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Abstract

The factorization property of graviton scattering amplitudes is reviewed and shown to be valid only if the "natural" value of the gyromagnetic ratio $g_S = 2$ is employed—independent of spin.

1 Introduction

Over the years there have been a number of speculations concerning the "natural" value of the g-factor (gyromagnetic ratio) for a particle of arbitrary spin. These basically fall into two categories. The first is the Belinfante conjecture, which asserts that the "natural" value is $g_S = 1/S$ for a particle of spin S[1]. This hypothesis reproduces the well-known Dirac value—$g_{S=\frac{1}{2}} = 2$—for a particle of spin 1/2 but predicts smaller numbers for particles of higher spin. A second proposal is that the value $g_S = 2$ is the "natural" value independent of spin[2]. Of course, in some sense any such speculation is somewhat metaphysical in that a "natural" value for the g-factor has no experimental basis for particles which participate in the strong interactions. One case which *does* have direct empirical support is that of the charged leptons which, carrying spin 1/2, agree with their Dirac value—$g_{\rm Dirac} = 2$—up to small electromagnetic corrections[3]. The only other case which has empirical support is that of the charged W-boson, which for reasons discussed below, has g_W=2 in the standard model[4]. The present experimental limits—$g = 2.20 \pm 0.20$[3]—are in agreement with the standard model prediction.

Below then in section 2 we briefly review the previous arguments in this regard, while in section 3 we present a new argument which favors the hypothesis $g_S = 2$—the factorization of gravitational amplitudes. A brief concluding section follows.

2 "Natural" g-factor

Every student learns in his/her first quantum mechanics course that the "natural" value for the g-factor of a spin-1/2 particle is its Dirac value—$g_{S=\frac{1}{2}} = 2$—and this result is strongly confirmed experimentally in the case of the charged leptons (e, μ, τ) up to small electromagnetic corrections[3]. Of course, for particles such as the proton, which possesses an experimental g-factor nearly three times this value, one is not surprised because the "natural" value of 2 is modified by large strong interaction corrections. In fact, this is the situation with nearly every other known particle—strong interaction corrections obliterate any underlying "bare" value of the g-factor, making any direct experimental confrontation impossible. Nevertheless, it is intriguing to speculate theoretically what this value might be. One of the first physicists to do so was Belinfante[1]. Using minimal substitution he calculated directly the g-factor for a charged particle carrying 3/2 and determined $g_{S=\frac{3}{2}} = 2/3$. Knowing the result for unit spin[5]—$g_{S=1} = 1$—he suggested that the "natural" value for particles of *arbitrary* spin S is $g_S = 1/S$. This proposal has become known as the "Belinfante conjecture" and has in fact been confirmed rigorously by later authors in the case that the electromagnetic interaction is introduced via minimal substitution[6].

Despite this theoretical confirmation there have developed a number of reasons to doubt the naturalness of Belinfante's suggestion. One is the feature that besides the charged leptons, the only other charged particle which does not have strong interactions—the $W^\pm$-boson—does *not* obey this prediction. Rather, in the standard model we have $g_{W^\pm} = 2$[4]. Since, as mentioned above, this number has been confirmed experimentally, it is important to understand where the difference from Belinfante's calculation comes about.

2.1 $W^\pm$ Boson

A neutral spin 1 field $\phi_\mu(x)$ having mass m is described by the Proca Lagrangian density, which is of the form[7]

$$\mathcal{L}(x) = -\frac{1}{4}U_{\mu\nu}(x)U^{\mu\nu}(x) + \frac{1}{2}m^2\phi_\mu(x)\phi^\mu(x) \tag{1}$$

where

$$U_{\mu\nu}(x) = i\partial_\mu\phi_\nu(x) - i\partial_\nu\phi_\mu(x) \tag{2}$$

is the spin 1 field tensor. If the particle has charge e, we can generate a gauge-invariant form of Eq. 1 by use of the well-known minimal substitution[8]—defining

$$\pi_\mu = i\partial_\mu - eA_\mu(x) \tag{3}$$

and

$$U_{\mu\nu}(x) = \pi_\mu\phi_\nu(x) - \pi_\nu\phi_\mu(x) \tag{4}$$

the charged Proca Lagrangian density becomes

$$\mathcal{L}(x) = -\frac{1}{2}U^\dagger_{\mu\nu}(x)U^{\mu\nu}(x) + m^2\phi^\dagger_\mu(x)\phi^\mu(x) \tag{5}$$

Introducing the left-right derivative

$$D(x)\overleftrightarrow{\nabla}F(x) \equiv D(x)\nabla F(x) - (\nabla D(x))F(x) \tag{6}$$

the single-photon component of the interaction can be written as

$$\mathcal{L}_{int}(x) = ieA^\mu(x)\phi^{\alpha\dagger}(x)[\eta_{\alpha\beta}\overleftrightarrow{\nabla}_\mu - \eta_{\beta\mu}\nabla_\alpha]\phi^\beta(x) + \eta_{\alpha\mu}(\nabla_\beta\phi^{\alpha\dagger}(x))\phi^\beta(x) \quad (7)$$

so that the on-shell matrix element of the electromagnetic current becomes

$$\frac{1}{\sqrt{4E_fE_i}} < p_f, \epsilon_B|j_\mu|p_i, \epsilon_A >= -\frac{e}{\sqrt{4E_fE_i}}\left[2P_\mu\epsilon_B^*\cdot\epsilon_A - \epsilon_{A\mu}\epsilon_B^*\cdot q + \epsilon_{B\mu}^*\epsilon_A\cdot q\right] \quad (8)$$

where we have used the property $p_f\cdot\epsilon_B^* = p_i\cdot\epsilon_A = 0$ for the Proca polarization vectors. If we now look at the spatial piece of this term we find

$$\frac{1}{\sqrt{4E_fE_i}} < p_f, \epsilon_B|\vec{\epsilon_\gamma}\cdot\vec{j}|p_i, \epsilon_A >\simeq \frac{e}{2m}\vec{\epsilon}_\gamma\times\vec{q}\cdot\hat{\epsilon}_B^*\times\hat{\epsilon_A} = \frac{e}{2m} < 1, m_f|\vec{S}|1, m_i > \cdot\vec{B} \quad (9)$$

where we have used the result that in the Breit frame for a nonrelativistically moving particle

$$i\hat{\epsilon}_B^*\times\hat{\epsilon}_A =< 1, m_f|\vec{S}|1, m_i > \quad (10)$$

which we recognize as representing a magnetic moment interaction with g=1. On the other hand if we take the time component of Eq. 8, we find, again in the Breit frame and a nonrelativistically moving system

$$\frac{1}{\sqrt{4E_fE_i}} < p_f, \epsilon_B|\epsilon_{0\gamma}j_0|p_i, \epsilon_A >\simeq -e\epsilon_{0\gamma}\left[\epsilon_B^*\cdot\epsilon_A + \frac{1}{2m}(\epsilon_{A0}\hat{\epsilon}_B^*\cdot\vec{q} - \epsilon_{B0}^*\hat{\epsilon}_A\cdot\vec{q})\right] \quad (11)$$

Using

$$\begin{aligned} \epsilon_A^0 &\simeq \frac{1}{2m}\hat{\epsilon}_A\cdot\vec{q}, \qquad \epsilon_B^0 \simeq -\frac{1}{2m}\hat{\epsilon}_B^*\cdot\vec{q} \\ \epsilon_B^*\cdot\epsilon_A &\simeq -\hat{\epsilon}_B^*\cdot\hat{\epsilon}_A - \frac{1}{2m^2}\hat{\epsilon}_B^*\cdot\vec{q}\hat{\epsilon}_A\cdot\vec{q} \end{aligned} \quad (12)$$

we observe that

$$\frac{1}{\sqrt{4E_fE_i}} < p_f, \epsilon_B|\epsilon_{0\gamma}j_0|p_i, \epsilon_A >\simeq e\epsilon_{0\gamma}\hat{\epsilon}_B^*\cdot\hat{\epsilon}_A \quad (13)$$

which is the expected electric monopole term—any electric quadrupole contributions have cancelled[9]. Overall then, Eq. 8 corresponds to a simple

E0 interaction with the charge accompanied by an M1 interaction with g-factor unity, which is consistent with the speculation by Belinfante that for a particle of spin S, $g = 1/S$[1].

Despite this suggestively simple result, however, Eq. 1 does *not* correctly describe the interaction of the charged W-boson field, due to the feature that the $W^\pm$ are components of an SU(2) vector field[4]. The proper Proca Lagrangian has the form

$$\mathcal{L}(x) = -\frac{1}{4}\vec{U}^\dagger_{\mu\nu}(x)\cdot\vec{U}^{\mu\nu}(x) + \frac{1}{2}m_W^2\vec{\phi}_\mu(x)\cdot\vec{\phi}^\mu(x) \tag{14}$$

where the field tensor $\vec{U}_{\mu\nu}(x)$ contains an additional term on account of gauge invariance

$$\vec{U}_{\mu\nu}(x) = \pi_\mu\vec{U}_\nu(x) - \pi_\nu\vec{U}_\mu(x) - ig\vec{U}_\mu(x)\times\vec{U}_\nu(x) \tag{15}$$

with g being the SU(2) electroweak coupling coupling constant. The Lagrange density Eq. 14 then contains the piece

$$\mathcal{L}_{int}(x) = -gW^{0\mu\nu}(x)(W_\mu^{+\dagger}(x)W_\nu^+(x) - W_\mu^{-\dagger}(x)W_\mu^-)(x) \tag{16}$$

among (many) others. However, in the standard model the neutral member of the W-triplet is a linear combination of Z^0 and photon fields[10]—

$$W_\mu^0 = \cos\theta_W Z_\mu^0 + \sin\theta_W A_\mu \tag{17}$$

and, since $g\sin\theta_W = e$, we have a term in the interaction Lagrangian

$$\mathcal{L}_{int}^{(1)}(x) = -eF_{\mu\nu}(x)(W_\mu^{+\dagger}(x)W_\nu^+(x) - W_\mu^{-\dagger}(x)W_\mu^-(x)) \tag{18}$$

which represents an additional interaction that must be appended to the convention Proca result. In the Breit frame and for a nonrelativistically moving system we have

$$\frac{1}{\sqrt{4E_fE_i}} < p_f,\epsilon_B|\vec{\epsilon_\gamma}\cdot\vec{j}^{(1)}|p_i,\epsilon_A > \simeq \frac{e}{2m_W}\vec{\epsilon}_\gamma\times\vec{q}\cdot\hat{\epsilon}_B^*\times\hat{\epsilon_A} = \frac{e}{2m_W} < 1,m_f|\vec{S}|1,m_i > \cdot\vec{B} \tag{19}$$

and

$$\frac{1}{\sqrt{4E_fE_i}} < p_f,\epsilon_B|j_0^{(1)}|p_i,\epsilon_A > \simeq -e\frac{1}{2m_W}(\epsilon_A^0\hat{\epsilon}_B^*\cdot\vec{q} - \epsilon_{B0}^*\hat{\epsilon}_A\cdot\vec{q}) = -\frac{e}{2m_W^2}\hat{\epsilon}_B^*\cdot\vec{q}\hat{\epsilon}_A\cdot\vec{q} \tag{20}$$

The first piece—Eq. 19—constitutes an additional magnetic moment and modifies the W-boson g-factor from its Belinfante value of unity to its standard model value of 2. Using

$$\frac{1}{2}(\epsilon^*_{Bi}\epsilon_{Aj}+\epsilon_{Ai}\epsilon^*_{Bj})-\frac{1}{3}\delta_{ij}\hat{\epsilon}^*_B\cdot\hat{\epsilon}_A =< 1,m_f|\frac{1}{2}(S_iS_j+S_jS_i)-\frac{2}{3}\delta_{ij}|1,m_i> \quad (21)$$

we observe that the second component—Eq. 20—implies the existence of a quadrupole moment of size $Q=-e/M_W^2$. Both of these results are well known predictions of the standard model for the charged vector bosons and the standard model prediction for the g-factor is experimentally confirmed—$g_W = 2.20 \pm 0.20$ [3].

Of course, a single example does not constitute a compelling case, but it has recently been suggested, from a number of viewpoints, that the "natural" value of the gyromagnetic ratio for a particle of *arbitrary* spin is g_S=2[2]. We shall briefly review these arguments below and then will present a argument which buttresses this assertion.

2.2 GDH Sum Rule

Perhaps the first author to suggest the importance of $g_S = 2$ was Weinberg who, in Brandeis lecture notes, examined the low energy limit of Compton scattering[11]. In this way he was able to generalize the Gerasimov-Drell-Hearn (GDH) sum rule, which relates a particle's anomalous magnetic moment to a weighted integral over its polarized photoabsorption cross sections[12]. In the case of spin 1/2 this was shown by GDH to have the form

$$\frac{2\pi\alpha}{M^2}\kappa^2 = \frac{1}{\pi}\int_0^\infty \frac{d\omega}{\omega}\Delta\sigma_s(\omega) \quad (22)$$

where κ is the nucleon anomalous magnetic moment and

$$\Delta\sigma_s(\omega) = \sigma_{\frac{3}{2}}(\omega) - \sigma_{\frac{1}{2}}(\omega)$$

is the difference between the cross section measured with the incoming photon and target polarizations parallel and antiparallel. The sum rule has been well tested and has been shown to work in the case of the nucleon[12]. Weinberg demonstrated that the sum rule can be generalized to arbitrary spin provided one defines the anomalous magnetic moment via—

$$\vec{\mu} = \frac{e\vec{S}}{2m}g_S(1+\kappa)$$

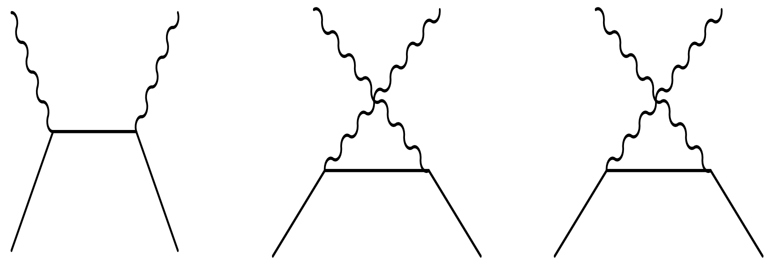

Figure 1: Diagrams relevant to Compton scattering.

with $g_S = 2$—independent of spin. From the perspective of the GDH sum rule then it is suggestive that the "natural" value of the g-factor is $g_S = 2$[11]. However, there also exists an argument from the realm of high energy Compton scattering[2].

2.3 Compton Scattering at High Energy

We next examine high energy Compton scattering from a basic spin-S target having mass m and charge e, and consider the case of spin one. As discussed above, the simple Proca interaction for a charged spin 1 system yields the Feynman rules for photon interactions[5]

$$\begin{aligned} PP\gamma: &= -ie\left\{(p_f+p_i)_\mu g_{\alpha\beta} - g_{\beta\mu}[gp_{f\alpha}-(g-1)p_{i\alpha}] - g_{\alpha\mu}[gp_{i\beta}-(g-1)p_{f\beta}]\right\} \\ PP\gamma\gamma: &= ie^2(2g_{\mu\nu}g_{\alpha\beta} - g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha}) \end{aligned} \quad (23)$$

where, for generality, we have included an anomalous moment (Pauli) interaction of the form

$$PP\gamma := -ie(g-1)F^{\mu\nu}(W_\mu^{+\dagger}W_\nu^{+} - W_\mu^{-\dagger}W_\nu^{-}) \quad (24)$$

Calculation of the three lowest order diagrams shown in Figure 1 then yields the result

$$\begin{aligned}
\mathrm{Amp}_{Compton}(S=1,g) &= e^2\{2\epsilon_A\cdot\epsilon_B^*\left[\frac{\epsilon_i\cdot p_i\epsilon_f^*\cdot p_f}{p_i\cdot k_i}-\frac{\epsilon_i\cdot p_f\epsilon_f^*\cdot p_i}{p_i\cdot k_f}-\epsilon_i\cdot\epsilon_f^*\right]\\
&- g\left[\epsilon_A\cdot[\epsilon_f^*,k_f]\cdot\epsilon_B^*\left(\frac{\epsilon_i\cdot p_i}{p_i\cdot k_i}-\frac{\epsilon_i\cdot p_f}{p_i\cdot k_f}\right)\right.\\
&- \left.\epsilon_A\cdot[\epsilon_i,k_i]\cdot\epsilon_B^*\left(\frac{\epsilon_f^*\cdot p_f}{p_i\cdot k_i}-\frac{\epsilon_f^*\cdot p_i}{p_i\cdot k_f}\right)\right]\\
&- g^2\left[\frac{1}{2p_i\cdot k_i}\epsilon_A\cdot[\epsilon_i,k_i]\cdot[\epsilon_f^*,k_f]\cdot\epsilon_B^*\right.\\
&- \left.\frac{1}{2p_i\cdot k_f}\epsilon_A\cdot[\epsilon_f^*,k_f]\cdot[\epsilon_i,k_i]\cdot\epsilon_B^*\right]\\
&- \frac{(g-2)^2}{m^2}\left[\frac{1}{2p_i\cdot k_i}\epsilon_A\cdot[\epsilon_i,k_i]\cdot p_i\epsilon_B^*\cdot[\epsilon_f^*,k_f]\cdot p_f\right.\\
&- \left.\frac{1}{2p_i\cdot k_f}\epsilon_A\cdot[\epsilon_f^*,k_f]\cdot p_i\epsilon_B^*\cdot[\epsilon_i,k_i]\cdot p_i\right]\}
\end{aligned}\qquad(25)$$

where we have defined

$$S\cdot[Q,R]\cdot T\equiv S\cdot QT\cdot R-S\cdot RT\cdot Q.$$

The interesting terms here are those on the last two lines, which are proportional to the factor $1/m^2$. They arise from the Born diagrams via the $k_\alpha k_\beta/m^2$ terms of the spin-one propagator

$$D_{\alpha\beta}(k)=\frac{i}{k^2-m^2}\times\left(-g_{\alpha\beta}+\frac{k_\alpha k_\beta}{m^2}\right)\qquad(26)$$

and reveal that if we take the limit as the mass becomes small the Compton amplitude will diverge, violating unitarity at a photon energy $\omega_i\sim m$ *unless the gyromagnetic ratio has the value* $g=2$, and this same condition can be shown to assure the absence of $1/m^2$ terms *for arbitrary spin*[2]! Again, this result is certainly suggests that the "natural" value of the g-factor is $g_S=2$, in agreement with the result found from the GDH sum rule.

2.4 Graviton Scattering and Factorization

Having reviewed "old" results[13], we now present a new argument which makes the case for $g=2$—factorization of graviton scattering amplitudes.

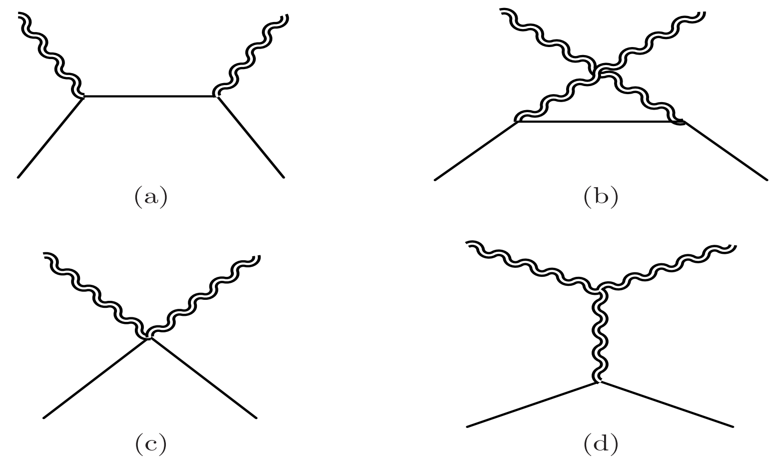

Figure 2: Diagrams relevant for gravitational Compton scattering.

Based on string theory arguments it has recently been pointed out that the elastic scattering of gravitons from a “bare” target of arbitrary spin should factorize[14]. This condition had been found earlier by Song et al. from gauge theory considerations[15]. That is, the graviton is a particle of spin 2 whose polarization tensor $\epsilon_{\mu\nu}$ can be written, in harmonic gauge, which we use henceforth, as a simple product of corresponding spin one photon polarization vectors—

$$\epsilon^{\pm 2}_{\mu\nu} = \epsilon^{\pm 1}_{\mu}\epsilon^{\pm 1}_{\nu}$$

The elastic scattering of gravitons from a target of arbitrary spin is constructed by summing the four diagrams shown in Figure 2, consisting of two Born pieces, a seagull term, and the graviton pole diagram. The factorization theorem asserts that for scattering from a target of spin S, the graviton scattering amplitude from an ”ideal” target particle of spin S can be written in the form

$$\epsilon^*_{f\alpha\beta}M^{\alpha\beta;\mu\nu}_{graviton}(S)\epsilon_{i\mu\nu} = \frac{\kappa^2}{8e^4}F\times[\epsilon^*_{f\alpha}A^{\alpha;\mu}_{Compton}(S)\epsilon_{i\mu}]\times[\epsilon^*_{f\beta}A^{\beta;\nu}_{Compton}(0)\epsilon_{i\nu}] \quad (27)$$

where $M^{\alpha\beta;\mu\nu}_{graviton}(S)$ is the elastic graviton scattering amplitude from a system of spin S, $A^{\alpha;\mu}_{Compton}(S)$ is the elastic Compton amplitude from a target of spin S and charge e, F is the kinematic factor

$$F = \frac{p_1\cdot k_1 p_1\cdot k_2}{k_1\cdot k_2} \quad (28)$$

and $\kappa^2 = 32\pi G$ is the gravitional coupling This is a remarkable result and dramatically simplifies the evaluation of graviton scattering. In the case of spin 0 the calculation is fairly straightforward. The four diagrams which must be included in order to satisfy gauge invariance are shown in Figure 2. From the Klein-Gordon Lagrangian

$$\mathcal{L}_{S=0} = \frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2\right) \tag{29}$$

we find the one- and two-graviton vertices[16]

$$\begin{aligned}
\tau_{\alpha\beta}(p,p') &= \frac{-i\kappa}{2}\left(p_\alpha {p'}_\beta + {p'}_\alpha p_\beta - \eta_{\alpha\beta}(p\cdot p' - m^2)\right)\\
\tau_{\alpha\beta,\gamma\delta}(p,p') &= i\kappa^2\left[I_{\alpha\beta,\rho\xi}I^{\xi}{}_{\sigma,\gamma\delta}\left(p^\rho p'^\sigma + p'^\rho p^\sigma\right)\right.\\
&- \frac{1}{2}\left(\eta_{\alpha\beta}I_{\rho\sigma,\gamma\delta} + \eta_{\gamma\delta}I_{\rho\sigma,\alpha\beta}\right)p'^\rho p^\sigma\\
&- \left.\frac{1}{2}\left(I_{\alpha\beta,\gamma\delta} - \frac{1}{2}\eta_{\alpha\beta}\eta_{\gamma\delta}\right)\left(p\cdot p' - m^2\right)\right]
\end{aligned} \tag{30}$$

while the triple graviton vertex has the form[17]

$$\begin{aligned}
\tau^{\mu\nu}_{\alpha\beta,\gamma\delta}(k,q) &= \frac{i\kappa}{2}\left\{P_{\alpha\beta,\gamma\delta}\left[k^\mu k^\nu + (k-q)^\mu(k-q)^\nu + q^\mu q^\nu - \frac{3}{2}\eta^{\mu\nu}q^2\right]\right.\\
&+ 2q_\lambda q_\sigma\left[I^{\lambda\sigma,}{}_{\alpha\beta}I^{\mu\nu,}{}_{\gamma\delta} + I^{\lambda\sigma,}{}_{\gamma\delta}I^{\mu\nu,}{}_{\alpha\beta} - I^{\lambda\mu,}{}_{\alpha\beta}I^{\sigma\nu,}{}_{\gamma\delta} - I^{\sigma\nu,}{}_{\alpha\beta}I^{\lambda\mu,}{}_{\gamma\delta}\right]\\
&+ [q_\lambda q^\mu(\eta_{\alpha\beta}I^{\lambda\nu,}{}_{\gamma\delta} + \eta_{\gamma\delta}I^{\lambda\nu,}{}_{\alpha\beta}) + q_\lambda q^\nu(\eta_{\alpha\beta}I^{\lambda\mu,}{}_{\gamma\delta} + \eta_{\gamma\delta}I^{\lambda\mu,}{}_{\alpha\beta})\\
&- q^2(\eta_{\alpha\beta}I^{\mu\nu,}{}_{\gamma\delta} + \eta_{\gamma\delta}I^{\mu\nu,}{}_{\alpha\beta}) - \eta^{\mu\nu}q^\lambda q^\sigma(\eta_{\alpha\beta}I_{\gamma\delta,\lambda\sigma} + \eta_{\gamma\delta}I_{\alpha\beta,\lambda\sigma})]\\
&+ [2q^\lambda(I^{\sigma\nu,}{}_{\alpha\beta}I_{\gamma\delta,\lambda\sigma}(k-q)^\mu + I^{\sigma\mu,}{}_{\alpha\beta}I_{\gamma\delta,\lambda\sigma}(k-q)^\nu\\
&- I^{\sigma\nu,}{}_{\gamma\delta}I_{\alpha\beta,\lambda\sigma}k^\mu - I^{\sigma\mu,}{}_{\gamma\delta}I_{\alpha\beta,\lambda\sigma}k^\nu)\\
&+ q^2(I^{\sigma\mu,}{}_{\alpha\beta}I_{\gamma\delta,\sigma}{}^\nu + I_{\alpha\beta,\sigma}{}^\nu I^{\sigma\mu,}{}_{\gamma\delta}) + \eta^{\mu\nu}q^\lambda q_\sigma(I_{\alpha\beta,\lambda\rho}I^{\rho\sigma,}{}_{\gamma\delta} + I_{\gamma\delta,\lambda\rho}I^{\rho\sigma,}{}_{\alpha\beta})]\\
&+ [(k^2 + (k-q)^2)\left(I^{\sigma\mu,}{}_{\alpha\beta}I_{\gamma\delta,\sigma}{}^\nu + I^{\sigma\nu,}{}_{\alpha\beta}I_{\gamma\delta,\sigma}{}^\mu - \frac{1}{2}\eta^{\mu\nu}P_{\alpha\beta,\gamma\delta}\right)\\
&- \left.(k^2\eta_{\gamma\delta}I^{\mu\nu,}{}_{\alpha\beta} + (k-q)^2\eta_{\alpha\beta}I^{\mu\nu,}{}_{\gamma\delta})]\right\}
\end{aligned} \tag{31}$$

where we have defined

$$I_{\alpha\beta,\mu\nu} = \frac{1}{2}(\eta_{\alpha\mu}\eta_{\beta\nu} + \eta_{\alpha\nu}\eta_{\beta\mu}) \tag{32}$$

and

$$P_{\alpha\beta,\mu\nu} = I_{\alpha\beta,\mu\nu} - \frac{1}{2}\eta_{\alpha\beta}\eta_{\mu\nu} \tag{33}$$

The other component which we require is the graviton propagator, which has the form in harmonic gauge

$$D_{\alpha\beta;\gamma\delta}(q) = \frac{iP_{\alpha\beta;\gamma\delta}}{q^2} \tag{34}$$

It is now straightforward (though tedious) to evaluate the four diagrams, yielding

Graviton Scattering: Spin 0

$$
\begin{aligned}
\text{Born}-\text{a}: &\qquad \text{Amp}_a(S=0) = -\kappa^2\frac{(\epsilon_i\cdot p_i)^2(\epsilon_f^*\cdot p_f)^2}{p_i\cdot k_i}\\
\text{Born}-\text{b}: &\qquad \text{Amp}_b(S=0) = \kappa^2\frac{(\epsilon_f^*\cdot p_i)^2(\epsilon_i\cdot p_f)^2}{p_i\cdot k_f}\\
\text{Seagull}: &\qquad \text{Amp}_c(S=0) = \kappa^2\left[\epsilon_f^*\cdot\epsilon_i(\epsilon_i\cdot p_i\epsilon_f^*\cdot p_f+\epsilon_i\cdot p_f\epsilon_f^*\cdot p_i)-\frac{1}{2}k_i\cdot k_f(\epsilon_f^*\cdot\epsilon_i)^2\right]\\
\text{g}-\text{pole}: &\qquad \text{Amp}_d(S=0) = \frac{\kappa^2}{2k_i\cdot k_f}\left[\epsilon_f^*\cdot p_f\epsilon_f^*\cdot p_i(\epsilon_i\cdot(p_i-p_f))^2\right.\\
&+\quad \epsilon_i\cdot p_i\epsilon_i\cdot p_f(\epsilon_f^*\cdot(p_i-p_f))^2\\
&+\quad \epsilon_i\cdot(p_i-p_f)\epsilon_f^*\cdot(p_f-p_i)(\epsilon_f^*\cdot p_f\epsilon_i\cdot p_i+\epsilon_f^*\cdot p_i\epsilon_i\cdot p_f)\\
&-\quad \epsilon_f^*\cdot\epsilon_i\big(\epsilon_i\cdot(p_i-p_f)\epsilon_f^*\cdot(p_f-p_i)(p_i\cdot p_f-m^2)\\
&+\quad k_i\cdot k_f(\epsilon_f^*\cdot p_f\epsilon_i\cdot p_i+\epsilon_f^*\cdot p_i\epsilon_i\cdot p_f)+\epsilon_i\cdot(p_i-p_f)(\epsilon_f^*\cdot p_fp_i\cdot k_f+\epsilon_f^*\cdot p_ip_f\cdot k_f)\\
&+\quad \epsilon_f^*\cdot(p_f-p_i)(\epsilon_i\cdot p_ip_f\cdot k_i+\epsilon_i\cdot p_fp_i\cdot k_i)\big)\\
&+\quad (\epsilon_f^*\cdot\epsilon_i)^2\Bigg(p_i\cdot k_ip_f\cdot k_i+p_i\cdot k_fp_f\cdot k_f-\frac{1}{2}(p_i\cdot k_ip_f\cdot k_f+p_i\cdot k_fp_f\cdot k_i)\\
&+\quad \left.\frac{3}{2}k_i\cdot k_f(p_i\cdot p_f-m^2)\Bigg)\right]
\end{aligned}
\tag{35}
$$

and when combined, one verifies that

$$
\begin{aligned}
\text{Amp}_a(S=0) \quad &+\quad \text{Amp}_b(S=0)+\text{Amp}_c(S=0)+\text{Amp}_d(S=0)\\
&=\quad \frac{\kappa^2}{8e^4}F\times[\text{Amp}_{Compton}(S=0)]^2
\end{aligned}
\tag{36}
$$

where

$$
\text{Amp}_{Compton}(S=0) = 2e^2\left[\frac{\epsilon_i\cdot p_i\epsilon_f^*\cdot p_f}{p_i\cdot k_i}-\frac{\epsilon_i\cdot p_f\epsilon_f^*\cdot p_i}{p_i\cdot k_f}-\epsilon_f^*\cdot\epsilon_i\right]
\tag{37}
$$

is the Compton scattering amplitude from a spinless target.

Similarly in the case of spin 1/2 we must use the vierbein ${e_a}^\mu$ in order to define the Dirac Lagrangian[16]

$$\mathcal{L}_{S=\frac{1}{2}} = \bar{\psi}(i\gamma^a {e_a}^\mu D_\mu - m)\psi \tag{38}$$

where the covariant derivative D_μ is given by

$$D_\mu = \partial_\mu + \frac{i}{4}\sigma^{ab}\omega_{\mu ab} \tag{39}$$

with

$$\omega_{\mu ab} = \frac{1}{2}{e_a}^\nu(\partial_\mu e_{b\nu} - \partial_\nu e_{b\mu}) - \frac{1}{2}{e_b}^\nu(\partial_\mu e_{a\nu} - \partial_\nu e_{a\mu}) + \frac{1}{2}{e_a}^\rho {e_b}^\sigma(\partial_\sigma e_{c\rho} - \partial_\rho e_{c\sigma}){e_\mu}^c \tag{40}$$

The resulting one and two graviton vertices are then:

$$\begin{aligned}
\tau_{\alpha\beta}(p,p') &= \frac{-i\kappa}{2}\left[\frac{1}{4}(\gamma_\alpha(p+p')_\beta + \gamma_\beta(p+p')_\alpha) - \eta_{\alpha\beta}(\frac{1}{2}(\not{p}+\not{p}') - m)\right] \\
\tau_{\alpha\beta,\gamma\delta}(p,p') &= i\kappa^2\left\{-\frac{1}{2}(\frac{1}{2}(\not{p}+\not{p}') - m)P_{\alpha\beta,\gamma\delta}\right. \\
&- \frac{1}{16}[\eta_{\alpha\beta}(\gamma_\gamma(p+p')_\delta + \gamma_\delta(p+p')_\gamma) \\
&+ \eta_{\gamma\delta}(\gamma_\alpha(p+p')_\beta + \gamma_\beta(p+p')_\alpha)] \\
&+ \frac{3}{16}(p+p')^\epsilon\gamma^\xi(I_{\xi\phi,\alpha\beta}{I^\phi}_{\epsilon,\gamma\delta} + I_{\xi\phi,\gamma\delta}{I^\phi}_{\epsilon,\alpha\beta}) \\
&- \left.\frac{i}{16}\epsilon^{\rho\sigma\eta\lambda}\gamma_\lambda\gamma_5({I_{\alpha\beta,\eta}}^\nu I_{\gamma\delta,\sigma\nu}{k'}_\rho - {I_{\gamma\delta,\eta}}^\nu I_{\alpha\beta,\sigma\nu}k_\rho)\right\}
\end{aligned} \tag{41}$$

The calculation is somewhat more challenging than in the case of spin 0 because of the Dirac algebra, but is still straightforward. Indeed, evaluating the same diagrams we find the individual contributions:

Graviton Scattering: Spin $\frac{1}{2}$

$$\text{Born}-\text{a}: \quad \text{Amp}_a(S=\frac{1}{2}) = -\kappa^2 \frac{\epsilon_f^*\cdot p_f \epsilon_i\cdot p_i}{8p_i\cdot k_i}\bar{u}(p_f)[\not{\epsilon}_f{}^*(\not{p}_i+\not{k}_i+m)\not{\epsilon}_i]u(p_i)$$

$$\text{Born}-\text{b}: \quad \text{Amp}_b(S=\frac{1}{2}) = \kappa^2 \frac{\epsilon_f^*\cdot p_i \epsilon_i\cdot p_f}{8p_i\cdot k_f}\bar{u}(p_f)[\not{\epsilon}_i(\not{p}_i-\not{k}_f+m)\not{\epsilon}_f^*]u(p_i)$$

$$\text{Seagull}: \quad \text{Amp}_c(S=\frac{1}{2}) = \kappa^2\bar{u}(p_f)\left[\frac{3}{16}\epsilon_f^*\cdot\epsilon_i(\not{\epsilon}_i\epsilon_f^*\cdot(p_i+p_f)+\not{\epsilon}_f^*\epsilon_i\cdot(p_i+p_f))\right.$$
$$\left. - \quad \frac{i}{16}\epsilon_f^*\cdot\epsilon_i\epsilon^{\rho\sigma\eta\lambda}\gamma_\lambda\gamma_5(\epsilon_{i\eta}\epsilon_{f\sigma}^*k_{f\rho}-\epsilon_{f\eta}^*\epsilon_{i\sigma}k_{i\rho})\right]u(p_i)$$

$$\text{g}-\text{pole}: \quad \text{Amp}_d(S=\frac{1}{2}) = \frac{\kappa^2}{16k_i\cdot k_f}\bar{u}(p_f)$$
$$\times \quad \left[\not{k}_i\epsilon_f^*\cdot\epsilon_i(-2\epsilon_i\cdot(p_i+p_f)\epsilon_f^*\cdot k_i+\epsilon_f^*\cdot\epsilon_i k_i\cdot(p_i+p_f))\right.$$
$$+ \quad \not{k}_f\epsilon_f^*\cdot\epsilon_i(-2\epsilon_f^*\cdot(p_i+p_f)\epsilon_i\cdot k_f+\epsilon_f^*\cdot\epsilon_i k_i\cdot(p_i+p_f))$$
$$+ \quad 4\not{\epsilon}_i[\epsilon_f^*\cdot k_i(\epsilon_i\cdot p_i\epsilon_f^*\cdot p_f-\epsilon_f^*\cdot p_i\epsilon_i\cdot p_f)+\epsilon_f^*\cdot\epsilon_i(\epsilon_f^*\cdot p_i p_f\cdot k_i-\epsilon_f^*\cdot p_f k_i\cdot p_i)]$$
$$+ \quad 4\not{\epsilon}_f^*[\epsilon_i\cdot k_f(\epsilon_i\cdot p_i\epsilon_f^*\cdot p_f-\epsilon_f^*\cdot p_i\epsilon_i\cdot p_f)+\epsilon_f^*\cdot\epsilon_i(\epsilon_i\cdot p_f k_f\cdot p_i-\epsilon_i\cdot p_i k_f\cdot p_f)]\Big]$$
$$\times \quad u(p_i) \tag{42}$$

Combining these terms, we reproduce again the factorization condition, but now in the form

$$\text{Amp}_a(S=\frac{1}{2}) + \text{Amp}_b(S=\frac{1}{2})+\text{Amp}_c(S=\frac{1}{2})+\text{Amp}_d(S=\frac{1}{2})$$
$$= \frac{\kappa^2}{8e^4}F*[\text{Amp}_{Compton}(S=0)]*[\text{Amp}_{Compton}(S=\frac{1}{2})] \tag{43}$$

where

$$\text{Amp}_{Compton}(S=\frac{1}{2}) = e^2\bar{u}(p_f)\left[\frac{\not{\epsilon}_f^*(\not{p}_i+\not{k}_i+m)\not{\epsilon}_i}{2p_i\cdot k_i}-\frac{\not{\epsilon}_i(\not{p}_f-\not{k}_i+m)\not{\epsilon}_f^*}{2p_f\cdot k_i}\right]u(p_i) \tag{44}$$

is the spin 1/2 Compton scattering amplitude.

Thus far we have verified factorization for spin 0 and spin 1/2, but we have learned nothing about conditions on the g-factor for particles of higher spin. This situation changes when we move to the case of spin 1, for which we use the Proca equation[7] discussed above. The one- and two-graviton vertices are then found to be

$$\begin{aligned}
\tau^{(1)}_{\beta,\alpha,\mu,\nu}(p_1,p_2) &= i\frac{\kappa}{2}\{(p_{1\mu}p_{2\nu}+p_{1\nu}p_{2\mu})\eta_{\alpha\beta}\\
&- p_{1\beta}(p_{2\mu}\eta_{\nu\alpha}+p_{2\nu}\eta_{\alpha\mu})\\
&- p_{2\alpha}(p_{1\mu}\eta_{\nu\beta}+p_{1\nu}\eta_{\beta\mu})\\
&+ (p_1\cdot p_2-m^2)(\eta_{\mu\alpha}\eta_{\nu\beta}+\eta_{\mu\beta}\eta_{\nu\alpha})\\
&- \eta_{\mu\nu}[(p_1\cdot p_2-m^2)\eta_{\alpha\beta}-p_{1\beta}p_{2\alpha}]\}\\
\tau^{(2)}_{\beta,\alpha,\mu,\nu,\rho,\sigma}(p_1,p_2) &= -i\frac{\kappa^2}{4}\{[p_{1\beta}p_{2\alpha}-\eta_{\alpha\beta}(p_1\cdot p_2-m^2)](\eta_{\mu\rho}\eta_{\nu\sigma}+\eta_{\mu\sigma}\eta_{\nu\rho}-\eta_{\mu\nu}\eta_{\rho\sigma})\\
&+ \eta_{\mu\rho}[\eta_{\alpha\beta}(p_{1\nu}p_{2\sigma}+p_{1\sigma}p_{2\nu})-\eta_{\alpha\nu}p_{1\beta}p_{2\sigma}-\eta_{\beta\nu}p_{1\sigma}p_{2\alpha}\\
&- \eta_{\beta\sigma}p_{1\nu}p_{2\alpha}-\eta_{\alpha\sigma}p_{1\beta}p_{2\nu}+(p_1\cdot p_2-m^2)(\eta_{\alpha\nu}\eta_{\beta\sigma}+\eta_{\alpha\sigma}\eta_{\beta\nu})]\\
&+ \eta_{\mu\sigma}[\eta_{\alpha\beta}(p_{1\nu}p_{2\rho}+p_{1\rho}p_{2\nu})-\eta_{\alpha\nu}p_{1\beta}p_{2\rho}-\eta_{\beta\nu}p_{1\rho}p_{2\alpha}\\
&- \eta_{\beta\rho}p_{1\nu}p_{2\alpha}-\eta_{\alpha\rho}p_{1\beta}p_{2\nu}+(p_1\cdot p_2-m^2)\eta_{\alpha\nu}\eta_{\beta\rho}+\eta_{\alpha\rho}\eta_{\beta\nu})]\\
&+ \eta_{\nu\rho}[\eta_{\alpha\beta}(p_{1\mu}p_{2\sigma}+p_{1\sigma}p_{2\mu})-\eta_{\alpha\mu}p_{1\beta}p_{2\sigma}-\eta_{\beta\mu}p_{1\sigma}p_{2\alpha}\\
&- \eta_{\beta\sigma}p_{1\mu}p_{2\alpha}-\eta_{\alpha\sigma}p_{1\beta}p_{2\mu}+(p_1\cdot p_2-m^2)(\eta_{\alpha\mu}\eta_{\beta\sigma}+\eta_{\alpha\sigma}\eta_{\beta\mu})]\\
&+ \eta_{\nu\sigma}[\eta_{\alpha\beta}(p_{1\mu}p_{2\rho}+p_{1\rho}p_{2\mu})-\eta_{\alpha\mu}p_{1\beta}p_{2\rho}-\eta_{\beta\mu}p_{1\rho}p_{2\alpha}\\
&- \eta_{\beta\rho}p_{1\mu}p_{2\alpha}-\eta_{\alpha\rho}p_{1\beta}p_{2\mu}+(p_1\cdot p_2-m^2)(\eta_{\alpha\mu}\eta_{\beta\rho}+\eta_{\alpha\rho}\eta_{\beta\mu})]\\
&- \eta_{\mu\nu}[\eta_{\alpha\beta}(p_{1\rho}p_{2\sigma}+p_{1\sigma}p_{2\rho})-\eta_{\alpha\rho}p_{1\beta}p_{2\sigma}-\eta_{\beta\rho}p_{1\sigma}p_{2\alpha}\\
&- \eta_{\beta\sigma}p_{1\rho}p_{2\alpha}-\eta_{\alpha\sigma}p_{1\beta}p_{2\rho}+(p_1\cdot p_2-m^2)(\eta_{\alpha\rho}\eta_{\beta\sigma}+\eta_{\beta\rho}\eta_{\alpha\sigma})]\\
&- \eta_{\rho\sigma}[\eta_{\alpha\beta}(p_{1\mu}p_{2\nu}+p_{1\nu}p_{2\mu})-\eta_{\alpha\mu}p_{1\beta}p_{2\nu}-\eta_{\beta\mu}p_{1\nu}p_{2\alpha}\\
&- \eta_{\beta\nu}p_{1\mu}p_{2\alpha}-\eta_{\alpha\nu}p_{1\beta}p_{2\mu}+(p_1\cdot p_2-m^2)(\eta_{\alpha\mu}\eta_{\beta\nu}+\eta_{\beta\mu}\eta_{\alpha\nu})]\\
&+ (\eta_{\alpha\rho}p_{1\mu}-\eta_{\alpha\mu}p_{1\rho})(\eta_{\beta\sigma}p_{2\nu}-\eta_{\beta\mu}p_{2\sigma})\\
&+ (\eta_{\alpha\sigma}p_{1\nu}-\eta_{\alpha\nu}p_{1\sigma})\eta_{\beta\rho}p_{2\mu}-\eta_{\beta\mu}p_{2\rho})\\
&+ (\eta_{\alpha\sigma}p_{1\mu}-\eta_{\alpha\mu}p_{1\sigma})(\eta_{\beta\rho}p_{2\nu}-\eta_{\beta\nu}p_{2\rho})\\
&+ (\eta_{\alpha\rho}p_{1\nu}-\eta_{\alpha\nu}p_{1\rho})(\eta_{\beta\sigma}p_{2\mu}-\eta_{\beta\mu}p_{2\sigma})\}
\end{aligned} \tag{45}$$

and the individual contributions from the four diagrams are somewhat more complex:

Graviton Scattering: Spin 1

$$\begin{aligned}
\text{Born}-\text{a}: \quad & \text{Amp}_a(S=1)=\kappa^2\frac{1}{2p_i\cdot k_i}[(\epsilon_i\cdot p_i)^2(\epsilon_f^*\cdot p_f)^2\epsilon_A\cdot\epsilon_B^* \\
- \quad & (\epsilon_f^*\cdot p_f)^2\epsilon_i\cdot p_i(\epsilon_A\cdot k_i\epsilon_B^*\cdot\epsilon_i+\epsilon_A\cdot\epsilon_i\epsilon_B^*\cdot p_i) \\
- \quad & (\epsilon_i\cdot p_i)^2\epsilon_f^*\cdot p_f(\epsilon_B^*\cdot\epsilon_f^*\epsilon_A\cdot p_f+\epsilon_B^*\cdot k_f\epsilon_A\cdot\epsilon_f^*) \\
+ \quad & \epsilon_i\cdot p_i\epsilon_f^*\cdot p_f\epsilon_i\cdot p_f\epsilon_A\cdot k_i\epsilon_B^*\cdot\epsilon_f^*+\epsilon_i\cdot p_i\epsilon_f^*\cdot p_f\epsilon_f^*\cdot p_i\epsilon_A\cdot\epsilon_i\epsilon_B^*\cdot k_f \\
+ \quad & (\epsilon_f^*\cdot p_f)^2\epsilon_B^*\cdot\epsilon_i\epsilon_A\cdot\epsilon_i p_i\cdot k_i+(\epsilon_i\cdot p_i)^2\epsilon_B^*\cdot\epsilon_f^*\epsilon_A\cdot\epsilon_f^* p_f\cdot k_f \\
+ \quad & \epsilon_i\cdot p_i\epsilon_f^*\cdot p_f(\epsilon_A\cdot k_i\epsilon_B^*\cdot k_f\epsilon_i\cdot\epsilon_f^*+\epsilon_B^*\cdot\epsilon_f^*\epsilon_A\cdot\epsilon_i p_i\cdot p_f) \\
- \quad & \epsilon_i\cdot p_i\epsilon_f^*\cdot p_i\epsilon_B^*\cdot\epsilon_f^*\epsilon_A\cdot\epsilon_i p_f\cdot k_f-\epsilon_f^*\cdot p_f\epsilon_i\cdot p_f\epsilon_A\cdot\epsilon_i\epsilon_B^*\cdot\epsilon_f^* p_i\cdot k_i \\
- \quad & \epsilon_i\cdot p_i\epsilon_A\cdot k_i\epsilon_B^*\cdot\epsilon_f^*\epsilon_f^*\cdot\epsilon_i p_f\cdot k_f-\epsilon_f^*\cdot p_f\epsilon_B^*\cdot k_f\epsilon_A\cdot\epsilon_i\epsilon_i\cdot\epsilon_f^* p_i\cdot k_i \\
+ \quad & \epsilon_A\cdot\epsilon_i\epsilon_B^*\cdot\epsilon_f^* p_i\cdot k_i p_f\cdot k_f\epsilon_i\cdot\epsilon_f^*-m^2\epsilon_B^*\cdot\epsilon_f^*\epsilon_A\cdot\epsilon_i\epsilon_f^*\cdot p_f\epsilon_i\cdot p_i] \\
\text{Born}-\text{b}: \quad & \text{Amp}_b(S=1)=-\kappa^2\frac{1}{2p_i\cdot k_f}[(\epsilon_f^*\cdot p_i)^2(\epsilon_i\cdot p_f)^2\epsilon_A\cdot\epsilon_B^* \\
+ \quad & (\epsilon_i\cdot p_f)^2\epsilon_f^*\cdot p_i(\epsilon_A\cdot k_f\epsilon_B^*\cdot\epsilon_f^*-\epsilon_A\cdot\epsilon_f^*\epsilon_B^*\cdot p_i) \\
+ \quad & (\epsilon_f^*\cdot p_i)^2\epsilon_i\cdot p_f(\epsilon_B^*\cdot k_i\epsilon_A\cdot\epsilon_i-\epsilon_B^*\cdot\epsilon_i\epsilon_A\cdot p_f) \\
- \quad & \epsilon_f^*\cdot p_i\epsilon_i\cdot p_f\epsilon_f^*\cdot p_f\epsilon_A\cdot k_f\epsilon_B^*\cdot\epsilon_i-\epsilon_f^*\cdot p_i\epsilon_i\cdot p_f\epsilon_i\cdot p_i\epsilon_A\cdot\epsilon_f^*\epsilon_B^*\cdot k_i \\
- \quad & (\epsilon_i\cdot p_f)^2\epsilon_B^*\cdot\epsilon_f^*\epsilon_A\cdot\epsilon_f^* p_i\cdot k_f-(\epsilon_f^*\cdot p_i)^2\epsilon_B^*\cdot\epsilon_i\epsilon_A\cdot\epsilon_i p_f\cdot k_i \\
+ \quad & \epsilon_f^*\cdot p_i\epsilon_i\cdot p_f(\epsilon_A\cdot k_f\epsilon_B^*\cdot k_i\epsilon_i\cdot\epsilon_f^*+\epsilon_B^*\cdot\epsilon_i\epsilon_A\cdot\epsilon_f^* p_i\cdot p_f) \\
+ \quad & \epsilon_f^*\cdot p_i\epsilon_i\cdot p_i\epsilon_B^*\cdot\epsilon_i\epsilon_A\cdot\epsilon_f^* p_f\cdot k_i+\epsilon_i\cdot p_f\epsilon_f^*\cdot p_f\epsilon_A\cdot\epsilon_f^*\epsilon_B^*\cdot\epsilon_i p_i\cdot k_f \\
- \quad & \epsilon_f^*\cdot p_i\epsilon_A\cdot k_f\epsilon_B^*\cdot\epsilon_i\epsilon_i\cdot\epsilon_f^* p_f\cdot k_i-\epsilon_i\cdot p_f\epsilon_B^*\cdot k_i\epsilon_A\cdot\epsilon_f^*\epsilon_f^*\cdot\epsilon_i p_i\cdot k_f \\
+ \quad & \epsilon_A\cdot\epsilon_f^*\epsilon_B^*\cdot\epsilon_i p_i\cdot k_f p_f\cdot k_i\epsilon_i\cdot\epsilon_f^*-m^2\epsilon_B^*\cdot\epsilon_i\epsilon_A\cdot\epsilon_f^*\epsilon_i\cdot p_f\epsilon_f^*\cdot p_i] \\
\text{seagull}: \quad & \text{Amp}_c(S=1)=-\frac{\kappa^2}{4}[(\epsilon_i\cdot\epsilon_f^*)^2(m^2-p_i\cdot p_f)\epsilon_A\cdot\epsilon_B^*+\epsilon_A\cdot p_f\epsilon_B^*\cdot p_i(\epsilon_i\cdot\epsilon_f^*)^2 \\
+ \quad & \epsilon_i\cdot p_i\epsilon_f^*\cdot p_f(2\epsilon_i\cdot\epsilon_f^*\epsilon_A\cdot\epsilon_B^*-2\epsilon_A\cdot\epsilon_2\epsilon_B^*\cdot\epsilon_1) \\
+ \quad & \epsilon_i\cdot p_f\epsilon_f^*\cdot p_i(2\epsilon_i\cdot\epsilon_f^*\epsilon_A\cdot\epsilon_B^*-2\epsilon_A\cdot\epsilon_i\epsilon_B^*\cdot\epsilon_f^*) \\
+ \quad & 2\epsilon_i\cdot p_i\epsilon_1\cdot p_f\epsilon_A\cdot\epsilon_f^*\epsilon_B^*\cdot\epsilon_f^*+2\epsilon_f^*\cdot p_f\epsilon_f^*\cdot p_i\epsilon_A\cdot\epsilon_i\epsilon_B^*\cdot\epsilon_i \\
- \quad & 2\epsilon_i\cdot p_i\epsilon_i\cdot\epsilon_f^*\epsilon_A\cdot p_f\epsilon_B^*\cdot\epsilon_f^*-2\epsilon_f^*\cdot p_f\epsilon_i\cdot\epsilon_f^*\epsilon_A\cdot\epsilon_i\epsilon_f^*\cdot p_i \\
- \quad & 2\epsilon_i\cdot p_f\epsilon_i\cdot\epsilon_f^*\epsilon_A\cdot\epsilon_f^*\epsilon_B^*\cdot p_i-2\epsilon_f^*\cdot p_i\epsilon_i\cdot\epsilon_f^*\epsilon_B^*\cdot\epsilon_i\epsilon_A\cdot p_f \\
- \quad & 2(m^2-p_f\cdot p_i)\epsilon_i\cdot\epsilon_f^*(\epsilon_A\cdot\epsilon_i\epsilon_B^*\cdot\epsilon_f^*+\epsilon_A\cdot\epsilon_f^*\epsilon_B^*\cdot\epsilon_i)]
\end{aligned} \tag{46}$$

and finally the (lengthy) graviton pole contribution is

$$
\begin{aligned}
\mathrm{g-pole}: \qquad & \mathrm{Amp}_d(S=1) = -\frac{\kappa^2}{16k_i\cdot k_f}\{\epsilon_B^*\cdot\epsilon_A[(\epsilon_i\cdot\epsilon_f^*)^2(4k_i\cdot p_ip_f\cdot k_i+4k_f\cdot p_ik_f\cdot p_f \\
- & \ 2(p_i\cdot k_ip_f\cdot k_f+p_f\cdot k_ip_i\cdot k_f)+6p_i\cdot p_fk_i\cdot k_f) \\
+ & \ 4((\epsilon_i\cdot k_f)^2\epsilon_f^*\cdot p_f\epsilon_f^*\cdot p_i+(\epsilon_f^*\cdot k_i)^2\epsilon_i\cdot p_i\epsilon_i\cdot p_f \\
+ & \ \epsilon_i\cdot k_f\epsilon_f^*\cdot k_i(\epsilon_i\cdot p_i\epsilon_f^*\cdot p_f+\epsilon_i\cdot p_f\epsilon_f^*\cdot p_i)) \\
- & \ 4\epsilon_i\cdot\epsilon_f^*(\epsilon_i\cdot k_f(\epsilon_f^*\cdot p_ip_f\cdot k_f+\epsilon_f^*\cdot p_fk_f\cdot p_i) \\
+ & \ \epsilon_f^*\cdot k_i(\epsilon_i\cdot p_ip_f\cdot k_i+\epsilon_i\cdot p_fp_i\cdot k_i)) \\
- & \ 4k_i\cdot k_f\epsilon_i\cdot\epsilon_f^*(\epsilon_i\cdot p_i\epsilon_f^*\cdot p_f+\epsilon_i\cdot p_f\epsilon_f^*\cdot p_i)-4p_i\cdot p_f\epsilon_i\cdot\epsilon_f^*\epsilon_i\cdot k_f\epsilon_f^*\cdot k_i] \\
- & \ (p_i\cdot p_f\epsilon_B^*\cdot\epsilon_A-\epsilon_B^*\cdot p_i\epsilon_A\cdot p_f)[10(\epsilon_i\cdot\epsilon_f^*)^2k_i\cdot k_f+4\epsilon_i\cdot\epsilon_f^*\epsilon_i\cdot k_f\epsilon_f^*\cdot k_i \\
- & \ 4(\epsilon_i\cdot\epsilon_f^*)^2k_i\cdot k_f-8\epsilon_i\cdot\epsilon_f^*\epsilon_i\cdot k_f\epsilon_f^*\cdot k_i] \\
+ & \ (p_i\cdot p_f-m^2)[(\epsilon_i\cdot\epsilon_f^*)^2(4\epsilon_A\cdot k_i\epsilon_B^*\cdot k_i+4\epsilon_A\cdot k_f\epsilon_B^*\cdot k_f \\
- & \ 2(\epsilon_A\cdot k_i\epsilon_B^*\cdot k_f+\epsilon_A\cdot k_f\epsilon_B^*\cdot k_i)+6\epsilon_B^*\cdot\epsilon_Ak_i\cdot k_f) \\
+ & \ 4[(\epsilon_i\cdot k_f)^2\epsilon_A\cdot\epsilon_f^*\epsilon_B^*\cdot\epsilon_f^*+(\epsilon_f^*\cdot k_i)^2\epsilon_A\cdot\epsilon_i\epsilon_B^*\cdot\epsilon_i \\
+ & \ \epsilon_i\cdot k_f\epsilon_f^*\cdot k_f(\epsilon_A\cdot\epsilon_i\epsilon_B^*\cdot\epsilon_f^*+\epsilon_A\cdot\epsilon_f^*\epsilon_B^*\cdot\epsilon_i)] \\
- & \ 4\epsilon_i\cdot\epsilon_f^*[\epsilon_i\cdot k_f(\epsilon_A\cdot\epsilon_f^*\epsilon_B^*\cdot k_f+\epsilon_B^*\cdot\epsilon_f^*\epsilon_A\cdot k_f) \\
+ & \ \epsilon_f^*\cdot k_i(\epsilon_A\cdot\epsilon_i\epsilon_B^*\cdot k_i+\epsilon_B^*\cdot\epsilon_i\epsilon_A\cdot k_i) \\
+ & \ k_i\cdot k_f(\epsilon_A\cdot\epsilon_i\epsilon_B^*\cdot\epsilon_f^*+\epsilon_B^*\cdot\epsilon_i\epsilon_A\cdot\epsilon_f^*)+\epsilon_A\cdot\epsilon_B^*\epsilon_i\cdot k_f\epsilon_f^*\cdot k_i]] \\
- & \ 2\epsilon_A\cdot p_f[(\epsilon_f^*\cdot\epsilon_i)^2[2\epsilon_B^*\cdot k_ip_i\cdot k_i+2\epsilon_B^*\cdot k_fp_i\cdot k_f+3\epsilon_B^*\cdot p_ik_i\cdot k_f \\
- & \ (\epsilon_B^*\cdot k_ip_i\cdot k_f+\epsilon_B^*\cdot k_fp_i\cdot k_i)] \\
+ & \ 2(\epsilon_i\cdot k_f)^2\epsilon_B^*\cdot\epsilon_f^*\epsilon_f^*\cdot p_i+2(\epsilon_f^*\cdot k_i)^2\epsilon_B^*\cdot\epsilon_i\epsilon_i\cdot p_i \\
+ & \ 2\epsilon_i\cdot k_f\epsilon_f^*\cdot k_i(\epsilon_B^*\cdot\epsilon_i\epsilon_f^*\cdot p_i+\epsilon_i\cdot p_i\epsilon_B^*\cdot\epsilon_f^*) \\
- & \ 2\epsilon_i\cdot\epsilon_f^*[\epsilon_i\cdot k_f(\epsilon_B^*\cdot\epsilon_f^*p_i\cdot k_f+\epsilon_f^*\cdot p_i\epsilon_B^*\cdot k_f) \\
+ & \ \epsilon_f^*\cdot k_i(\epsilon_B^*\cdot\epsilon_ip_i\cdot k_i+\epsilon_B^*\cdot k_i\epsilon_i\cdot p_i)] \\
- & \ 2k_i\cdot k_f\epsilon_i\cdot\epsilon_f^*(\epsilon_B^*\cdot\epsilon_i\epsilon_f^*\cdot p_i+\epsilon_B^*\cdot\epsilon_f^*\epsilon_i\cdot p_i)-2\epsilon_B^*\cdot p_i\epsilon_i\cdot\epsilon_f^*\epsilon_i\cdot k_f\epsilon_f^*\cdot k_i] \\
- & \ 2\epsilon_B^*\cdot p_i[(\epsilon_f^*\cdot\epsilon_i)^2[2\epsilon_A\cdot k_ip_f\cdot k_i+2\epsilon_A\cdot k_fp_f\cdot k_f+3\epsilon_A\cdot p_fk_i\cdot k_f \\
- & \ (\epsilon_A\cdot k_ip_f\cdot k_f+\epsilon_A\cdot k_fp_f\cdot k_f)] \\
+ & \ 2(\epsilon_i\cdot k_f)^2\epsilon_A\cdot\epsilon_f^*\epsilon_f^*\cdot p_f+2(\epsilon_f^*\cdot k_i)^2\epsilon_A\cdot\epsilon_i\epsilon_i\cdot p_f \\
+ & \ 2\epsilon_i\cdot k_f\epsilon_f^*\cdot k_i(\epsilon_A\cdot\epsilon_i\epsilon_f^*\cdot p_f+\epsilon_i\cdot p_f\epsilon_A\cdot\epsilon_f^* \\
- & \ 2\epsilon_i\cdot\epsilon_f^*[\epsilon_i\cdot k_f(\epsilon_A\cdot\epsilon_f^*p_f\cdot k_f+\epsilon_f^*\cdot p_f\epsilon_A\cdot k_f) \\
+ & \ \epsilon_f^*\cdot k_i(\epsilon_A\cdot\epsilon_ip_f\cdot k_i+\epsilon_A\cdot k_i\epsilon_i\cdot p_f)] \\
- & \ 2k_i\cdot k_f\epsilon_i\cdot\epsilon_f^*(\epsilon_A\cdot\epsilon_i\epsilon_f^*\cdot p_f+\epsilon_A\cdot\epsilon_f^*\epsilon_i\cdot p_f)-2\epsilon_A\cdot p_f\epsilon_i\cdot\epsilon_f^*\epsilon_i\cdot k_f\epsilon_f^*\cdot k_i]\}
\end{aligned} \tag{47}
$$

Nevertheless, when combined (after considerable effort) one finds once again the factorization condition to be valid, this time in the form

$$\begin{aligned}\text{Amp}_a(S=1) &+ \text{Amp}_b(S=1)+\text{Amp}_c(S=1)+\text{Amp}_d(S=1)\\ &= \frac{\kappa^2}{8e^4}F*[\text{Amp}_{Compton}(S=0)]*[\text{Amp}_{Compton}(S=1,g=2)]\end{aligned} \tag{48}$$

where $\text{Amp}_{Compton}(S=1,g=2)$ is the Compton scattering amplitude quoted earlier in Eq. 25 *with the g-factor set equal to 2*.

From the gravitational side of Eq, 48, the full graviton scattering amplitude might naively be expected to contain terms proportional to $1/m^2$ from the Born diagrams and the piece of the spin 1 propagator proportional to $1/m^2$. However, this does *not* occur, as can be seen from the half-off-shell form of the single graviton coupling given above

$$\begin{aligned}< p_2,\lambda|T_{\mu\nu}|p_1,\epsilon_A > &= \epsilon_{A\lambda}(p_{2\mu}p_{1\nu}+p_{1\nu}p_{2\mu})+g_{\mu\nu}p_{1\lambda}\epsilon_A\cdot p_2\\ &- p_{1\lambda}(p_{2\mu}\epsilon_{A\nu}+p_{2\nu}\epsilon_{A\mu})-\epsilon_A\cdot p_2(p_{1\mu}g_{\lambda\nu}+p_{1\nu}g\lambda\mu)\\ &+ (p_2\cdot p_1-m^2)(g_{\lambda\mu}\epsilon_{A\nu}+\epsilon_{A\mu}g_{\lambda\nu}-g_{\mu\nu}\epsilon_{A\lambda})\end{aligned} \tag{49}$$

Setting $p_2=p_1+k$ and contracting with the intermediate state momentum $(p_1+k)^\lambda$ we find a result proportional to m^2—

$$(p_1+k)^\lambda < p_1+k_1,\lambda|T_{\mu\nu}|p_1,\epsilon_A >= m^2(g_{\mu\nu}\epsilon_A\cdot k_1-\epsilon_{A\mu}(p_1+k_1)_\nu-\epsilon_{A\nu}(p_1+k_1)_\mu) \tag{50}$$

This term cancels the factor $1/m^2$ from the spin one propagator so that no term proportional to $1/m^2$ survives in the Born amplitude and this vanishing of terms which diverge as $m\to 0$ can be shown to be a general property regardless of the spin of the target. If we acknowledge the validity of the factorization result, then the vanishing of $1/m^2$ terms in the gravitational amplitude can only result from the vanishing of such terms in the corresponding Compton amplitude, which we have already argued occurs only if the value $g=2$ is chosen, so from a new standpoint—-factorization of graviton scattering amplitudes—we see again that the "natural" value for the g-factor is $g_S=2$.

3 Conclusions

Above we have examined the question of the "natural" value for the g-factor of a particle of spin S. Although the simple minimal substitution gives rise to the Belinfante conjecture $g_S = 1/S$[1], we pointed out that more recent studies have suggested a correctness of a universal value—$g_S = 2$—independent of spin. We first pointed out that this arises from the well known features:

i) the standard model g-factor of the charged W-boson is 2

ii) the GDH sum rule provides a measure of the quantity $(g_S - 2)^2$ in the case of arbitrary spin[12]. If we use this sum rule to *define* the anomalous magnetic moment then clearly the “natural” value for the gyromagnetic ratio is $g_S = 2$[11]

iii) in high energy Compton scattering from a target of arbitrary spin the choice of a gyromagnetic ratio different from 2 leads to terms which are divergent in the small mass limit[2] and which violate unitarity at photon energies $\omega \sim m$.

We then presented a new argument for the correctness of this assertion—factorization in graviton scattering. Gauge invariance and string theory arguments make the case that the graviton scattering amplitude for a “bare” target having spin S should factor into pieces proportional to the product of the Compton scattering amplitude for spin 0 times the Compton scattering amplitude for spin S times a universal kinematic factor. Since the graviton scattering amplitude does not contain terms involving the inverse mass squared of the target particle, the same must be true for the Compton amplitudes, but this is true only if the gyromagnetic ratio has the value 2.

Again we emphasize that there is little experimental content in this prediction—except for the well-verified cases of the charged e, μ, τ leptons and the charged W boson, all of which carry $g \simeq 2$ in the standard model. Nevertheless, the question of the existence of a “natural” value for the g-factor is an intriguing one, to which we have provided new input.

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$$\frac{dS^\mu}{d\tau} = \frac{eg}{2m}F^{\mu\nu}S_\nu + \frac{e}{2m}(g-2)\frac{dx^\mu}{d\tau}F^{\nu\lambda}S_\nu\frac{dx_\lambda}{d\tau} \tag{51}$$

which simplifies for $g = 2$ independent of the spin of the particle. However, this is not at the same level as the quantum arguments given above.

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