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DISCRETE HOLOMORPHIC GEOMETRY I. DARBOUX TRANSFORMATIONS AND SPECTRAL CURVES

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Abstract. Finding appropriate notions of discrete holomorphic maps and, more generally, conformal immersions of discrete Riemann surfaces into 3–space is an important problem of discrete differential geometry and computer visualization. We propose an approach to discrete conformality that is based on the concept of holomorphic line bundles over "discrete surfaces", by which we mean the vertex sets of triangulated surfaces with bi–colored set of faces. The resulting theory of discrete conformality is simultaneously Möbius invariant and based on linear equations. In the special case of maps into the 2–sphere we obtain a reinterpretation of the theory of complex holomorphic functions on discrete surfaces introduced by Dynnikov and Novikov.

As an application of our theory we introduce a Darboux transformation for discrete surfaces in the conformal 4–sphere. This Darboux transformation can be interpreted as the space– and time–discrete Davey–Stewartson flow of Konopelchenko and Schief. For a generic map of a discrete torus with regular combinatorics, the space of all Darboux transforms has the structure of a compact Riemann surface, the spectral curve. This makes contact to the theory of algebraically completely integrable systems and is the starting point of a soliton theory for triangulated tori in 3– and 4–space devoid of special assumptions on the geometry of the surface.

1. Introduction

The notions of discrete Riemann surfaces and discrete conformal maps are important recurring themes in discrete geometry. In computer graphics, discrete conformal parameterizations and their approximations are used as texture mappings and for the construction of geometric images. In mathematical physics, discrete Riemann surfaces occur in the discretization of physical models such as conformal field theories and statistical mechanics which involve smooth Riemann surfaces. In surface geometry, the concept of discrete conformality is fundamental in the description of discrete analogues of special surface classes including minimal and, more generally, constant mean or Gaussian curvature surfaces.

Whereas discrete models generally have smooth limits there are no consistent procedures to "discretize" smooth models (which is reminiscent to the problem of "quantizing" a classical theory). There are currently several approaches to discrete conformality for maps into the complex plane including the Möbius invariant approach modeled on circle packings or patterns which goes back to Koebe and more recently Thurston, cf. [34], Möbius invariant polygon evolutions driven by constant cross–ratio conditions [3, 4, 19] and linear approaches modelled on discretizations of the Cauchy–Riemann equation, see e.g. Dynnikov and Novikov [14, 15], Mercat [29], Kenyon [22]. For a comparison to the circle packing approach and integrable system interpretation, see Bobenko, Mercat, and

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Suris [2]. Discrete conformal maps into space are often modeled on discrete versions of conformal curvature lines and thus apply only to the restricted class of isothermic surfaces, cf. [3, 4, 5]. The conformality condition is again expressed by cross–ratio conditions, that is by non–linear difference equations on the vertex positions.

The theory of discrete holomorphicity developed in the present paper provides an approach to conformal geometry of discrete surfaces which applies equally to maps into the plane and higher dimensional target spaces and is Möbius invariant, given by linear equations, and not restricted to special surface classes. Our approach is based on the concept of holomorphic line bundles over discrete surfaces. The discrete conformal maps are then ratios of holomorphic sections of such line bundles. From this point of view, the only difference between planar conformal maps, i.e., holomorphic functions, and conformal maps into 3– and 4–space is whether the holomorphic line bundles in question are complex or quaternionic. The relation between the Möbius geometry of discrete surfaces and discrete holomorphic line bundles is provided by a discrete version of the Kodaira correspondence. A vantage point of this approach to discrete holomorphic geometry is that the definition of holomorphic line bundles does not require an a priori notion of discrete Riemann surfaces about which we have nothing to say. This is indicated by the smooth theory where any linear, first–order elliptic differential operator between complex or quaternionic line bundles over an oriented 2–dimensional manifold M defines a complex structure on M and renders the line bundles holomorphic.

The notion of holomorphic line bundles over discrete surfaces requires an additional combinatorial structure which, in the context of discrete conformality, was introduced by Dynnikov and Novikov [14, 15] and appears also in Colin de Verdière [12] and previously in unpublished notes by Thurston: in the present paper a *discrete surface* M is the vertex set of a triangulated smooth surface with a bi–coloring of the faces into black and white triangles, see Figure 1. A complex or quaternionic *line bundle* W over such a discrete

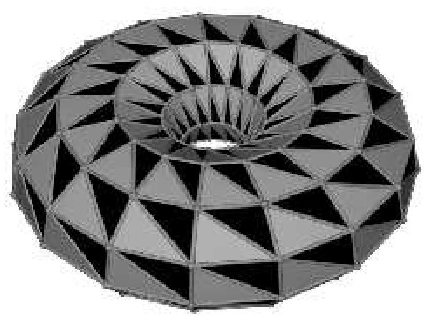

FIGURE 1. A discrete torus immersed into 3–space.

surface M is then a family of 1–dimensional complex or quaternionic vector spaces W_p parameterized by the vertices $p \in M$. A holomorphic structure on a line bundle W over a discrete surface is an analogue of a linear, first–order elliptic differential operator acting on the space of sections

$$\Gamma(W) = \prod_{p \in M} W_p$$

of the bundle W. A first order linear operator acts on the "1–jet" of a section by which we mean the collection of its restrictions to the triangles. Ellipticity is encoded by taking only half of the 1–jet, namely the restrictions to black triangles. If we had taken all triangles, we would have defined a connection on the bundle W which encodes both a holomorphic and an anti–holomorphic structure. Thus, a holomorphic structure is given by assigning to every black triangle b a 2–dimensional linear subspace

$$U_b \subset \Gamma(W_{|b}) = \prod_{p \in b} W_p \tag{1.1}$$

of local holomorphic sections. A section $\psi \in \Gamma(W)$ is *holomorphic* if $\psi_{|b} \in U_b$ for each black triangle b of M. The space of holomorphic sections is denoted by $H^0(W)$.

An important example of a holomorphic line bundle over a discrete surface M is obtained by any map $f\colon M \to \mathbb{KP}^1$, where $\mathbb{KP}^1$ denotes the Riemann sphere $S^2 = \mathbb{CP}^1$ or the conformal 4–sphere $S^4 = \mathbb{HP}^1$. If $L \subset V$ is the pull–back by f of the tautological bundle over $\mathbb{KP}^1$, that is $L_p = f(p) \subset \mathbb{K}^2$, where V is the trivial $\mathbb{K}^2$–bundle over M, then the line bundle V/L has a unique holomorphic structure such that the constant sections of V project to holomorphic sections of V/L: the linear space $U_b \subset \Gamma(V/L_{|b})$ prescribed on a black triangle b is the image of the canonical projection

$$\pi\colon \mathbb{K}^2 \to \prod_{p \in b} \mathbb{K}^2/L_p = \Gamma(V/L_{|b}). \tag{1.2}$$

For U_b to be 2–dimensional we need that the restriction of f to each black triangle is non–constant.

The space of holomorphic sections of V/L contains the 2–dimensional linear system $H \subset H^0(V/L)$ obtained by projection of all constant sections of V. As in the smooth theory, this linear system H determines the original map f up to Möbius transformations as a ratio of independent holomorphic sections in H. This last is an instance of the Kodaira correspondence: given a holomorphic line bundle W over a discrete surface M and a 2–dimensional linear system $H \subset H^0(W)$ without base points, we obtain a map $f\colon M \to \mathbb{P}H$ whose value at a vertex $p \in M$ is the line $L_p \subset H$ of sections $\psi \in H$ vanishing at p. Thus, $W \cong H/L$ with H denoting the trivial H–bundle over M and where the holomorphic structure on H/L is the one induced by the map f.

The notion of discrete holomorphicity proposed by Dynnikov and Novikov [14, 15] for maps into $\mathbb{C}$ is equivalent to the complex case of our theory. Our point of view based on the Kodaira correspondence immediately reveals the Möbius invariance of discrete holomorphicity, a fact not readily visible in [14, 15].

Our exposition of discrete holomorphic geometry is very much in the spirit of the smooth quaternionic holomorphic geometry developed in [30, 11, 16, 7, 8]. Several of its key concepts and formulae carry over verbatim to the discrete setting. The approach based on holomorphic line bundles and Kodaira correspondence emphasizes the similarity between the complex and quaternionic case, that is between holomorphic maps into $S^2 = \mathbb{CP}^1$ and conformal surface theory in $S^4 = \mathbb{HP}^1$.

As the main application of discrete holomorphic geometry given in this paper we shed light on the relationship between discrete surface theory in S^4 and integrable systems which is well–established in the smooth situation [23, 35, 24, 10, 6]. Central to our discussion of the integrable system aspect of conformal surface theory is the notion of Darboux transformation for discrete surfaces in the conformal 4–sphere S^4. Iterating this Darboux transformation gives rise to a discrete flow and, if the underlying surface is a discrete torus, the space of all Darboux transforms is parameterized by a Riemann surface of finite genus, the spectral curve of the discrete torus in $\mathbb{KP}^1$.

Darboux transforms appear, just like in the smooth case, as prolongations of holomorphic sections: let $f\colon M \to \mathbb{KP}^1$ be a map whose restriction to each black triangle is non–constant. By (1.2) any holomorphic section $\psi \in H^0(V/L)$ of the induced holomorphic line bundle V/L over M has a unique "lift" $\hat{\psi}\colon M' \to \mathbb{K}^2$ defined on the set M' of black triangles such that $\pi\hat{\psi}_b = \psi_{|b}$. Of course, $\hat{\psi}$ is constant if $\psi \in H$ is contained in the linear system $H \subset H^0(V/L)$ induced by f. On the other hand, if ψ is a holomorphic section with monodromy $h \in \operatorname{Hom}(\Gamma, \mathbb{K}_*)$, that is $\gamma^*\psi = \psi h_\gamma$ for every deck transformation $\gamma \in \Gamma$, then $\hat{\psi}$ has the same monodromy h and thus, generally, is not constant. Provided that ψ is not identically zero on any of the black triangles, we obtain a well–defined map $f^\sharp\colon M' \to \mathbb{KP}^1$ given by $f^\sharp(b) = \hat{\psi}_b\mathbb{K}$ which we call a *Darboux transform* of f.

From this description we see that the Darboux transforms of a discrete surface arise from solutions to a linear equation. Alternatively, we can characterize the Darboux transformation by a non–linear, Möbius invariant zero curvature relation which can be expressed as a multi–ratio condition. This condition already appeared in a number of instances, including the characterization of integrable triangular lattices investigated by Bobenko, Hoffmann and Suris [1] and in the space and time discrete versions of the Kadomtsev–Petviashvili (KP) and Davey–Stewartson (DS) equation introduced by Konopelchenko and Schief [25, 26]. From our point of view the former correspond to 3–periodic sequences of Darboux transforms and the latter to a sequence of iterated Darboux transforms. This is consistent with the smooth theory of conformal surfaces in S^4, where the Darboux transformation can be seen as a time discretization of the Davey–Stewartson flow which approximates the smooth flow [6].

To obtain completely integrable systems requires, like in the smooth case, the discrete surface $M = T^2$ to be a 2–torus. We show that the space of all Darboux transforms $f^\sharp$ of a sufficiently generic discrete torus $f\colon M \to S^4$ with regular combinatorics can be given the structure of a compact Riemann surface, the *spectral curve* Σ of f. This curve is Möbius invariant and encodes the constants of motion of the above mentioned discrete evolution equations. In the smooth case the spectral curve plays an important role in the study of conformally immersed tori in 3– and 4–space [36, 18, 7, 8] and also in the context of the Willmore problem [33].

Since the spectral curve Σ parameterizes Darboux transforms of the discrete torus f there is map

$$F\colon M' \times \Sigma \to S^4$$

assigning to $\sigma \in \Sigma$ the Darboux transform $F(-,\sigma)\colon M' \to S^4$. This map has a unique lift $\hat{F}\colon M' \times \Sigma \to \mathbb{CP}^3$ that is holomorphic in Σ and satisfies $F = \pi \circ \hat{F}$ where $\pi\colon \mathbb{CP}^3 \to S^4$ denotes the twistor fibration. Thus, a discrete torus in S^4, which is just a finite set of points with regular combinatorics, gives rise to a family of algebraic curves parameterized over the black triangles $b \in M'$. Since holomorphic curves in $\mathbb{CP}^3$ project to super conformal Willmore surfaces in S^4, we also obtain a M'–family of super conformal Willmore surfaces $F(b,-)\colon \Sigma \to S^4$.

A fundamental property of the Darboux transformation which lies at the heart of integrability is Bianchi permutability: given two Darboux transforms $f^\sharp$ and $f^\flat$ of a discrete surface $f\colon M \to S^4$ with regular combinatorics, there is a common Darboux transform $\hat{f}$ of $f^\sharp$ and $f^\flat$. Bianchi permutability implies that the spectral curve Σ of f is preserved under the Darboux transformation. Moreover, it can be used to show that the family of algebraic curves $\hat{F}$ in $\mathbb{CP}^3$ "linearizes" in the Jacobian of Σ.

We conclude the paper by introducing a discrete flow on polygons in S^4 in terms of a constant cross–ratio condition on the generated quadrilaterals. For polygons in S^2, a reduction of the S^4 case, this cross–ratio evolution was developed in [3, 4, 19, 31]. Reducing this flow to polygons in $\mathbb{R}^3$ gives the doubly discrete smoke ring flow [20, 21, 32] (up to translation of the 3–plane). By thinking of a polygon as a discrete *thin cylinder* this flow in S^4 is given by iterated Darboux transforms. For closed polygons we thus have a spectral curve and the polygon flow linearizes on its Jacobian. The corresponding discrete evolution equations, $1+1$–reductions of the discrete Davey–Stewartson equation, are the discrete Korteweg–de Vries (KdV) equation for polygons in S^2 and the discrete non–linear Schrödinger (NLS) equation for $\mathbb{R}^3$.

2. Holomorphic line bundles over discrete surfaces

The approach to discrete conformality proposed in the present paper is based on the concept of holomorphic line bundles over discrete surfaces. The idea behind this approach is that both the intrinsic and extrinsic conformal and holomorphic geometry can be encoded in the language of holomorphic line bundles and linear systems of holomorphic sections. In the first part of the section we recall the basic notions of the smooth theory of holomorphic line bundles over Riemann surfaces, in the second part we develop the discrete counterparts. A crucial ingredient in the definition of holomorphic line bundles over discrete surfaces is the combinatorial structure of a triangulation with black and white colored faces. The use of such additional combinatorial data in the context of discrete holomorphicity appears previously in Dynnikov and Novikov [14, 15], Colin de Verdière [12] and also in unpublished notes by Thurston.

2.1. **Smooth theory (complex version).** A *holomorphic structure* on a complex line bundle W over a Riemann surface M is given by a so called $\bar\partial$–operator, a complex linear, first–order differential operator

$$\bar\partial\colon \Gamma(W) \to \Gamma(\bar K W)$$

satisfying the Leibniz rule

$$\bar\partial(\psi f) = \bar\partial(\psi) f + \psi \bar\partial(f) \qquad (2.1)$$

for all real and therefore, by complex linearity, for all complex functions f. Here, $\bar K W$ denotes the bundle of 1–forms with values in W that satisfy $*\omega = -i\omega$ with $*$ denoting the complex structure on T^*M. A section $\psi \in \Gamma(W)$ is called *holomorphic* if $\bar\partial\psi = 0$ and the space of holomorphic sections is denoted by $H^0(W)$. Holomorphic sections are thus defined as solutions to a linear, first–order elliptic partial differential equation. The equation $\bar\partial\psi = 0$ describing holomorphicity can be seen as half of an equation $\nabla\psi = 0$ describing parallel sections, because every holomorphic structure can be complemented to a connection $\nabla = \partial + \bar\partial$ satisfying $\bar\partial = \nabla'' := \frac{1}{2}(\nabla + i*\nabla)$ by choosing an anti–holomorphic structure ∂.

What we mean by the statement that holomorphic line bundles encode the intrinsic conformal geometry of Riemann surfaces is that the complex structure on the surface M itself may be recovered from the first–order elliptic operator $\bar\partial$:

Lemma 2.1. *Let $A\colon \Gamma(W) \to \Gamma(\tilde{W})$ be a complex linear, first–order elliptic differential operator between sections of complex line bundles W and $\tilde{W}$ over a surface M. Then there exists a unique complex structure on M such that $\tilde{W} \cong \bar{K}W$ and A becomes a $\bar{\partial}$–operator satisfying the Leibniz rule* (2.1)*.*

Proof. The complex structure on M is given by the unique complex structure $*$ on the bundle T^*M that makes the symbol $\sigma(A)\colon T^*M \to \mathrm{Hom}(W, \tilde{W})$ a complex anti–linear operator which then induces an isomorphism $\tilde{W} \cong \bar{K}W$. This definition of $*$ makes sense because, by ellipticity of the operator A, its symbol $\sigma(A)$ is an injective bundle morphism from the real rank 2 vector bundle T^*M to the complex line bundle $\mathrm{Hom}(W, \tilde{W})$.

The Leibniz rule obviously holds for constant functions f. By definition of the symbol and the isomorphism $\tilde{W} \cong \bar{K}W$, for every point $p \in M$ the Leibniz rule holds for real functions f vanishing at p. Therefore, it holds for all real functions and, by complex linearity, for all complex functions. □

In addition to the complex structure on M, a first–order elliptic operator between line bundles W and $\tilde{W} \cong \bar{K}W$ over M defines a complex holomorphic structure on W. This additional data is essential for encoding the extrinsic geometry of holomorphic maps from the Riemann surface M into complex projective space $\mathbb{CP}^n$: given a Riemann surface M, there is a 1–1–correspondence between

- projective equivalence classes of holomorphic curves $f\colon M \to \mathbb{CP}^n$ and
- isomorphy classes of holomorphic line bundles W on M with $n+1$–dimensional linear system $H \subset H^0(W)$ of holomorphic sections without basepoints.

This correspondence between holomorphic curves and linear systems $H \subset H^0(W)$ is called *Kodaira correspondence*. Because this version of Kodaira correspondence is of minor importance for the present paper, we skip further details and refer the Reader to the literature on complex algebraic geometry, e.g. [17]. Instead, in the following section we give a detailed treatment of Kodaira correspondence for conformal immersions into $S^4 = \mathbb{HP}^1$.

2.2. **Smooth theory (quaternionic version).** We briefly describe now the quaternionic versions of the concepts discussed in the preceding section. A detailed introduction to quaternionic holomorphic line bundles over Riemann surfaces can be found in [16].

Let W be a quaternionic line bundle with complex structure $J \in \Gamma(\mathrm{End}(W))$ over a Riemann surface M. A *holomorphic structure* on W is given by a quaternionic linear, first–order differential operator

$$D\colon \Gamma(W) \to \Gamma(\bar{K}W)$$

satisfying the Leibniz rule

$$D(\psi f) = D(\psi)f + (\psi df)'' \qquad \text{with} \qquad (\psi df)'' = \frac{1}{2}(\psi df + J\psi * df) \tag{2.2}$$

for all real and hence quaternionic functions f. A section $\psi \in \Gamma(W)$ is called *holomorphic* if $D\psi = 0$. Thus, as in the complex case, holomorphic sections are defined by a linear, first–order elliptic partial differential equation. Moreover, the holomorphicity equation $D\psi = 0$ can again be seen as one half of a parallelity equation $\nabla\psi = 0$ for a quaternionic connection ∇ satisfying $D = \nabla''$. The quaternionic analogue to Lemma 2.1 is:

Lemma 2.2. *Let $A\colon \Gamma(W) \to \Gamma(\tilde{W})$ be a quaternionic linear, first–order elliptic operator between quaternionic line bundles W and $\tilde{W}$ over an (oriented) surface M. Then there are unique complex structures on M, W and $\tilde{W}$ such that $\tilde{W} \cong \bar{K}W$ and A is a quaternionic holomorphic structure satisfying the Leibniz rule* (2.2).

Proof. The symbol $\sigma(A)\colon T^*M \to \mathrm{Hom}(W, \tilde{W})$ is a injective bundle morphisms from the real rank 2 vector bundle T^*M to the real rank 4 bundle $\mathrm{Hom}(W, \tilde{W})$. Up to sign, there are unique complex structures J and $\tilde{J}$ on W and $\tilde{W}$ such that the image of $\sigma(A)$ is the rank 2 bundle of elements $B \in \mathrm{Hom}(W, \tilde{W})$ satisfying $\tilde{J}B = BJ$. Moreover, there is a unique complex structure $*$ on T^*M compatible with the orientation and a unique choice of J and $\tilde{J}$ such that $\sigma(A)\colon T^*M \to \mathrm{Hom}(W, \tilde{W})$ satisfies $*\sigma(A) = \tilde{J}\sigma(A) = \sigma(A)J$.

As in the proof of Lemma 2.1 one can check that A as an operator from W to $\tilde{W} \cong \bar{K}W$ satisfies the Leibniz rule (2.2). □

An important application of quaternionic holomorphic line bundles over Riemann surfaces is the Kodaira correspondence for conformal immersions into the 4–sphere, a 1–1–correspondence between

- Möbius equivalence classes of immersions $f\colon M \to S^4 = \mathbb{HP}^1$ of a smooth surface M into the conformal 4–sphere and
- isomorphy classes of quaternionic holomorphic line bundles W over M with a 2–dimensional linear system $H \subset H^0(W)$ without Weierstrass points (see below).

In order to describe this 1–1–correspondence we identify maps $f\colon M \to S^4$ from M into the 4–sphere with line subbundles $L \subset V$ of the trivial quaternionic $\mathbb{H}^2$–bundle V over M. The quaternionic holomorphic line bundle corresponding to an immersion is then the quotient bundle $W = V/L$ equipped with the unique holomorphic structure such that all projections to V/L of constant sections of V are holomorphic. The 2–dimensional linear system $H \subset H^0(W)$ obtained by projecting all constant sections has no Weierstrass points in the following sense: for every $p \in M$, the space of sections $\psi \in H$ that vanish at p is 1–dimensional and the vanishing is of first order (the latter because f is immersed). Let

$$L = \begin{pmatrix} f \\ 1 \end{pmatrix} \mathbb{H},$$

where $f\colon M \to \mathbb{H}$ is the representation of the immersion in an affine chart. Then, the holomorphic sections ψ and φ obtained by projecting the first and second basis vector of $\mathbb{H}^2$ to the quotient line bundle $V/L \cong \mathbb{H}^2/L$ satisfy

$$\varphi = -\psi f.$$

The affine representation f of the immersion is thus the quotient of two holomorphic sections in the linear system H and changing the basis ψ, φ of H amounts to changing the affine representation of the immersion by a fractional linear transformations.

Conversely, given a holomorphic line bundle W over M together with a 2–dimensional linear system $H \subset H^0(W)$ that has no Weierstrass points, the line bundle L defined by $L_p = \ker(\mathrm{ev}_p)$ with $\mathrm{ev}_p\colon H \to W_p$ denoting the evaluation at p is a conformal immersion $f\colon M \to S^4 = \mathbb{HP}^1 \cong \mathbb{P}(H)$. Affine representations of this immersion f are obtained by taking quotients of holomorphic sections ψ, $\varphi \in H$.

2.3. **Complex and quaternionic holomorphic line bundles over discrete surfaces.** The basis of the discrete holomorphic geometry developed in this paper is the concept of holomorphic line bundles over discrete surfaces. Our definition of holomorphic line bundles assumes that the discrete surface is equipped with the additional combinatorial structure of a bi–colored triangulation. The idea to use such combinatorial data in the context of discrete holomorphicity appears previously in Dynnikov and Novikov [14, 15] and Colin de Verdière [12].

Definition 2.3. A *discrete surface* M is the vertex set $\mathcal{V}$ of a triangulation $(\mathcal{V}, \mathcal{E}, \mathcal{F})$ of an oriented smooth surface whose set of faces $\mathcal{F}$ is equipped with a bi–coloring, that is, the faces $\mathcal{F}$ of the triangulation are decomposed $\mathcal{F} = \mathcal{B} \dot{\cup} \mathcal{W}$ into "black" and "white" triangles such that two triangles of the same color never share an edge in $\mathcal{E}$.

The existence of such bi–coloring is equivalent to the property that every closed "thick path" of triangles in M (i.e., every closed path in the dual cellular decomposition M^*) has even length. By triangulation we mean a regular cellular decomposition all of whose faces are triangles, where regular means that the glueing map on the boundary of each 1– and 2–cell is injective. In other words, there are no identifications among the three vertices and edges of a triangle, an assumption that will be necessary in the definition of holomorphic line bundles below. Note that we do not assume triangulations to be simplicial ("strongly regular") cellular decompositions.

A discrete surface M is *compact* if its set of vertices if finite or, equivalently, if the underlying smooth surface is compact. We call a discrete surface *connected* or *simply connected* if the underlying smooth surface is connected or simply connected. Similarly, by *(universal) covering* of a discrete surface M we mean the vertex set of the triangulation induced on a covering (the universal covering) of the underlying smooth surface.

Before we define holomorphic structures on complex or quaternionic line bundles over discrete surfaces, recall that in the smooth case holomorphic structures are given by linear, first–order elliptic differential operators whose kernels describe the space of holomorphic sections. The discrete analogue to linear, first–order differential equations being difference equation defined on the faces of the triangulation, it is natural to define discrete holomorphic structures by imposing linear equations on the restrictions of sections to the faces. Reflecting the fact that the elliptic operators defining holomorphic structures in the smooth case are the "half" of a connection, i.e., can be written as $\bar{K}$–part ∇'' of a connection ∇, it is natural to define holomorphic structure on line bundles over discrete surfaces by imposing linear equations on half of the faces only.

Definition 2.4. Let W be a (complex or quaternionic) line bundle over a discrete surface M, that is, over the vertex set of a triangulation of an oriented surface with black and white colored faces. A *holomorphic structure* on W is given by assigning to each black triangle $b = \{u, v, w\} \in \mathcal{B}$ a 2-dimensional space of sections $U_b \subset \Gamma(W_{|b})$ with the property that a section $\psi \in U_b$ vanishing at two of the vertices of b has to vanish at all three of them. A section $\psi \in \Gamma(W)$ is called *holomorphic* if $\psi_{|b} \in U_b$ for every black triangle $b \in \mathcal{B}$. The space of holomorphic sections of W is denoted by $H^0(W)$.

Similar to the smooth case, there are essentially two different links between the geometry of immersions of discrete surfaces into 4–space and discrete quaternionic holomorphic geometry: the Möbius geometric one via Kodaira correspondence (introduced in the following section) and a Euclidean one based on the concept of Weierstrass representation (to be discussed in the forthcoming paper [9]).

2.4. **Kodaira correspondence for immersions of discrete surfaces into S^4.** As in the smooth case, there is a 1–1–correspondence between Möbius equivalence classes of immersions $f\colon M \to S^4 = \mathbb{HP}^1$ of a discrete surface M and certain 2–dimensional linear systems of sections of a quaternionic holomorphic line bundle over M. Replacing quaternions by complex numbers yields the analogous correspondence between maps into $\mathbb{CP}^1$ and linear systems of holomorphic sections of complex line bundles.

Definition 2.5. A map $f\colon M \to S^4 = \mathbb{HP}^1$ from a discrete surface M into the 4–sphere S^4 is called an *immersion* if and only if $f_p \neq f_q$ for adjacent points p, $q \in M$.

In the following, we identify maps from a discrete surface M into S^4 with the corresponding line subbundles $L \subset V$ of a trivial $\mathbb{H}^2$–bundle V over M. An immersions is thus a line subbundle $L \subset V$ with the property that $L_p \neq L_q$ for adjacent points p, $q \in M$.

The concept of holomorphic structures on line bundles over discrete surfaces allows to formulate the *Kodaira correspondence* relating the Möbius geometry of immersions to quaternionic holomorphic geometry in close analogy to the smooth case. Kodaira correspondence is a 1–1–correspondence between

- Möbius equivalence classes of immersions $f\colon M \to S^4 = \mathbb{HP}^1$ of a discrete surface M and
- isomorphy classes of quaternionic holomorphic line bundles W over M equipped with a 2–dimensional linear system $H \subset H^0(W)$ satisfying (2.3).

The holomorphic line bundle corresponding to an immersion $L \subset V$ of a discrete surface is obtained by exactly the same construction as in the smooth case: it is the bundle $W = V/L$ equipped with the unique holomorphic structure such that constant sections $\hat\psi \in \Gamma(V)$ project to holomorphic sections $\psi = \pi\hat\psi$ of $W = V/L$. This holomorphic structure is given by assigning to each black triangle $b = \{u, v, w\} \in \mathcal{B}$ the space of sections

$$U_b \subset \Gamma(V/L_{|b}) \cong V/L_u \oplus V/L_v \oplus V/L_w$$

obtained by projection of constant sections $\hat\psi \in \Gamma(V)$. The 2–dimensional linear system $H \subset H^0(W)$ corresponding to the immersion L is the space of sections of $W = V/L$ obtained by projection of constant sections of V. The property of $H \subset H^0(W)$ reflecting the fact that f is an immersion is

$$\text{if } \psi \in H \subset H^0(W) \text{ satisfies } \psi_p = \psi_q = 0 \text{ for adjacent } p, q \in M \text{ then } \psi = 0. \tag{2.3}$$

This condition is the discrete analogue to the property that the 2–dimensional linear system $H \subset H^0(W)$ has no Weierstrass points, i.e., that sections vanishing to second order at one point have to vanish identically.

Conversely, a 2–dimensional linear system $H \subset H^0(W)$ of a quaternionic holomorphic line bundle W over a discrete surface M that satisfies (2.3) gives rise to an immersion $L \subset V$ of M into $\mathbb{HP}^1 \cong \mathbb{P}(H)$, where V denotes the trivial H–bundle $M \times H$ over M and $L_p = \ker(\mathrm{ev}_p)$ with $\mathrm{ev}_p\colon H \to W_p$ the evaluation of a section at $p \in M$.

In coordinates Kodaira correspondence is written by the same explicit formulae as in the smooth case (cf. [16, 11]): every 2–dimensional linear system $H \subset H^0(W)$ satisfying (2.3) has a basis of sections ψ, $\varphi \in H$ with ψ nowhere vanishing such that there is a function $f\colon M \to \mathbb{H}$ satisfying $\varphi = -\psi f$. With respect to such a basis ψ, φ the immersion into

$\mathbb{HP}^1 \cong \mathbb{P}(H)$ obtained from the linear system H via Kodaira correspondence is

$$L = \begin{pmatrix} f \\ 1 \end{pmatrix} \mathbb{H}.$$

Changing the basis of H amounts to changing f by a fractional linear transform and therefore corresponds to a Möbius transform of S^4.

In Sections 3 and 4 we show how the holomorphic structure on V/L related to an immersion of a discrete surface via Kodaira correspondence naturally leads to the concept of Darboux transformation and spectral curve for immersions of discrete surfaces and tori.

2.5. **Three generations of cellular decompositions with bi–colored faces.** Despite the far reaching analogies between discrete and smooth holomorphic geometry, the discrete theory has an interesting additional aspect which is not visible in the smooth theory because it vanishes when passing to the continuum limit: given a discrete surface M or, more generally, the vertex set $\mathcal{V}$ of a regular cellular decomposition $(\mathcal{V}, \mathcal{E}, \mathcal{F})$ of a smooth surface equipped with a bi–coloring $\mathcal{F} = \mathcal{B} \dot{\cup} \mathcal{W}$ of its faces (with regular meaning that all glueing maps are injective), there are two other regular cellular decompositions M' and M'' of the same underlying smooth surface which are again equipped with bi–colorings of their faces.

The cellular decomposition M' is obtained by taking $\mathcal{B}$ as the set of vertices, $\mathcal{W}$ as the set of "black" faces, and $\mathcal{V}$ as the set of "white" faces with the following combinatorics: the vertices of a face $w \in \mathcal{W}$ of M' are the elements $b \in \mathcal{B}$ that touch w (wrt. M) and two such vertices b_1, b_2 a connected by an edge of the face w of M' if the share a vertex $v \in \mathcal{V}$ (wrt. M). Similarly, the vertices of a face $v \in \mathcal{V}$ of M' are the elements $b \in \mathcal{B}$ that contain v (wrt. M) and two such vertices b_1, b_2 are connected by an edge of M' if they touch a common white faces $w \in \mathcal{W}$ (wrt. M).

Applying the same construction to the decomposition M' leads to the third cellular decomposition M''. Thus, by cyclically permuting the roles of $\mathcal{V}$, $\mathcal{B}$ and $\mathcal{W}$ we obtain three generations M, M' and M'' of cellular decomposition of the same surface. The sequence of cellular decompositions obtained by successively applying the above construction is actually three–periodic, because by applying the construction to M'' one gets back to the initial cellular decomposition M: in order to see this, we introduce yet another cellular decomposition of the surface, the triangulation whose set of vertices consists of $\mathcal{V}$, $\mathcal{B}$ and $\mathcal{W}$ and whose edges correspond to all possible incidences of the cellular decomposition M, namely

- the vertex $v \in \mathcal{V}$ is contained in $b \in \mathcal{B}$ or $w \in \mathcal{W}$ (wrt. M) and
- the faces $b \in \mathcal{B}$ and $w \in \mathcal{W}$ touch along an edge (wrt. M).

This triangulation has a tri–coloring of its vertices and is the stellar subdivision of the cellular composition M^* dual to M: the dual cellular decomposition M^* inherits a bi–coloring of its vertices from the black and white coloring of M's faces; the third color corresponds to the additional vertices of the stellar subdivision.

Conversely, every triangulation of a surface with tri–colored set of vertices gives rise to a cellular decomposition with black and white coloring of its faces (see Figure 2): one color takes the role of the vertices, the other two become faces. The edges of the triangulation correspond to the different types of incidences, i.e., that of a vertex lying on a face and that of two faces intersecting along an edge.

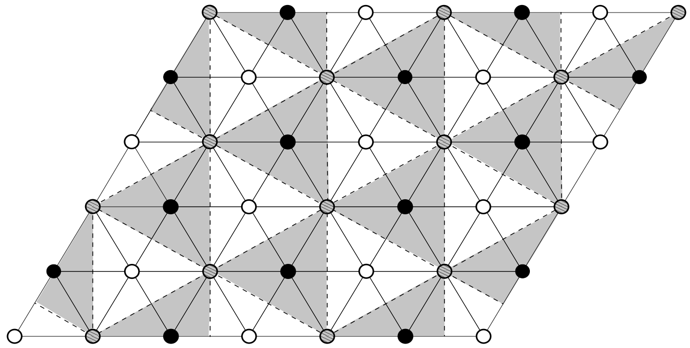

FIGURE 2. A triangulation with black and white colored vertices obtained from a regular tri–colored triangulation.

By cyclically permuting the role of the three colors we obtain the sequence M, M' and M'' of cellular decompositions with black and white colored faces which is therefore three–periodic under the above construction, that is $M = M'''$.

Although the present paper focuses on the case that M is a discrete surface as defined in Section 2.3, i.e., the vertex set of a triangulation with bi–colored set of faces, discrete holomorphicity can be defined in the context of more general bi–colored cellular decompositions: in [9] we introduce a Weierstrass representation for immersions $f\colon M \to \mathbb{R}^4 = \mathbb{H}$ defined on the vertex set $M = \mathcal{V}$ of an arbitrary cellular decomposition with bi–colored faces using "paired" holomorphic line bundles over the faces. This representation is the direct analogue to the Weierstrass representation [30] for smooth surfaces in $\mathbb{R}^4$.

The assumption that the black faces $\mathcal{B}$ are triangles is necessary for the Kodaira correspondence between immersions of discrete surfaces and linear systems of holomorphic line bundles, see Section 2.4. It is equivalent to the property that, in the above triangulation with tri–colored vertices, all black vertices have valence six.

That the white faces $\mathcal{W}$ are triangles is equivalent to six–valence of the white vertices in the tri–colored triangulation and will be necessary for the multi ratio characterization (3.3) of the Darboux transformation is Section 3.

When dealing with iterated Darboux transforms (as in Sections 3.3 and 4.4) we have to assume that, in addition to M, also the cellular decompositions M' and M'' have the property that all their "black" faces are triangles or, equivalently, that all vertices in the above tri–colored triangulation have valence six. Since M is a triangulation this is equivalent to the six–valence of all vertices of M. This motivates the following definition.

Definition 2.6. A discrete surface M is said to be of *regular combinatorics* if every vertex in the underlying triangulation has valence six.

If M is a discrete surface with regular combinatorics, the cellular decompositions M' and M'' are also bi–colored triangulations with regular combinatorics.

Every discrete surface with regular combinatorics is equivalent to the equilateral triangulation of the plane or a quotient thereof. In particular, corresponding to the three edges of triangles in the regular triangulation of the plane, a discrete surface with regular combinatorics has three distinguished *directions*. Moreover, for discrete surfaces with regular combinatorics it makes sense to define a *row* of vertices, black triangles or white triangles as the vertices that, with respect to the equilateral triangulation of the plane, are contained on a straight line, or as the triangles touching a straight line, respectively.

The Euler–formula shows that the only compact discrete surfaces with regular combinatorics are discrete tori obtained as quotient of the regular triangulation of the plane by some lattice $\Gamma \cong \mathbb{Z}^2$. The asymptotic analysis of spectra of quaternionic holomorphic line bundles over discrete tori in Section 4 will be considerably simplified by the assumption that the discrete torus has regular combinatorics.

3. The Darboux transformation

We introduce a Darboux transformation for immersions of discrete surfaces into S^4. Like the Darboux transformation for immersions of smooth surfaces is a time–discrete version of the Davey–Stewartson flow, see [6], one expects the Darboux transformation of discrete surfaces to be a space– and time–discrete version of the Davey–Stewartson flow. This is indeed the case: for immersed discrete surfaces with regular combinatorics, the Darboux transformation can be interpreted as the space– and time–discrete Davey–Stewartson flow introduced by Konopelchenko and Schief [26]. The case of immersions into $S^2 = \mathbb{CP}^1$ is a special reduction of the theory corresponding to the double discrete KP equation [25].

3.1. **Definition of the Darboux transformation.**

The following considerations show how discrete holomorphic geometry, in particular the Kodaira correspondence for immersions of discrete surfaces, naturally leads to the Darboux transformation for immersions of discrete surfaces.

Let $f\colon M \to S^4 = \mathbb{HP}^1$ be an immersion of a simply connected discrete surface M and denote by $L \subset V$ the corresponding line subbundle of the trivial $\mathbb{H}^2$–bundle V over M. By definition of the holomorphic structure on the quaternionic line bundle V/L related to the immersion via Kodaira correspondence (see Section 2.4), the restriction $\psi|_b$ of a holomorphic section $\psi \in H^0(V/L)$ to the vertices v_1, ..., v_3 of a black triangle $b \in \mathcal{B}$ is obtained by projecting a vector $\hat\psi_b \in \mathbb{H}^2$ to the fibers $(V/L)|_{v_j}$. Because $\hat\psi_b$ is uniquely determined by ψ, every holomorphic section $\psi \in H^0(V/L)$ gives rise to a section $\hat\psi$ of the trivial $\mathbb{H}^2$–bundle over the set $\mathcal{B}$ of black triangles.

Definition 3.1. Let $f\colon M \to S^4$ be an immersed discrete surface and V/L the corresponding quaternionic holomorphic line bundle. A section $\hat\psi$ of the trivial $\mathbb{H}^2$–bundle over $\mathcal{B}$ is the *prolongation* of a holomorphic section $\psi \in H^0(V/L)$ if and only if

$$\psi_v = \pi\hat\psi_b$$

for every $b \in \mathcal{B}$ and $v \in \mathcal{V}$ contained in b, where π denotes the canonical projection to V/L.

The prolongation $\hat\psi$ of a holomorphic section ψ is constant if ψ is contained in the 2–dimensional linear system $H \subset H^0(V/L)$ related to f by Kodaira correspondence. For a holomorphic section $\psi \in H^0(V/L)$ not contained in H, the prolongation $\hat\psi$ is non–constant and, provided ψ does not vanish identically on any of the black triangles, $L^\sharp = \hat\psi\mathbb{H}$ defines a map from the set $\mathcal{B}$ of black triangles into S^4. Exactly as in the smooth case [7], Darboux transforms of L will be the maps $L^\sharp$ locally given by $L^\sharp = \hat\psi\mathbb{H}$ for $\hat\psi$ the prolongation of a nowhere vanishing local holomorphic section ψ of V/L. The non–vanishing of ψ is reflected in the condition

$$L_b^\sharp \neq L_{v_i} \text{ for every black triangle } b = \{v_1, v_2, v_3\} \text{ and } i = 1, ..., 3. \tag{3.1}$$

We derive now a zero curvature condition characterizing the line subbundles $L^\sharp$ that locally can be obtained by prolongation from nowhere vanishing holomorphic sections among all line subbundles with (3.1) of the trivial $\mathbb{H}^2$–bundle over the black triangles $\mathcal{B}$. For this we will in the following consider $\mathcal{B}$ as the vertex set of the cellular decomposition M' of the underlying surface (see Section 2.5). Let $L^\sharp$ be a line subbundle of the trivial $\mathbb{H}^2$–bundle over the set of vertices $\mathcal{B}$ of M'. If the bundle $L^\sharp$ satisfies (3.1) it carries a connection defined by

$$P_{b_2 b_1} \colon L^\sharp_{b_1} \xrightarrow{\pi} (V/L)_v \xrightarrow{\pi^{-1}} L^\sharp_{b_2}, \tag{3.2}$$

where $b_1,\ b_2 \in \mathcal{B}$ are two black triangles connected by an edge of M' (i.e., touching a common white triangle), where $v \in \mathcal{V}$ is the vertex contained in b_1 and b_2, and where $\pi \colon L^\sharp_{b_i} \to (V/L)_v$ are the projections to the quotient bundle.

By definition, a (local) section $\varphi \in \Gamma(L^\sharp)$ is the prolongation of a holomorphic section of V/L if and only if $\varphi_{b_1} \equiv \varphi_{b_2} \bmod L_v$ for all black triangles $b_1,\ b_2 \in \mathcal{B}$ sharing a vertex $v \in \mathcal{V}$. This is equivalent to φ being parallel with respect to the connection (3.2) on the bundle $L^\sharp$. The local existence of a non–trivial parallel section $\varphi \in \Gamma(L^\sharp)$ is equivalent to flatness of the connection (3.2) on $L^\sharp$.

The curvature of the connection (3.2) on a bundle $L^\sharp$ over M' that satisfies (3.1) is the 2–form assigning to each face of the cellular decomposition M' the holonomy around that face. By definition of the connection (3.2) the holonomy is automatically trivial around the faces of M' that correspond to the vertices of the original triangulation M. Thus, the only condition for the connection to be flat is that its holonomy is trivial when going around the faces of M' that correspond to the white triangles of M. We show now how to express the triviality of these holonomies as a multi ratio condition. For this we use the standard embedding $\mathbb{H} \subset \mathbb{HP}^1$ and we write L and $L^\sharp$ as

$$L_v = \begin{pmatrix} x_v \\ 1 \end{pmatrix} \mathbb{H} \qquad \text{and} \qquad L_b^\sharp = \begin{pmatrix} x_b \\ 1 \end{pmatrix} \mathbb{H}$$

with $x_v,\ x_b \in \mathbb{H}$. The connection (3.2) then becomes

$$P_{b_j b_i} \begin{pmatrix} x_{b_i} \\ 1 \end{pmatrix} = \begin{pmatrix} x_{b_j} \\ 1 \end{pmatrix} (x_{b_j} - x_v)^{-1}(x_{b_i} - x_v)$$

and, denoting the vertices and black triangles as in Figure 3, the holonomy around a white triangle (with clockwise orientation) is

$$P_{13,12} P_{12,23} P_{23,13} \begin{pmatrix} x_{13} \\ 1 \end{pmatrix} = \begin{pmatrix} x_{13} \\ 1 \end{pmatrix} (x_{13} - x_1)^{-1}(x_{12} - x_1)(x_{12} - x_2)^{-1}$$
$$(x_{23} - x_2)(x_{23} - x_3)^{-1}(x_{13} - x_3).$$

Hence, the holonomy $P_{13,12} P_{12,23} P_{23,13}$ is trivial if and only if the multi ratio around the hexagon indicated by the arrows in Figure 3 is

$$M_6(x_1, x_{12}, x_2, x_{23}, x_3, x_{13}) = -1, \tag{3.3}$$

where the multi ratio of an ordered six–tuple of quaternions $x_1, ..., x_6 \in \mathbb{H}$ is defined by

$$M_6(x_1, x_2, x_3, x_4, x_5, x_6) = (x_1 - x_2)(x_2 - x_3)^{-1}(x_3 - x_4)(x_4 - x_5)^{-1}(x_5 - x_6)(x_6 - x_1)^{-1}.$$

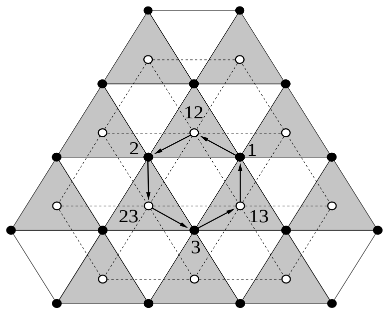

FIGURE 3. Combinatorics of the Darboux transformation.

Note that (unlike suggested by Figure 3) we do not assume here that the vertices of M have valence six: around the faces of M' that correspond to vertices of M (and which are not necessarily triangles) the holonomy of the connection (3.2) is always trivial (regardless of the valence of the vertex). The triviality of the holonomy around a face of M' that corresponds to a white triangle of M (and therefore is a triangle) is always equivalent to the multi ratio condition (3.3).

The following theorem summarizes the preceding discussion.

Theorem 3.2. *Let M be a discrete surface, i.e., the vertex set of a black and white triangulated smooth surface, and let M' be the cellular decomposition whose vertices are the black triangles (see Section 2.5). Let $f\colon M \to S^4$ be an immersion of a discrete surface M into $S^4 = \mathbb{HP}^1$ and let $f^\sharp\colon M' \to S^4$ be a map from M' to S^4 such that the corresponding line subbundles L and $L^\sharp$ of the trivial $\mathbb{H}^2$–bundles over M and M' respectively satisfy* (3.1)*. Then, the following conditions are equivalent:*

i) *around each white triangle of M, the multi ratio condition* (3.3) *is satisfied for the hexagon indicated in Figure 3,*
ii) *the connection* (3.2) *on the bundle $L^\sharp$ is flat, and*
iii) *locally the bundle $L^\sharp$ can be obtained by prolonging a holomorphic section of V/L (which needs to be nowhere vanishing since we assume* (3.1) *to be satisfied).*

Definition 3.3. A *Darboux transform* of an immersion $f\colon M \to S^4$ of a discrete surface M into $S^4 = \mathbb{HP}^1$ is a map $f^\sharp\colon M' \to S^4$ with the property that the corresponding line subbundles L and $L^\sharp$ of the trivial $\mathbb{H}^2$–bundles over M and M' satisfy (3.1) and the conditions of the preceding theorem.

The situation is completely analogous to the smooth case [7]: quaternionic holomorphic geometry provides a correspondence between solutions to a non–linear zero curvature equation of Möbius geometric origin on the one hand and solution to the linear equation describing quaternionic holomorphicity on the other hand. Locally, one direction of this correspondence is realized by projecting parallel sections of $L^\sharp$ to the quotient V/L; the other direction is realized by prolonging nowhere vanishing holomorphic sections of V/L.

It should be noted that the Darboux transformation for surfaces in $\mathbb{HP}^1$ is directed: as one can easily see from i) of Theorem 3.2, if $f^\sharp$ is an immersed Darboux transform of f, then $f^\perp$ is a Darboux transform of $(f^\sharp)^\perp$, where $\perp$ denotes the dual immersion into $(\mathbb{HP}^1)^*$. For any isomorphism $\mathbb{HP}^1 \cong (\mathbb{HP}^1)^*$, the transformation $\perp$ is an orientation reversing conformal diffeomorphism. In particular, for the right identification the transformation $\perp$ corresponds to quaternionic conjugation.

3.2. **Example.** As an example we briefly discuss a special case of our theory. Let W be the complex holomorphic line bundle over the vertices of the regular triangulation of the plane all of whose holomorphic sections are of the form $\psi = \varphi f$, where φ is a trivializing holomorphic section and f is a complex function that maps all black triangles to positive equilateral triangles in $\mathbb{C}$. Figures 4 shows two maps f_i, $i = 1, 2$ that are obtained as quotient of the form $f_i = \psi_i/\varphi$ from local holomorphic sections ψ_i, $i = 1, 2$ of W.

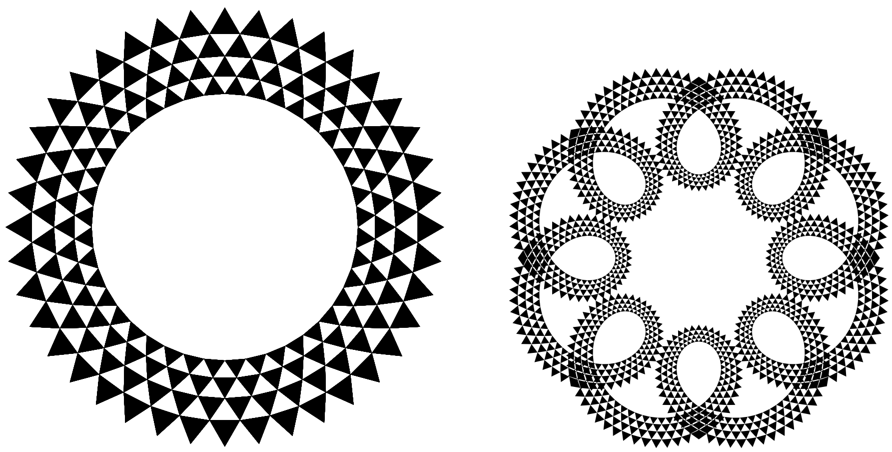

FIGURE 4. Two maps f_1 and f_2 into the plane with equilateral black triangles.

Figure 5 shows the Darboux transform of f_1 obtained by prolonging the holomorphic section $\psi_2 = \varphi f_2$, where f_1 here is the 5–fold covering of the "circle" and 4 is the maximal number of "loops" that can be "added".

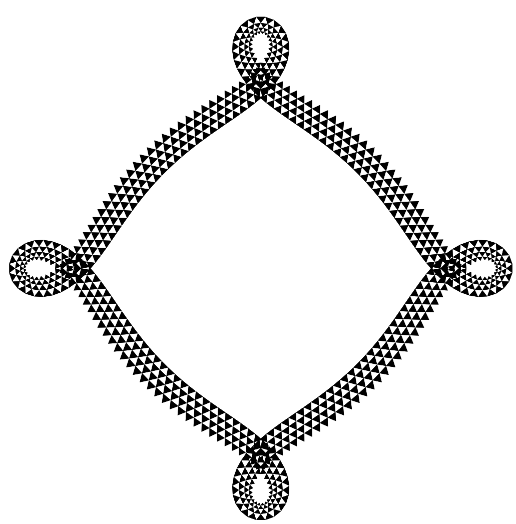

FIGURE 5. A Darboux transform of f_1.

3.3. **Iterated Darboux transforms and Davey–Stewartson flow.** We show now how a solution to the discrete Davey–Stewartson equation of Konopelchenko and Schief [26] can be interpreted as a sequence of iterated Darboux transforms.

For relating iterated Darboux transforms to the discrete Davey–Stewartson equation it is sufficient to reinterpret the underlying combinatorics, because Darboux transforms and the space– and time–discrete Davey–Stewartson flow are both defined by the same kind of multi ratio condition: a map $x\colon \mathbb{Z}^3 \to \mathbb{H}$ a solution to the discrete Davey–Stewartson equation of [26] if and only if it satisfies the multi ratio equation

$$M_6(x_1, x_{12}, x_2, x_{23}, x_3, x_{13}) = -1 \tag{3.4}$$

on each cube of the $\mathbb{Z}^3$–lattice with notation as indicated in Figure 6.

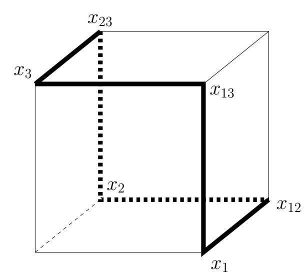

FIGURE 6. The space– and time–discrete Davey–Stewartson flow is defined by the multi ratio condition $M_6(x_1, x_{12}, x_2, x_{23}, x_3, x_{13}) = -1$ on the cubes of a $\mathbb{Z}^3$–lattice.

In order to interpret a solution to equation (3.4) on the $\mathbb{Z}^3$–lattice as a sequence of iterated Darboux transforms, we project the $\mathbb{Z}^3$–lattice to the plane perpendicular to the vector $(1, 1, 1)$. This yields a regular triangulation of the plane whose set of vertices has a natural tri–coloring: two vertices of the triangulation are given the same color if they have preimages with the same distance from the plane. This is illustrated in Figure 7 which shows the image of one cube of the $\mathbb{Z}^3$–lattice under the projection to the plane and the tri–coloring of its vertices.

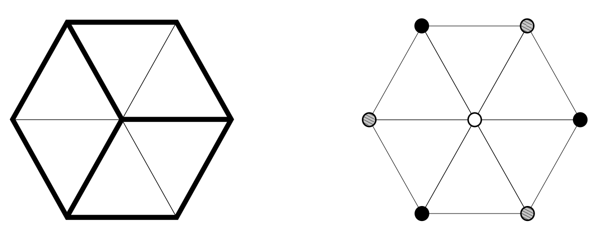

FIGURE 7. Planar projection of a cube and the tri–coloring of its vertices.

As explained in Section 2.5, such a triangulation with tri–colored set of vertices gives rise to a three periodic sequence M, M' and M'' of cellular decompositions with black and white colored faces. In our case M, M' and M'' are triangulations: each of the colors can be taken as the set of vertices of a triangulation whose black triangles correspond to the next height–level of $\mathbb{Z}^3$ over the plane and whose white triangles correspond to two height levels above. (As an example, see Figure 2 in Section 2.5 which shows the triangulation M whose set of vertices $\mathcal{V}$ are the grey vertices of the tri–colored triangulation, whose black faces $\mathcal{B}$ are the black vertices, and whose white faces $\mathcal{W}$ are the white vertices.)

By definition, a Darboux transform of an immersion $f\colon M \to S^4$ of a discrete surface M is defined on the "black triangulation" M'. Similarly, a Darboux transform of an immersion of M' is defined on the "white triangulation" M'' and Darboux transforming an immersion of M'' yields a map defined on the initial triangulation M.

It is immediately clear from the multi ratio equations (3.3) and (3.4), that—under the correspondence between the different height–levels of the $\mathbb{Z}^3$–lattice over the plane perpendicular to $(1,1,1)$ and the three periodic sequence M, M' and M'' of triangulations with back and white colored faces—a solution to the Davey–Stewartson equation corresponds to a sequence of iterated Darboux transforms: for every cube of the $\mathbb{Z}^3$–lattice, the multi ratio is taken along the fat line indicated in Figure 6 which projects to the hexagon bounding Figure 7 and, in the triangulations with bi–colored faces, corresponds to a hexagon as indicated in Figure 3.

4. The spectral curve

For immersions of compact surfaces, a natural question is whether the space of all Darboux transforms has any interesting the structure. In the following we show how, for generic immersions of a discrete torus with regular combinatorics, this space can be desingularized to a Riemann surface of finite genus, the spectral curve of the immersion. This spectral curve has similar properties as in the smooth case. In particular, it has a canonical geometric realization as a family of algebraic curves in $\mathbb{CP}^3$. By construction the spectral curve does not change under Möbius transformations and, as a consequence of Bianchi permutability, it is preserved under the evolution by Darboux transforms.

4.1. **Definition and properties of the spectral curve.** The idea to study the spectral curve as an invariant of immersed tori goes back to Taimanov [36] and Grinevich and Schmidt [18]. A Möbius invariant approach to the spectral curve of smooth tori in S^4 is developed in our papers [7, 8]. In the following we present the analogous theory for discrete tori.

As in the smooth case, Darboux transforms of an immersed discrete surface $f\colon M \to S^4$ correspond to nowhere vanishing holomorphic sections with monodromy of the quaternionic holomorphic line bundle V/L: by definition, a Darboux transform $f^\sharp\colon M' \to S^4$ of an immersion $f\colon M \to S^4$ is a line subbundle $L^\sharp$ of the trivial $\mathbb{H}^2$–bundle over M' that satisfies (3.1) and has the property that the induced connection on $L^\sharp$ is flat. In case that the underlying surface has non–trivial topology, parallel sections of this flat connection on $L^\sharp$ are in general not defined on the discrete surface M' itself, but are sections with holonomy on its universal covering. Projecting such a parallel section with holonomy of $L^\sharp$ to the quotient bundle V/L yields a nowhere vanishing holomorphic section with monodromy of V/L, that is, a holomorphic section of the pull back of V/L to the universal covering of M which is equivariant with respect to a representation $h \in \operatorname{Hom}(\Gamma, \mathbb{H}_*)$ of the group of deck transformations Γ in the sense that

$$\gamma^*\psi = \psi h_\gamma \tag{4.1}$$

for all deck transformations $\gamma \in \Gamma$. Taking conversely a line bundle $L^\sharp$ spanned by the prolongation of a nowhere vanishing holomorphic section with monodromy of V/L yields a global Darboux transform $f^\sharp$ of f. This proves the following proposition.

Proposition 4.1. *There is a bijective correspondence between global Darboux transforms of an immersed discrete surface $f\colon M \to S^4 = \mathbb{HP}^1$ and nowhere vanishing holomorphic sections with monodromy of the quaternionic holomorphic line bundle V/L up to scale.*

Scaling a holomorphic section ψ with monodromy by a quaternion $\lambda \in \mathbb{H}_*$ yields the section $\psi\lambda$ with conjugated monodromy $\lambda^{-1}h\lambda$, but does not change the corresponding Darboux transform. The space of Darboux transforms of an immersed discrete surface $f\colon M \to S^4$ is thus fibered over the conjugacy classes of representations $\mathrm{Hom}(\Gamma, \mathbb{H}_*)/\mathbb{H}_*$ that are possible for holomorphic sections with monodromy of V/L. This suggests, as a first step to understanding the structure of the space of Darboux transforms, to investigate the monodromies of holomorphic sections and motivates the following definition.

Definition 4.2. The *quaternionic spectrum* of a quaternionic holomorphic line bundle W over a discrete surface is the subset $\mathrm{Spec}_{\mathbb{H}}(W) \subset \mathrm{Hom}(\Gamma, \mathbb{H}_*)/\mathbb{H}_*$ of conjugacy classes of quaternionic representations of the group Γ of deck transformations that arise as multiplier of holomorphic sections with monodromy of W.

If the underlying surface M is a discrete torus, its group of deck transformations Γ is abelian such that every conjugacy class of representations $\mathrm{Hom}(\Gamma, \mathbb{H}_*)/\mathbb{H}_*$ contains a complex representations $h \in \mathrm{Hom}(\Gamma, \mathbb{C}_*)$ which is unique up to complex conjugation $h \mapsto \bar{h} = j^{-1}hj$. For the study of Darboux transforms of an immersed discrete torus it is therefore sufficient to consider complex representations $h \in \mathrm{Hom}(\Gamma, \mathbb{C}_*)$, because every Darboux transform can be obtained from a holomorphic section ψ with complex monodromy $h \in \mathrm{Hom}(\Gamma, \mathbb{C}_*)$ of V/L. This complex representation $h \in \mathrm{Hom}(\Gamma, \mathbb{C}_*)$ is unique up to complex conjugation: if ψ is a holomorphic section with monodromy h, the section ψj has monodromy $\bar{h}$.

Definition 4.3. The set of complex representations that are possible as multiplier of holomorphic sections with monodromy is called the *spectrum*

$$\mathrm{Spec}(W) = \{h \in \mathrm{Hom}(\Gamma, \mathbb{C}_*) \mid \exists \text{ holomorphic } \psi \neq 0 \text{ with } \gamma^*\psi = \psi h_\gamma \text{ for all } \gamma \in \Gamma\}$$

of a quaternionic holomorphic line bundle W over a discrete torus M.

The spectrum $\mathrm{Spec}(W)$ is invariant under the real involution $\rho(h) = \bar{h}$ and the quaternionic spectrum is the quotient $\mathrm{Spec}_{\mathbb{H}}(W) = \mathrm{Spec}(W)/\rho$.

The spectrum $\mathrm{Spec}(V/L)$ of the quaternionic holomorphic line bundle V/L corresponding to an immersed discrete torus $f\colon M \to \mathbb{HP}^1$ is never empty, because by Kodaira correspondence the space of sections with trivial monodromy is at least quaternionic 2–dimensional so that $h = 1 \in \mathrm{Spec}(V/L)$. For arbitrary quaternionic holomorphic line bundles W over discrete tori one can prove (similar to Part 1 in the proof of Theorem 4.5) that the spectrum $\mathrm{Spec}(W) \subset \mathrm{Hom}(\Gamma, \mathbb{C}_*) \cong \mathbb{C}_* \times \mathbb{C}_*$ extends to an algebraic subset of $\mathbb{C} \times \mathbb{C}$ described by the vanishing of one polynomial function, the determinant of a polynomial family of square matrices: in the system of complex linear equations characterizing holomorphic sections with a given monodromy, the number of variables $2|\mathcal{V}|$ equals the number of equations $2|\mathcal{B}|$, because, for black and white triangulated tori, Eulers formula $|\mathcal{V}| - |\mathcal{E}| + |\mathcal{B}| + |\mathcal{W}| = 0$ together with $|\mathcal{E}| = 3|\mathcal{B}| = 3|\mathcal{W}|$ implies

$$|\mathcal{V}| = |\mathcal{B}| = |\mathcal{W}| \tag{4.2}$$

with $|\mathcal{V}|$, $|\mathcal{E}|$, $|\mathcal{B}|$ and $|\mathcal{W}|$ denoting the number of vertices, edges, black and white triangles. Provided the determinant of the corresponding polynomial family of square matrices is not constant, the spectrum is thus a 1–dimensional algebraic set and can be normalized to a Riemann surface of finite genus.

Definition 4.4. Let W be a quaternionic holomorphic line bundle over a discrete torus M with 1–dimensional spectrum $\mathrm{Spec}(W)$. The *spectral curve* of W is the Riemann surface Σ normalizing $\mathrm{Spec}(W)$. Under the normalization map $h\colon \Sigma \to \mathrm{Spec}(W)$, the involution ρ of $\mathrm{Spec}(W)$ lifts to an anti–holomorphic involution $\rho\colon \Sigma \to \Sigma$.

The spectral curve of an immersed discrete torus $f\colon M \to S^4$ for which the holomorphic line bundle V/L has 1–dimensional spectrum $\mathrm{Spec}(V/L)$ is defined as the spectral curve of V/L.

The spectral curve as a geometric invariant of tori was first introduced by Taimanov, Grinevich and Schmidt [36, 18] for immersions of smooth tori into $\mathbb{R}^3$. It is defined using Floquet theory for periodic partial differential operators, see [27, 28]. The discrete analogue of Floquet theory used to define the spectrum and spectral curve of quaternionic holomorphic line bundles over discrete tori can immediately be generalized to an arbitrary linear ("difference") operator acting on the sections of a vector bundle over the vertex set of a cellular decomposition of the torus (with values in another bundle over the vertex set of another cellular decomposition). The spectrum of such operators is always an algebraic subset of $\mathrm{Hom}(\Gamma, \mathbb{C}_*)$. In case it is 1–dimensional one can therefore define a spectral curve of finite genus by normalizing the spectrum.

In the following we will not pursue this general discussion, but investigate the spectra $\mathrm{Spec}(W)$ of holomorphic line bundles W over discrete tori M with regular combinatorics.

Theorem 4.5. *For a discrete torus M with regular combinatorics, the spectrum $\mathrm{Spec}(W)$ of a quaternionic holomorphic line bundle W over M is always a 1–dimensional algebraic set such that the spectral curve Σ of W is defined. It is the Riemann surface of finite genus that normalizes $h\colon \Sigma \to \mathrm{Spec}(W)$ the spectrum and carries an anti–holomorphic involution $\rho\colon \Sigma \to \Sigma$ with $h \circ \rho = \bar{h}$ which extends to the compactification $\bar{\Sigma}$.*

If W satisfies the genericity assumption (4.5) below, there is a complex holomorphic line bundle $\mathcal{L} \to \Sigma$ whose fiber $\mathcal{L}_\sigma$ over $\sigma \in \Sigma$ is contained in the space of holomorphic sections with monodromy h^σ of W with equality away from a finite subset of Σ. The pullback of $\mathcal{L}$ under the anti–holomorphic involution ρ is $\rho^*\mathcal{L} = \mathcal{L}j$ with j denoting the multiplication of sections by the quaternion j. In particular, the involution ρ has no fixed points.

The theorem is a discrete version of Theorem 2.6 in [8]. Its proof is given in Section 4.2. As in [8], the main work in the proof is to investigate the asymptotics of $\mathrm{Spec}(W)$ and the asymptotics for large monodromies of its holomorphic sections with monodromy.

There are several essential differences to the smooth case [8] when the spectral curve has always one or two ends, at most two connected components, the minimal dimension of the spaces of holomorphic sections with monodromy $h \in \mathrm{Spec}(W)$ is one, and generically the spectral genus is infinite. For a quaternionic holomorphic line bundle W over a discrete torus M with regular combinatorics,

- the spectral curve always has finite genus,
- the spectral curve has many ends and may have many connected components (cf. Corollary 4.15 and Lemma 4.12) and
- it is possible that the minimal dimension of the spaces of holomorphic sections with monodromy $h \in \mathrm{Spec}(W)$ is greater than one (see Section 4.3).

For example, a "homogeneous" torus with regular combinatorics in S^4, the orbit of a finite group of Möbius transformations, has a connected spectral curve and three pairs of ends which correspond to the three directions of the triangulation. This shows that, in the space of immersed discrete tori in S^4 with regular combinatorics, there is a Zariski open subset of tori with irreducible spectral curve.

If $W = V/L$ is the quaternionic holomorphic line bundle induced by a sufficiently generic immersed torus $f\colon M \to S^4$, the line bundle $\mathcal{L} \to \Sigma$ of Theorem 4.5 allows to geometrically interpret the points of the spectral curve Σ as Darboux transforms of f. For all but finitely many points $h \in \mathrm{Spec}(V/L)$ in the spectrum there is then a unique Darboux transform corresponding to holomorphic sections with monodromy h. The quotient Σ/ρ of the spectral curve Σ by the involution ρ can thus be thought of as a parameter space for the Darboux transforms of f. More precisely:

Theorem 4.6. *Let $f\colon M \to S^4$ be an immersion of a discrete torus M with regular combinatorics for which the bundle V/L satisfies the genericity assumption* (4.5) *and has irreducible spectral curve. Taking prolongations of elements in the fibers of the bundle $\mathcal{L}$ (see Theorem 4.5) yields a map*

$$\hat{F}\colon M' \times \Sigma \to \mathbb{CP}^3$$

which is holomorphic in the second variable. The composition $F = \pi \circ \hat{F}$ with the twistor projection $\pi\colon \mathbb{CP}^3 \to \mathbb{HP}^1$ has the property that $f^\sharp = F(-,\sigma)\colon M' \to \mathbb{HP}^1$, for all but finitely many points $\sigma \in \Sigma$, is the unique Darboux transform of f that belongs to holomorphic section with monodromy h^σ or $\bar{h}^\sigma = h^{\rho\sigma}$ of V/L. In particular, $\hat{F}$ is compatible with the fixed point free anti–holomorphic involution ρ in the sense that

$$\hat{F}(b, \rho\sigma) = \hat{F}(b, \sigma)j$$

for all $b \in M' = \mathcal{B}$ and all $\sigma \in \Sigma$ with j denoting multiplication by the quaternion j seen as an anti–holomorphic involution of $\mathbb{CP}^3$.

The map $\hat{F}$ uniquely extends to $M' \times \bar{\Sigma}$ as a family of algebraic curves $\hat{F}\colon M' \times \bar{\Sigma} \to \mathbb{CP}^3$. For every row of black triangles of M there is a unique pair of points at infinity ∞, $\rho\infty \in \bar{\Sigma}\backslash\Sigma$ such that for each b in that row

$$f(v) = F(b, \infty) = F(b, \rho\infty), \tag{4.3}$$

where v denotes the upper vertex of b. Conversely, every point at infinity $\infty \in \bar{\Sigma}\backslash\Sigma$ belongs to one row of black triangles with the property that the value of f at the upper vertex v of each black triangle b in that row is given by (4.3)*.*

The theorem is a discrete version of Theorem 4.2 in [7] and will be proven in Section 4.2.

Remark 4.7. Let $f\colon M \to S^4$ be an immersion of a discrete torus M with regular combinatorics that satisfies (4.5) and has irreducible spectral curve. Then:

i) For fixed $\sigma \in \Sigma$, the map $f^\sharp\colon M' \to \mathbb{HP}^1$ defined by $f^\sharp = F(-,\sigma)$ is a Darboux transform of f, except σ is one of the finitely many $\sigma \in \Sigma$ for which the elements in $\mathcal{L}_\sigma$ are holomorphic sections with monodromy of V/L that have a zero. The maps $f^\sharp$ obtained for σ contained in this finite subset of Σ are called *singular Darboux transforms*. Note that, over black triangles $b \in M' = \mathcal{B}$ on which the elements of $\mathcal{L}_\sigma$ vanish identically, such singular Darboux transform $f^\sharp$ is not obtained as a prolongation of elements in $\mathcal{L}_\sigma$, but as the limit of prolongations of elements in $\mathcal{L}_{\sigma'}$ for $\sigma' \to \sigma$.

ii) By construction, a Darboux transform $f^\sharp$ of f can be obtained as $f^\sharp = F_\sigma$ for some $\sigma \in \Sigma$ unless the nowhere vanishing holomorphic sections with monodromy of V/L corresponding to $f^\sharp$ belong to one of the finitely many monodromies for which the space of holomorphic sections is higher dimensional.

iii) For fixed $b \in M' = \mathcal{B}$, the map $F_b = F(b, -)$ is a (possibly branched) super–conformal Willmore immersion

$$F_b \colon \bar{\Sigma} \to \mathbb{HP}^1.$$

For every vertex v of the triangle b, there is a pair of points at infinity ∞, $\rho\infty \in \bar{\Sigma}\backslash\Sigma$ such that

$$f(v) = F_b(\infty) = F_b(\rho\infty).$$

4.2. **Proof of Theorems 4.5 and 4.6.** The strategy for proving Theorems 4.5 and 4.6 is similar to the smooth case [8, 7]. As in the smooth case, the following proposition (cf. Proposition 3.1 of [8]) is essential for the proof:

Proposition 4.8. *Let D_x be a holomorphic family of Fredholm operators depending on a parameter $x \in X$ in a connected complex manifold X. Then, the minimal kernel dimension is generic and attained away from an analytic subset $Y \subset X$. If the manifold X is complex 1–dimensional, the vector bundle $\mathcal{K}_x = \ker(D_x)$ defined over $X\backslash Y$ holomorphically extends through the isolated set Y of points with higher dimensional kernel. If the index of the operators D_x is zero, the set of $x \in X$ for which D_x is invertible is locally given as the vanishing locus of one holomorphic function.*

The proof of Theorem 4.5 is divided in two parts: in Part 1 we define a family D_h of linear operators that depends holomorphically on $h \in \mathrm{Hom}(\Gamma, \mathbb{C}_*)$ and has the property that the kernel of D_h is isomorphic to the space of holomorphic sections with monodromy h of W. In Part 2 we investigate the asymptotics of $\mathrm{Spec}(W)$ under the assumption that the underlying discrete torus has regular combinatorics. The minimal kernel dimension of the family of operators D_h is then zero and the subset $Y = \mathrm{Spec}(W)$ of $X = \mathrm{Hom}(\Gamma, \mathbb{C}_*)$ is non–empty and hence 1–dimensional. It is an algebraic subset given by one polynomial equation, because D_h is a polynomial family of operators between finite dimensional spaces of the same dimension. The spectrum $\mathrm{Spec}(W)$ can thus be normalized to a spectral curve Σ of finite genus. Applying the proposition again to D_h seen as a family of operators parametrized over the 1–dimensional manifold Σ yields a holomorphic vector bundle $\mathcal{L} \to \Sigma$ (possibly with rank depending on the connected component) such that, for every $\sigma \in \Sigma$, the fiber $\mathcal{L}_\sigma$ is contained in the space of holomorphic sections with monodromy h^σ with equality away from finitely many points. Under the genericity assumption (4.5) we show, by investigating the asymptotics of holomorphic sections for large monodromies, that the minimal kernel dimension of D_h on each component of Σ is one. The bundle $\mathcal{L}$ is thus a holomorphic line bundle.

Although the family of operators D_h defined in Part 1 of the proof immediately generalizes to discrete tori with arbitrary combinatorics, the asymptotic analysis in Part 2 depends essentially on the assumption that the torus has regular combinatorics: an important ingredient in Part 2 is an adapted bases of the lattice and a compatible fundamental domain for discrete tori with regular combinatorics. This reduces the problem of finding holomorphic sections with monodromy of W to the study of eigenlines of a family of operators that is polynomial in one complex variable. The methods used in Part 2 of the proof of Theorem 4.5 are the main difference to the smooth case [8], where the 1–dimensionality of $\mathrm{Spec}(W)$ and $\mathcal{L}$ is proven by asymptotic comparison to the “vacuum” case of quaternionic holomorphic line bundles obtained by doubling complex holomorphic line bundles.

The proof of Theorem 4.6 is similar to the smooth case [7]: taking prolongations of the elements in the fibers of the bundle $\mathcal{L} \to \Sigma$ yields an extrinsic realization of Σ as a family $\hat{F}\colon M' \times \Sigma \to \mathbb{CP}^3$ of algebraic curves parametrized over M'. The asymptotics of sections of $\mathcal{L}$ for large monodromies shows that $\hat{F}$ extends to a family of algebraic curves.

Proof of Theorem 4.5 – Part 1: For quaternionic holomorphic line bundles W over discrete tori with regular combinatorics we define now a holomorphic family of operators D_h depending on $h \in \mathrm{Hom}(\Gamma, \mathbb{C}_*)$ with the property that the kernel of D_h is isomorphic to the space of holomorphic sections with monodromy h of W. Recall that a discrete torus M has regular combinatorics if all its vertices have valence six or, equivalently, if M is the quotient of the black and white colored, equilateral triangulation of the plane by some lattice $\Gamma \cong \mathbb{Z}^2$.

Lemma 4.9. *Let M be a discrete torus with regular combinatorics. For each direction in the regular triangulation of the plane there is a positive basis γ, η of the lattice Γ with the property that γ is parallel to the edges corresponding to the direction and that the black triangles touching γ lie on its positive side, see Figure 8.*

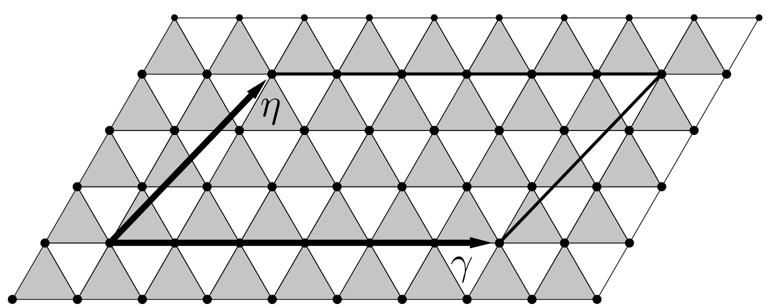

FIGURE 8. An adapted basis of the lattice.

Proof. Because M has only finitely many points, for each direction of the triangulation there is a unique smallest $\gamma \in \Gamma$ that is parallel to the corresponding edges of the triangles and has the property that black triangles touching γ lie on its positive side. With respect to an arbitrary basis $\tilde{\gamma}$, $\tilde{\eta}$ the vector γ takes the form $\gamma = \tilde{\gamma}a + \tilde{\eta}b$ with a, $b \in \mathbb{Z}$ relatively prime. Hence, there are c, $d \in \mathbb{Z}$ with $ad - bc = 1$ such that γ together with $\eta = \tilde{\gamma}c + \tilde{\eta}d$ form a positive basis of the lattice. This proves that each of the three possible γ can be complemented to a positive basis γ, η of the lattice with η unique up to adding multiples of γ. $\square$

Definition 4.10. A basis γ, η of the lattice with the properties of Lemma 4.9 is called *adapted*. An adapted basis is called *normalized* if the angle between γ and η is smaller or equal $\pi/3$ and the angle between γ and $\eta - \gamma$ is greater than $\pi/3$. We call the number of points of a torus in the direction of γ the *length* n of the torus with respect to the adapted basis γ, η and the number of rows above γ its *thickness* m.

The choice of an adapted basis γ, $\eta \in \Gamma$ fixes an isomorphism $h \mapsto (h(\gamma), h(\eta))$ from $\mathrm{Hom}(\Gamma, \mathbb{C}_*)$ to $\mathbb{C}_* \times \mathbb{C}_*$. This introduces coordinates $(\mu, \lambda) \in \mathbb{C}_* \times \mathbb{C}_*$ on $\mathrm{Hom}(\Gamma, \mathbb{C}_*)$, where μ and λ are the monodromies in the γ and η directions, that is $\mu = h(\gamma)$ and $\lambda = h(\eta)$.

In order to define the family of operators we fix now a normalized adapted basis γ, η of the lattice Γ. Moreover, by choosing a vertex v_0 of the universal covering of M, we fix a *compatible fundamental domain* of M consisting of all vertices of the regular triangulation that are of the form $v_0 + t_1\gamma + t_2\eta$ with $t_i \in [0, 1[$ as indicated by the fat black points in Figure 9.

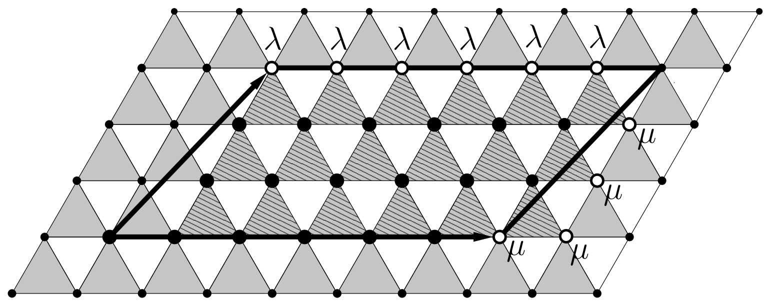

FIGURE 9. Fundamental domain of a discrete torus M with regular combinatorics.

To reflect the dependence on these choices, our family of operators is denoted by $D_{\mu,\lambda}$, where $(\mu, \lambda) \in \mathbb{C}_* \times \mathbb{C}_*$ are the coordinates on $\mathrm{Hom}(\Gamma, \mathbb{C}_*)$ introduced by γ, η. The operator $D_{\mu,\lambda}$ is defined on the direct sum of the lines W over the fat black dots in Figure 9 (i.e., over the vertices in the fundamental domain) and takes values in the trivial $\mathbb{H}$–bundle over the shaded black triangles. It is the composition of

- the complex (but not quaternionic) linear operator that maps a section of W defined over the fat black dots to its unique extension as a section with monodromy corresponding to $(\mu, \lambda) \in \mathbb{C}_* \times \mathbb{C}_* \cong \mathrm{Hom}(\Gamma, \mathbb{C}_*)$ and
- a non–trivial choice (fixed independent of $(\mu, \lambda) \in \mathbb{C}_* \times \mathbb{C}_*$) of quaternionic linear equations $W_p \oplus W_q \oplus W_r \to \mathbb{H}$ defining holomorphicity on the shaded black triangles $b = \{p, q, r\}$.

By definition of $D_{\mu,\lambda}$ we have:

Lemma 4.11. *The kernel of $D_{\mu,\lambda}$ is isomorphic to the space of holomorphic sections with monodromy h of W, where $h \in \mathrm{Hom}(\Gamma, \mathbb{C}_*)$ is the multiplier whose coordinates with respect to the chosen adapted basis γ, η are $(\mu, \lambda) \in \mathbb{C}_* \times \mathbb{C}_*$, i.e., $\mu = h(\gamma)$ and $\lambda = h(\eta)$.*

The family $D_{\mu,\lambda}$ of complex linear operators is polynomial (of order one) in μ and λ, because it only involves the extension of the section over the fat black points in Figure 9 to the fat white points (which is obtained by multiplication with μ or λ from the values of the section at Γ–related fat black points, as indicated by the μ's and λ's in Figure 9). The operators $D_{\mu,\lambda}$ have index $\mathrm{Index}(D_{\mu,\lambda}) = 0$, because they are operators between finite dimensional complex vector spaces of the same dimension $2|\mathcal{V}| = 2|\mathcal{B}|$, see (4.2). In the coordinates $(\mu, \lambda) \in \mathbb{C}_* \times \mathbb{C}_*$, the spectrum is thus an algebraic set $\mathrm{Spec}(W) \subset \mathrm{Hom}(\Gamma, \mathbb{C}_*)$ given as the vanishing locus of one polynomial function, the determinant of the family of operators $D_{\mu,\lambda}$.

The fact that the determinant of the family $D_{\mu,\lambda}$ is polynomial is an essential difference to the smooth theory where one has to deal with a holomorphic family of Fredholm operators whose determinant is transcendental. As a consequence, the spectral curve of immersed discrete tori is always a Riemann surface of finite genus while for immersions of smooth tori it can have infinite genus.

Proof of Theorem 4.5 – Part 2: We investigate now the asymptotics of the spectrum $\mathrm{Spec}(W)$ and of the spaces of holomorphic sections for large monodromies $h \in \mathrm{Spec}(W)$. Compared to the smooth case, the "asymptotic analysis" is simplified significantly by the fact that the polynomial family of operators $D_{\mu,\lambda}$ depending on $(\mu, \lambda) \in \mathbb{C}_* \times \mathbb{C}_*$ immediately extends to $\mathbb{C} \times \mathbb{C}$. For understanding the asymptotics it turns out to be sufficient to study this extension along the line $(\mu, \lambda) \in \mathbb{C} \times \{0\}$.

Because the restriction of a holomorphic section to a black triangle is uniquely determined by its value on two of the vertices, a section in the kernel of $D_{\mu,\lambda}$ is uniquely determined by its values on the lowest row of fat black points in Figure 9. It is helpful to see the black triangles touching γ as arrows pointing in the propagation direction for the "evolution" of "initial data" of holomorphic sections. In fact, the values of a holomorphic section in $\ker(D_{\mu,\lambda})$ over the fat black points in Figure 9 can be obtained recursively from the values on the lowest row of fat black points by successive extension to the row above: for fixed $\mu \in \mathbb{C}$, there is a complex linear operator $T_0(\mu)$ mapping the initial data given on the lowest row (together with its extension to the white points in the lowest row by multiplication with μ) to the direct sum of the fibers of W over the fat black points in the row above such that the resulting section on the vertices of the lowest row of shaded black triangles is holomorphic. There is an operator $T_1(\mu)$ mapping this data again to the row above etc. Finally, for m the thickness of the torus with respect to the adapted basis γ, η, there is an operator $T_{m-1}(\mu)$ mapping the upper row of fat black points to the row of white points and then, under the identification with respect to the lattice vector η, to the lowest row of fat black points. Hence, for every $\mu \in \mathbb{C}$ we obtain a complex linear endomorphism

$$H(\mu) = T_{m-1}(\mu) \cdot T_{m-2}(\mu) \cdots T_1(\mu) \cdot T_0(\mu)$$

of the direct sum of the fibers of W over the fat black points in the lowest row of Figure 9. By construction, the operator $D_{\mu,\lambda}$ has a non–trivial kernel if and only if λ is an eigenvalue of the endomorphism $H(\mu)$. The spectrum of W is thus given by the set of $(\lambda, \mu) \in \mathbb{C}_* \times \mathbb{C}_*$ for which $\det(\lambda - H(\mu)) = 0$. In particular, it is a branched covering of the μ–plane $\mathbb{C}_*$ with the points corresponding to eigenvalue $\lambda = 0$ removed. This proves the following lemma.

Lemma 4.12. *If W is quaternionic holomorphic line bundle over a discrete torus M with regular combinatorics, the algebraic subset $\mathrm{Spec}(W) \subset \mathrm{Hom}(\Gamma, \mathbb{C}_*) \cong \mathbb{C}_* \times \mathbb{C}_*$ is 1–dimensional such that it is possible to define the spectral curve Σ of W as the normalization of its spectrum $\mathrm{Spec}(W)$. The number of connected components of Σ is at most $2 \cdot n_{max}$, where n_{max} denotes the maximal length of M, cf. Definition 4.10.*

The following two lemmas are essential for understanding the asymptotic behavior of holomorphic sections near the ends $\bar{\Sigma} \backslash \Sigma$ of Σ.

Lemma 4.13. *Let Σ be the spectral curve of a quaternionic holomorphic line bundle over a discrete torus with regular combinatorics. For each end $\infty \in \bar{\Sigma} \backslash \Sigma$ of the spectral curve Σ, there is a unique normalized adapted basis γ, η of the lattice Γ for which the meromorphic functions $\mu = h(\gamma)$ and $\lambda = h(\eta)$ satisfy*

$$\mu(\infty) \in \mathbb{C}_* \qquad \textit{and} \qquad \lambda(\infty) = 0.$$

With respect to the other two normalized adapted bases, the corresponding meromorphic function satisfies $\mu(\infty) = \infty$ and $\mu(\infty) = 0$, respectively.

Proof. Denote by γ_i, η_i, $i = 1,...,3$ the three possible normalized adapted bases with numbering as in Figure 10.

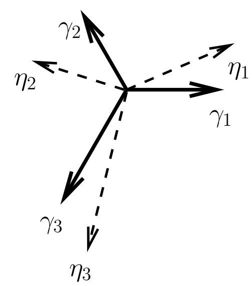

FIGURE 10. The three normalized adapted bases γ_i, η_i, $i = 1, ..., 3$ of the lattice.

There are l_1, ..., $l_3 \in \mathbb{N}$ such that $l_1\gamma_1 + l_2\gamma_2 + l_3\gamma_3 = 0$. Therefore, the meromorphic functions $\mu_i = h(\gamma_i)$ on Σ describing the monodromies in direction of γ_i satisfy

$$\mu_1^{l_1} \cdot \mu_2^{l_2} \cdot \mu_3^{l_3} = 1. \tag{4.4}$$

Because ∞ is a point at infinity, not all $\mu_i(\infty)$, $i = 1,...,3$ can be in $\mathbb{C}_*$ and, by (4.4), at least one of the $\mu_i(\infty)$, $i = 1,...,3$ has to be zero and one has to be infinity. Without loss of generality we can assume that $\mu_1(\infty) = \infty$ and

- $\mu_2(\infty) \in \mathbb{C}_*$ (Case 1) or
- $\mu_2(\infty) = 0$ (Case 2).

In Case 1, we obtain that $\lambda_2(\infty) = 0$ and $\mu_3(\infty) = 0$ while $\lambda_3(\infty) = 0$ or $\lambda_3(\infty) \in \mathbb{C}_*$ (depending on whether the angle between γ_3 and η_3 is smaller or equal $\pi/3$). This proves the statement for Case 1.

To complete the proof we show that Case 2 is impossible as it implies $\lambda_2(\infty) = 0$. This is obvious if the angle between γ_2 and η_2 is $\pi/3$, because then the operator $T_0(0)$ from Part 2 defined with respect to γ_2, η_2 has no kernel and thus $D_{0,0}$ has no kernel. If the angle between γ_2 and η_2 is smaller than $\pi/3$, we pass to the discrete torus $\tilde{M}$ obtained as quotient of the regular triangulation by $\tilde{\Gamma} = \mathrm{Span}_{\mathbb{Z}}\{\gamma_1, \gamma_2\}$ for which $\tilde{\gamma}_2 = \gamma_2$, $\tilde{\eta}_2 = -\gamma_1$ is a normalized adapted basis. Because holomorphic sections with monodromy of W give rise to holomorphic sections with monodromy of the pullback $\tilde{W}$ of W to the covering $\tilde{M}$ of M, the spectral curve Σ of W embeds into that of $\tilde{W}$. With respect to $\tilde{\gamma}_1$ and $\tilde{\eta}_2$, the image of the point ∞ under this embedding has coordinates $0 = \tilde{\mu} = \tilde{h}(\tilde{\gamma}_1)$ and $0 = \tilde{\lambda} = \tilde{h}(\tilde{\eta}_2)$. But this is impossible by the above argument, because the angle between $\tilde{\gamma}_1$ and $\tilde{\eta}_2$ is $\pi/3$. □

Lemma 4.14. *Let W be a quaternionic holomorphic line bundle over a discrete torus M with regular combinatorics. Counted with multiplicities there are $2m$ points $(\mu, 0) \in \mathbb{C}_* \times \mathbb{C}$ for which $D_{\mu,0}$ has a non–trivial kernel, where m is the thickness of the torus with respect to the adapted basis chosen to define $D_{\mu,\lambda}$.*

Proof. The operator $D_{\mu,0}$ has a non–trivial kernel if and only if one of the operators $T_i(\mu)$, $i = 0$, ..., $m - 1$ has a non–trivial kernel. For $\mu = 0$, only $T_0(\mu)$ can have a kernel: the sections with support in the fat black points of the lowest row that are not contained in one of the shaded black triangles. For $\mu \in \mathbb{C}_*$, the operator $T_i(\mu)$ has a non–trivial kernel if there is a holomorphic section defined on the vertices of the i^{th} row of shaded black triangles that vanishes on all upper vertices of the triangles and has horizontal monodromy μ. Because on every black triangle there is a quaternionic 1–dimensional space of holomorphic sections vanishing on the upper vertex, such holomorphic section is parallel with respect to a quaternionic connection on the restriction of W to the vertices in the i^{th} row and μ, $\bar{\mu}$ are the complex eigenvalues of the quaternionic holonomy of this connection, in the following called *horizontal holonomies*. In other words, the operator $T_i(\mu)$ with $\mu \in \mathbb{C}_*$ is invertible if and only if μ is not one of the horizontal holonomies of the quaternionic connection on the restriction of W to the i^{th} row of fat black points induced by the holomorphic structure. □

Lemma 4.13 shows that the ends $\bar{\Sigma}\backslash\Sigma$ of Σ are divided into three different types corresponding to the three different directions of the regular triangulation on the torus: for each of the three normalized adapted bases γ, η one type of ends $\infty \in \bar{\Sigma}\backslash\Sigma$ is characterized by the property that the meromorphic functions $\mu = h(\gamma)$ and $\lambda = h(\gamma)$ take values $\mu(\infty) \in \mathbb{C}_*$ and $\lambda(\infty) = 0$ while the other two types are characterized by $\mu(\infty) = 0$ and $\mu(\infty) = \infty$, respectively. The meromorphic function μ is a branched covering (see above) and hence non–constant on the components of $\bar{\Sigma}$ such that each component contains at least one end of every type.

The family $D_{\mu,\lambda}$, $(\mu, \lambda) \in \mathbb{C}_* \times \mathbb{C}_*$ of operators defined by the choice of an adapted basis and a compatible fundamental domain extends to $(\mu, \lambda) \in \mathbb{C} \times \mathbb{C}$ and the first type of ends is mapped to those $(\mu, 0) \in \mathbb{C}_* \times \{0\}$ for which $D_{\mu,0}$ has a non–trivial kernel. Under the **genericity assumption** that the horizontal holonomies (see the proof of Lemma 4.14)

$$\mu_0, \bar{\mu}_0, ..., \mu_{m-1}, \bar{\mu}_{m-1} \text{ are mutually distinct for every normalized adapted basis,} \tag{4.5}$$

the horizontal holonomies are in 1–1–correspondence to the ends of the respective type. This shows:

Corollary 4.15. *Let Σ be the spectral curve of a holomorphic line bundle W over a discrete torus with regular combinatorics for which* (4.5) *is satisfied. The number of ends $\bar{\Sigma}\backslash\Sigma$ of Σ is $2(m_1 + m_2 + m_3)$, where m_i denotes the thickness of the torus with respect to the normalized adapted basis γ_i, η_i. Every connected component of $\bar{\Sigma}$ contains at least three ends, one corresponding to every direction of the triangulation.*

The assumption (4.5) implies that the kernel of $D_{\mu,0}$ is 1–dimensional if μ is one of the horizontal holonomies. In particular, by Proposition 4.8, the minimal and therefore generic kernel dimension of $D_{\mu,\lambda}$ is one on every connected component when $D_{\mu,\lambda}$ is seen as a family of operators parametrized over $\Sigma' = \{\sigma \in \bar{\Sigma} \mid \mu(\sigma) \in \mathbb{C}, \lambda(\sigma) \in \mathbb{C}\}$. This implies that the kernels of $D_{\mu,\lambda}$ extend through the isolated points in Σ at which the kernel is higher dimensional to a holomorphic line bundle $\mathcal{L} \to \Sigma$. Although the family of operators $D_{\mu,\lambda}$ depends on the choice of adapted basis and fundamental domain, by Lemma 4.11 every fiber $\mathcal{L}_\sigma$ is isomorphic to a space of holomorphic sections with monodromy h^σ of W.

Lemma 4.16. *If a quaternionic holomorphic line bundle W over a discrete torus with regular combinatorics satisfies* (4.5)*, there is a unique complex holomorphic line bundle $\mathcal{L} \to \Sigma$ with the property that the fiber $\mathcal{L}_\sigma$ over $\sigma \in \Sigma$ is a subspace of the space of holomorphic sections with monodromy h^σ of W that, for all but finitely many points $\sigma \in \Sigma$, coincides with the space of holomorphic sections with the given monodromy. The pullback of $\mathcal{L}$ under the anti–holomorphic involution $\rho\colon \Sigma \to \Sigma$ is $\rho^*\mathcal{L} = \mathcal{L}j$. In particular, ρ has no fixed points.*

Proof. The existence and uniqueness of the line bundle $\mathcal{L}$ is explained in the preceding discussion. The number of points with higher dimensional kernel is finite because, under the normalization map $h\colon \Sigma \to \operatorname{Spec}(W)$, it is mapped to a subset of the finite set of singular points of the algebraic curve $\operatorname{Spec}(W)$.

For generic $\sigma \in \Sigma$, the spaces of holomorphic section with monodromy h^σ and $h^{\rho\sigma}$ of W are complex 1–dimensional. Multiplying a non–trivial element $\psi \in \mathcal{L}_\sigma \backslash \{0\}$ by the quaternionic j then yields a non–trivial holomorphic section ψj with monodromy $h^{\rho\sigma} = \bar{h}^\sigma$ which spans $\mathcal{L}_{\rho\sigma}$. This implies $\rho^*\mathcal{L} = \mathcal{L}j$. In particular, although complex conjugation on $\operatorname{Spec}(W)$ leaves all real representations fixed, its lift to the anti–holomorphic involution $\rho\colon \Sigma \to \Sigma$ has no fixed points. □

Proof of Theorem 4.6: The idea behind the definition of $F(-,\sigma)$ and $\hat{F}(-,\sigma)$ is essentially the same as in the smooth case [7]. For generic immersions $f\colon M \to S^4$ of discrete tori with regular combinatorics, we show that all but finitely many points $\sigma \in \Sigma$ of the spectral curve give rise to a 1–dimensional space of holomorphic sections with monodromy h^σ of V/L which are nowhere vanishing and hence define a unique Darboux transform of f. The maps $\hat{F}$ and F are then defined by the projective lines obtained from prolonged non–trivial elements in the fibers of $\mathcal{L}$, where $\mathcal{L} \to \Sigma$ is the holomorphic line bundle whose fiber $\mathcal{L}_\sigma$ over σ is a subspace of the holomorphic sections with monodromy h^σ of V/L.

In contrast to the smooth case, for immersions of discrete tori with regular combinatorics it is not always true that the generic dimension of the spaces of holomorphic sections with monodromy is one, see Section 4.3. In Theorem 4.6 we therefore restrict to immersions with the property that $W = V/L$ satisfies the genericity assumption (4.5). This assures the 1–dimensionality of $\mathcal{L} \to \Sigma$ and, in case the spectral curve is irreducible, implies that non–trivial elements in a generic fiber of $\mathcal{L}$ are nowhere vanishing holomorphic sections with monodromy of V/L (which is important for defining $\hat{F}$ and F). Moreover, the assumption (4.5) enables us to derive the asymptotics of F.

We first define $\hat{F}$ on $M' \times \Sigma$. By Lemma 4.16, away from a finite set $S_1 \subset \Sigma$ of points at which the space of holomorphic sections with the corresponding monodromy h^σ is higher dimensional, the fiber $\mathcal{L}_\sigma$ of the holomorphic line bundle $\mathcal{L} \to \Sigma$ coincides with the space of holomorphic section with monodromy h^σ.

Lemma 4.17. *Let Σ be the spectral curve of an immersed discrete torus with* (4.5)*. If Σ is irreducible, there is a discrete set $S_2 \subset \Sigma$ such that the non–trivial elements of $\mathcal{L}_\sigma$ for $\sigma \in \Sigma\backslash S_2$ are nowhere vanishing holomorphic sections with monodromy h^σ of V/L.*

Lemma 4.17 (and Theorem 4.6) holds more generally under the assumption that the quotient Σ/ρ of the spectral curve under the fixed point free anti–holomorphic involution ρ is connected, i.e., that Σ is either connected or the direct sum of two connected Riemann surfaces interchanged under ρ. The latter happens for generic immersed tori in $\mathbb{CP}^1$, cf. Lemma 4.20. It would be interesting to know whether the assumption that Σ/ρ is connected can be dropped from Lemma 4.17 and Theorem 4.6.

Proof. We choose a normalized adapted basis of the lattice Γ and a compatible fundamental domain. Such choice defines a family $D_{\mu,\lambda}$ of operators whose kernels describe the holomorphic sections with monodromy, see Lemma 4.11.

Assume S_2 does not exist. Then, because Σ is connected, there is a vertex v_0 in the fundamental domain (i.e., a fat black dot in Figure 9) that is a zero for all holomorphic sections in $\mathcal{L}_\sigma$, $\sigma \in \Sigma$. But this contradicts (4.5): assume the compatible fundamental domain of the torus is chosen such that the vertex v_0 is contained in the upper row of fat black dots in Figure 9. Denote $\infty \in \bar{\Sigma}\backslash\Sigma$ one of the two ends for which $\mu(\infty) \in \mathbb{C}_*$ is one of the two horizontal holonomies corresponding to the upper row of black triangles. Then $D_{\mu(\infty),0}$ has a 1–dimensional kernel whose elements are sections that vanish identically on the upper row of white dots in Figure 9 and therefore have no zeroes on the upper row of fat black dots. This contradicts the assumption that all elements in $\mathcal{L}_\sigma$, $\sigma \in \Sigma$ vanish at v_0. □

A local holomorphic section of $\mathcal{L}$ is a complex holomorphic family $\sigma \mapsto \psi^\sigma$ of quaternionic holomorphic sections with monodromy h^σ of V/L. With respect to a fundamental domain compatible with a normalized adapted basis, for every shaded black triangle b in Figure 9, the prolongation $\sigma \mapsto \hat{\psi}_b^\sigma$ is a holomorphic map to $\mathbb{C}^4 = (\mathbb{H}^2, \mathbf{i})$. Projectively, this yields a map $\hat{F}\colon M' \times \Sigma \to \mathbb{CP}^3$ which is holomorphic in the second variable: if $\sigma \mapsto \hat{\psi}_b^\sigma$ vanishes at some point $\tilde{\sigma} \in S_2$ (that is, if $\psi^{\tilde{\sigma}}$ vanishes simultaneously at all three vertices of b) we define $\hat{F}(b, \tilde{\sigma})$ as the line described by the first non–trivial element in the Taylor expansion of $\sigma \mapsto \hat{\psi}_b^\sigma$ at $\tilde{\sigma}$.

Denote $F = \pi \circ \hat{F}$ the composition of $\hat{F}$ with the twistor projection $\pi\colon \mathbb{CP}^3 \to \mathbb{HP}^1$. By construction, for all $\sigma \in \Sigma\backslash S_2$ the map $f^\sharp = F(-, \sigma)$ is a Darboux transform of f. For $\sigma \in S_2$ it is a singular Darboux transform as defined in i) of Remark 4.7. Similar to $\mathcal{L} \to \Sigma$, the map $\hat{F}$ clearly does not depend on the choice of adapted basis and fundamental domain. The quaternionic symmetry $\rho^*\mathcal{L} = \mathcal{L}j$ of the bundle $\mathcal{L}$, see Lemma 4.16, implies

$$\hat{F}(b, \rho\sigma) = \hat{F}(b, \sigma)j. \tag{4.6}$$

In order to extend $\hat{F}$ through an end $\infty \in \bar{\Sigma}\backslash\Sigma$ we chose the unique normalized adapted basis γ, η of the lattice with respect to which $\mu(\infty) \in \mathbb{C}_*$ and $\lambda(\infty) = 0$, where $\mu = h(\gamma)$ and $\lambda = h(\eta)$, see Lemma 4.13. For every compatible fundamental domain the family of operators $D_{\mu,\lambda}$ extends to $\mathbb{C} \times \mathbb{C}$ and, because ∞ is a point at infinity, the operator $D_{\mu(\infty),0}$ has a non–trivial kernel. Without loss of generality we can therefore assume that the compatible fundamental domain is chosen such that the operator $T_{m-1}(\mu(\infty))$ is not invertible. We extend the map $\hat{F}$ to ∞ by taking the projective lines described by the prolongation of a nowhere vanishing local holomorphic section of $\mathcal{L}'$ defined near ∞, where $\mathcal{L}' \to \Sigma'$ denotes the kernel bundle of $D_{\mu,\lambda}$ seen as a family of operators parametrized over $\Sigma' = \{\sigma \in \bar{\Sigma} \mid \mu(\sigma) \in \mathbb{C}, \lambda(\sigma) \in \mathbb{C}\}$. By our choice of fundamental domain the sections in the 1–dimensional kernel of $D_{\mu(\infty),0}$ do not vanish on the $(m-1)^{th}$ row of fat black dots in Figure 9, but its holomorphic extension to the (white) dots in the m^{th} row above vanishes identically. This implies

$$f(v) = F(b, \infty) = F(b, \rho\infty)$$

for b a shaded black triangles b in the upper row and v its upper vertex. The fact that for every choice of adapted basis the kernel bundle $\mathcal{L}$ extends to Σ' proves that the points corresponding to singular Darboux transforms cannot accumulate at infinity. This implies:

Corollary 4.18. *The subset $S_2 \subset \Sigma$ of points $\sigma \in \Sigma$ for which the holomorphic sections in $\mathcal{L}_\sigma$ have zeroes is finite.*

4.3. **Spectral curves of small tori.** We discuss the spectral curves of immersions $f\colon M \to S^4$ of discrete tori M with regular combinatorics and vertex set consisting of three or four points. In particular, we give an example of a discrete torus in S^4 with regular combinatorics for which every $\sigma \in \Sigma$ gives rise to a 2–dimensional space of holomorphic sections with monodromy h^σ of V/L.

By definition, the minimal number of vertices of a discrete surface is three, because the triangulation of the underlying smooth surface is assumed to be regular. Every discrete torus with three vertices has regular combinatorics: its number of black triangles is three by (4.2) and, because every triangle is assumed to have three distinct vertices, every vertex has valence six. Because three is a prime number, for every adapted basis the thickness of the torus (see Definition 4.10) has to be one. Therefore, every discrete torus with three vertices is a "thin torus" as shown in Figure 11.

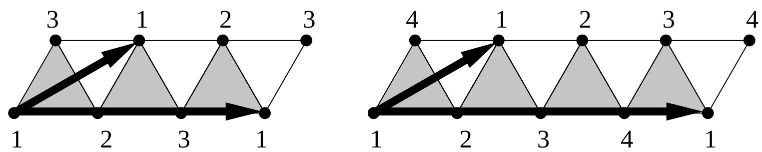

FIGURE 11. Thin tori with three and four vertices.

A discrete torus with four vertices not necessarily has regular combinatorics. Up to isomorphism, there are two discrete tori with non–regular combinatorics: the torus obtained by "adding" a vertex of valence two to the discrete torus with three vertices and the torus with two vertices of valence four and two vertices of valence eight (obtained by gluing opposite edges of a square that is triangulated by mutually joining the midpoints of all four edges by straight lines).

Remark 4.19. The discrete torus M with four vertices one of which has valence two is "isospectral" to the torus $\hat{M}$ with three vertices: let more generally M be a discrete torus obtained by adding a vertex of valence two to another discrete torus $\hat{M}$. For every quaternionic holomorphic line bundle W over M and every monodromy h, the space of holomorphic section with monodromy h of W is then canonically isomorphic to the space of holomorphic sections with the same monodromy of the restriction of W to $\hat{M}$.

A discrete torus with four points and regular combinatorics can either be a thin torus as in Figure 11, that is, it admits an adapted basis for which its thickness is one, or the thickness for every adapted basis is two (the only prime factor of four) and the torus is of the form shown in Figure 12.

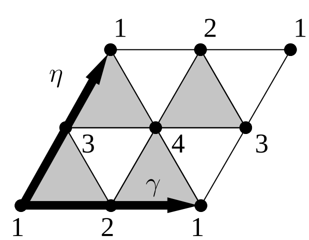

FIGURE 12. "Thicker" torus with four vertices.

For immersions of discrete tori with three or four vertexes we can always assume, after applying a Möbius transformation, that the immersion takes values in $\mathbb{CP}^1$. By the following lemma, an immersion of a discrete torus with values in $\mathbb{CP}^1$ has a decomposable spectral curves.

Lemma 4.20. *Let $f\colon M \to \mathbb{CP}^1 \subset \mathbb{HP}^1$ be an immersion of a discrete torus M with regular combinatorics that takes values in the 2–sphere $S^2 = \mathbb{CP}^1$. Then, the spectral curve Σ of f can be decomposed $\Sigma = \Sigma_1 \dot\cup \Sigma_2$ into two Riemann surfaces which are interchanged under the anti–holomorphic involution ρ. Moreover, all Darboux transforms $f^\sharp = F(-,\sigma)$ corresponding to points $\sigma \in \Sigma$ take values in $\mathbb{CP}^1$.*

Proof. As in Sections 4.1 and 4.2 (with obvious adaptations like replacing quaternions by complex number etc.) one can define a spectral curve Σ_1 for an immersion f with values in $\mathbb{CP}^1$ as the normalization of the possible monodromies of holomorphic sections of the complex holomorphic line bundle $\hat V/\hat L$ over M obtained from f by complex Kodaira correspondence, where $\hat V$ is trivial $\mathbb{C}^2$–bundle over M and $\hat L$ the line subbundle corresponding to f.

The quaternionic holomorphic line bundle V/L corresponding to f via quaternionic Kodaira correspondence is then the quaternionification of $\hat V/\hat L$. All vector spaces and operators in Section 4.2 have thus a natural direct sum decomposition with respect to this quaternionification. In particular, the $\mathbb{CP}^1$–spectral curve Σ_1 of $\hat V/\hat L$ is embedded into the $\mathbb{HP}^1$–spectral curve Σ of V/L and $\Sigma = \Sigma_1 \dot\cup \Sigma_2$, where $\Sigma_2 = \rho\Sigma_1 \cong \bar\Sigma_1$ is the image of Σ_1 under the involution ρ. □

As explained in Section 5, immersions of thin tori as in Figure 11 are polygons in $\mathbb{CP}^1$. By Remark 5.3 and Theorem 5.4, the $\mathbb{HP}^1$–spectral curve Σ of such a thin torus in $\mathbb{CP}^1 \subset \mathbb{HP}^1$ is the double of the $\mathbb{CP}^1$–spectral curve of the corresponding polygon in $\mathbb{CP}^1$. Moreover, by Remark 5.3, the dimension of the space of holomorphic sections with monodromy h^σ of V/L for generic $\sigma \in \Sigma$ is one.

In the remainder of this section we investigate the spectral curve of an arbitrary immersion $f\colon M \to \mathbb{CP}^1 \subset \mathbb{HP}^1$ of the torus M in Figure 12. By Lemma 4.20, its $\mathbb{HP}^1$–spectral curve is the double of its $\mathbb{CP}^1$–spectral curve. It is therefore sufficient to determine this $\mathbb{CP}^1$–spectral curve.

In the affine coordinate $\mathbb{C} \to \mathbb{CP}^1$ $x \mapsto [x, 1]$ the immersion is given by four mutually disjoint points x_1, ..., $x_4 \in \mathbb{C}$. We trivialize the bundle $\hat V/\hat L$ corresponding to the immersion by the holomorphic section $\varphi = \pi e_1$ with π denoting the canonical projection to the quotient $\hat V/\hat L$ and e_1 denoting the first basis vector of $\mathbb{C}^2$ seen as a section of the trivial bundle $\hat V$. Over a black triangle b with vertices p, q and r, the holomorphic structure on $\hat V/\hat L$ is given by the linear form

$$\alpha = \left(\frac{x_r - x_q}{x_p - x_q}\,,\, \frac{x_r - x_p}{x_q - x_p}\,,\, -1 \right) \tag{4.7}$$

acting on $\Gamma(\hat V/\hat L)_{|b} = (\hat V/\hat L)_{|p} \oplus (\hat V/\hat L)_{|q} \oplus (\hat V/\hat L)_{|r} \cong \mathbb{C}\oplus\mathbb{C}\oplus\mathbb{C}$ with isomorphism induced by φ.

Using (4.7) we compute now the operator $H(\mu)$ defined in Part 2 of the proof of Theorem 4.5 (see Section 4.2): the operator $T_0(\mu)$ maps a section $(y_1, y_2) \in \mathbb{C}^2$ over the first and second vertex in the lowest row of Figure 12 to the section

$$\left(\frac{x_3 - x_2}{x_1 - x_2} y_1 + \frac{x_3 - x_1}{x_2 - x_1} y_2\,,\, \frac{x_4 - x_1}{x_2 - x_1} y_2 + \frac{x_4 - x_2}{x_1 - x_2} y_1 \mu \right)$$

over the first and second vertex in the middle row. Similarly, the operator $T_1(\mu)$ maps a section $(y_3, y_4) \in \mathbb{C}^2$ over the first and second vertex in the middle row of Figure 12 to the section

$$\left(\frac{x_1 - x_4}{x_3 - x_4} y_3 + \frac{x_1 - x_3}{x_4 - x_3} y_4\,,\ \frac{x_2 - x_3}{x_4 - x_3} y_4 + \frac{x_2 - x_4}{x_3 - x_4} y_3 \mu\right)$$

over the first and second vertex in the lower row. The composition $H(\mu) = T_1(\mu) \circ T_0(\mu)$ is

$$(y_1, y_2) \mapsto \frac{(x_1 - x_3)(x_2 - x_4)\mu - (x_2 - x_3)(x_1 - x_4)}{(x_1 - x_2)(x_3 - x_4)} (y_1, y_2).$$

Hence, for every $\mu \in \mathbb{C}_*$ there is a unique eigenvalue λ of $H(\mu)$ and there is a unique value $\mu \in \mathbb{C}_*$ for which this eigenvalue λ is zero. The spectral curve Σ of $f\colon M \to S^4 = \mathbb{HP}^1$ is therefore the double of the three–punctured projective plane $\mathbb{CP}^1 \backslash \{\infty_1, \infty_2, \infty_3\}$. Because λ is always a double eigenvalue of $H(\mu)$, the bundle $\mathcal{L} \to \Sigma$ describing the holomorphic sections with given monodromy of V/L is a rank two bundle.

4.4. **Bianchi permutability.** A Bianchi type permutability theorem usually states something like: given two transformations $f^\sharp$ and $f^\flat$ of f, there is a common transformation $\tilde f$ of both $f^\sharp$ and $f^\flat$ that can be computed algebraically. This may be visualized by

$$\begin{tikzcd}
 & f^\flat \arrow[dr, rightsquigarrow] & \\
f \arrow[ur, rightsquigarrow] \arrow[dr, rightsquigarrow] & & \tilde f. \\
 & f^\sharp \arrow[ur, rightsquigarrow] &
\end{tikzcd}$$

Recall that iterated Darboux transforms of an immersion $f\colon M \to S^4$ of a discrete surface M are only defined if M has regular combinatorics, because all three cellular decompositions M, M' and M'' (see Section 2.5) need to be triangulations: while the original immersions f is defined on M, the Darboux transforms $f^\sharp$ and $f^\flat$ are defined on M' and the iterated Darboux transform $\tilde f$ is defined on M''.

Theorem 4.21. *Let $f\colon M \to S^4$ be an immersion of a discrete surface M with regular combinatorics and let $f^\sharp$, $f^\flat\colon M' \to S^4$ be two immersed Darboux transforms of f with $f^\sharp(b) \neq f^\flat(b)$ for all $b \in M'$. Then, there is a map $\tilde f\colon M'' \to S^4$ that simultaneously is a Darboux transform of $f^\sharp$ and $f^\flat$.*

Proof. By Proposition 4.1, corresponding to the Darboux transforms $f^\sharp$ and $f^\flat$ there are holomorphic sections $\psi^\sharp$ and $\psi^\flat$ with monodromy of V/L whose prolongations $\hat\psi^\sharp$ and $\hat\psi^\flat$ span the line subbundles $L^\sharp$ and $L^\flat$ of the trivial $\mathbb{H}^2$–bundle given by $f^\sharp$ and $f^\flat$. The idea of the proof is to show that—with the right interpretation—the same formula

$$\hat\varphi = \hat\psi^\flat + \hat\psi^\sharp \chi \tag{4.8}$$

as in the smooth case defines the prolongation $\hat\varphi$ of a holomorphic section φ of $V/L^\sharp$. Recall that the prolongations $\hat\psi^\flat$ and $\hat\psi^\sharp$ are defined over the vertices of M', i.e., over the black triangles of M. For (4.8) to make sense we need $\hat\varphi$ to be a well defined section with monodromy of the trivial $\mathbb{H}^2$–bundle over the black triangles of M' alias white triangles of M.

The proof is straightforward once we have explained how to make sense of (4.8). We do this by proving that there is a unique quaternionic function χ defined on the white triangles of the universal covering of $\tilde{M}$ with the property that

$$\hat{\psi}_1^\flat + \hat{\psi}_1^\sharp \chi_w = \hat{\psi}_2^\flat + \hat{\psi}_2^\sharp \chi_w = \hat{\psi}_3^\flat + \hat{\psi}_3^\sharp \chi_w$$

for every white triangle $w \in \tilde{\mathcal{W}}$, where the black triangles adjacent to w are denoted by 1,...,3, see Figure 13.

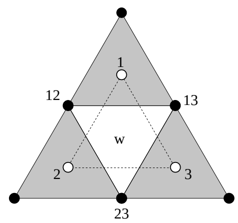

FIGURE 13. A white triangle with adjacent vertices and black triangles.

Existence and uniqueness of χ_w follows from the fact that for $i,\, j \in \{1,2,3\}$, $i \neq j$

$$(*) \qquad \hat{\psi}_i^\flat + \hat{\psi}_i^\sharp \chi_{ij} = \hat{\psi}_j^\flat + \hat{\psi}_j^\sharp \chi_{ij}$$

is equivalent to

$$(**) \qquad \hat{\psi}_i^\flat - \hat{\psi}_j^\flat = (\hat{\psi}_j^\sharp - \hat{\psi}_i^\sharp)\chi_{ij}.$$

Because $L^\sharp$ is immersed and both sides of $(**)$ are contained in the line $L_{v_{ij}} \subset \mathbb{H}^2$, where v_{ij} denotes the vertex of M between the black triangles i and j (see Figure 13), there is a unique $\chi_{ij} \in \mathbb{H}$ such that $(*)$ holds. To see that χ_{ij} coincides for all $i,\, j \in \{1,2,3\}$, $i \neq j$, note that, since $\hat{\psi}^\flat$ and $\hat{\psi}^\sharp$ are prolongations of holomorphic sections with monodromy of V/L,

$$\hat{\psi}_i^\flat + \hat{\psi}_i^\sharp \chi_{12} \equiv \hat{\psi}_3^\flat + \hat{\psi}_3^\sharp \chi_{12} \quad \mod \quad L_{i3}$$

for $i = 1$ and $i = 2$. Because $L_{13} \oplus L_{23} = \mathbb{H}^2$ and $\hat{\psi}_1^\flat + \hat{\psi}_1^\sharp \chi_{12} = \hat{\psi}_2^\flat + \hat{\psi}_2^\sharp \chi_{12}$ we indeed have $\chi_{i3} = \chi_{12}$ for $i = 1,\, 2$ such that χ_{ij} depends only on the white triangle w. It therefore makes sense to defines $\chi_w := \chi_{ij}$ such that (4.8) becomes

$$\hat{\psi}_w = \hat{\psi}_1^\flat + \hat{\psi}_1^\sharp \chi_w = \hat{\psi}_2^\flat + \hat{\psi}_2^\sharp \chi_w = \hat{\psi}_3^\flat + \hat{\psi}_3^\sharp \chi_w$$

and in particular yields a well defined section $\hat{\varphi}$ of the trivial $\mathbb{H}^2$–bundle over M'' which, by $(**)$, has the same monodromy as $\hat{\psi}^\flat$. By construction, $\hat{\varphi}$ is the prolongation of a holomorphic section φ of $V/L^\sharp$ which is nowhere vanishing, because $L^\sharp \oplus L^\flat$. In particular, $\hat{\varphi}$ is nowhere vanishing and $\tilde{L} = \hat{\varphi}\mathbb{H}$ defines a Darboux transform of $f^\sharp$.

Since $f^\flat$ is immersed, equation $(**)$ implies that the function χ is nowhere vanishing and therefore $\varphi\chi^{-1} = \hat{\psi}^\flat \chi^{-1} + \hat{\psi}^\sharp$ is also a well defined section of the trivial $\mathbb{H}^2$–bundle over M''. It has the same monodromy as $\hat{\psi}^\sharp$ and is the prolongation of a nowhere vanishing holomorphic section of $V/L^\flat$. This shows that $\tilde{L}$ is also a Darboux transform of $f^\flat$. $\square$

Formula (4.8) in the proof of the preceding theorem does also prove the following lemma, because for defining χ via $(**)$ it is sufficient that $\hat{\psi}^\sharp$ corresponds to an immersed Darboux transform and $\hat{\psi}^\flat$ is the prolongation of a holomorphic section with monodromy.

Lemma 4.22. *Let $f\colon M \to S^4$ be an immersion of a discrete surface M with regular combinatorics and $f^\sharp\colon M' \to S^4$ an immersed Darboux transform of f. For every holomorphic sections $\psi^\flat$ with monodromy $h^\flat$ of V/L, there is a holomorphic section φ with the same monodromy $h^\flat$ of $V/L^\sharp$.*

Together with Corollary 4.15 this implies the invariance of the spectral curve under Darboux transformations.

Theorem 4.23. *The spectral curve Σ of an immersion $f\colon M \to S^4$ of a discrete torus M with regular combinatorics that satisfies* (4.5) *is preserved under Darboux transforms.*

5. Polygons as thin cylinders

We show now that a discrete curve in the conformal 4–sphere S^4, provided it is a polygon in the sense that the images of every three consecutive points are mutually disjoint, can be treated as an immersion of a discrete surface of special type called a thin cylinder. In particular, a closed polygon can be treated as an immersion of a thin torus, a special type of discrete torus. We introduce a Darboux transformation for immersed curves in S^4. In case of polygons this transformation coincides with the Darboux transformation of Section 3 applied to the corresponding thin cylinders in S^4.

5.1. A Darboux transformation for discrete curves in S^4.

We generalize the Darboux transformation for curves in $S^2 = \mathbb{CP}^1$ [3, 4, 19, 31] to curves in $S^4 = \mathbb{HP}^1$. As a special case this includes, up to translations of $\mathbb{R}^3 = \operatorname{Im}\mathbb{H}$ by adding real numbers, the doubly discrete smoke ring flow introduced by Hoffmann [20, 21] for arc length parametrized curves in $\mathbb{R}^3 = \operatorname{Im}\mathbb{H}$ and generalized in [32] to arbitrary curves in $\mathbb{R}^3$.

By *discrete curve* we mean a map $\gamma\colon I \cap \mathbb{Z} \to S^4$ defined on the intersection of some interval $I \subset \mathbb{R}$ with $\mathbb{Z}$. In order to simplify notation, we adopt the convention to drop indices and denote the points γ_n on a curve and their successors γ_{n+1}, γ_{n+2},... simply by γ and γ_+, γ_{++},... For example, we call a discrete curve *immersed* if $\gamma \neq \gamma_+$ at all points and we call it a *polygon* if in addition $\gamma \neq \gamma_{++}$.

An immersed curve η in $S^2 = \mathbb{CP}^1$ is a *Darboux transform* of an immersed curve γ if all quadrilaterals spanned by edges of the curve γ and the corresponding edges of the transformed curve η have the same cross–ratio, see Figure 14, that is, if

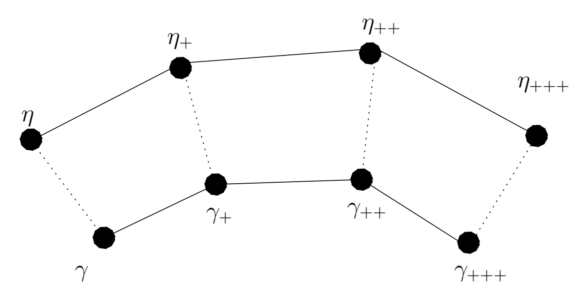

Figure 14. A Darboux transform of a discrete curve.

$$M_4(\gamma, \eta_+, \gamma_+, \eta) = \lambda \tag{5.1}$$

for some constant $\lambda \in \mathbb{C}_*$, where M_4 denotes the cross–ratio

$$M_4(z_1, z_2, z_3, z_4) = (z_1 - z_2)(z_2 - z_3)^{-1}(z_3 - z_4)(z_4 - z_1)^{-1}.$$

Note that the cross–ration $M_4(z_1, z_2, z_3, z_4)$ is the image of z_4 under the unique projective transformation $z \mapsto M_4(z_1, z_2, z_3, z)$ that maps z_1, z_2 and z_3 to the points ∞, 1, 0.

The Darboux transform η is uniquely determined by the cross ratio $\lambda \in \mathbb{C}_*$ together with an initial value, say η_0: the other points of η are determined by the recursion formula

$$\eta_+ = (P + \lambda\, Q)\eta, \tag{5.2}$$

where P and Q denote the projections from $\mathbb{C}^2$ to the summands of the splitting given by γ and γ_+. To check this, assume that γ, γ_+ and η have homogeneous coordinates $[1, 0]$, $[0, 1]$ and $[\lambda, 1]$. Then

$$P + \lambda Q = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}$$

and the point η_+ given by (5.2) has homogeneous coordinates $[\lambda, \lambda] = [1, 1]$ such that the cross–ration condition (5.1) is indeed satisfied.

For generalizing this cross–ration evolution to curves in $S^4 = \mathbb{HP}^1$ one has to deal with the problem that prescribing 3 points in S^4 and a non–real cross–ratio does only determine a unique forth point if one additionally prescribes an oriented 2–sphere. This problem can be overcome by adapting the recursion formula (5.2) to the quaternionic setting: in order to allow the use of complex spectral parameters $\lambda \in \mathbb{C}_*$ we view $\mathbb{H}^2$ as the 4–dimensional complex vector space $(\mathbb{H}^2, \mathbf{i})$ with right multiplication by complex numbers $\mathbb{C} = \mathrm{Span}_{\mathbb{R}}\{1, \mathbf{i}\} \subset \mathbb{H}$. Then (5.2) can be seen as a recursion formula for homogeneous lifts $\hat{\eta}$ of the Darboux transform η:

Definition 5.1. A curve η in $S^4 = \mathbb{HP}^1$ is a *Darboux transform* of a curve γ in $S^4 = \mathbb{HP}^1$ if there is $\lambda \in \mathbb{C}_*$ and a homogeneous lift $\hat{\eta}$ of η to $\mathbb{H}^2$ that satisfies

$$\hat{\eta}_+ = (P + \lambda\, Q)\hat{\eta}, \tag{5.3}$$

where P and Q denote the (quaternionic linear) projections operators from $\mathbb{H}^2$ to the summands of the splitting given by γ and γ_+ and where λ stands for the complex (but not quaternionic) linear operator obtained by multiplication with $\lambda \in \mathbb{C}_*$ on the space $(\mathbb{H}^2, \mathbf{i})$, in other words $\hat{\eta}_+ = P\hat{\eta} + Q\hat{\eta}\lambda$.

Because (5.3) is not a quaternionic linear equation, the initial value that determines a Darboux transform is not the point $\eta = \hat{\eta}\mathbb{H} \in \mathbb{HP}^1$, but a "twistor lift" $\hat{\eta}\mathbb{C} \in \mathbb{CP}^3$. This twistor lift $\hat{\eta} \in \mathbb{CP}^3$ has the following geometric interpretation: there is a unique oriented 2–sphere through γ, γ_+ and η obtained as the image of the complex line $\hat{\gamma}\mathbb{C} \oplus \hat{\gamma}_+\mathbb{C}$ under the twistor projection $\pi\colon \mathbb{CP}^3 \to \mathbb{HP}^1$, where $\hat{\gamma}$ and $\hat{\gamma}_+$ are homogeneous lifts of γ and γ_+ with $\hat{\eta} = \hat{\gamma} + \hat{\gamma}_+$. Obviously, the point η_+ obtained by (5.3) is also contained in this 2–sphere such that the cross–ration of the four points is a well defined complex number and, by the above argument for the complex case, indeed η_+ is the unique forth point on that oriented 2–sphere with $M_4(\gamma, \eta_+, \gamma_+, \eta) = \lambda$. The recursion formula (5.3) defines not only η_+, but a twistor lift $\hat{\eta}_+$ and hence a new oriented 2–sphere through γ_+, γ_{++} and η_+, a new point η_{++} on that 2–sphere and so forth... In addition to the Darboux transform η our construction yields a congruence of oriented 2–spheres along the edges of the curves. The intersection of two consecutive 2–spheres describes the initial curve γ and the Darboux transform η.

In case that $\gamma\colon \mathbb{Z} \to S^4$ is a *closed* curve with period n one is mainly interested in closed Darboux transformations. These are obtained by taking as initial conditions for Darboux transforms with parameter $\lambda \in \mathbb{C}_*$ the eigenlines of the holonomy

$$H(\lambda) = (P_{n-1} + \lambda\, Q_{n-1}) \cdot (P_{n-2} + \lambda\, Q_{n-2}) \cdot \ldots \cdot (P_1 + \lambda\, Q_1) \cdot (P_0 + \lambda\, Q_0). \tag{5.4}$$

A natural thing to do when coming across such polynomial family $\lambda \mapsto H(\lambda)$ of endomorphisms is to study its eigenspaces and their dependence on λ. The vanishing locus $\det(\mu - H(\lambda)) = 0$ of the characteristic polynomial is an algebraic curve and the normalization of $\{(\mu, \lambda) \in \mathbb{C}_* \times \mathbb{C}_* \mid \det(\mu - H(\lambda)) = 0\}$ is the *spectral curve* Σ of the closed immersed discrete curve γ. Proposition 4.8 implies that the eigenspaces of $H(\lambda)$ extend to holomorphic vector bundles over the components of the Riemann surface Σ.

For a generic closed curve in S^4 on expects the holonomies $H(\lambda)$ to have simple eigenvalues for all but finitely many parameters λ. The eigenspaces then extend to an "eigenline bundle" over Σ and the spectral curve parametrizes the closed Darboux transforms of γ. In particular, the spectral curve Σ is then a 4–fold branched covering of the λ–plane.

As in the case of discrete tori in S^4, the spectral curve of a closed curve in S^4 has an anti–holomorphic involution $\rho\colon \Sigma \to \Sigma$. It covers $\lambda \mapsto \bar{\lambda}$ and is induced by $H(\bar{\lambda}) = j^{-1} H(\lambda) j$, where j denotes the complex anti–linear operator given by right multiplication with the quaternion j.

Theorem 5.2. *Let γ be a closed polygon in S^4 for which the holonomy $H(\lambda)$ generically has four simple eigenvalues. The spectral curve of γ has three pairs ∞_+, $\rho\infty_+$, ∞_0, $\rho\infty_0$, ∞_-, and $\rho\infty_-$ of points at infinity such that, for each point of the curve, the twistor projection of the eigenline curve evaluated at the respective point at infinity is γ_+, γ and γ_-.*

Proof. For investigating the asymptotics of the spectral curve Σ it is sufficient to study the coefficients with lowest and highest order of the polynomial $H(\lambda)$. The lowest order coefficient of $H(\lambda)$ is

$$H(0) = P_{n-1} \cdot P_{n-2} \cdots P_1 \cdot P_0.$$

The subspaces L_1 and L_{n-1} are invariant under $H(0)$ which vanishes on L_1 and acts non–trivial on L_{n-1}, because γ is assumed to by a polygon.

For curves of even length, the highest order coefficient is

$$\begin{aligned} H_{max} &= Q_{n-1} \cdot P_{n-2} \cdots P_2 \cdot Q_1 \cdot P_0 + P_{n-1} \cdot Q_{n-2} \cdots Q_2 \cdot P_1 \cdot Q_0 = \\ &= Q_{n-1} \cdot Q_{n-3} \cdots Q_3 \cdot Q_1 \cdot P_0 + Q_{n-2} \cdot Q_{n-4} \cdots Q_2 \cdot Q_0, \end{aligned}$$

because $Q_{l+1} \cdot Q_l = 0$ and $P_{l+1} \cdot Q_l = Q_l$. The endomorphism H_{max} leaves L_0 invariant and maps L_1 to L_{n-1}. Because γ is a polygon, its restriction to both summands L_0 and L_1 is non–trivial and H_{max} is invertible.

For curves of odd length, the highest order term is

$$H_{max} = Q_{n-1} \cdot Q_{n-2} \cdots Q_2 \cdot Q_0.$$

It vanishes on L_0 and its restriction to L_1 is non–trivial (because γ is a polygon) and maps L_1 to L_0. In particular, H_{max} is nil–potent. □

For curves of odd length, the last part of the proof suggests that the spectral curve has two branch points, ∞_0 and $\rho\infty_0$, over $\lambda = \infty$ such that there are no other points at infinity apart from those described in Theorem 5.2.

Remark 5.3. Analogous to Lemma 4.20, the spectral curve of a closed, immersed discrete curve in $S^2 = \mathbb{CP}^1$, when seen as a curve in S^4, consists of two copies of the $\mathbb{CP}^1$–spectral curve which are interchanged under ρ. For polygons in $S^2 = \mathbb{CP}^1$, as in the proof of Theorem 5.2 the asymptotics at $\lambda = 0$ shows that the holonomy, which in this case reduces to a family of endomorphisms of a complex rank two vector space, generically has simple eigenvalues. In particular, a closed polygon γ in $\mathbb{CP}^1$ always admits an eigenline bundle over its spectral curve which parametrizes the eigenlines of $H(\lambda)$ and hence closed Darboux transforms of γ.

5.2. **Thin cylinders.** In order to interpret the parameter domain of a discrete curve as a *thin cylinder*, a special type of discrete surface, we take one row of black and white triangles in the regular triangulation of the plane and identify the lower left point of each black triangle with the upper right point of the white triangle right of it. Figure 15 shows as an example the fundamental domain of a *thin torus* that is parameter domain for closed curves of period n.

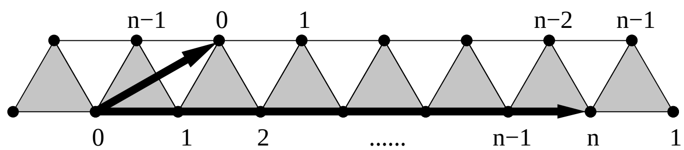

FIGURE 15. A thin torus with n points.

An immersion of a thin cylinder is an immersed discrete curve γ which is a polygon in the sense that every three consecutive points of γ are mutually different, that is, in addition to $\gamma \neq \gamma_+$ the curve satisfies $\gamma \neq \gamma_{++}$ for all points.

If M is a thin cylinder, then M' and M'' introduced in Section 2.5 are also thin cylinders. The numbering of the vertices as in Figure 15 induces a numbering of the black and white triangles (and hence of the vertices of M' and M''): a black triangle gets the same number than its lower left vertex while a white triangle gets the same number than its upper right vertex. With this numbering convention, all maps from M, M' and M'' to S^4 are curves defined on the same parameter domain. We prove now that a map $f^\sharp \colon M' \to S^4$ is the Darboux transform of an immersion $f \colon M \to S^4$ of a thin cylinder M if the curve corresponding to $f^\sharp$ is a Darboux transform of the polygon corresponding to f:

Theorem 5.4. *Let $\gamma\colon I \cap \mathbb{Z} \to S^4$ be a polygon in S^4. A discrete curve $\eta\colon I \cap \mathbb{Z} \to S^4$ is a Darboux transform of γ if and only if the thin cylinder in S^4 corresponding to η is a Darboux transform of the immersed thin cylinder corresponding to γ. In particular, the spectral curve of a closed polygon γ in S^4 coincides with the spectral curve of the corresponding thin torus in S^4.*

Proof. Let $\hat\psi$ be the prolongation of a holomorphic section with complex monodromy on the universal covering of a thin cylinder and let $\lambda \in \mathbb{C}_*$ be its "vertical" monodromy (i.e., the monodromy in direction of the lattice vector that identifies the lower left vertex of a black triangle with the upper right vertex of the white triangle right of it). As in Figure 16, we denote by $\hat\psi$, $\hat\psi_+$,... the section $\hat\psi$ on the black triangles with lower left corner γ, γ_+,... along one row of black triangles.

Because the projections of $\hat\psi_+$ and $\hat\psi\lambda$ to $(V/L)_{|\gamma}$ coincide, we have

$$(*) \qquad \hat\psi_+ \equiv \hat\psi\lambda \mod L_{|\gamma},$$

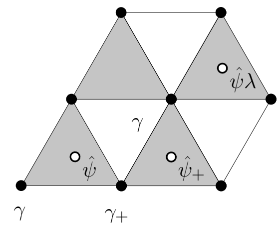

FIGURE 16. Darboux transformation of a piece of a thin cylinder.

and, because the projections of $\hat{\psi}_+$ and $\hat{\psi}$ to $(V/L)_{|\gamma_+}$ coincide, we have

$$(**) \qquad \hat{\psi}_+ \equiv \hat{\psi} \mod L_{|\gamma_+}.$$

Together, equations $(*)$ and $(**)$ are equivalent to

$$\hat{\psi}_+ = P\hat{\psi} + Q\hat{\psi}\lambda$$

with P and Q denoting the quaternionic linear projections from V to the summands of the splitting induced by γ and γ_+.

□

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