Bayesian Model Selection and Parameter Estimation for Gravitational Wave Signals from Binary Black Hole Coalescences

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BAYESIAN MODEL SELECTION AND PARAMETER ESTIMATION FOR GRAVITATIONAL WAVE SIGNALS FROM BINARY BLACK HOLE COALESCENCES

A Thesis Presented by

ALEXANDER L. LOMBARDI

Submitted to the Graduate School of the University of Massachusetts Amherst in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

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Physics
BAYESIAN MODEL SELECTION AND PARAMETER ESTIMATION FOR GRAVITATIONAL WAVE SIGNALS FROM BINARY BLACK HOLE COALESCENCES

A Thesis Presented
by
ALEXANDER L. LOMBARDI

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DEDICATION

In loving memory of

Lisa Robichaud

The epitome of living life to the fullest. Thank you for always inspiring me to get the most out of every experience.
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I must start by thanking my amazing advisor, Laura Cadonati. From our first meeting in my freshman year, she has inspired me as a scientist and as a person. I think she might be a superhero. Thank you for the enumerable opportunities and for all of your support over the last seven years.

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Finally, I want to thank all of my family and friends, especially mom and Josh for your endless, lifelong support. And thank you Amanda, for holding my hand through it all.
In his theory of General Relativity, Einstein describes gravity as a geometric property of spacetime, which deforms in the presence of mass and energy. The accelerated motion of masses produces deformations, which propagate outward from their source at the speed of light. We refer to these radiated deformations as gravitational waves. Over the past several decades, the goal of the Laser Interferometer Gravitational-wave Observatory (LIGO) has been the search for direct evidence of gravitational waves from astrophysical sources, using ground based laser interferometers. As LIGO moves into its Advanced era (aLIGO), the direct detection of gravitational waves is inevitable. With the technology at hand, it is imperative that we have the tools to analyze the detector signal and examine the interesting astrophysical properties of the source. Some of the main targets of this search are coalescing compact binaries. In this thesis, I describe and evaluate bhextractor, a data analysis algorithm that uses Principal Component Analysis (PCA) to identify the main features of a set of
gravitational waveforms produced by the coalescence of two black holes. Binary Black Hole (BBH) systems are expected to be among the most common sources of gravitational waves in the sensitivity band of aLIGO. However, the gravitational waveforms emitted by BBH systems are not well modeled and require computationally expensive Numerical Relativity (NR) simulations. bhextractor uses PCA to decompose a catalog of available NR waveforms into a set of orthogonal Principal Components (PCs), which efficiently select the major common features of the waveforms in the catalog and represent a portion of the BBH parameter space. From these PCs, we can reconstruct any waveform in the catalog, and construct new waveforms with similar properties. Using Bayesian analysis and Nested Sampling, one can use bhextractor to classify an arbitrary BBH waveform into one of the available catalogs and estimate the parameters of the gravitational wave source.
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CHAPTER 1
INTRODUCTION

In 1687, Sir Isaac Newton published his *Philosophiæ Naturalis Principia Mathematica*. This work became the foundation of natural philosophy in the western world for the duration of the eighteenth and nineteenth centuries, and the laws therein are still regarded as some of the most important fundamental principles of physics. Among the great achievements in the *Principia* is Newton’s law of universal gravitation. The law states that there is an attractive force between every massive object in the universe and every other massive object that is proportional to the product of the masses of the two objects and inversely proportional to the square of the distance between them. In its modern form, this may be represented mathematically by:

\[ F = G \frac{m_1 m_2}{r^2} \]

(1.1)

where \( G \approx 6.673 \times 10^{-11} \) m³ kg⁻¹ s⁻² is the universal gravitational constant, measured empirically. While this theory remains an excellent approximation in the measurement of gravitational effects between bodies, we know that this is not the whole story. Newton’s theory was successful in predicting the existence of Neptune in our solar system, but was unable to explain the anomalous perihelion precession of Mercury. Furthermore, the theory required instantaneous action at a distance, inconsistent with Albert Einstein’s special theory of relativity, which states that nothing can travel faster than the speed of light. Einstein was able to resolve these inconsistencies in his 1916 publication of the theory of General Relativity, which incorporated gravity into his relativistic model of the universe. The theory of General Relativity
describes gravity as a geometric property of spacetime, which becomes curved due to the presence of mass and energy. General Relativity was able to unify Newtonian gravity with the precepts of special relativity, and account for the discrepancies where Newton’s theory fell short.

One of the most important predictions made by the theory of General Relativity is the existence of gravitational waves. Gravitational waves are perturbations in spacetime generated by the accelerated motion of masses. These waves propagate out from their source at the speed of light. General relativity also predicts that these perturbations will be weak; so weak that Einstein believed they would never be detectable [2]. The first indirect evidence of gravitational waves came from the observation of the decay of the orbital period of PSR 1913+16, a pulsar in a binary neutron star system (see figure 1.1). The pulsar was discovered in 1974 by Russell Hulse and Joseph Taylor at the University of Massachusetts Amherst. For this work, they received a Nobel Prize in 1993. Their observations showed compelling evidence that the binary system was losing energy due to the emission of gravitational waves. The rate of this period shift strongly matched that which was predicted by general relativity, thus indirectly demonstrating the existence of gravitational waves.

The main goal in the past several decades has been the direct detection of gravitational waves. From its inception, the field of experimental astrophysics has been limited to the electromagnetic spectrum. Direct detection of gravitational waves will illuminate properties of our universe which are impossible to see on the electromagnetic spectrum. Leading developments in this effort is the Laser Interferometer Gravitational-Wave Observatory (LIGO) and its collaborators. This chapter serves an introduction to gravitational waves and the LIGO detectors.
Figure 1.1: The decay of the orbital period of the Hulse-Taylor pulsar from 1975 to 2003 due to the emission of gravitational waves; note that the observations agree exceptionally well with the predictions of general relativity [14].
1.1 Introduction to Gravitational Waves

Gravitational waves are a direct product of Einstein’s theory of gravitation. Just as Maxwell’s equations in electrodynamics predicted wave solutions, so do Einstein’s field equations from his theory of General Relativity. With this picture in mind, Newton’s instantaneous action at a distance model of gravity is no longer satisfactory. Rather, we must envision matter and energy as existing in a four-dimensional fabric of spacetime, which deforms in the presence of mass. Unlike the prediction made by the Newtonian model, these effects are not necessarily localized to a massive object. As objects move through spacetime, these deformations propagate from their source as gravitational waves. Like electromagnetic waves, this information about the curvature of spacetime travels at the speed of light. The fundamental concepts that govern gravitational wave production follow directly from Einstein’s field equations. The Einstein field equations are:

\[
G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{1.2}
\]

where \( T_{\mu\nu} \) is the four-dimensional stress energy tensor, which represents the mass and energy in a given spacetime, and \( g_{\mu\nu} \) and \( R_{\mu\nu} \) are four dimensional metric tensors which describe the particular arrangement of mass and energy. In the weak field approximation, we can ignore the nonlinear contributions to the spacetime metric \( g_{\mu\nu} \). This approximation is often called the linearized theory of gravitation.

In special relativity, space and time are joined to create a four-dimensional coordinate system known as spacetime, in which we must consider a new concept of distance called the spacetime interval \( ds \). This interval is invariant; it is measured to be the same by all observers. The spacetime interval between two neighboring points is defined as:

\[
ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2 = \sum_{\mu,\nu=0}^{3} \eta_{\mu\nu}dx^\mu dx^\nu \tag{1.3}
\]
where the indices $\mu$ and $\nu$ range from 0 to 3 to represent the time and space coordinates $t$, $x$, $y$, and $z$, respectively. $\eta_{\mu\nu}$ is the Minkowski metric. In Cartesian coordinates, this is:

\[
\eta_{\mu\nu} = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

(1.4)

The Minkowski metric represents a flat spacetime, but this is an incomplete picture. We know that spacetime, in general, may be curved. For the systems that we expect to be able to detect in aLIGO, this curvature is a very small perturbation to Minkowski space. The more spacetime metric $g_{\mu\nu}$ can then be treated as the sum of this flat background and a small perturbation $h_{\mu\nu}$ such that in the weak field approximation we get:

\[
g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad |h_{\mu\nu}| \ll 1
\]

(1.5)

The assumption that $h_{\mu\nu}$ is small allows us to ignore anything higher than first order in this quantity, such that we are left with the linearized version of general relativity. Equation 1.5 does not completely describe the coordinate system in spacetime. There may be other coordinate systems such that the spacetime metric can still be written as the Minkowski metric plus a small perturbation, but the perturbation may be different. It is essential that we choose a particular gauge in which this metric is invariant. It turns out that want to use the transverse traceless gauge, or “TT gauge”. In this coordinate system, and in the weak field approximation, Einstein’s field equation becomes the wave equation [10]:

\[
\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}\right) h_{\mu\nu} = 0.
\]

(1.6)
Then elements of $h_{\mu\nu}$ have wave solutions of the form $h(2\pi ft - \mathbf{k} \cdot \mathbf{x})$, where $f = |\mathbf{k}|/2\pi c$, which is a plane wave propagating in the $\hat{k}$ direction at the speed of light, $c$. In the TT gauge, assuming a gravitational wave propagating in the $\hat{k}$ direction, the perturbed metric tensor has the form [10]:

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & b & -a & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(1.7)

We can then write this metric as the linear combination of two basis tensors, which we call $\hat{h}_+$ and $\hat{h}_\times$ (“h-plus” and “h-cross”), $h = a\hat{h}_+ + b\hat{h}_\times$, where

$$\hat{h}_+ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(1.8)

and

$$\hat{h}_\times = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(1.9)

Gravitational waves are transverse and quadrupolar in nature. This means that as they propagate, gravitational waves alternately stretch space along one transverse direction while contracting it along the orthogonal direction in the same plane. Due to their quadrupolar nature, gravitational waves have two polarizations which we represent by $\hat{h}_+$ and $\hat{h}_\times$, and one polarization becomes the other when its principal axis is rotated $45^\circ$. Figure 1.2 shows the effect of these two polarizations on a ring of
freely falling test masses. Generally, gravitational waves from astrophysical sources will be a combination of both polarizations.

Unlike their electromagnetic equivalent, gravitational waves move through space nearly unperturbed. Unfortunately, this is due to the fact that they interact only very weakly with matter, which makes their effects extremely difficult to detect. The coupling coefficient \((8\pi G)/c^4\) from 1.2 is an extremely small value, which describes a very rigid spacetime. For an idea of scale, the amplitude shown in figure 1.2 is approximately \(h = 0.5\), however gravitational waves passing through Earth have an amplitude \(h \approx 10^{-20}\). Because of these weak effects, it would be impossible to measure waves of terrestrial origin. With the LIGO interferometers, we can only hope to measure gravitational waves from astrophysical sources. In the next section, I describe sources which are of particular interest to LIGO.

### 1.2 Gravitational Wave Sources

As the theory suggests, all massive objects affect the curvature of spacetime, but not all objects are equally likely to produce gravitational waves detectable by the
LIGO interferometers. Plausible sources of gravitational waves can be classified by
their morphology into two main categories: short duration, and long duration. In
the long duration category are continuous wave and stochastic signals, while the
main subcategories of short duration signals are known as bursts and compact binary
coalescence (CBC).

1.2.1 Continuous Waves

Continuous wave signals are emitted by sources that exhibit periodic motion in
a steady frequency that persists for a period longer than the observation time. One
example of such a source is a non-axisymmetric, rotating neutron star. Because their
frequency is stable over the time scale of observation, the signal is well modeled.

1.2.2 Stochastic Background

The universe is filled with massive objects which emit gravitational waves. The
incoherent superposition of gravitational waves from these sources result in a stochas-
tic background of gravitational radiation. In addition to the many contributions from
discrete sources, stochastic gravitational waves may also be signals from the early uni-
verse, akin to the Cosmic Microwave Background Radiation in the electromagnetic
spectrum. This signal is persistent and much longer than the measurement time, but
unlike continuous wave signals, these waveforms are stochastic in nature and therefore
may be analyzed statistically, but cannot be well modeled. While certainly interest-
ing, this type of signal would be extremely difficult to measure with ground-based
interferometers.

1.2.3 Bursts

There are some signals which are very short in duration relative to the mea-
surement time. These signals are classified as transients. Bursts are non-repeating
transient events that have a duration on the order of milliseconds. One source of this
type is a supernova. A supernova is the explosion of a star due to the gravitational collapse of its core, or due to the reignition of nuclear fusion. A supernova is so violent that it outshines entire galaxies, and the energy radiated is as much as an average star emits over its lifespan. Gamma ray bursts are also a source of this type. Some sources of this type are well modeled, while others are more mysterious. A major challenge with burst sources is their short duration. It is difficult to predict when such an event will occur with any degree of accuracy, so it is necessary to record as much data as possible, and to develop clever ways to look for this type of signal within that data.

1.2.4 Compact Binary Coalescence

Another major source of transient gravitational waves are the coalescence of compact binary system of black holes and/or neutron stars. The systems of interest are binary black holes (BBH), black hole - neutron star (BHNS), and binary neutron stars (BNS). The Hulse-Taylor pulsar mentioned in a previous section is an example from this category. The focus of this thesis is on model selection for binary black hole systems. These waveforms and the properties of BBH systems will be discussed in the next chapter.

Much of the physics of these systems is a mystery at this point. One of the main goals of LIGO is to detect these signals directly in order to to illuminate the mechanisms behind them. In the next section, I will introduce the reader to the detectors used to make these measurements.

1.3 LIGO: The Laser Interferometer Gravitational-Wave Observatory

Since Hulse and Taylor’s discovery, the focus in the field of gravitational physics has been to directly detect gravitational waves. Although other methods have been
tried (see [13]), large scale, ground-based interferometers currently provide the greatest sensitivity to the type of signals we expect, and thus offer the best chance for a direct detection. Gravitational waves deform spacetime, changing the physical distance between free test masses. We also know that this change is extremely small ($\sim 10^{-18}$ meters) [10]. These properties make interferometers an ideal measuring instrument for gravitational waves. Ground-based, km-scale laser-interferometers have been built around the world, creating an international network of detectors searching for gravitational waves from astrophysical sources. There are two detectors in the United States that make up LIGO. One is located in Livingston, Louisiana, and one located in Hanford, Washington: both have 4 km arms. VIRGO is a 3 km interferometer near Pisa, Italy. The British-German GEO600 is a 600 m detector located near Hannover, Germany. Currently in the beginning stages are the underground Japanese detector called KAGRA with 3 km arms, and LIGO India, which will be identical to the US detectors. The existence of multiple detectors spread across the globe is necessary to provide coincidence information about gravitational waves and to determine the direction of the source. With only one detector it would be impossible to point at the source of the waves. Having multiple detectors also allows us to reject environmental and instrumental artifacts in the data by requiring coincident signals in all detectors. None of these detectors has produced a detection at the present time, but they’ve all achieved huge improvements in sensitivity compared to earlier generations of detectors, and the continued progression of technology ensures an event rate suitable to claim detection in the Advanced LIGO era.

1.3.1 Ground-Based Interferometers

As gravitational waves pass through our solar system, they will stretch and contract earth in the aforementioned $\hat{h}_+ \text{ and } \hat{h}_\times$ polarizations. The effect is extremely small with large wavelength, so we require an instrument that is very long and sen-
sitive to small changes in length. For this we use laser interferometers, employing a method very similar to Michelson and Morley in their famous experiment to detect the luminiferous aether.

In an interferometer, a light source produces a beam, which is incident on a beam splitter. This splits the light into two equal parts that travel down orthogonal paths in the L-shaped arms of the interferometer. The two arms are precisely equal in length and have mirrors at the far ends of the arms. The beam in each arm cavity is reflected by the end mirror and returns along its original path. The two beams are recombined at the beam splitter and the waves are superimposed. A photodetector is positioned to collect the superimposed signal. Since the speed of light in a vacuum (which we produce in the arm cavities) is constant, we may tune the system such that the beams interfere completely destructively and the signal appears dark to the photodetector. We call this operating on a dark fringe. However, as a gravitational wave passes through the earth, the arm lengths change (see figure 1.3), and the beams produce an interference pattern that can be detected by the photodetector. Analysis of this pattern will allow us to determine the difference in these arm lengths with high precision.

If we choose two test masses a length $L$ apart, a passing gravitational wave will produce a strain

$$\frac{\Delta L}{L} = \frac{h}{2}$$

(1.10)

of approximately $10^{-21}$. So if we have detectors with $L \sim 1$ km arms, our interferometer must be capable of measuring a change in length of $10^{-18}$ meters. This is an extremely simplified explanation of interferometry, but these are the basic principles that inspire the LIGO interferometers. If our interferometer is sufficiently long, and noise is adequately attenuated, we will be able to see the tiny deformation of the
earth in this strain signal as gravitational waves propagate through the detector as seen in figure 1.3.

Over the past several years there has been heavy reconstruction and commissioning of new detectors at the same sites, which will comprise Advanced LIGO. The rest of this section will detail the properties of the Advanced LIGO (aLIGO) detectors as they were simulated in the work that follows.

The LIGO experiment consists of two detectors located at two different sites in the United States. The observatory in Hanford, Washington is home to the 4 km-long detector dubbed “H1”. The observatory in Livingston Parish, Louisiana contains one 4 km-long detector called “L1”. These detectors are essentially identical in design. Plans are under way to put another identical detector in India, but construction has not yet begun. The original Initial LIGO detectors were designed at the California Institute of Technology and the Massachusetts Institute of Technology with funding from the National Science Foundation. The detectors were constructed during the 1990’s and took continuous data from November 2005 through September 2007. These detectors were designed to be sensitive to gravitational waves in the 40-7000 Hz band and reach a strain amplitude as low as $10^{-21}$. The current generation of detectors,
dubbed “Advanced LIGO”, utilizes the existing sites and infrastructure from Initial LIGO. However, many adjustments have been made to increase sensitivity, including higher laser power, improved optics, and improved seismic isolation. These improvements should increase sensitivity by about a factor of 10 while widening the range of detectable frequencies. All of the LIGO detectors are based on the Fabry-Perot Michelson interferometer configuration. To improve the sensitivity of the detectors, they are designed to be very long, but as quiet space is scarce and the earth’s curvature limit the length of the detector, it is beneficial to also reflect the beam several times in the arm cavity, extending the effective arm length. Strain sensitivity was also increased by using partially transparent mirrors to build a power-recycling cavity and a signal recycling cavity (new to aLIGO). The basic layout of the LIGO detectors is shown in figure 1.4.

1.3.2  Noise sources and Sensitivity

As the expected gravitational wave signal is extremely small, identifying and mitigating noise sources is essential to achieving a detection. The detection bandwidth for the Advanced LIGO detectors ranges from 10 Hz to 7 kHz. Across this band, there are essentially three regions where different noise sources are dominant.

At low frequencies (<50Hz), seismic noise is dominant. Seismic noise is caused by vibrations in the earth’s surface due to earthquakes, tides, anthropogenic sources, etc. This noise is reduced using sophisticated isolation systems, including fused silica fiber suspensions and actively controlled optical isolation platforms.

For frequencies 50 - 150 Hz, the noise curve is dominated by thermal noise due to Brownian motion in the optics and suspensions. In the interferometer test masses, this is mostly due to the optic coatings, which make the mirrors highly reflective. The coatings are designed to minimize this effect by confining it to a narrow bandwidth. The suspension fibers are designed to be very thin in the middle and thicker near the
Figure 1.4: Advanced LIGO optical configuration. ITM: input test mass; ETM: end test mass; ERM: end reaction mass; CP: compensation plate; PRM: power recycling mirror; PR2/PR3: power recycling mirror 2/3; BS: 50/50 beam splitter; SRM: signal recycling mirror; SR2/SR3: signal recycling mirror 2/3; FI: Faraday isolator; $\varphi_m$: phase modulator; PD: photodetector. The laser power numbers correspond to full-power operation. All of the components shown, except the laser and phase modulator, are mounted in the LIGO ultra-high vacuum system on seismically isolated platforms. [6]
ends, where they are fused to the test masses. Since this is the frequency band we are most interested in for detections, this geometry is used to make the violin modes of the suspensions high (510 Hz fundamental) and the vertical stretching mode low (9 Hz).

For frequencies greater than 150 Hz, the quantum or “shot” noise is the largest contributor. Shot noise is the effect of statistical fluctuations in detected light intensity at the photodetector. Also in this regime is radiation pressure due to the momentum transferred to the mirrors as photons are reflected. Concessions must be made here, as increasing laser power reduces shot noise but increases radiation pressure.

There are many other noise sources which have been omitted here for brevity. Figure 1.5 shows the projected strain noise spectrum for the nominal Advanced LIGO mode of operation. Clearly, the ability to detect certain gravitational waveforms varies with frequency. In the region of highest sensitivity, when the detector is in the nominal mode, the binary neutron star range is $\sim 190$ Mpc [6].
Figure 1.5: Contributions to total noise for the nominal (high power, broadband) mode of operation of Advanced LIGO. [6]
CHAPTER 2
GRAVITATIONAL WAVES FROM BINARY BLACK HOLE SYSTEMS

Some of the main targets of ground based gravitational wave astronomy are coalescing compact binaries. In this thesis, we look specifically at binary black hole (BBH) systems. While there is no direct evidence of such systems as of yet, theoretical models suggest their formation. What’s more, the gravitational waveforms produced by these systems are expected to contain many cycles in the sensitive band of the LIGO detectors, making them ideal candidates for study in this experiment.

2.1 BBH Astrophysics and Properties

As with all massive objects, binary black holes rotate around each other due to their mutual gravitational attraction. Unlike a conservative Newtonian two-body system, the BBH system has a dynamic quadrupole and higher order mass moment, which results in radiation of gravitational waves (GW). As the system produces GW, it loses energy such that the two black holes move closer together until they eventually coalesce into a single black hole. The signal from the coalescence of a binary black hole has a rich frequency spectrum. It is convenient to separate this signal into three distinct phases, which we call Inspiral, Merger, and Ringdown. When the two black holes are far from each other, the orbit can be well described by a post-Newtonian approximation. This is the Inspiral phase. As they rotate and emit GW, they lose energy and move closer together to conserve angular momentum. As they move closer together the emission of GW increases in frequency; this is known as a “chirp”. When
the black holes become very close together, the equations of general relativity become highly non-linear. Eventually the black holes will fall into each other, forming a single black hole. This is known as the Merger phase. There is no complete analytical model for the Merger phase, and only numerical solutions to Einstein's field equations can model this merger. The resulting black hole is in an excited state and emits quasi-normal mode GW radiation (QNR). This signal can be decomposed into a basis of spin-weighted spherical harmonics. Full treatment of these modes is outside the scope of this thesis, but they can be well approximated as a damped sinusoid. This is the final Ringdown phase of the BBH system. These stages are illustrated in figure 2.1.

Figure 2.1: The Inspiral, Merger, and Ringdown phases of the BBH evolution. Each phase must be treated separately in models. [3]
Table 2.1: Compact binary coalescence rates per Mpc$^3$ per Myr. [1]

<table>
<thead>
<tr>
<th>Source</th>
<th>$R_{\text{low}}$</th>
<th>$R_{\text{re}}$</th>
<th>$R_{\text{high}}$</th>
<th>$R_{\text{max}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>NS-NS (Mpc$^{-3}$ Myr$^{-1}$)</td>
<td>0.01</td>
<td>1</td>
<td>10</td>
<td>50</td>
</tr>
<tr>
<td>NS-BH (Mpc$^{-3}$ Myr$^{-1}$)</td>
<td>$6 \times 10^{-4}$</td>
<td>0.03</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>BH-BH (Mpc$^{-3}$ Myr$^{-1}$)</td>
<td>$1 \times 10^{-4}$</td>
<td>0.005</td>
<td>0.3</td>
<td></td>
</tr>
</tbody>
</table>

2.2 Detection Rates

While the GW waveform produced by BBH systems spans the sensitivity band of ground based detectors through each of the three phases, all such systems are not equally detectable. We can classify BBH systems by their total mass where we typically use units of solar mass, $M_\odot$. If the BBH system has a total mass of $2 - 30 M_\odot$, we call it a stellar mass BBH. Masses greater than $10^4 M_\odot$ are called super massive black holes (SMBH). Everything in between ($30 - 1000 M_\odot$) is called an intermediate mass black hole (IMBH). All of the waveforms considered in this study fall into this category. Table 2.1 shows the expected rates of compact binary coalescences per Myr per Mpc$^3$. Slightly more interesting are the rates in Table 2.2, which give the expected rate at which LIGO could be able to make a detection. Plausible rates are converted into detection rates, optimal horizon distances of 33 Mpc / 445 Mpc are assumed for NS-NS inspirals in the Initial / Advanced LIGO-Virgo networks. For NS-BH inspirals, horizon distances of 70 Mpc / 927 Mpc are assumed. For BH-BH inspirals, horizon distances of 161 Mpc / 2187 Mpc are assumed. These distances correspond to a choice of $1.4 M_\odot$ for NS mass and $10 M_\odot$ for BH mass. For a full discussion of how these rates were estimated, see the LSC paper [1]. These rates are encouraging, as we can reasonably expect to detect compact binary coalescences in the Advanced LIGO-Virgo era.
Table 2.2: Detection rates for compact binary coalescence sources. [1]

<table>
<thead>
<tr>
<th>IFO</th>
<th>Source</th>
<th>$N_{low}$ yr$^{-1}$</th>
<th>$N_{re}$ yr$^{-1}$</th>
<th>$N_{high}$ yr$^{-1}$</th>
<th>$N_{max}$ yr$^{-1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>NS-NS</td>
<td>$2 \times 10^{-4}$</td>
<td>0.02</td>
<td>0.2</td>
<td>0.6</td>
</tr>
<tr>
<td>Initial</td>
<td>NS-BH</td>
<td>$7 \times 10^{-5}$</td>
<td>0.004</td>
<td>0.1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>BH-BH</td>
<td>$2 \times 10^{-4}$</td>
<td>0.007</td>
<td>0.5</td>
<td></td>
</tr>
<tr>
<td>Advanced</td>
<td>NS-NS</td>
<td>0.4</td>
<td>40</td>
<td>400</td>
<td>1000</td>
</tr>
<tr>
<td></td>
<td>NS-BH</td>
<td>0.2</td>
<td>10</td>
<td>300</td>
<td></td>
</tr>
<tr>
<td></td>
<td>BH-BH</td>
<td>0.4</td>
<td>20</td>
<td>1000</td>
<td></td>
</tr>
</tbody>
</table>
CHAPTER 3
BHEXTRACTOR

The coalescence of binary black hole (BBH) systems is one of the most promising sources of gravitational waves (GW), detectable by Advanced LIGO and its partners. Detection of these sources will open a new era of gravitational wave astrophysics. The gravitational wave signature from these sources can be used to determine the physical properties of the source, as well as deduce interesting physics, previously hidden to other observation methods. In order to study these properties, we need a set of tools, that can extract the gravitational wave signal from noisy detector data, and reveal identifying features. Despite progress in numerical simulations of the coalescence of binary black hole systems, the simulation of highly asymmetric spinning systems and the construction of accurate physical templates remain challenging and computationally expensive. Available numerical relativity waveforms span an increasing portion of the physical parameter space of unequal mass, spin and precession binary black holes, but these simulations can take a week or more to run. Furthermore the number of these simulations necessary to fully span the parameter space is immense, making it computationally prohibitive to search over all of these simulations in parameter estimation and matched filtering. In this chapter, I describe an algorithm to reduce catalogs of waveforms with similar parameters into a set of principal components, which represent a certain region of the binary black hole parameter space. Using Bayesian analysis and nested sampling, one can use this algorithm to classify an arbitrary BBH waveform into one of these regions and estimate the parameters of the GW
source. The name we’ve assigned to algorithm, and the collection of tools therein, is bhextractor.

### 3.1 How bhextractor Works

bhextractor is a set of algorithms, written mostly in Python and C, and utilizing many of the tools in the LIGO Algorithm Library (LAL). This work closely follows a similar analysis algorithm called the Supernova Model Evidence Extractor (SMEE) [8], the goal of which is to determine the explosion mechanism of a core collapse supernova, given a simulated GW signal. In the following subsections, I describe the steps of a Bayesian data analysis algorithm which classifies detected GW signals from binary black holes as belonging to one of a set of signal catalogs, representing, e.g., different mass ratios, spins, etc.

#### 3.1.1 Waveform Catalogs

We start with a set of numerical relativity (NR) waveforms produced by the MAYA code of the Georgia Institute of Technology [12], which uses the Einstein Toolkit\(^1\). The simulations generate a Weyl Scalar, \(\Psi_4\), decomposed into spin-weighted spherical harmonics that is converted to strain. These NR waveforms are sorted into three distinct catalogs, each with different physical parameters. The catalogs are called Q-series, HR-series, and RO3 series. The Q-series contains 13 non-spinning, unequal-mass simulations. We use 15 waveforms from the HR-series, a set of unequal-mass, equal spin simulations, with initial spin parallel to the initial angular momentum. Finally, the RO3-series is a set of 20 unequal-mass simulations with the lower mass black hole spin aligned to the initial angular momentum (z-axis) and the other black hole at a tilt angle \(\theta\) with the z-axis in the xz-plane; these systems are precessing. The tilt angles are defined for a specific separation of the black holes at one instant in

\(^1\)http://www.einsteintoolkit.org/
Table 3.1: Physical parameters for the three waveform catalogs used in this study.

<table>
<thead>
<tr>
<th>Catalog Name</th>
<th>Q</th>
<th>HR</th>
<th>RO3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass Ratio, $q = m_1/m_2$</td>
<td>1-2.5</td>
<td>1-4</td>
<td>1.5-4</td>
</tr>
<tr>
<td>Spin magnitude, $a$</td>
<td>0.0</td>
<td>0.0</td>
<td>0.9</td>
</tr>
<tr>
<td>Tilt angle, $\theta$</td>
<td>0.0</td>
<td>0.0</td>
<td>45° - 270°</td>
</tr>
<tr>
<td>N waveforms</td>
<td>13</td>
<td>15</td>
<td>20</td>
</tr>
</tbody>
</table>

the evolution of the binary system, and change in time. See table 3.1 for the physical parameters of each catalog. The waveforms for the Q-series catalog are plotted in figure 3.1. For reasons described in the next chapter, we represent each waveform, not only in terms of strain $h_+$ and $h_x$, but also in terms of amplitude and phase. The waveforms for the other catalogs are plotted in Appendix A. Since there is no functional form to relate the waveforms, we use Principal Component Analysis to determine the differences in each catalog and how to best capture their signatures on the data.

3.1.2 Principal Component Analysis

The waveform catalogs give us a small subset of the available parameter space with a wide range. While the waveforms in each catalog are very different in detail, they share strong, general features. Using principal component analysis (PCA) we can decompose the waveform catalogs into a set of orthonormal principal components (PC) that essentially describe these major features. There are many different algorithms to derive the principal components; in this study, we opt for a standard Singular Value Decomposition (SVD)[7] using scikit-learn\(^2\), a machine learning Python toolkit. For a catalog of $n$ waveforms $\{h_i\}_{i=1,...,n}$ with $m$ samples, we build a matrix $H$ whose columns correspond to each of the waveforms. We then factorize the resulting $m \times n$ matrix $H$ such that:

\(^2\)http://www.scikit-learn.org/
Figure 3.1: The full catalog of time series waveforms in the Q-series. $h_+$ is shown in blue and $h_x$ is in green.
$H = USV^T$, \hspace{1cm} (3.1)

where $U$ is an $m \times m$ matrix whose columns are the eigenvectors of $HH^T$ and $V$ is an $n \times n$ matrix whose columns are eigenvectors of $H^TH$. $S$ is an $m \times n$ matrix, where the diagonal entries $\{S_{jj}\}_{j=1,\ldots, n}$ correspond to the square root of the $j$th eigenvalue. The columns of $U$ are the principal components for the catalog, ranked by eigenvalue. The first column is the first PC, which represents the most significant features of the catalog, and the axis of the parameter space with the largest variance. The second PC has the second most significant features, etc. The catalog waveforms can be reconstructed as a linear combination of the PCs, weighted by their eigenvalues:

$$h_i \approx \sum_{j=1}^{k} U_j \beta_j,$$

(3.2)

where $h_i$ is the catalog waveform, $u_j$ is the $j$th PC and $\beta_j$ is the corresponding coefficient, which is found by projecting $h_i$ onto $u_j$. The sum over $k$ PCs is an approximation of the catalog waveform since, in general, $k < n$. When $k = n$ the waveform is constructed perfectly, and many of the waveforms can be perfectly reconstructed with fewer than $n$ PCs, which is a computational advantage. Indeed, if one chooses the $\beta$ coefficients correctly, it is possible to reconstruct waveforms that are not contained in the original catalog, but share similar properties. So by applying PCA to our waveform catalog, we gain the ability to explore a larger parameter space, and reduce the computational cost of NR simulation. For plots of all of the principal components for each waveform catalog, see Appendix B.

3.1.3 Nested Sampling

A nested sampling algorithm is a computational approach to computing the Bayesian statistical probability that a set of data matches a certain model. This Bayesian approach has two distinct applications: model selection and parameter estimation. Given some data, and two or more models that might describe the data, one can use
this technique to determine the preferred model from the ratio of posterior probabilities for each model:

\[ O_{1,2} = \frac{p(M_1|I) p(D|M_1, I)}{p(M_2|I) p(D|M_2, I)}, \]  

(3.3)

where \( p() \) is probability, \( M_{1,2} \) are the models being compared, \( D \) is the data, and \( I \) is information that you know to be true about the system. This is the posterior odds ratio, often called the ‘Bayes factor’, which follows directly from Bayes’ theorem:

\[ p(M_i|D, I) p(D|I) = p(D|M_i, I) p(M_i|I). \]  

(3.4)

Succinctly, Eq. 3.4 can be read as, “the posterior times the evidence is equal to the likelihood times the prior.” The posterior is the probability that the model \( M_i \) is true, given the data \( D \). The likelihood is the probability of measuring the data, given that the model is true. The prior is the probability that the model is true based on previous knowledge and experiments. Finally, the evidence is a normalization constant that is often omitted to simplify calculations, and in the case of Eq. 3.3, we see that it will cancel out, as we have no reason to prefer one model over another in this case. If the Bayes factor is greater than 1, model 1 is preferred, and if is is less than 1, model 2 is preferred. A Bayes factor \( \sim 1 \), does not suggest a strong preference for either model being compared. Figure 3.2 shows the performance of the model selection algorithm. Here, waveforms from the HR catalog were added to 50 different noise realizations, shaped by the aLIGO noise curve (1.5). Note that most of the Bayes factors are distributed in the negative range. As the spin \( a \) increases, the algorithm shows a stronger preference over the Q catalog, which has no spin.

Model selection is useful when you want to know which hypothesis best fits your data, but this Bayesian approach can also be used for parameter estimation. That is, if we can use model selection to choose the correct model for our data, we can then use that model to determine the parameters of the system. The algorithm is
Figure 3.2: Distribution of Bayes factors for HR waveforms. Each pair of boxes in the figure corresponds to the sample of Bayes factors for the 50 different noise realizations. The boxes denote the interquartile range of the distribution, the red lines indicate the median value and the whiskers show the outliers within 1.5 times the interquartile range. The x-axis indicates the physical parameters of the injection performed, where \( q \) is the mass ratio and \( a \) is spin; for the HR catalog, the mass ratio and spin magnitudes are varied. The two \( a = 0 \) systems are seen to be difficult to distinguish from the Q catalog which is not surprising since the Q catalog contains waveforms for non-spinning systems.[5]
fairly simple. You start by choosing $N$ points in your parameter space $\{\theta_i\}_{i=1,\ldots,N}$, uniformly sampled from the prior distribution, then use those parameters in your model to construct $N$ templates. For each of these templates, you compute the likelihood, i.e. how well your template matches the data. Each point $\theta_i$ is used to define a likelihood contour in the parameter space. The point with the lowest likelihood $L^*$ is removed and replaced with a new point sampled uniformly prior, where the prior has now been narrowed to exclude points with likelihood less than $L^*$. This process continues for a user specified number of iterations. The algorithm eventually converges on parameter values, but the number of iterations required is problem dependent. The result is a set of posterior samples that give us a distribution of the recovered parameter. Implementation of this algorithm varies widely. We follow the methods originally developed by J. Skilling [11].

3.2 Mass Scaling in bhextractor

There are many different parameters that determine the shape of the BBH waveforms. Among these is the total mass of the system. We could generate many catalogs of waveforms for different masses, but this would be extremely computationally expensive, and is unnecessary, since we know how the waveforms scale with mass, assuming all other parameters stay the same. Instead of generating many catalogs, we add to the algorithm, a way to scale the waveforms in the catalog to any total mass we choose. Up until now, the algorithm has only been successful with the total mass assumed to be known and constant (for our catalogs $M_{\text{total}} = 250M_\odot$). Accordingly we also add a mass dimension for our algorithm to search over. To first order in the time domain, the amplitude of the GW signal from a BBH source scales linearly with mass [9], i.e. if you double the mass, you double the amplitude of the GW. The same is true in the frequency domain. On the time-axis, the waveform expands linearly with mass, such that the time step between each point doubles in duration when the
mass of the system is doubled. This amounts to the entire waveform shifting to lower frequencies. In practice this means creating a new time axis for the waveform and interpolating to the time samples of the original time axis. Figure 3.3 shows how this scaling looks in practice. Since the inspiral, merger, and ringdown phases all have slightly different behavior, this scaling is only an approximation, and can easily be refined to tune the algorithm in the future. With mass scaling included, the parameter estimation algorithm takes the following steps:

1. Make an “injection”

   (a) Take a waveform from one of the catalogs
   
   (b) Scale the waveform to the desired mass
   
   (c) Add the waveform to simulated aLIGO detector noise

2. Perform Nested Sampling

   (a) Sample a point on the parameter space \( \theta = \{M, \beta_1, \beta_2, \ldots\} \)
   
   (b) Construct the template waveform
       
       i. \( h(\theta) = \sum_{j=1}^{k} P C_j \beta_j \)
       
       ii. Scale \( h \) with mass
   
   (c) Compute the likelihood \( p(D|\theta) \) that the injection matches your template

3. Repeat step 2, moving to regions of greater likelihood in the parameter space as described in section 3.1.3.

As a byproduct of computing the likelihood in this process, we are left with posterior samples, which give us a distribution for each of the parameters in our template. By looking at histograms of these posterior samples, we can determine the most likely value for each parameter and compute the statistical uncertainty of our measurement. Figure 3.4 shows the nested samples for the total mass of a 400
$M_\odot$ Q catalog waveform, injected at SNR = 15. The prior range was set at between 250 and 500 $M_\odot$. Notice that the nested samples fill the prior range to start, but eventually narrow to a small region of the parameter space. Figure 3.5 shows the posterior samples for this same event. In this well behaved example, the samples are clearly peaked around the injection mass as we expect.

3.3 Evaluating the Algorithm

3.3.1 Match

Naturally, we would like to be able to check how well our algorithm is performing. Before we can test the parameter estimation, we first want to make sure that our PCA was done properly and that we can use the PCs to accurately reconstruct the waveforms in our catalogs. This is easy in principle: we project the waveforms onto the matrix of PCs, to find the nominal $\beta$ values for each waveform. Then using equation 3.2, we reconstruct each waveform from the PCs and $\beta$ values, and compare that reconstruction to the original waveform. The quantity that we use to measure the degree of similarity between the original and reconstruction is called “match’. The match, $\mu$, is essentially the normalized dot product between the original catalog waveform $h_{\text{cat}}$ and the reconstructed waveform $h_{\text{rec}}$:

\begin{align*}
\mu &= \sum_{j=1}^{k} \beta_j \\
&= \sum_{j=1}^{k} (||h_{\text{cat}}|| \cdot u_j) \beta_j \\
&= ||h_{\text{cat}}|| \cdot \sum_{j=1}^{k} \beta_j u_j \\
&= ||h_{\text{cat}}|| \cdot ||h_{\text{rec}}||
\end{align*}  

(3.5)
Figure 3.3: Examples of mass scaling done in time domain (top) and frequency domain (bottom). In both plots, the red curve is the 250 $M_\odot$ waveform, and the blue curve is the same waveform scaled by a factor of 2 in amplitude and time to a total mass of 500 $M_\odot$. 
Figure 3.4: Nested samples for the total mass of a 400 $M_\odot$, SNR = 15, Q catalog injection
Figure 3.5: Posterior samples for the total mass of a 400 $M_\odot$, SNR = 15, Q catalog injection
where \( k \) is the number of PCs used in the reconstruction and \( u_j \) is the \( j \)th principal component. Using this definition, if the waveform is perfectly reconstructed then \( \mu = 1 \), and conversely, if the original waveform is orthogonal to the reconstruction, then \( \mu = 0 \). As an example, we can use the match to show how well or mass scaling procedure works. Figure 3.6 shows the match for two sets of waveforms. The top plot shows two waveforms from Q-series, generated at 250 and 350 \( M_\odot \). These are both NR waveforms that haven’t been scaled by the algorithm, but as they different masses, their match is predictably low. The lower plot compares the NR simulated waveform with the reconstruction, in which we take the 250 \( M_\odot \) waveform from the first plot and scale it as in section 3.2. In this case the match is very nearly 1, but not perfect, as this scaling is an approximation.

### 3.3.2 Explained Variance

Another quantity that is useful in assessing our PCA is “explained variance”. The waveforms in a given catalog collectively represent some portion of the parameter space as defined in table 3.1. While these waveforms share similar features, they all have different details that contribute to the total variance \( \sigma \) of the catalog. The total variance in the catalog is defined as the sum of the variances of the individual waveforms. This quantity is simply the trace of the covariance matrix \( H \), since the diagonal elements \( h_n \) contain the variances. PCA replaces the waveforms with orthogonal PCs, such that the covariance matrix \( U \) is diagonal and sorted in decreasing order. The trace of the covariance matrix is the same such that the total variance is conserved. As a result, the first principal component is that which has the largest variance (eigenvalue) \( u_n \). The ratio of this value to the total variance is called explained variance, \( \xi \). Using equation 3.1 we define the total variance as:
Figure 3.6: Top: The match between two waveforms from Q-series, generated at 250 \( \text{M}_\odot \) and 350 \( \text{M}_\odot \) using NR simulation. Bottom: The match between the 350 \( \text{M}_\odot \) catalog waveform, and the 250 \( \text{M}_\odot \) waveform scaled to 350 \( \text{M}_\odot \).
\[
\sigma = \text{Tr}(H) = h_1 + \ldots + h_j \\
= \text{Tr}(UHUT) = \text{Tr}(UU^T H) \\
= \text{Tr}(U) = u_1 + \ldots + u_j
\] (3.6)

Then the fraction of the total variance explained by \( k \) principal components is:

\[
\xi = \frac{1}{\sigma} \sum_{j=1}^{k} u_j
\] (3.7)

When \( \xi = 1 \), the total variance of the catalog is reproduced by the PCs, and when \( \xi = 0 \), the PCs are not representing the catalog at all. In the next chapter, we use explained variance along with match to evaluate the quality of our PCA.
CHAPTER 4
RESULTS

In making our principal components from the waveform catalogs, we must make a choice about which representation of the waveforms to use. The obvious choice would be to use the two polarizations of strain, $h_+$ and $h_\times$. However if we treat the waveform amplitude and phase separately, and make the principal components from these, we see some improvement of the algorithm performance. Specifically, an accurate reconstruction of the catalog waveforms requires fewer PCs when using amplitude and phase, rather than strain. As a measure of this, figure 4.1 shows the explained variance of the HR catalog as a function of number of PCs, using PCs from $h_+$ and $h_\times$ (top), and using PCs from amplitude and phase (bottom). This shows that when using amplitude and phase PCs, fewer principal components explain more of the variance in the catalog. This is true for the RO3 catalog as well, however there is a slight disadvantage for the Q catalog. Since the Q catalog has much simpler waveforms, we do not gain anything by using the amplitude and phase PCs. For figures of explained variance for all catalogs, see Appendix C. As it is beneficial for the majority of waveforms, in the rest of this analysis, we use amplitude and phase PCs.

To look more deeply at the individual waveforms in the catalog, I also computed the match for each waveform as a function of number of PCs used in the reconstruction. Figure 4.2 shows these values plotted again for the HR catalog. Notice that the reconstructions all eventually have a perfect match, given enough PCs. Also notice that, in general, we can use fewer than half of the available PCs and still get an
Figure 4.1: Top: explained variance in the HR catalog as a function of number of PCs, using strain principal components. Bottom: using amplitude and phase principal components.
accurate reconstruction, which is the point of using PCA in the first place. Similar plots for the Q and RO3 catalogs can be found in Appendix D.

Finally, we want to test how well the parameter estimation algorithm recovers the total mass of the system, and other parameters including the PC coefficients ($\beta_j$). There are many input variables that will determine how well the algorithm performs. For example, using more PCs will give you a more accurate reconstruction, but will take much more CPU time. The mass and SNR of the injected signal will also effect the computation time and accuracy of the reconstruction. To assess the overall performance of bhextractor, we compare results for injections over a range of masses, SNRs, and number of PCs used in the waveform reconstruction. In this analysis we use waveforms from the Q catalog, scaled to each combination of total mass $M = \{250, 300, 400, 500\} M_\odot$, and SNR = \{10, 20, 30, 40, 50\}. At each combination of mass and SNR, we inject a Q catalog waveform into 50 different noise realizations, for 1000 total events, and we look at the statistics of the recovered mass from the posterior samples using 3 PCs. Figure 4.3 shows the results of these 1000 events, evenly distributed over the different masses and SNRs. These results agree well with what we expect. That is, the recovered mass is generally very close to the injected mass. The error bars 250 and 500 $M_\odot$ do not overlap with the target, which is not surprising since the samples are not generated outside of this range.

The mass posterior samples in figure 3.5 are a good example of the types of distributions we see in the majority of the recovered parameters. Sometimes the recovered distributions are not quite as convincing, and will have more than one peak in their histogram, as pictured in figure 4.4. One may also want to see if there is any correlation between parameters. bhextractor includes some helpful plotting tools to compare these variables. Figure 4.5 shows the posterior samples for mass plotted against the first PC coefficient in amplitude and phase. From this we see that the
Figure 4.2: Top: Match of HR catalog reconstructed waveform as a function of number of PCs used in the reconstruction, using strain principal components. Bottom: using amplitude and phase principal components.
Figure 4.3: Recovered system total mass as a function of SNR. 3 PCs and 1024 live points were used. The boxes denote the interquartile range of the distribution, the red lines indicate the median value and the whiskers show the outliers within 1.5 the interquartile range.
total mass and amplitude coefficient are well localized, while the phase coefficient doesn’t seem to do much in the reconstruction.

We have shown that the algorithm behaves in the way we expect, with some satisfying initial results. In it’s present state, bhextractor can search over the PC coefficients and the total mass of the system, but there are many other parameters of interest that need to be added to this algorithm. Work will continue on bhextractor to include these parameters in the analysis. The mass scaling algorithm will also need to be refined to reflect different features of the inspiral, merger and ringdown in the BBH waveforms. Additional studies need to be conducted to gather more statistics at each mass and SNR, and to see how the performance is effected when using more principal components.
Figure 4.5: Posterior samples for the total mass of a 400 $M_\odot$, SNR = 20, Q catalog injection
APPENDIX A

WAVEFORM CATALOGS

In this appendix are plots for the waveform catalogs used in this analysis. For each catalog (Q, HR, and RO3), the left column shows the original numerical relativity time series for both strain polarizations, $h_+$ and $h_\times$. The names assigned to each waveform are also given. For reasons discussed in chapter 4, it is beneficial to use the instantaneous amplitude and phase rather than the strain polarizations. The amplitude and phase are plotted in the center and right columns, respectively.
Figure A.1: The full catalog of time series waveforms in the Q-series. $h_+$ is shown in blue and $h_\times$ is in green.
Figure A.2: The full catalog of time series waveforms in the HR-series. $h_+$ is shown in blue and $h_\times$ is in green.
Figure A.3: The full catalog of time series waveforms in the RO3-series. $h_+$ is shown in blue and $h_\times$ is in green.
Performing PCA on the waveform catalogs in appendix A, gives us an orthogonal set of principal components (PCs) that represent each catalog as described in section 3.1.2. In this appendix are plots of for each PC in the three catalogs (Q, HR, and RO3). The PCs for $h_+$, instantaneous amplitude, and instantaneous phase are plotted separately. In each case the PCs are ranked in descending order by their eigenvalue, which corresponds to the explained variance (see section 3.3.2) of the PC.
Figure B.1: Principal components for Q-series: $h_+$
Figure B.2: Principal components for Q-series: amplitude
Figure B.3: Principal components for Q-series: phase
Figure B.4: Principal components for HR-series: $h_+$
Figure B.5: Principal components for HR-series: amplitude
Figure B.6: Principal components for HR-series: phase
Figure B.7: Principal components for RO3-series: $h_+$
Figure B.8: Principal components for RO3-series: amplitude
Figure B.9: Principal components for RO3-series: phase
APPENDIX C

EXPLAINED VARIANCE

Explained variance (see section 3.3.2) is a measure of how well each principal component represents the waveforms in a given catalog. We use this quantity to make a choice about which representation of the waveforms to use in this analysis, as described in chapter 4. In this appendix are plots of explained variance as a function of the number of PCs used in the waveform reconstruction, for each catalog. The amplitude and phase PCs are plotted separately from the strain PCs. In each case, note that we do not require all of the PCs to fully describe the catalog (explained variance=1). This is the advantage of the PCA method described in this thesis.
Figure C.1: Q-series: explained variance as a function of number of PCs, using strain principal components.
Figure C.2: Q-series: explained variance as a function of number of PCs, using amplitude and phase principal components
Figure C.3: HR-series: explained variance as a function of number of PCs, using strain principal components
Figure C.4: HR-series: explained variance as a function of number of PCs, using amplitude and phase principal components
Figure C.5: RO3-series: explained variance as a function of number of PCs, using strain principal components
Figure C.6: RO3-series: explained variance as a function of number of PCs, using amplitude and phase principal components
APPENDIX D
MATCHES

In the nested sampling process (see section 3.1.3), we build a template waveform from the principal components and compare it to the injected waveform (i.e. the gravitational wave signal). As a another assessment of our PCA algorithm, we use a quantity called match (see section 3.3.1) to measure how well our PCs reconstruct the original catalog waveforms. In this appendix are plots of the match as a function of the number of PCs used in the reconstruction for each waveform in the three catalogs. Here, a match of 1 means that the two waveforms are the same, and a match of 0 indicates that the waveform and the reconstruction are orthogonal.
Figure D.1: Q-series: match of reconstructed waveform as a function of number of PCs used in the reconstruction, using strain principal components
Figure D.2: Q-series: match of reconstructed waveform as a function of number of PCs used in the reconstruction, using amplitude and phase principal components.
Figure D.3: HR-series: match of reconstructed waveform as a function of number of PCs used in the reconstruction, using strain principal components
Figure D.4: HR-series: match of reconstructed waveform as a function of number of PCs used in the reconstruction, using amplitude and phase principal components
Figure D.5: RO3-series: match of reconstructed waveform as a function of number of PCs used in the reconstruction, using strain principal components
Figure D.6: RO3-series: match of reconstructed waveform as a function of number of PCs used in the reconstruction, using amplitude and phase principal components
BIBLIOGRAPHY


