smith for giving an excellent eighth grade algebra class and being a good friend to my mother.

I also thank the Universities of Massachusetts and Michigan for their generous support. Especially, I thank graduate program director Siman Wong for keeping tabs on me and his genuine interest in my success. I thank Dick Canary, Lizhen Ji, and Ralf Spatzier for their encouragement. I thank the NSF for their support through grant DMS - 0602191.

On a personal level, I thank my fellow graduate students for inspiring conversations, my parents for their support and providing me many opportunites, and most importantly, I thank Julee for her commitment and love.

ABSTRACT

GEOMETRY OF SATAKE AND TOROIDAL COMPACTIFICATIONS

SEPTEMBER 2010

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In [JM02, §14], Ji and MacPherson give new constructions of the Borel–Serre and reductive Borel–Serre compactifications [BS73, Zuc82] of a locally symmetric space $\Gamma\backslash X$. They use equivalence classes of eventually distance minimizing (EDM) rays to describe the boundaries of these compactifications. The primary goal of this thesis is to construct the Satake compactifications of $\Gamma\backslash X$ [Sat60a] using finer equivalence relations on EDM rays. To do this, we first construct the Satake compactifications of the global symmetric space X [Sat60b] with equivalence classes of geodesics in X. We then define equivalence relations on EDM rays using geometric properties of their lifts in X. We show these equivalence classes are in one-to-one correspondence with the points of the Satake boundary.

As a secondary goal, we outline the construction of the toroidal compactifications of Hilbert modular varieties [Hir71, Ehl75] using a larger class of "toric

curves" and equivalence relations that depend on the compactifications' defining combinatorial data.

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CHAPTER 1

INTRODUCTION

1.1 Preliminaries

In this thesis, we study symmetric and locally symmetric spaces and their compactifications. Locally symmetric spaces appear in many contexts. For example, they arise frequently as moduli spaces in algebraic geometry. The prototypical example of a symmetric space of noncompact type is the complex upper half plane $\mathfrak{H}$:

$$\mathfrak{H} = \{z \in \mathbb{C} \mid \mathfrak{Im}(z) > 0\}.$$

The group $SL_2(\mathbb{Z})$ acts on $\mathfrak{H}$ by fractional linear transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}.$$

Let $\Gamma \subset SL_2(\mathbb{Z})$ be a finite-index subgroup. Then the quotient $\Gamma\backslash\mathfrak{H}$ is called an *(open) modular curve*. Topologically, this quotient is a punctured Riemann surface, and is a typical example of a locally symmetric space of noncompact type. For instance, if $\Gamma = SL_2(\mathbb{Z})$, then the resulting quotient is $\mathbb{P}^1$ with a single puncture.

It is useful and informative to compactify locally symmetric spaces. The boundary components added on at infinity often reflect deeper structure of the space.

For example, when a locally symmetric space is a moduli space, a compactification's points at infinity sometimes correspond to degenerations of the objects parametrized by the points in the interior.

We consider the example $\Gamma\backslash\mathfrak{H}$. Since $\Gamma\backslash\mathfrak{H}$ is topologically a punctured Riemann surface, there are two natural ways to compactify it: we can fill in the punctures with points to obtain a closed manifold, or we can add circles at each puncture to obtain a manifold with boundary. Historically, the first compactification appears in work of Fricke–Klein and Poincaré. One way to build it is to first adjoin the cusps $\mathbb{Q} \cup \{i\infty\}$ to $\mathfrak{H}$ with an appropriate topology. The action of $SL_2(\mathbb{Z})$ on $\mathfrak{H}$ extends to an action on $\mathbb{R} \cup \{i\infty\}$ that takes $\mathbb{Q} \cup \{i\infty\}$ to itself. After taking the quotient of the partial compactification $\mathfrak{H} \cup \mathbb{Q} \cup \{i\infty\}$, we have filled the punctures with points.

The second compactification is an example of a general construction introduced by Borel–Serre [BS73]. For a general locally symmetric space $\Gamma\backslash X$, this compactification is denoted $\overline{\Gamma\backslash X}^{BS}$. In our example, it is obtained by first attaching copies of the real line $\mathbb{R}$ at each of the cusps $\mathbb{Q} \cup \{i\infty\}$ and then taking the quotient by Γ. The resulting quotient attaches S^1 boundary components at the punctures.

There are other compactifications one finds in the literature, although since $\mathfrak{H}$ is such a simple example, these compactifications typically coincide for $\Gamma\backslash\mathfrak{H}$. For instance, the compactification by adjoining points to $\Gamma\backslash\mathfrak{H}$ also realizes the *reductive Borel–Serre* [Zuc82], *maximal Satake* [Sat60a], *Baily–Borel* [BB66], and the *toroidal* compactifications [KKMSD73, AMRT75] of $\Gamma\backslash\mathfrak{H}$. These different compactifications are related by the notion of *dominance*: a compactification $\overline{\Gamma\backslash X}^1$ is said to dominate another compactification $\overline{\Gamma\backslash X}^2$ if the identity map on $\Gamma\backslash X$ extends to a continuous surjective map from $\overline{\Gamma\backslash X}^1$ to $\overline{\Gamma\backslash X}^2$. If the extended map is not a homeomorphism, then $\overline{\Gamma\backslash X}^1$ is said to *strictly dominate* $\overline{\Gamma\backslash X}^2$. For the quotient $\Gamma\backslash\mathfrak{H}$, the reductive

Borel–Serre point boundaries are obtained by collapsing the S^1 boundaries of the Borel–Serre compactifications to points. Hence the Borel–Serre compactification $\overline{\Gamma\backslash\mathfrak{H}}^{BS}$ strictly dominates the reductive Borel–Serre compactification $\overline{\Gamma\backslash\mathfrak{H}}^{RBS}$.

One can ask why so many different compactifications are necessary. It turns out that a given compactification may have certain advantages over another one. For example, the Baily–Borel compactification, which is defined only for Hermitian locally symmetric spaces, has the advantage of being a projective algebraic variety. On the other hand, it is usually highly singular. The toroidal compactifications are also algebraic and provide resolutions of the singularities of the Baily–Borel compactification, but have the drawback that their construction depends on an auxillary choice of combinatorial data. The Borel–Serre compactification is a manifold with corners and is useful in the study of cohomology of arithmetic groups. However, from a differential geometric perspective, the Borel–Serre is not so nice, since the invariant metric on the locally symmetric space degenerates on the boundary of $\overline{\Gamma\backslash X}^{BS}$. This is not the case with the reductive Borel–Serre compactification. Here the metric extends to be nondegenerate on the boundary, but now $\overline{\Gamma\backslash X}^{RBS}$ is no longer a manifold with corners and is singular.

In [JM02] Ji–MacPherson gave a new construction of the Borel–Serre and reductive Borel–Serre compactifications. Their method uses equivalence classes of geodesics in $\Gamma\backslash\mathfrak{H}$ that "go to infinity" to describe the boundary points. This work was motivated in part by work of Karpelevič [Kar67], who used equivalence classes of geodesics to build compactifications of global symmetric spaces.

The construction is easy to describe for the quotients $\Gamma\backslash\mathfrak{H}$. The directed geodesics that "go to infinity" are exactly the projections of geodesics in $\mathfrak{H}$ having limit points in the set $\mathbb{Q}\cup\{i\infty\}$. The points of the boundary of the Borel–Serre compactification $\overline{\Gamma\backslash\mathfrak{H}}^{BS}$ are in one to one correspondence with these projected geodesics. For example, when $\Gamma = SL_2(\mathbb{Z})$, the projections of the vertical line geodesics

$$\tilde{\gamma}(t) = a + e^t i, \ a \in \mathbb{R}$$

correspond to the single S^1 boundary component.

In general, the boundary components of the *reductive Borel–Serre compactification* are obtained by collapsing certain factors of the Borel–Serre boundary. In the description with geodesics, this collapsing is reflected by placing certain equivalence relations on the geodesics that describe the Borel–Serre compactifications. Specifically, we identify geodesics that become arbitrarily close as they approach infinity. For example, the projected vertical line geodesics in $SL_2(\mathbb{Z})\backslash\mathfrak{H}$ form a single equivalence class that corresponds to the unique cusp one adds to form $\overline{SL_2(\mathbb{Z})\backslash\mathfrak{H}}^{RBS}$.

1.2 Maximal Satake Compactification

The main goal of this thesis is to give new constructions of the *Satake compactifications* [Sat60a] via equivalence relations on geodesics. Up to isomorphism, there are a finite number of Satake compactifications that form a partially ordered set, where the order relation is dominance. These compactifications were originally defined using the finite dimensional representation theory of Lie groups. There is a unique maximal Satake compactification that Zucker proved is dominated by the Borel–Serre and reductive Borel–Serre compactifications [Zuc83]. For example, consider the symmetric space $SL_3(\mathbb{R})/SO(3)$. There are three distinct Satake compactifications, one maximal and two minimal. The maximal Satake is constructed using the adjoint representation of $SL_3(\mathbb{R})$, while the two nonmaximal compactifications are constructed using the standard and dual standard representations.

As in the case of $\Gamma\backslash\mathfrak{H}$, there is a unique Satake compactification that is isomorphic to the reductive Borel–Serre compactification. The first interesting example where the maximal Satake and reductive Borel–Serre compactifications differ is the *Hilbert modular surface.* Let $k/\mathbb{Q}$ be a real quadratic extension. The field k is isomorphic to $\mathbb{Q}(\sqrt{d})$, where d is a squarefree positive integer, and comes equipped with two real embeddings, denoted $a \mapsto a^{(i)}$, $i = 1, 2$. Let $\mathcal{O}_k$ be the ring of integers of k. The group $SL_2(\mathcal{O}_k)$ acts on the product of upper half planes $\mathfrak{H} \times \mathfrak{H}$ using the embeddings to act by fractional linear transformations in each factor:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (z_1, z_2) = \left(\frac{a^{(1)} z_1 + b^{(1)}}{c^{(1)} z_1 + d^{(1)}}, \frac{a^{(2)} z_2 + b^{(2)}}{c^{(2)} z_2 + d^{(2)}} \right), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathcal{O}_k).$$

If $\Gamma \subset SL_2(\mathcal{O}_k)$ is a finite-index subgroup, then the quotient $\Gamma\backslash\mathfrak{H} \times \mathfrak{H}$ is called a Hilbert modular surface.

There is a natural compactification of $\Gamma\backslash\mathfrak{H} \times \mathfrak{H}$ obtained by adjoining cusps to $\mathfrak{H} \times \mathfrak{H}$ that mimics the standard compactification of the modular curve. The cusps are attached to the product of upper half planes using the two real embeddings of k. Specifically, we attach a cusp at the point $\{i\infty, i\infty\}$, and for every $c \in k$, we attach the point $(c^{(1)}, c^{(2)}) \in \mathbb{R} \times \mathbb{R}$ to $\mathfrak{H} \times \mathfrak{H}$. The action of Γ extends to the cusps, and one can show that there are finitely many Γ-orbits. Thus one can form a compactification of $\Gamma\backslash\mathfrak{H} \times \mathfrak{H}$ by adjoining finitely many points. For example, when $\Gamma = SL_2(\mathcal{O}_k)$, the quotient $\Gamma\backslash\mathfrak{H} \times \mathfrak{H}$ has $Cl(k)$ punctures, where $Cl(k)$ is the class number of k. This is the unique Satake compactification of the Hilbert modular surface.

The construction of the reductive Borel–Serre compactification $\overline{\Gamma\backslash\mathfrak{H} \times \mathfrak{H}}^{RBS}$ resembles the Borel–Serre compactification of the modular curve, in that we attach a copy of $\mathbb{R}$ at each of the cusps of $\mathfrak{H} \times \mathfrak{H}$ and take the quotient by Γ. The boundary of the resulting compactification replaces the points in the unique Satake compactification with circles. We remark that the circle boundary components of the Borel–Serre compactification of the modular curve and $\overline{\Gamma\backslash\mathfrak{H}\times\mathfrak{H}}^{RBS}$ arise from different structural features of the groups $SL_2(\mathbb{Q})$ and $SL_2(k)$ and are not the "same" S^1. The Borel–Serre S^1 boundary comes from the nilpotent radical of $SL_2(\mathbb{R})$, whereas the reductive Borel–Serre S^1 boundary is related to the fact that $\mathcal{O}_k$ has units of infinite order.

We now explain how to construct the maximal Satake compactification of the Hilbert modular surface using geodesics in the spirit of [JM02]. Let $\tilde{\gamma}$ be a geodesic in $\mathfrak{H}\times\mathfrak{H}$ whose projection in $SL_2(\mathcal{O}_k)\backslash\mathfrak{H}\times\mathfrak{H}$ goes to infinity. Consider the set $C(\tilde{\gamma})$ of geodesics in $\mathfrak{H}\times\mathfrak{H}$ that remain at a fixed distance from $\tilde{\gamma}$. This set, originally introduced by Karpelevič [Kar67], is called the *congruence bundle* of $\tilde{\gamma}$. Karpelevič showed that $C(\tilde{\gamma})$ is a metric space, where the distance between two geodesics $\tilde{\gamma}_1, \tilde{\gamma}_2 \in C(\tilde{\gamma})$ is defined by

$$D(\tilde{\gamma}_1, \tilde{\gamma}_2) := \lim_{t\to\infty} \inf_{s\in\mathbb{R}} d(\tilde{\gamma}_1(t), \tilde{\gamma}_2(s)).$$

For example, consider the geodesic

$$\tilde{\gamma}(t) = (ie^{t/\sqrt{2}}, ie^{t/\sqrt{2}}).$$

This goes to the cusp $\{i\infty, i\infty\}$ and goes to infinity in any quotient. The congruence bundle $C(\tilde{\gamma})$ consists of geodesics in the subset $\{(z_1, z_2) \in \mathfrak{H}\times\mathfrak{H} \mid \mathfrak{Re}(z_i) = 0\}$ that, up to reparametrization, take the form

$$(i\lambda e^{t/\sqrt{2}}, i\lambda^{-1}e^{t/\sqrt{2}}),$$

where $\lambda \in \mathbb{R}_{>0}$. Let $\tilde{\gamma}_j(t) = (i\lambda_j e^{t/\sqrt{2}}, i\lambda_j^{-1}e^{t/\sqrt{2}})$, $j = 1, 2$ be two geodesics in $C(\tilde{\gamma})$, and write $\lambda_j = e^{a_j}$ where $a_j \in \mathbb{R}$. Then we have $D(\tilde{\gamma}_1, \tilde{\gamma}_2) = |a_2 - a_1|/\sqrt{2}$.

The projections of the geodesics in $C(\tilde{\gamma})$ all go to infinity in the quotient $\Gamma\backslash\mathfrak{H}\times\mathfrak{H}$. The action by Γ identifies each geodesic with infinitely many others in $C(\tilde{\gamma})$. As a whole, these projected geodesics cover an S^1 boundary component of the reductive Borel–Serre compactification $\overline{SL_2(\mathcal{O}_k)\backslash\mathfrak{H}\times\mathfrak{H}}^{RBS}$.

To describe the point boundary component of the Satake compactification, we introduce a new equivalence relation that we call the *C-relation.* Consider the group of isometries on $C(\tilde{\gamma})$, and let $\tilde{\gamma}'$ be any element in $C(\tilde{\gamma})$. Isometries T whose *displacement distance*

$$D(T(\tilde{\gamma}'),\tilde{\gamma}')$$

is a fixed constant for all $\tilde{\gamma}'$ are called *Clifford translations*. We call two geodesics γ_1,γ_2 in $SL_2(\mathcal{O}_k)\backslash\mathfrak{H}\times\mathfrak{H}$ C-related if there exists a Clifford translation T that takes $\tilde{\gamma}_1$ to $\tilde{\gamma}_2$.

To see how this relation works, consider the geodesics $\tilde{\gamma}_j(t)$, $j=1,2$ in the congruence bundle $C(\tilde{\gamma})$. The translation T on $C(\tilde{\gamma})$ that takes $\tilde{\gamma}_1$ to $\tilde{\gamma}_2$ is given by

$$T(\tilde{\gamma}_1)=(e^{a_2-a_1}(i\lambda_1 e^{t/\sqrt{2}}),e^{a_1-a_2}(i\lambda_1^{-1}e^{t/\sqrt{2}}))=\tilde{\gamma}_2.$$

In this example it is clear all geodesics in $C(\tilde{\gamma})$ are equivalent under this relation. Hence this equivalence class accounts for the $\{i\infty,i\infty\}$ boundary point. Examples of two C-related geodesics are pictured in Figure 1. The open quadrant in this figure represents the unique two dimensional maximal flat in which both $\tilde{\gamma}_1,\tilde{\gamma}_2$ reside. For a general locally symmetric space $\Gamma\backslash X$, a congruence bundle $C(\tilde{\gamma})$ may have isometries that are not Clifford translations. As we will see in the next subsection, there are examples where $C(\tilde{\gamma})$ has no translations.

1.3 Nonmaximal Satake Compactifications

As previously mentioned, for a general locally symmetric space there are finitely many Satake compactifications up to isomorphism. The most basic examples where

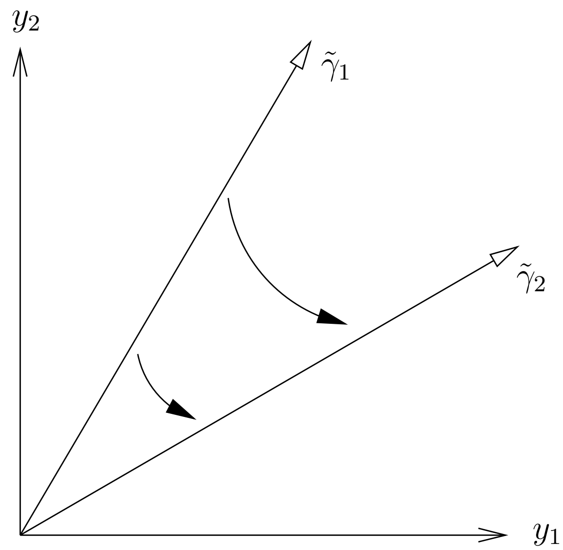

Figure 1: C-related geodesics

nonisomorphic Satake compactification arise are quotients of $SL_3(\mathbb{R})/SO(3)$ by a finite-index subgroup $\Gamma \subset SL_3(\mathbb{Z})$. In this case the maximal Satake compactification is isomorphic to the reductive Borel–Serre compactification, and thus the description of the maximal Satake compactification using geodesics is the same as that for the reductive Borel–Serre (cf. [JM02, §14]). Therefore we consider the nonmaximal Satake compactifications of $\Gamma\backslash SL_3(\mathbb{R})/SO(3)$. As mentioned on p.4, up to equivalence there are two such compactifications, associated to the standard and dual standard representations of $SL_3(\mathbb{R})$. We will consider the compactification associated to the standard representation.

In general, the boundary components of nonmaximal Satake compactifications are obtained by collapsing boundary components of the maximal Satake. We can detect this for the standard Satake compactification of $\Gamma\backslash SL_3(\mathbb{R})/SO(3)$ using the diagonal subgroup in $SL_3(\mathbb{R})$. This follows the approach of Casselman [Cas97] that provides an alternative description of the Satake boundary.

In the following description, let X denote the symmetric space $SL_3(\mathbb{R})/SO(3)$ and x_0 the basepoint corresponding to $SO(3)$. Let A denote the subgroup of diagonal matrices $\text{diag}(a_1, a_2, a_3) \subset SL_3(\mathbb{R})$, where $a_i > 0$ and $\prod a_i = 1$. The A-orbit through x_0 is a submanifold $Ax_0 \subset X$ called a *maximal flat*. By definition, a flat in a symmetric space is a complete, totally geodesic submanifold isometric to some flat Euclidean space $\mathbb{R}^n$.

In the maximal Satake compactification, the closure of Ax_0 has a hexagonal boundary. The boundary is decomposed into six open line segments and six points, which respectively correspond to the six maximal and six minimal parabolic subgroups of $SL_3(\mathbb{R})$ that contain A. In the nonmaximal Satake compactification, the closure of Ax_0 has a triangular boundary. None of the minimal parabolic subgroups contribute to the triangular boundary. Three of the open line segments corresponding to maximal parabolic subgroups collapse to points, while the three other open segments remain unchanged.

The flat Ax_0 is not the unique maximal flat that passes through x_0. To describe the Satake compactifications we use the whole family of flats passing through x_0. The collapsing in the boundary of the maximal flats reflects the collapsing of the boundary components when passing between different Satake compactifications. It turns out that the geodesics in these maximal flats fill the Satake boundaries. This suggests that appropriate equivalence relations on these geodesics can be used to build the boundary. To explain how this works, we need to introduce new notation.

There is a correspondence [JM02, Theorem 10.18] between geodesics in $\Gamma\backslash X$ that go to infinity and certain parabolic subgroups of $SL_3(\mathbb{R})$. Each maximal flat is partitioned into *chambers* that correspond to the parabolic subgroups that intersect the the torus A nontrivially. From the previous example, we saw that the twelve components that comprise the hexagonal boundary of $\overline{Ax_0}$ in the maximal Satake compactification are in one to one correspondence with the six maximal parabolics and six minimal parabolics that contain A. Each chamber of Ax_0 corresponds to two maximal parabolic subgroups and one minimal parabolic that is contained both maximal parabolics. Using the language of chambers, we introduce a new equivalence relation that we call the *μ-relation*. The μ-relation is defined in three steps:

1. As a first step, we need to identify geodesics in the same chamber that go to the same point at infinity.

2. Next, there are geodesic rays in a given maximal flat that lie in different chambers that need to be identified.

3. Finally, we identify geodesics in arbitrary flats that pass through the base-point x_0.

These relations are complicated since geodesics corresponding to different parabolic subgroups must be identified. This phenomenon is not present in the construction of the Borel–Serre, reductive Borel–Serre, or ma ximal Satake compactifications.

We refer the interested reader to later chapters for more detailed information about these relations. Here we consider examples of standard geodesics and how they behave under the relations in steps (1) and (2). Let

$$\tilde{\gamma}_1(t) = \operatorname{diag}(e^t, e^t, e^{-2t})x_0,\ \tilde{\gamma}_2(t) = \operatorname{diag}(e^t, 1, e^{-t})x_0$$

$$\tilde{\gamma}_3(t) = \operatorname{diag}(e^{2t}, e^{-t}, e^{-t})x_0,\ \tilde{\gamma}_4(t) = \operatorname{diag}(e^t, e^{-t}, 1)x_0.$$

The diagram on the left of Figure 2 shows all of these geodesics as they are contained in the standard maximal flat corresponding to the diagonal subgroup of $SL_3(\mathbb{R})$. The geodesics $\tilde{\gamma}_1(t), \tilde{\gamma}_2(t), \tilde{\gamma}_3(t)$ are all contained in the standard chamber corresponding to parabolic subgroups that contain the minimal subgroup of upper triangular matrices. The geodesics $\tilde{\gamma}_3(t), \tilde{\gamma}_4(t)$ are also contained in a common chamber corresponding to a different minimal parabolic subgroup of $SL_3(\mathbb{R})$. It is important to note that a geodesic can be contained in multiple chambers. Under the relation in step (1), projections of $\tilde{\gamma}_2(t)$ and $\tilde{\gamma}_3(t)$ are equivalent in the context of the standard chamber. In the context of the adjacent nonstandard chamber, $\tilde{\gamma}_3(t)$ and $\tilde{\gamma}_4(t)$ are equivalent. We emphasize that the relation in step (1) does not identify $\tilde{\gamma}_2(t)$ and $\tilde{\gamma}_4(t)$ since they do not live in the same chamber. Under the relation in step (2), projections of $\tilde{\gamma}_2(t)$ and $\tilde{\gamma}_4(t)$ are now identified. The geodesic $\tilde{\gamma}_1(t)$ is not equivalent to any of the others. The relations involved in steps (1) and (2) are portrayed in the diagram on the right of Figure 2.

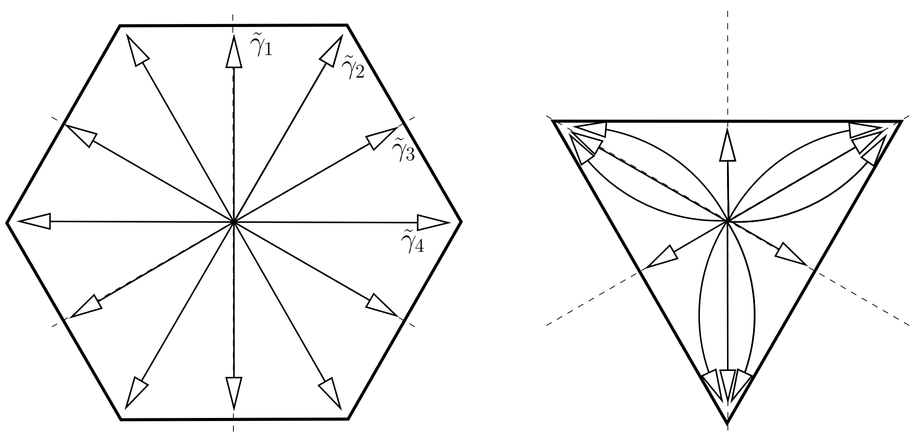

Figure 2: Convergence of geodesics in a maximal flat

We introduce an infinite class of geodesics that show how the relation in step (3) works. Let

$$g \in \begin{pmatrix} 1 & 0 \\ 0 & SO(2) \end{pmatrix}$$

be a matrix in $SL_3(\mathbb{R})$. The conjugates $g\tilde{\gamma}_2(t)g^{-1}$ are an infinite family of geodesics whose limit points represent the cusps at infinity of an upper half plane boundary component in the maximal Satake compactification. The unlabelled geodesic in Figure 3 is an example of a conjugate. These cusps correspond to the dashed circle in Figure 3. The relation in step (1) identifies $\tilde{\gamma}_3$ to all the conjugates, however it does not identify any of the conjugates with each other. The relation in step (2) makes equivalent each conjugate with only one other conjugate. To be precise, conjugates with antipodal limit points in the closure of the $\mathfrak{H}$ boundary viewed in the disc model are made equivalent. Finally, the relation in step (3) makes equivalent all conjugates of $\tilde{\gamma}_2(t)$.

The μ-relation is fundamentally different from the C-relation. We see this difference by examining other geodesics in the standard chamber that the relation in step (1) makes equivalent. Consider the infinite family of geodesics $\tilde{\gamma}_{3,\lambda}(t) = \text{diag}(\lambda^{-2}e^{2t}, \lambda e^{-t}, \lambda e^{-t})$ where $\lambda \in \mathbb{R}_{>0}$. Geometrically, we think of this family as those geodesics in the standard maximal flat obtained by shifting the basepoint of $\tilde{\gamma}_3$. All of these geodesics are contained in $C(\tilde{\gamma}_3)$ and are identified under the relation in step (1). It is useful to note that any isometry of $C(\tilde{\gamma}_3)$ that takes one element of this family to another is not a translation on $C(\tilde{\gamma}_3)$. This is the reason these geodesics are not equivalent under the relation that constructs the maximal Satake compactification.

1.4 Toroidal Compactifications

The second goal of this thesis is to construct a new set of curves and devise appropriate equivalence relations to construct the toroidal boundary of the Hilbert modular varieties. Unlike the other compactifications we have considered, the construction of the toroidal compactifications involves a noncanonical choice of data Σ. This results in an infinite family of nonisomorphic compactifications. We explain

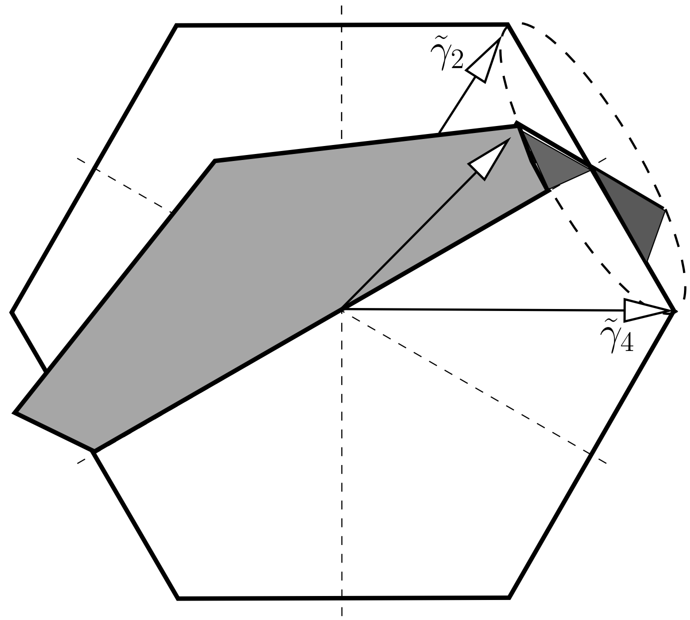

Figure 3: Three equivalent geodesics under the relation in step (3)

our work for the Hilbert modular surface $\Gamma\backslash\mathfrak{H}\times\mathfrak{H}$. For this particular example, there is a canonical choice Σ that gives a smooth compactification, denoted $\overline{\Gamma\backslash\mathfrak{H}\times\mathfrak{H}}^{tor}_{\Sigma}$. The boundaries of the toroidal compactifications are quite different from the boundaries of other compactifications we have mentioned. Topologically, the boundary consists of a finite number of complex projective lines $\mathbb{P}^1$'s that intersect transversally.

Unlike the previously considered compactifications, the limit points of geodesics do not fill the boundary. More precisely, there are points p in the toroidal boundary such that no geodesic γ satisfies $\lim_{t\to\infty}\gamma(t) = p$ in the compactification. To see this, choose a connected component of the boundary and let n be the number of $\mathbb{P}^1$'s. Then the geodesics that head for this component only map to n points and n copies of S^1. In particular, each intersection point is reached by certain geodesics. Topologically, we think of a $\mathbb{P}^1$ boundary component as the disjoint union of $\mathbb{C}^\times$

and two points $\{\infty\}$ and $\{0\}$. The copies of S^1 are thought of as the unit circles in the $\mathbb{C}^\times$ components.

To describe a set of curves that will map to all points in the toroidal boundary, we consider the boundary component over the cusp $(i\infty, i\infty)$. Let $\Gamma = SL_2(\mathcal{O}_k)$, and let

$$\tilde{\gamma}(t) = (ie^{t/\sqrt{2}}, ie^{t/\sqrt{2}}). \tag{1.4.1}$$

The geodesics of the form

$$\tilde{\gamma}_{(a,b)}(t) = (a + ie^{t/\sqrt{2}}, b + ie^{t/\sqrt{2}}) \tag{1.4.2}$$

become arbitrarily close to $\tilde{\gamma}$ as $t \to \infty$ and they fill out a real submanifold homeomorphic to S^1 that is contained in a $\mathbb{P}^1$ boundary component. To fill the rest of the boundary component, we can use projections of curves that take the form

$$\tilde{c}(t) = (i(t + \lambda_1), i(t + \lambda_2)) \tag{1.4.3}$$

and the corresponding family of curves

$$\tilde{c}_{(a,b)}(t) = (a + i(t + \lambda_1), b + i(t + \lambda_2)). \tag{1.4.4}$$

It is not hard to see that these curves fill out the boundary. The question remains, how can we describe this set of curves using information from the geometry of $\Gamma \backslash \mathfrak{H} \times \mathfrak{H}$ or the underlying space $\mathfrak{H} \times \mathfrak{H}$?

A natural place to find curves on a symmetric space is to look at the solutions of the *Killing vector fields*. By definition, a Killing vector field is a vector field whose flow transformations are isometries of the underlying space. In the case of $\mathfrak{H}$, there are three basic Killing vector fields that correspond to the standard basis elements of $\mathfrak{sl}_2(\mathbb{R})$. We can compute these vector fields by investigating the action of certain one parameter subgroups of isometries on $\mathfrak{H}$. Consider the following subgroups of $SL_2(\mathbb{R})$:

$$\left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \right\}.$$

We let each subgroup act on $z = x + iy \in \mathfrak{H}$ by fractional linear transformations. If we take the derivative of each transformation at $t = 0$, we obtain the vector fields

$$\left\{ \frac{\partial}{\partial x},\ x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y},\ x^2 - y^2\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y} \right\}.$$

Every Killing vector field on $\mathfrak{H}$ can be written as an $\mathbb{R}$-linear combination of these vector fields.

Curves that arise as solutions to Killing vector fields have unexpected convergence to the cusps of $\mathfrak{H}$. For instance, consider the integral curves of the three basic Killing vector fields. The integral curves passing through the point i are

$$t + i, ie^t, \text{ and } i/(it + 1).$$

It is clear that the limit points of the above curves are

$$\{\infty\}, \{0, \infty\}, \text{ and } \{0\}$$

respectively. We see that nongeodesic curves may have unbounded distance as they approach infinity, but still converge to the same limit points in $\overline{\mathfrak{H}}$.

The Killing vector fields of $\mathfrak{H} \times \mathfrak{H}$ arise from the Killing vector fields on each factor. Unfortunately, even the integral curves of Killing vector fields in $\mathfrak{H} \times \mathfrak{H}$ do not map to every point on the toroidal boundary. To define a vector field on $\mathfrak{H} \times \mathfrak{H}$ whose integral curves fill the toroidal boundary, we define an action of the compact torus $\mathbb{T}^2 \simeq S^1 \times S^1$ on the tangent bundle $\mathrm{T}(\mathfrak{H} \times \mathfrak{H})$. For any point $(z_1, z_2) \in \mathfrak{H} \times \mathfrak{H}$, we consider the basis

$$\left\{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_2} \right\}$$

of the tangent space $\mathrm{T}_{(z_1,z_2)}(\mathfrak{H} \times \mathfrak{H})$ where $z_j = x_j + iy_j$, $j = 1, 2$. A vector field X in the space of vector fields $\mathfrak{X}(\mathfrak{H} \times \mathfrak{H})$ takes the form $X_1 + X_2$ where

$$X_j = f_j(z_j)\frac{\partial}{\partial x_j} + g_j(z_j)\frac{\partial}{\partial y_j},\ j = 1, 2$$

are vector fields in each factor.

We can finally define the action of $\mathbb{T}^2$ on $\mathrm{T}(\mathfrak{H} \times \mathfrak{H})$. For an element $(e^{i\theta_1}, e^{i\theta_2}) \in \mathbb{T}^2$ and a point-vector pair $((z_1, z_2), \vec{v}) \in \mathrm{T}(\mathfrak{H} \times \mathfrak{H})$, define

$$(e^{i\theta_1}, e^{i\theta_2}) \cdot ((z_1, z_2), \vec{v}) = ((z_1, z_2), e^{i\theta_1} \cdot \vec{v_1} + e^{i\theta_2} \cdot \vec{v_2})$$

where $\vec{v_1}$ and $\vec{v_2}$ are the contributions to $\vec{v}$ from each $\mathrm{T}\mathfrak{H}$ factor under the identification $\mathrm{T}(\mathfrak{H} \times \mathfrak{H}) \simeq \mathrm{T}\mathfrak{H} \times \mathrm{T}\mathfrak{H}$. Now we write $\vec{v_j}$ in the basis $\{\partial/\partial x_j, \partial/\partial y_j\}$ corresponding to the tangent space $\mathrm{T}_{z_j}\mathfrak{H}$. Finally, the action $e^{i\theta_j} \cdot \vec{v_j}$ simply rotates $\vec{v_j}$ by the angle θ_j. This rotation is depicted in Figure 4.

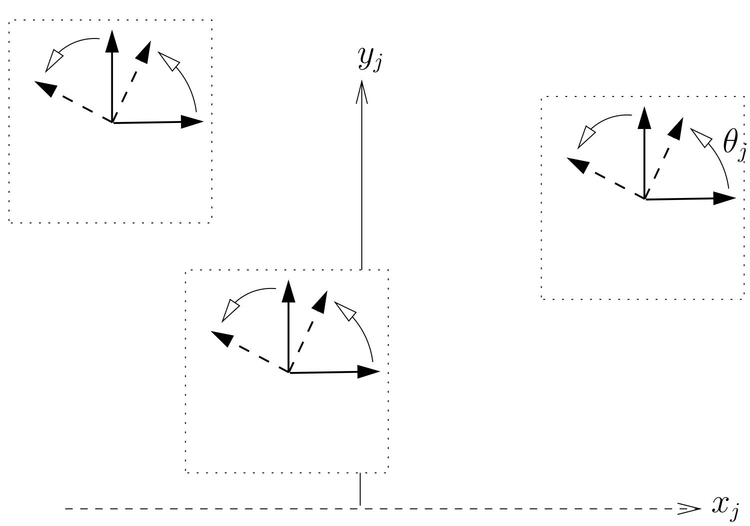

Figure 4: Torus action on $\mathrm{T}\mathfrak{H}$ factor

We now describe how the curves $\tilde{c}(t)$ (1.4.3) associated to the geodesic $\tilde{\gamma}(t)$ (1.4.1) arise in the context of Killing vector fields and the torus action. Consider the vector fields that arise by acting on the set of Killing vector fields by $\mathbb{T}^2$. These vector fields can be parametrized by $\mathbb{R}^6 \times \mathbb{T}^2$. We disregard the trivial vector field corresponding to the point $(0, \ldots, 0) \in \mathbb{R}^6$, and call two vector fields equivalent if they have the same $\mathbb{T}^2$ component and their $\mathbb{R}^6$ components are equal up to a positive scalar. The resulting equivalence classes are parametrized by $\mathbb{S}^5 \times \mathbb{T}^2$. We will prove there is a unique equivalence class with the property that when restricted to the maximal flat

$$\{(z_1, z_2) \in \mathfrak{H} \times \mathfrak{H} \mid \mathfrak{Re}(z_i) = 0\},$$

any of its integral curves $c(t)$ remain in the flat and satisfy the geometric condition

$$\lim_{t \to \infty} d(\tilde{\gamma}(t), c(t)) = 0.$$

The curves $c(t)$ are exactly those curves $\tilde{c}(t)$ mentioned in the beginning of the paragraph. Figure 5 gives examples of three special EDM geodesic lifts (rays with white heads) and their associated toric curves (rays with black heads). The curves associated to any one of these special geodesics converge to real submanifold diffeomophic to $\mathbb{R}$ that is contained in a $\mathbb{P}^1$ boundary component of the toroidal compactification as depicted in Figure 5. All other curves $\tilde{c}_{(a,b)}(t)$ (1.4.4) associated to the vertical line geodesics $\tilde{\gamma}_{(a,b)}(t)$ (1.4.2) are described similarly.

The projections of both the geodesics $\tilde{\gamma}(t)$ and their corresponding curves $\tilde{c}(t)$ give a rich enough collection to describe the boundary. However, for a point p in the boundary, there are an infinite number of curves in this collection that converge to p. We define an equivalence relation on this collection, the Σ-*relation*, and use it to construct a series of further relations such that the defined equivalence classes are in one to one correspondence with points in the toroidal boundary. We briefly describe these relations below.

The definition of the compactification $\overline{\Gamma\backslash\mathfrak{H}\times\mathfrak{H}}_{\Sigma}^{tor}$ involves the choice of a polyhedral fan Σ (in fact, Σ represents data that defines a polyhedral fan corresponding to each rational cusp). The faces of the cones of this fan correspond to finite collections of geodesics in each flat. The Σ-relation uses familiar concepts from convex and metric geometry, namely convex hulls, orthogonal projections, and footpoints, to associate toric curves in the same maximal flat. Since curves in different flats map to the same boundary point p, we need further relations on Σ-equivalence classes. These relations use information given by the data Σ that is overlooked by the Σ-relation. In their definitions, the t, N' and V-relations use the geometry of lattice points obtained by embedding $\mathcal{O}_k$ into $\mathbb{R}^n$ where n is the degree of the extension $k/\mathbb{Q}$. In particular, the t and N'-relations correspond with the entire lattice, while the V-relation concerns with the group of units in $\mathcal{O}_k$. We give precise definitions of these relations in Chapter 6.

1.5 Plan of the Thesis

In the second chapter, we introduce the general language of symmetric and locally symmetric spaces. Of particular importance are the descriptions of the global symmetric space X in terms of the real (2.1.1) and rational (2.3.1) horospherical decompositions. These decompositions are of great importance to this thesis on two levels. First, the factors of each decomposition provide a description of the boundary components of several compactifications using the uniform method [BJ07]. Second, they provide means to describe geodesics that satisfy a certain distance minimizing property in the global and locally symmetric spaces.

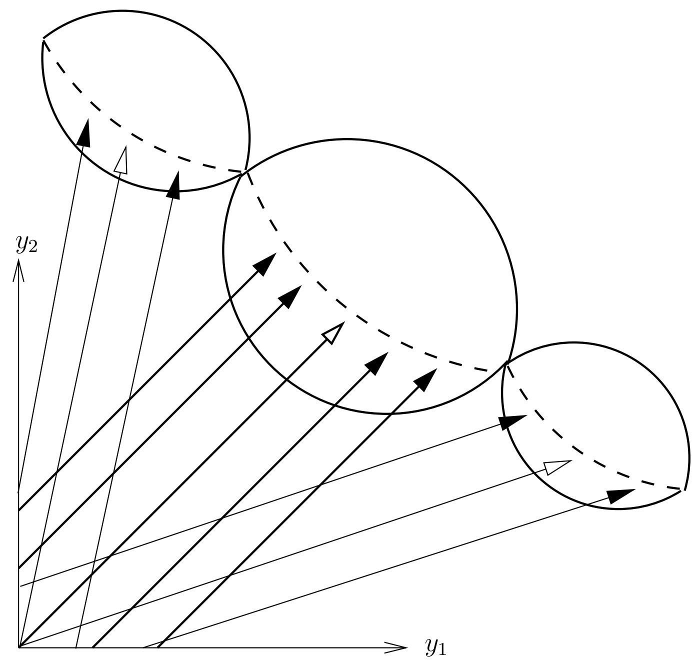

Figure 5: Toric curves in a maximal flat

In the context of locally symmetric spaces, the geodesics γ that satisfy the distance condition

$$d(\gamma(t_1), \gamma(t_2)) = |t_2 - t_1|$$

for $t_1, t_2 \gg 0$ are called *eventually distance minimizing*, or briefly *EDM* geodesics (Definition 2.33). Finally, we describe the results of Ji–MacPherson that construct certain compactifications using equivalence classes of distance minimizing geodesics in locally symmetric spaces. In particular, we describe the *RL-relation* and the *NRL-relation* (Definitions 2.44 and 2.45) that they use to build the Borel–Serre and reductive Borel–Serre compactifications.

In the third chapter, we describe the Satake compactifications of global symmetric spaces (Definition 2.6) using equivalence classes of geodesics. This has two purposes. First, it is interesting in its own right, as it completes the picture Karpelevič paints of the Satake compactifications as being dominated by the Karpelevič compactification [Kar67]. Second, in chapter 4 we will apply these equivalence relations when we construct Satake compactifications of locally symmetric spaces. To describe the maximal (resp. nonmaximal) Satake compactifications we define the C-relation (Definition 3.4) (resp. μ-relation (Definition 3.38)) on geodesics in the symmetric space X.

In the fourth chapter we complete the primary goal of the thesis, namely the construction of Satake compactifications of locally symmetric spaces using EDM geodesics. We first consider the maximal Satake compactification. We define a new equivalence relation on EDM geodesics, the *C-relation*, and apply it with results of Ji–MacPherson to define first a *$NRLC$-relation* (Definition 4.4) and then a *NRC-relation* (Definition 4.8). The equivalence classes associated to these relations are the same, and both describe the maximal Satake compactification. We also give an alternative approach by defining another equivalence relation, the *C_ϵ-relation* (Definition 4.14), that does not involve passing to the global space.

We conclude the fourth chapter by describing a one-to-one correspondence between certain equivalence classes of EDM geodesics and the boundary points of nonmaximal Satake compactifications $\overline{\Gamma\backslash X}^S_\tau$. To do this, we use the μ-relation from Chapter 3 to define the *$NRLC\tau$-relation* (Definition 4.23) on EDM geodesics. This relation makes equivalent geodesics in different NRLC equivalence classes that are projections of μ-related geodesics. Much like the $NRLC$-relation, the $NRLC\tau$-relation uses an equivalence relation on geodesics in the global space in its definition. Finally, we show the NRLCμ-equivalence classes are in one to one correspondence with the points in the nonmaximal Satake boundary.

In the fifth chapter, we turn to toroidal compactifications. We introduce the resolution of singularities of the Hilbert modular surface developed by Hirzebruch [Hir71, Hir73] and generalized by Ehlers [Ehl75] to Hilbert modular varieties. We describe the topology [Oda88, Nam80, Ji98] of the toroidal embeddings used in the definition of the toroidal compactifications.

In the sixth chapter, we describe the toroidal compactifications of the Hilbert modular varieties using a general class of curves that contain the EDM geodesics as a subset. For each EDM geodesic there is an infinite family of such *toric curves* (Definition 6.4). We describe these curves using the language of *torus actions* (Definition 6.1) and *Killing vector fields* (Definition 6.1.4). Toroidal compactifications are defined using certain combinatorial data, denoted Σ. We define a series of equivalence relations on toric curves, the last of which is called the V-relation (Definition 6.25) whose equivalence classes are in one to one correspondence with the points in the toroidal boundary.

CHAPTER 2

BACKGROUND FOR SATAKE COMPACTIFICATIONS

In this chapter, we introduce our main objects of study: global and locally symmetric spaces and their Satake compactifications. We also introduce certain classes of geodesics on these spaces that play a key role in our investigation.

In §2.1 we introduce global symmetric spaces of noncompact type, denoted X, and their Satake compactifications. We outline the original construction of these compactifications [Sat60b], achieved by embedding X into a certain compact space and taking the closure of the image. For our purposes, a more recent construction of Borel–Ji [BJ07] is useful. This method builds the compactifications by attaching a collection of boundary components at infinity. The topology on this disjoint union of boundary components is described by specifying how unbounded sequences converge (cf. [JM02, §5]). We introduce the real horospherical decomposition of X which is used to define the attached boundary components. These constructions are called the *embedding* and *uniform* method, respectively.

In §2.2 we introduce distance minimizing (DM) geodesics in global symmetric spaces. These are exactly the directed unit speed geodesics. We briefly mention two compactifications of Karpelevič that were constructed using these geodesics (the geodesic and Karpelevič compactifications). This work of Karpelevič and more recent work of Hattori [Hat92] were influential in the work of Ji–MacPherson [JM02] to which the author is indebted. We introduce several equivalence relations on DM geodesics and describe how the associated equivalence classes are parametrized by certain factors of the real horospherical decomposition.

In §2.3 we introduce locally symmetric spaces, denoted $\Gamma\backslash X$, and their Satake, Borel–Serre, and reductive Borel–Serre compactifications. We briefly describe the construction of the Borel–Serre and reductive Borel–Serre compactifications by the uniform method. We give a detailed description of the maximal Satake compactification and its topology by the uniform method. We use this description in our construction appearing in §4.1.

In §2.4 we introduce eventually distance minimizing (EDM) geodesics in $\Gamma\backslash X$. Certain equivalence classes of these geodesics have been used [JM02, §14] to describe the Borel–Serre and reductive Borel–Serre compactifications. We discuss these equivalence relations and how they were combined to construct the compactifications.

2.1 Global Symmetric Spaces and their Compactifications

Global symmetric spaces have been a central object of study for the past century. They are a class of manifolds with a certain symmetry property.

Definition 2.1. [Hel78, p.205] A Riemannian manifold X is called a *global symmetric space* if each $p \in X$ is an isolated fixed point of an involutive isometry s_p of X.

These spaces were completely classified by Elie Cartan in the early part of the 20th century. Their classification depends on his earlier classification of simple Lie algebras over $\mathbb{R}$.

We have the following decomposition theorem of symmetric spaces. It is commonly referred to as the DeRham decomposition.

Proposition 2.2. ([Hel78, 4.2], [Ebe96, 1.2]) Let X be a simply connected Riemannian global symmetric space. Then X decomposes uniquely into the product

$$X = X_0 \times X_- \times X_+,$$

where X_0 is a Euclidean space, and X_- and X_+ are Riemannian global symmetric spaces of the compact and noncompact type, respectively.

Following the assumption in [BJ07], which enables the description of the Satake compactifications using the uniform method, we let G be an adjoint connected semisimple Lie group. For a maximal compact subgroup $K \subset G$, the geometric quotient $X = G/K$ is a symmetric space of noncompact type.

There is another important decomposition of X that is induced by a decomposition of the group G. For the Lie algebra $\mathfrak{g}$ of G we have a Cartan decomposition $\mathfrak{g} = \mathfrak{p} + \mathfrak{k}$. Let $\mathfrak{a} \subset \mathfrak{p}$ be any maximal abelian subspace. We have a set of simple roots α_i that evaluate elements of $\mathfrak{a}$. Let $\mathfrak{a}^+$ be the open set in $\mathfrak{a}$ upon which all simple roots evaluate positively. Let $A = \exp \mathfrak{a}$, $A^+ = \exp \mathfrak{a}^+$, and $\overline{A^+}$ equal the closure of A^+ in G.

Theorem 2.3. [Hel78, p.402] Let G be any connected semisimple Lie group with Lie algebra $\mathfrak{g}$. Then we have

$$G = K\overline{A^+}K.$$

That is, each $g \in G$ can be written $g = k_1 a k_2$ where $k_1, k_2 \in K$ and $a \in \overline{A^+}$. Moreover, $a = a^+(g)$ is unique.

This polar decomposition is also defined for certain reductive Lie groups.

Definition 2.4. [Kna96, p.72] Consider the unitary subgroup $U(n) \subset GL_n(\mathbb{C})$ and the set of Hermitian $n \times n$ matrices $\mathcal{H}_n$. For $GL_n(\mathbb{C})$, the map

$$U(n) \times \mathcal{H}_n \to GL_n(\mathbb{C})$$

given by $(k, X) \mapsto ke^X$ is a homeomorphism. For any X, we have an eigendecomposition $X = UAU^*$ where $UU^* = Id$ and A is diagonal. The inverse map is the *polar decomposition* of $GL_n(\mathbb{C})$. This induces a polar decomposition KAK of any reductive subgroup of $GL_n(\mathbb{C})$.

We will use this decomposition to describe the connection between the real horospherical decompostion and the μ-relation (Definition 3.38).

In [Sat60b], Satake initiated the study of compactifications of the global symmetric space X. More recently, in [Zuc83, GJT98, BJ07], the relationship between the Satake compactifications and alternative descriptions of the compactifications have been studied. In fact, up to isomorphism, there are only finitely many Satake compactifications with one maximal compactification that dominates all others. Below we look at these compactifications in certain examples.

2.1.1 Classical construction of Satake compactifications

Example 1. The prototypical example of a global symmetric space is the complex upper half plane $X = \mathfrak{H}$. This can be thought of as the quotient G/K where $G = SL_2(\mathbb{R})$ and $K = SO(2)$. Up to isomorphism, there is one Satake compactification of $\mathfrak{H}$. We construct this compactification by first considering the map that takes elements of $SL_2(\mathbb{R})$ into the set of symmetric 2×2 matrices via $g \mapsto gg^t$. Specifically we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} a^2+b^2 & ac+bd \\ ac+bd & c^2+d^2 \end{pmatrix}.$$

One can easily see that the kernel of this map is exactly the compact subgroup $SO(2)$, and that the image is contained in the closed cone of positive semidefinite matrices. If we projectivize the set of nonzero symmetric matrices and take the closure of the image of $SL_2(\mathbb{R})$, we obtain a compactification of $\mathfrak{H}$. The boundary of this compactification consists of $\mathbb{R} \cup \{i\infty\}$ when viewed in the upper half plane model, or perhaps more easily identifiable as S^1 in the disc model.

In the following, we trace the exposition in [BJ06, I.4]. Satake's approach of compactifying a general symmetric space X consists of first defining a standard compactification of the space $\mathcal{P}_n = PSL_n(\mathbb{C})/PSU(n)$, then embedding X into this standard compactification, and finally taking the closure of the embedding. We explain this process below.

We can identify $\mathcal{P}_n$ with the space of positive definite Hermitian matrices of determinant one by mapping $PSL_n(\mathbb{C})$ into $\mathcal{H}_n$ via $g \mapsto gg^*$, where g^* denotes the conjugate transpose of g. For any nonzero matrix $A \in \mathcal{H}_n$, let $[A]$ denote the image of A in the projectivization $P(\mathcal{H}_n)$. This gives a map

$$i : \mathcal{P}_n \to P(\mathcal{H}_n)$$

via $A \mapsto [A]$. This map yields a compactification of $\mathcal{P}_n$ essentially made by attaching equivalence classes of positive semidefinite Hermitian matrices as made precise in the following definition.

Definition 2.5. The closure of $i(\mathcal{P}_n)$ in the compact space $P(\mathcal{H}_n)$ is a $PSL_n(\mathbb{C})$-equivariant compactification of $\mathcal{P}_n$ called the *standard Satake compactification of* $\mathcal{P}_n$ and denoted by $\overline{\mathcal{P}_n}^S$.

Following the example in [BJ06, I.4.2], we consider the standard Satake compactification of $PSL_2(\mathbb{C})/PSU(2)$.

Example 2. We can identify the symmetric space $\mathcal{P}_2$ with real hyperbolic three dimensional space $\mathbb{H}^3$. Consider the upper half space model of $\mathbb{H}^3$:

$$\mathbb{H}^3 = \{(z,t) \mid z = x + yi \in \mathbb{C}, t \in \mathbb{R}_{>0}\}.$$

Think of $\mathbb{H}^3$ as a subset of the quaternions $\mathbb{H}$ under the map

$$(x+iy,t) \mapsto (x, y \cdot i, t \cdot j, 0 \cdot k).$$

We have an action of $SL_2(\mathbb{C})$ on $\mathbb{H}$ by fractional linear transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot w = \frac{aw+b}{cw+d}$$

that acts transitively on the image $\mathbb{H}^3$. The stabilizer of the point $(0, 0 \cdot i, 1 \cdot j, 0 \cdot k)$ is the compact subgroup $SU(2) \subset SL_2(\mathbb{C})$. If we consider only elements of $SL_2(\mathbb{C})$ that take the form

$$\left\{ \begin{pmatrix} a & b/a \\ 0 & a^{-1} \end{pmatrix} \middle| \, b \in \mathbb{C}, a \in \mathbb{R}_{>0} \right\}$$

we easily see this subgroup also acts transitively on $\mathbb{H}^3$. The closure of this subgroup in $P(\mathcal{H}_2)$ consists of equivalence classes of matrices that take the form

$$\left\{ \begin{pmatrix} |b|^2 & b \\ b & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \middle| \, b \in \mathbb{C} \right\}.$$

As a set, the boundary of $\overline{\mathcal{P}_2}^S$ consists of the points at infinity $\mathbb{C} \cup \{\infty\}$ in the upper half space model.

We continue our exposition of the Satake compactifications by examining how an embedding of X into $\overline{\mathcal{P}_n}^S$ is chosen. In the examples involving the groups $SL_2(\mathbb{R})$ and $SL_2(\mathbb{C})$, we used maps that took an element g to gg^t and gg^* respectively. For any adjoint semisimple Lie group G, we chose an irreducible faithful projective representation

$$\tau : G \to PSL_n(\mathbb{C})$$

that satisfies $\tau(\theta(g)) = (\tau(g)^*)^{-1}$, where θ is the Cartan involution on G associated with K, and $g \mapsto (g^*)^{-1}$ is the Cartan involution on $PSL_n(\mathbb{C})$ associated with $PSU(n)$. In our previous examples, the representation was the identity map and the Cartan involution was the inverse conjugate transpose.

With the above conditions on the representation, the map

$$i_\tau : X \to \mathcal{P}_n$$

defined by $gK \mapsto \tau(g)\tau(g)^*$ is a totally geodesic embedding of X. By taking an appropriate closure, we define the Satake compactifications.

Definition 2.6. The closure of $i_\tau(X)$ in $\overline{\mathcal{P}_n}^S$ is called the *Satake compactification associated with the representation* τ, and denoted by $\overline{X}^S_\tau$.

We reexamine the example of $\mathfrak{H}$.

Example 3. If we take the standard representation of $G = SL_2(\mathbb{R})$ into $PSL_2(\mathbb{C})$, we can identify the embedding of $\mathfrak{H}$ with the set of matrices

$$\left\{ \begin{pmatrix} a & b/a \\ 0 & a^{-1} \end{pmatrix} \middle| \, b \in \mathbb{R}, a \in \mathbb{R}_{>0} \right\}.$$

The closure of the embedding in $\overline{\mathcal{P}_2}^S$ will correspond to the equivalence classes determined by the matrices

$$\left\{ \begin{pmatrix} b^2 & b \\ b & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \middle| \, b \in \mathbb{R} \right\}$$

in $P(\mathcal{H}_2)$. These boundary components correspond exactly to the set $\mathbb{R} \cup \{\infty\}$ in the upper half space model.

2.1.2 Construction of Satake compactifications using the uniform method

In [BJ07] there is a new construction of the Satake compactifications using the uniform method. The real horospherical decomposition of X is fundamental to this description. This decomposition can be thought of as a generalization of the upper half plane model of the hyperbolic plane. Before describing the real horospherical decomposition, we must first define the Langlands decomposition (cf. [BJ06, p.32]). For a real parabolic subgroup P of G we have the Langlands decomposition of P

$$P = N_P A_P M_P$$

into its nilpotent radical N_P and its Levi subgroup $A_P M_P$, which is stable under the Cartan involution θ associated with K. The group A_P is referred to as the split component of P associated with K. Let $K_P = M_P \cap K$. Then K_P is a maximal compact subgroup of M_P, and the quotient $X_P = M_P/K_P$ is a symmetric space of noncompact type called the *boundary symmetric space*.

The Langlands decomposition of P induces the horospherical decomposition of X with respect to P:

$$X = N_P \times A_P \times X_P. \tag{2.1.1}$$

The choice of a maximal compact subgroup $K \subset G$ corresponds to the choice of a basepoint $x_0 \in X$. For example, when $X = \mathfrak{H}$ the choice of $SO(2) \subset SL_2(\mathbb{R})$ corresponds to the basepoint $i \in \mathfrak{H}$. In the literature, when using the horospherical decomposition coordinates, the basepoint x_0 is sometimes attached to the X_P component (often we simply omit it).

The group G acts on X and preserves the real horospherical decomposition. Since the Cartan decomposition implies $G = PK$, the action of G on X can be described by the actions of P and K on X. For any two elements $g, h \in G$, denote (left) conjugation by ${}^g h = ghg^{-1}$. Similarly denote right conjugation by

$h^g = g^{-1}hg$. There is an equivariant P-action on the horospherical decomposition $N_P \times A_P \times X_P$

$$n_0 a_0 m_0 (n, a, z) = (n_0 ({}^{a_0 m_0} n), a_0 a, m_0 z), \tag{2.1.2}$$

where $p = n_0 a_0 m_0$ is the Langlands decomposition of $p \in P$. There is an action of K on X given by

$$k \cdot (n, a, z) = ({}^k n, {}^k a, k \cdot z) \in N_{{}^kP} \times A_{{}^kP} \times X_{{}^kP}, \tag{2.1.3}$$

where

$$k \cdot z = k \cdot m K_P = {}^k m \, {}^k K_P \in X_{{}^kP}. \tag{2.1.4}$$

This action takes the horospherical decomposition with respect to P to the horospherical decomposition with respect to ${}^kP$.

For an arbitrary parabolic subgroup Q, we have the following result that describes the boundary component X_Q in terms of a horospherical decomposition.

Lemma 2.7. [BJ06, I.1.22] For every pair of real parabolic subgroups $P \subset Q$, there is a unique parabolic subgroup P' of M_Q such that

$$N_P = N_Q N_{P'}, M_{P'} = M_P, A_P = A_Q A_{P'},$$

which implies that

$$X_Q = N_{P'} \times A_{P'} \times X_P.$$

Definition 2.8. The decomposition described in the previous lemma

$$X_Q = N_{P'} \times A_{P'} \times X_P$$

is called the *relative horospherical decomposition* for the pair $P \subset Q$.

The boundary symmetric spaces X_P are used in the construction of the maximal Satake compactification in [BJ07]. As a set, the maximal Satake is the disjoint union of X and all boundary symmetric spaces X_P.

2.1.3 Topology of $\overline{X}_{max}$

Here we discuss the topology of $\overline{X}_{max}$ in terms of convergent sequences to explain how geodesics converge to the maximal boundary.

Let P be a real parabolic subgroup and let X_P be the boundary symmetric space in the real horospherical decompostion of X with respect to P. As a set, we have

$$\overline{X}_{max} = X \cup \coprod_P X_P.$$

The following description of convergent sequences appears in [BJ06, I.10.2]. The set of roots $\Phi(P, A_P)$ of the adjoint action of the Lie algebra $\mathfrak{a}_P$ of the real split torus on the Lie algebra of the nilpotent radical $\mathfrak{n}_P$ plays a key role. We refer the interested reader to [BJ06, I.1.10] for a more detailed treatment of the notation. Each root in $\Phi(P, A_P)$ can be thought of as a character of A_P. For $a \in A_P$ and $\alpha \in \Phi(P, A_P)$, let

$$a^\alpha = \exp \alpha(\log a). \tag{2.1.5}$$

Fix a boundary component X_P and let $z_\infty \in X_P$. An unbounded sequence y_j in X converges to z_∞ if and only if y_j can be written in the form $y_j = k_j n_j a_j z_j$, where $k_j \in K, n_j \in N_P, a_j \in A_P, z_j \in X_P$ satisfy

1. $k_j \to e$, where e is the identity element,
2. for all $\alpha \in \Phi(P, A_P)$, $a_j^\alpha \to \infty$,
3. $n_j^{a_j} \to e$,
4. $z_j \to z_\infty$.

We remark that this is only part of what one needs to topologize $\overline{X}_{max}$. The interested reader may refer to [BJ06, I.10.2] for a description of the convergence of unbounded sequences in the boundary components. We use the above description to prove the following:

Lemma 2.9. Every geodesic ν converges to a boundary point in $\overline{X}^S_{max}$.

Proof. Every geodesic ν takes the form $\nu(t) = (n, a\exp(tH), z) \in N_P \times A_P \times X_P$, where $H \in \mathfrak{a}_P^+(\infty)$ is a unit length vector in the Weyl chamber $\mathfrak{a}_P^+$ and $\log a \perp H$ for some parabolic subgroup P. A sequence y_j on ν takes the form $y_j = k_j n_j a_j z_j$ where $k_j = e, n_j = n, z_j = z$ and $a_j = a\exp(t_j H)$ where $t_j \to \infty$. Since k_j, z_j are fixed we only show conditions 2 and 3 above.

For condition 2, since $H \in \mathfrak{a}_P^+(\infty)$ and $\log a \perp H$, we have $a^\alpha = 1$. Therefore, $a_j^\alpha = \exp(t_j H)^\alpha$. Since $\exp(t_j H)^\alpha = \exp\alpha(t_j H)$ and $\alpha(H) > 0$, we have $a_j^\alpha \to \infty$ as $j \to \infty$. Thus, condition 2 holds.

For condition 3, we recall by definition,

$$n_j^{a_j} = n^{a\exp(t_j H)} = \exp(-t_j H)a^{-1}na\exp(t_j H) = a^{-1}\exp(-t_j H)n\exp(t_j H)a.$$

Since $\exp(-t_j H)n\exp(t_j H) \to e$ as $j \to \infty$, we have $n_j^{a_j} \to e$ as $j \to \infty$. Thus, condition 3 holds. Therefore ν converges to $z_\infty \in e(P)$. □

2.1.4 The μ-reduction

The boundary components in the nonmaximal Satake compactifications are closely related to those of the maximal Satake. They are indexed by a subset of the maximal boundary components and are obtained by collapsing certain factors of these maximal components. For example, in the space $X = SL_3(\mathbb{R})/SO(3)$, boundary components of the maximal Satake consist of copies of $\mathfrak{H}$ and points $\mathbb{R} \cup \{i\infty\}$ in the closure of $\mathfrak{H}$. In the nonmaximal Satake compactifications certain copies of $\mathfrak{H}$ and all the corresponding boundary points are collapsed down to a single point.

In the description that follows, we disregard the boundary components corresponding to the boundary points and include one point at infinity corresponding to the collapsed $\mathfrak{H}$ component.

There is a further decomposition of the boundary symmetric space X_P that depends on how certain characters (or roots) of the split component A_P interact with a weight μ contained in the dual space $\mathfrak{a}_P^*$ of the Lie algebra $\mathfrak{a}_P$ corresponding to A_P. Each X_P in the maximal Satake boundary splits into two boundary symmetric spaces

$$X_P = X_I \times X_{I'}, \tag{2.1.6}$$

where we define the spaces $X_I, X_{I'}$ below. As a set, the Satake compactification corresponding to the weight μ will then be defined as the disjoint union

$$\overline{X}_\mu = X \cup \coprod_{\mu-\text{saturated } P} X_I \tag{2.1.7}$$

where only factors of certain μ-saturated parabolic subgroups contribute to the boundary. We follow the exposition appearing in [BJ06, I.11.1] and explain this decomposition.

We choose and fix a minimal parabolic subgroup P_0. From now on, we refer to P_0 as the standard minimal parabolic subgroup. Let $\mathfrak{a}_{P_0}^+$ denote the corresponding positive Weyl chamber and $\overline{\mathfrak{a}_{P_0}^+}$ its closure. These are both contained in the maximal abelian subalgebra $\mathfrak{a}_{P_0}$. Let $\mathfrak{a}_{P_0}^*$ be the dual of $\mathfrak{a}_{P_0}$, let $\mathfrak{a}_{P_0}^{*+}$ be the dual of the positive chamber $\mathfrak{a}_{P_0}^+$, and let $\overline{\mathfrak{a}_{P_0}^{*+}}$ be the dual of the closed positive chamber $\overline{\mathfrak{a}_{P_0}^+}$. We fix a dominant weight μ contained in $\overline{\mathfrak{a}_{P_0}^{*+}}$.

Let $\Delta(P_0, A_{P_0})$ denote the set of simple roots corresponding to P_0. For any subset $J \subset \Delta(P_0, A_{P_0})$ we can consider the group $A_{P_{0,J}} = \{a \in A_{P_0} \mid a^\alpha = 1 \text{ for all } \alpha \in J\}$, where $a^\alpha = \exp \alpha(\log a)$. The group $A_{P_{0,J}}$ is the real split torus of a standard parabolic subgroup that we denote $P_{0,J}$. We use subsets of $\Delta(P_0, A_{P_0})$ to explain the collapsing of standard boundary symmetric spaces below.

Definition 2.10. A subset $J \subset \Delta(P_0, A_{P_0})$ is called *μ-connected* if $J \cup \{\mu\}$ cannot be written as a disjoint union $J_1 \cup J_2$ such that elements in J_1 are perpendicular to elements in J_2 with respect to an invariant positive-definite inner product on $\mathfrak{a}_{P_0}^{*+}$.

On the level of groups, $P_{0,J}$ is called *μ-connected* if J is μ-connected.

Example 4. For examples of μ-connected subsets of simple roots we refer the interested reader to the very nice expository paper of Goresky and papers of Casselman and Saper [Gor05, Cas97, Sap04].

Definition 2.11. For any subset $J \subset \Delta(P_0, A_{P_0})$, consider the set

$$J' = \{\alpha \in \Delta(P_0, A_{P_0}) \mid \alpha \perp J \cup \{\mu\}\}.$$

The union $K = J \cup J'$ is called the *μ-saturation* of J.

On the level of groups, $P_{0,K}$ is called the *μ-saturation* of $P_{0,J}$. The μ-saturation is unique. We now come to the main definition of this subsection.

Definition 2.12. For a standard parabolic subgroup $P_{0,J}$, let I_J be the maximal μ-connected subset of J. The subset I_J is called the *μ-reduction* of J, and the maximal μ-connected standard parabolic $P_{0,I_J} \subset P_{0,J}$ is called the *standard μ-reduction* of $P_{0,J}$.

As noted in [BJ06, I.11.1], the μ-reduction is not unique. For example, for any $q \in P_{0,J} - P_{0,I_J}$ the parabolic ${}^qP_{0,I_J}$ is also a μ-reduction of $P_{0,J}$. We use μ-saturated and standard μ-reduced parabolic subgroups to define the nonmaximal boundary components. We first define the splitting of the boundary symmetric space X_P appearing in equation 2.1.6.

Let $P_{0,J}$ be a standard parabolic subgroup. Let I_J be the largest μ-connected subset of J and let I'_J be the subset of J perpendicular to $I_J \cup \{\mu\}$. We have $J = I_J \cup I'_J$. This gives a splitting of the boundary symmetric space $X_{P_{0,J}}$ corresponding to $P_{0,J}$:

$$X_{P_{0,J}} = X_{P_{0,I_J}} \times X_{P_{0,I'_J}}. \tag{2.1.8}$$

This splitting 2.1.8 will be used to determinine an equivalent formulation of the μ*-relation* (Definition 3.38) in terms of the real horospherical decomposition.

We can now define the nonmaximal boundary components. Let $Q = P_{0,J}$ be a μ-saturated standard parabolic subgroup. Define the boundary component $e(Q)$ corresponding to Q as $e(Q) = X_{P_{0,I_J}}$. For any μ-saturated standard parabolic subgroup we specify a particular μ-reduction that corresponds to a standard parabolic. We let the boundary symmetric space corresponding to this μ-reduced standard parabolic be the boundary component.

For a general parabolic subgroup Q, there exists an element $k \in K$ and a unique standard parabolic $P_{0,J}$ such that $Q = {}^kP_{0,J}$. When $P_{0,J}$ is μ-saturated we define the corresponding boundary component to be $e(Q) = X_{{}^kP_{0,I_J}}$. The group Q is conjugate to the same standard parabolic for multiple elements $k \in K$. As proved in [BJ06, I.11.3], the definition of this boundary component is independent of the choice of k.

As a set, the nonmaximal Satake compactification is the disjoint union of X and all the boundary components $e(Q)$ corresponding to μ-saturated parabolics Q.

2.1.5 Topology of $\overline{X}_\mu$

In this subsection, we describe the topology on $\overline{X}_\mu$ in terms of convergent sequences. Specifically, we consider the convergence of unbounded sequences in X. The references for this description are [BJ06, §I.11.10] and [BJ07, §5]. There is a slight error in [BJ06, §I.11.10] that is resolved in [BJ07, §5].

To ease notation, we stop using the symbol ${}^kP_{0,J}$ to refer to a general parabolic subgroup. Instead we let Q denote a μ-saturated parabolic and P_I its μ-reduction. The boundary component $e(Q)$ is denoted X_{P_I} or simply X_I. In our new notation, as a set we have

$$\overline{X}_\mu = X \cup \coprod_{\mu-\text{saturated } Q} X_I.$$

To describe the convergence of interior points of X to boundary points, we use horospherical decompositions corresponding to nonsaturated parabolic subgroups R as well. Consider a μ-saturated parabolic subgroup Q that contains a minimal parabolic subgroup P. Let P_I be the μ-connected reduction of Q. Assume that $P \subset P_I$. For any parabolic subgroup R satisfiying $P_I \subseteq R \subseteq Q$, write $R = P_{J'}$, where $J' = I \cup I'$ with I' perpendicular to I. The space X_R decomposes as $X_{P_I} \times X_{P_{I'}}$ and the horospherical decomposition

$$X = N_R \times A_R \times X_R$$

can be refined to

$$X = N_R \times A_R \times X_{P_I} \times X_{P_{I'}}. \tag{2.1.9}$$

A topology on $\overline{X}_\mu$ is given as follows. An unbounded sequence y_j converges to a boundary point $z_\infty \in X_I$ if there exists a parabolic subgroup R (as above) such that in the refined decomposition (2.1.9) we have $k_j \cdot y_j = k_j \cdot (n_j a_j z_j z'_j)$, where the factors satisfy the following:

1. the image of k_j in the quotient $K/K \cap \mathcal{Z}(X_I)$ converges to the identity coset,
2. for $\alpha \in \Delta(Q, A_P)$, $a_j^\alpha \to \infty$, while for $\alpha \in \Delta(R, A_P) - \Delta(Q, A_P)$, a_j^α is bounded from below,
3. $n_j^{a_j} \to e$,
4. $z_j \to z_\infty$, and
5. z'_j is bounded.

The interested reader may refer to [BJ07, §5] for a description of the convergence of unbounded sequences in the boundary components.

The following lemma plays a role in our description of $\overline{X}_\mu$ with geodesics.

Lemma 2.13. Every geodesic ν converges to a boundary point in $\overline{X}_\mu$.

Proof. By Lemma 2.9, every geodesic converges to a boundary point in $\overline{X}^S_{max}$. Since [BJ06, I.11.15] shows that for any $\overline{X}_\mu$, the identity map on X extends to a continuous surjective map from $\overline{X}^S_{max}$ to $\overline{X}_\mu$ (i.e. $\overline{X}^S_{max}$ dominates $\overline{X}_\mu$), it follows that any geodesic converges in $\overline{X}_\mu$. We can also show this directly as in Lemma 2.9. □

2.1.6 Examples

We now give examples illustrating the definitions in this chapter.

Example 5. Consider the upper half plane $X = \mathfrak{H}$. We have $X = SL_2(\mathbb{R})/SO(2)$. When P is the standard parabolic subgroup of upper triangular matrices, we have

$$N_P = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \quad X_P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_P = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix},$$

where $a \in \mathbb{R}$ and $\lambda \in \mathbb{R}_{>0}$.

There is a unique Satake compactification, and the boundary components are points. The boundary component $\{i\infty\}$ is represented by the identity matrix X_P above, and the components in bijection with the points of $\mathbb{R}$ are those boundary symmetric spaces defined by other real parabolic subgroups (namely conjugates of the standard parabolic by elements in $SL_2(\mathbb{R})$).

Example 6. The simplest example of a higher rank symmetric space is the product of upper half planes (by higher rank, we mean the dimension of the largest split component A_P is larger than one). Consider $X = \mathfrak{H} \times \mathfrak{H} = SL_2(\mathbb{R})/SO(2) \times SL_2(\mathbb{R})/SO(2)$. If we let $P \subset SL_2(\mathbb{R})$ be the standard parabolic from Example 5, the three standard parabolic subgroups take the form $P \times P$, $SL_2(\mathbb{R}) \times P$, and $P \times SL_2(\mathbb{R})$. Once again, there is a unique Satake compactification of X (since the representation must be irreducible). The boundary symmetric spaces take the form $\{*\} \times \{*\}$, $\mathfrak{H} \times \{*\}$, and $\{*\} \times \mathfrak{H}$ respectively for each type of parabolic, where $\{*\}$ is a point. Since the boundary symmetric space of the standard minimal parabolic $P \times P$ is a point, the horospherical decomposition of X with respect to $P \times P$

consists of the factors

$$N_{P\times P} = \left\{ \begin{pmatrix} 1 & a_1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & a_2 \\ 0 & 1 \end{pmatrix} \right\} \text{ and } A_{P\times P} = \left\{ \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^{-1} \end{pmatrix}, \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_2^{-1} \end{pmatrix} \right\},$$

where $a_i \in \mathbb{R}$ and $\lambda_i \in \mathbb{R}_{>0}$.

Example 7. Now we consider an irreducible higher-rank example. Let $X = SL_3(\mathbb{R})/SO(3)$. We can identify this space with the set of positive definite symmetric 3×3 matrices modulo homotheties. We have three standard parabolic subgroups and three different Satake compactifications. Let P_1, P_2 and P_3 denote the parabolic subgroups of $SL_3(\mathbb{R})$ with the block forms

$$\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}, \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix}, \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}$$

respectively. The factors of the horospherical decompositions are

$$N_{P_1} = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}, X_{P_1} \sim \{*\}, A_{P_1} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix},$$

$$N_{P_2} = \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}, X_{P_2} \simeq \mathfrak{H} \times \{*\}, A_{P_2} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix},$$

and

$$N_{P_3} = \begin{pmatrix} 1 & a & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, X_{P_3} \simeq \{*\} \times \mathfrak{H}, A_{P_3} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_3 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

respectively, where $a, b, c \in \mathbb{R}$ and $\lambda_i \in \mathbb{R}_{>0}$. The boundary symmetric spaces are therefore points and copies of $\mathfrak{H}$. The maximal Satake compactification has a contribution from each parabolic subgroup. In the boundary, two $\mathfrak{H}$s corresponding to the maximal standard parabolic subgroups P_2, P_3 share a point in their boundaries at corresponding to the point contributed by P_1. This reveals the inductive nature of the Satake compactifications, in that noncompact boundary components are compactified by other boundary components. In each of the nonmaximal Satake compactifications, there is no contribution from the minimal parabolic subgroups. One of the boundary symmetric spaces corresponding to a maximal parabolic will collapse to a point, leaving a collapsed boundary of $\mathfrak{H}$s and points. In Satake's original construction, the three representations that yield the distinct compactifications are the adjoint, standard, and dual to standard representations.

2.2 DM Geodesics in Global Symmetric Spaces

In a global symmetric space X, unit speed directed curves $\nu(t)$ that satisfy the distance relationship

$$d(\nu(t_1), \nu(t_2)) = |t_2 - t_1|$$

for all $t_1, t_2 \in \mathbb{R}$ are called *distance minimizing (DM) rays.* They are exactly the unit speed directed geodesics in X. A detailed study of the geometry of these geodesics is given in [Kar67]. One result of this study is the construction of the geodesic and Karpelevič compactifications of X. In [BGS85, Ebe96], there is a more general discussion about the geometry of geodesics in manifolds of nonpositive curvature (of which global symmetric spaces are a special case). In [Ner98, Ner03], Neretin extends the original study in [Kar67] and constructs several other compactifications of certain symmetric spaces using these geodesics.

In the following, we describe several equivalence relations on geodesics in X. Many of these relations were originally described in [Kar67]. All of the sets of equivalence classes can be described using certain factors of the horospherical decomposition. The relationship between equivalence classes and horospherical decompositions relies on the G-invariance of the metric on X and the G-action on the horospherical decomposition (2.1.2)–(2.1.3).

This following relation was used to define ideal boundary points in the geodesic compactification in [Kar67].

Definition 2.14. Two geodesics ν_1, ν_2 are called *equivalent*, denoted $\nu_1 \sim \nu_2$, if $\lim_{t\to\infty} \sup d(\nu_1(t), \nu_2(t)) < \infty$.

This defines an equivalence relation on geodesics in X; we denote the class containing ν by $[\nu]$. Ji and MacPherson [JM02] use the following notation to denote this equivalence class. The set $F(\nu)$ of geodesics equivalent to ν

$$F(\nu) = \{\ \nu' \mid \nu' \in [\nu]\} \tag{2.2.1}$$

is called the *finite bundle* associated to ν. We use both notations.

The relation gives our first example of how a geometric condition on geodesics can be characterized using the horospherical decomposition.

Proposition 2.15. For a geodesic ν, there exists a parabolic subgroup P such that $\nu(t) = (n, a\exp(tH), z) \in N_P \times A_P \times X_P$, where H is a unit length velocity vector in the positive Weyl chamber $\mathfrak{a}_P^+$ of $\mathfrak{a}_P$ (where $\exp \mathfrak{a}_P = A_P$).

Proof. This follows from the theory of pencils in symmetric spaces. Maximal totally flat geodesic submanifolds of X through a basepoint x_0 take the form $\exp \mathfrak{a} x_0$, where $\mathfrak{a}$ is a maximal abelian subalgebra of the Lie algebra $\mathfrak{g}$ of G. Geodesics are one dimensional totally flat geodesic submanifolds. Therefore they are contained in some maximal submanifold. □

Using the description of ν in the horospherical decomposition we have the following:

Lemma 2.16. [BJ06, I.2.10-12] Two geodesics ν_1, ν_2 in X are equivalent if and only if they have the form $\nu_i(t) = (n_i, a_i \exp(tH_i), z_i) \in N_P \times A_P \times X_P$ with $H_1 = H_2$.

This yields the following parametrization:

Proposition 2.17. [BJ06, I.2.15] Let $\nu(t) = (u, z, a\exp(tH)) \in N_P \times A_P \times X_P$. The family of geodesics $F(\nu)$ is parametrized by $N_P \times X_P \times \langle H \rangle^{\perp}$.

There is a finer equivalence relation among geodesics in a finite bundle that identifies geodesics whose distance vanishes in the limit.

Definition 2.18. Two geodesics ν_1, ν_2 are called *nil related (N-related)*, denoted $\nu_1 \overset{N}{\sim} \nu_2$, if $\lim_{t\to\infty} \inf_{s\in\mathbb{R}} d(\nu_1(t), \nu_2(s)) = 0$.

This is an obvious equivalence relation on geodesics; we denote the equivalence class containing ν by $[\nu]_N$. The set $N(\nu)$ of geodesics N-related to ν

$$N(\nu) = \{ \nu' \mid \nu' \in [\nu]_N \} \tag{2.2.2}$$

is called the *nil bundle* associated to ν. As in Lemma 2.16, N-related geodesics have a common structure in the horospherical decomposition.

Lemma 2.19. [BJ06, I.2.29] Two geodesics ν_1, ν_2 in X are N-related if and only if they have the form $\nu_i(t) = (n_i, a_i \exp(tH_i), z_i) \in N_P \times A_P \times X_P$ with $z_1 = z_2$, $a_1 = a_2$, and $H_1 = H_2$.

Proposition 2.20. Let $\nu(t) = (u, z, a\exp(tH)) \in N_P \times A_P \times X_P$. The family of geodesics $N(\nu)$ is parametrized by N_P.

We have a similar description of N-equivalence classes using factors of the horospherical decomposition.

Proposition 2.21. [BJ06, I.2.29] Let $\nu(t) = (u, z, a\exp(tH)) \in N_P \times A_P \times X_P$. The set of N-equivalence classes in $F(\nu)$ is parametrized by $X_P \times \langle H\rangle^{\perp}$.

Following [JM02, 1.6,14.6], the set of N-equivalence classes in Proposition 2.21 is called the *metric link*, denoted $S(\nu)$. The metric link is a metric space with distance determined by

$$\lim_{t\to\infty} \inf_{s\in\mathbb{R}} d(\nu_1(t), \nu_2(s)), \tag{2.2.3}$$

where ν_1, ν_2 are representatives of two N-equivalence classes.

For a geodesic ν, consider the geodesics in $F(\nu)$ that remain at a fixed distance from ν. This set

$$C(\nu) = \{\ \nu' \in F(\nu) \mid d(\nu(t), \nu'(t)) \text{ constant}\} \tag{2.2.4}$$

is called the *congruence bundle* associated to ν. As in [JM02, p.536], for a geodesic ν in X, the metric link $S(\nu)$ can be identified with the congruence bundle $C(\nu)$:

Lemma 2.22. Two geodesics ν_1, ν_2 in X are in the same congruence bundle if and only if they have the form $\nu_i(t) = (n_i, a_i\exp(tH_i), z_i) \in N_P \times A_P \times X_P$ with $n_1 = n_2$ and $H_1 = H_2$.

Proof. If ν_1, ν_2 are in the same congruence bundle, then it is obvious they are equivalent. Therefore $H_1 = H_2$. Suppose that $n_1 \neq n_2$. Without loss of generality, suppose that $z_1 = z_2 = Id$, $a_1 = a_2 = Id$, and $n_2 = Id$. Then we have

$d(\nu_1(t), \nu_2(t)) = d(\exp(-tH_1)n_1 \exp(tH_1)x_0, x_0)$, where x_0 is the basepoint corresponding to K. This distance is different for every value of t (unless $n_1 = Id$). This is a contradiction, and so $n_1 = n_2$. By the G-invariance of the metric, it is clear that any two geodesics of this form are contained in the same congruence bundle. □

Since the metric link can be parametrized by $X_P \times \langle H \rangle^\perp$, Lemma 2.22 shows that the geodesics in $C(\nu)$ are in bijection with the N-equivalence classes in $F(\nu)$.

The congruence bundle is a metric space with the extra structure of a Riemannian symmetric space. That is, we can identify $C(\nu)$ with $X_P \times \langle H \rangle^\perp$ as described in the last paragraph. We recall the DeRham decomposition for a Riemannian symmetric space (Proposition 2.2). It is clear that in the realization $X_P \times \langle H \rangle^\perp$ of $C(\nu)$, the Euclidean space $X_0 = \langle H \rangle^\perp \in \mathfrak{a}_P$ and the noncompact space $X_+ = X_P$. These observations will be useful in Chapter 3 when we prove that the Clifford relation is an equivalence relation on geodesics in $C(\nu)$.

2.3 Locally Symmetric Spaces and their Compactifications

The primary goal of this thesis is to describe certain compactifications of locally symmetric spaces. The standard definition of such spaces is provided below.

Definition 2.23. [Hel78, p. 200] A Riemannian manifold is called a *locally symmetric space* if for each element p there exists a normal neighborhood of p on which the geodesic symmetry with respect to p is an isometry.

The locally symmetric spaces we consider use the theory of *algebraic groups* in their definition. Let $\mathbb{C}[x_{ij}, D^{-1}]$ denote the coordinate ring of the group $GL_n(\mathbb{C})$. Here, the variables x_{ij} are the entries of an indeterminate matrix and D is the polynomial $\det(x_{ij})$. A linear algebraic group $\mathbf{G}$ is a subgroup of $GL_n(\mathbb{C})$ with the structure of an affine algebraic variety defined by an ideal $I \subset \mathbb{C}[x_{ij}, D^{-1}]$. Moreover, the group multiplication and inversion for $\mathbf{G}$ are morphisms of algebraic varieties. We say $\mathbf{G}$ is defined over a field $k \subset \mathbb{C}$ if the ideal I is defined by polynomials over k. The group $\mathbf{G}$ is called *connected* if it is connected when viewed as an algebraic variety. The group $\mathbf{G}$ is called *semisimple* if its maximal connected solvable normal subgroup is trivial.

We follow the assumptions in [BJ06, III.11] and [BJ07], which allow us to use the uniform method and certain results on EDM geodesics in [JM02]. Let $\mathbf{G}$ be a connected semisimple linear algebraic group defined over $\mathbb{Q}$. The real locus $G = \mathbf{G}(\mathbb{R})$ is a semisimple Lie group. For a maximal compact subgroup $K \subset G$, the quotient $X = G/K$ is a symmetric space of noncompact type. Let $\Gamma \subset \mathbf{G}(\mathbb{Q})$ be an arithmetic subgroup. Then the quotient $\Gamma\backslash X$ is a locally symmetric space.

Compactifications of $\Gamma\backslash X$ have been a popular area of study for the past half century. Historically, the earliest such compactifications appeared in the work of Fricke–Klein and Poincaré. These were the compactifications of quotients of $\Gamma\backslash\mathfrak{H}$ by adjoining cusps. Two nice introductions to the contemporary theory of compactifications are [Gor05, Gun06].

Satake initiates the modern theory of compactifications of $\Gamma\backslash X$ in [Sat60a]. In [BB64, BB66], Baily and Borel show that when $\Gamma\backslash X$ is a Hermitian locally symmetric space, one of the minimal Satake compactifications is an algebraic variety. Borel and Serre [BS73] define a compactification of $\Gamma\backslash X$ that is a manifold with corners. The boundary components in the Borel–Serre compactification are fiber bundles composed of locally symmetric space bases with compact fibers. In [Zuc82], Zucker defines the reductive Borel–Serre compactification. This is obtained by collapsing the fibers in the Borel–Serre boundary. The maximal Satake, Borel–Serre, and reductive Borel–Serre compactifications have all been constructed [BJ06] using the uniform method.

For $\mathbf{G}$, as above, a closed subgroup $\mathbf{P} \subset \mathbf{G}$ is called a *parabolic subgroup* if $\mathbf{G}/\mathbf{P}$ is a projective variety. If a parabolic subgroup $\mathbf{P}$ is defined over $\mathbb{Q}$, then it is called a *rational parabolic subgroup*. The real locus $P = \mathbf{P}(\mathbb{R})$ is a parabolic subgroup of $G = \mathbf{G}(\mathbb{R})$. For a rational parabolic subgroup $\mathbf{P}$, let $\mathbf{N_P}$ be the *unipotent radical* of $\mathbf{P}$ with $N_P = \mathbf{N_P}(\mathbb{R})$, let $\mathbf{L_P} = \mathbf{N_P}\backslash\mathbf{P}$ be the *Levi quotient* of $\mathbf{P}$ with $L_P = \mathbf{L_P}(\mathbb{R})$, let $\mathbf{S_P}$ be the $\mathbb{Q}$-split center of $\mathbf{L_P}$, and let $A_\mathbf{P}$ be the connected component of the identity in $\mathbf{S_P}(\mathbb{R})$. Finally let $\mathbf{M_P}$ be the intersection of the kernels of the squared characters of $\mathbf{L_P}$,

$$\mathbf{M_P} = \bigcap_{\chi \in X(\mathbf{L_P})} \ker \chi^2,$$

where $M_\mathbf{P} = \mathbf{M_P}(\mathbb{R})$. The real locus of the Levi quotient decomposes as the product

$$L_P \simeq A_\mathbf{P} \times M_\mathbf{P}.$$

To define the rational horospherical decomposition of $P = \mathbf{P}(\mathbb{R})$, we need to lift L_P and its subgroups $A_\mathbf{P}$ and $M_\mathbf{P}$ into P. Let x_0 denote the basepoint in the symmetric space X that corresponds with the maximal compact subgroup K in the description $X = G/K$. The Cartan involution θ of G corresponding to K extends to an involution θ of $\mathbf{G}$. There is a unique Levi subgroup $\mathbf{L}_{\mathbf{P},x_0} \subset \mathbf{G}$ stable under θ. The projection

$$\pi_P : \mathbf{L}_{\mathbf{P},x_0} \to \mathbf{N_P}\backslash\mathbf{P}$$

gives an isomorphism of $\mathbf{L}_{\mathbf{P},x_0}$ onto $\mathbf{L_P}$. Let $i_{x_0} = (\pi_P|_{L_{P,x_0}})^{-1}$ where $L_{P,x_0} = \mathbf{L}_{\mathbf{P},x_0}(\mathbb{R})$. The map i_{x_0} gives an isomorphism of L_P onto L_{P,x_0}. Let $A_{\mathbf{P},x_0}$ and $M_{\mathbf{P},x_0}$ denote the images of $A_\mathbf{P}$ and $M_\mathbf{P}$ under i_{x_0}.

The lift $i_{x_0}(L_P)$ gives rise to the rational Langlands decompostion of P (cf. [BJ06, III.1.3])

$$P = N_P \times A_{\mathbf{P},x_0} \times M_{\mathbf{P},x_0}.$$

In turn, the rational Langlands decompostion of P gives rise to the rational horospherical decompostion of X

$$X = N_P \times A_{\mathbf{P},x_0} \times X_{\mathbf{P},x_0} \tag{2.3.1}$$

where $X_{\mathbf{P},x_0} = M_{\mathbf{P},x_0}/K \cap M_{\mathbf{P},x_0}$.

Definition 2.24. The rational horospherical decomposition of X corresponding to a rational parabolic subgroup $\mathbf{P} \subset \mathbf{G}$ and basepoint x_0 is given by (2.3.1).

Although the algebraic groups $\mathbf{L_P}, \mathbf{M_P}, \mathbf{S_P}$ are all defined over $\mathbb{Q}$, the lifts $\mathbf{L}_{\mathbf{P},x_0}, \mathbf{M}_{\mathbf{P},x_0}, \mathbf{S}_{\mathbf{P},x_0}$ need not be. However, we have the following proposition:

Proposition 2.25. [BJ06, III.1.11] For any rational parabolic subgroup $\mathbf{P}$, there exists a basepoint $x_1 \in X$ and a lift map i_{x_1} such that $\mathbf{L}_{\mathbf{P},x_1}, \mathbf{M}_{\mathbf{P},x_1}, \mathbf{S}_{\mathbf{P},x_1}$ are algebraic groups defined over $\mathbb{Q}$ and i_{x_1} is a morphism defined over $\mathbb{Q}$.

The proof of this proposition relies on the fact that one can obtain the basepoint x_1 by considering N_P-translates nx_0 of any basepoint x_0. From now on, we supress the baspoint x_0 involved in the definition of the horospherical decomposition unless it is necessary.

We have the following relative horospherical decomposition of the rational boundary symmetric space $X_{\mathbf{Q}}$:

Lemma 2.26. [BJ06, III.1.16] For any pair of rational parabolic subgroups $\mathbf{P} \subset \mathbf{Q}$, there exists a unique rational parabolic subgroup $\mathbf{P}'$ of $\mathbf{M_Q}$ such that

$$X_{\mathbf{Q}} = N_{P'} \times A_{P'} \times X_{\mathbf{P}}.$$

This decomposition plays a vital role in the description of the boundary components. The rational boundary symmetric space $X_{\mathbf{P}}$ splits as a product of non-compact and Euclidean factors

$$X_{\mathbf{P}} = X_P \times \exp \mathfrak{a}_{\mathbf{P}}^{\perp}. \tag{2.3.2}$$

This splitting gives a *refined rational horospherical decomposition*

$$X = N_P \times X_P \times \exp \mathfrak{a}_{\mathbf{P}}^{\perp} \times A_{\mathbf{P}}. \tag{2.3.3}$$

2.3.1 Uniform method for Borel–Serre and reductive Borel–Serre

In [BJ06, III.9–10], Borel and Ji use the uniform method to construct the Borel–Serre and reductive Borel–Serre compactifications. We briefly describe these compactifications below. However, we do not describe the topology of these spaces.

The boundary components are easily described for each compactification in terms of the rational horospherical decomposition. For each rational parabolic subgroup $\mathbf{P}$, the boundary component $e(\mathbf{P})$ of the Borel–Serre compactification is

$$e(\mathbf{P}) = N_P \times X_{\mathbf{P}}. \tag{2.3.4}$$

Similarly, the boundary of the reductive Borel–Serre is

$$e(\mathbf{P}) = X_{\mathbf{P}}. \tag{2.3.5}$$

The Borel–Serre partial compactification ${}_{\mathbb{Q}}\overline{X}^{BS}$ is the set

$${}_{\mathbb{Q}}\overline{X}^{BS} = X \cup \coprod_{\mathbf{P}} N_P \times X_{\mathbf{P}} \tag{2.3.6}$$

with a topology described in terms of convergent sequences. For an arithmetic subgroup Γ the quotient $\Gamma\backslash {}_{\mathbb{Q}}\overline{X}^{BS}$ is isomorphic to the Borel–Serre compactification of $\Gamma\backslash X$. Similarly, the reductive Borel–Serre partial compactification ${}_{\mathbb{Q}}\overline{X}^{RBS}$ is the set

$$ {}_{\mathbb{Q}}\overline{X}^{RBS} = X \cup \coprod_{\mathbf{P}} X_{\mathbf{P}}, \tag{2.3.7} $$

again with a convergent sequence topology. The quotient $\Gamma\backslash {}_{\mathbb{Q}}\overline{X}^{RBS}$ is isomorphic to the reductive Borel–Serre compactification of $\Gamma\backslash X$.

2.3.2 Geometric rationality of Satake compactifications

We recall the original construction (Definition 2.6) of the Satake compactifications. In the following subsection we define what it means for a compactification $\overline{X}^S_\tau$ to be geometrically rational with respect to a rational structure determined by the action of an arithmetic group Γ.

Not every representation τ of G gives rise to a compactification of $\Gamma\backslash X$. We need the compactification $\overline{X}^S_\tau$ to be *geometrically rational* (Definition 2.31). That is, we need the boundary components X_I that meet certain distinguished open sets in X to have a particular structure with respect to the rational structure of X. The framework of geometric rationality was first described in [Sat60a] for certain examples. Baily–Borel [BB66, §3.5] defined a more general notion in the context of a quotient by a discrete subgroup Γ (not necessarily arithmetic). We introduce two objects to describe the definition of Baily–Borel. For a boundary component X_I, let its *normalizer* $\mathcal{N}(X_I)$ be the subgroup

$$ \mathcal{N}(X_I) = \{g \in G \mid g \cdot X_I = X_I\}. \tag{2.3.8} $$

Let its *centralizer* $\mathcal{Z}(X_I)$ be the following subgroup of $\mathcal{N}(X_I)$:

$$ \mathcal{Z}(X_I) = \{g \in G \mid g \cdot z = z \text{ for all } z \in X_I\}. \tag{2.3.9} $$

We consider the quotient $\mathcal{N}(X_I)/\mathcal{Z}(X_I)$ and let N_Q denote the unipotent radical of $\mathcal{N}(X_I)$.

Definition 2.27. The boundary component X_I is called Γ-rational if

1. the quotient $\Gamma_{N_Q}\backslash N_Q$ is compact and
2. the projection of $\Gamma \cap \mathcal{N}(X_I)$ is discrete in $\mathcal{N}(X_I)/\mathcal{Z}(X_I)$.

We are concerned only with the case that Γ is arithmetic. In the context of Hermitian symmetric spaces, Γ-rationality is equivalent to the linear algebraic group $\mathcal{N}(X_I) \otimes \mathbb{C}$ being defined over $\mathbb{Q}$. For non-Hermitian examples, the rationality of $\mathcal{N}(X_I) \otimes \mathbb{C}$ only implies part (1) of the Γ-rationality criterion. In [BB66, §3.6], equivalent formulations of part (2) in the above definition are given when the projection of $\Gamma \cap \mathcal{N}(X_I)$ in $\mathcal{N}(X_I)/\mathcal{Z}(X_I)$ is of *arithmetic type* (cf. [BB66, §3.4]). One of these formulations is described in [BJ06].

Remark 2.28. We remark that topologizing the Satake compactification of $\Gamma\backslash X$ is subtle. Satake's original description can be found in [Sat60a]. Topologizing the compactification in terms of convergent sequences is an open problem that we plan to consider in future work. For now we use Satake's original definition and restrict our attention to geometrically rational compactifications.

The following three definitions help to define this notion of geometric rationality.

Definition 2.29. A boundary component X_I of $\overline{X}^S_\tau$ is called *Siegel rational* if ΓX_I meets the closure of some Siegel sets of rational parabolic subgroups of $\mathbf{G}$.

We refer the interested reader to [BJ06, III.1.17] for the definition of a Siegel set.

Definition 2.30. A boundary component X_I of $\overline{X}^S_\tau$ is called *weakly rational* if its normalizer $\mathcal{N}(X_I)$ is the real locus of a rational parabolic subgroup $\mathbf{P} \subset \mathbf{G}$.

We use weak rationality to define a rational boundary component.

Definition 2.31. A boundary component X_I of $\overline{X}^S_\tau$ is called *rational* if it is weakly rational and the centralizer $\mathcal{Z}(X_I)$ contains a cocompact subgroup Z that is a normal subgroup of Q and is the real locus of an algebraic group $\mathbf{Z}$ defined over $\mathbb{Q}$.

Using the above terminology, we define geometric rationality.

Definition 2.32. A Satake compactification $\overline{X}^S_\tau$ is called *geometrically rational* if every Siegel rational boundary component is rational.

Matters of geometric rationality are further discussed in [Cas97, Sap04]. Both papers require knowledge of the classification of algebraic semisimple groups appearing in [Tit66].

2.3.3 Uniform method for the maximal Satake compactification

For any rational parabolic subgroup $\mathbf{P}$, we define the associated maximal Satake boundary component $e(\mathbf{P}) = X_P$, where $P = \mathbf{P}(\mathbb{R})$. Define the maximal Satake partial compactification to be

$$_{\mathbb{Q}}\overline{X}^S_{max} = X \cup \coprod_{\mathbf{P}} e(\mathbf{P}) = X \cup \coprod_{\mathbf{P}} X_P.$$

Note that the product is taken over all $\mathbb{Q}$-parabolic subgroups; there is no saturation condition. We define the topology on $_{\mathbb{Q}}\overline{X}^S_{max}$ using convergent sequences. These sequences are described using two analogues of the rational horospherical decomposition of X. The first is the *refined rational horospherical decomposition*

$$X = N_P \times A_{\mathbf{P}} \times X_P \times \mathfrak{a}^\perp_{\mathbf{P}}. \tag{2.3.10}$$

Here $\mathfrak{a}^\perp_{\mathbf{P}}$ is the orthogonal complement of $\mathfrak{a}_{\mathbf{P}}$ inside $\mathfrak{a}_P$.

The second is a description of the boundary symmetric space X_Q. For any pair of rational parabolic subgroups $\mathbf{P}$, $\mathbf{Q}$, with $\mathbf{P} \subset \mathbf{Q}$, $\mathbf{P}$ determines a unique rational parabolic subgroup $\mathbf{P}'$ of $\mathbf{M_Q}$. Similarly, the real parabolic subgroup P determines a unique real parabolic subgroup P'' of M_Q. The real parabolic P'' is contained in $P' = \mathbf{P}'(\mathbb{R})$. As a result, we can write

$$X_Q = N_{P'} \times A_{\mathbf{P}'} \times X_P \times \mathfrak{a}_{P''}^{\mathbf{P}'}. \tag{2.3.11}$$

Here $\mathfrak{a}_{P''}^{\mathbf{P}'}$ is the orthogonal complement of $\mathfrak{a}_{\mathbf{P}'}$ in $\mathfrak{a}_{P''}$. The description of the topology is as follows:

1. For any rational parabolic subgroup $\mathbf{P}$, an unbounded sequence y_j in X converges to $z_\infty \in e(\mathbf{P}) = X_P$ if and only if in the decomposition $y_j = (n_j, a_j, z_j, a_j^\perp) \in N_P \times A_\mathbf{P} \times X_P \times \mathfrak{a}_\mathbf{P}^\perp$, the coordinates satisfy the following conditions:

   (a) $z_j \to z_\infty$ in X_P,

   (b) for all $\alpha \in \Phi(P, A_\mathbf{P})$, $a_j^\alpha \to \infty$.

2. For any pair of rational parabolic subgroups $\mathbf{P}$, $\mathbf{Q}$, with $\mathbf{P} \subset \mathbf{Q}$, a sequence of points y_j in $e(\mathbf{Q}) = X_Q$ converges to a point $z_\infty \in e(\mathbf{P}) = X_P$ if and only if in the decomposition $y_j = (n_j, a_j', z_j, a_j'') \in X_Q = N_{P'} \times A_{\mathbf{P}'} \times X_P \times \mathfrak{a}_{P''}^{\mathbf{P}'}$, we have:

   (a) $z_j \to z_\infty$ in X_P,

   (b) for all $\alpha \in \Phi(P', A_{\mathbf{P}'})$, $(a_j')^\alpha \to \infty$.

These are two typical convergent sequences. Combinations of them yield general convergent sequences.

Example 8. Consider the five dimensional symmetric space $X = SL_3(\mathbb{R})/SO(3)$. Table 1 gives the dimension of the boundary components in each of the three compactifications described above. For a nonmaximal geometrically rational compactification of X (e.g. that defined by the standard representation), one of the boundary modular curves collapses to a point, and we disregard the component corresponding to the minimal rational parabolic $\mathbf{P}_1$.

Compactification	$\dim e(\mathbf{P_2})$	$\dim e(\mathbf{P_1})$	$\dim e(\mathbf{P_3})$
Borel-Serre	4	3	4
Reduced Borel-Serre	2	0	2
Maximal Satake	2	0	2

Table 1: Dimensions of Boundary Components when $X = SL_3(\mathbb{R})/SO(3)$

2.4 EDM Geodesics in Locally Symmetric Spaces

Geodesics in $\Gamma\backslash X$ behave differently than in their global symmetric space counterparts. For example, they may be closed, self intersect, or reenter a compact region in

finitely many times. This is not surprising since $\Gamma\backslash X$ is typically not simply connected.

In [Sie64], Siegel first notes the importance of understanding which geodesics in $\Gamma\backslash X$ "go to infinity". Detailed characterizations of these geodesics in certain special cases appear in [Hat92, Leu96]. The general characterization of such geodesics in terms of a distance minimizing property appears in [JM02]. We quote a number of results from this paper.

Consider the projection map

$$\pi : X \longrightarrow \Gamma\backslash X \tag{2.4.1}$$

that sends a symmetric space into an arithmetic quotient. Distance in $\Gamma\backslash X$ is determined by this map and distance in X as follows:

$$d_{\Gamma\backslash X}(\pi(p), \pi(q)) = \inf_{g\in\Gamma} d_X(p, gq). \tag{2.4.2}$$

Geodesics in $\Gamma\backslash X$ arise as projections of geodesics in X. Using this notion of distance we have analogues of DM geodesics encountered in the last subsection.

Definition 2.33. A geodesic $\gamma(t) \in \Gamma\backslash X$ is called *eventually distance minimizing* (EDM) if there exists a number $t_0 \gg 0$ such that for any $t_1, t_2 \geq t_0$ we have $d(\gamma(t_1), \gamma(t_2)) = |t_2 - t_1|$.

We use the following terminology for geodesics in X that project to EDM geodesics.

Definition 2.34. Let $\tilde{\gamma}$ be a geodesic in X whose projection $\pi(\tilde{\gamma})$ is an EDM geodesic. We call $\tilde{\gamma}$ an *EDM lift* or simply a *lift*.

The following two theorems completely characterize EDM geodesics in terms of the rational horosperical deomposition.

Theorem 2.35. [JM02, 10.18] Any EDM geodesic γ in $\Gamma\backslash X$ has a lift of the form $\tilde{\gamma}(t) = (u, z, a\exp(tH)) \in N_P \times X_{\mathbf{P}} \times A_{\mathbf{P}}$.

Remark 2.36. In the literature, elements of the nilpotent radical N_P are denoted by either u or n. We will use both notations.

Theorem 2.37. [JM02, 10.20] Two EDM lifts $\tilde{\gamma}_1, \tilde{\gamma}_2$ with horospherical decompositions $\tilde{\gamma}_i(t) = (n_i, a_i \exp(tH_i), z_i)$ project to the same EDM geodesic in $\Gamma\backslash X$ up to reparametrization if and only if $H_1 = H_2$, $\log a_1 - \log a_2 \in \langle H_1 \rangle$, and $(n_1, z_1) = g(n_2, z_2)$ for some $g \in \Gamma_P$.

Motivated by results in [Hat92] that describe the boundaries of the Borel–Serre and geodesic compactifications when $\mathbf{G} = \mathbf{SL_3}$, we have the following descriptions of the Borel–Serre and reductive Borel–Serre compactifications for an arbitrary $\Gamma\backslash X$. Let $\mathrm{rk}_{\mathbb{Q}}(\mathbf{G})$ denote the dimension of the largest $\mathbb{Q}$-split torus in $\mathbf{G}$. Since $\Gamma\backslash X$ is noncompact, necessarily we have $\mathrm{rk}_{\mathbb{Q}}(\mathbf{G}) > 0$.

Proposition 2.38. [JM02, 14.2] If $\mathrm{rk}_{\mathbb{Q}}(\mathbf{G}) = 1$, then the set of EDM geodesics in $\Gamma\backslash X$ corresponds bijectively to $\partial(\overline{\Gamma\backslash X}^{BS})$ through the map $\gamma \mapsto \lim_{t\to\infty} \gamma(t)$.

We can define an N-relation among EDM geodesics in the same way we did for geodesics in X.

Proposition 2.39. [JM02, 14.5] If $\mathrm{rk}_{\mathbb{Q}}(\mathbf{G}) = 1$, then the set of N-equivalence classes corresponds bijectively to $\partial(\overline{\Gamma\backslash X}^{RBS})$ through the map $\gamma \mapsto \lim_{t\to\infty} \gamma(t)$.

Example 9. Let $\mathbf{G} = \mathbf{SL_2}$. EDM geodesics that approach $\{i\infty\}$ in an arithmetic quotient of $\mathfrak{H}$ are projections of vertical geodesics. The limit points of vertical geodesics in X are in bijection with the horizontal horocycle. We think of the limit points of EDM geodesics in the Borel–Serre compactification as an S^1 boundary component that is a quotient of the nilpotent radical N_P. The N-relation identifies all EDM geodesics with limit point $\{i\infty\}$. This single equivalence class corresponds to a point boundary component of the reductive Borel–Serre compactification.

The next example uses Weil's restriction of scalars functor. This is a fundamental tool when constructing examples of symmetric spaces whose $\mathrm{rk}_{\mathbb{Q}}(\mathbf{G})$, $\mathrm{rk}_{\mathbb{R}}(\mathbf{G})$, and absolute rank $\mathrm{rk}_{\mathbb{C}}(\mathbf{G})$ differ. Here $\mathrm{rk}_{\mathbb{R}}(\mathbf{G})$ and $\mathrm{rk}_{\mathbb{C}}(\mathbf{G})$ are the dimensions of the maximal $\mathbb{R}$-split and $\mathbb{C}$-split tori, respectively. A nice exposition of this construction is available in [PR94].

Example 10. Let $\mathbf{G} = Res_{k/\mathbb{Q}}(\mathbf{SL_2})$, where $k/\mathbb{Q}$ is a real quadratic extension. We have $\mathbf{G}(\mathbb{Q}) = SL_2(k)$. The corresponding symmetric space $X = \mathfrak{H} \times \mathfrak{H}$ has $\mathrm{rk}_{\mathbb{Q}}(\mathbf{G}) = 1$ and $\mathrm{rk}_{\mathbb{R}}(\mathbf{G}) = 2$. The Borel–Serre boundary components are 3-manifolds that fiber over the circle, and the reductive Borel–Serre boundaries are the S^1 bases of these bundles. The nontriviality of the reductive Borel–Serre boundary shows that not all geodesics heading to infinity in the same direction are N-related. Indeed, unlike the modular curve example, there are geodesics in X that stay at a finite distance and project to EDM geodesics in the Hilbert modular surface $\Gamma\backslash X$.

The congruence bundle $C(\gamma)$ of an EDM geodesic γ was defined in [JM02, 14.6]. Like the EDM geodesic definition (Definition 2.33) it has a fixed distance condition:

$$C(\gamma) = \{\ \gamma' \mid d(\nu(t), \nu'(t)) = c \text{ a constant, for } t \gg 0\}. \tag{2.4.3}$$

The congruence bundle can be identified with the product

$$C(\gamma) \cong \Gamma_{M_{\mathbf{P}}}\backslash X_{\mathbf{P}} \times \mathrm{Span}(H)^{\perp} \tag{2.4.4}$$

of a boundary locally symmetric space and Euclidean factor. Ji and MacPherson introduce the following definition as a means to describe the Borel–Serre compactifications in higher rank.

Definition 2.40. [JM02, 14.8] For any EDM geodesic γ, the *rank* of γ is defined as

$$r(\gamma) = \max\{\ k \in \mathbb{Z} \mid \text{there exists a faithful isometric action of } \mathbb{R}^{n-1} \text{ on } C(\gamma)\}.$$

This leads to the following equivalence relation.

Definition 2.41. [JM02, 14.8] An EDM geodesic γ' in $C(\gamma)$ is called *linearly related* (L-related) to γ if γ, γ' belong to one $\mathbb{R}^{n-1}$ orbit of $C(\gamma)$ of the isometric action of $\mathbb{R}^{n-1}$, where $n = r(\gamma)$.

In fact, the L-relation restricts to an equivalence relation on $F(\gamma)$.

Definition 2.42. [JM02, 14.10.(3)] The dimension of the quotient $F(\gamma)/L$ is called the *mobility degree* of the EDM geodesic γ.

The mobility degree is used to identify geodesics with velocity vectors originating from the Lie algebra of the same $\mathbb{Q}$-split component $A_{\mathbf{P}}$. The following equivalence relation identifies L-equivalence classes of some geodesics whose distance is unbounded in the limit.

Definition 2.43. Two L-equivalence classes $[\gamma_0]_L, [\gamma_1]_L$ are *rotationally related* (R-related) if there exist representatives γ_0, γ_1 and a family of EDM geodesics γ_s connecting them such that the mobility degree of γ_s does not change and $d(\gamma_{s_1}(t), \gamma_{s_2}(t)) = c|s_2 - s_1|t$ when $t \geq 0$, where c is some constant.

We can define a new relation on EDM geodesics using the last definition.

Definition 2.44. Two EDM geodesics γ_1, γ_2 are called *RL-related* if their L-equivalence classes $[\gamma_0]_L, [\gamma_1]_L$ are R-related.

Finally we can combine the N-relation with the RL-relation:

Definition 2.45. Two EDM geodesics γ_1, γ_2 are *NRL-related* if there exists an EDM geodesic γ' such that γ' is RL-related to γ_1, and γ' is N-related to γ_2.

In Table 2, we give a summary of how the horospherical decomposition of two EDM geodesic lifts are related if their projections are in the same equivalence class.

This relation identifies EDM geodesics with the same rational boundary symmetric space contribution. Similarly, we have the following:

Proposition 2.46. The set of RL-equivalence classes corresponds bijectively to $\partial(\overline{\Gamma\backslash X}^{BS})$ through the map $\gamma \mapsto \lim_{t\to\infty} \gamma(t)$.

Proposition 2.47. The set of NRL-equivalence classes corresponds bijectively to $\partial(\overline{\Gamma\backslash X}^{RBS})$ through the map $\gamma \mapsto \lim_{t\to\infty} \gamma(t)$.

Relation	Horospherical decomposition
N	$z_1 = z_2, a_1 = a_2, H_1 = H_2$
L	$H_1 = H_2$
RL	$n_1 = n_2, z_1 = z_2$
NRL	$z_1 = z_2$

Table 2: Relations among EDM geodesics

The basic plan of attack employed by Ji and MacPherson to describe the Borel–Serre and reductive Borel–Serre compactifications is as follows.

1. Use the uniform method to describe the compactification $\overline{\Gamma\backslash X}$ as a set.

2. Describe the topology of $\overline{\Gamma\backslash X}$ in terms of convergent subsequences.

3. Determine the set of geodesics that go to infinity.

4. Construct equivalence relations on this set of geodesics that are in one to one correspondence with points in the boundary.

The equivalence relations identify EDM geodesics whose lifts have identical factors of their horospherical decomposition. Figure 2 gives a list of such equalities. We employ the same basic strategy for the Satake compactifications. For the non-maximal Satake compactifications of the locally symmetric space we will not use a topology defined using convergent sequences, rather we use the original Satake topology. In the case of toroidal compactifications of Hilbert modular varieties, we focus on the latter steps in the above plan.

CHAPTER 3

A GEOMETRIC CONSTRUCTION OF THE SATAKE BOUNDARY OF GLOBAL SYMMETRIC SPACES

This chapter contains our first results, the constuctions of the maximal Satake compactification (Theorem 3.17) and nonmaximal Satake compactifications (Theorem 3.44) of the global space X using geodesics. Borel and Ji define a topology on compactifications by specifying how unbounded sequences in X converge to points in the boundary and how unbounded sequences in one boundary component converge to points in another boundary component. We use this construction to verify certain equivalence classes of geodesics in X are in one to one correspondence with points in the Satake boundary.

For the maximal Satake compactification, we use the N-relation (Definition 2.18) and a global symmetric space analogue of the RL-relation (Definition 2.44) called the RC-relation (Definition 3.11) to describe the boundary. For the nonmaximal Satake compactifications, we introduce the μ-relation (Definition 3.38), which in association with the N and RC-relations describes the nonmaximal boundary.

3.1 The Maximal Satake Compactification

As noted in Chapter 1, there have been many compactifications defined that are equivalent to the Satake compactifications or the maximal Satake in particular. In [Moo64], Moore showed the first such equivalence by proving the Satake compactifications are isomorphic to the Furstenberg compactifications [Fur63]. In [Mos73, §4], Mostow gave a geometric construction of the maximal Furstenburg boundary. This is the collection of lowest dimensional components of the Satake boundary. This was, in essence, the first geometric description of a part of the Satake boundary. Recently, many compactifications have been constructed that are isomorphic to the maximal Satake. These include the subgroup, subalgebra, and dual-cell compactifications of X. See [BJ06, I.17,19] for more details.

In this section, we use equivalence classes of geodesics to construct the maximal Satake compactification. As discussed in [JM02, Remark 14.22], if defined similarly in the global space, the L-relation will identify all geodesics in $C(\nu)$. A finer relation is needed to construct the maximal Satake boundary. We address this relation in the next subsection.

3.1.1 The Clifford Relation

For any metric space (M, d), let $I(M)$ be the set of isometries of M.

Definition 3.1. [Hel78, p.278] For $\phi \in I(M)$, the *displacement function* d_ϕ is defined by

$$d_\phi(x) = d(x, \phi(x)),\ x \in M.$$

Definition 3.2. [Hel78, p.278] An isometry ϕ is called a *Clifford translation* if its displacement function d_ϕ is constant.

The following theorem of Wolf characterizes the Clifford translations on simply connected manifolds of nonpositive curvature. This is a more general class of spaces that include symmetric spaces of noncompact type. It uses a generalization of the standard de Rham decomposition of a global symmetric space (Proposition 2.2).

Theorem 3.3. [Wol64] Let M be a simply connected manifold of nonpositive curvature, and let $M = M_0 \times M_1$ be the Riemannian product decomposition of M, where M_0 is the Euclidean factor and M_1 is the product of the compact and noncompact factors. Let $\phi \in I(M) = I(M_0) \times I(M_1)$.

The following are equivalent:

1. ϕ is a Clifford translation of M.

2. $\phi = T \times \{1\}$ in $I(M) = I(M_0) \times I(M_1)$ where T is an ordinary translation of the Euclidean factor M_0.

The following is the main definition of this subsection. Recall that $C(\nu)$ is the congruence bundle (2.2.4) associated to the geodesic ν. As mentioned in [Kar67],$C(\nu)$ is a metric space with distance function D (2.2.3).

Definition 3.4. Two geodesics ν_1, ν_2 in the same congruence bundle $C(\nu)$ are called *Clifford related* (C-related) if there exists a Clifford translation ϕ of $C(\nu)$ such that $\phi(\nu_1) = \nu_2$.

This relation is the symmetric space analogue to the L-relation (Definition 2.41). We characterize the C-relation in terms of the horospherical decomposition as follows.

Lemma 3.5. Two geodesics $\nu_1, \nu_2 \in C(\nu)$ are C-related if and only if they have the form $\nu_i(t) = (n_i, a_i \exp(tH_i), z_i) \in N_P \times A_P \times X_P$ with $n_1 = n_2$, $z_1 = z_2$ and $H_1 = H_2$.

Proof. Since ν_1, ν_2 are in $C(\nu)$, Lemma 2.22 implies $n_1 = n_2$ and $H_1 = H_2$. If ν_1, ν_2 are C-related but $z_1 \neq z_2$, then there is a Clifford isometry that does not fix X_P. This contradicts Theorem 3.3, and thus $z_1 = z_2$.

To prove the other direction, assume $n_1 = n_2$, $z_1 = z_2$ and $H_1 = H_2$. Then by Lemma 2.22, the geodesics ν_1, ν_2 are in the same congruence bundle $C(\nu)$. Consider the isometry ϕ of $C(\nu)$ that takes $\nu'(t) = (n, a\exp(tH)z,) \in C(\nu)$ to $\nu''(t) = (n, \exp(H_a + (H_2 - H_1))\exp(tH), z)$ where $H_a = \log a$, $H_1 = \log a_1$, and $H_2 = \log a_2$. This is an isometry whose displacement function stays fixed and satisfies $\phi(\nu_1) = \nu_2$. Therefore ν_1, ν_2 are C-related. □

Lemma 3.6. The C-relation is an equivalence relation on geodesics in $C(\nu)$.

Proof. This follows easily from the above lemma. Indeed, from the description of the RC-relation in coordinates above, it is immediate that it is an equivalence relation. □

Remark 3.7. All of the relations we construct in chapters 3 and 4 will have a similar description in terms of the real or rational horospherical decomposition. Essentially, all of the work to prove the relation is an equivalence relation is carried out in these lemmata. Thus, from now on, we omit these proofs and place a QED symbol after the statements.

For any geodesic ν in X, let $[\nu]_C$ denote the C-related equivalence class containing ν. Recall that $F(\nu)$ is the finite bundle (2.2.1) associated to the geodesic ν.

Lemma 3.8. The C-relation partitions the finite bundle $F(\nu)$ into a union of C-equivalence classes.

Proof. Let $\nu(t) = (n, a\exp(tH), z) \in N_P \times A_P \times X_P$. Then $F(\nu)$ consists of all $\nu'(t) = (n', a'\exp(tH'), z')$ such that $H' = H$. Lemma 2.2 then implies that the restriction of the C-relation to $F(\nu)$ partitions it into equivalence classes. □

Now we will combine the C-relation with relations analogous to the N and R-relations of geodesics in the local space [JM02]. As in Definition 2.42, we have a similar notion of mobility degree in the global symmetric space.

Definition 3.9. The dimension of the quotient $F(\nu)/C$ is called the *mobility degree* of ν.

As in the proof of [JM02, 14.15], which asserts that the mobility degree of an EDM geodesic is equal to $\dim N_P + \dim X_{\mathbf{P}}$, we have that the mobility degree of the geodesic ν is equal to $\dim N_P + \dim X_P$. Here, P is the unique real parabolic subgroup for which ν takes the form $\nu(t) = (n, a\exp(tH), z) \in N_P \times A_P \times X_P$.

Definition 3.10. Two C-equivalence classes $[\nu_0]_C, [\nu_1]_C$ are *R-related* if there exist representatives ν_0, ν_1 and a family of geodesics $\nu_s(t)$ connecting them such that the mobility degree of $\nu_s(t)$ does not change and $d(\nu_{s_1}(t), \nu_{s_2}(t)) = c|s_1 - s_2|t$ when $t \geq 0$, where c is some constant.

Definition 3.11. Two geodesics ν_0, ν_1 are *RC-related* if their C-equivalence classes $[\nu_0]_C, [\nu_1]_C$ are R-related.

We have a description of the RC-relation in terms of the horospherical decomposition.

Lemma 3.12. Two geodesics ν_0, ν_1 are RC-related if and only if they have the form $\nu_i(t) = (n_i, a_i\exp(tH_i), z_i) \in N_P \times A_P \times X_P$ with $n_0 = n_1$ and $z_0 = z_1$.

Proof. If ν_0, ν_1 are RC-related this implies that there exist representatives ν_0', ν_1' in the C-equivalence classes $[\nu_0]_C, [\nu_1]_C$ so that $d(\nu_0'(t), \nu_1'(t)) = ct$. In particular, $d(\nu_0'(0), \nu_1'(0)) = 0$, therefore ν_0', ν_1' have the same basepoint. Since the horospherical decomposition for a particular real parabolic subgroup is unique, this implies that the components $n_0' = n_1', z_0' = z_1'$ and $a_0' = a_1'$. That is, the geodesics differ only in their velocity vectors H_i'. Since ν_0', ν_1' are representative of C-equivalence classes, we have $n_0 = n_0', z_0 = z_0'$ and $n_1 = n_1', z_1 = z_1'$. Therefore, $n_0 = n_1$ and $z_0 = z_1$.

For the other direction, suppose both ν_0, ν_1 have the above form. By Lemma 3.5, we can find two representatives ν_0', ν_1' in $[\nu_0]_C, [\nu_1]_C$ with equal split components $a_0' = a_1'$. By the same lemma, the nilpotent and boundary symmetric space components of the representatives will be identical. We chose the family of geodesics $\nu_s'(t) = (n_0, a_0' \exp(tH_s), z_0) \in N_P \times A_P \times X_P$, where $H_s = sH_1 + (1-s)H_0$. This family satisfies the required distance condition, and therefore ν_0, ν_1 are RC-related.

□

Lemma 3.13. The RC-relation is an equivalence relation on geodesics in X.

□

Definition 3.14. Two geodesics ν_1, ν_2 are *NRC-related* if there exists a geodesic ν' such that ν' is RC-related to ν_1 and ν' is N-related to ν_2.

We have a description of the NRC-relation in terms of the horospherical decomposition.

Lemma 3.15. Two geodesics ν_1, ν_2 are NRC-related if and only if they have the form $\nu_i(t) = (n_i, a_i \exp(tH_i), z_i) \in N_P \times A_P \times X_P$ with $z_1 = z_2$.

Proof. If ν_1, ν_2 are NRC-related, then by Lemma 2.19, there is a geodesic ν' such that $z' = z_2$, $a' = a_2$, and $H' = H_2$. By Lemma 3.12, we have $n' = n_1$ and $z' = z_1$. Thus we have $z_1 = z_2$. If ν_1, ν_2 have the form $\nu_i(t) = (n_i, a_i \exp(tH_i), z_i) \in N_P \times A_P \times X_P$ with $z_1 = z_2$, then consider $\nu'(t) = (n_1, a_2 \exp(tH_2), z_1)$. This geodesic ν' is N-related to ν_1 and RC-related to ν_2, and then by Lemmata 2.19 and 3.12, the geodesics ν_1, ν_2 are NRC-related. □

Relation	Horospherical decomposition
N	$z_1 = z_2, a_1 = a_2, H_1 = H_2$
C	$H_1 = H_2$
RC	$n_1 = n_2, z_1 = z_2$
NRC	$z_1 = z_2$

Table 3: Relations among geodesics in X

Lemma 3.16. The NRC-relation is an equivalence relation on geodesics in X.

□

Theorem 3.17. The set of NRC-equivalence classes of geodesics corresponds bijectively to $\partial(\overline{X}^S_{max})$ through the map $\nu \mapsto \lim_{t\to\infty} \nu(t)$.

Proof. By Lemma 2.9 every geodesic converges to a boundary point in $\overline{X}^S_{max}$. The theorem then follows from the conclusion in Lemma 3.15. □

Remark 3.18. Wolf's theorem suggests that we may be able to define an analogue of the maximal Satake compactification for manifolds of nonpositive curvature. We plan to investigate this in future work.

Remark 3.19. In [BJ06, I.15], Borel and Ji define the real Borel–Serre partial compactification ${}_{\mathbb{R}}\overline{X}^{BS}$. Following the Borel–Ji attachment method, for a real parabolic subgroup P the corresponding boundary component $e(P) = N_P \times X_P$. As noted in [BJ06, I.15.7], the N_P factor causes the noncompactness. It seems plausible that $\partial({}_{\mathbb{R}}\overline{X}^{BS})$ can be constructed with RC-equivalence classes of geodesics, although we have not checked the details.

In the same section, the authors also define the real reductive Borel–Serre partial compactification ${}_{\mathbb{R}}\overline{X}^{RBS}$. Similarly, it seems plausible that we can construct ${}_{\mathbb{R}}\overline{X}^{RBS}$ using NRC-equivalence classes of geodesics. We plan to pursue this in future work.

3.2 The Non-Maximal Satake Compactification

In this section, we devise an equivalence relation that describes the non-maximal Satake compactifications. There is a great deal more terminology and notation involved in the description of these compactifications by means of the uniform method (as defined in §2.1.4).

We use the relative horospherical decomposition of a boundary symmetric space (Definition 2.8) extensively. In particular, each factor in the splitting $X_Q = X_I \times X_{I'}$ has its own horospherical decomposition with respect to subgroups of factors $M_I, M_{I'}$ respectively.

3.2.1 Weyl chambers

The equivalence relation in the next subsection is defined using equivalence relations on geodesics in what we call *closed Weyl chambers*. Since the term Weyl chamber takes different meanings in the literature, we define our use below. First we recall the definitions of Weyl chambers (in the standard sense) and maximal flats.

Let x_0 be the basepoint corresponding to the maximal compact subgroup $K \subset G$. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K, respectively. The Lie algebra $\mathfrak{g}$ has a unique Cartan involution θ that fixes $\mathfrak{k}$. The -1 eigenspace of θ is denoted $\mathfrak{p}$. The maximal abelian subalgebras $\mathfrak{a}$ of $\mathfrak{p}$ are conjugate under K. A nonzero linear form λ on a subalgebra $\mathfrak{a}$ is called a *root* if the space

$$\mathfrak{g}_\lambda = \{V \in \mathfrak{g} \mid [H, V] = \lambda(H)V, \textit{ for all } H \in \mathfrak{a}\}$$

is not trivial.

Each root α determines a root hyperplane $H_\alpha \subset \mathfrak{a}$ on which α evaluates trivially. The complements of root hyperplanes in $\mathfrak{a}$ are called the *(open) Weyl chambers.* The choice of a particular Weyl chamber determines an ordering on the set of roots. Given a particular Weyl chamber, the roots that determine the boundary hyperplanes of the chamber and satisfy the property that given any two such roots α_1, α_2 if the sum $\alpha_1 + \alpha_2$ is also a root, then the sum also determines a boundary hyperplane of the chamber are called *positive roots.* The set of positive roots that cannot be written as the sum of two distinct positive roots are called *simple roots.* The underlying Weyl chamber is called the *positive Weyl chamber* with respect to this set of simple roots.

For any subalgebra $\mathfrak{a}$, the subset $(\exp \mathfrak{a})x_0 = Ax_0$ is a *maximal flat totally geodesic subspace* of X. All tori A corresponding to maximal flats passing through x_0 are conjugate under K.

Let $P_0 \subset G$ denote the standard minimal parabolic subgroup. There is a standard maximal torus $\exp \mathfrak{a} = A$ that intersects P_0 nontrivially. There is a correspondence between standard parabolic subgroups and subsets of positive roots as described in [BJ06, I.1.3]. We denote the set of simple roots corresponding to P_0 by $\Delta(P_0, A_{P_0})$. Indeed, minimal parabolics correspond to open Weyl chambers.

Let $\mathfrak{a}_{P_0}^+ \subset \mathfrak{a}$ denote the open Weyl chamber corresponding to P_0 where $\mathfrak{a}$ is the standard maximal abelian subalgebra. For any subset $I \subset \Delta(P_0, A_{P_0})$, there is a unique parabolic subgroup P_I containing P_0 such that

$$A_{P_I} = \{a \in A_{P_0} \mid a^\alpha = 1,\ \alpha \in I\}$$

is the split component of P_I with respect to the basepoint x_0. We let $\Delta(P_I, A_{P_I})$ denote the set $\Delta(P_0, A_{P_0}) - I$. In what follows, the I may not be referenced and the set of roots will appear as $\Delta(P, A_P)$.

Definition 3.20. We say that a geodesic ν is in the *standard closed Weyl chamber* if it takes the form $\nu(t) = a\exp(tH)x_0$ where $a \in (\exp\mathfrak{a})x_0$ and $H \in \overline{\mathfrak{a}_{P_0}^+}$.

Consider an arbitrary maximal flat $(\exp\mathfrak{a}')x_0$ passing through x_0 and a parabolic subgroup P whose split component A_P has nontrivial intersection with $\exp\mathfrak{a}'$. More generally, we have the following definition:

Definition 3.21. We say that a geodesic ν is in a *closed Weyl chamber* if it takes the form $\nu(t) = a\exp(tH)x_0$ where $a \in (\exp\mathfrak{a}')x_0$ and $H \in \overline{\mathfrak{a}_{P'}^+}$ where P' is some minimal parabolic subgroup such that $A_{P'} \subset \exp\mathfrak{a}'$.

The following lemma is a consequence of work in [Cas97, Ji97] that investigates the closure of maximal flat geodesic submanifolds in the Satake compactifications. We give a proof of this using the relative horospherical decomposition. Integral to this proof is the polar decomposition (Definition 2.3).

Lemma 3.22. Any point in the maximal Satake boundary can be realized as the limit point of a geodesic in some maximal flat through x_0.

Proof. Since any parabolic subgroup is conjugate under K to a standard parabolic, it is enough to prove that all points in the boundary components corresponding to the standard parabolic subgroups can be realized as limit points of geodesics. Let P be such a standard parabolic and let $(\exp \mathfrak{a})x_0$ be the standard maximal flat through x_0 in X. It is clear that geodesics in $(\exp \mathfrak{a})x_0$ will map to all points in the intersection of the closure of $(\exp \mathfrak{a})x_0$ and the boundary component X_P. We must find geodesics in flats through x_0 that map to the other points of X_P. Thus given an element $z_\infty \in X_P$, we must find a geodesic in a maximal flat through x_0 whose limit point is z_∞.

To do this, we use the polar decomposition $M_P = K_P \overline{A}^+ K_P$ and the action of K on the horospherical decomposition $N_P \times A_P \times X_P$:

$$k \cdot (n, a\exp(tH), mx_0) = ({}^k n, {}^k a \exp(t\mathrm{Ad}(k)H), {}^k mx_0).$$

By the polar decomposition, there exist elements $k, k' \in K_P$ and $a' \in \exp \mathfrak{a} \cap M_P$ such that ${}^k a' k k' = z_\infty \in X_P$. Consider the geodesic $\tilde{\gamma}(t) = (n, a\exp(tH), (n', a', z'))$ where $(n', a', z') \in N_{P'} \times A_{P'} \times X_{P'}$ and $n = a = n' = z' = Id$, and pick an H such that $\lim_{t\to\infty} \tilde{\gamma}(t) = a' \in X_P$. Then, by the action of K described above, $k' \cdot \tilde{\gamma}(t) \to z_\infty$ as $t \to \infty$. Therefore any point in the maximal Satake boundary can be realized as a limit point of a geodesic in some maximal flat through x_0.

□

3.2.2 Plan for constructing equivalence relations

The nonmaximal Satake compactifications are unique from a geodesic perspective in that we must identify geodesics that cannot be written in the standard form in the horospherical decomposition with respect to the same parabolic subgroup. Moreover, there are an infinite number of geodesics with this property that must belong to the same equivalence class.

To tackle this problem, we first form equivalence relations only on geodesics that are contained in certain maximal flats. These maximal flats are those that all pass through a fixed basepoint x_0. This is a three step process. We

1. identify certain geodesics in the same closed Weyl chamber, then
2. identify certain geodesics in adjacent closed Weyl chambers, and finally
3. identify certain geodesics different closed Weyl chambers.

By Lemma 3.22, the geodesics in these maximal flats fill the maximal Satake boundary. Therefore, we are able to describe the nonmaximal boundary components with these geodesics and the relations formed by the above identifications. However, our previous description of the maximal Satake compactification does not use an equivalence relation restricted to only a certain type of geodesic. Therefore we will define an equivalence relation on all geodesics by saying that two NRC-equivalence classes are μ-related if there exist representatives that are identified in the sense of the closed Weyl chambers above. Lastly, we will say two geodesics are NRCμ-related if their NRC-equivalence classes are μ-related.

3.2.3 Geodesics in the same closed Weyl chamber

Let μ be a dominant weight as in §2.1.4. In this subsection, we define the μ-relation on geodesics in the same closed Weyl chamber. We first define some combinatorial data on geodesics in the standard closed Weyl chamber.

Definition 3.23. For a geodesic $\nu(t) = a \exp(tH)x_0$ in the standard closed Weyl chamber, let

1. $\Delta_H = \{\alpha \in \Delta(P_0, A_{P_0}) \mid \alpha(H) = 0\}$, and

2. I_H the maximal μ-connected subset of Δ_H.

Remark 3.24. We refer the reader to Definition 2.10 to remind themselves of the definition of a μ-connected subset.

The set Δ_H is exactly the subset of roots $\Delta(P, A_P) \subset \Delta(P_0, A_{P_O})$ where H is a vector in $\mathfrak{a}_P^+(\infty)$. We can think of roots as characters on A_P. For a root in $\Delta(P_0, A_{P_0})$ and an element $a \in A_P$, we recall $a^\alpha = \exp \alpha(\log a) \in \mathbb{R}_{>0}$. We now define the main tool used to identify geodesics under the μ-relation. Let $r = \mathrm{rk}_{\mathbb{R}}(\mathbf{G})$.

Definition 3.25. For a geodesic $\nu(t) = a \exp(tH)x_0$ in the standard closed Weyl chamber, let the *root profile* $(\nu)_\mu$ be the r-tuple of nonnegative real numbers whose i^{th}-entry is

$$(\nu)_{\mu,i} = \begin{cases} a^{\alpha_i} & \text{if } \alpha_i \in I_H, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 3.26. Two geodesics ν_1, ν_2 in the same standard closed Weyl chamber are *μ-related* if $(\nu_1)_\mu = (\nu_2)_\mu$.

In the next lemma, we give an equivalent definition of the μ-relation on geodesics in the standard closed Weyl chamber using the horospherical decomposition. As before, this enables the proof that the μ-relation is an equivalence relation.

Lemma 3.27. Let R_1, R_2 be standard parabolic subgroups with the same standard parabolic μ-reduction P which are also contained in a common standard parabolic μ-saturation Q. Two geodesics ν_1, ν_2 in the standard closed Weyl chamber are μ-related if and only if they take the form $\nu_i(t) = (n_i, a_i \exp(tH_i), z_i, z_i') \in N_{R_i} \times A_{R_i} \times X_{R_i,I} \times X_{R_i,I'}$ with $z_1 = z_2$.

Proof. If two geodesics are μ-related, then their root profiles are identical. Therefore, $I_{H_1} = I_{H_2}$, so we call this set I_H. Having identical root profiles implies that standard μ-reductions of standard parabolics involved in the refined horospherical decompositions (2.1.9) of the geodesics are the same. Thus, the geodesics take the above form with $X_{R_1,I} = X_{R_2,I}$. It remains to show that $z_1 = z_2$. Let $a_{(I,i)}$ and $a_{(I'_i,i)}$ be the contributions in the relative horospherical decompositions of the split factors of the boundary symmetric space for each ν_i. Suppose the root profiles are the same, but the $a_{(I,i)}$ components are different. Consider $(a_{I,1}a_{I,2}^{-1})^{\alpha}$, where $\alpha \in I_H$. Since $I_H \subset \Delta_H$, each α evaluates trivially. The set I_H provides a diffeomorphism between the split component $A_I \subset X_{R_i,I}$ and $(\mathbb{R}_{>0})^r$ where $r = |I_H|$ (cf. [BJ06, I.1.10]). The image of $a_{I,1}a_{I,2}^{-1}$ under this diffeomorphism is the identity, and this can only happen if $a_{I,1} = a_{I,2}$. Since the other factor of the relative horospherical decomposition of factors in the splitting are trivial, this implies that $z_1 = z_2$.

The converse implication can be proved similarly. One uses the assumption preceeding Lemma 3.26 to establish the decomposition of $\mathfrak{a} = \mathfrak{a}^I \oplus \mathfrak{a}^{I'_i} \oplus \mathfrak{a}_{J_i}$ as in [BJ06, I.11.3]. Since the geodesics are in the standard closed Weyl chamber and they have the same μ-reduction, the root profiles will evaluate to zero in the same positions, Since $z_1 = z_2$, the nonzero parts of the root profile arise from evaluation on the same $\mathfrak{a}^I$ term. Thus, the lemma is proved. □

We use the following Proposition to define a μ-relation on geodesics in the same nonstandard closed Weyl chamber. It uses the notion of the *adjoint action* of K on the Lie algebra $\mathfrak{g}$ which is defined by $k \cdot x = kxk^{-1}$ for $k \in K$ and $x \in \mathfrak{g}$.

Proposition 3.28. [Ebe96, 2.8.3] Any two Weyl chambers are conjugate under an element of $Ad(K)$. Moreover, the orbit of a vector under the action of $Ad(K)$

intersects each maximal abelian subspace a finite number of times and each Weyl chamber exactly once.

A certain subquotient of K permutes the open Weyl chambers contained in any maximal abelian subalgebra.

Definition 3.29. Let A be equal to $\exp \mathfrak{a}$ for some maximal abelian subalgebra. Let $\mathcal{N}(A) = \{g \in G \mid gag^{-1} \in A \text{ for all } a \in A\}$ and $\mathcal{Z}(A) = \{g \in G \mid gag^{-1} = a \text{ for all } a \in A\}$. The *Weyl group* W is the quotient group $(\mathcal{N}(A) \cap K)/(\mathcal{Z}(A) \cap K)$.

Definition 3.30. Two geodesics ν_1, ν_2 in the same closed Weyl chamber are *μ-related* if their conjugates in the standard closed Weyl chamber are μ-related in the sense of Definition 3.26.

Lemma 3.31. The μ-relation on geodesics in the same closed Weyl chamber is well defined.

Proof. Consider a geodesic $\nu(t) = a \exp(tH) x_0$ in the standard closed Weyl chamber. We first check that the root profile is well defined up to a change in parametrization. Consider the geodesic $\nu'(t) = a \exp(H') \exp(tH)$ where $H' \perp H$. For every $\alpha \in I_H$, the coordinates $a^\alpha, (a \exp(H'))^\alpha$ of the root profiles of ν and ν' are equal since $(a \exp(H'))^\alpha = a^\alpha \exp(H')^\alpha = a^\alpha \exp \alpha(H) = a^\alpha$. Therefore the root profile is well defined up to change in parametrization. We make the following arguments to show the root profile of a geodesic in any closed Weyl chamber is well defined.

Any geodesic that takes the form $\nu(t) = a \exp(tH) x_0$ can be thought of as the exponentiation of a line $\mathcal{L}$ in $\mathfrak{a}$. Suppose a point on $\mathcal{L}$ intersects some open Weyl chamber wall nontrivially. By Proposition 3.28, the image of this point is unique in the standard open Weyl chamber, and so the image of $\mathcal{L}$ is unique in the standard open Weyl chamber. Therefore the image of ν in the standard closed Weyl chamber is unique, thus the root profile is well defined for ν.

On the other hand, suppose $\mathcal{L}$ does not intersect any open Weyl chamber wall. This implies that ν passes through the basepoint x_0. The root profile will thus consist of entries of 0's and 1's, with the 1's corresponding to the roots that evaluate trivially on the defining velocity vector. Therefore the root profile is well defined for ν.

Therefore the μ-relation on a closed Weyl chamber is well defined since the defining root profile is well defined on geodesics in the standard closed Weyl chamber. □

Lemma 3.32. Let R_1 and R_2 be parabolic subgroups that contain a common minimal parabolic and have a common μ-saturation Q. Two geodesics ν_1, ν_2 in the same closed Weyl chamber are μ-related if and only if they take the form $\nu_i(t) = (n_i, a_i \exp(tH_i), z_i, z_i') \in N_{R_i} \times A_{R_i} \times X_{R_i,I} \times X_{R_i,I'}$ with $z_1 = z_2$.

Proof. If ν_1, ν_2 are μ-related, by Lemmata 3.27 and 3.28, we have $z_1 = z_2$. In the other direction, since R_1 and R_2 have a common μ-saturation Q, they share the same μ-reduction. So the boundary symmetric spaces $X_{R_i,I}$ are equal. It follows that the root profiles of ν_1, ν_2 have the same zero entries. To show the nonzero entries of the root profiles are equal when $z_1 = z_2$, we examine how conjugation by K takes ν_1, ν_2 to the standard closed Weyl chamber. This action of K (2.1.3) on the spilt factors of the refined horospherical decomposition is defined in [BJ06, I.11.7]. Since $z_1 = z_2$ the corresponding coordinates $k \cdot z_1$ and $k \cdot z_2$ are also equal in the standard maximal flat through x_0. These coordinates are the only pieces of the $k \cdot \nu_i(0)$ that the root profiles detects. Thus, the nontrivial entries are equal and the ν_i are μ-related. □

Lemma 3.33. The μ-relation on closed Weyl chambers is an equivalence relation.
□

We use this equivalence relation to develop the next equivalence relation on adjacent closed Weyl chambers. We proceed as before; we first look at geodesics in the standard closed Weyl chamber, then treat the general case. In fact, we need the more general version of the μ-relation on (nonstandard) closed Weyl chambers to develop the relation that deals with a geodesic in the standard closed Weyl chamber.

3.2.4 Geodesics in adjacent closed Weyl chambers

In the following, we put an equivalence relation on geodesics that are in the same maximal flat which passes through the basepoint x_0. We compare geodesics in a closed Weyl chamber with geodesics in an *adjacent closed Weyl chamber*.

Definition 3.34. We say geodesics ν_1, ν_2 contained in closed Weyl chambers are contained in *adjacent closed Weyl chambers* if they are in the same maximal flat and if the spaces $\overline{\mathfrak{a}^+_{P'_1}}$ and $\overline{\mathfrak{a}^+_{P'_2}}$ that containing the velocity vectors of ν_1, ν_2 intersect nontrivially.

The relation we define on these geodesics is a generalization of the μ-relation on geodesics in the same closed Weyl chamber.

Definition 3.35. Let ν_1 be a geodesic in the standard closed Weyl chamber. Let ν_1, ν_2 be geodesics in the standard maximal flat that are in adjacent closed Weyl chambers. The geodesics ν_1, ν_2 are called *μ-related* if there exists a geodesic ν' in the standard closed Weyl chamber that is μ-related to both ν_1 and ν_2 (in the sense of Definition 3.30).

Definition 3.36. Two geodesics ν_1, ν_2 in adjacent closed Weyl chambers are called μ*-related* if there exists a geodesic ν' that is μ-related to both ν_1 and ν_2 in the sense of Definition 3.30.

We formulate this μ-relation in terms of the horospherical decomposition.

Lemma 3.37. Let R_1, R_2 be parabolic subgroups that are conjugate under $k \in K$ to a standard parabolic P_1 and a parabolic subgroup P_2, respectively, such that for some $k' \in K \cap \mathcal{Z}(X_P)$ we have $P \subset P_1$, ${}^{k'}P_2 \subset Q$, where Q is a μ-saturated standard parabolic, P is the standard parabolic that is a μ-reduction of Q, and X_P is the standard μ-reduction of X_Q. Two geodesics ν_1, ν_2 in adjacent closed Weyl chambers are μ-related if and only if they take the form $\nu_i(t) = (n_i, a_i \exp(tH_i), z_i, z_i') \in N_{R_i} \times A_{R_i} \times X_{R_i,I} \times X_{R_i,I'}$ with $z_1 = z_2$.

Proof. Suppose ν_1, ν_2 in adjacent closed Weyl chambers are μ-related. Then they can be written in terms of the horospherical decomposition with respect to groups of the form R_i above. The geodesic ν' can be written as $\nu'(t) = (n, a \exp(tH), z, z_i') \in N_{{}^kP} \times A_{{}^kP} \times X_{{}^kP_I} \times X_{{}^kP_{I'}}$ where ${}^kP$ is common μ-reduction of R_i. By Lemma 3.32 we have $z = z_1$ and $z = z_2$. Therefore $z_1 = z_2$.

In the other direction, Suppose ν_1, ν_2 take the given form with $z_1 = z_2$. It is clear that ν_1, ν_2 are contained in adjacent closed Weyl chambers. When we conjugate back to the standard closed Weyl chamber we take the geodesic ν' contained in the standard flat that takes the form $\nu'(t) = (n, a \exp(tH), k \cdot z_1, z_i') \in N_P \times A_P \times X_{P_I} \times X_{P_{I'}}$ where P is the standard μ-reduction as above. Then ν' has the same root profile as the conjugates of ν_1, ν_2 in the standard closed Weyl chamber. Therefore, ν_1, ν_2 are μ-related. □

3.2.5 Geodesics in different closed Weyl chambers

We give a definition of the μ-relation between any geodesics that are contained in flats that pass through x_0.

Definition 3.38. Two geodesics ν_1, ν_2 are called *μ-related* if there exists a collection of geodesics $\{\nu'_1, \ldots, \nu'_n\}$ such that ν_1 is μ-related to ν'_1, ν'_i is μ-related to ν'_{i+1}, for $i = 1, \ldots, n-1$, and ν'_n is μ-related to ν_2 (all in the sense of Definition 3.36).

Lemma 3.39. Let R_1, R_2 be parabolic subgroups that have the same μ-saturation Q. Two geodesics in different closed Weyl chambers ν_1, ν_2 are μ-related if and only if they take the form $\nu_i(t) = (n_i, a_i \exp(tH_i), z_i, z'_i) \in N_{R_i} \times A_{R_i} \times X_{R_i,I} \times X_{R_i,I'}$ where $z_1 = z_2$.

Proof. Suppose ν_1, ν_2 in different closed Weyl chambers are μ-related. Then there exist a collection of geodesics $\{\nu'_1, \ldots, \nu'_n\}$ where ν'_i is μ-related to ν'_{i+1} in the sense of Definition 3.36. Therefore, by Lemma 3.37, each ν_i, ν'_i has the same z coordinate. Therefore we have $z_1 = z_2$.

In the other direction, suppose the geodesics ν_1, ν_2 take the form above. Now we must find a collection of geodesics ν'_i connecting ν_1, ν_2 by the μ-relation on adjacent closed Weyl chambers. In fact we find a single geodesic ν' that does the job. We consider the general form that a geodesic contained in a maximal flat through x_0 may take.

In the standard flat through x_0 geodesics take the form $\nu(t) = (Id, a' \exp(tH), a_I, a_{I'})$ in a horospherical decomposition of X with respect to a parabolic subroup that intersects the flat nontrivially. Since other maximal flats are conjugate to the standard flat by conjugation by elements of K, any geodesic in a maximal flat through x_0 takes the form $(Id,\ {}^k a' \exp(tAd(k)H),\ {}^k a_I,\ {}^k a_{I'})$ as described in [BJ06, I.11.8].

Consider the family of geodesics in the standard maximal flat that are fixed by the $K_{I'}$-action. Examples of geodesics in this family include those with trivial $a_{I'}, a'$ coordinates with velocity vectors that satisfy $Ad(k_{I'})H = H$. Velocity vectors contained in $\mathfrak{a}_Q^+(\infty)$ satisfy this second condition. There may be an infinite number of such velocity vectors as there exist nonmaximal μ-saturated subgroups (cf. [BJ06, Remark I.11.6]).

All the μ-connected reductions of Q are conjugate under $\mathcal{Z}(e(Q)) = \mathcal{Z}(X_{P_{0,I}}) = N_Q A_{P_{0,J}} M_{P_{0,I'}}$. Thus all flats through x_0 that intersect a space X_I corresponding to μ-reduced parabolic subgroup are conjugate by elements in $K_{J\setminus I}$.

For each of the geodesics ν_i consider the geodesic $\nu'(t) = (Id, Id \exp(tH), z_1, Id)$ in the refined real horospherical decomposition corresponding to Q. This is obtained by acting on a geodesic $(Id, Id \exp(tH), a_1, Id)$ in the standard flat by an appropriate element $k_I \in K_I$. The action of $K_{J\setminus I}$ fixes ν'. There exist elements $k_i \in K_{J\setminus I}$ so that ν_i is contained in the flat $k_i \cdot K_I(\exp \mathfrak{a})x_0$ where $(\exp \mathfrak{a})x_0$ is the standard maximal flat.

Thus, we have found maximal flats through x_0 that contain both ν' and ν_i, $i = 1, 2$. By Lemma 3.37, ν' is μ-related to both ν_i in the sense of Definition 3.36. Therefore ν_i are μ-related in the sense of Definition 3.38. □

3.2.6 $NRC\mu$-relation

Here we use the μ-relation from the previous subsection to describe the nonmaximal boundary.

Definition 3.40. Two NRC-equivalence classes are called *μ-related* if there exist representatives ν_1, ν_2 that are μ-related (in the sense of Definition 3.38).

Definition 3.41. Two geodesics ν_1, ν_2 are called *NRCμ-related* if their NRC-equivalence classes are μ-related.

Let Q be a μ-saturated parabolic subgroup containing a minimal parbolic subgroup P. Let P_I be the μ-connected reduction of Q and consider two parabolic subgroups R_1, R_2 such that $P_I \subset R_i \subset Q$.

Lemma 3.42. Two geodesics ν_1, ν_2 are NRCμ-related if and only if they have the form $\nu_i(t) = (n_i, a_i \exp(tH_i), z_i, z_i') \in \ N_{R_i} \times A_{R_i} \times X_{R_{i,I}} \times X_{R_{i,I'}}$ with $z_1 = z_2$.

Proof. Suppose ν_1, ν_2 are NRCμ-related. By Lemma 3.15, the NRC-equivalence classes fix the boundary symmetric space coordinate in each horospherical decomposition. Therefore the splitting of the boundary symmetric space determines a fixed pair (z_i, z_i') in each symmetric space. The next statement is a consequence of Lemma 3.39. The fact that representatives are μ-related allows us to write ν_1, ν_2 in the horospherical decompositions with respect to parabolics R_i as defined above where $z_1 = z_2$.

In the other direction, suppose ν_1, ν_2 takes the above form. Since $z_1 = z_2$, we have that ν_1, ν_2 are μ-related by Lemma 3.39. Of course, ν_i is a representative of its own NRC-equivalence class, and therefore ν_1, ν_2 are NRCμ-related. □

Lemma 3.43. The $NRC\mu$-relation is an equivalence relation on geodesics in X. □

Theorem 3.44. The set of NRCμ-related equivalence classes of geodesics corresponds bijectively to $\partial(\overline{X}_\mu)$ through the map $\nu \mapsto \lim_{t\to\infty} \nu(t)$.

Proof. By the definition of the topology of $\overline{X}_\mu$ in terms of convergent sequences, every geodesic converges to a limit point in $\overline{X}_\mu$. Then the proposition follows from the conclusion of Lemma 3.42. □

The proposition below shows the two definitions of the Satake compactification are equivalent. Let τ be a representation of G with highest weight μ_τ. Assume that μ_τ and μ are situated on the same Weyl chamber face.

Proposition 3.45. [BJ06, I.11.18] For any Satake compactification $\overline{X}^S_\mu$, the identity map on X extends to a homeomorphism $\overline{X}_{\mu_\tau} \to \overline{X}^S_\mu$.

The following corollary follows immediately by Theorem 3.43 and Proposition 3.44.

Corollary 3.46. The set of NRCμ-related equivalence classes of geodesics corresponds bijectively to $\partial(\overline{X}^S_\tau)$ through the map $\nu \mapsto \lim_{t\to\infty} \nu(t)$.

We will use this result to describe the nonmaximal Satake compactifications in the next chapter.

CHAPTER 4

A GEOMETRIC CONSTRUCTION OF THE SATAKE BOUNDARY OF LOCALLY SYMMETRIC SPACES

In this chapter, we describe the Satake boundary of $\Gamma\backslash X$ using equivalence classes of EDM geodesics. We recall the construction of the maximal Satake compactification of $\Gamma\backslash X$ using the uniform method [BJ06, §III.11]. We use the NRL-relation and C-relation to define an equivalence relation on EDM geodesics whose classes are in one to one correspondence with the maximal Satake boundary. We recall Satake's original construction of the nonmaximal Satake compactifications and remark on properties of the dominating maps from the maximal Satake to nonmaximal Satake compactifications. We then show that the μ-relation in association with the NRL and C-relations describes the nonmaximal boundary.

Remark 4.1. There are several reasons we use the μ-relation from the previous chapter to define a relation on EDM geodesics in $\Gamma\backslash X$ instead of using a relation defined with the $\mathbb{Q}$-rank or maximal $\mathbb{Q}$-split flats. Specifically, we do not use maximal $\mathbb{Q}$-split flats through a basepoint x_0 because these will not cover the maximal Satake boundary. This point is not apparent in certain $\mathbb{Q}$-rank one examples, but it is clear in the case $\mathbf{G} = \mathbf{SL_3}$. On the other hand, if we consider all maximal

$\mathbb{Q}$-split flats (not just those that pass through x_0) their closures will cover the maximal Satake boundary. However, there is some difficulty in showing an analagous root profile yields a well defined μ-relation. Lastly, if we use the $\mathbb{Q}$-roots to define a root profile, when $\mathrm{rk}_{\mathbb{Q}}(\mathbf{G}) = 1$ this is not enough to distinguish limit points in boundary components that are not 0-dimensional.

Inspired by the assumption in [JM02, 9.16], there may be promise in using maximal $\mathbb{Q}$-split flats that pass through some ϵ-ball centered at x_0. These flats cover the maximal Satake boundary, but we do not pursue this approach here.

4.1 The Maximal Satake Compactification

There is also a notion of equivalence on EDM geodesics. Two EDM geodesics are equivalent if

$$\lim_{t\to\infty} \sup d(\gamma_1(t), \gamma_2(t)) < \infty. \tag{4.1.1}$$

This relation is used in [JM02, §9] to construct the geodesic compactification on Riemannian manifolds. We can also use it to describe the Satake compactification of certain Hermitian locally symmetric spaces.

Proposition 4.2. If $\mathrm{rk}_{\mathbb{Q}}(\mathbf{G}) = 1$ and $\Gamma\backslash X$ is a tube domain (*i.e.* X can be expressed as a sum $V + C$ where V is a finite dimensional real vector space and $C \subset V$ is a homogeneous self-adjoint cone), then the set of equivalence classes of EDM geodesics corresponds bijectively to $\partial(\overline{\Gamma\backslash X}^{BB})$ through the map $\gamma \mapsto \lim_{t\to\infty} \gamma(t)$.

Proof. As discussed in [AMRT75, p. 162], the rational boundary components are 0-dimensional. Therefore the Bailey–Borel compactification is obtained by adjoining cusps. On the other hand, we know that the boundary components for the geodesic compactification for $\mathbb{Q}$-rank 1 Hermitian locally symmetric spaces are cusps [JM02, Theorem in §1.5]. Since 4.1.1 constructs the geodesic compactification, this completes the proof. □

4.1.1 The $NRLC$-Relation

We use the NRL and C relations to describe the maximal Satake in the general case.

Definition 4.3. Two NRL equivalence classes $[\gamma_1]_{NRL}, [\gamma_2]_{NRL}$ are *C-related* if there exist representatives γ_1', γ_2' with lifts $\tilde{\gamma}_1', \tilde{\gamma}_2' \in X$ that are C-related (in the sense of Definition 3.4).

Definition 4.4. Two EDM geodesics γ_1, γ_2 are *$NRLC$-related* if their NRL equivalence classes $[\gamma_1]_{NRL}, [\gamma_2]_{NRL}$ are C-related.

Lemma 4.5. Two EDM geodesics γ_1, γ_2 are $NRLC$-related if and only if they have lifts of the form $\tilde{\gamma}_i(t) = (n_i, z_i, a_i^{\perp}, a_i \exp(tH_i)) \in N_P \times X_P \times \exp \mathfrak{a}_{\mathbf{P}}^{\perp} \times A_{\mathbf{P}}$ with $z_1 = z_2$.

Proof. This requires translation between real and rational horospherical decompositions of X. We choose a basepoint $x_0 = K$ that gives a rational horospherical decomposition of X with respect to $\mathbf{P}$. We define the C-relation on geodesics in X that live in maximal flats through x_0. Now the real horospherical decomposition $X = N_P \times A_P \times X_P$ and the refined rational horospherical decomposition $X = N_P \times X_P \times \exp \mathfrak{a}_{\mathbf{P}}^{\perp} \times A_{\mathbf{P}}$ have identical factors N_P and X_P and we have $A_P = \exp \mathfrak{a}_{\mathbf{P}}^{\perp} \times A_{\mathbf{P}}$. Also we have $X_{\mathbf{P}} = X_P \times \exp \mathfrak{a}_{\mathbf{P}}^{\perp}$.

Suppose γ_1, γ_2 are $NRLC$-related. Then by Definition 4.4, there exist lifts $\tilde{\gamma}_1', \tilde{\gamma}_2'$ that are C-related. By Lemma 3.5, $\tilde{\gamma}_1', \tilde{\gamma}_2'$ have equal n, z, H coordinates in the real horospherical decomposition with respect to P. By [JM02, Lemma 14.18], lifts of elements in a NRL-equivalence classes all share the same $X_{\mathbf{P}}$ coordinate. For each element of $X_{\mathbf{P}}$, the splitting $X_{\mathbf{P}} = X_P \times \exp \mathfrak{a}_{\mathbf{P}}^{\perp}$ determines a unique pair $(z, a^{\perp})$. Therefore all lifts of $[\gamma_1]_{NRL}$ and $[\gamma_2]_{NRL}$ have the same z coordinate. This implies, $z_1 = z_2$ in the refined rational horospherical decomposition.

In the other direction, suppose γ_1, γ_2 have lifts with $z_1 = z_2$. We can find lifts of representatives in the NRL equivalence classes with equal n, z, H coordinates in the real horospherical decomposition. These lifts are C-related by Lemma 3.5. Therefore, γ_1, γ_2 are $NRLC$-related. □

Lemma 4.6. The $NRLC$-relation is an equivalence relation on EDM geodesics. □

Theorem 4.7. The set of $NRLC$-related equivalence classes of EDM geodesics corresponds bijectively to $\partial(\overline{\Gamma\backslash X}^S_{max})$ through the map $\gamma \mapsto \lim_{t\to\infty} \gamma(t)$.

Proof. By the convergence of unbounded sequences in ${}_{\mathbb{Q}}\overline{X}^S_{max}$ in §2.3.3 and the classification of EDM geodesics in Theorem 2.35, it follows that that every EDM geodesic in $\Gamma\backslash X$ converges to a boundary point of $\overline{\Gamma\backslash X}^S_{max}$. Therefore the proposition follows from the conclusion of Lemma 4.6. □

4.1.2 The NRC-Relation

Here we provide an alternate definition for two EDM geodesics to be C-related. We will not need the results of this subsection in the sequel, and we do not provide proofs of most statments. We only include this since it may be of interest to some readers. The general discussion resumes in §4.2.

Definition 4.8. Two geodesics γ_1, γ_2 in the same congruence bundle $C(\gamma)$ are called *C-related* if there exist lifts $\tilde{\gamma}_1, \tilde{\gamma}_2$ in the same congruence bundle $C(\tilde{\gamma})$ and a Clifford translation T of $C(\tilde{\gamma})$ such that $T(\tilde{\gamma}_1) = \tilde{\gamma}_2$.

Lemma 4.9. Two geodesics $\gamma_1, \gamma_2 \in C(\gamma)$ are C-related if and only if they have lifts of the form $\tilde{\gamma}_i(t) = (n_i, z_i, a_i^{\perp}, a_i \exp(tH_i)) \in N_P \times X_P \times \exp \mathfrak{a}_{\mathbf{P}}^{\perp} \times A_{\mathbf{P}}$ with $n_1 = n_2$, $z_1 = z_2$ and $H_1 = H_2$.

Proof. If γ_1, γ_2 are C-related this implies they have lifts in the same congruence bundle, therefore they have lifts with $n_1 = n_2$ and $H_1 = H_2$. Suppose $z_1 \neq z_2$, then the existence of a Clifford translation taking one lift to the other would contradict Theorem 3.3. Therefore $z_1 = z_2$. If there are lifts that take the form above, then it is clear they are contained in the same congruence and have a Clifford translation taking one to the other. Therefore the EDM geodesics γ_1, γ_2 are C-related. □

Lemma 4.10. The C-relation is an equivalence relation on EDM geodesics in $\Gamma \backslash X$.

□

We consider the restriction of the C-relation to the finite bundle $F(\gamma)$. The quotient $F(\gamma)/C$ can be identified with the space $\Gamma_{X_P} \backslash X_P$.

Definition 4.11. The dimension of $F(\gamma)/C$ is called the *extended mobility degree* of γ.

Definition 4.12. Two C-equivalence classes $[\gamma_0]_C, [\gamma_1]_C$ are *R-related* if there exist representatives γ_0, γ_1 and a family of geodesics $\gamma_s(t)$ connecting them such that the mobility degree of $\gamma_s(t)$ does not change and $d(\gamma_{s_1}(t), \gamma_{s_2}(t)) = c|s_1 - s_2|t$ when $t \geq 0$, where c is some constant.

Definition 4.13. Two geodesics γ_0, γ_1 are *RC-related* if their C-equivalence classes $[\gamma_0]_C, [\gamma_1]_C$ are R-related.

We can combine RC-equivalence with N-equivalence to define a NRC-equivalence relation. Although this does not make use of the L-relation, the equivalence classes are in one to one correspondence with the maximal Satake boundary.

4.1.3 Local Clifford Relation

In the following we generalize the definition of the Clifford relation. This defnition may give means to define the maximal Satake compactification without explicitly using a relation on geodesics in X. Once again, this subsection is not needed in the future, and is only here for the interested reader.

Definition 4.14. Let γ_1, γ_2 be EDM geodesics in the same congruence bundle within a fixed distance ϵ of each other. We call γ_1, γ_2 *C_ϵ-related* if there exists an isometry ϕ whose displacement function on any closed ϵ-ball containing γ_1, γ_2 is constant and $\phi(\gamma_1) = \gamma_2$.

The following definition generalizes the notion of rank in Definition 14.8 of [JM02].

Definition 4.15. For any EDM geodesic γ, the *ϵ-rank* of γ is defined as

$$r_\epsilon(\gamma) = \max\{\ k \in \mathbb{Z} \mid \text{there exists a faithful isometric action of } (-\epsilon, \epsilon)^{n-1} \text{ on } C(\gamma)\}$$

for some $\epsilon > 0$.

Here the definition of action is slightly altered. When we check the composition of action axiom we consider only $\epsilon_1, \epsilon_2 \in (-\epsilon, \epsilon)^{n-1}$ so that $|\epsilon_1 + \epsilon_2| < \epsilon$.

4.2 The Non-Maximal Satake Compactifications

In this section, we put an additional relations on the $NRLC$-equivalence relation to construct the boundary of the nonmaximal Satake compactification. First, we define the Satake compactification as Satake had [Sat60a]. We refer the interested reader to [BJ06, III.3.3] for a treatment using modern notation.

Let $X_{P(Q_i)}$ be the boundary symmetric spaces corresponding to the m μ-saturated standard rational parabolic subgroups. Under the assumption of geometric rationality, whose definition was first approached with respect to Siegel sets ([Sat60a], cf. [BJ06, III.3.4]), the boundary can be defined as the following union

$$\partial^* X = \cup_{i=1}^m \Gamma X_{P(Q_i)}.$$

Proposition 4.16. [BJ06, III.3.9] If a Satake compactification $\overline{X}_\tau^S$ is geometrically rational, then the following construction gives a Hausdorff compactification of $\Gamma\backslash X$,

$$\overline{\Gamma\backslash X}_\tau^S = \Gamma\backslash(X \cup \partial^* X) = \Gamma\backslash X \cup \coprod_{i=1}^m \Gamma_{X_{P(Q_i)}} \backslash X_{P(Q_i)},$$

where $\mathbf{Q}_1, \ldots, \mathbf{Q}_m$ are representatives of Γ-conjugacy classes of μ_τ-saturated parabolic subgroups of $\mathbf{G}$.

Proposition 4.17. [BJ06, III.11.10] Let $\overline{X}_\tau^S$ be a maximal Satake compactification. If $\overline{X}_\tau^S$ is geometrically rational, the induced compactification $\overline{\Gamma\backslash X}_\tau^S$ is isomorphic to the compactification $\overline{\Gamma\backslash X}_{max}^S$.

Proposition 4.18. [BJ06, III.15.2] The maximal Satake compactification $\overline{\Gamma\backslash X}_{max}^S$ dominates all other Satake compactifications $\overline{\Gamma\backslash X}_\tau^S$, if $\overline{X}_\tau^S$ is geometrically rational and $\overline{\Gamma\backslash X}_\tau^S$ is defined. Moreover, the inverse images of the dominating maps are Satake compactifications of lower-dimensional locally symmetric spaces.

Remark 4.19. In the proof of the above proposition, it is claimed that for a pair of geometrically rational Satake compactifcations $\overline{X}^S_{\tau_1}, \overline{X}^S_{\tau_2}$ with dominance relation $\overline{X}^S_{\tau_1} \to \overline{X}^S_{\tau_2}$, the inverse image of a boundary component $X_P \in \overline{X}^S_{\tau_2}$ splits as a product $X_P \times \overline{X_{P'}}^S$, where $X_P \times X_{P'}$ is the largest boundary component of $\overline{X}^S_{\tau_1}$ that maps to X_P. Moreover, the Γ action induces actions of discrete subgroups on X_P and $X_{P'}$, which defines a locally symmetric space $\Gamma_{X_{P'}}\backslash X_{P'}$, whose Satake compactification induced from $\overline{X_{P'}}^S$ is the inverse image over the points in $\Gamma_{X_P}\backslash X_P$ in $\overline{\Gamma\backslash X}^S_{\tau_2}$. This uses geometric rationality in an essential way.

Lemma 4.20. Every EDM geodesic converges to a boundary point in $\overline{\Gamma\backslash X}^S_{\tau}$.

Proof. By Theorem 4.7, we know that any EDM geodesic γ converges to a boundary point in the maximal Satake boundary. By Proposition 4.18, we know that the maximal Satake compactification dominates all nonmaximal Satake compactifications. The dominating maps are continuous maps that map any EDM geodesic to itself and any limit point $lim_{t\to\infty}\gamma(t) \in \Gamma_{X_P}\backslash X_P$ to the point $\pi(z) \in \Gamma_{X_I}\backslash X_I$ where $(z, z') \in X_I \times X_{I'} = X_P$ is the splitting of a lift of the limit point $lim_{t\to\infty}$. The continuity of the dominance map implies that the limit $lim_{t\to\infty}$ in $\overline{\Gamma\backslash X}^S_{\tau}$ is equal to $\pi(z)$. Therefore, every EDM geodesic converges to a boundary point. □

Lemma 4.21. Every point in the boundary of a nonmaximal Satake compactification $\overline{\Gamma\backslash X}^S_{\tau}$ is the limit point of some EDM geodesic.

Proof. For a point $z_\infty \in \partial\overline{\Gamma\backslash X}^S_{\tau}$, consider the fiber of the dominating map from the maximal Satake to $\overline{\Gamma\backslash X}^S_{\tau}$. For any point $\tilde{z}_\infty$ in this fiber, there exists an EDM geodesic $\gamma(t)$ that converges to this point (by Lemma 3.22). The domainating map is continuous and the identity on $\Gamma\backslash X$. Therefore the limit of the image of $\gamma(t)$ in $\overline{\Gamma\backslash X}^S_{\tau}$ is z_∞. □

4.2.1 $NRLC\tau$ -relation

Now we define an equivalence relation that describes the boundary of nonmaximal Satake compactifications.

Definition 4.22. Two NRLC equivalence classes $[\gamma_1]_{NRLC}, [\gamma_2]_{NRLC}$ are *τ-related* if there exist representatives γ_1', γ_2' with lifts $\tilde{\gamma}_1', \tilde{\gamma}_2' \in X$ that are μ-related in the sense of Definition 3.38.

Definition 4.23. Two EDM geodesics γ_1, γ_2 are *NRLCτ-related* if their NRLC equivalence classes $[\gamma_1]_{NRLC}, [\gamma_2]_{NRLC}$ are τ-related.

Let $\mathbf{P}_i, i = 1, 2$ be rational parabolic subgroups whose real loci P_i have the same μ-saturation.

Lemma 4.24. Two geodesics γ_1, γ_2 are *$NRLC\tau$-related* if and only if they have lifts of the form $\tilde{\gamma}_i(t) = (u_i, z_i, z_i', a_i^{\perp}, a_i \exp(tH_i)) \in N_{P_i} \times X_{P_{i,I}} \times X_{P_{i,I'}} \times \exp \mathfrak{a}_{\mathbf{P}_i}^{\perp} \times A_{\mathbf{P}}$ with $z_1 = z_2$.

Proof. We must be carful here identifying lifts that correspond to different parabolic subgroups with the μ-relation. The reason is that not every basepoint will yield a rational horospherical decomposition with respect to the rational parabolic $\mathbf{P}$. Recall that Proposition 2.25 rectifies this problem. So for each rational parabolic we can find an appropriate basepoint. The μ-relation identifies geodesics corresponding to different parabolic subgroups. The calculation of root profiles all takes place in a standard closed Weyl chamber. It suffices to choose a basepoint x_0 that yields rational algebraic groups $\mathbf{L}_{\mathbf{P}_0,x_0}, \mathbf{M}_{\mathbf{P}_0,x_0}, \mathbf{S}_{\mathbf{P}_0,x_0}$ where $\mathbf{P}_0$ is the standard minimal rational parabolic subgroups. This baspoint will guarantee rational algebraic groups $\mathbf{L}_{\mathbf{P},x_0}, \mathbf{M}_{\mathbf{P},x_0}, \mathbf{S}_{\mathbf{P},x_0}$ for any standard rational parabolics $\mathbf{P}$ containing $\mathbf{P}_0$.

By Lemma 4.5, elements of a $NRLC$-equivalence class all have the same boundary symmetric space coordinate in the refined rational horospherical decomposition. If representative lifts are μ-related, then by Lemma 3.39, we have $z_1 = z_2$ where the geodesics are written in the rational horospherical decompositions as above.

In the other direction, suppose two geodesics have lifts take the above form. Since $z_1 = z_2$ and the defining parabolics P_i have the same μ-saturation, they themselves are the representative lifts in the $NRLC$-equivalence classes that are μ-related. Therefore, the geodesics γ_1, γ_2 are *NRLCμ-related*. $\square$

Lemma 4.25. The $NRLC\tau$-relation is an equivalence relation on EDM geodesics. $\square$

Now we must show that geodesics in these equivalence classes all map to the same point in the Satake compactification $\overline{\Gamma\backslash X}^S_\tau$. We rely on the description of the fiber of the dominating map from the maximal Satake to $\overline{\Gamma\backslash X}^S_\tau$.

Lemma 4.26. EDM geodesics γ_1, γ_2 are NRLCτ-related if and only if $\lim_{t\to\infty} \gamma_1(t) = \lim_{t\to\infty} \gamma_2(t)$ in $\overline{\Gamma\backslash X}^S_\tau$.

Proof. If γ_i are NRLCτ-related, then we know there exist EDM lifts $\tilde{\gamma}_i$ and appropriate parabolic subgoups (with the same μ-saturation) such that $z_1 = z_2$ in their respective horospherical decompositions. Since the dominating map from the maximal Satake is continuous and is the identity on $\Gamma\backslash X$ the EDM's both converge to z_1 in $\overline{\Gamma\backslash X}^S_\tau$. Therefore $\lim_{t\to\infty} \gamma_1(t) = \lim_{t\to\infty} \gamma_2(t)$ in $\overline{\Gamma\backslash X}^S_\tau$.

If the two EDM's converge to the same boundary point in $\overline{\Gamma\backslash X}^S_\tau$, then there exist representative EDM lifts that converge to boundary points in the maximal Satake compactification with the same μ-saturated component and possibly different non-μ-saturated component (by Remark 4.19). Therefore, there exist parabolic subgroups with the same μ-saturation such that the lifts have a horospherical decomposition with $z_1 = z_2$. Therefore, the EDM's γ_i are NRLCτ-related. $\square$

Theorem 4.27. The set of NRLCτ-related equivalence classes of EDM geodesics corresponds bijectively to $\partial(\overline{\Gamma\backslash X}_\mu)$ through the map $\gamma \mapsto \lim_{t\to\infty} \gamma(t)$.

Proof. By Lemma 4.20, every EDM geodesic in $\Gamma\backslash X$ converges to a boundary point of $\overline{\Gamma\backslash X}^S_\mu$. Therefore the theorem follows from the conclusions of Lemmata 4.25 and 4.26. $\square$

CHAPTER 5

BACKGROUND FOR TOROIDAL COMPACTIFICATIONS

In his thesis [Hir53], Hirzebruch showed how to resolve the quotient singularities of the Hilbert modular surface $\Gamma\backslash\mathfrak{H}^2$ using a method involving continued fractions. The Baily–Borel compactification $\overline{\Gamma\backslash\mathfrak{H}^2}^{BB}$ has additional singularites at the cusps. In [Hir71], Hirzebruch devised a similar continued fractions method that resolved these cuspidal singularities. The Baily–Borel compactification [BB66] is defined for any Hermitian locally symmetric space and usually is very singular. Historically, Igusa [Igu66] first resolved the Baily–Borel compactification of certain quotients of the Siegel upper half space. Following Hirzebruch, Elhers resolved the singularities of the general Hilbert modular variety in [Ehl75]. Using the theory of toroidal embeddings developed in [KKMSD73], a general resolution of singularities of the Baily–Borel compactification was given in [AMRT75]. Concurrently, Satake [Sat73] gave a resolution of the cusp singularities in the $\mathbb{Q}$-rank 1 case.

In this chapter, we focus on the work of Ehlers. We describe his resolution of singularities of the Hilbert modular variety and comment on the canonical resolution of the Hilbert modular surface due to Hirzebruch. We convert much of the language in [Ehl75] to that appearing in [Ji98]. We do this to facilitate arguments in Chapter 6 that use the T-orbit decomposition of the toroidal embedding boundary [Ji98, 4.2].

5.1 Resolution of $\overline{\Gamma\backslash\mathfrak{H}^n}^{BB}$

The boundary components of the Baily–Borel compactifications $\overline{\Gamma\backslash\mathfrak{H}^n}^{BB}$ are points. The link of any point in the boundary is a $\mathbb{T}^n$ bundle over $\mathbb{T}^{n-1}$, where $\mathbb{T}^n$ is the compact torus $(\mathbb{S}^1)^n$. When $n > 1$ it is known [Chr63] that the cusp is singular. There may also exist quotient singularities in the interior of $\overline{\Gamma\backslash\mathfrak{H}^n}^{BB}$ corresponding to points in $\mathfrak{H}^n$ with nontrivial stabilizers in Γ. These singularities are not of interest to us, and we focus on Ehler's procedure to resolve the cusp singularities of $\overline{\Gamma\backslash\mathfrak{H}^n}^{BB}$.

Let $k/\mathbb{Q}$ be a totally real extension of degree n. k comes equipped with n real embeddings, $\mu \mapsto \mu^{(j)}$, $j = 1, \ldots n$. Let $N \subset K$ be a complete $\mathbb{Z}$-module of rank n. N acts on $\mathbb{C}^n$ by translations defined coordinatewise by

$$\mu \cdot (z_j) = (z_j + \mu^{(j)}).$$

The quotient $T = \mathbb{C}^n/N$ is isomorphic to $(\mathbb{C}^*)^n$. For a basis $\{\mu_1, \ldots, \mu_n\}$ of N, this isomorphism is given by

$$\psi_{\{\mu_1 \ldots \mu_n\}} : T \to (\mathbb{C}^*)^n,\ z \bmod N \mapsto (u_1, \ldots, u_n)$$

where $(u_1, \ldots, u_n)$ are determined by

$$e^{2\pi i z_j} = u_1^{\mu_1^{(j)}} \ldots u_n^{\mu_n^{(j)}}.$$

A different choice of basis $\{\rho_1, \dots \rho_n\}$ of N gives the following commutative diagram

$$\begin{array}{ccc} T & \xrightarrow{\psi_{\{\mu_1 \dots \mu_n\}}} & (\mathbb{C}^*)^n \\ \Vert & & \downarrow \Psi \\ T & \xrightarrow{\psi_{\{\rho_1 \dots \rho_n\}}} & (\mathbb{C}^*)^n \end{array}$$

where $\Psi = \psi_{\{\rho_1 \dots \rho_n\}} \circ \psi^{-1}_{\{\mu_1 \dots \mu_n\}}$. Let $(a_{ij}) \in GL_n(\mathbb{Z})$ be the matrix that sends $\{\mu_1, \dots, \mu_n\}$ to $\{\rho_1, \dots \rho_n\}$. Then Ψ is given by

$$(u_1, \dots, u_n) \mapsto (u_1^{a_{11}} \dots u_n^{a_{1n}}, \dots, u_1^{a_{n1}} \dots u_n^{a_{nn}}).$$

We can extend Ψ to points $(u_1, \dots, u_n) \in \mathbb{C}^n$ satisfying $u_j \neq 0$ if $a_{ij} < 0$ for any i. Let this extension be denoted $\tilde{\Psi}$. The topological space $Y_{\tilde{\Psi}}$ obtained by glueing two copies of $\mathbb{C}^n$ together with $\tilde{\Psi}$ has the Hausdorff separation property.

Let $U_N^+ \subset k$ be the group of totally positive units that satisfy $uN = N$. Let $V \subset U_N^+$ be a finite index subgroup. Topologically, the Baily–Borel compactification $\overline{\Gamma \backslash \mathfrak{H}^n}^{BB}$ is the disjoint union $\Gamma \backslash \mathfrak{H}^n \cup \{\text{cusps}\}$. Each cusp of the Baily–Borel compactification $\overline{\Gamma \backslash \mathfrak{H}^n}^{BB}$ determines a unique pair (N, V) as follows. Consider the semidirect product

$$\Gamma(N, V) = N \rtimes V = \left\{ \left(\begin{array}{cc} v & n \\ 0 & 1 \end{array} \right) \in GL_2(K) \middle| v \in V,\ n \in N \right\}.$$

The group $\Gamma(N, V)$ acts on $\mathfrak{H}^n$ by translations

$$(z_j) \mapsto (v^{(j)} z_j + n^{(j)}).$$

We can find a neighborhood of the distinguished cusp $\{\infty\}$ such that in this neighborhood, the Baily–Borel looks like $\Gamma(N, V) \backslash \mathfrak{H}^n \cup \{\infty\}$ for a suitable pair (N, V). So to resolve the singularity at this cusp, we build a compactification

$$\Gamma(Id, V) \backslash (\Gamma(N, Id) \backslash \mathfrak{H}^n) \cup F(\Sigma).$$

The boundary component $F(\Sigma)$ is defined below. To resolve the other cusps, we fix a cusp F and use an element $g \in SL_2(K)$ that maps F to $\{\infty\}$. Let B denote the standard Borel subgroup of $SL_2(K)$. The quotient $(g\Gamma g^{-1} \cap B)\backslash\mathfrak{H}^2$ describes the Baily–Borel in a suitable neighborhood of F. The group $g\Gamma g^{-1} \cap B$ determines a new pair (N', V'). We then attach a toroidal boundary corresponding to (N', V'). This process is continued until all Γ-equivalence classes of rational cusps are accounted for. Note that different rational cusps may have nonisomorphic boundary components associated to each.

Now we define the polyhedral cone decompositions Σ used to resolve the cusp singularities. Let $E = k \otimes_{\mathbb{Q}} \mathbb{R}$. There is a distinguished copy of $k \subset E$ corresponding to the elements $x \otimes_{\mathbb{Q}} 1$ where $x \in K$. Thus, there is a canonically isomorphism $i : E \to \mathbb{R}^n$ that sends $x \otimes_{\mathbb{Q}} 1 \in k$ to $(x^{(1)}, \ldots, x^{(n)})$.

Remark 5.1. The tensor product $k \otimes_{\mathbb{Q}} \mathbb{R}$ is denoted by $N \otimes \mathbb{R}$ in [Ji98]. Therefore, we will use E, $k \otimes_{\mathbb{Q}} \mathbb{R}$, and $N \otimes \mathbb{R}$ interchangeably.

Definition 5.2. A subset σ of $N \otimes \mathbb{R}$ is a *strongly convex rational polyhedral cone* if there exists a finite number of elements $n_1, \ldots, n_s$ in N such that

$$\sigma = \{a_1 n_1 + \ldots + a_s n_s \mid a_i \geq 0\}$$

and σ contains no lines.

From now on, we consider only strongly convex cones. We can describe σ dually using linear forms: there exist linear forms $l_1, \ldots, l_k$ so that

$$\sigma = \{x \in N \otimes \mathbb{R} \mid l_1(x) \geq 0, \ldots, l_k(x) \geq 0\}.$$

We consider partial polyhedral cone decompositions Σ of E that satisfy:

1. $|\Sigma| = (E \cap i^{-1}(0, \infty)^n) \cup \{0\}$.
2. If $\sigma \cap \sigma' \neq \emptyset$, then the intersection is a face contained in both σ and σ'.
3. If $\sigma \in \Sigma$, then so are all faces of σ.
4. V preserves Σ.
5. For $\sigma \in \Sigma$ and $v \in V$, $\dim \sigma \cap \sigma v \leq 1$.
6. Modulo V here are finitely many cones.

From now on, we consider cone decompositions that are built using the lattice $N \subset E$. Any such cone σ has one dimensional faces that are rays emanating from the origin passing through points of N. Let $\Sigma^{(n)}$ be the set of n-dimensional cones in Σ. Each cone $\sigma \subset \Sigma^{(n)}$ determines a basis of N. This basis consists of the first nontrivial points of $N \subset E$ through which the defining one dimensional faces pass. Recall from our discussion in the previous two pages that we have a set of coordinates $(u)_\sigma$ on the copy of $(\mathbb{C}^*)^n$ corresponding to σ. For a cone σ' adjacent to σ, we have an extended map $\tilde{\Psi}$ to copies of $\mathbb{C}^n$ containing $(\mathbb{C}^*)^n_\sigma$ and $(\mathbb{C}^*)^n_{\sigma'}$. We glue copies of $\mathbb{C}^n$ to obtain the space

$$X_\Sigma = \bigcup_{\sigma \in \Sigma^{(n)}} (\mathbb{C}^n)_\sigma / \sim .$$

Each face $\tau \subset \sigma$ is defined by a set $\{\mu_1, \ldots, \mu_r\} \in N$. Let $\mathbf{St}\tau$ denote the collection of cones $\{\sigma\}$ (not necessarily full dimensional) that contain τ. For each τ we have a codimension r space $F(\tau) \subset X_\Sigma$ such that $F(\tau) \subset \cup_{\sigma \in \mathbf{St}\tau \cap \Sigma^{(n)}} (\mathbb{C}^n)_\sigma$ and $F(\tau) \cap (\mathbb{C}^n)_\sigma = \{(u)_\sigma \mid u_1 = \ldots = u_r = 0\}$. To define $F(\tau)$, we first consider the rank r $\mathbb{Z}$-submodule $N_\tau \subset N$. This is the submodule that is generated by the elements of N through which the one-dimensional edges of τ pass. We then use the quotient $E/(N_\tau \otimes_{\mathbb{Q}} \mathbb{R})$ and its projection map $pr : E \to E/(N_\tau \otimes_{\mathbb{Q}} \mathbb{R})$ to define the cone decomposition Σ_τ using points in the rank $n - r$ $\mathbb{Z}$-module N/N_τ. We have $\Sigma_\tau = \{pr(\sigma) \mid \sigma \in \mathbf{St}\tau\}$. We then let $F(\tau)$ be X_{Σ_τ} defined as above. We let

$$F(\Sigma) = \bigcup_{\dim \tau = 1} F(\tau).$$

We write $z \in X_\Sigma - F(\Sigma)$ in the coordinates $(u)_\sigma$ corresponding to $\mathbb{C}^n_\sigma$. The map

$$\pi : X_\Sigma - F(\Sigma) \to T$$

that takes

$$\pi(u) = \frac{1}{2\pi i} \begin{pmatrix} \mu_1^{(1)} \log u_1 + \ldots + \mu_n^{(1)} \log u_n \\ \vdots \\ \mu_1^{(n)} \log u_1 + \ldots + \mu_n^{(n)} \log u_n \end{pmatrix}.$$

yields an isomorphism

$$X_\Sigma - F(\Sigma) \simeq T.$$

Consider the union

$$\tilde{X} = \pi^{-1}(N \backslash \mathfrak{H}^n) \cup F(\Sigma) \subset X_\Sigma.$$

For an element $v \in V$ and $z \in X_\Sigma$, we have $\pi(v \cdot z) = v \cdot \pi(z)$. By Lemmata 1 and 2 in [Ehl75], the group V acts freely and properly discontinuously on $X \subset X_\Sigma$. It follows that $Y = V \backslash \tilde{X}$ is a complex manifold and one can prove the following:

Theorem 5.3. The isomorphism π extends to a continuous map $\hat{\pi} : Y \to \Gamma(N, V) \backslash \mathfrak{H}^n$ that takes $F(\Sigma)$ to $\{\infty\}$. Moreover, $\hat{\pi}$ is holomorphic and a resolution of the singular point $\{\infty\}$ where $\hat{\pi}^{-1}(\{\infty\}) = V \backslash F(\Sigma)$ is a compact manifold.

Example 11. When $n = 2$ the module N has a canonical infinite set of bases. Any such basis arises as a consecutive pair of lattice points on the boundary of the convex hull of $N \cap E^+$. The resulting space $\tilde{X}$ is the union of $\mathfrak{H}^2$ and an infinite chain of $\mathbb{P}^1$ boundary components that interesect transversely over each cusp. Modulo the V-action there are only finitely many $\mathbb{P}^1$'s.

5.2 Topology of the Resolution

We can also describe the boundary $F(\Sigma)$ using the *T-orbit decomposition* that appears in [Ji98]. For a partial polyhedral cone decomposition Σ in $N \otimes \mathbb{R}$, we have a partial compactification of T. This compactification is called the *toroidal embedding* associated with T and is denoted by $\overline{T}_\Sigma$.

We follow the expostion and notation in [Ji98] to describe the structure and topology of toroidal embeddings. We use this description in the identifications in the next chapter.

Let σ be a cone in a rational polyhedral cone decomposition $\Sigma \subset N \otimes \mathbb{R}$. Consider the real subspace $\mathrm{Span}(\sigma) = \sigma + (-\sigma) \subset N \otimes \mathbb{R}$. The complex subspace $\mathrm{Span}_{\mathbb{C}}(\sigma) = \mathrm{Span}(\sigma) \otimes \mathbb{C}$ acts on T by translation.

Definition 5.4. Let the boundary component associated with σ, $O(\sigma)$, be the quotient

$$T/\mathrm{Span}_{\mathbb{C}}(\sigma) = (\mathbb{C}^r/\mathrm{Span}_{\mathbb{C}}(\sigma))/N'.$$

Where N' is the rank $n - r$ $\mathbb{Z}$-module N/N_σ.

Remark 5.5. For an arbitrary cone τ, the closure $\overline{O(\tau)}$ in $\partial\overline{\Gamma\backslash X^{tor}_\Sigma}$ equals $F(\tau)$.

The toroidal embedding $\overline{T}_\Sigma$ is defined by

$$\overline{T}_\Sigma = T \cup \bigsqcup_{\sigma \in \Sigma, \sigma \neq \{0\}} O(\sigma)$$

with the following topology. A sequence $z_j = x_j + iy_j$ in T converges to a point $z_\infty \in O(\sigma)$ for some $\sigma \in \Sigma$ if and only if for the defining linear functionals $l_1, \ldots . l_k$ of σ the following equations are satisfied:

1. $l_1(x_j) \to \infty, \ldots, l_p(x_j) \to \infty$ as $j \to \infty$, while $l_{p+1}(x_j), \ldots, l_k(x_j)$ are bounded.

2. The projection of z_j in $O(\sigma)$ converges to the point z_∞.

Let $\mathbf{Q}$ be a rational parabolic subgroup. In general, we take $\mathbf{Q}$ to be maximal, but since $\mathrm{rk}_{\mathbb{Q}}(\mathbf{G}) = 1$ it follows that all rational parabolics are maximal.

Under the refined real Langlands decomposition used to describe the Baily–Borel boundary [Ji98, §3.2], the group $Q = \mathbf{Q}(\mathbb{R})$ factors as

$$Q = N_Q G_{Q,h} G_{Q,l} A_Q.$$

In our example,

$$N_Q \cong \mathbb{R}^n, G_{Q,h} \cong G_{Q,l} \cong \{Id\}, A_Q \cong \mathbb{R}^n.$$

Consider the projection map

$$Q = N_Q G_{Q,h} G_{Q,l} A_Q \to G_{Q,l}.$$

Let $\Gamma_Q = \Gamma \cap Q$ and let $\Gamma_{Q,l}$ denote the projection of Γ_Q onto $G_{Q,l}$.

In our example, $\Gamma_{Q,l}$ is trivial. There is a splitting of the nilpotent radical $N_Q = U_Q \times V_Q$ where U_Q is the center of N_Q and $V_Q = N_Q/U_Q$. For Hilbert modular varieties, we have $U_Q = N_Q$. If we let Γ_{N_Q} denote the intersection $\Gamma \cap N_Q$, then Γ_{N_Q} is a torsion free lattice of rank n in the group N_Q. Restrict the projection

$$Q = N_Q G_{Q,h} G_{Q,l} A_Q \to G_{Q,l}$$

to the group Γ_Q, and let Γ'_Q denote the kernel of this restricted projection.

Remark 5.6. The choice of N and V in the exposition of Ehler's paper above corresponds to the groups Γ_{N_Q} and Γ'_Q/Γ_{N_Q} in Ji's paper (in the context of Hilbert modular varieties).

Consider the lattice $N = \Gamma_{N_Q}$ in $N_Q \otimes \mathbb{R} = \Gamma_{N_Q} \otimes \mathbb{R} \cong \mathbb{R}^n$.

Then $T = \mathbb{C}^n/N$ is isomorphic to a complex torus $(\mathbb{C}^*)^n$. We can write T in the form $(N \otimes \mathbb{R})/N \times i(N \otimes \mathbb{R})$.

The following lemma indicates how EDM rays converge to the toroidal boundary. To do so, it uses the above decomposition of T.

Lemma 5.7. [Ji98, Lemma 4.2.1] Write $T = (N \otimes \mathbb{R})/N \times i(N \otimes \mathbb{R})$. For any point $x \in (N \otimes \mathbb{R})/N$ and a ray $c(t)$ in $i(N \otimes \mathbb{R})$ which starts from the origin and is contained in the interior σ° of a cone $\sigma \in \Sigma$, then the ray $x + ic(t)$ converges to a boundary point in $O(\sigma) \in T_\Sigma$ as $t \to \infty$. If σ has codimension zero in $N \otimes \mathbb{R}$, then any two such rays whose imaginary parts are contained in the interior σ° converge to the same boundary point.

The above lemma is used fundamentally in the identifications of the final chapter. We introduce various refinements of the boundary components $O(\sigma)$ of the T-orbit decomposition.

Recall the underlying complex space $\mathbb{C}^n = N_\mathbb{C} = N \otimes \mathbb{C}$. The torus T is obtained by quotienting by the lattice N

$$T = N_\mathbb{C}/N \simeq (\mathbb{C}^*)^n \simeq (N \otimes \mathbb{R}/N) \times i(N \otimes \mathbb{R}).$$

If we quotient $\mathbb{C}^n$ by the complex vector space $\mathrm{Span}_\mathbb{C}(\sigma)$, we obtain the following surjective map

$$N_\mathbb{C}/\mathrm{Span}_\mathbb{C}(\sigma) \to T/\mathrm{Span}_\mathbb{C}(\sigma).$$

Let the kernel of this map be denoted by N'. We have the following identification

$$O(\sigma) = T/\mathrm{Span}_{\mathbb{C}}(\sigma) = (\mathbb{C}^n/\mathrm{Span}_{\mathbb{C}}(\sigma))/N'.$$

Remark 5.8. This is the same N' introduced in the previous section.

We can restrict to the real and imaginary parts of $\mathbb{C}^n$ to form the identification

$$O(\sigma) \simeq (\mathfrak{Re}(\mathbb{C}^n)/\mathfrak{Re}(\mathrm{Span}_{\mathbb{C}}(\sigma)))/N' \times i(\mathfrak{Im}(\mathbb{C}^n)/\mathfrak{Im}(\mathrm{Span}_{\mathbb{C}}(\sigma))). \tag{5.2.1}$$

We use this parts of this decomposition to build a one to one correspondence between equivalence classes of certain curves and the toroidal boundary in the next chapter.

CHAPTER 6

A GEOMETRIC CONSTRUCTION OF THE TOROIDAL BOUNDARY OF HILBERT MODULAR VARIETIES

To describe the toroidal boundary in the spirit of [JM02], EDM geodesics will not suffice. There are points in the boundary that are not reached by EDM's. However, EDM geodesics do map to every boundary component in the T-orbit decomposition. We define a family of *toric curves* (Definition 6.4) corresponding to every EDM geodesic. These families of curves reach every point in the toroidal boundary. We define an equivalence relation on *projected toric curves* whose equivalence classes are in one to one correspondence with the toroidal boundary. To do this, we define a series of equivalence relations.

The Σ-*relation* (Definition 6.11) is defined on toric curves that lie in the same maximal flat. The t-*relation* (Definition 6.15) is defined on Σ-equivalence classes that lie in different maximal flats. The N'-*relation* (Definition 6.20) is defined on t-equivalence classes that correspond to the same cone σ in the polyhedral decomposition Σ. The final relation, the V-*relation* (Definition 6.25) is defined on N'-equivalence classes that correspond to different cones.

6.1 Toric curves

In this section, we define the toric curves associated to an EDM lift $\tilde{\gamma}$. To do this, we first define an action of the compact torus $\mathbb{T}^n = (\mathbb{S}^1)^n$ on the tangent bundle $\mathrm{T}(\mathfrak{H}^n)$. We apply this action to the set of Killing vector fields of $\mathfrak{H}^n$ to obtain a new family of vector fields. The toric curves associated to $\tilde{\gamma}$ are the integral curves of particular vector fields in this new family.

The motivation for defining a torus action on $\mathrm{T}(\mathfrak{H}^n)$ came from the lecture notes that accompanied the 2004 Park City short course given by Robert MacPherson. For a compact torus $\mathbb{T}^n$, we have the following definition:

Definition 6.1. [Mac07, 1.5] A *space with a torus action* is a (Hausdorff) topological space M together with a self map $M \longrightarrow M$ for every element $t \in \mathbb{T}^n$, notated $m \mapsto t \cdot m$, such that composition of homeomorphisms corresponds to multiplication in the group, $t_1(t_2 \cdot m) = (t_1 \cdot t_2) \cdot m$, and $(t, m) \mapsto t \cdot m$ is jointly continuous in m and t.

The tangent bundle $\mathrm{T}(\mathfrak{H}^n)$ of $\mathfrak{H}^n$ is a Hausdorff topological space. By definition, $\mathrm{T}(\mathfrak{H}^n)$ is the disjoint union of the tangent spaces $\mathrm{T}_{gK}(\mathfrak{H}^n)$ over all points $gK \in \mathfrak{H}^n$. The tangent space $\mathrm{T}_{gK}(\mathfrak{H}^n)$ is equal to the product of the tangent spaces $\mathrm{T}_{g_1K_1}(\mathfrak{H}) \times \ldots \times \mathrm{T}_{g_nK_n}(\mathfrak{H})$ in each factor of $\mathfrak{H}^n$.

We define an action of $\mathbb{T}^n$ on $\mathrm{T}(\mathfrak{H}^n)$ by describing how $\mathbb{T}^n$ acts on each tangent space. For a point-vector pair $(gK, \vec{v}) \in \mathrm{T}_{gK}(\mathfrak{H}^n)$ and $(e^{i\theta_1}, \ldots, e^{i\theta_n}) \in \mathbb{T}^n$, the torus action

$$(e^{i\theta_1}, \ldots, e^{i\theta_n}) \cdot (gK, \vec{v}) = (gK, e^{i\theta_j} \cdot \vec{v}_j) \tag{6.1.1}$$

fixes the point gK and rotates the vector $\vec{v}_j$ in the jth factor of $\mathrm{T}_{g_1K_1}(\mathfrak{H}) \times \ldots \times \mathrm{T}_{g_nK_n}(\mathfrak{H})$ by an angle of θ_j.

Let $\mathfrak{X}(\mathfrak{H}^n)$ denote the set of vector fields on $\mathfrak{H}^n$. A vector field $X \in \mathfrak{X}(\mathfrak{H}^n)$ is a section of the tangent bundle $\mathrm{T}(\mathfrak{H}^n)$. We now describe how the torus action on $\mathrm{T}(\mathfrak{H}^n)$ restricts to act on an element in $\mathfrak{X}(\mathfrak{H}^n)$. Any vector field $X \in \mathfrak{X}(\mathfrak{H}^n)$ takes the form

$$X = \sum_{j=1}^{n} f_j \frac{\partial}{\partial x_j} + g_j \frac{\partial}{\partial y_j} \tag{6.1.2}$$

where f_j, g_j are functions in the coordinates (x_j, y_j) and $\partial/\partial x_j, \partial/\partial y_j$ are standard basis elements of the jth factor of the tangent bundle. Let X_j denote the vector field in the jth factor. For $(e^{i\theta_j}) \in \mathbb{T}^n$ and a vector field $X = \sum X_j$, we have

$$(e^{i\theta_j}) \cdot X = \sum (f_j \cos\theta_j - g_j \sin\theta_j) \frac{\partial}{\partial x} + (f_j \sin\theta_j + g_j \cos\theta_j) \frac{\partial}{\partial y}. \tag{6.1.3}$$

As previously mentioned, EDM geodesics do not map to every point in the toroidal boundary. EDM geodesics are projections of geodesics in $\mathfrak{H}^n$. Geodesics in any symmetric space can be thought of as integral curves of a class of vector fields closely related to the group of isometries on the space. We define this class of vector fields.

Definition 6.2. (cf. [Ebe96, 2.1]) A vector field X on a symmetric space is said to be a *Killing vector field* if its flow transformations are isometries of the symmetric space.

We may identify the set of Killing vector fields of a symmetric space with the set of left invariant vector fields on the connected component of the isometry group of the symmetric space.

To calculate the Killing vector fields on $\mathfrak{H}^n$ it is enough to calculate the Killing vector fields on each $\mathfrak{H}$ factor. One can easily compute the three generators

$$\left\{ (x^2 - y^2) \frac{\partial}{\partial x} + 2xy \frac{\partial}{\partial y},\ x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y},\ \frac{\partial}{\partial x} \right\}$$

of the Lie algebra of Killing vectors on $\mathfrak{H}$. Any Killing vector field on $\mathfrak{H}$ is an $\mathbb{R}$-linear combination of these generators. Thus, the Killing vector fields of $\mathfrak{H}$ take the form

$$X = (a(x^2 - y^2) + bx + c)\frac{\partial}{\partial x} + ((2ax + b)y)\frac{\partial}{\partial y} \tag{6.1.4}$$

with $a, b, c \in \mathbb{R}$ arbitrary constants. We can find the integral curves of the Killing vector fields. Following [Lov05], when $a \neq 0$ the integral curves of X take the form

$$\begin{aligned} z(t) &= -\tfrac{k}{a}\tfrac{\alpha \cosh kt - \sinh kt}{\alpha \sinh kt - \cosh kt} - \tfrac{b}{2a} && \text{if } \tfrac{b^2}{4} - ac > 0, \\ z(t) &= -\tfrac{k}{a}\tfrac{\alpha \cos kt + \sin kt}{\alpha \sin kt - \cos kt} - \tfrac{b}{2a} && \text{if } \tfrac{b^2}{4} - ac < 0, \\ z(t) &= -\tfrac{1}{at+\alpha} - \tfrac{b}{2a} && \text{if } \tfrac{b^2}{4} - ac = 0. \end{aligned}$$

where $k = \sqrt{|b^2/4 - ac|}$ and α is an element in $\mathfrak{H}$. When $a = 0$ and $b \neq 0$ the integral curves take the form $z(t) = x(t) + iy(t)$ where

$$\begin{aligned} x(t) &= (x + \tfrac{c}{b})e^{bt} - \tfrac{c}{b} \\ y(t) &= ye^{bt} \end{aligned}$$

for a fixed $z = x + iy$ in $\mathfrak{H}$. For example, when $a = c = 0$ and $b \neq 0$ the integral curves of x are those representing dilation from the origin. These are also the curves that stay at a fixed distance from the vertical line geodesic ie^t. When $a = b = 0$ and $c \neq 0$ the integral curves of x are horocycles given by lines with constant imaginary part.

It turns out that even the projections of integral curves of Killing vector fields will not map to every point in the toroidal boundary. So we consider integral curves of vector fields obtained by the $\mathbb{T}^n$-action on the set of Killing vector fields. We call this the set of *toric Killing vector fields*. Projections of integral curves of toric Killing vector fields map to every point on the toroidal boundary. This set of curves is very large. We put some geometric conditions on the integral curves of toric Killing vector field with respect to an EDM lift $\tilde{\gamma}$. These conditions will define a reasonable set of toric curves corresponding to $\tilde{\gamma}$.

Each EDM lift $\tilde{\gamma}$ is contained in a unique maximal flat of $\mathfrak{H}^n$. For example, the vertical line geodesics are contained in the subsets of $\mathfrak{H}^n$ with fixed real entries. We let $nAx_0 = n \cdot (\exp \mathfrak{a})x_0$ denote the maximal flat containing $\tilde{\gamma}$. For a lift $\tilde{\gamma}$, we are interested in toric Killing vector fields X that satisfy the following properties:

1. When restricted to nAx_0, X has integral curves all of which stay in nAx_0.

2. All of the integral curves in nAx_0 are N-related to $\tilde{\gamma}$.

We introduced the N-relation (Definition 2.18) for geodesics, but it makes sense to define this distance relation on arbitrary curves.

As an example, consider the EDM lift $\tilde{\gamma}(t) = (ie^{t/\sqrt{n}}, \ldots, ie^{t/\sqrt{n}})$ contained in the maximal flat $nAx_0 = \{z \in \mathfrak{H}^n \mid \mathfrak{Re}(z_j) = 0\}$. We identify the toric Killing vector fields that satisfy the two properties above. When we restrict a Killing vector field to nAx_0 the fields take the form

$$X = \sum (c_j - a_j y_j^2)\frac{\partial}{\partial x_j} + b_j y_j \frac{\partial}{\partial y_j}.$$

The toric Killing vector fields take the form

$$\sum ((c_j - a_j y_j^2)\cos\theta_j - b_j y_j \sin\theta_j)\frac{\partial}{\partial x_j} + ((c_j - a_j y_j^2)\sin\theta_j + b_j y_j \cos\theta_j)\frac{\partial}{\partial y_j}.$$

For the integral curves of such a toric Killing vector field to be contained in nAx_0, the component of the $\frac{\partial}{\partial x_j}$ must vanish. That is we have, $(c_j - a_j y_j^2)\cos\theta_j - b_j y_j \sin\theta_j = 0$ for all y_j. If $\cos\theta_j \sin\theta_j \neq 0$ this only occurs when $a_j = b_j = c_j = 0$. If $\cos\theta_j = 0$ then $b_j = 0$ and if $\sin\theta_j = 0$ then $a_j = c_j = 0$. The resulting vector fields are

$$(c_j - a_j y_j^2)\frac{\partial}{\partial y_j} \text{ and } b_j y_j \frac{\partial}{\partial y_j} \tag{6.1.5}$$

respectively. So we have determined the vector fields that satisfy condition 1. We now find the vector fields in (6.1.5) whose integral curves are N-related to $\tilde{\gamma}$. We must have $\cos\theta_j = 0$ and $a_j = 0$ for all j. Also, we have $c_i \sin\theta_i = c_j \sin\theta_j \neq 0$, for all $i \neq j$. The curves we described carve out the appropriate paths, but there are still too many of them. We have described a set of curves with no regard to parametrization. In the setting of EDM geodesics this problem was resolved by considering only unit speed geodesics. Analogously, we consider only *normalized* toric curves, i.e. those satisfying $c_i \sin\theta_i = c_j \sin\theta_j = 1$.

Consider another EDM lift $\tilde{\gamma}$ in the maximal flat $nAx_0 = \{z \in \mathfrak{H}^n \mid \mathfrak{Re}(z_j) = 0\}$. The vector fields whose integral curves satisfy properties (1) and (2) take a similar form, only with $c_i \sin\theta_i = \lambda c_j \sin\theta_j$ for some positive constant λ that depends on the slope of $\tilde{\gamma}$ in nAx_0. We have the following lemma:

Lemma 6.3. Consider an EDM lift $\tilde{\gamma} \in \mathfrak{H}^n$ contained in its unique maximal flat nAx_0. There is a unique toric Killing vector field $X_{\tilde{\gamma}}$ of $\mathfrak{H}^n$ (up to parametrization of the c_j and sign of $\sin\theta_j$) that satisfy the properties (1) and (2) above.

Proof. This result was proved for the EDM lift $\tilde{\gamma}(t) = (ie^{t/\sqrt{n}}, \ldots, ie^{t/\sqrt{n}})$ in the previous paragraphs. We can translate any EDM lift to this canonical lift by an isometry and argue analogously. □

We use the previous lemma to define the set of toric curves corresponding to $\tilde{\gamma}$.

Definition 6.4. For an EDM lift $\tilde{\gamma} \in \mathfrak{H}^n$ the *toric curves* $\tilde{c}(t)$ associated to $\tilde{\gamma}$ are the integral curves of $X_{\tilde{\gamma}}$ defined in the previous lemma.

Let π denote the projection map $\pi : \mathfrak{H}^n \to \Gamma\backslash\mathfrak{H}^n$. For some toric curve $\tilde{c}(t)$, a *projected toric curve* takes the form $c(t) = \pi(\tilde{c}(t))$. We consider these curves to be the analogue of EDM geodesics in the context of toroidal compactifications. This is because of the following:

Theorem 6.5. Projected toric curves $c(t)$ map to every point in the toroidal boundary $\partial(\overline{\Gamma\backslash\mathfrak{H}^n}_{\Sigma}^{tor})$.

Proof. This follows from the decription of how unbounded sequences in the locally symmetric space converge to the toroidal boundary. For a boundary component $O(\sigma)$, the EDM lifts that converge to $O(\sigma)$ are exactly those that are rays emanating from the origin the are in the relative interior of σ and their translates by the unipotent radical N_P. The toric curves that converge to points in $O(\sigma)$ are "translates" of the lifts described in the previous sentence. For a point $z_\infty \in O(\sigma)$, convergence to this point depends on three things. The unbounded nature of certain functionals (which any toric curve or EDM lift converging to the component $O(\sigma)$ will satisfy), the boundedness of other linear functionals (which all toric curves satisfy since the functionals evaulate to a constant based on the translated nature of the curve), and the convergence to z_∞ in the projection to $O(\sigma)$ (which an infinite family of toric curves will satisfy). The toric curves that satisfy the last condition include any whose constant real x_j contributions to z_j (arising from the unipotent radical) projects to the appropriate $(x_j)_\infty$ in the decomposition $z_\infty = (z_j)_\infty = (x_j)_\infty + i(y_j)_\infty$, and whose unbounded y_j contributions project to $(y_j)_\infty$. This second projection of the imaginary part is entirely dependent on the bounded linear functionals. Since these functions evaluate to constants an infinite family of such toroidal curves in each maximal flat can be found that satisfy this property. Therefore, any point z_∞ in the toroidal boundary is the limit point of some toric curve. □

6.2 Σ-relation

We have an appropriate class of curves that map to every point in the toroidal boundary. From the linear functionals involved in the description of convergent sequences, it is clear that several equivalence relations are needed to find a one to one correspondence between certain classes of toric curves and toroidal boundary points. We define an equivalence relation on toric curves $\tilde{c}(t)$ that are contained in the same maximal flat. We use definitions from convex and metric geometry (cf. [BGS85, §1]) to define the relation. We will use this relation in the next subsection to define the t-relation on projected toric curves.

Let X be a Riemannian manifold of nonpositive curvature. Symmetric spaces of noncompact type are examples of such manifolds.

Definition 6.6. A subset W of X is called *convex*, if for $p, q \in W$ there is (up to parametrization) a unique shortest geodesic from p to q in V and this geodesic is contained in W.

Any flat in $\mathfrak{H}^n$, maximal or otherwise, is a convex subset of $\mathfrak{H}^n$. In particular, for a maximal flat $nAx_0 = n\cdot(\exp \mathfrak{a})x_0$ containing the maximal rational flat $n\cdot(\exp \mathfrak{a}_{\mathbf{Q}})x_0$, the codimension one flat $n \cdot (\exp \mathfrak{a}_{\mathbf{Q}}^{\perp})x_0 \subset nAx_0$ is a convex subset.

Definition 6.7. Let $W \subset X$ be a convex subset. For a subset $S^* \subset W$, the *convex hull* $ch(S^*)$ of S^* is the smallest convex subset of W which contains S^*.

We consider finite sets of points S^* in $n \cdot (\exp \mathfrak{a}_{\mathbf{Q}}^{\perp})x_0$ and their convex hulls $ch(S^*)$.

Definition 6.8. Let $W \subset X$ be a convex subset and $K \subset W$ be a compact subset. For $p \in W$ let $r(p)$ be the radius of the minimal ball centered at p that contains K. Then there is a unique $p \in W$ so that $r(p)$ is minimal. This point p is contained in $\overline{ch}(K)$ and is called the *center* of K.

For our relation, we consider compact subsets K that are the convex hulls $ch(S^*)$ of sets S^* with a finite number of points contained in the convex subset $n\cdot(\exp\mathfrak{a}_{\mathbf{Q}}^{\perp})x_0$.

Definition 6.9. Let $W \subset X$ be a convex subset, $W_0 \subset W$ a closed convex subset. For any point $p \in W$ there is a unique point $\phi_{W_0}(p) \in W_0$ of minimal distance to p. This point $\pi_{W_0}(p) \in W_0$ is called the *footpoint* of p on W_0.

We consider sets in $n\cdot(\exp\mathfrak{a}_{\mathbf{Q}}^{\perp})x_0$ whose footpoint is the center of some convex hull $ch(S^*)$.

For any full dimensional cone σ, there is a set of lifts $\{\tilde{\gamma}_1, \dots, \tilde{\gamma}_n\}$ that correspond to the one dimensional faces of σ. Let S be a subset of $\{\tilde{\gamma}_1, \dots, \tilde{\gamma}_n\}$. Consider the set S^* of intersection points $S \cap n \cdot (\exp\mathfrak{a}_{\mathbf{P}}^{\perp})x_0$. Let $ch(S^*)$ denote the convex hull the point set S^* in $n \cdot (\exp\mathfrak{a}_{\mathbf{P}}^{\perp})x_0$. Let $ch(S^*)^{\circ}$ denote the relative interior of the convex hull. We define an equivalence relation on toric curves $\tilde{c}_1, \tilde{c}_2$ associated to EDM lifts $\tilde{\gamma}_1, \tilde{\gamma}_2$ where the intersection points $\tilde{\gamma}_i \cap n \cdot (\exp\mathfrak{a}_{\mathbf{P}}^{\perp})x_0$ are contained in $ch(S^*)^{\circ}$.

Let x be the center of $ch(S^*)$ Consider the set

$$L_x = \{x' \in n \cdot (\exp\mathfrak{a}_{\mathbf{P}}^{\perp})x_0 \mid \phi_{ch(S^*)}(x') = x\}.$$

Lemma 6.10. L_x is a closed convex subset of $n \cdot (\exp\mathfrak{a}_{\mathbf{P}}^{\perp})x_0$.

Proof. Since $n \cdot (\exp\mathfrak{a}_{\mathbf{P}}^{\perp})x_0$ is isometric to a copy of $\mathbb{R}^{n-1}$ with the flat Euclidean metric, we know $ch(S^*)$ is isometric to a closed convex set in some subspace $\mathbb{R}^k \subset \mathbb{R}^{n-1}$ where $k = |S^*|-1$ and L_x is isometric to the perpendicular subspace in $\mathbb{R}^{n-1}$. This perpendicular subspace is a closed convex set, thus L_x is a closed convex set. □

By the previous lemma, we can define footpoints associated to projections on L_x. We use these footpoints in the following main definition:

Definition 6.11. Let $\tilde{\gamma}_1, \tilde{\gamma}_2$ be two EDM lifts whose intersection with $n\cdot(\exp \mathfrak{a}_{\mathbf{P}}^{\perp})x_0$ is contained in the relative interior $ch(S^*)^\circ$. Let $\tilde{c}_1, \tilde{c}_2$ be toric curves associated to $\tilde{\gamma}_1, \tilde{\gamma}_2$ and let $x_{(\tilde{c}_i)}$ denote the intersection points $\tilde{c}_i \cap \mathfrak{n} \cdot (\exp \mathfrak{a}_{\mathbf{P}}^{\perp})x_0$. We call $\tilde{c}_1, \tilde{c}_2$ are called Σ-*related* if $\phi_{L_x}(x_{(\tilde{c}_1)}) = \phi_{L_x}(x_{(\tilde{c}_2)})$.

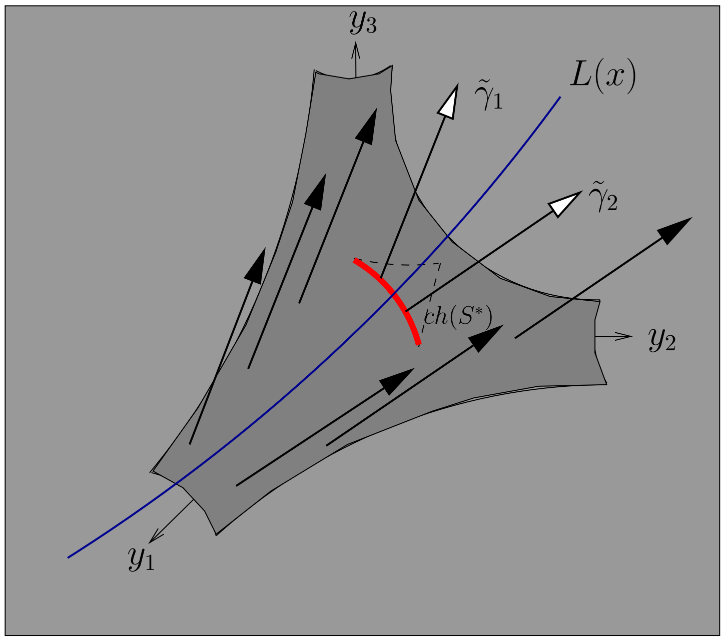

Figure 6: Σ-relation for a Hilbert modular threefold

Figure 6 gives examples of two EDM lifts $\tilde{\gamma}_i$, $i = 1, 2$ and their associated toric curves. The three curves on the left of $L(x)$ correspond to $\tilde{\gamma}_1$ and the three curves on the right of $L(x)$ correspond to $\tilde{\gamma}_2$. In this example, the toric the lifts $\tilde{\gamma}_i$ are Σ-related. In the figure, the left most toric curve associated to $\tilde{\gamma}_1$ and the left most toric curve associated to $\tilde{\gamma}_2$ are Σ-related, however, the right most toric curve associated to $\tilde{\gamma}_1$ and the right most toric curve associated to $\tilde{\gamma}_2$ are not Σ-related.

Lemma 6.12. The Σ-relation is an equivalence relation on toric curves.

Proof. The equality in the previous definition makes this clear. □

Unlike the equivalence relations encountered in other chapters, we do not use the horospherical decomposition to show the Σ-relation is an equivalence relation. This is because toric curves associated to an EDM lift do not have a "nice" description in terms of the horospherical decomposition. They do have a simple description when one considers an embedding of the symmetric space into a complex vector space. Toric curves are then visualized as (real) one dimensional rays. This was described previously in Lemma 5.7.

Recall that the boundary component $O(\sigma)$ corresponding to a face σ has the following quotient decomposition

$$O(\sigma) \simeq (\mathfrak{Re}(\mathbb{C}^n)/\mathfrak{Re}(\mathrm{Span}_{\mathbb{C}}(\sigma)))/N' \times i(\mathfrak{Im}(\mathbb{C}^n)/\mathfrak{Im}(\mathrm{Span}_{\mathbb{C}}(\sigma))).$$

The Σ-relation is defined on toric curves in the same maximal flat. In the above description of $O(\sigma)$ we think of this as fixing the $\mathfrak{Re}(\mathbb{C}^n)$ component.

Lemma 6.13. The set of Σ-related equivalence classes of toric curves corresponds bijectively to the set $\mathfrak{Re}(\mathbb{C}^n) + i(\mathfrak{Im}(\mathbb{C}^n)/\mathfrak{Im}(\mathrm{Span}_{\mathbb{C}}(\sigma)))$ through the map $\tilde{c}(t) \to \lim_{t\to\infty} \tilde{c}(t)$.

Proof. Two toric curves $\tilde{c}_i$ being Σ-related implies that they are in the same maximal flat and furthermore, they are "translates" of EDM lifts $\tilde{\gamma}_i$ that are on the relative interior of some facet σ. These conditions alone guarantee that the defining linear functionals $l_1, \ldots, l_k$ will diverge and the "real coordinate" in the projection to $O(\sigma)$ will converge, as $t \to \infty$. The equality of the projection ϕ onto L_x guarantees that the remaining functionals $l_{p+1}, \ldots, l_k$ stay bounded. In fact, since the toric curves are "translates" of EDM geodesics the functionals evaluate to constants. □

Remark 6.14. Whe we refer to "translates" in the previous paragraph, we mean linear translations of rays emanating from the origin with respect to the Euclidean metric on the space $N \otimes \mathbb{R}$.

The relations in the next sections will correspond to the quotient of $\mathfrak{Re}(\mathbb{C}^n)$ by $\mathfrak{Re}(\mathrm{Span}_{\mathbb{C}}(\sigma))$, the quotient of $\mathfrak{Re}(\mathbb{C}^n)/\mathfrak{Re}(\mathrm{Span}_{\mathbb{C}}(\sigma))$ by N', and the quotient of the boundary components $O(\sigma)$ by Γ'_P/Γ_{N_P}. These last two identifications can be thought of when $n = 2$ as forming the infinite chains of $\mathbb{P}^1$'s at the boundary and identifying these chains to give finite chains, respectively.

6.3 t-Relation

We define an equivalence relation on Σ-equivalence classes of toric curves in different maximal flats. All of these maximal flats are translates of each other by elements of the nilpotent radical.

For any toric curve $\tilde{c}$ corresponding to a particular cusp, we have an associated pair (σ, n) where σ is a face in the cone decomposition Σ corresponding to the parabolic subgroup Q and n is an element of N_Q. To be explicit, let $\tilde{c}_i,\ i = 1, 2$ be two toric curves that are associated to the same face σ but different elements $n_i \in N_Q$.

Definition 6.15. Two Σ-equivalence classes $[\tilde{c}_i]_\Sigma,\ i = 1, 2$ are *t-related* if there exist representatives $\tilde{c}_i$ such that $proj(n_1) = proj(n_2) \in \mathfrak{Re}(\mathrm{Span}_{\mathbb{C}}(\sigma))$ where $\tilde{c}_i \in n_i A x_0$.

Lemma 6.16. The t-relation is an equivalence relation on Σ-equivalence classes.

Proof. The equality in the previous definition makes this clear. □

We now define the t-relation on toric curves.

Definition 6.17. Two toric curves $\tilde{c}_1, \tilde{c}_2$ are *t-related* if their Σ-equivalence classes $[\tilde{c}_i]_\Sigma$ are t-related.

Lemma 6.18. The t-relation is an equivalence relation on toric curves.

Proof. This follows from Lemma 6.16. □

Lemma 6.19. The set of t-related equivalence classes of toric curves corresponds bijectively to the set $\mathfrak{Re}(\mathbb{C}^n)/\mathfrak{Re}(\mathrm{Span}_\mathbb{C}(\sigma)) + i(\mathfrak{Im}(\mathbb{C}^n)/\mathfrak{Im}(\mathrm{Span}_\mathbb{C}(\sigma)))$ through the map $\tilde{c}(t) \to \lim_{t\to\infty} \tilde{c}(t)$.

Proof. The proof is clear based on the definition of the boundary component as a quotient. □

6.4 N'-Relation

The following relation relies only on the projection map $\pi : X \to \Gamma \backslash X$.

Definition 6.20. Two t-equivalence classes $[\tilde{c}_i]_t$, $i = 1, 2$ corresponding to the same cone σ are *N'-related* if there exist representatives $\tilde{c}_i$ so that $\pi(\tilde{c}_1) = \pi(\tilde{c}_2)$.

Lemma 6.21. The N'-relation is an equivalence relation on t-equivalence classes.

Proof. The equality in the previous definition makes this clear. □

We now define the N'-relation on toric curves.

Definition 6.22. Two toric curves $\tilde{c}_1, \tilde{c}_2$ are *N'-related* if their t-equivalence classes $[\tilde{c}_i]_t$ are N'-related.

Lemma 6.23. The N'-relation is an equivalence relation on toric curves.

Proof. The follows from the previous lemma. $\square$

Lemma 6.24. The set of N'-related equivalence classes of toric curves corresponds bijectively to the set $O(\sigma) = \mathfrak{Re}(\mathbb{C}^n)/\mathfrak{Re}(\mathrm{Span}_{\mathbb{C}}(\sigma))/N' \times i(\mathfrak{Im}(\mathbb{C}^n)/\mathfrak{Im}(\mathrm{Span}_{\mathbb{C}}(\sigma)))$ through the map $\tilde{c}(t) \to \lim_{t\to\infty} \tilde{c}(t)$.

Proof. The N'-relation is an equivalence relation on t-equivalence classes corrresponding to the same facet σ. It is clear that two toric curves that are Γ_{U_Q} translates of each other project to the same curve in the quotient. On the other hand two toric curves that project to the same curve must be Γ_{U_Q} translates. If the translational element projects nontrivially to the quotient Γ_Q/Γ_{U_Q}, then the element arises from the group of units which does not preserve facets. Therefore, two toric curves associated to the same facet project to the same curve if and only if they are Γ_{U_Q} translates. Some toric curves that have this property have been identified by the t-relation, namely those toic curves that are translates by an element of Γ_{U_Q} corresponding to a point in $\mathfrak{Re}(\mathrm{Span}_{\mathbb{C}}(\sigma))$. The remaining identifications correspond to elements in Γ_{U_Q} corresponding to lattice points in N'. $\square$

The relation we defined has equivalence classes that are in one to one correspondence with what can be thought of as a partial toroidal compactification. However, the original toroidal construction was not done in the same manner as the new viewpoint of Borel–Ji. Comparing these two approaches is a topic of future interest.

6.5 V-Relation

This last relation identifies certain infinite families of varieties in the boundary components and also identifies entire compact boundary components with one another. This identification is done simultaneously, where the identification inside particular boundary components arises elements of Γ_Q that correspond to the group of units $\mathcal{O}_k$ and the identification of entire boundary components correspond to parabolic subgroups that have the same Γ conjugate representative.

Definition 6.25. Two N'-equivalence classes $[\tilde{c}_i]_t,\ i = 1, 2$ corresponding to different cones σ_i are *V-related* if there exist representatives $\tilde{c}_i$ so that $\pi(\tilde{c}_1) = \pi(\tilde{c}_2)$.

Lemma 6.26. The V-relation is an equivalence relation on N'-equivalence classes.

Proof. This quickly follows from the equality in the previous definition. □

We now define the V-relation on toric curves.

Definition 6.27. Two toric curves $\tilde{c}_1, \tilde{c}_2$ are *V-related* if their N'-equivalence classes $[\tilde{c}_i]_{N'}$ are V-related.

Lemma 6.28. The V-relation is an equivalence relation on toric curves.

Proof. This follows from the previous lemma. □

We have our final theorem:

Theorem 6.29. The set of V-equivalence classes of toric curves corresponds bijectively to $\partial(\overline{\Gamma\backslash\mathfrak{H}^n}^{tor}_{\Sigma})$ through the map $c(t) \mapsto \lim_{t\to\infty} c(t)$.

Proof. There are two ways that representative toric curves corresponding to different facets can project to the same curve in the locally symmetric quotient. On the one hand, if the isometry preserves the cusp neighborhood, then it corresponds to an element of the group of units. On the other hand, it will correspond to an element in Γ that maps one cusp to another. These cusps are said to be in the same Γ conjugacy class. The N'-equivalence classes are in bijective correspondence with infinite union of boundary components $O(\sigma)$. As a set, the toroidal boundary is defined as the union $\coprod_{i=1}^{m} \coprod_{j=1}^{n} V_i \backslash O(\sigma_{i,j})$ where V_i are the elements arising from the ring of units in the split component A_{P_i} of a parabolic P_i representing a Γ-conjugacy class. The $\sigma_{i,j}$ are a finite collection corresponding to each Γ-conjugacy class. The finite number of associated boundary components $O(\sigma_{i,j})$ have identifications only under the action of V_i. Therefore two toric curves whose boundary points are identified under the V_i action must also be identified. This only happens when representatives are V_i conjugate. Therefore the lemma is proved. □

Remark 6.30. The construction of the toroidal compactification is fundamentally different from that of the other compactifications mentioned. In future work with Lizhen Ji, we hope to construct a compactification isomorphic to a toroidal compactification by attaching boundary components at infinity and quotienting by Γ all at once. As originally defined one must consider two levels of quotienting. In the case of Hilbert modular varieties, first with the groups Γ_{N_P} and then with the group $\Gamma_P/\Gamma_{N_P} \simeq V$.

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