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EQUIVARIANT \((K-)\)HOMOLOGY OF AFFINE GRASSMANNIAN AND TODA LATTICE

ROMAN BEZRUCA VNIKOV, MICHAEL FINKELBERG, AND IVAN MIRKOVIĆ

1. Introduction

1.1. Let \(G\) be an almost simple complex algebraic group, and let \(\text{Gr}_G\) be its affine Grassmannian. Recall that if we set \(\mathcal{O} = \mathbb{C}[\![t]\!]\), \(\mathcal{F} = \mathbb{C}(\!(t)\!)\), then \(\text{Gr}_G = G(\mathcal{F})/G(\mathcal{O})\).

It is well-known that the subgroup \(\Omega_K\) of polynomial loops into a maximal compact subgroup \(K \subset G\) projects isomorphically to \(\text{Gr}_G\); thus \(\text{Gr}_G\) acquires the structure of a topological group. An algebro-geometric counterpart of this structure is provided by the convolution diagram \(G(\mathcal{F}) \times_{G(\mathcal{O})} \text{Gr}_G \rightarrow \text{Gr}_G\).

It allows one to define the convolution of two \(G(\mathcal{O})\) equivariant geometric objects (such as sheaves, or constrictible functions) on \(\text{Gr}_G\). A famous example of such a structure is the category of \(G(\mathcal{O})\) equivariant perverse sheaves on \(\text{Gr}\) (“Satake category” in the terminology of Beilinson and Drinfeld); this is a semi-simple abelian category, and convolution provides it with a symmetric monoidal structure. By results of [10], [19], [2] this category is identified with the category of (algebraic) representations of the Langlands dual group.

The starting point for the present work was the observation that a similar definition works in another setting, yielding a monoidal structure on the category of \(G(\mathcal{O})\) equivariant perverse coherent sheaves on \(\text{Gr}_G\): existence of this t-structure is due to the fact that dimensions of all \(G(\mathcal{O})\)-orbits inside a given component of \(\text{Gr}_G\) are of the same parity, cf. [3]. The resulting monoidal category turns out to be non-symmetric, though its Grothendieck ring \(K^{G(\mathcal{O})}(\text{Gr}_G)\) is commutative. One of the results of this paper is a computation of this ring. Along with \(K^{G(\mathcal{O})}(\text{Gr}_G)\) we compute its “graded version”, the ring \(H^{G(\mathcal{O})}(\text{Gr})\) of equivariant homology of \(\text{Gr}\), where the algebra structure is again provided by convolution.\(^1\) (The ring \(H^{G(\mathcal{O})}(\text{Gr}_G)\) was essentially computed by Dale Peterson [20], cf. also [15].)

To describe the answer, let \(\hat{G}\) be the Langlands dual group to \(G\), and let \(\hat{\mathfrak{g}}\) be its Lie algebra. Consider the universal centralizers \(\mathcal{Z}_\theta^\hat{G}\) and \(\mathcal{Z}_G^\hat{G}\); if we denote by \(C_{\hat{G},\hat{\mathfrak{g}}} \subset \hat{G} \times \hat{\mathfrak{g}}\) (resp. \(C_{\hat{G},\hat{\mathfrak{g}}} \subset \hat{G} \times \hat{G}\)) the locally closed subvariety formed by all the pairs \((g, x)\) such that \(\text{Ad}_g(x) = x\) and \(x\) is regular (resp. all the pairs \((g_1, g_2)\) such that \(\text{Ad}_{g_1}g_2 = g_2\) and

\(^1\)The two rings are related via the Chern character homomorphism from \(K^{G(\mathcal{O})}(\text{Gr})\) to the completion of \(H^{G(\mathcal{O})}(\text{Gr})\).
$g_2$ is regular), then $\mathfrak{z}_G^\mathcal{G}$ (resp. $\mathfrak{z}_G^\mathcal{G}$) is the categorical quotient $C_{\mathcal{G},\mathfrak{g}}/\mathcal{G}$ (resp. $C_{\mathcal{G},\mathcal{G}}/\mathcal{G}$) with respect to the diagonal adjoint action of $\mathcal{G}$.

We identify Spec $\left( \mathcal{H}^{G(O)}(\text{Gr}_G) \right)$ with $\mathfrak{z}_G^\mathcal{G}$. Also, we identify Spec $\left( K_{G(O)}(\text{Gr}_G) \right)$ with a variant of $\mathfrak{z}_G^\mathcal{G}$ (the isomorphism Spec $\left( K_{G(O)}(\text{Gr}_G) \right) \simeq \mathfrak{z}_G^\mathcal{G}$ holds true iff $G$ is of type $E_8$).

Notice that $\mathfrak{z}_G^\mathcal{G}$ inherits a canonical symplectic structure as a hamiltonian reduction of the cotangent bundle $T^*\mathcal{G}$. Also, $\mathfrak{z}_G^\mathcal{G}$ inherits a canonical Poisson structure as a q-Hamiltonian reduction of the q-Hamiltonian $\mathcal{G}$-space internal fusion double $D(\mathcal{G})$ (see [1]); this Poisson structure is in fact symplectic iff $\mathcal{G}$ is simply connected (that is, $G$ is adjoint).

The corresponding Poisson structures on $K^{G(O)}(\text{Gr}_G)$, $H^{G(O)}(\text{Gr}_G)$ come from a deformation of these commutative algebras to non-commutative algebras $H^\bullet_{G(O)} \times G_m(\text{Gr}_G)$ (resp. $K^{G(O)} \times G_m(\text{Gr}_G)$); here $G_m$ acts on $\text{Gr}_G$ by loop rotation. We conjecture that the non-commutative algebra $H^\bullet_{G(O)} \times G_m(\text{Gr}_G)$ can also be obtained from the ring of differential operators on $\mathcal{G}$ by quantum Hamiltonian reduction.

The space $\mathfrak{z}_G^\mathcal{G}$ contains an open piece $\mathfrak{z}(\mathcal{G})$ which for $\mathcal{G}$ adjoint (that is, for $G$ simply connected) is a complexification of the Kostant’s phase space of the classical Toda lattice ([14], Theorem 2.6). We remark in passing that Toda lattice also appears in the (apparently related) computations by Givental, Kim and others of quantum cohomology of flag varieties (see e.g. [13]).

Our computation should be compared with (and is to a large extent inspired by) [10] where equivariant cohomology $H_{G(O)}(\text{Gr}_G)$ were computed\(^2\) in terms of the $\mathcal{G}$. (The precise relation between the two computations is spelled out in Remark 2.13).

The second main object considered in the paper is another derived category of coherent sheaves with a convolution monoidal structure, namely the derived category $D^b\mathcal{Coh}^{G(O)}_{\Lambda_G}(T^*\text{Gr})$ of $G(O)$-equivariant coherent sheaves on the cotangent bundle of $\text{Gr}_G$ supported on the union $\Lambda_G$ of conormal bundles to the $G(O)$-orbits (the definition of involved objects requires extra work since $\text{Gr}_G$ is infinite dimensional). (In this case we do not find a $t$-structure compatible with convolution, so all we get is a monoidal triangulated category). Notice that the singular support of a $G(O)$-equivariant $D$-module on $\text{Gr}_G$ is an object of $\text{Coh}^{G(O)}_{\Lambda_G}(T^*\text{Gr})$, thus this category can be considered a “classical limit” of the (derived) Satake category. We compute the Grothendieck ring of $D^b\mathcal{Coh}^{G(O)}_{\Lambda_G}(T^*\text{Gr})$ identifying its spectrum with $(T \times \check{T})/W$, where $T \subset G$, and $\check{T} \subset \check{G}$ are Cartan subgroups. This is a singular variety birationally equivalent to Spec $\left( K^{G(O)}(\text{Gr}_G) \right)$. Unlike the latter, the former remains unchanged if we replace $G$ by $\check{G}$. This motivates a conjecture that the corresponding triangulated monoidal categories for $G$ and $\check{G}$ are equivalent. The conjecture is compatible with a “classical

\(^2\)Another description for $H_{G(O)}(\text{Gr}_G)$ is provided by a general result of [16]; in fact, its extension from [17] gives also an answer for $K^{G(O)}(\text{Gr}_G)$, and a similar technique can be applied to compute $H^{G(O)}(\text{Gr}_G)$. However, this form of the answer does not make the relation to the (dual) group geometry explicit.
limit” of the geometric Langlands conjecture of Beilinson and Drinfeld (see 7.9 below for a more precise statement of the conjecture).

Finally, we remark that the convolution of $G(O)$-equivariant perverse coherent sheaves is closely related to the fusion product of $G(O)$-modules introduced by B. Feigin\footnote{The relation between convolution and fusion was known to B. Feigin since 1997.} [6] (see Section 8). In fact, our desire to understand the category $\mathcal{P}^G(O)(Gr_G)$, and the work [6] of B. Feigin and S. Loktev, was one of the motivations for the present work.

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2. Notations and statements of the results

2.1. Kostant slices. $G$ is an almost simple algebraic group with the Lie algebra $\mathfrak{g}$. We choose a principal $\mathfrak{sl}_2$ triple $(e, h, f)$ in $\mathfrak{g}$. Let $\phi: \mathfrak{sl}_2 \to \mathfrak{g}$ (resp. $\Phi: SL_2 \to G$) be the corresponding homomorphism. We denote by $e_G$ (resp. $f_G$) the image $\Phi \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (resp. $\phi \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$). We denote by $\mathcal{Z}(e)$ the centralizer of $e$ in $\mathfrak{g}$, and by $Z(e)$ (resp. $Z^0(e)$) the centralizer of $e$ (equivalently, of $e_G$) in $G$ (resp. its neutral connected component). We denote by $\Sigma_e \subset \mathfrak{g}$ (resp. $\Sigma_G \subset G$) the Kostant slice $\mathcal{Z}(e) + f$ (resp. $Z^0(e) \cdot f_G$). It is known that $\Sigma_e \subset \mathfrak{g}^{reg}$ (resp. $\Sigma_G \subset G^{reg}$), and the projection to the categorical quotient $\Sigma_e \hookrightarrow \mathfrak{g} \twoheadrightarrow \mathfrak{g}/Ad_G = t/W$ induces an isomorphism $\Sigma_e \simeq t/W$. Similarly, if $G$ is simply connected, the projection to the categorical quotient $\Sigma_G \hookrightarrow G \twoheadrightarrow G//Ad_G = T/W$ induces an isomorphism $\Sigma_G \simeq T/W$.

2.2. The universal centralizers. We consider the locally closed subvariety $C_{e,G} \subset \mathfrak{g} \times \mathfrak{g}$ (resp. $C_{e,G} \subset \mathfrak{g} \times G$, $C_{e,G} \subset G \times \mathfrak{g}$, $C_{e,G} \subset G \times G$) formed by all the pairs $(x_1, x_2)$ such that $[x_1, x_2] = 0$ and $x_2$ is regular (resp. all the pairs $(x, g)$ such that $Ad_g(x) = x$ and $g$ is regular; all the pairs $(g, x)$ such that $Ad_g(x) = x$ and $x$ is regular; all the pairs $(g_1, g_2)$ such that $Ad_{g_1}(g_2) = g_2$ and $g_2$ is regular). The categorical quotients with respect to the diagonal adjoint action of $G$ are denoted respectively $C_{e,G}//G = \mathfrak{g}^e$, $C_{e,G}//G = \mathfrak{g}^e$, $C_{e,G}//G = \mathfrak{g}^{reg}$, $C_{e,G}//G = \mathfrak{g}^{reg}$. The projections to the second (regular) factor are denoted by $\varpi: \mathfrak{g}^e \to \mathfrak{g}^{reg}//G = t/W$, $\varpi: \mathfrak{g}^{reg} \to \mathfrak{g}^{reg}//G = T/W$. In all the four cases $\varpi$ is flat.

We consider the restrictions of our centralizer varieties to the Kostant slices: $C_{e,G}^\Sigma = C_{e,G} \cap (\mathfrak{g} \times \Sigma_G)$, $C_{e,G}^\Sigma = C_{e,G} \cap (\mathfrak{g} \times \Sigma_G)$, $C_{e,G}^\Sigma = C_{e,G} \cap (G \times \Sigma_G)$, $C_{e,G}^\Sigma = C_{e,G} \cap (G \times \Sigma_G)$.
Then the locally closed embedding $C^\Sigma_{g,g} \hookrightarrow C_{g,g} \to \mathfrak{Z}_\theta$ induces an isomorphism $C^\Sigma_{g,g} \simeq \mathfrak{Z}_\theta$. Similarly, we have isomorphisms $C^\Sigma_{G,g,G} \simeq \mathfrak{Z}_G$ and (for simply connected $G$) $C^\Sigma_{g,G} \simeq \mathfrak{Z}_G$. Thus both $\mathfrak{Z}_\theta \to t/W$ and $\mathfrak{Z}_G \to T/W$ (for simply connected $G$) are the sheaves of abelian Lie algebras, while both $\mathfrak{Z}_G \to t/W$ and $\mathfrak{Z}_G \to T/W$ (for simply connected $G$) are the sheaves of abelian Lie groups.

2.3. Isogenies. The center $Z(G)$ acts naturally on $\mathfrak{Z}_G$ (resp. $\mathfrak{Z}_G^g$) by $z(x,g) = (x, zg)$ (resp. $z(g,x) = (zg,x)$). The center $Z(G)$ acts on $\mathfrak{Z}_G^g$ on both sides: $z_1(g_1,g_2)z_2 = (z_1g_1,z_2g_2)$. Let $\tilde{G}$ denote the universal cover of $G$. Then the fundamental group $\pi_1(G)$ is embedded into $Z(\tilde{G})$, and we have $\mathfrak{Z}_G = \pi_1(G)\backslash\mathfrak{Z}_G^g$, $\mathfrak{Z}_G^g = \pi_1(G)\backslash\mathfrak{Z}_G$, $\mathfrak{Z}_G = \pi_1(G)\backslash\mathfrak{Z}_G^g/\pi_1(G)$.

2.4. Symplectic structures. We fix an invariant identification $g \simeq g^*$, hence $t \simeq T^*$. Then $g \times g$ gets identified with $g \times g^* = T^*g$ (the cotangent bundle), and $g \times g$ gets identified with $G \times g^* = T^*G$. After this $\mathfrak{Z}_G^g$ (resp. $\mathfrak{Z}_G^g$) can be viewed as a hamiltonian reduction of $T^*g$ (resp. $T^*G$); thus it inherits a canonical symplectic structure.

Identifying $g \times G$ with $G \times g^* = T^*G$ we can view $\mathfrak{Z}_G^g$ as a hamiltonian reduction of $T^*G$ as well; thus it inherits a canonical Poisson structure. Note that $\mathfrak{Z}_G^g$ is smooth and symplectic iff $G$ is simply connected. We have symplectic isomorphisms $\mathfrak{Z}_G^g \simeq T^*(t/W)$, and (in case $G$ is simply connected) $\mathfrak{Z}_G^g \simeq T^*(T/W)$.

Note that $\mathfrak{Z}_G^g$ and $\mathfrak{Z}_G^g$ share a common open piece $\mathfrak{Z}(G)$ formed by the classes of pairs $(g, x)$ where both $g$ and $x$ are regular. The canonical symplectic structures agree on $\mathfrak{Z}_G \supset \mathfrak{Z}(G) \subset \mathfrak{Z}_G^g$. Note also that for adjoint $G$ the space $\mathfrak{Z}(G)$ contains (a complexification of) the Kostant’s phase space $\mathfrak{Z}(G)$ of the classical Toda lattice [14], and the embedding $\mathfrak{Z}(G) \hookrightarrow \mathfrak{Z}_G^g$ is given by the Theorem 2.6 of loc. cit.

A. Alexeev, A. Malkin and E. Meinrenken introduced in [1] Example 6.1 the q-Hamiltonian $G$-space internal fusion double $D(G)$. Its q-Hamiltonian reduction is $\mathfrak{Z}_G^g$, so it inherits a canonical Poisson structure. For a simply connected $G$ the space $\mathfrak{Z}_G^g$ is smooth and symplectic.

2.5. Affine blow-ups. The set of roots of $G$ (resp. $\tilde{G}$) is denoted by $R$ (resp. $\tilde{R}$). We will view $\alpha \in R$ (resp. $\tilde{\alpha} \in \tilde{R}$) as a homomorphism $t \to \mathbb{C}$ (resp. $t \to \mathbb{C}$) or as a homomorphism $T \to \mathbb{C}^*$ (resp. $\tilde{T} \to \mathbb{C}^*$) depending on a context. Also, for a root $\alpha \in R$ we denote by $1^\alpha$ (resp. $2^\alpha$) the linear function on $t \times t$ obtained as a composition of $\alpha$ with the projection of $\alpha$ on the first (resp. second) factor.

We consider the following affine blow-up of $t \times t$ at the diagonal walls: $\mathfrak{B}_g^G = \text{Spec}(\mathbb{C} \{ t \times t, \frac{1}{1^\alpha} \}, \alpha \in R)$). We also set $\mathfrak{B}_g^G = \text{Spec}(\mathbb{C}[t \times T, \frac{1}{1^\alpha}], \alpha \in R)$; $\mathfrak{B}_G^G = \text{Spec}(\mathbb{C}[T \times T, \frac{1}{1^\alpha}], \alpha \in R)$; $\mathfrak{B}_G^G = \text{Spec}(\mathbb{C}[T \times T, \frac{1}{1^\alpha}], \alpha \in R)$; and let $\mathfrak{B}_g^G = \mathfrak{B}_g^G/W$, $\mathfrak{B}_G^G = \mathfrak{B}_G^G/W$, $\mathfrak{B}_G^G = \mathfrak{B}_G^G/W$ (thus $\mathfrak{B}_g^G = \text{Spec}(\mathbb{C}[t \times t, \frac{1}{1^\alpha} \alpha \in R])W$, etc.). We denote by $\varpi$ the projection of $\mathfrak{B}$ to the second factor; thus we have $\varpi : \mathfrak{B}_g^G \to t/W$, $\mathfrak{B}_G^G \to T/W$.
2.6. **Poisson structures.** We have the canonical trivializations of the tangent bundles $\mathcal{T}(t \times t) = (t \times t) \times (t \times t)$, $\mathcal{T}(T \times T) = (T \times T) \times (t \times t)$, $\mathcal{T}(T \times t) = (T \times t) \times (t \times t)$. Making use of the identification $t = t^* \simeq t$ we obtain the $W$-invariant symplectic structures on the above varieties. Thus the above affine blow-ups carry the rational Poisson structures (regular off the discriminants $D \subset \mathcal{B}$).

**Proposition 2.7.** The Poisson structure on $\mathcal{B}_G^G - D$ (resp. $\mathcal{B}_G^G - D$, $\mathcal{B}_G^G - D$, $\mathcal{B}_G^G - D$) extends to the global Poisson structure; it is a symplectic structure if the corresponding variety is smooth.

**Proposition 2.8.** We are in the setup of 2.5.

a) $\varpi$ is flat if $G$ is simply connected;

b) There are natural identifications $\mathcal{B}_G^G \simeq \mathcal{Z}_G$, $\mathcal{B}_G^G \simeq \mathcal{Z}_G$, $\mathcal{B}_G^G \simeq \mathcal{Z}_G$ commuting with $\varpi$.

c) If $G$ is simply laced and adjoint, we have an identification $\mathcal{B}_G^G \simeq \mathcal{Z}(G) \setminus \mathcal{Z}_G$ commuting with $\varpi$;

d) If $G$ is simply laced and simply connected, we have an identification $\mathcal{B}_G^G \simeq \mathcal{Z}^G_G / \mathcal{Z}(G)$ commuting with $\varpi$;

e) The above identifications respect the Poisson structures.

2.9. **Flat group sheaves.** We consider the functor $\mathcal{F}_G^G$ on the category Flat$_{t/W}$ of schemes flat over $t/W$ to the category of sets, sending a test scheme $S$ to the set of $W$-invariant morphisms $\left(\text{Mor}(S \times_{t/W} t, t)\right)^W$. Similarly, we consider the functor $\mathcal{F}_G^G$ on the category Flat$_{T/W}$ sending a test scheme $S$ to the set of $W$-invariant morphisms $\left(\text{Mor}(S \times_{T/W} t, T)\right)^W$. Also, we consider the functor $\mathcal{F}_G^G$ on the category Flat$_{t/W}$ sending a test scheme $S$ to the set of $W$-invariant morphisms $\left(\text{Mor}(S \times_{t/W} t, T)\right)_0^W \subset \left(\text{Mor}(S \times_{t/W} t, T)\right)^W$ subject to the condition (cf. [5] 4.2)

\[
\alpha \left( f(\alpha^{-1}(0)) \right) = 1 \quad \forall \alpha \in R.
\]

(note that the $W$-invariance condition automatically implies $\alpha \left( f(\alpha^{-1}(0)) \right) = \pm 1 \quad \forall \alpha \in R$.)

Furthermore, we consider the functor $\mathcal{F}_G^G$ on the category Flat$_{T/W}$ sending a test scheme $S$ to the set of $W$-invariant morphisms $\left(\text{Mor}(S \times_{T/W} T, T)\right)_0^W \subset \left(\text{Mor}(S \times_{T/W} T, T)\right)^W$ subject to the condition

\[
\alpha \left( f(\alpha^{-1}(1)) \right) = 1 \quad \forall \alpha \in R.
\]

(note that the $W$-invariance condition automatically implies $\alpha \left( f(\alpha^{-1}(1)) \right) = \pm 1 \quad \forall \alpha \in R$.)

Finally, we consider the functor $\mathcal{F}_G^G$ on the category Flat$_{T/W}$ sending a test scheme $S$ to the set of $W$-invariant morphisms $\left(\text{Mor}(S \times_{T/W} T, \hat{T})\right)_0^W \subset \left(\text{Mor}(S \times_{T/W} T, \hat{T})\right)^W$ subject to the condition

\[
\tilde{\alpha} \left( f(\alpha^{-1}(1)) \right) = 1 \quad \forall \alpha \in R.
\]
(note that the $W$-invariance condition automatically implies $\bar{\alpha}(f(\alpha^{-1}(1))) = \pm 1 \forall \alpha \in R$.)

The following Proposition is a generalization of [5] 11.6.

**Proposition 2.10.** Assume that $G$ is simply connected. The functor $\mathfrak{F}_G^G$ (resp. $\mathfrak{F}_G^C$, $\mathfrak{F}_G^G$, $\mathfrak{F}_G^C$) is representable by the scheme $\mathfrak{F}_G^G$ (resp. $\mathfrak{F}_G^G$, $\mathfrak{F}_G^G$, $\mathfrak{F}_G^C$).

2.11. **Equivariant Borel-Moore Homology.** For the definition of convolution in equivariant Borel-Moore Homology we refer the reader to [4] 2.7, 8.3 or [18] Chapter 2.

We have $H^G_G(pt) = H^G_G(pt) = \mathbb{C}[t/W]$, and $H^G_G(pt) = H^G_G(pt) = \mathbb{C}[t/W][\hbar]$ where $\hbar$ is the generator of $H^2_G(pt)$. We will consider the $\mathbb{C}[t/W]$-algebra (resp. $\mathbb{C}[t/W][\hbar]$-algebra) (with respect to convolution) $H^G_G$ (resp. $H^G_G \times \mathbb{G}_m(\mathfrak{grG})$). Note that setting $h = 0$ in $H^G_G \times \mathbb{G}_m(\mathfrak{grG})$ we obtain $H^G_G(\mathfrak{grG})$: indeed for any group $H$, a space $X$ with an $H \times \mathbb{G}_m$ action, and an $H \times \mathbb{G}_m$-equivariant complex $\mathcal{F}$ on $X$ we have a long exact sequence

\[ \cdots \to H^{i-2}_{H \times \mathbb{G}_m}(X, \mathcal{F}) \xrightarrow{h} H^i_{H \times \mathbb{G}_m}(X, \mathcal{F}) \to H^i_H(X, \mathcal{F}) \to H^{i-1}_{H \times \mathbb{G}_m}(X, \mathcal{F}) \to \cdots \]

coming from the principal $\mathbb{G}_m$-bundle $E(H \times \mathbb{G}_m) \times_H X \to E(H \times \mathbb{G}_m) \times_H \mathbb{G}_m$; if the space of $H \times \mathbb{G}_m$-equivariant cohomology is $h$-torsion free, then we get $H^*_H(X, \mathcal{F}) = H^*(X, \mathcal{F})|_{h=0}$.

**Theorem 2.12.** a) The algebra $H^G_G(\mathfrak{grG})$ is commutative;

b) Its spectrum together with the projection onto $t/W = t/W$ is naturally isomorphic to $\mathfrak{F}_G^G \cong \mathbb{C}[t/W]$.

c) The Poisson structure on $H^G_G(\mathfrak{grG})$ arising from the $h$-deformation $H^G_G \times \mathbb{G}_m(\mathfrak{grG})$, corresponds under the above identification to the Poisson structure of 2.4 on $\mathfrak{F}_G^G$.

**Remark 2.13.** The equivariant cohomology ring $H^*_G_G(\mathfrak{grG}, \mathbb{C}) = H^*_G_G(\mathfrak{grG})$ was computed by V. Ginzburg [10]. More precisely, the projection to the second (regular) factor $\mathfrak{F}_G^G \to \mathfrak{f}^G/\mathbb{G} = t/W$ makes $\mathfrak{F}_G^G$ a sheaf of abelian Lie algebras. V. Ginzburg identifies $H^*_G_G(\mathfrak{grG})$ with the global sections of the relative universal enveloping algebra $U_{t/W}(\mathfrak{F}_G^G)$. One can easily check that this result is compatible with our Theorem 2.12(b) as follows. For a group scheme $A$ over a base $S$ one has a natural pairing $U(a) \times \mathfrak{g}(A) \to \mathfrak{g}(S)$ where $U(a)$ is the enveloping (over $\mathfrak{g}(S)$) of the Lie algebra of $A$; the pairing sends $(\xi, \eta)$ to $\xi(\eta)$ restricted to the identity of $A$. On the other hand, for a compact (or ind-compact) $H$-space $X$ we have a pairing $H^*_H(X) \times H^*_H(X) \to H^*_H(pt)$ induced by the action of cohomology on homology, and the push-forward map in Borel-Moore homology $H^*_H(X) \to H^*_H(pt)$. The isomorphisms of [10] and of Theorem 2.12 take the first pairing into the second one.

2.14. **Equivariant K-theory.** For the definition of convolution in equivariant K-theory we refer the reader to Chapter 5 of [4].

We have $K^G_G(pt) = \mathbb{C}[t/W]$, and $K^G_G \times \mathbb{G}_m(pt) = \mathbb{C}[t/W][q^{\pm 1}]$. We will consider the $\mathbb{C}[t/W]$-algebra (resp. $\mathbb{C}[t/W][q^{\pm 1}]$-algebra) (with respect to convolution)
Theorem 2.15. \( a) \) The algebra \( K^G(\mathcal{O})(\text{Gr}_G) \) (resp. \( K^G(\mathcal{O}) \times G_m(\text{Gr}_G) \)) is commutative; 
\( b) \) Its spectrum together with the projection onto \( T/W \) is naturally isomorphic to \( \mathcal{B}_G \cong T/W \); 
\( c) \) The Poisson structure on \( K^G(\mathcal{O})(\text{Gr}_G) \) arising from the \( q \)-deformation \( K^G(\mathcal{O}) \times G_m(\text{Gr}_G) \), corresponds under the above identification to the Poisson structure of 2.7 on \( \mathcal{B}_G \) in case the latter variety is smooth, i.e. \( G \) is simply connected.

3. Calculations in rank 1

In this section \( G = SL_2 \), and \( \bar{G} = PGL_2 \). The Weyl group \( W = \mathbb{Z}/2\mathbb{Z} \), the Cartan torus \( T = G_m = \mathbb{C}^* \) with a coordinate \( z \), and the only simple root \( \alpha(z) = z^2 \). The dual torus \( \bar{T} = G_m = \mathbb{C}^* \) with a coordinate \( t \), and \( \bar{\alpha}(t) = t \). The Cartan Lie algebra \( t = \mathbb{C} \) with a coordinate \( x = \alpha(x) \). We fix a \( \sqrt{-1} \).

3.1. \( \mathfrak{z}_G \) and \( \mathcal{B}_G \). We choose the standard \( \mathfrak{sl}_2 \)-triple \( e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \), \( h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \), \( f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \). Then the Kostant slice \( \Sigma_G = \left\{ \begin{pmatrix} a-1 & a-2 \\ 1 & 1 \end{pmatrix}, a \in \mathbb{C} \right\} \).

One checks that a matrix \( \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \) commutes with \( \begin{pmatrix} a-1 & a-2 \\ 1 & 1 \end{pmatrix} \) iff
\[
\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \sqrt{-1} \begin{pmatrix} (1-a)c + b & (2-a)c \\ -c & b - c \end{pmatrix}
\]
for \( b, c \in \mathbb{C} \). Then the condition \( \det = 1 \) reads as
\[
1 = abc - b^2 - c^2.
\]

Thus, \( \mathfrak{z}_G \) is identified with a hypersurface \( S \) in \( A^3 \) given by the equation (4). The left (resp. right) multiplication by \( -1 \in Z(SL_2) \) is an involution \( \iota \) (resp. \( j \)) on \( S \) given by \( \iota(a, b, c) = (a, -b, -c) \) (resp. \( j(a, b, c) = (-a, b, -c) \)). Hence, \( \mathfrak{z}_G = \iota(S)/j \).

Generically, we can diagonalize two commuting matrices simultaneously, that is, there is \( g \in SL_2 \) such that \( g \sqrt{-1} \begin{pmatrix} (1-a)c + b & (2-a)c \\ -c & b - c \end{pmatrix} g^{-1} = \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix} \) and
\[
g \begin{pmatrix} a-1 & a-2 \\ 1 & 1 \end{pmatrix} g^{-1} = \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}
\]
for some \( y, z \in G_m = \mathbb{C}^* = T \) defined up to simultaneous inversion. Then we have
\[
a = z + z^{-1}, \quad b = -\frac{\sqrt{-1}}{2} \left( y + y^{-1} + \frac{(y - y^{-1})(z + z^{-1})}{z - z^{-1}} \right), \quad c = -\sqrt{-1} \frac{y - y^{-1}}{z - z^{-1}}.
\]

We conclude that \( \mathbb{C}[S] = \mathbb{C}[y^{\pm 1}, z^{\pm 1}, \frac{y - y^{-1}}{z - z^{-1}}]^W \) where the nontrivial element \( w \in W \) acts by \( w(y, z) = (y^{-1}, z^{-1}) \). We can rewrite \( \mathbb{C}[y^{\pm 1}, z^{\pm 1}, \frac{y - y^{-1}}{z - z^{-1}}]^W \) as \( \mathbb{C}[y^{\pm 1}, z^{\pm 1}, \frac{2}{z - z^{-1}}]^W \) to manifest its coincidence with \( \mathbb{C}[\mathcal{B}_G] \). All in all, we have \( \mathcal{B}_G \simeq S \simeq \mathfrak{z}_G \). Since we can
identify $\tilde{T}$ with $T/Z(G)$, the identifications $\mathfrak{B}_G^G \simeq S/\mathfrak{z}$, $\mathfrak{B}_G^{\mathfrak{z}} \simeq \mathfrak{z}/\mathfrak{s}$, $\mathfrak{B}_G^{\mathfrak{z}} \simeq \mathfrak{z}/\mathfrak{s}/\mathfrak{j} \simeq \mathfrak{z}_G^{\mathfrak{z}}$ follow immediately.

3.2. $\mathfrak{z}_G^G$ and $\mathfrak{B}_G^G$. The Kostant slice $\Sigma_\mathfrak{z} = \left\{ \left( \begin{array}{cc} 0 & \delta \\ 1 & 0 \end{array} \right), \delta \in \mathbb{C} \right\}$. One checks that a matrix

$$\left( \begin{array}{cc} x_{11} & x_{12} \\ x_{21} & x_{22} \end{array} \right)$$

commutes with $\left( \begin{array}{cc} 0 & \delta \\ 1 & 0 \end{array} \right)$ iff

$$\left( \begin{array}{cc} x_{11} & x_{12} \\ x_{21} & x_{22} \end{array} \right) = \left( \begin{array}{cc} \xi & \delta \eta \\ \eta & \zeta \end{array} \right)$$

for $\xi, \eta \in \mathbb{C}$. Then the condition $\det = 1$ reads as

$$1 = \xi^2 - \delta \eta^2. \tag{6}$$

Thus, $\mathfrak{z}_G^G$ is identified with a hypersurface $S'$ in $\mathbb{A}^3$ given by the equation (6). The action of $-1 \in Z(SL_2)$ is an involution $\mathfrak{i}$ on $S'$ given by $\mathfrak{i}(\delta, \xi, \eta) = (\delta, -\xi, -\eta)$. Hence, $\mathfrak{z}_G^G = \mathfrak{z}/\mathfrak{s}'$.

Generically, we can diagonalize two commuting matrices simultaneously, that is, there is $g \in SL_2$ such that $g \left( \begin{array}{cc} \xi & \delta \eta \\ \eta & \zeta \end{array} \right) g^{-1} = \left( \begin{array}{cc} y & 0 \\ 0 & y^{-1} \end{array} \right)$ and $g \left( \begin{array}{cc} 0 & \delta \\ 1 & 0 \end{array} \right) g^{-1} = \left( \begin{array}{cc} x & 0 \\ 0 & -x \end{array} \right)$ for some $y \in \mathbb{G}_m = \mathbb{C}^* = T$, $x \in \mathbb{C} = t$, defined up to $(y, x) \mapsto (y^{-1}, -x)$. Then we have

$$\delta = x^2, \quad \xi = \frac{y + y^{-1}}{2}, \quad \eta = \frac{y - y^{-1}}{2x}. \tag{6}$$

We conclude that $\mathbb{C}[S'] = \mathbb{C}[y^\pm 1, x, \frac{y-y^{-1}}{x}]^W$ where the nontrivial element $w \in W$ acts by $w(y, x) = (y^{-1}, -x)$. We can rewrite $\mathbb{C}[y^\pm 1, x, \frac{y-y^{-1}}{x}]^W$ as $\mathbb{C}[y^\pm 1, x, \frac{y^2-1}{x}]^W$ to manifest its coincidence with $\mathbb{C}[\mathfrak{B}_G^G]$. All in all, we have $\mathfrak{B}_G^G \simeq S' \simeq \mathfrak{z}_G^G$. Since we can identify $\tilde{T}$ with $T/Z(G)$, the identification $\mathfrak{B}_G^G \simeq \mathfrak{z}/\mathfrak{s}' \simeq \mathfrak{z}_G^G$ follows immediately.

3.3. $\mathfrak{z}_G^\mathfrak{z}$ and $\mathfrak{B}_G^\mathfrak{z}$. Recall the Kostant slice $\Sigma_\mathfrak{z} = \left\{ \left( \begin{array}{cc} a - 1 & a - 2 \\ 1 & 1 \end{array} \right), \ a \in \mathbb{C} \right\}$. One checks that a traceless matrix

$$\left( \begin{array}{cc} x_{11} & x_{12} \\ x_{21} & -x_{11} \end{array} \right)$$

commutes with $\left( \begin{array}{cc} a - 1 & a - 2 \\ 1 & 1 \end{array} \right)$ iff

$$\left( \begin{array}{cc} x_{11} & x_{12} \\ x_{21} & -x_{11} \end{array} \right) = \zeta \left( \begin{array}{cc} 2 - a & 4 - 2a \\ -2 & a - 2 \end{array} \right)$$

for $\zeta \in \mathbb{C}$.

Thus, $\mathfrak{z}_G^\mathfrak{z}$ is identified with $\mathbb{A}^2$ with coordinates $a, \zeta$. The action of $-1 \in Z(SL_2)$ is an involution $\mathfrak{j}$ on $\mathbb{A}^2$ given by $\mathfrak{j}(a, \zeta) = (-a, -\zeta)$. Hence, $\mathfrak{z}_G^\mathfrak{z} = \mathbb{A}^2/\mathfrak{j}$.

Generically, we can diagonalize two commuting matrices simultaneously, that is, there is $g \in SL_2$ such that $g \zeta \left( \begin{array}{cc} 2 - a & 4 - 2a \\ -2 & a - 2 \end{array} \right) g^{-1} = \left( \begin{array}{cc} x & 0 \\ 0 & -x \end{array} \right)$ and

$$g \left( \begin{array}{cc} a - 1 & a - 2 \\ 1 & 1 \end{array} \right) g^{-1} = \left( \begin{array}{cc} z & 0 \\ 0 & z^{-1} \end{array} \right)$$

for some $x \in \mathbb{C} = t$, $z \in \mathbb{G}_m = \mathbb{C}^* = T$ defined up to $(x, z) \mapsto (-x, z^{-1})$. Then we have

$$a = z + z^{-1}, \quad \zeta = \frac{x}{z - z^{-1}}.$$
We conclude that $\mathbb{C}[A^2] = \mathbb{C}[x, z^{\pm 1}, \frac{1}{z - z^{-1}}]W$ where the nontrivial element $w \in W$ acts by $w(x, z) = (-x, z^{-1})$. We can rewrite $\mathbb{C}[x, z^{\pm 1}, \frac{1}{z - z^{-1}}]W$ as $\mathbb{C}[x, z^{\pm 1}, \frac{x}{z - z^{-1}}]W$ to manifest its coincidence with $\mathbb{C}[\mathfrak{B}_G]$. All in all, we have $\mathfrak{B}_G^{\circ} \simeq \mathbb{A}^2 \simeq \mathfrak{B}_G^\circ$. Since we can identify $\check{T}$ with $T/Z(G)$, the identification $\mathfrak{B}_G^{\circ} \simeq \mathbb{A}^2/j \simeq \mathfrak{B}_G^\circ$ follows immediately.

3.4. $\mathfrak{B}_G^\circ$ and $\mathfrak{B}_G^\circ$. Recall the Kostant slice $T = \{(0, 0, x) : x \in \mathbb{C}\}$. One checks that a traceless matrix $\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & -x_{11} \end{pmatrix}$ commutes with $\begin{pmatrix} 0 & \delta \\ \theta & 0 \end{pmatrix}$ if $\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & -x_{11} \end{pmatrix} = \begin{pmatrix} 0 & \delta \theta \\ \theta & 0 \end{pmatrix}$ for $\theta \in \mathbb{C}$. Thus, $\mathfrak{B}_G^\circ$ is identified with $\mathbb{A}^2$ with coordinates $\delta, \theta$.

Generically, we can diagonalize two commuting matrices simultaneously, that is, there is $g \in SL_2$ such that $g \begin{pmatrix} 0 & \delta \theta \\ \theta & 0 \end{pmatrix} g^{-1} = \begin{pmatrix} u & 0 \\ 0 & -u \end{pmatrix}$ and $g \begin{pmatrix} 0 & \delta \theta \\ \theta & 0 \end{pmatrix} g^{-1} = \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix}$ for some $u, x \in \mathbb{C} = t$, defined up to $(u, x) \leftrightarrow (-u, -x)$. Then we have

$$\delta = x^2, \ \theta = \frac{u}{x}.$$ We conclude that $\mathbb{C}[A^2] = \mathbb{C}[u, x, \frac{1}{x}]W$ where the nontrivial element $w \in W$ acts by $w(u, x) = (-u, -x)$. Hence we get an identification $\mathfrak{B}_G^\circ \simeq \mathbb{A}^2 \simeq \mathfrak{B}_G^\circ$.

3.5. $\mathfrak{B}_G^G$ and $\mathfrak{B}_G^G$. Recall the setup of Proposition 2.10. We will prove that the functor $\mathfrak{B}_G^G$ is representable by the scheme $\mathfrak{B}_G^G$; the other parts of the Proposition are proved absolutely similarly, as well as the Proposition for $G$ replaced by $\check{G}$. For a scheme $S$ flat over $t/W$ we will denote by $S_t$ the cartesian product $S \times_t W$. Our usual coordinate $x$ on $t$ gives rise to the same named function on $S_t$. The nontrivial element $w \in W$ acts by the involution of $S_t$. Finally, we denote by $(\mathfrak{B}_G^G)_S$ the affine blow-up of $S \times_t T$, that is $S \times_t \mathfrak{B}_G^G$. Clearly, $w$ acts as an involution of $(\mathfrak{B}_G^G)_S$.

Note that the condition (1) is void in the case under consideration. Given a $w$-equivariant morphism $f : S_t \to T = \mathbb{G}_m$ we see that $f^2 - 1$ is divisible by $x$, hence $f$ lifts uniquely to a section $\check{f}$ of $(\mathfrak{B}_G^G)_S$ over $S_t$. Evidently, $\check{f}$ is $w$-invariant. If we consider $\check{f}$ as a closed subscheme of $(\mathfrak{B}_G^G)_S$, then $\check{f}/W$ is a closed subscheme of $(\mathfrak{B}_G^G)_S/W = S \times_t \mathfrak{B}_G^G$ which is the graph of a morphism $\check{f} : S \to \mathfrak{B}_G^G$.

Conversely, given a morphism $\check{f} : S \to \mathfrak{B}_G^G$ we consider its graph $\Gamma_{\check{f}}$ as a closed subscheme of $S \times_t \mathfrak{B}_G^G$, and then the cartesian product $\Gamma_{\check{f}} \times S \times_t \mathfrak{B}_G^G$ $(\mathfrak{B}_G^G)_S$ is a section $\check{f}$ of $(\mathfrak{B}_G^G)_S$ over $S_t$. Evidently, $\check{f}$ gives rise to a $w$-equivariant function $f : S_t \to T$.

3.6. A basis in equivariant $K$-theory. We recall a few standard facts about the affine Grassmannians $\text{Gr}_G$ and $\text{Gr}_\check{G}$. The $G(O)$-orbits (equivalently, $\check{G}(O)$-orbits) on $\text{Gr}_G$ are numbered by nonnegative integers and denoted by $\text{Gr}_G^n, n \in \mathbb{N}$. The orbits $\text{Gr}_{G, 2n}, n \in \mathbb{N}$, form a connected component of $\text{Gr}_G$ equal to $\text{Gr}_G$. The open embedding of an orbit into its closure will be denoted by $j_n : \text{Gr}_G^n \hookrightarrow \overline{\text{Gr}_G^n}$ or simply by $j$ if no confusion is likely. We have $\dim \text{Gr}_G^n = n$; in particular, $\text{Gr}_G^0$ is a point.
We have $K^G(O)(\text{Gr}_{G,0}) = \text{Rep}(G)$ with a basis $\mathbf{v}(n)$, $n \in \mathbb{N}$, formed by the classes of irreducible $G$-modules $\mathcal{V}(n)$. Also, $K^G(O)(\text{Gr}_{G,0}) = \text{Rep}(G) \subset \text{Rep}(G)$ has a basis $\mathbf{v}(2n)$, $n \in \mathbb{N}$.

For $m > 0$ the $G(O)$-equivariant line bundles in $\text{Gr}_{G,m}$ are numbered by integers and denoted by $\mathcal{L}(n)_m$. Among them, the $\mathcal{O}(O)$-equivariant line bundles are exactly $\mathcal{L}(2n)_m$, $n \in \mathbb{Z}$. We define $\mathcal{V}(n)_m$ as $j_!\mathcal{L}(n)_m[\frac{m}{2}]$, that is, the (nonderived) direct image to the orbit closure placed in the homological degree $-\frac{m}{2}$. Note that since the complement $\overline{\text{Gr}}_{G,m} - \text{Gr}_{G,m}$ has codimension 2, the above direct image is a coherent sheaf. The degree shift will become clear later. The class $[\mathcal{L}(n)_m]$ in $K^G(O)(\text{Gr}_{G})$ will be denoted by $\mathbf{v}(n)_m$. Thus, it is natural to denote $\mathbf{v}(n)$ above by $\mathbf{v}(n)_0$; we will keep both names.

The collection $\{\mathbf{v}(n)_m : n \in \mathbb{N} \text{ if } m = 0; n \in \mathbb{Z} \text{ if } m \in \mathbb{N} - 0\}$ forms a basis in $K^G(O)(\text{Gr}_{G})$. Among this collection, all the $\mathbf{v}(n)_m$ with $n$ even (resp. $m$ even) form a basis in $K^G(O)(\text{Gr}_{G})$ (resp. $K^G(O)(\text{Gr}_{G})$).

3.7. Convolution: commutativity. In this subsection $G$ is an arbitrary semisimple group. We prove 2.15 (a). We refer the reader to [7] for the basics of Beilinson-Drinfeld Grassmannian. Recall that $\text{Gr}_{G}^B \xrightarrow{\pi} \mathbb{A}^1$ is a flat ind-scheme such that $\pi^{-1}(\mathbb{A}^1 - 0) = (\mathbb{A}^1 - 0) \times \text{Gr}_G \times \text{Gr}_G$, while $\pi^{-1}(0) = \text{Gr}_G$. We also have the deformed convolution diagram $\text{Gr}_{G}^B,\text{conv} \xrightarrow{\Pi} \text{Gr}_{G}^B$ such that $\Pi$ is an isomorphism over $\mathbb{A}^1 - 0$, while over $0 \in \mathbb{A}^1$ our $\Pi$ is the usual convolution diagram $G(F) \times_{G(O)} \text{Gr}_G \xrightarrow{\Pi_2} \text{Gr}_G$.

Given two $G(O)$-equivariant complexes of coherent sheaves $\mathcal{A}, \mathcal{B}$ on $\text{Gr}_G$, we can form their “deformed convolution” complex $\mathcal{A} \overset{\pi}{\boxtimes} \mathcal{B}$ on $\text{Gr}_{G}^B,\text{conv}$ such that over $\mathbb{A}^1 - 0$ it is isomorphic to $\mathcal{O}_{\mathbb{A}^1-0} \boxtimes \mathcal{A} \boxtimes \mathcal{B}$, while over $0 \in \mathbb{A}^1$ it is isomorphic to the usual twisted product $\mathcal{A} \boxtimes \mathcal{B}$ on the convolution diagram $G(F) \times_{G(O)} \text{Gr}_G$. In addition, if $\mathcal{A}, \mathcal{B}$ are coherent sheaves, then $\mathcal{A} \overset{\pi}{\boxtimes} \mathcal{B}$ is flat over $\mathbb{A}^1$. It implies that in the $K$-group the class $[\mathcal{A} \boxtimes \mathcal{B}]$ is the specialization (see [4] 5.3) of the class $[\mathcal{O}_{\mathbb{A}^1-0} \boxtimes \mathcal{A} \boxtimes \mathcal{B}]$ in the family $\text{Gr}_{G}^B,\text{conv}$ and, also the class $[\mathcal{A} \boxtimes \mathcal{B}] = [\Pi_2],([\mathcal{A} \boxtimes \mathcal{B}])$ is the specialization of the class $[\mathcal{O}_{\mathbb{A}^1-0} \boxtimes \mathcal{A} \boxtimes \mathcal{B}]$ in the family $\text{Gr}_{G}^B,\text{conv} \xrightarrow{\pi} \mathbb{A}^1$. Hence the desired commutativity.

3.8. Convolution: relations. We return to the setup of 3.6. Note that $\text{Gr}_{G,1} \simeq \mathbb{P}^1$, and $\mathcal{V}(n)_1$ is the line bundle $\mathcal{O}(n)$ on $\mathbb{P}^1$. The twisted product $\mathcal{V}(n)_1 \times \mathcal{V}(l)_1$ is the line bundle $\mathcal{O}(n,l)$ on the 2-dimensional subvariety $\mathcal{H}_2 \subset G(F) \times_G \text{Gr}_G$ isomorphic to the Hirzebruch surface $\mathbb{P}(\mathcal{O}(2) \oplus 0)$ over $\mathbb{P}^1$. The projection $\Pi_0 : \mathcal{H}_2 \to \text{Gr}_{G,2}$ is the contraction of the $-2$-section $\mathbb{P}^1 \dashrightarrow \mathcal{H}_2$.

Now it is easy to compute $\mathbf{v}(n)_1 \star \mathbf{v}(n)_1 = \mathbf{v}(2n)_2$, $\mathbf{v}(1)_1 \star \mathbf{v}(-1)_1 = \mathbf{v}(0)_2 + 1$. Taking into account the evident relation $\mathbf{v}(1)_0 \star \mathbf{v}(0)_1 = \mathbf{v}(1)_1 + \mathbf{v}(-1)_1$ we arrive at

$$\mathbf{v}(1)_0 \star \mathbf{v}(0)_1 \star \mathbf{v}(1)_2 = \mathbf{v}(1)_2 \star \mathbf{v}(1)_1 + \mathbf{v}(0)_1 \star \mathbf{v}(0)_1 + 1.$$  

A moment of reflection shows that $K^G(O)(\text{Gr}_G)$ is generated as algebra by $\mathbf{v}(1)_0$, $\mathbf{v}(0)_2 = \mathbf{v}(0)_1 \star \mathbf{v}(0)_1$, $\mathbf{v}(2)_2 = \mathbf{v}(1)_1 \star \mathbf{v}(1)_1$, $\mathbf{v}(1)_2 = \mathbf{v}(1)_1 \star \mathbf{v}(0)_1$ (one has to use that $\mathbf{v}(k)_{2l} \star \mathbf{v}(n)_{2m} = \mathbf{v}(k + n)_{2l+2m}$ plus the terms supported
on the smaller orbits). Similarly, $K^{G(O)}(\text{Gr}_G)$ is generated as algebra by $v(2)_0 = v(1)_0 \ast v(1)_0 - 1$, $v(0)_1$, $v(2)_2 = v(1)_1 \ast v(1)_1$, $v(2)_1 = v(1)_1 \ast v(1)_0 - v(0)_1$.

Note that both algebras $K^{G(O)}(\text{Gr}_G)$ and $K^{G(O)}(\text{Gr}_\tilde{G})$ lie in the vector space $K^{G(O)}(\text{Gr}_G)$, and their intersection is the common subalgebra $K^{G(O)}(\text{Gr}_G)$. The tensor product algebra $K^{G(O)}(\text{Gr}_G) \otimes_{K^{G(O)}(\text{Gr}_G)} K^{G(O)}(\text{Gr}_G)$ can be identified as a vector space with $K^{G(O)}(\text{Gr}_G)$, and then it is generated by the three basic elements $v(1)_0, v(0)_1, v(1)_1$ subject to the only relation (7).

The comparison of equations (7) and (4) shows that the assignment $a \mapsto v(1)_0, b \mapsto v(0)_1, c \mapsto v(1)_1$ establishes an isomorphism $\mathbb{C}[S] \simeq K^{G(O)}(\text{Gr}_G)$. It identifies the spectrum of $K^{G(O)}(\text{Gr}_G)$ with $\mathfrak{s}/J \simeq \mathfrak{g}_G^c$, and the spectrum of $K^{G(O)}(\text{Gr}_\tilde{G})$ with $S/J \simeq \mathfrak{g}_G^c$.

### 3.9. Iwahori-equivariant $K$-theory

Let $I \subset G(O)$ be the Iwahori subgroup. The space $K^T(\text{Gr}_G) = K^T(\text{Gr}_G) = K^{T(O)}(\text{Gr}_G) = K(T(O) \backslash T(F)/G(O))$ is equipped with the two commuting actions: $K(T(O) \backslash T(F)/T(O))$ acts by convolutions on the left, and $K^G(\text{Gr}_G) = K^{G(O)}(\text{Gr}_G) = K(G(O) \backslash G(F)/G(O))$ acts by convolutions on the right. Also, $W$ acts on $K^T(\text{Gr}_G)$ commuting with the right action of $K^G(\text{Gr}_G)$.

The algebra $K^G(\text{Gr}_G)$ normalizes the action of $K(T(O) \backslash T(F)/T(O))$ and induces the natural (diagonal) action of $W$ on $\mathbb{C}[T \times T]$.

Our aim in this subsection is to identify the $K(T(O) \backslash T(F)/T(O)) \times W - K^G(\text{Gr}_G)$-bimodule $K^T(\text{Gr}_G)$ with the $\mathbb{C}[T \times T] \times W - \mathbb{C}[\mathfrak{g}_G^c]$-bimodule $\mathfrak{g}_G^c$ (and similarly for $G$ replaced by $\tilde{G}$). As in 3.8, it suffices to identify the $K(T(O) \backslash T(F)/T(O)) \times W - K^G(\text{Gr}_G)$-bimodule $K^T(\text{Gr}_G)$ with the $\mathbb{C}[T \times T] \times W - \mathbb{C}[\mathfrak{g}_G^c]$-bimodule $\mathfrak{g}_G^c$.

Note that $K^G(\text{Gr}_G) \subset K^T(\text{Gr}_G)$, and the $K^G(\text{Gr}_G)$-module $K^T(\text{Gr}_G)$ is free of rank 2 with the generators $1, z$ where $z$ is the generator of $K^T(pt) = \mathbb{C}[T]$ (so that, e.g. $v(1)_0 = z + z^{-1}$). Furthermore, $\mathbb{C}[y^{z \pm 1}, z^{z \pm 1}] = \mathbb{C}[T \times T] = K(T(O) \backslash T(F)/T(O)) \subset K^T(\text{Gr}_G)$, and one can check

$$y^{-1} = \sqrt{-1}(u_0 - u_2), \quad y = \sqrt{-1}(v(0)_1 - v(2)_1 + u_2 - u_0)$$

where $u_0 \in K^T(\text{Gr}_G)$ (resp. $u_2$) is the class of the irreducible skyscraper sheaf supported at the one-point Iwahori orbit in $\text{Gr}_{G,1} = \mathbb{P}^1$ with the trivial action of $T$ (resp. with the action of $T$ given by $z^2$), and placed in the homological degree $-\frac{1}{2}$. Hence

$$y + y^{-1} = \sqrt{-1}(2v(0)_1 - v(1)_0 \ast v(1)_1), \quad y - y^{-1} = \sqrt{-1}(z - z^{-1})v(1)_1.$$  

Comparing (9) with (5) we get the desired identification of the $K(T(O) \backslash T(F)/T(O)) \times W - K^G(\text{Gr}_G)$-bimodule $K^T(\text{Gr}_G)$ with the $\mathbb{C}[y^{z \pm 1}, z^{z \pm 1}] \times W - \mathbb{C}[y^{z \pm 1}, z^{z \pm 1}, \frac{y - y^{-1}}{z - z^{-1}}]$-bimodule $\mathbb{C}[y^{z \pm 1}, z^{z \pm 1}, \frac{y - y^{-1}}{z - z^{-1}}]$. 


3.10. **Borel-Moore Homology.** For an arbitrary semisimple $G$ one proves the commutativity of $H_*^{G(O)}(\text{Gr}_G)$ (Theorem 2.12 a) exactly as in 3.7 using the Beilinson-Drinfeld Grassmannian and the specialization in Borel-Moore Homology (see [4] 2.6.30).

For $\tilde{G} = PGL_2$, let us denote by $\delta \in H^1_{\tilde{G}(O)}(pt, \mathbb{Z}) = H^1_{\tilde{G}^\vee(O)}(pt, \mathbb{Z})$ the generator of the equivariant (co)homology. Furthermore, we denote by $\eta$ (resp. $\xi$) the generator of $H_{-2}^{\tilde{G}(O)}(\text{Gr}_{\tilde{G},1}, \mathbb{Z})$ (resp. the generator of $H^2_{\tilde{G}(O)}(\text{Gr}_{\tilde{G},1}, \mathbb{Z})$). Then it is easy to see that $\delta, \xi, \eta$ generate $H^{\tilde{G}(O)}_*(\text{Gr}_{\tilde{G}})$ (while $\delta, \xi^2, \eta^2, \xi \eta$ generate the subalgebra $H^{G(O)}_*(\text{Gr}_G)$), and we claim that

$$1 = \xi^2 - \delta \eta^2.$$  

In effect, this is an equality in $H^0_{\tilde{G}(O)}(\text{Gr}_{\tilde{G},2})$. Since $\text{Gr}_{\tilde{G},2}$ is rationally smooth, $H^0_{\tilde{G}(O)}(\text{Gr}_{\tilde{G},2}) = H^4_{\tilde{G}^\vee(O)}(\text{Gr}_{\tilde{G},2})$. Let us denote by $B\text{Gr}_{\tilde{G},2} \to B\tilde{G}(O)$ the associated fibre bundle over the classifying space of $\tilde{G}$ with the fiber $\text{Gr}_{\tilde{G},2}$. Then $1 \in H^1_{\tilde{G}(O)}(\text{Gr}_{\tilde{G},2}) = H^4(B\text{Gr}_{\tilde{G},2})$ is the Poincaré dual class of the codimension 2 cycle $B\tilde{G}(O) = B\text{Gr}_{\tilde{G},0} \to B\text{Gr}_{\tilde{G},2}$, and $\delta \eta^2 = p^* \delta$.

Recall the convolution morphism $\Pi_0 : \mathcal{H}_2 \to \text{Gr}_{\tilde{G},2}$ of 3.8. This is a morphism of $\tilde{G}(O)$-varieties, and we denote by $\Pi_0 : B\mathcal{H}_2 \to B\text{Gr}_{\tilde{G},2}$ the corresponding morphism of associated fibre bundles. Note that (additively) $H^*(B\mathcal{H}_2) = H^* (B\text{Gr}_{\tilde{G},1}) \otimes H^*(B\tilde{G}(O))$.

In the other cases being similar. Till the further notice $G$ is assumed simply connected.

4. **Centralizers and blow-ups**

The aim of this section is a proof of Proposition 2.8. We will consider $\mathfrak{B}_G^T$ and $\mathfrak{Z}_G^T$, the other cases being similar. Till the further notice $G$ is assumed simply connected.

**Lemma 4.1.** $\varpi : \mathfrak{B}_G^T \to T/W$ is flat.

**Proof.** It suffices to prove that the first projection of $\mathfrak{B}_G^T$ to $T$ is smooth (recall that $\mathfrak{B}_G^T$ is defined as $\text{Spec} (\mathbb{C}[T \times T, \frac{1}{\alpha - 1}], \alpha \in R)$). In effect, then $\mathbb{C}[T \times T, \frac{1}{\alpha - 1}]$, $\alpha \in R$] is a flat $\mathbb{C}[T]$-module; hence it is a flat $\mathbb{C}[T]^W$-module (since $\mathbb{C}[T]$ is free over $\mathbb{C}[T]^W$, see [21]). Finally, $\mathbb{C}[T \times T, \frac{1}{\alpha - 1}]$, $\alpha \in R$] is a direct summand of a flat $\mathbb{C}[T]^W$-module $\mathbb{C}[T \times T, \frac{1}{\alpha - 1}]$, $\alpha \in R$]; hence it is flat.

The affine blow-up $\mathfrak{Z}_G^T$ is the result of the following successive blow up of $T \times T$. We choose an ordering $\alpha_1, \ldots, \alpha_\nu$ of the set of positive roots $R^+$. We define $\mathfrak{B}_1$ as the
blow up of $T \times T$ at the diagonal wall $1\alpha_1 = 2\alpha_1 = 1$ with the proper preimage of the divisor $1\alpha_1 = 1$ removed. We define $\mathfrak{B}_2$ as the blow up of $\mathfrak{B}_1$ at the proper transform of the diagonal wall $1\alpha_2 = 2\alpha_2 = 1$ with the proper preimage of the divisor $1\alpha_2 = 1$ removed. Going on like this we construct $\mathfrak{B}_i$; evidently, it coincides with $\mathfrak{B}_G^G$.

Note that at each step the center of the blow-up is smooth over the corresponding wall $2\alpha_i = 1$ in $T$ by the following Claim. Thus the desired flatness assertion follows inductively from the

Claim. Let $p : X \to Y$ be a smooth morphism of smooth varieties; let $X' \subset X$ be a subvariety such that $Y' = f(X') \subset Y$ is a smooth hypersurface, and $p : X' \to Y'$ is also smooth. Then the blow-up $\text{Bl}_X' X$ with the proper preimage of the divisor $p^{-1}(p(X'))$ removed is smooth over $Y$.

The smoothness is checked in the formal neighbourhoods of points by direct calculation in coordinates. This completes the proof of the lemma.

4.2. The simultaneous resolution. Recall that $\{(g, B) : g \in B\} = G \overset{\nu}{\to} G$ is the Grothendieck simultaneous resolution; here $B$ is a Borel subgroup, and $p(g, B) = g$. We also have the projection $g : G \to T$ to the abstract Cartan, which we identify with $T$; namely, $g(g, B) = g \mod \text{rad}(B)$. The preimage $p^{-1}(\Sigma_G) \subset G$ is identified with $T$ by $g$. We denote by $\mathfrak{Z}_G^G \subset G \times G$ the subset of triples $(g_1, g_2, B)$ such that $Ad_{g_1} = g_2$ and $(g_2, B) \in p^{-1}(\Sigma_G)$. Note that necessarily $g_1 \in B$ (as well as $g_2 \in B$); hence we have the projections $g_1, g_2 : \mathfrak{Z}_G^G \to T$; namely, $g_1(g_1, g_2, B) = g_1 \mod \text{rad}(B)$.

The natural projection $\mathfrak{Z}_G^G \to \mathfrak{Z}_G^G$ (forgetting $B$) is a Galois $W$-covering. Finally, $g_2 : \mathfrak{Z}_G^G \to T$ is flat.

4.3. The proof of Proposition 2.8. In order to identify $\mathfrak{Z}_G^G$ and $\mathfrak{B}_G^G$ it suffices to identify their Galois $W$-coverings $\mathfrak{Z}_G^G \to T$ and $\mathfrak{B}_G^G \to T$ in an equivariant way. Let $D \subset T$ denote the discriminant, so that $T - D = T^{\text{reg}}$. Let $\Delta \in \mathbb{C}[T]^W$ denote the product $\prod_{\alpha \in R}(\alpha - 1)$, so that $D$ is the divisor cut out by $\Delta$.

Evidently, both $\mathfrak{Z}_G^G|_{T^{\text{reg}}}$ and $\mathfrak{B}_G^G|_{T^{\text{reg}}}$ are isomorphic to $T \times T^{\text{reg}}$. Hence both $\mathbb{C}[\mathfrak{Z}_G^G]$ and $\mathbb{C}[\mathfrak{B}_G^G]$ are the flat $\mathbb{C}[T]$-modules embedded into $\mathbb{C}[T \times T](\Delta^{-1})$. We must prove that the identification of $\mathfrak{Z}_G^G|_{T^{\text{reg}}}$ and $\mathfrak{B}_G^G|_{T^{\text{reg}}}$ extends to the identification over the whole $T$. To this end it suffices to check that the identification extends over the codimension 1 points of $T$ (indeed, for a flat quasi-coherent sheaf $\mathcal{F}$ on a normal irreducible scheme we have $\mathcal{F} \cong j_*j^*\mathcal{F}$ if $j$ is an open imbedding with complement of codimension 2). Let $g \in T$ be a regular point of $D$; that is, $g$ is a semisimple element of $G$ such that the centralizer $Z(g)$ has semisimple rank 1.

We must construct an isomorphism between localizations $(\mathfrak{Z}_G^G)_q$ and $(\mathfrak{B}_G^G)_q$ which is compatible with the above isomorphism at the generic point. To this end note that the embedding of reductive groups $Z(g) \hookrightarrow G$ (note that $Z(g)$ is connected since $G$ is
simply connected) induces the morphisms \( ZG/(g) \to ZG \) and \( ZG/(g) \to ZG \) which become isomorphisms after localizations: \( ZG/(g) \to ZG \) and \( ZG/(g) \to ZG \). Now the desired identification \( ZG/(g) \to ZG \) follows from the calculations in 3.1.

This completes the identification \( ZG \cong ZG \) for a simply connected \( G \). Evidently, this identification respects the left and right actions of the center \( Z(G) \), so the isomorphism for an arbitrary \( G \) follows from the one for its universal cover. The other isomorphisms in 2.8 (b) are proved in a similar way.

To prove 2.8 (c), (d) it suffices to notice that the minimal level (viewed as a \( W \)-equivariant homomorphism \( T \to \Tilde{T} \)) for a simply laced simply connected \( G \) identifies \( \Tilde{T} \) with \( T/Z(G) \); also, \( \Tilde{G} = G/Z(G) \).

5. \( W \)-IN Variant SECTIONS AND BLOW-UPS

The aim of this section is a proof of Proposition 2.10. We concentrate on the last statement, the other being completely similar.

Let \( T^{reg} \subset T \), \( T^{reg}_\alpha \subset T \) be the open subschemes defined by \( T^{reg} = \{ t \mid \alpha(t) \neq 1 \} \) for all roots \( \alpha \}; \( T^{reg}_\alpha = \{ t \mid \beta(t) \neq 1 \} \) for all roots \( \beta \neq \alpha \}; and \( \Tilde{T} = \bigcup_\alpha T^{reg}_\alpha \) (thus \( \Tilde{T} \) has codimension 2 in \( T \)) (where the empty subscheme in a curve is considered to be of codimension 2)). Notice that since \( G \) is simply connected the action of \( W \) on \( T^{reg} \) is free.

We start with a

Lemma 5.1. The map \( ZG \times_T \Tilde{T} \to ZG/W \times_T \Tilde{T} \) is an isomorphism.

Proof Let \( X \to Y \) be a flat morphism of semi-separated (which means that the diagonal embedding is affine) schemes of finite type over a characteristic zero field, and let a finite group \( W \) act on \( X, Y \) so that the map is \( W \)-equivariant. Assume that \( Y \) is flat over \( Y/W \). We then claim that the map \( X \to X/W \times_{Y/W} Y \) is an isomorphism provided that for every Zariski point \( y \in Y \) the action of \( \text{Stab}_W(y) \) on the scheme-theoretic fiber \( X_y \) is trivial (here \( X/W, Y/W \) stand for categorical quotients). To check this claim we can assume \( X \) is affine: by semi-separatedness every \( W \)-invariant subset in \( X \) has a \( W \)-invariant affine neighborhood. Let us first assume also that \( Y/W \) is a point; then (by replacing \( Y \) by its connected component, and \( W \) by the stabilizer of that component) we can assume that \( Y \) is nilpotent. Then \( O_X \) is free over \( O_Y \), and the generators of \( O_X \) as an \( O_Y \) module can be chosen to be \( W \)-invariant (by semi-simplicity of the \( W \) action on \( O_X \), and triviality of the \( W \)-action on \( O_X \otimes_{O_Y} k \); since \( O_Y^W = k \) (where \( k \) is the base field) we see that \( O_Y^W \otimes_{O_Y} \iso O_X \) as claimed. Now for a general \( Y \) we see that the morphism in question is a morphism of flat schemes of finite type over \( Y/W \), which induces an isomorphism on every fiber; and such a morphism is necessarily an isomorphism.

Now it remains to check that the above conditions hold for \( X = ZG \times_T \Tilde{T}, Y = T \). For \( y \in T^{reg} \) the stabilizer of \( y \) is trivial, so there is nothing to check. Consider now
Proposition 5.2. Let $S \to T/W$ be a flat morphism, and set $\phi : S \times_{T/W} T^{reg}/W \to (T \times T)/W$ be a $T^{reg}/W$-morphism. Then the following are equivalent:

(i) $\phi$ extends to a morphism $S \times_{T/W} \hat{T}/W \to \mathfrak{B}_G^G \times_{T/W} \hat{T}$.

(ii) $\phi$ extends to a morphism $S \to \mathfrak{B}_G^G$.

(iii) For every $\alpha \in R$ the morphism $\phi \times \text{id}_{T^{reg}} : S \times_{T/W} T^{reg} \to \hat{T} \times T^{reg}$ extends to a morphism $S \times_{T/W} T^{reg}_\alpha \to \hat{T} \times T^{reg}_\alpha$ such that (3) holds.

(iv) $\phi \times \text{id}_{T^{reg}} : S \times_{T/W} T^{reg} \to \hat{T} \times T^{reg}$ extends to a morphism $S \times_{T/W} T \to \hat{T} \times T$, such that (3) holds for every $\alpha \in R$.

Proof It is enough to assume that $S$ is affine. Indeed, a morphism from $S$ extends if its restriction to every affine open in $S$ does, because compatibility on intersections follows from uniqueness of such an extension; this uniqueness follows from flatness: if $S$ is flat affine, then tensoring the injection $\mathcal{O} \to j_*\mathcal{O}$ with $\mathcal{O}_S$ we get an imbedding $\mathcal{O}_S \hookrightarrow j_*j^*\mathcal{O}_S$, where $j$ stands for the imbedding $T^{reg}/W \to T/W$, or $T^{reg} \to T$. So we will assume $S$ affine from now on.

(iv) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (i) are obvious.

To check that (iii) $\Rightarrow$ (iv) we tensor (over $\mathcal{O}_{T/W}$) the exact sequence of $\mathcal{O}_T$-modules

\begin{equation}
0 \to \mathcal{O} \to \mathcal{O}_{T^{reg}} \to \bigoplus_{\alpha} (\mathcal{O}_{T^{reg}_\alpha}/\mathcal{O}_T)
\end{equation}

with $\mathcal{O}_S$. The resulting exact sequence shows that a regular function on $S \times_{T/W} T^{reg}$ extends to a regular function on $S \times_{T/W} T$ iff it extends to $S \times_{T} T^{reg}_\alpha$ for all $\alpha$. Applying this observation to $(\phi \times \text{id})^*(f|_{\hat{T} \times T^{reg}})$ for each regular function $f$ on $\hat{T} \times T$ we see that (iii) implies extendability of $\phi \times \text{id}$ to $S \times_{T/W} T$. It is also clear that (3) holds if it holds on $\hat{T}$.

Verification of (i) $\Rightarrow$ (ii) is similar (with (11) replaced by the $W$-invariant part of (11)).

It remains to check (i) $\iff$ (iii). If (i) holds, i.e. $\phi$ extends to a map $S \times_{T/W} \hat{T}/W \to \mathfrak{B}_G^G \times_{T/W} \hat{T}$ then we can take the fiber product of this map with $\text{id}_{\hat{T}}$ over $T/W$. By Lemma 5.1 it yields a map $S \times_{T/W} \hat{T} \to \mathfrak{B}_G^G \times_{T} \hat{T}$, which can be composed with the projection $\mathfrak{B}_G^G \to \hat{T} \times T$ to produce a map $S \times_{T/W} \hat{T} \to \hat{T} \times \hat{T}$. It is clear that this map satisfies (3), because the image of the map $\mathfrak{B}_G^G \to \hat{T} \times T$ intersected with $\hat{T} \times \text{Ker}(2\alpha)$ is contained in $\text{Ker}(1\hat{\alpha}) \times T$. 

$y \in T^{reg}_\alpha$, $y \notin T^{reg}$. Then the stabilizer of $y$ is $\{1, s_\alpha\}$. The ring of functions on $\mathfrak{B}_G^G$ is generated by $1\hat{\lambda}$, $2\mu$, $t_\alpha$ where $\hat{\lambda}$, $\mu$ run over weights of $T$, $T$ respectively, $\alpha \in R^+$, and $t_\alpha(2\alpha - 1) = 1\hat{\alpha} - 1$. We have $s_\alpha(1\hat{\lambda}) = 1\hat{\lambda} \cdot (1\hat{\alpha})^{(-\alpha, \lambda)}$, $s_\alpha(2\mu) = 2\mu \cdot (2\alpha)^{(-\mu, \alpha)}$, and $s_\alpha(t_\alpha) = t_\alpha \cdot \frac{2\alpha}{\alpha}$. On the fiber we have $2\alpha = 1$, hence $1\hat{\alpha} = 1$, so the action of $s_\alpha$ on the fiber is trivial. □
Conversely, if (iii) holds then restricting the given map \( S \times_{T/W} T \rightarrow T \times T \) to \( S \times_{T/W} (\text{Ker}(\alpha) \cap T) \) we get a map into \( \text{Ker}(\tilde{\alpha}) \times T \) (this is immediate from (3)). This means that the map lifts to a map into \( \mathfrak{B}_G^{\mathfrak{d}} \). Replacing both the source and the target by their quotients by \( W \) we get the map required in (i). \( \Box \)

6. \( K \)-theory and blow-ups

The aim of this section is a proof of Proposition 2.15. Recall that 2.15 (a) was already proved in 3.7. \( G \) is assumed simply connected till the further notice.

6.1. Reminder on the affine Grassmannians. Let \( X = X_G \) be the lattice of characters of \( T \), and let \( Y = Y_G \) be the lattice of cocharacters of \( G \). Note that \( X_G = Y_G \), \( Y_G = X_G \). Let \( X^+ \subset X \) (resp. \( Y^+ \subset Y \)) be the cone of dominant weights (resp. dominant coweights). It is well known that the \( G(O) \)-orbits in \( \text{Gr}_G \) are numbered by the dominant coweights: \( \text{Gr}_G = \bigsqcup_{\lambda \in Y^+} \text{Gr}_{G,\lambda} \). The adjacency relation of orbits corresponds to the standard partial order on coweights: \( \text{Gr}_{G,\lambda} \supseteq \bigsqcup_{\mu \leq \lambda} \text{Gr}_{G,\mu} \). The open embedding \( \text{Gr}_{G,\lambda} \hookrightarrow \overline{\text{Gr}_{G,\lambda}} \) will be denoted by \( j_{\lambda} \) or simply by \( j \) if no confusion is likely. The dimension \( \dim(\text{Gr}_{G,\lambda}) = (2\rho, \lambda) \) where \( 2\rho = \sum_{\alpha \in R^+} \alpha \), and \( (,): X \times Y \rightarrow \mathbb{Z} \) is the canonical perfect pairing.

Recall that the \( T \)-fixed points in \( \text{Gr}_G \) are naturally numbered by \( Y \); a point \( \mu \) lies in an orbit \( \text{Gr}_{G,\lambda} \) iff \( \mu \) lies in the \( W \)-orbit of \( \lambda \). Each \( G(O) \)-orbit \( \text{Gr}_{G,\lambda} \) is partitioned into Iwahori orbits isomorphic to affine spaces and numbered by \( \mu \in W\lambda \). Hence the basics of [4] Chapter 5 are applicable in our situation.

In particular, \( K^T(\text{Gr}_{G,\lambda}) \) is a free \( K^T(pt) \)-module, and \( K^{G(O)}(\text{Gr}_{G,\lambda}) = K^G(\text{Gr}_{G,\lambda}) \) is a free \( K^G(pt) \)-module (recall that \( K^T(pt) = \mathbb{C}[T] \), and \( K^G(pt) = \mathbb{C}[T/W] \)). Moreover, the natural map \( K^T(pt) \otimes_{K^G(pt)} K^G(\text{Gr}_{G,\lambda}) \rightarrow K^T(\text{Gr}_{G,\lambda}) \) is an isomorphism, and \( K^G(\text{Gr}_{G,\lambda}) = K^T(\text{Gr}_{G,\lambda}) W \), cf. [4] 6.1.22.

Since \( K^{G(O)}(\text{Gr}_G) = K^T(\text{Gr}_G) \) (resp. \( K^{G(O)}(\text{Gr}_G) = K^G(\text{Gr}_G) \)) is filtered by the support in \( G(O) \)-orbit closures, with the associated graded \( \bigoplus_{\lambda \in Y^+} K^T(\text{Gr}_{G,\lambda}) \) (resp. \( \bigoplus_{\lambda \in Y^+} K^G(\text{Gr}_{G,\lambda}) \)), we arrive at the following

\textbf{Lemma 6.2.} \( K^{G(O)}(\text{Gr}_G) = K^T(\text{Gr}_G) \) is a flat \( K^T(pt) \)-module, and \( K^{G(O)}(\text{Gr}_G) = K^G(\text{Gr}_G) \) is a flat \( K^G(pt) \)-module. Moreover, the natural map \( K^T(pt) \otimes_{K^G(pt)} K^G(\text{Gr}_G) \rightarrow K^T(\text{Gr}_G) \) is an isomorphism, and \( K^G(\text{Gr}_G) = (K^T(\text{Gr}_G))^W \).

6.3. Localization. The space \( K^T(\text{Gr}_G) = K^{G(O)}(\text{Gr}_G) = K(T(O) \setminus T(F)/G(O)) \) is equipped with the two commuting actions: \( K(T(O) \setminus T(F)/T(O)) \) acts by convolutions on the left, and \( K^G(\text{Gr}_G) = K^{G(O)}(\text{Gr}_G) = K(G(O) \setminus G(F)/G(O)) \) acts by convolutions on the right. Also, \( W \) acts on \( K^T(\text{Gr}_G) \) commuting with the right action of \( K^G(\text{Gr}_G) \). Clearly, the algebra \( K(T(O) \setminus T(F)/T(O)) \) is isomorphic to \( \mathbb{C}[\hat{T} \times T] \). The action of \( W \) on \( K^T(\text{Gr}_G) \) normalizes the action of \( K(T(O) \setminus T(F)/T(O)) \) and induces the natural (diagonal) action of \( W \) on \( \mathbb{C}[\hat{T} \times T] \).
Let \( g \) be a general (regular) element of \( T \). Then the fixed point set 
\((\text{Gr}_G)^g = (\text{Gr}_G)^T = Y\) coincides with the image of the embedding \( \text{Gr}_T \hookrightarrow \text{Gr}_G \). 
According to Thomason Localization Theorem (see e.g. [4] 5.10), after localization, 
\((K^T(\text{Gr}_G))_g\) becomes a free rank one \((K(T(\mathcal{O})) / T(\mathcal{F})/T(\mathcal{O}))_g\)-module. 
This means that after restriction to \( T^{reg} \subset T = \text{Spec}(K^T(pt)) \) we have an isomorphism \( K^T(\text{Gr}_G)|_{T^{reg}} \cong C[\hat{T} \times T]|_{T^{reg}} \) compatible with the natural \( W \)-actions. The localized algebra \( K^G(\text{Gr}_G)|_{T^{reg}/W} \) is embedded into 
\( (\text{End}_{K(T(\mathcal{O})) / T(\mathcal{F})/T(\mathcal{O})})|_{T^{reg}} (K^T(\text{Gr}_G)|_{T^{reg}}))_W \). According to Lemma 6.2, 
\( K^G(\text{Gr}_G) = (K^T(\text{Gr}_G))^W \); hence this embedding is an isomorphism, and we have 
\( K^G(\text{Gr}_G)|_{T^{reg}/W} \cong C[\hat{T} \times T]|_{T^{reg}}^W \).

Hence both \( C[\mathfrak{B}_G]^g \) and \( K^G(\text{Gr}_G)_g \) are the flat \( C[T]^W \)-modules embedded into 
\( C[\hat{T} \times T]|_{\Delta^{-1}} \) (see 4.3). We must prove that the identification of \( C[\mathfrak{B}_G]^g|_{T^{reg}/W} \) and 
\( K^G(\text{Gr}_G)|_{T^{reg}/W} \) extends to the identification over the whole \( T/W \). To this end it suffices to check that the identification extends over the codimension 1 points of \( T/W \). Let \( g \in T/W \) be a regular point of \( D \); that is, \( g \) is represented by a semisimple element of \( G \) such that the centralizer \( Z(g) \) has semisimple rank 1.

We must prove that the localizations \( C[\mathfrak{B}_G]^g \) and \( (K^G(\text{Gr}_G))_g \) are isomorphic. To this end it suffices to identify \( C[\hat{T} \times T, \frac{1}{\alpha - 1} \in R]_g \) (which we denote by \( C[\mathfrak{B}_G]^g \) for short) and \((K^T(\text{Gr}_G))_g\). Note that the embedding of reductive groups \( Z(g) \hookrightarrow G \) (the neutral connected component) induces the isomorphism \( \text{Gr}_{Z(g)} = (\text{Gr}_G)^g \hookrightarrow \text{Gr}_G \). 
According to Thomason Localization Theorem, we have an isomorphism of localizations 
\((K^T(\text{Gr}_{Z(g)}))_g \cong (K^T(\text{Gr}_G))_g \). Finally, the isomorphism \( K^T(\text{Gr}_{Z(g)}) \cong C[\mathfrak{B}_{Z(g)}]_g \) follows from the calculations in 3.8, 3.9, and together with the evident isomorphism of localizations 
\( C[\mathfrak{B}_{Z(g)}]_g \cong C[\mathfrak{B}_G]^g \) establishes the desired isomorphism 
\((K^T(\text{Gr}_G))_g \cong C[\mathfrak{B}_G]^g \).

This completes the proof of 2.15 (b).

6.4. Comparison of Poisson structures. In order to compare the Poisson structures on \( K^G(\text{Gr}_G) \) and \( C[\mathfrak{B}_G]^g \) it suffices to identify them on the open subset 
\( K^G(\text{Gr}_G)|_{T^{reg}/W} = C[\mathfrak{B}_G]^g|_{T^{reg}/W} = C[\hat{T} \times T]|_{T^{reg}}^W \). The space 
\( K^{T \times G_m}(\text{Gr}_G) = K^{T(\mathcal{O}) \times G_m}(\text{Gr}_G) = K(T(\mathcal{O}) \ltimes G_m) / G(\mathcal{F}) \ltimes G_m / G(\mathcal{O}) \ltimes G_m) \) 
is equipped with the two commuting actions: \( K(T(\mathcal{O}) \ltimes G_m) / T(\mathcal{F}) \ltimes G_m / T(\mathcal{O}) \ltimes G_m) \) acts by convolutions on the left, and 
\( K^{G \times G_m}(\text{Gr}_G) = K^{G(\mathcal{O}) \times G_m}(\text{Gr}_G) = K(G(\mathcal{O}) \ltimes G_m) / G(\mathcal{F}) \ltimes G_m / G(\mathcal{O}) \ltimes G_m) \) 
acts by convolutions on the right. Also, \( W \) acts on \( K^{T(\mathcal{O}) \times G_m}(\text{Gr}_G) \) commuting with the right action of \( K^{G(\mathcal{O}) \times G_m}(\text{Gr}_G) \). Clearly, the algebra
\( K(T(O) \times \mathbb{G}_m \setminus T(F) \times \mathbb{G}_m / T(O) \times \mathbb{G}_m) \) is isomorphic to the group algebra \( \mathbb{C}[\Gamma] \) of the following Heisenberg group \( \Gamma \).

It is a \( \mathbb{Z} \)-central extension of \( Y \times X \) with the multiplication (written multiplicatively)

\[
(\langle q \rangle, e^{\lambda_1}, e^{\mu_1}) \cdot (\langle q \rangle, e^{\lambda_2}, e^{\mu_2}) = (\langle q \rangle, e^{\lambda_1 + \lambda_2}, e^{\mu_1 + \mu_2})
\]

where \( \langle , \rangle : X \times Y \to \mathbb{Z} \) is the canonical perfect pairing.

Finally, the action of the Weyl group \( W \) on \( K(T(O) \times \mathbb{G}_m \setminus T(F) \times \mathbb{G}_m / T(O) \times \mathbb{G}_m) \) induces the natural (diagonal) action of \( W \) on \( \mathbb{C}[\Gamma] \). From this we deduce, exactly as in 6.3, that \( K^{\mathbb{G}(O) \times \mathbb{G}_m}(\mathbb{G}_G)[T_{\text{reg}}/W] \simeq \mathbb{C}[\Gamma][T_{\text{reg}}/W] \). It follows that the Poisson structure on \( K^{\mathbb{G}(O) \times \mathbb{G}_m}(\mathbb{G}_G)[T_{\text{reg}}/W] \) coincides with the standard Poisson structure on \( \mathbb{C}[\tilde{T} \times T_{\text{reg}}] \).

This completes the proof of 2.15 (c).

6.5. The case of non simply connected \( G \). For general \( G \) let \( \tilde{G} \) denote its universal cover, and let \( \tilde{T} \) stand for the Cartan of \( \tilde{G} \). Note that the dual torus is \( \tilde{T} / \pi_1(G) \).

As in 6.3, we have \( K^{\mathbb{G}(G)}(\mathbb{G}_G) = (\text{End}_{K(T(O) \setminus T(F)/T(O))}(K(T(G))) W, W \) so it suffices to identify the \( K(T(O) \setminus T(F)/T(O)) \times W = \mathbb{C}[\tilde{T} \times T] \times W \)-module \( K(T(G)) \) with \( \mathbb{C}[\tilde{T} \times T, \frac{\alpha - 1}{\alpha - 1}, \alpha \in R] = \text{Spec} \mathbb{C}[\tilde{G}] \). We do this by reduction to the known case of \( \tilde{G} \).

Evidently, the \( K(T(O) \setminus T(F)/T(O)) \times W = \mathbb{C}[\tilde{T} \times T] \times W \)-module \( K^{\mathbb{G}(\tilde{G})}(\mathbb{G}_G) \) equals \( \mathbb{C}[\tilde{T} \times T] \times W \otimes_{\mathbb{C}[\tilde{T} / \pi_1(\tilde{G}) \times T]} K^{\mathbb{G}(\tilde{G})} \). On the other hand, it follows from 6.3 that the \( K(T(O) \setminus T(F)/T(O)) \times W = \mathbb{C}[(\tilde{T} / \pi_1(\tilde{G})) \times T] \times W \)-module \( K^{\mathbb{G}(\tilde{G})}(\mathbb{G}_G) \) equals the invariants of \( \pi_1(\tilde{G}) \) in \( K^{\mathbb{G}(\tilde{G})}(\mathbb{G}_G) \), that is \( \mathbb{C}[(\tilde{T} / \pi_1(\tilde{G})) \times T, \frac{\alpha - 1}{\alpha - 1}, \alpha \in R] \).

This completes the proof of 2.15 for general \( G \).

6.6. Borel-Moore Homology and blow-ups. Theorem 2.12 is proved absolutely parallely to the proof of Theorem 2.15.

7. Computation of \( K_{\mathbb{G}(O)}(\Lambda) \).  

7.1. The affine Grassmannian Steinberg variety. We denote by \( u \subset g(0) \) (resp. \( U \subset G(0) \)) the nilpotent (resp. unipotent) radical. It has a filtration \( u = u^{(0)} \supset u^{(1)} \supset \ldots \) by congruence subalgebras. The trivial (Tate) vector bundle \( g(F) \) with the fiber \( g(F) \) over \( G_G \) has a structure of an ind-scheme. It contains a profinite dimensional vector subbundle \( u \) whose fiber over a point \( g \in G_G \) is represented by a compact subalgebra in \( g(F) \). The trivial vector bundle \( g(F) = g(F) \times G_G \) also contains a trivial vector subbundle \( u \times G_G \).

We will call \( \mathcal{U} \) the cotangent bundle of \( G_G \), and we will call the intersection \( \Lambda := \mathcal{U} \cap (u \times G_G) \) the affine Grassmannian Steinberg variety. It has a structure of an ind-scheme of ind-finite type. Namely, if \( p \) stands for the natural projection \( \Lambda \to G_G \), then \( \Lambda_{\leq \lambda} := p^{-1}(G_G, G_G) \) is a scheme of finite type, and \( \Lambda = \bigcup \Lambda_{\leq \lambda} \).
Note that for a fixed $\tilde{\lambda}$ and $l \gg 0$ the intersection of fibers of $u$ over all points of $\text{Gr}_{G,\tilde{\lambda}}$ (as vector subspaces of $g(F)$) contains $u^{(l)}$. Thus $u^{(l)}$ acts freely (by fiberwise translations) on $\Lambda_{\leq \tilde{\lambda}}$, and the quotient is a scheme of finite type, to be denoted by $\Lambda^l_{\leq \tilde{\lambda}}$.

For $k > l$ we have evident affine fibrations $p^l_k : \Lambda^k_{\leq \tilde{\lambda}} \to \Lambda^l_{\leq \tilde{\lambda}}$, and $\Lambda_{\leq \tilde{\lambda}}$ coincides with the inverse limit of this system.

Similarly, the total space of the vector bundle $u$ (to be denoted by the same symbol) is a union of infinite type schemes $u_{\leq \tilde{\lambda}}$, and for fixed $\lambda$ and $l \gg 0$, the scheme $u_{\leq \lambda}$ is the inverse limit of affine fibrations $p^l_k : u_{\leq \lambda}^k \to u_{\leq \lambda}^l (k > l)$. Note that the proalgebraic group $G(O)$ acts on all the above schemes, and the fibrations $p^l_k$ are $G(O)$-equivariant.

A $G(O)$-equivariant coherent sheaf $F$ on $u$ is by definition supported on some $u_{\leq \lambda}$. There, it is defined as a collection of $G(O)$-equivariant sheaves $F^l$ on $u_{\leq \lambda}$ for $l \gg 0$ together with isomorphisms $(p^l_k)^* F^l \simeq F^k$. We will consider the $G(O)$-equivariant coherent sheaves on $u$ supported on $\Lambda$, and $D^b \text{Coh}^{G(O)}_\Lambda(u)$ stands for the derived category of such sheaves, and $K^{G(O)}(\Lambda)$ stands for the $K$-group of such sheaves.

7.2. Convolution in $D^b \text{Coh}^{G(O)}_\Lambda(u)$. We have a principal $G(O)$-bundle $G(F) \to \text{Gr}_G$.

Given a $G(O)$-(ind)-scheme $A$ we can form an associated bundle $\tilde{A} = G(F) \times_{G(O)} A \to \text{Gr}_G$. Given a coherent $G(O)$-equivariant sheaf $F$ on $A$ we can form an associated sheaf $\tilde{F}$ on $\tilde{A}$ as $G(O)$-invariants in the direct image of $\mathcal{O}_{G(F)} \boxtimes F$ from $G(F) \times A$ to $G(F) \times_{G(O)} A$. If $A = \text{Gr}_G$, apart from the natural projection $p_1 : \tilde{A} \to \text{Gr}_G$, we have a multiplication map $G(F) \times_{G(O)} \text{Gr}_G \to \text{Gr}_G$, to be denoted $p_2$. Then $(p_1, p_2)$ identifies $\tilde{\text{Gr}}_G$ with $\text{Gr}_G \times \text{Gr}_G$.

Furthermore, $\tilde{u}$ is a vector bundle over $\tilde{\text{Gr}}_G = \text{Gr}_G \times \text{Gr}_G$ which is naturally identified with $p^2_2 u$. Thus we have an ind-proper morphism $p_2 : \tilde{u} \to u$.

Note that both $\tilde{u} = p^2_2 u$ and $p^1_1 u$ are subbundles in the trivial (Tate) vector bundle $g(F)$ over $\text{Gr}_G \times \text{Gr}_G$ with the fiber $g(F)$. Their intersection is naturally identified with $\Lambda$. In particular, we have an embedding $\tilde{\Lambda} \subset p^1_1 u \oplus p^2_2 u$, and an ind-proper morphism $p_2 : \tilde{\Lambda} \to u$.

Hence given $G(O)$-equivariant coherent sheaves $\mathcal{F}, \mathcal{G}$ on $\Lambda$ we can consider the $G(O)$-equivariant complex $\mathcal{F} \ast \mathcal{G} := (p_2)_*(p^1_1 \mathcal{F} \boxtimes \mathcal{G})$ (tensor product over the structure sheaf of the profinite dimensional vector bundle $p^1_1 u \oplus p^2_2 u$). Clearly, $\mathcal{F} \ast \mathcal{G}$ is supported on $\Lambda$. Hence we get a convolution operation on $D^b \text{Coh}^{G(O)}_\Lambda(u)$ and on $K^{G(O)}(\Lambda)$ once we check that $p^2_2 \mathcal{F} \boxtimes \mathcal{G}$ is bounded.

To this end, note that $\mathcal{G}$ is flat over the first copy of $\text{Gr}_G$, and for some $\tilde{\lambda}$ the sheaf $\mathcal{F}$ is supported on $\Lambda_{\leq \tilde{\lambda}}$, so the tensor product $p^1_1 \mathcal{F} \boxtimes \mathcal{G}$ can actually be computed over the structure sheaf of $(p^1_1 u \oplus p^2_2 u)_{\text{Gr}_G \times \text{Gr}_G} = u_{\leq \tilde{\lambda}} \times u \subset u \times u = p^1_1 u \oplus p^2_2 u$. That is, $p^1_1 \mathcal{F} \boxtimes \mathcal{G}$ is the direct image of $p^1_1 \mathcal{F}_{u_{\leq \tilde{\lambda}} \times u} \boxtimes \mathcal{G}_{u_{\leq \tilde{\lambda}} \times u}$ under the closed embedding $u_{\leq \tilde{\lambda}} \times u \hookrightarrow u \times u$. On the other hand, $p^1_1 \mathcal{F}$ is flat over the second copy of $\text{Gr}_G$, while the support of $\mathcal{G}$ intersected with $u_{\leq \tilde{\lambda}} \times u$ is contained in $u_{\leq \tilde{\lambda}} \times u_{\leq \tilde{\mu}}$ for some $\tilde{\mu}$. Hence the
tensor product \( p_1^* \mathcal{F} \otimes \tilde{\mathcal{G}} \) can actually be computed over the structure sheaf of \( \mathbb{G}_{\leq \lambda} \times \mathbb{G}_{\leq \mu} \).

There exists \( l \gg 0 \) such that the diagonal fiberwise action of \( (0) \) on \( \mathbb{G}_{\leq \lambda} \times \mathbb{G}_{\leq \mu} \) is free, and both \( p_1^* \mathcal{F} \) and \( \tilde{\mathcal{G}} \) restricted to \( \mathbb{G}_{\leq \lambda} \times \mathbb{G}_{\leq \mu} \) are \( (0) \)-equivariant, that is, they are lifted from the sheaves on \( \mathbb{G}_{\leq \lambda} \times \mathbb{G}_{\leq \mu} / \mathbb{G} =: V \); we abuse notation by keeping the same names for these sheaves. So the tensor product \( p_1^* \mathcal{F} \otimes \tilde{\mathcal{G}} \) can actually be computed as the tensor product of coherent sheaves over the structure sheaf of the profinite dimensional vector bundle \( V \) over the finite dimensional scheme \( \overline{\text{Gr}}_{G, \lambda} \times \overline{\text{Gr}}_{G, \mu} \).

Now there exists a vector subbundle \( V' \subset V \) such that the quotient \( \overline{\mathbf{V}} := V / V' \) is a finite dimensional vector bundle, \( p_1^* \mathcal{F} \) is lifted from \( \overline{\mathbf{V}} \), and the support of \( \tilde{\mathcal{G}} \) in \( V \) projects isomorphically onto its image in \( \overline{\mathbf{V}} \). Moreover, recall that \( p_1^* \mathcal{F} \) is flat over \( \overline{\text{Gr}}_{G, \mu} \), while \( \tilde{\mathcal{G}} \) is flat over \( \overline{\text{Gr}}_{G, \lambda} \). Clearly, in this situation \( p_1^* \mathcal{F} \otimes \tilde{\mathcal{G}} \in D^b(V) \). This explains why \( G(\mathcal{O}) \)-equivariant coherent sheaves \( \mathcal{F}, \mathcal{G} \) on \( \Lambda \) the tensor product \( p_1^* \mathcal{F} \otimes \tilde{\mathcal{G}} \) is a bounded complex of coherent sheaves on \( p_1^* \mathcal{U} \oplus p_2^* \mathcal{U} \) supported on \( \Lambda \). Hence the same is true for the bounded complexes of \( G(\mathcal{O}) \)-equivariant coherent sheaves \( \mathcal{F}, \mathcal{G} \) on \( \mathcal{U} \) supported on \( \Lambda \). Thus, \( D^b(\text{Coh}^G_{\Lambda}(\mathcal{U})) \) is closed with respect to convolution.

**Theorem 7.3.** \( K^{G(\mathcal{O})}(\Lambda) \) is a commutative algebra isomorphic to \( \mathbb{C}[\bar{T} \times T]^W \).

**Remark 7.4.** Since \( \Lambda_G \) is an affine Grassmannian analogue of the classical Steinberg variety, this result agrees well with the geometric realization of the Cherednik double affine Hecke algebra in [8], [23]. In effect, \( K^{G(\mathcal{O})}(\Lambda_G) \) is the spherical subalgebra of the Cherednik algebra with both parameters trivial: \( q = t = 1 \).

### 7.5. Bialynicki-Birula stratifications

The proof of Theorem 7.3 uses the following lemma on \( K \)-theory of cellular spaces. Let \( M \) be a normal quasiprojective variety equipped with a torus \( H \)-action with finitely many fixed points. We assume that \( M \) is equipped with an \( H \)-invariant stratification \( M = \bigsqcup_{\mu \in M^H} M_{\mu} \) such that each stratum \( M_{\mu} \) contains exactly one \( H \)-fixed point \( \mu \), and \( M_{\mu} \) is isomorphic to an affine space. For \( \mu \in M^H \) we denote by \( j_{\mu} : M_{\mu} \hookrightarrow M \) the locally closed embedding of the corresponding stratum. We denote by \( i_{\mu} : \mu \hookrightarrow M_{\mu} \) the closed embedding of an \( H \)-fixed point in the corresponding stratum, or in the whole of \( M \) when no confusion is likely. We denote by \( \mu \leq \nu \) the closure relation of strata. We denote by \( M_{\leq \mu} \subset M \) the union \( \bigcup_{\nu \leq \mu} M_{\nu} \).

Given an \( H \)-equivariant closed embedding of \( M \) into a smooth \( H \)-variety \( M' \) (for the existence see [22]) we denote by \( T^* M \) the restriction of the cotangent bundle \( T^* M' \) to \( M \subset M' \). We denote by \( i : M \hookrightarrow T^* M \) the embedding of the zero section. We also denote by \( i_{\mu} \) the closed embedding of the conormal bundle \( T^*_M \) to \( T^* M \) when no confusion is likely. Finally, we denote by \( \mathcal{L}' \) the union of conormal bundles \( \bigcup_{\mu} T^*_M M' \), and \( j \) stands for the closed embedding \( \mathcal{L}' \hookrightarrow T^* M \). We denote by \( \mathcal{L}'_{\leq \mu} \subset \mathcal{L}' \) the union \( \bigcup_{\nu \leq \mu} T^*_M \) to \( M' \); it is a closed subvariety of \( \mathcal{L}' \). It has a closed subvariety \( \mathcal{L}'_{\leq \mu} := \bigcup_{\nu \leq \mu} T^*_M \).
For $\mu \in M^H$ we have an embedding $i_{\mu*} : K^H(\mu) \hookrightarrow K^H(M)$. We have an embedding $j_* : K^H(\mathcal{L}') \hookrightarrow K^H(T^*M) \cong K^H(M)$. Indeed, the exact sequences (see [4] Chapter 5)

$$0 \to K^H(\mathcal{L}'_{\leq \mu}) \to K^H(\mathcal{L}'_{\leq \mu}) \to K^H(T_{M_\mu}^*M') \to 0,$$

$$0 \to K^H(T^*M'|_{M_\mu}) \to K^H(T^*M'|_{M_\mu}) \to K^H(T^*M'|_{M_\mu})$$

give rise to the support filtrations on $K^H(\mathcal{L}')$ and $K^H(T^*M)$ with associated graded $\bigoplus_{\mu \in M^H} K^H(T_{M_\mu}^*M')$ and $\bigoplus_{\mu \in M^H} K^H(T^*M'|_{M_\mu})$. Now $j_*$ is strictly compatible with the support filtrations and clearly injective on the associated graded.

Note that the image $j_*(K^H(\mathcal{L}')) \subset K^H(M)$ is independent of the choice of the closed embedding $M \hookrightarrow M'$. In effect, given another embedding $M \hookrightarrow \tilde{M}$, we can consider the diagonal embedding $M \hookrightarrow M'' := M' \times M$. Clearly, we have a projection $p : T^*M''|_M \to T^*M'|_M$ which realizes $T^*M''|_M$ as a vector bundle over $T^*M'|_M$.

Moreover, if we denote by $\mathcal{L}''$ the union of conormal bundles $\bigcup_{\mu} T_{M_\mu}^*M'' \subset T^*M''|_M$ then $\mathcal{L}'' = p^{-1}\mathcal{L}'$. This shows that the images of $K^H(\mathcal{L}')$ and $K^H(\mathcal{L}'')$ in $K^H(M)$ coincide, and thus $j_*(K^H(\mathcal{L}')) \subset K^H(M)$ is well-defined.

**Lemma 7.6.** In $K^H(M)$ we have an equality $j_*(K^H(\mathcal{L}')) = \bigoplus_{\mu} i_{\mu*}(K^H(\mu))$.

**Proof** Let $K^H(D_M)$ stand for the $K$-group of weakly $H$-equivariant $D$-modules on $M'$ supported on $M \subset M'$. Given such a $D$-module and passing to associated graded with respect to a good filtration, we obtain an $H$-equivariant coherent sheaf on $T^*M$, and this way one obtains a homomorphism $SS : K^H(D_M) \to K^H(T^*M) \cong K^H(M)$ (see e.g. [11]). Let $\delta_\mu$ stand for a $\delta$-function $D$-module at the point $\mu \in M^H$ with its obvious $H$-equivariance. Then, evidently, $SS(\delta_\mu)$ generates $i_{\mu*}(K^H(\mu))$ as a module over $K^H(pt)$. Moreover, $\{SS(j_{\mu*}O_{M_\mu})$, $\mu \in M^H\}$ forms a basis of $j_*(K^H(\mathcal{L}'))$.

In effect, the closed embedding $\mathcal{L}'_{\leq \mu} \hookrightarrow \mathcal{L}'_{\leq \mu}$ gives rise to the exact sequence

$$0 \to K^H(\mathcal{L}'_{\leq \mu}) \to K^H(\mathcal{L}'_{\leq \mu}) \to K^H(T_{M_\mu}^*M') \to 0$$

(see [4] Chapter 5), and the image of $SS(j_{\mu*}O_{M_\mu})$ in $K^H(T_{M_\mu}^*M')$ clearly generates it.

So it is enough to check the equality in $K^H(T^*M)$:

$$SS(\delta_\mu) = SS(j_{\mu*}O_{M_\mu}) : (-1)^{\dim M_\mu} \det(T_{M_\mu}M_\mu)$$

where $\det(T_{M_\mu}M_\mu)$ is the character of $H$ (thus an invertible element of $K^H(pt) = \mathbb{C}[H]$) acting in the determinant of the tangent bundle of $M_\mu$ at $\mu$.

To this end note that restriction to the $H$-fixed points gives rise to an embedding $\oplus_{\nu} i_{\nu*} : K^H(T^*M) \hookrightarrow \oplus_{\nu} K^H(\nu)$. This is checked by induction in $\nu$ using the exact sequences

$$0 \to K^H(T^*M'|_{M_\mu}) \to K^H(T^*M'|_{M_\mu}) \to K^H(T^*M'|_{M_\mu}) \to 0.$$

It is clear that for $\nu = \mu$ the restrictions $i_{\nu*}$ of the LHS and RHS of (12) coincide. We are going to check that for $\nu \neq \mu$ the restrictions $i_{\nu*}$ of the LHS and RHS of (12) both vanish. Evidently, $i_{\nu*} SS(\delta_\mu) = 0$. 


Recall that \( i_\nu \) also stands for the closed embedding \( T_\nu^* M' \hookrightarrow T^* M \), so we just have to check that \( i_\nu^* SS(j_\mu^! \mathcal{O}_{M_\mu}) = 0 \in K^H(T_\nu^* M') \). Note that the functor of global sections of \( H \)-equivariant coherent sheaves on the vector space \( T_\nu^* M' \) gives rise to an embedding \( \Gamma : K^H(T_\nu^* M') \hookrightarrow \mathbb{Z}^{X^*(H)} \) where \( X^*(H) \) stands for the lattice of characters of \( H \). Now for a \( D \)-module \( \mathcal{F} \) we have \( \Gamma(i_\nu^* SS \mathcal{F}) = i_\nu^* \mathcal{F} \) where \( i_\nu^* \mathcal{F} \) stands for the fiber at \( \nu \in M \) of the \( H \)-equivariant quasicoherent \( \mathcal{O}_M \)-module \( \mathcal{F} \). Finally, for \( \mathcal{F} = j_\mu^! \mathcal{O}_{M_\mu} \) and \( \nu \neq \mu \) we have \( i_\mu^* j_\mu^! \mathcal{O}_{M_\mu} = 0 \). This completes the proof of the lemma.

### 7.7. Bialynicki-Birula stratification of \( \text{Gr}_G \)

We consider the stratification of \( \text{Gr}_G \) by the Iwahori orbits \( \text{Gr}_G = \bigsqcup_{\mu \in Y} \text{Gr}_G^\mu \). This is a refinement of the stratification by the \( G(\mathcal{O}) \)-orbits: \( \text{Gr}_{G,\lambda} = \bigsqcup_{\mu \in B} \text{Gr}_G^\mu \). Let us denote by \( n \supset u \) the nilpotent radical of the Iwahori subalgebra in \( g(\mathcal{F}) \). The union of conormal bundles to the Iwahori orbits is the following subvariety \( \Lambda_I \) of the cotangent bundle \( \mathfrak{u}^* \) by definition, \( \Lambda_I := \mathfrak{u} \cap (n \times \text{Gr}_G) \).

We have a closed embedding \( \Lambda \hookrightarrow \Lambda_I \). Lemma 7.6 allows us to compute \( K^T(\Lambda_I) \hookrightarrow K^T(\Lambda_I) \hookrightarrow K^T(\mathfrak{u}) = K^T(\text{Gr}_G) \). Note that \( W \) acts naturally on both \( K^T(\Lambda) \) and \( K^T(\text{Gr}_G) \), and the embedding \( K^T(\Lambda) \hookrightarrow K^T(\text{Gr}_G) \) is \( W \)-equivariant. Also, \( (K^T(\Lambda_I))^W = K^G(\Lambda) = K^G(\mathcal{O}(\Lambda)) \). Hence, the image of the embedding \( K^G(\mathcal{O}(\Lambda)) \hookrightarrow K^T(\Lambda_I) \approx \mathbb{C}[\hat{T} \times T] \subset K^T(\text{Gr}_G) \) lies in the invariants of the diagonal \( W \)-action on \( \mathbb{C}[\hat{T} \times T] \). Thus to prove Theorem 7.3 we must check that the image of this embedding contains \( \mathbb{C}[\hat{T} \times T]^W \).

We have projections \( \pi : \Lambda \to \text{Gr}_G \), and \( \pi_I : \Lambda_I \to \text{Gr}_G \). For \( \lambda \in Y^+ \) we denote by \( \Lambda_{<\lambda} \) (resp. \( \Lambda_{\leq \lambda}, \Lambda_{\leq \lambda} \)) the preimage \( \pi^{-1}(\text{Gr}_{G,\lambda}) \) (resp. \( \pi^{-1}(\overline{\text{Gr}_{G,\lambda}}), \pi^{-1}(\overline{\text{Gr}_{G,\lambda}} - \text{Gr}_{G,\lambda}) \)). For \( \lambda \in Y^+ \) we denote by \( \Lambda_{I,\lambda} \) (resp. \( \Lambda_{I,\leq \lambda}, \Lambda_{I,\leq \lambda} \)) the preimage \( \pi_{I}^{-1}(\text{Gr}_{G,\lambda}) \) (resp. \( \pi_{I}^{-1}(\overline{\text{Gr}_{G,\lambda}}), \pi_{I}^{-1}(\overline{\text{Gr}_{G,\lambda}} - \text{Gr}_{G,\lambda}) \)). Clearly, \( \Lambda_{<\lambda} \) (resp. \( \Lambda_{I,\leq \lambda} \)) is closed in \( \Lambda_{\leq \lambda} \) (resp. \( \Lambda_{I,\leq \lambda} \)), with the open complement \( \Lambda_{\lambda} \) (resp. \( \Lambda_{I,\lambda} \)). In \( K \)-groups we have exact sequences (see [4] Chapter 5)

\[
0 \to K^T(\Lambda_{<\lambda}) \to K^T(\Lambda_{\leq \lambda}) \to K^T(\Lambda_{\lambda}) \to 0,
\]

\[
0 \to K^T(\Lambda_{I,\leq \lambda}) \to K^T(\Lambda_{I,\leq \lambda}) \to K^T(\Lambda_{I,\lambda}) \to 0.
\]

Thus we obtain a support filtration on \( K^T(\Lambda_I) \) (resp. \( K^T(\Lambda) \)) with associated graded \( \bigoplus_{\lambda \in Y^+} K^T(\Lambda_{I,\lambda}) \) (resp. \( \bigoplus_{\lambda \in Y^+} K^T(\Lambda_{\lambda}) \)).

We have the embeddings \( K^T(\Lambda_{I,\lambda}) \hookrightarrow K^T(\Lambda_{I,\lambda}) \hookrightarrow K^T(\mathfrak{u}(\text{Gr}_G)) \approx K^T(\text{Gr}_G) \). The Weyl group \( W \) acts naturally both on \( K^T(\Lambda_{\lambda}) \) and \( K^T(\text{Gr}_G) \), and to prove Theorem 7.3 it suffices to check that the image of \( (K^T(\Lambda_{I,\lambda}))^W \) in \( K^T(\Lambda_{I,\lambda}) \) contains (equivalently, coincides with) the intersection \( K^T(\Lambda_{I,\lambda}) \cap (K^T(\text{Gr}_G))^W \).
To this end recall that $\text{Gr}_{G,\lambda}$ can be $G$-equivariantly identified with the total space $\tilde{B}$ of a vector bundle over a certain partial flag variety $B$ of the group $G$ (the quotient $G/P_{\lambda}$ by a parabolic subgroup depending on $\lambda$). The Borel subgroup $B \subset G$ acts on $B$ with finitely many orbits numbered by the cosets of parabolic Weyl subgroup $W_{\lambda} = W/W_{\lambda}$; we have $B = \bigsqcup_{w \in W_{\lambda}} B_w$. Let us denote by $\mathcal{L} \subset T^*B$ the union of conormal bundles $\mathcal{L} = \bigsqcup_{w \in W_{\lambda}} T^*_B T^* B_w$. Let us also denote by $\tilde{B}_w$ the preimage of $B_w$ in $\tilde{B}$ (it coincides with a certain Iwahori orbit $\text{Gr}^0_{G,\lambda} \subset \text{Gr}_{G,\lambda} = \tilde{B}$). We define $\tilde{\mathcal{L}} := \bigsqcup_{w \in W_{\lambda}} T^*_B T^* B \subset T^*\tilde{B}$.

Then there exists a $G$-equivariant profinite dimensional vector bundle $\mathcal{V} \subset T^*\tilde{B}$ such that $\mathcal{V} \simeq \mathcal{V}|_{\tilde{\mathcal{L}}} \simeq \Lambda_{I,\lambda}$, and under this isomorphism we have $\mathcal{V}|_{\tilde{\mathcal{L}}} \simeq \Lambda_{\tilde{\mathcal{L}}}$, and under this isomorphism we have $\mathcal{V}|_{\tilde{\mathcal{L}}} \simeq \Lambda_{I,\lambda}$. Thus to prove Theorem 7.3 it is enough to check that the image of $(K^T(\mathcal{B}))^W$ in $K^T(T^*\tilde{B})$ contains the intersection $K^T(\mathcal{L}) \cap (K^T(T^*\tilde{B}))^W$. Equivalently, we have to check that the image of $(K^T(\mathcal{B}))^W$ in $K^T(T^*\tilde{B})$ contains the intersection $K^T(\mathcal{L}) \cap (K^T(T^*\tilde{B}))^W$.

This is the subject of the following lemma.

**Lemma 7.8.** Let $i : B \hookrightarrow T^*B$ denote the embedding of the zero section, and let $j : \mathcal{L} \hookrightarrow T^*B$ denote the natural closed embedding. Then $i_*(K^T(\mathcal{B}))^W$ coincides with $\text{Im}(j_* : K^T(\mathcal{L}) \hookrightarrow K^T(T^*B)) \cap (K^T(T^*B))^W$.

**Proof.** For $w \in W_{\lambda}$ we denote by $w \in B_w \subset B$ the corresponding $T$-fixed point. We denote by $i_w$ the closed embedding $T_w B \hookrightarrow T^*B$ (and also the closed embedding $w \hookrightarrow B$, when the confusion is unlikely), and we denote by $i_{w*}$ the closed embedding $w \hookrightarrow T^*B$. According to Lemma 7.6, the image of $j_* : K^T(\mathcal{L}) \hookrightarrow K^T(T^*B)$ coincides with the image of $\bigoplus_{w \in W_{\lambda}} i_{w*} : \bigoplus_{w \in W_{\lambda}} K^T(T^*_w B) \hookrightarrow K^T(T^*B)$. We have an embedding $\bigoplus_{w \in W_{\lambda}} i_{w*} : K^T(T^*_w B) \hookrightarrow \bigoplus_{w \in W_{\lambda}} K^T(w)$, and similarly an embedding $\bigoplus_{w \in W_{\lambda}} i_{w*} : K^T(B) \hookrightarrow \bigoplus_{w \in W_{\lambda}} K^T(w)$.

Clearly, the $W$-invariants project injectively into any direct summand: $K^G(B) = (K^T(B))^W \xrightarrow{i_w*} K^T(w)$ (resp. $K^G(T^*B) = (K^T(T^*B))^W \xrightarrow{i_{w*}} K^T(w)$) for any $w \in W_{\lambda}$. Thus it suffices to check that for any $w \in W_{\lambda}$ we have a coincidence $\text{Im}(i_w* i_{w*}) : K^T(T^*_w B)^W \hookrightarrow K^T(w) = \text{Im}(i_{w*} i_{w*} : K^G(\mathcal{B}) \rightarrow K^T(w))$. Note that if $w = e$ (the identity coset of $W_{\lambda}$ in $W$), then the image $i_{e*}(K^T(e))^W \subset K^T(e)$ (resp. $i_{e*}(K^T(T^*_w B))^W \subset K^T(e)$) coincides with $(K^T(e))^W \subset \mathbb{C}[T]^W_{\lambda}$. Moreover, under identification $K^T(T^*_w B) = K^T(e) = \mathbb{C}[T]$, we have $K^T(T^*_w B) \cap (K^T(T^*_w B))^W = \mathbb{C}[T]^W_{\lambda}$.

Identifying both $K^T(T^*_w B)$ and $K^T(e)$ with $\mathbb{C}[T]$, the map $i_{e*}i_{w*}$ is a multiplication by the product $\Delta_1 = \prod_{k=1}^{\dim \mathbb{B}} (1 - \chi_k)$ where $\chi_k$ run through the characters of $T$ in the tangent space $T_e(T^*_w B) = T^*_w B$. Furthermore, identifying $K^G(\mathcal{B})$ with $\mathbb{C}[T]^W_{\lambda}$, and $K^T(e)$ with $\mathbb{C}[T]$, the map $i_{e*}i_{w*}$ is a multiplication by the product $\Delta_2 = \prod_{k=1}^{\dim \mathbb{B}} (1 - \chi_k')$ where $\chi'_k$ run through the characters of $T$ in the tangent space $T_e B$. We can arrange the characters $\chi'_k$ so that we have $\chi'_k = \chi_k^{-1}$. Then we see that $\Delta_1 = \Delta_2 \prod_{k=1}^{\dim \mathbb{B}} (-\chi_k)$, so they differ by an invertible function, hence the corresponding images coincide: $\Delta_1 \cdot \mathbb{C}[T]^W_{\lambda} = \Delta_2 \cdot \mathbb{C}[T]^W_{\lambda}$.

This completes the proof of the lemma along with Theorem 7.3.
7.9. In this subsection we describe (without striving for high precision) a conjectural picture motivating Theorem 7.3.

We hope that the isomorphism $K^{G(O)}(\Lambda_G) = \mathbb{C}[\hat{T} \times T]^W = \mathbb{C}[T \times T]^W = K^{G(O)}(\Lambda_G)$ lifts to an equivalence of monoidal categories $F : D^b\text{Coh}^{G(O)}_{\Lambda_G}(\mathcal{U}_G) \simeq D^b\text{Coh}^{\tilde{G}(O)}(\mathcal{U}_{\tilde{G}})$. The conjectural equivalence $F$ is related to the Langlands correspondence in the following way.

Recall that the conjectural (for $G = GL(n)$ mostly proven in [9]) geometric Langlands correspondence is an equivalence of triangulated categories between the derived category of $D$-modules on the stack $\text{Bun}_G$ of $G$-bundles on a given smooth projective curve $C$, and the derived category of coherent sheaves on the stack of $\tilde{G}$ local systems on the same curve. One might expect its “classical limit” to be an equivalence between the derived categories of coherent sheaves $L : D(T^* \text{Bun}_G) \simeq D(T^* \text{Bun}_{\tilde{G}})$ where $T^* \text{Bun}_G$ is the cotangent bundle to the moduli stack of $G$-bundles on $C$. Given a point $c \in C$, and identifying $\mathcal{O}$ with the algebra of functions on the formal neighbourhood of $c$, one gets an action of $D^b\text{Coh}^{G(O)}(\mathcal{U}_G)$ on $D(T^* \text{Bun}_G)$. The “classical limit” of the Hecke eigen-property of geometric Langlands correspondence (see [2]) should be stated in terms of this action; it should say that the global equivalence $L$ is compatible with our local equivalence $F$.

8. PERVERSE SHEAVES AND FUSION

We refer the reader to [3] for the definition of perverse equivariant coherent sheaves and related objects.

8.1. Recall the setup of 6.1. Note that all the $G(O)$-orbits in a connected component of $\text{Gr}_G$ have dimensions of the same parity. Thus it makes sense to consider the middle perversity function $p(\text{Gr}_{G,\lambda}) = -\frac{1}{2} \dim(\text{Gr}_{G,\lambda}) = -\langle \rho, \lambda \rangle$. It is obviously strictly monotone and comonotone, but at some connected components of $\text{Gr}_G$ it takes values in half-integers. This means that we consider equivariant complexes formally placed in half-integer homological degrees. The theory of [3] defines the artinian abelian category $\mathcal{P}^{G(O)}(\text{Gr}_G)$ of perverse $G(O)$-equivariant coherent sheaves (with respect to the above middle perversity). Let $D^{b, G(O)}(\text{Gr}_G)$ denote the bounded derived category of $G(O)$-equivariant coherent sheaves on $\text{Gr}_G$ (with the same convention that the complexes at “odd” connected components are placed in half-integer homological degrees).

Given two complexes $\mathcal{F}, \mathcal{G} \in D^{b, G(O)}(\text{Gr}_G)$ we have their convolution $\mathcal{F} \star \mathcal{G} \in D^{b, G(O)}(\text{Gr}_G)$. Recall that $\mathcal{F} \star \mathcal{G} = \Pi_0(\mathcal{F} \otimes \mathcal{G})$ where $\Pi_0 : G(\mathcal{F}) \times G(\mathcal{O}) \text{Gr}_G \rightarrow \text{Gr}_G$ is the convolution diagram, and $\mathcal{F} \otimes \mathcal{G}$ is the twisted product of $\mathcal{F}$ and $\mathcal{G}$ on $G(\mathcal{F}) \times G(\mathcal{O}) \text{Gr}_G$.

**Proposition 8.2.** The convolution preserves perverse sheaves: for $\mathcal{F}, \mathcal{G} \in \mathcal{P}^{G(O)}(\text{Gr}_G)$ we have $\mathcal{F} \star \mathcal{G} \in \mathcal{P}^{G(O)}(\text{Gr}_G)$.

**Proof** Denote the projection $G(\mathcal{F}) \rightarrow G(\mathcal{F})/G(\mathcal{O}) = \text{Gr}_G$ by $p$, and consider a stratification $G(\mathcal{F}) \times G(\mathcal{O}) \text{Gr}_G = \bigsqcup_{\lambda, \mu \in \mathcal{Y}^+} p^{-1}(\text{Gr}_{G,\lambda}) \times G(\mathcal{O}) \text{Gr}_{G,\mu}$. Clearly, $\mathcal{F} \otimes \mathcal{G}$ is smooth (locally free) along this stratification, and perverse (with respect to the middle perversity). According to [19] 2.7, the map $\Pi_0$ is stratified semismall with respect to
the above stratification. Now the perversity of \( \Pi_{0*}(\mathcal{F} \otimes \mathcal{G}) \) follows in the same manner as in the constructible case, cf. loc. cit.

### 8.3. The absence of commutativity constraint

According to Proposition 8.2, \( \mathcal{P}^{G(O)}(\text{Gr}_{G}) \) acquires the structure of abelian artinian monoidal category. Moreover, according to 2.15 (a), its \( K \)-ring is commutative. Nevertheless, \( \mathcal{P}^{G(O)}(\text{Gr}_{G}) \) admits no commutativity constraint, as can be seen in the following example.

We recall the setup of 3.6, and consider \( \text{Gr}_{PGL_2} \). One can check that there are the nonsplit exact sequences in \( \mathcal{P}^{PGL_2(O)}(\text{Gr}_{PGL_2}) \):

\[
0 \to V(0)_0 \to V(0)_1 \ast V(-2)_1 \to V(-2)_2 \to 0 \\
0 \to V(-2)_2 \to V(-2)_1 \ast V(0)_1 \to V(0)_0 \to 0
\]

Thus \( V(0)_1 \ast V(-2)_1 \) and \( V(-2)_1 \ast V(0)_1 \) are nonisomorphic.

### 8.4. \( G(O) \ltimes \mathbb{G}_m \)-equivariant sheaves and fusion

The orbits of \( G(O) \ltimes \mathbb{G}_m \) on \( \text{Gr}_G \) coincide with the \( G(O) \)-orbits, so one can consider the abelian artinian monoidal category \( \mathcal{P}^{G(O) \ltimes \mathbb{G}_m}(\text{Gr}_G) \) of \( G(O) \ltimes \mathbb{G}_m \)-equivariant coherent perverse sheaves on \( \text{Gr}_G \).

For \( \mathcal{F} \in \mathcal{P}^{G(O) \ltimes \mathbb{G}_m}(\text{Gr}_G) \) we have \( R\Gamma(\text{Gr}_G, \mathcal{F}) \in D^b(G(O) \ltimes \mathbb{G}_m - \text{mod}) \).

B. Feigin and S. Loktev define (under certain restrictions) in [6] the fusion product

\[
V_1 \ast \ldots \ast V_k \in G(O) \ltimes \mathbb{G}_m - \text{mod}
\]

of \( G(O) \ltimes \mathbb{G}_m \)-modules \( V_1, \ldots, V_k \). We recall some of their results in case \( G = PGL_2 \).

Let \( V(n) \) be the \( n+1 \)-dimensional \( G(O) \ltimes \mathbb{G}_m \)-module factoring through \( G(O) \ltimes \mathbb{G}_m \to G \times \mathbb{G}_m \to G \). Recall the irreducible \( PGL_2(O) \)-equivariant perverse sheaf \( V(n)_m \) introduced in 3.6. It can be lifted to the same named \( PGL_2(O) \ltimes \mathbb{G}_m \)-equivariant perverse sheaf, where the action of \( \mathbb{G}_m \) in the fiber over a \( \mathbb{G}_m \)-fixed point in the orbit \( \text{Gr}_{PGL_2,m} \) is set trivial. In particular, \( R\Gamma(\text{Gr}_{PGL_2}, V(n)_1) = V(n)[\frac{n}{2}] \) for \( n \geq 0 \).

Now we can reformulate Theorem 2.5 of [6] as follows.

**Proposition 8.5.** Let \( n_1 \geq n_2 \geq \ldots \geq n_k \). Then

(a) \( R\Gamma(\text{Gr}_{PGL_2}, V(n_1)_1 \ast \ldots \ast V(n_k)_1) \) is concentrated in degree \(-\frac{k}{2}\).

(b) \( R\Gamma(\text{Gr}_{PGL_2}, V(n_1)_1 \ast \ldots \ast V(n_k)_1)[\frac{k}{2}] \cong V(n_k)_1 \ast \ldots \ast V(n_1)_1 \).

### 8.6. Multiplication table

According to Proposition 8.5, the calculation of fusion product in \( K(G(O) \ltimes \mathbb{G}_m - \text{mod}) \) is closely related to the ring structure of \( K^{G(O) \ltimes \mathbb{G}_m}(\text{Gr}_G) \). Let us formulate the recurrence relations in \( K^{G(O) \ltimes \mathbb{G}_m}(\text{Gr}_G) \), compare [6], end of section 2.1. So \( v(n)_m \) is the class of \( V(n)_m \) in \( K^{G(O) \ltimes \mathbb{G}_m}(\text{Gr}_G) \). We assume that \( n \geq 0 \).

\[
q^{-l} v(l + n + 1)_0 \ast v(l)_1 = q^{-2l} v(2l + n + 1)_2 + q^2 v(n + 1)_0 + q^4 v(n - 3)_0 + \ldots
\]

(the last summand being \( q^n v(0)_0 \) if \( n \) is even, and \( q^{n-1} v(1)_0 \) if \( n \) is odd.)

\[
q^{-l-2} v(l + n - 1)_0 \ast v(l)_1 = q^{-2l-2} v(2l - n + 1)_2 + q^{-2} v(n - 2)_0 + q^{-4} v(n - 4)_0 + \ldots
\]

(the last summand being \( q^{-n} v(0)_0 \) if \( n \) is even, and \( q^{-n+1} v(1)_0 \) if \( n \) is odd.)

\[
v(l + 1)^a_1 \ast v(l)_1^b = q^{\frac{1}{2}(a(1-a)+(l+a)(1-a-b))} v(a + l + a + b)_a+b
\]
References


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