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Asymptotic stability of small solitons in the discrete nonlinear Schrödinger equation in one dimension

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Abstract

Asymptotic stability of small solitons in one dimension is proved in the framework of a discrete nonlinear Schrödinger equation with septic and higher power-law nonlinearities and an external potential supporting a simple isolated eigenvalue. The analysis relies on the dispersive decay estimates from Pelinovsky & Stefanov (2008) and the arguments of Mizumachi (2008) for a continuous nonlinear Schrödinger equation in one dimension. Numerical simulations suggest that the actual decay rate of perturbations near the asymptotically stable solitons is higher than the one used in the analysis.

1 Introduction

Asymptotic stability of solitary waves in the context of continuous nonlinear Schrödinger equations in one, two, and three spatial dimensions was considered in a number of recent works (see Cuccagna \[4\] for a review of literature). Little is known, however, about asymptotic stability of solitary waves in the context of discrete nonlinear Schrödinger (DNLS) equations.

Orbital stability of a global energy minimizer under a fixed mass constraint was proved by Weinstein \[24\] for the DNLS equation with power nonlinearity

\[
i u_n + \Delta du_n + |u_n|^{2p} u_n = 0, \quad n \in \mathbb{Z}^d,
\]

where \(\Delta_d\) is a discrete Laplacian in \(d\) dimensions and \(p > 0\). For \(p < \frac{2}{d}\) (subcritical case), it is proved that the ground state of an arbitrary energy exists, whereas for \(p \geq \frac{2}{d}\) (critical and supercritical cases), there is an energy threshold, below which the ground state does not exist.

Ground states of the DNLS equation with power-law nonlinearity correspond to single-humped solitons, which are excited in numerical and physical experiments by a single-site initial data with sufficiently large amplitude \[11\]. Such experiments have been physically realized in optical settings with both focusing \[7\] and defocusing \[13\] nonlinearities. We would like to consider long-time dynamics of the ground states and prove their asymptotic stability under some assumptions on the spectrum of the linearized DNLS equation. From the beginning, we would like to work in the space of one spatial dimension \((d = 1)\) and to add an external potential \(V\) to the DNLS equation. These specifications are motivated by physical applications.
(see, e.g., the recent work of [16] and references therein for a relevant discussion). We hence write the main model in the form

\[ i\dot{u}_n = (-\Delta + V_n)u_n + \gamma|u_n|^{2p}u_n, \quad n \in \mathbb{Z}, \]

where \( \Delta u_n := u_{n+1} - 2u_n + u_{n-1} \) and \( \gamma = 1 \) (\( \gamma = -1 \)) for defocusing (focusing) nonlinearity. Besides physical applications, the role of potential \( V \) in our work can be explained by looking at the differences between the recent works of Mizumachi [15] and Cuccagna [5] for a continuous nonlinear Schrödinger equation in one dimension. Using an external potential, Mizumachi proved asymptotic stability of small solitons bifurcating from the ground state of the Schrodinger operator \( H_0 = -\partial_x^2 + V \) under some assumptions on the spectrum of \( H_0 \). He needed only spectral theory of the self-adjoint operator \( H_0 \) in \( L^2 \) since spectral projections and small nonlinear terms were controlled in the corresponding norm. Pioneering works along the same lines are attributed to Soffer–Weinstein [20, 21, 22], Pillet & Wayne [19], and Yao & Tsai [25, 26, 27]. Compared to this approach, Cuccagna proved asymptotic stability of nonlinear space-symmetric ground states in energy space of the continuous nonlinear Schrödinger equation with \( V \equiv 0 \). He had to invoke the spectral theory of non-self-adjoint operators arising in the linearization of the nonlinear Schrödinger equation at the ground state, following earlier works of Buslaev & Perelman [11, 2, 3], Buslaev & Sulem [3], and Gang & Sigal [8, 9].

Since our work is novel in the context of the DNLS equation, we would like to simplify the spectral formalism and to focus on nonlinear analysis of asymptotic stability. This is the main reason why we work with small solitons bifurcating from the ground state of the discrete Schrodinger operator \( H = -\Delta + V \). We will make use of the dispersive decay estimates obtained recently for operator \( H \) by Stefanov & Kevrekidis [23] (for \( V \equiv 0 \)), Komech, Kopylova & Kunze [12] (for compact \( V \)), and Pelinovsky & Stefanov [18] (for decaying \( V \)). With more efforts and more elaborate analysis, our results can be generalized to large solitons with or without potential \( V \) under some restrictions on spectrum of the non-self-adjoint operator associated with linearization at the nonlinear ground state.

From a technical point of view, many previous works on asymptotic stability of solitary waves in continuous nonlinear Schrödinger equations address critical and supercritical cases, which in \( d = 1 \) corresponds to \( p \geq 2 \). Because the dispersive decay in \( l^1 - l^{\infty} \) norm is slower for the DNLS equation, the critical power appears at \( p = 3 \) and the proof of asymptotic stability of discrete solitons can be developed for \( p \geq 3 \). The most interesting case of the cubic DNLS equation for \( p = 1 \) is excluded from our consideration. To prove asymptotic stability of discrete solitons for \( p \geq 3 \), we extend the pointwise dispersive decay estimates from [18] to Strichartz estimates, which allow us for a better control of the dispersive parts of the solution. The nonlinear analysis follows the steps in the proof of asymptotic stability of continuous solitons by Mizumachi [15].

In addition to analytical results, we also approximate time evolution of small solitons numerically in the DNLS equation (11) with \( p = 1, 2, 3 \). Not only we confirm the asymptotic stability of discrete solitons in all the cases but also we find that the actual decay rate of perturbations near the small soliton is faster than the one used in our analytical arguments.

The article is organized as follows. The main result for \( p \geq 3 \) is formulated in Section 2. Linear estimates are derived in Section 3. The proof of the main theorem is developed in Section 4. Numerical illustrations for \( p = 1, 2, 3 \) are discussed in Section 5. Appendix A gives proofs of technical formulas used in Section 3.
Acknowledgement. When the paper was essentially complete, we became aware of a similar work of Cuccagna & Tarulli [6], where asymptotic stability of small discrete solitons of the DNLS equation (1) was proved for $p \geq 3$.

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2 Preliminaries and the main result

In what follows, we use bold-faced notations for vectors in discrete spaces $l^1_s$ and $l^2_s$ on $\mathbb{Z}$ defined by their norms

$$\|u\|_{l^1_s} := \sum_{n \in \mathbb{Z}} (1 + n^2)^{s/2} |u_n|, \quad \|u\|_{l^2_s} := \left(\sum_{n \in \mathbb{Z}} (1 + n^2)^{s} |u_n|^2\right)^{1/2}.$$ Components of $u$ are denoted by regular font, e.g. $u_n$ for $n \in \mathbb{Z}$.

We shall make the following assumptions on the external potential $V$ defined on the lattice $\mathbb{Z}$ and on the spectrum of the self-adjoint operator $H = -\Delta + V$ in $l^2$.

(V1) $V \in l^1_{2\sigma}$ for a fixed $\sigma > \frac{5}{2}$.

(V2) $V$ is generic in the sense that no solution $\psi_0$ of equation $H\psi_0 = 0$ exists in $l^2_{-\sigma}$ for $\frac{1}{2} < \sigma \leq \frac{3}{2}$.

(V3) $V$ supports exactly one negative eigenvalue $\omega_0 < 0$ of $H$ with an eigenvector $\psi_0 \in l^2$ and no eigenvalues above 4.

The first two assumptions (V1) and (V2) are needed for the dispersive decay estimates developed in [18]. The last assumption (V3) is needed for existence of a family $\phi(\omega)$ of real-valued decaying solutions of the stationary DNLS equation

$$(-\Delta + V_n)\phi_n(\omega) + \gamma \phi_n^{2p+1}(\omega) = \omega\phi_n(\omega), \quad n \in \mathbb{Z},$$

near $\omega = \omega_0 < 0$. This is a standard local bifurcation of decaying solutions in a system of infinitely many algebraic equations (see [14] for details).

Lemma 1 (Local bifurcation of stationary solutions). Assume that $V \in l^\infty$ and that $H$ has an eigenvalue $\omega_0$ with a normalized eigenvector $\psi_0 \in l^2$ such that $\|\psi_0\|_{l^2} = 1$. Let $\epsilon := \omega - \omega_0$, $\gamma = +1$, and $\epsilon_0 > 0$ be sufficiently small. For any $\epsilon \in (0, \epsilon_0)$, there exists an $\epsilon$-independent constant $C > 0$ such that the stationary DNLS equation (2) admits a solution $\phi(\omega) \in C^2([\omega_0, \omega_0 + \epsilon_0], l^2)$ satisfying

$$\left\|\phi(\omega) - \frac{\epsilon^{1/p} \psi_0}{\|\psi_0\|_{l^{2p+2}}}\right\|_{l^2} \leq C\epsilon^{1 + \frac{1}{2p}}.$$ Moreover, the solution $\phi(\omega)$ decays exponentially to zero as $|n| \to \infty$. 

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Remark 1. Because of the exponential decay of \( \phi(\omega) \) as \( |n| \to \infty \), the solution \( \phi(\omega) \) exists in \( l^2_{\sigma} \) for all \( \sigma \geq 0 \). In addition, since \( \|\phi\|_{l^1} \leq C_{\sigma}\|\phi\|_{l^2_{\sigma}} \), for any \( \sigma > \frac{1}{2} \), the solution \( \phi(\omega) \) also exists in \( l^1 \).

Remark 2. The case \( \gamma = -1 \) with the local bifurcation to the domain \( \omega < \omega_0 \) is absolutely analogous. For simplification, we shall develop analysis for \( \gamma = +1 \) only.

To work with solutions of the DNLS equation (1) for all \( t \in \mathbb{R}_+ \) starting with some initial data at \( t = 0 \), we need global well-posedness of the Cauchy problem for (1). Because \( H \) is a bounded operator from \( l^2 \) to \( l^2 \), global well-posedness for (1) follows from simple arguments based on the flux conservation equation
\[
i \frac{d}{dt} |u_n|^2 = u_n(\bar{u}_{n+1} + \bar{u}_{n-1}) - \bar{u}_n(u_{n+1} + u_{n-1})
\]
(3)
and the contraction mapping arguments (see [17] for details).

Lemma 2 (Global well-posedness). Fix \( \sigma \geq 0 \). For any \( u_0 \in l^2_{\sigma} \), there exists a unique solution \( u(t) \in C^1(\mathbb{R}_+, l^2_{\sigma}) \) such that \( u(0) = u_0 \) and \( u(t) \) depends continuously on \( u_0 \).

Remark 3. Global well-posedness holds also on \( \mathbb{R}_- \) (and thus on \( \mathbb{R} \)) since the DNLS equation (1) is a reversible dynamical system. We shall work in the positive time intervals only.

Equipped with the results above, we decompose a solution to the DNLS equation (1) into a family of stationary solutions with time varying parameters and a radiation part using the substitution
\[
u(t) = e^{-i\theta(t)} (\phi(\omega(t)) + z(t)) ,
\]
(4)
where \( (\omega, \theta) \in \mathbb{R}^2 \) represents a two-dimensional orbit of stationary solutions \( u(t) = e^{-i\theta - i\omega t} \phi(\omega) \) (their time evolution will be specified later) and \( z(t) \in C^1(\mathbb{R}_+, l^2_{\sigma}) \) solves the time-evolution equation in the form
\[
i \dot{z} = (H - \omega)z - (\dot{\theta} - \omega)(\phi(\omega) + z) - i\omega \partial_\omega \phi(\omega) + N(\phi(\omega) + z) - N(\phi(\omega)),
\]
(5)
where \( H = -\Delta + V, \ [N(\psi)]_n = \gamma |\psi_n|^{2p} \psi_n \), and \( \partial_\omega \phi(\omega) \) exists thanks to Lemma [11]. The linearized time evolution at the stationary solution \( \phi(\omega) \) involves operators
\[
L_- = H - \omega + W, \quad L_+ = H - \omega + (2p + 1)W,
\]
where \( W_n = \gamma \phi_n^{2p}(\omega) \) and \( W \) decays exponentially as \( |n| \to \infty \) thanks to Lemma [11]. The linearized time evolution in variables \( v = \text{Re}(z) \) and \( w = \text{Im}(z) \) involves a symplectic structure which can be characterized by the non-self-adjoint eigenvalue problem
\[
L_- v = -\lambda w, \quad L_- w = \lambda v.
\]
Using Lemma [11] we derive the following result.

Lemma 3 (Double null subspace). For any \( \epsilon \in (0, \epsilon_0) \), the linearized eigenvalue problem (2) admits a double zero eigenvalue with a one-dimensional kernel, isolated from the rest of the spectrum. The generalized kernel is spanned by vectors \( (0, \phi(\omega)), (\partial_\omega \phi(\omega), 0) \in l^2 \) satisfying
\[
L_- \phi(\omega) = 0, \quad L_+ \partial_\omega \phi(\omega) = \phi(\omega).
\]
If \((v, w) \in l^2\) is symplectically orthogonal to the double subspace of the generalized kernel, then
\[
\langle v, \phi(\omega) \rangle = 0, \quad \langle w, \partial_\omega \phi(\omega) \rangle = 0,
\]
where \(\langle u, v \rangle := \sum_{n \in \mathbb{Z}} u_n \bar{w}_n\).

**Proof.** By Lemma 1 in [18], operator \(H\) has the essential spectrum on \([0, 4]\). Because of the exponential decay of \(W\) as \(|n| \to \infty\), the essential spectrum of \(L_+ + L_-\) is shifted by \(-\omega \approx -\omega_0 > 0\), so that the zero point in the spectrum of the linearized eigenvalue problem (6) is isolated from the continuous spectrum and other isolated eigenvalues. The geometric kernel of the linearized operator \(L = \text{diag}(L_+, L_-)\) is one-dimensional for \(\epsilon \in (0, \epsilon_0)\) since \(L_- \phi(\omega) = 0\) is nothing but the stationary DNLS equation (2) whereas \(L_+\) has an empty kernel thanks to the perturbation theory and Lemma [1]. Indeed, for a small \(\epsilon \in (0, \epsilon_0)\), we have
\[
\langle \psi_0, L_+ \psi_0 \rangle = 2p\gamma \epsilon + O(\epsilon^2) \neq 0.
\]
By the perturbation theory, a simple zero eigenvalue of \(L_+\) for \(\epsilon = 0\) becomes a positive eigenvalue for \(\epsilon > 0\) (if \(\gamma = +1\)). The second (generalized) eigenvector \((-\partial_\omega \phi(\omega), 0)\) is found by direct computation thanks to Lemma [1]. It remains to show that the third (generalized) eigenvector does not exist. If it does, it would satisfy the equation
\[
L_- w_0 = -\partial_\omega \phi(\omega).
\]
However,
\[
\langle \phi(\omega), \partial_\omega \phi(\omega) \rangle = \frac{1}{2} \frac{d}{d\omega} ||\phi(\omega)||_{l^2} = \frac{\frac{1}{p} - 1}{2p||\psi_0||_{l^{2p+2}}^2} (1 + O(\epsilon)) \neq 0
\]
for \(\epsilon \in (0, \epsilon_0)\) by Lemma [1]. Therefore, no \(w_0 \in l^2\) exists.

To determine the time evolution of varying parameters \((\omega, \theta)\) in the evolution equation (5), we shall add the condition that \(z(t)\) is symplectically orthogonal to the two-dimensional null subspace of the linearized problem (6). To normalize the eigenvectors uniquely, we set
\[
\psi_1 = \frac{\phi(\omega)}{||\phi(\omega)||_{l^2}}, \quad \psi_2 = \frac{\partial_\omega \phi(\omega)}{||\partial_\omega \phi(\omega)||_{l^2}}
\]
and require that
\[
\langle \text{Re} z(t), \psi_1 \rangle = \langle \text{Im} z(t), \psi_2 \rangle = 0.
\]
By Lemma [1], both eigenvectors \(\psi_1\) and \(\psi_2\) are locally close to \(\psi_0\), the eigenvector of \(H\) for eigenvalue \(\omega_0\), in any norm, e.g.,
\[
||\psi_1 - \psi_0||_{l^2} + ||\psi_2 - \psi_0||_{l^2} \leq C \epsilon,
\]
for some \(C > 0\). Although the vector field of the time evolution problem (5) does not lie in the orthogonal complement of \(\psi_0\), that is in the absolutely continuous spectrum of \(H\), the difference is small for small \(\epsilon > 0\). We shall prove that the conditions (8) define a unique decomposition (4).
Lemma 4 (Decomposition). Fix $\epsilon > 0$ and $\delta > 0$ be sufficiently small. Assume that there exists $T = T(\epsilon, \delta)$ and $C_0 > 0$, such that $u(t) \in C^1([0, T], l^2)$ satisfies
\[
\|u(t) - \phi(\omega_0 + \epsilon)\|_{l^2} \leq C_0 \delta e^{\frac{\epsilon}{T}},
\]
uniformly on $[0, T]$. There exists a unique choice of $(\omega, \theta) \in C^1([0, T], \mathbb{R}^2)$ and $z(t) \in C^1([0, T], l^2)$ in the decomposition (4) provided the constraints (8) are met. Moreover, there exists $C > 0$ such that
\[
|\omega(t) - \omega_0 - \epsilon| \leq C \delta \epsilon, \quad |\theta(t)| \leq C \delta, \quad \|z(t)\|_{l^2} \leq C \delta e^{\frac{\epsilon}{T}},
\]
uniformly on $[0, T]$.

Proof. We write the decomposition (4) in the form
\[
z = e^{i\theta} (u - \phi(\omega_0 + \epsilon)) + \left( e^{i\theta} \phi(\omega_0 + \epsilon) - \phi(\omega) \right).
\]
First, we show that the constraints (8) give unique values of $(\omega, \theta)$ satisfying bounds (11) uniformly in $[0, T]$ provided the bound (10) holds. To do so, we rewrite (8) and (12) as a fixed-point problem $F(\omega, \theta) = 0$, where $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is given by
\[
F(\omega, \theta) = \begin{bmatrix}
\langle \text{Re}(u - \phi(0)) e^{i\theta}, \psi_1 \rangle + \langle \phi(0) \cos \theta - \phi(\omega), \psi_1 \rangle \\
\langle \text{Im}(u - \phi(0)) e^{i\theta}, \psi_2 \rangle + \langle \phi(0) \sin \theta, \psi_2 \rangle
\end{bmatrix},
\]
where $\phi(0) := \phi(\omega_0 + \epsilon)$. We note that $F$ is $C^1$ in $(\theta, \omega)$ thanks to Lemma 1. Direct computations give the vector field
\[
F(\omega_0 + \epsilon, 0) = \begin{bmatrix}
\langle \text{Re}(u - \phi(0)), \psi_1^{(0)} \rangle \\
\langle \text{Im}(u - \phi(0)), \psi_2^{(0)} \rangle
\end{bmatrix}
\]
and the Jacobian $DF(\omega_0 + \epsilon, 0) = D_1 + D_2$ with
\[
D_1 = \begin{bmatrix}
-\langle \partial_\omega \phi(0), \psi_1^{(0)} \rangle & 0 \\
0 & \langle \phi(0), \psi_2^{(0)} \rangle
\end{bmatrix},
D_2 = \begin{bmatrix}
\langle \text{Re}(u - \phi(0)), \partial_\omega \psi_1^{(0)} \rangle & -\langle \text{Im}(u - \phi(0)), \psi_1^{(0)} \rangle \\
\langle \text{Im}(u - \phi(0)), \partial_\omega \psi_2^{(0)} \rangle & \langle \text{Re}(u - \phi(0)), \psi_2^{(0)} \rangle
\end{bmatrix},
\]
where $\psi_{1,2}^{(0)} = \psi_{1,2}(\omega_0 + \epsilon)$ and $\partial_\omega \psi_{1,2}^{(0)} = \partial_\omega \psi_{1,2}(\omega_0 + \epsilon)$. Thanks to the bound (10) and the normalization of $\psi_{1,2}$, there exists an $(\epsilon, \delta)$-independent constant $C_0 > 0$ such that
\[
\|F(\omega_0 + \epsilon, 0)\| \leq C_0 \delta e^{\frac{\epsilon}{T}}.
\]
On the other hand, $DF(\omega_0 + \epsilon, 0)$ is invertible for small $\epsilon > 0$ since
\[
|(D_1)_{11}| \geq C_1 \delta e^{\frac{\epsilon}{T} - 1}, \quad |(D_1)_{22}| \geq C_2 \delta e^{\frac{\epsilon}{T}},
\]
and
\[
|(D_2)_{11}| + |(D_2)_{21}| \leq C_3 \delta e^{\frac{\epsilon}{T} - 1}, \quad |(D_2)_{12}| + |(D_2)_{22}| \leq C_4 \delta e^{\frac{\epsilon}{T}},
\]
for some \((\epsilon, \delta)\)-independent constants \(C_1, C_2, C_3, C_4 > 0\). By the Implicit Function Theorem, there exists a unique root of \(F(\omega, \theta) = 0\) near \((\omega_0 + \epsilon, 0)\) for any \(u(t)\) satisfying (11) such that

\[
|\omega(t) - \omega_0 - \epsilon| \leq C\delta\epsilon, \quad |\theta(t)| \leq C\delta,
\]

for some \(C > 0\). Moreover, if \(u(t) \in C^1([0, T], l^2)\), then \((\omega, \theta) \in C^1([0, T], \mathbb{R}^2)\). Finally, existence of a unique \(z(t)\) and the bound \(\|z(t)\|_{l^2} \leq C\delta\epsilon^{\frac{1}{2p}}\) follow from the representation (12) and the triangle inequality.

Assuming \((\omega, \theta) \in C^1([0, T], \mathbb{R}^2)\) at least locally in time and using Lemma 4, we define the time evolution of \((\omega, \theta)\) from the projections of the time evolution equation (5) with the symplectic orthogonality conditions (8). The resulting system is written in the matrix–vector form

\[
A(\omega, z) \begin{bmatrix} \omega \\ \theta - \omega \end{bmatrix} = f(\omega, z),
\]

(13)

where

\[
A(\omega, z) = \begin{bmatrix} \langle \partial_\omega \phi(\omega), \psi_1 \rangle - \langle \text{Re}z, \partial_\omega \psi_1 \rangle & \langle \text{Im}z, \psi_1 \rangle \\ \langle \text{Im}z, \partial_\omega \psi_2 \rangle & \langle \phi(\omega) + \text{Re}z, \psi_2 \rangle \end{bmatrix}
\]

and

\[
f(\omega, z) = \begin{bmatrix} \langle \text{ImN}(\phi + z) - Wz, \psi_1 \rangle \\ \langle \text{ReN}(\phi + z) - N(\phi) - (2p + 1)Wz, \psi_2 \rangle \end{bmatrix}.
\]

Using an elementary property for power functions

\[
||a + b||_{2^p}(a + b) - |a|^{2^p}a| \leq C_p(|a|^{2^p}|b| + |b|^{2^p+1}),
\]

for some \(C_p > 0\), where \(a, b \in \mathbb{C}\) are arbitrary, we bound the vector fields of (5) and (13) by

\[
||N(\phi(\omega) + z) - N(\phi(\omega))||_{l^2} \leq C \left(||\phi(\omega)||_{l^2}^{|2^p|}||z||_{l^2} + ||z||_{l^2}^{2^p+1}\right),
\]

(14)

\[
||f(\omega, z)|| \leq C \sum_{j=1}^{2^p} \left(||\phi(\omega)||_{l^2}^{2^p-1}||\psi_j||_{l^2}||z||_{l^2} + ||\psi_j||_{l^2}||z||_{l^2}^{2^p+1}\right),
\]

(15)

for some \(C > 0\), where the pointwise multiplication of vectors on \(l^2\) is understood in the sense

\[
(|\phi||\psi|)_n = \phi_n\psi_n.
\]

By Lemmas 1 and 4, \(A(\omega, z)\) is invertible for a small \(z \in l^2\) and a small \(\epsilon \in (0, \epsilon_0)\) so that solutions of system (13) satisfy the estimates

\[
|\dot{\omega}| \leq C\epsilon^{2^p-\frac{1}{2}} \left(||\psi_1||_{l^2}||z||_{l^2} + ||\psi_2||_{l^2}||z||_{l^2}\right),
\]

(16)

\[
|\dot{\theta} - \omega| \leq C\epsilon^{1-\frac{1}{2}} \left(||\psi_1||_{l^2}||z||_{l^2} + ||\psi_2||_{l^2}||z||_{l^2}\right),
\]

(17)

for some \(C > 0\) uniformly in \(||z||_{l^2} \leq C_0\epsilon^{\frac{1}{2p}}\) for some \(C_0 > 0\). 


Remark 4. The estimates (16) and (17) show that if \( \|z\|_{l^2} \leq C\delta \frac{1}{\varepsilon} \) for some \( C > 0 \), then

\[
|\omega(t) - \omega(0)| \leq C\delta^2 \varepsilon^2, \quad \left| \theta(t) - \int_0^t \omega(t') dt' \right| \leq C\delta^2 \varepsilon,
\]

uniformly on \([0, T]\) for any fixed \( T > 0 \). These bounds are smaller than bounds (11) of Lemma 4. They become comparable with bounds (11) for larger time intervals \([0, T]\), where \( T \leq C\delta \varepsilon \) for some \( C > 0 \). Our main task is to extend these bounds globally to \( T = \infty \).

By the theorem on orbital stability in [24], the trajectory of the DNLS equation (1) originating from a point in a local neighborhood of the stationary solution \( \phi(\omega(0)) \) remains in a local neighborhood of the stationary solution \( \phi(\omega(t)) \) for all \( t \in \mathbb{R}_+ \). By a definition of orbital stability, for any \( \mu_0 > 0 \) there exists a \( \nu_0 > 0 \) such that if \( |\omega(0) - \omega_0| \leq \nu_0 \) then \( |\omega(t) - \omega_0| \leq \mu_0 \) uniformly on \( t \in \mathbb{R}_+ \). Therefore, there exists a \( \delta(\varepsilon) \) for each \( \varepsilon \in (0, \varepsilon_0) \) such that \( T(\varepsilon, \delta) = \infty \) for any \( \delta \in (0, \delta(\varepsilon)) \) in Lemma 4. To prove the main result on asymptotic stability, we need to show that the trajectory approaches to the stationary solution \( \phi(\omega_\infty) \) for some \( \omega_\infty \in (\omega_0, \omega_0 + \varepsilon_0) \). Our main result is formulated as follows.

**Theorem 1** (Asymptotic stability in the energy space). Assume (V1)–(V3), fix \( \gamma = +1 \) and \( p \geq 3 \). Fix \( \varepsilon > 0 \) and \( \delta > 0 \) be sufficiently small and assume that \( \theta(0) = 0 \), \( \omega(0) = \omega_0 + \varepsilon \), and

\[
||u(0) - \phi(\omega_0 + \varepsilon)||_{l^2} \leq C_0 \delta \varepsilon \frac{1}{\varepsilon},
\]

for some \( C_0 > 0 \). Then, there exist \( \omega_\infty \in (\omega_0, \omega_0 + \varepsilon_0) \), \( (\omega, \theta) \in C^1(\mathbb{R}_+, \mathbb{R}^2) \), and a solution \( u(t) \in X := C^1(\mathbb{R}_+, l^2) \cap L^6(\mathbb{R}_+, l^\infty) \) to the DNLS equation (7) such that

\[
\lim_{t \to \infty} \omega(t) = \omega_\infty, \quad ||u(t) - e^{-i\theta(t)} \phi(\omega(t))||_X \leq C\delta \varepsilon^{1/(2p)}.
\]

Theorem 1 is proved in Section 4. To bound solutions of the time-evolution problem (5) in the space \( X \) (intersected with some other spaces of technical nature), we need some linear estimates, which are described in Section 3.

3 Linear estimates

We need several types of linear estimates, each is designed to control different nonlinear terms of the vector field of the evolution equation (5). For notational convenience, we shall use \( L^p_t \) and \( l^p_n \) to denote \( L^p \) space on \( t \in [0, T] \) and \( l^p \) space on \( n \in \mathbb{Z} \), where \( T > 0 \) is an arbitrary time including \( T = \infty \). The notation \(< n > = (1 + n^2)^{1/2}\) is used for the weights in \( l^p_n \) norms. The constant \( C > 0 \) is a generic constant, which may change from one line to another line.

3.1 Decay and Strichartz estimates

Under assumptions (V1)–(V2) on the potential, the following result was proved in [18].

**Lemma 5** (Dispersive decay estimates). Fix \( \sigma > \frac{\gamma}{2} \) and assume (V1)–(V2). There exists a constant \( C > 0 \) depending on \( V \) such that

\[
||\langle n \rangle^{-\sigma} e^{-itH} P_{a.c.}(H)f||_{l^2_n} \leq C(1 + t)^{-3/2} ||\langle n \rangle^\sigma f||_{l^6_n}, \quad (18)
\]

\[
||e^{-itH} P_{a.c.}(H)f||_{l^\infty_n} \leq C(1 + t)^{-1/3} ||f||_{l^6_n}, \quad (19)
\]

for all \( t \in \mathbb{R}_+ \), where \( P_{a.c.}(H) \) is the projection to the absolutely continuous spectrum of \( H \).
estimates. Similar to the continuous case, these estimates are only needed in one dimension, where the norm in V resolvent on B satisfies the time evolution problem (5) cannot be closed.

Remark 5. Unlike the continuous case, the upper bound (19) is non-singular as t \to 0 because the discrete case always enjoys an estimate \|f\|_{l^2_n} \leq \|f\|_{l^2_k} \leq \|f\|_{l^1_n}.

Using Lemma 5 and Theorem 1.2 of Keel-Tao [10], the following corollary transfers pointwise decay estimates into Strichartz estimates.

Corollary 1 (Discrete Strichartz estimates). There exists a constant C > 0 such that
\[\left\|e^{-itH} P_{a.c.}(H)f\right\|_{L^6_t l^\infty \cap L^\infty_t l^2_n} \leq C\|f\|_{l^2_n},\]  
\[\left\|\int_0^t e^{-i(t-s)H} P_{a.c.}(H)g(s)ds\right\|_{L^6_t l^\infty \cap L^\infty_t l^2_n} \leq C\|g\|_{L^1_t l^4_n},\]  
where the norm in \(L^p t^q_n\) is defined by
\[\|f\|_{L^p t^q_n} = \left(\int_{\mathbb{R}^+} (\|f(t)\|_{l^q_n})^p \, dt\right)^{1/p}.

3.2 Time averaged estimates

To control the evolution of the varying parameters \((\omega, \theta)\), we derive additional time averaged estimates. Similar to the continuous case, these estimates are only needed in one dimension, because the time decay provided by the Strichartz estimates is insufficient to guarantee time integrability of \(\dot{\omega}(t) - \omega(t)\) bounded from above by the estimates [16] and [17]. Without the time integrability of these quantities, the arguments on the decay of various norms of \(z(t)\) satisfying the time evolution problem [5] cannot be closed.

Lemma 6. Fix \(\sigma > \frac{5}{2}\) and assume (V1) and (V2). There exists a constant C > 0 depending on V such that
\[\left\|\int_{\mathbb{R}^+} e^{-itH} P_{a.c.}(H)f dt\right\|_{L^2_t l^2_n} \leq C\|f\|_{l^2_n},\]  
\[\left\|\int_{\mathbb{R}^+} e^{-itH} P_{a.c.}(H)F(s) dt\right\|_{l^2_n} \leq C\|F\|_{l^1_t l^2_n},\]  
\[\left\|\int_0^t e^{-i(t-s)H} P_{a.c.}(H)F(s)ds\right\|_{L^\infty_t l^2_n} \leq C\|F\|_{L^1_t l^2_n},\]  
\[\left\|\int_0^t e^{-i(t-s)H} P_{a.c.}(H)F(s)ds\right\|_{L^\infty_t l^2_n} \leq C\|F\|_{L^1_t l^2_n},\]  
\[\left\|\int_0^t e^{-i(t-s)H} P_{a.c.}(H)F(s)ds\right\|_{L^\infty_t l^2_n} \leq C\|F\|_{L^1_t l^2_n}.

To proceed with the proof, let us set up a few notations. First, introduce the perturbed resolvent \(R_V(\lambda) := (H - \lambda)^{-1}\) for \(\lambda \in \mathbb{C}\setminus[0, 4]\). We proved in [18, Theorem 1] that for any fixed \(\omega \in (0, 4)\), there exists \(R_+^V(\omega) = \lim_{\epsilon \to 0} R(\omega \pm i\epsilon)\) in the norm of \(B(\sigma, -\sigma)\) for any \(\sigma > \frac{1}{2}\), where \(B(\sigma, -\sigma)\) denotes the space of bounded operators from \(l^2_n\) to \(l^2_{-\sigma}\).

Next, we recall the Cauchy formula for \(e^{itH}\)
\[e^{-itH} P_{a.c.}(H) = \frac{1}{\pi} \int_0^4 e^{-i\omega t} \text{Im} R_V(\omega) d \omega = \frac{1}{2i} \int_0^4 e^{-i\omega t} \left[R^+(\omega) - R^-(\omega)\right] d \omega,\]  
where \(R^+(\omega) = \lim_{\epsilon \to 0} R(\omega + i\epsilon)\) and \(R^-(\omega) = \lim_{\epsilon \to 0} R(\omega - i\epsilon)\).
where the integral is understood in norm $B(\sigma, -\sigma)$. We shall parameterize the interval $[0, 4]$ by $\omega = 2 - 2\cos(\theta)$ for $\theta \in [-\pi, \pi]$.

Let $\chi_0, \chi \in C_0^\infty$: $\chi_0 + \chi = 1$ for all $\theta \in [-\pi, \pi]$, so that
\[
\text{supp}\chi_0 \subset [-\theta_0, \theta_0] \cup (-\pi, -\pi + \theta_0) \cup (\pi - \theta_0, \pi)
\]
and
\[
\text{supp}\chi \subset [\theta_0/2, \pi - \theta_0/2] \cup [-\pi + \theta_0/2, -\theta_0/2],
\]
where $0 < \theta_0 \leq \frac{\pi}{4}$. Note that the support of $\chi$ stays away from both 0 and $\pi$. Following Mizumachi [15], the proof of Lemma [6] relies on the technical lemma.

**Lemma 7.** Assume (V1) and (V2). There exists a constant $C > 0$ such that
\[
\sup_{n \in \mathbb{Z}} \| \chi R_\nu^\pm(\omega) f \|_{L_2^\infty(0, 4)} \leq C \| f \|_{l_n^2}, \quad \text{for } (23), (25), (26)
\]
\[
\sup_{n \in \mathbb{Z}} \| \langle n \rangle^{-\sigma/2} \chi R_\nu^\pm(\omega) f \|_{L_2^\infty(0, 4)} \leq C \| f \|_{l_n^2}.
\]

The proof of Lemma 7 is developed in Appendix A. Using Lemma 7, we can now prove Lemma 6.

**Proof of Lemma 6.** Let us first show (24), since it can be deduced from (18), although, it can also be viewed (and proved) as a dual of (22) as well. Indeed, (23) is equivalent to (22). For (25), we apply the so-called averaging principle, which tells us that to prove (27), it is sufficient to show it for $F(t) = \delta(t - t_0)f$, where $f \in l_n^2$ and $\delta(t - t_0)$ is Dirac’s delta-function. Therefore, we obtain
\[
\| \langle n \rangle^{-\sigma/2} \chi R_\nu^\pm(\omega) f \|_{L_2^\infty(0, 4)} \leq C \| f \|_{l_n^2},
\]
where in the last step, we have used (22).

For (26), we argue as follows. Define

$$TF(t) = \int_\mathbb{R} e^{-i(t-s)H} P_{a.c.}(H) F(s) ds$$

$$= e^{-itH} P_{a.c.}(H) \left( \int_\mathbb{R} e^{-isH} P_{a.c.}(H) F(s) ds \right)$$

$$= e^{-itH} P_{a.c.}(H) f,$$

where $f = \int_\mathbb{R} e^{-isH} P_{a.c.}(H) F(s) ds$. By an application of the Strichartz estimate (20) and subsequently (23), we obtain

$$\|TF\|_{L^4_t L^\infty_x L^2_y} \leq C\|f\|_{L^2_y} \leq \|<n>^3 F\|_{L^3_x L^2_y} \leq C\|<n>^3 F\|_{L^2_x L^2_y} \leq C\|<n>^3 F\|_{L^2_x L^2_y},$$

where in the last two steps, we have used Hölder’s inequality and the fact that $L^2_n$ and $L^2_t$ commute. Now, by the Christ-Kiselev’s lemma (e.g. Theorem 1.2 in [10]), we conclude that the estimate (26) applies to $\int_0^t e^{-i(t-s)H} P_{a.c.}(H) F(s) ds$, similar to $TF(t)$. To complete the proof of Lemma 7, it only remains to prove (22). Let us write

$$e^{-itH} P_{a.c.}(H) = \chi e^{-itH} P_{a.c.}(H) + \chi_0 e^{-itH} P_{a.c.}(H)$$

Take a test function $g(t)$ such that $\|g\|_{L^1_t L^2_y} = 1$ and obtain

$$|\langle \chi e^{-itH} P_{a.c.}(H) f, g(t) \rangle_{n,t}| = \frac{1}{\pi} \left| \int_0^4 \langle \chi \text{Im} R_V(\omega) f, \int_\mathbb{R} e^{-it\omega} g(t) dt \rangle_{n} d\omega \right|$$

$$\leq C \int_0^4 \langle \|\chi R_V(\omega)f\|, |g(\omega)| \rangle_{n} d\omega$$

$$\leq C\|\chi R_V(\omega)f\|_{L^1_x L^2_y(0,4)} \|g\|_{L^2_x L^2_y(0,4)}.$$

By Plancherel’s theorem, $\|\hat{g}\|_{L^1_x L^2_y(0,4)} \leq \|\hat{g}\|_{L^1_x L^2_y(\mathbb{R})} \leq \|g\|_{L^1_t L^2_y} = 1$. Using (28), we obtain

$$\|\chi e^{-itH} P_{a.c.}(H) f\|_{L^1_x L^2_y} = \sup_{\|g\|_{L^1_t L^2_y} = 1} \left| \langle \chi e^{-itH} P_{a.c.}(H) f, g(t) \rangle_{n,t} \right| \leq C\|f\|_{L^2_y}.$$

Similarly, using (29) instead of (28), one concludes

$$\left\|<n>^3 \chi_0 e^{-itH} P_{a.c.}(H) f\right\|_{L^1_x L^2_y} = \sup_{\|g\|_{L^1_t L^2_y} = 1} \left| \langle \chi_0 e^{-itH} P_{a.c.}(H) f, g(t) \rangle_{n,t} \right| \leq C\|f\|_{L^2_y}.$$

Combining the two estimates, we obtain (22).

4 Proof of Theorem [1]

Let $y(t) = e^{-i\theta(t)} z(t)$ and write the time-evolution problem for $y(t)$ in the form

$$i\dot{y} = H y + g_1 + g_2 + g_3,$$
where
\[ g_1 = \left( N(\phi + ye^{-i\theta}) - N(\phi) \right) e^{-i\theta}, \quad g_2 = -(\dot{\theta} - \omega) \phi e^{-i\theta}, \quad g_3 = -i\dot{\omega} \partial_\omega \phi(\omega)e^{-i\theta}. \]

Let \( P_0 = \langle \cdot, \psi_0 \rangle \psi_0, \) \( Q = (I - P_0) \equiv P_{a,c}(H), \) and decompose the solution \( y(t) \) into two orthogonal parts
\[ y(t) = a(t)\psi_0 + \eta(t), \]
where \( \langle \psi_0, \eta \rangle = 0 \) and \( a(t) = \langle y(t), \psi_0 \rangle. \) The new coordinates \( a(t) \) and \( \eta(t) \) satisfy the time evolution problem
\[
\begin{align*}
    i\dot{a} &= \omega_0 a + \langle g, \psi_0 \rangle, \\
    i\dot{\eta} &= H\eta + Qg
\end{align*}
\]
where \( g = \sum_{j=1}^3 g_j. \) The time-evolution problem for \( \eta \equiv P_{a,c}(H)\eta \) can be rewritten in the integral form as
\[
\eta(t) = e^{-itH}Q\eta(0) - i \int_0^t e^{-i(t-s)H}Qg(s)ds, \quad (30)
\]
Fix \( \sigma > \frac{5}{2} \) and introduce the norms
\[
\begin{align*}
    M_1 &= \|\eta\|_{L_t^1L_x^\infty}, \\
    M_2 &= \|\eta\|_{L_t^\inftyL_x^c}, \\
    M_3 &= \| n > -\sigma \eta \|_{L_t^\infty L_x^1}, \\
    M_4 &= \|a\|_{L_t^1}, \\
    M_5 &= \|a\|_{L_t^\infty}, \\
    M_6 &= \|\omega - \omega(0)\|_{L_t^\infty}
\end{align*}
\]
where the integration in \( L_t^p \) is performed on an interval \([0, T]\) for any \( T \in (0, \infty). \) Our goal is to show that \( \dot{\omega} \) and \( \dot{\theta} - \omega \) are in \( L_t^1, \) while the norms above satisfy an estimate of the form
\[
\sum_{j=1}^5 M_j \leq C\|y(0)\|_{l_\sigma^2} + C\left( \sum_{j=1}^6 M_j \right)^2, \quad (31)
\]
and
\[
M_6 \leq C\epsilon^{2-\frac{1}{T}}(M_3 + M_4)^2, \quad (32)
\]
for some \( T \)-independent constant \( C > 0 \) uniformly in \( \sum_{j=1}^6 M_j \leq C\delta \epsilon^{\frac{1}{4T}}, \) where small positive values of \( (\epsilon, \delta) \) are fixed by the initial conditions \( \omega(0) = \omega_0 + \epsilon \) and \( \|y(0)\|_{l_\sigma^2} \leq C_0 \delta \epsilon^{\frac{1}{4T}} \) for some \( C_0 > 0. \) The estimate \((31)\) and \((32)\) allow us to conclude, by elementary continuation arguments, that
\[
\sum_{j=1}^5 M_j \leq C\|y(0)\|_{l_\sigma^2} \leq C\delta \epsilon^{\frac{1}{4T}}
\]
and \( |\omega(t) - \omega_0 - \epsilon| \leq C\delta^2 \epsilon^2 \) uniformly on \([0, T]\) for any \( T \in (0, \infty). \) By interpolation, \( a \in L_t^q \) so that \( z(t) \in L_t^0([0, T], l_n^{\infty}). \) Theorem then holds for \( T = \infty. \) In particular, since \( \omega(t) \in L_t^1(\mathbb{R}_+) \) and \( |\omega(t) - \omega_0 - \epsilon| \leq C\delta^2 \epsilon^2, \) there exists \( \omega_\infty := \lim_{t \to \infty} \omega(t) \) so that \( \omega_\infty \in (\omega_0, \omega_0 + \epsilon_0). \) In addition, since \( z(t) \in L_t^0(\mathbb{R}_+, l_n^{\infty}), \) then
\[
\lim_{t \to \infty} \|u(t) - e^{-i\theta(t)}\phi(\omega(t))\|_{l_n^\infty} = \lim_{t \to \infty} \|z(t)\|_{l_n^\infty} = 0.
\]
Estimates for $M_6$: By the estimate (16), we have
\[
\int_0^T |\dot{\omega}| \, dt \leq C e^{2-\frac{1}{2}} \| < n >^{-\sigma} |\psi|^2 \|_{L^1_t} \| < n >^{2\sigma} \psi_1 \|_{L^2_t} + \| < n >^{2\sigma} \psi_2 \|_{L^2_t} \\
\leq C e^{2-\frac{1}{2}} \| < n >^{-\sigma} |\psi|^2 \|_{L^1_t} \leq C e^{2-\frac{1}{2}} (M_3 + M_4)^2,
\]
where we have used the fact that $\psi_1$ and $\psi_2$ decay exponentially as $|n| \to \infty$. As a result, we obtain
\[
M_6 \leq \| \dot{\omega} \|_{L^1_t} \leq C e^{2-\frac{1}{2}} (M_3 + M_4)^2.
\]
Similarly, we also obtain that
\[
\int_0^T |\dot{\theta} - \omega| \, dt \leq C e^{1-\frac{1}{2}} (M_3 + M_4)^2.
\]

Estimates for $M_4$ and $M_5$: We use the projection formula $a = \langle y, \psi_0 \rangle$ and recall the orthogonality relation (33), so that
\[
\langle z, \psi_0 \rangle = \langle \text{Re} z, \psi_0 - \psi_1 \rangle + i \langle \text{Im} z, \psi_0 - \psi_2 \rangle.
\]
By Lemma 11 and definitions of $\psi_{1,2}$ in (31), we have
\[
\| < n >^{2\sigma} (\psi_0 - \psi_{1,2}) \|_{L^2_t} \leq C |\omega - \omega_0|
\]
for some $C > 0$. Provided $\sigma > \frac{1}{2}$, we obtain
\[
M_4 = \| \langle y, \psi_0 \rangle \|_{L^1_t} \leq \| \langle \text{Re} z, \psi_0 - \psi_1 \rangle \|_{L^2_t} + \| \langle \text{Im} z, \psi_0 - \psi_2 \rangle \|_{L^2_t} \\
\leq \| < n >^{-2\sigma} z \|_{L^1_t} \| \langle \text{Re} z, \psi_0 - \psi_1 \rangle \|_{L^2_t} \leq C \| < n >^{-\sigma} \psi \|_{L^2_t} \| < n >^{2\sigma} (\psi_0 - \psi_1) \|_{L^2_t} + \| < n >^{2\sigma} (\psi_0 - \psi_2) \|_{L^2_t} \\
\leq C \| < n >^{-\sigma} \psi \|_{L^2_t} \| < n >^{2\sigma} (\psi_0 - \psi_1) \|_{L^2_t} + \| < n >^{2\sigma} (\psi_0 - \psi_2) \|_{L^2_t} \leq C (M_3 + M_4) M_6
\]
and, similarly,
\[
M_5 = \| \langle y, \psi_0 \rangle \|_{L^2_t} \leq \| \langle \text{Re} z, \psi_0 - \psi_1 \rangle \|_{L^2_t} + \| \langle \text{Im} z, \psi_0 - \psi_2 \rangle \|_{L^2_t} \leq C (M_3 + M_5) M_6.
\]

Estimates for $M_3$: The free solution in the integral equation (30) is estimated by (22) as
\[
\| < n >^{-\sigma} e^{-itH} Q \eta(0) \|_{L^1_t L^2} \leq \| < n >^{-3/2} e^{-itH} Q \eta(0) \|_{L^1_t L^2} \leq C \| \eta(0) \|_{L^2_t}.
\]
Since $\dot{\omega}$ and $\dot{\theta} - \omega$ are $L^1_t$ thanks to the estimates above, we treat the terms of the integral equation (30) with $g_2$ and $g_3$ similarly. By (25), we obtain
\[
\| < n >^{-\sigma} \int_0^t e^{-i(t-s)H} Q g_{2,3}(s) ds \|_{L^2_t L^2_t} \leq C \| g_{2,3} \|_{L^1_t L^2_t} \\
\leq C \| \dot{\omega} \|_{L^2_t} \| \dot{\theta} - \omega \|_{L^2_t} + \| \dot{\omega} \|_{L^2_t} \| \partial_\omega \phi(\omega) \|_{L^2_t} \\
\leq C e^{1-\frac{1}{2}} (M_3 + M_4)^2.
\]

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On the other hand, using the bound \( L^4 \) on the vector field \( g_1 \), we estimate by \( 24 \) and \( 25 \)

\[
\| < n >^{-\sigma} \int_0^t e^{-i(t-s)H} Q g_1(s) \, ds \|_{L^n_t L^2_x} \leq C(\| < n >^{\sigma} |\phi(\omega)|^{2p} |z| \|_{L^2_t L^2_x} + \| |z|^{2p+1} \|_{L^{2p+1}_t L^{2p+1}_x})
\]

\[
\leq C \left( \| < n >^{-\sigma} y \|_{L^n_t L^2_x} \| < n >^{\sigma} |\phi(\omega)|^{2p} \|_{L^n_t L^2_x} + \| a \|_{L^{2p+1}_t L^{2p+1}_x} + \| a \|_{L^{2p+1}_t L^{2p+1}_x} + \| \eta \|_{L^{2p+1}_t L^{2p+1}_x} \right)
\]

\[
\leq C \left( (M_3 + M_4) M_6 + M_4^2 M_5^{2p-1} + \| \eta \|_{L^{2p+1}_t L^{2p+1}_x} \right),
\]

where we have

\[
\| a \|_{L^{2p+1}_t L^{2p+1}_x} \leq \| a \|_{L^\infty_t L^2_x} \| a \|_{L^2_x}^2.
\]

and

\[
\| < n >^{\sigma} |\phi(\omega)|^{2p} \|_{L^n_x} \leq C \| \omega - \omega_0 \|_{L^n_x},
\]

the latter estimate follows from Lemma 1.

To deal with the last term in the estimate, we use the Gagliardo-Nirenberg inequality, that is, for all \( 2 \leq r, w \leq \infty \) such that \( \frac{6}{r} + \frac{2}{w} \leq 1 \), there is a \( C > 0 \) such that

\[
\| \eta \|_{L^r_t L^w_x} \leq C \left( \| \eta \|_{L^\infty_t L^n_x} + \| \eta \|_{L^n_t L^\infty_x} \right) = C(M_1 + M_2).
\]

If \( p \geq 3 \), then \( ((2p+1), 2(2p+1)) \) is a Strichartz pair satisfying \( \frac{6}{2p+1} + \frac{2}{2(2p+1)} \leq 1 \) and hence, combining all previous inequalities, we have

\[
M_3 \leq C \left( \| \eta(0) \|_{L^2_x} + e^{1-\frac{1}{r}} (M_3 + M_4)^2 + (M_3 + M_4) M_6 + M_4^2 M_5^{2p-1} + (M_1 + M_2)^{2p+1} \right),
\]

which agrees with the estimate \( 31 \) for any \( p \geq 3 \).

**Estimates for \( M_1 \) and \( M_2 \):** With the help of \( 20 \), the free solution is estimated by

\[
\| e^{-iH t} Q \eta(0) \|_{L^6_t L^{\infty}_x \cap L^{\infty}_t L^2_x} \leq C \| \eta(0) \|_{L^2_x}.
\]

With the help of \( 21 \), the nonlinear terms involving \( g_{2,3} \) are estimated by

\[
\left\| \int_0^t e^{-i(t-s)H} Q g_{2,3}(s) \, ds \right\|_{L^6_t L^{\infty}_x \cap L^{\infty}_t L^2_x} \leq C \| g_{2,3} \|_{L^1_t L^2_x} \leq C e^{1-\frac{1}{r}} (M_3 + M_4)^2.
\]

The nonlinear term involving \( g_1 \) is estimated by the sum of two computations thanks to the bound \( L^4 \). The first computation is completed with the help of \( 26 \),

\[
\left\| \int_0^t e^{-i(t-s)H} Q |\phi(\omega)|^{2p} |y| \, ds \right\|_{L^6_t L^{\infty}_x \cap L^{\infty}_t L^2_x} \leq C \| < n >^{3} |\phi(\omega)|^{2p} |y| \|_{L^2_t L^2_x} \leq C (M_3 + M_4) M_6,
\]

\[
\| < n >^{3+\sigma} |\phi(\omega)|^{2p} \|_{L^{\infty}_n L^2_x} \leq C (M_3 + M_4) M_6,
\]

\[
\| < n >^{-\sigma} y \|_{L^n_t L^2_x} \leq C (M_3 + M_4) M_6,
\]

\[
\| \eta \|_{L^4_t L^4_x} \leq C (M_3 + M_4) M_6.
\]
whereas the second computation is completed with the help of (21),

\[
\left\| \int_0^t e^{-i(t-s)H} Q|y|^{2p+1} ds \right\|_{L^2L^\infty L^2} \leq C\|y\|^{2p+1}_{L^2L^\infty L^{2p+1}n} \leq C\|y\|^{2p+1}_{L^2L^\infty L^{2p+1}n} \leq C(M_2M_3^{2p-1} + (M_1 + M_2)^{2p+1}) ,
\]

provided \( p \geq 3 \) holds. We conclude that the estimates for \( M_1 \) and \( M_2 \) are the same as the one for \( M_3 \).

## 5 Numerical results

We now add some numerical computations which illustrate the asymptotic stability result of Theorem \[ \text{[1]} \]. In particular, we shall obtain numerically the rate, at which the localized perturbations approach to the asymptotic state of the small discrete soliton. One advantage of numerical computations is that they are not limited to the case of \( p \geq 3 \) (which is the realm of our theoretical analysis above), but can be extended to arbitrary \( p \geq 1 \). In what follows, we illustrate the results for \( p = 1 \) (the cubic DNLS), \( p = 2 \) (the quintic DNLS), and \( p = 3 \) (the septic DNLS).

Let us consider the single-node external potential with \( V_n = -\delta_{n,0} \) for any \( n \in \mathbb{Z} \). This potential is known (see Appendix A in [12]) to have only one negative eigenvalue at \( \omega_0 < 0 \), the continuous spectrum at \([0, 4]\), and no resonances at 0 and 4, so it satisfies assumptions (V1)–(V3). Explicit computations show that the eigenvalue exists at \( \omega_0 = 2 - \sqrt{5} \) with the corresponding eigenvector \( \psi_{0,n} = e^{-\kappa|n|} \) for any \( n \in \mathbb{Z} \), where \( \kappa = \text{arcsinh}(2^{-1}) \). The stationary solutions of the nonlinear difference equation (2) exist in a local neighborhood of the ground state of \( H = -\Delta + V \), according to Lemma \[ \text{[1]} \]. We shall consider numerically the case \( \gamma = -1 \), for which the stationary solution bifurcates to the domain \( \omega < \omega_0 \). Figure \[ \text{[1]} \] illustrates the stationary solutions for \( p = 1 \) and two different values of \( \omega \), showcasing its increased localization (decreasing width and increasing amplitude), as \( \omega \) deviates from \( \omega_0 \) towards the negative domain.

In order to examine the dynamics of the DNLS equation (1) we consider single-node initial data \( u_n = A\delta_{n,0} \) for any \( n \in \mathbb{Z} \), with \( A = 0.75 \), and observe the temporal dynamics of the solution \( u(t) \). The resulting dynamics involves the asymptotic relaxation of the localized perturbation into a discrete soliton after shedding of some “radiation”. This dynamics was found to be typical for all values of \( p = 1, 2, 3 \). In Figure \[ \text{2} \] upon suitable subtraction of the phase dynamics, we illustrate the approach of the wave profile to its asymptotic form in the \( l^\infty \) norm. The asymptotic form is obtained by running the numerical simulation for sufficiently long times, so that the profile has relaxed to the stationary state. Using a fixed-point algorithm, we identify the stationary state with the same \( l^2 \) norm (as the central portion of the lattice) and confirm that the result of further temporal dynamics is essentially identical to the stationary state. Subsequently the displayed \( l^\infty \) norm of the deviation from the asymptotic profile is computed, appropriately eliminating the phase by using the gauge invariance of the DNLS equation (1).

We have found from Figure \[ \text{2} \] in the cases \( p = 3 \) (top panel), \( p = 2 \) (middle panel) and \( p = 1 \) (bottom panel) that the approach to the stationary state follows a power law which is well approximated as \( \propto t^{-3/2} \). The dashed line on all three figures represents such a decay in
Figure 1: Two profiles of the stationary solution of the nonlinear difference equation (2) for $V_n = -\delta_{n,0}, p = 1$, and for $\omega = -2$ (solid line with circles) and $\omega = -5$ (dashed line with stars).

each of the cases. We note that the decay rate observed in numerical simulations of the DNLS equation (1) is faster than the decay rate $\propto t^{-1/6-p}$ for any $p > 0$ in Theorem 1.

A Proof of Lemma 7

For the proof of Lemma 7 we will have to show both the “high frequency” estimate (28) and the “low frequency” estimate (29). To simplify notations, we drop the bold-face font for vectors on $\mathbb{Z}$ in the appendix.

A.1 Proof of (28)

Recall the finite Born series representation of $R_V$

$$R(\omega) = R_0(\omega) - R_0(\omega)V R_0(\omega) + R_0(\omega)V R(\omega) V R_0(\omega),$$

which is basically nothing but the resolvent identity iterated twice. We have shown in [18] that for the “sandwiched resolvent” $G_{U,W}(\omega) = U R_V(\omega) W$, we have the bounds (see estimate (33) in [18])

$$\sup_{\theta \in [-\pi, \pi]} \sum_m |G_m(\omega)| + \left| \frac{d}{d\theta} G_m(\omega) \right| \leq C \|U\|_{l_\sigma^2} \|W\|_{l_\sigma^2},$$

for any fixed $\sigma > \frac{5}{2}$, where $\omega = 2 - 2 \cos(\theta)$.

For the three pieces arising from (33), similar arguments apply. Starting with the free
Figure 2: Evolution for $p = 3$ (top), 2 (middle), 1 (bottom) of $\|u(t) - e^{-i\theta(t)\phi(\omega_\infty)}\|$ as a function of time in a log-log scale (solid) and comparison with a $t^{-3/2}$ power law decay (dashed) as a guide to the eye.
resolvent term, we have

$$\sup_{n \in \mathbb{Z}} \int_0^4 \chi |(R_0^\pm(\omega)f)_n|^2 d\omega \leq C \sup_{n \in \mathbb{Z}} \int_{-\pi}^{\pi} \frac{\chi}{\sin(\theta)} \left| \sum_{m \in \mathbb{Z}} e^{i\theta|m-n|} f_m \right|^2 d\theta \leq C \sup_{n \in \mathbb{Z}} \int_{\theta_0/2, \pi-\theta_0/2}^{\theta_0/2, \pi-\theta_0/2} \left( \left| \sum_{m \geq n} e^{i\theta m} f_m \right|^2 + \left| \sum_{m \leq n} e^{-i\theta m} f_m \right|^2 \right) d\theta.$$

Introducing the sequence

$$(g^n)_m := \begin{cases} f_m & m \geq n \\ 0 & m < n \end{cases}$$

we see that the last expression is simply $C(\|g^n\|_{L^2[\theta_0/2, \pi-\theta_0/2]}^2 + \|f - g^n\|_{L^2[\theta_0/2, \pi-\theta_0/2]}^2)$, which is equal by Plancherel's identity to

$$C\|g^n\|_{L^2}^2 + \|f - g^n\|_{L^2}^2 \leq 2C\|f\|_{L^2}^2.$$

For the second piece in (33), we use that $\|R_0^\pm(\omega)\|_{L^1 \to L^\infty} \leq C/\sin(\theta)$ and $|\sin(\theta)| \geq C_0$ on $[\theta_0/2, \pi-\theta_0/2]$ for some $C_0 > 0$, to conclude

$$\sup_{n \in \mathbb{Z}} \|\chi R_0^\pm(\omega)V R_0^\pm(\omega)f\|_{L^2_{\omega}(0,4)}^2 \leq \int_{-\pi}^{\pi} \frac{\chi}{\sin^2(\theta)} \left( \sum_{n \in \mathbb{Z}} |V_n| \|R_0^\pm(\omega)f_n\| \right)^2 d\theta \leq C\|V\|_{L^1} \sup_{n \in \mathbb{Z}} \int_{-\pi}^{\pi} \chi |(R_0^\pm(\omega)f)_n|^2 d\theta,$$

by the triangle inequality. At this point, we have reduced the estimate to the previous case, provided that $V \in L^1$.

For the third piece in (33), we make use of (34). We have, similar to the previous estimate,

$$\sup_{n \in \mathbb{Z}} \|\chi R_0^\pm(\omega)V R_0^\pm(\omega)V R_0^\pm(\omega)f\|_{L^2_{\omega}(0,4)}^2 =$$

$$\sup_{n \in \mathbb{Z}} \|\chi R_0^\pm(\omega)V R_0^\pm(\omega)|V|^{1/2} \text{sgn}(V) V^{1/2} R_0^\pm(\omega)f\|_{L^2_{\omega}(0,4)}^2 =$$

$$\sup_{n \in \mathbb{Z}} \|\chi R_0^\pm(\omega)G_{V,|V|^{1/2} \text{sgn}(V)} |V|^{1/2} R_0^\pm(\omega)f\|_{L^2_{\omega}(0,4)}^2 \leq C\|V\|_{L^2_{\omega}}^2 \|V\|_{L^2_{\omega}}^{1/2} \|V\|_{L^2_{\omega}}^{1/2} \sup_{n \in \mathbb{Z}} \int_{-\pi}^{\pi} \chi |(R_0^\pm(\omega)f)_n|^2 d\theta,$$

where in the last inequality, we have again reduced the estimate to the first case.

### A.2 Proof of (29)

We only consider the interval $[-\theta_0, \theta_0]$ in the compact support of $\chi_0(\theta)$ since the arguments for other intervals are similar. Following the algorithm in [15] and the formalism in [18], we let $\psi^\pm(\theta)$ be two linearly independent solutions of

$$\psi_{n+1} + \psi_{n-1} + (\omega - 2)\psi_n = V_n \psi_n, \quad n \in \mathbb{Z}, \quad (35)$$
according to the boundary conditions $|\psi_n^\pm - e^{\mp in\theta}| \to 0$ as $n \to \pm \infty$. Let $\psi_n^\pm(\theta) = e^{\mp in\theta}\Psi_n^\pm(\theta)$ for all $n \in \mathbb{Z}$. Using the Green function representation, we obtain

$$
\Psi_n^+(\theta) = 1 - \frac{i}{2\sin\theta} \sum_{m=n}^{\infty} \left(1 - e^{-2i\theta(m-n)}\right) V_m \Psi_m^+(\theta),
$$

$$
\Psi_n^-(\theta) = 1 - \frac{i}{2\sin\theta} \sum_{m=-\infty}^{n} \left(1 - e^{-2i\theta(n-m)}\right) V_m \Psi_m^-(\theta).
$$

The discrete Green function for the resolvent operators $R^\pm(\omega)$ has the kernel

$$
[R_V^\pm(\omega)]_{n,m} = \begin{cases} 
\frac{1}{W(\theta_\pm)} \left( \psi_n^+(\theta_\pm)\psi_m^-(\theta_\pm) \right) & \text{for } n \geq m \\
\psi_n^-(\theta_\pm)\psi_m^+(\theta_\pm) & \text{for } n < m
\end{cases}
$$

where $\theta_- = -\theta_+, \theta_- \in [0, \pi]$ for $\omega \in [0, 4]$, and $W(\theta) = W[\psi^+, \psi^-] = \psi_n^+\psi_{n+1}^- - \psi_{n+1}^+\psi_n^-$ is the discrete Wronskian, which is independent of $n \in \mathbb{Z}$. We need to estimate

$$
\|\chi_0 R_V^\pm(\omega)f\|_{L^2_+(0,4)}^2 = \int_{-\pi}^{\pi} \frac{2\sin\theta d\theta}{2\omega^2} \left( \sum_{m=-\infty}^{n-1} \psi_n^-(\theta)\psi_m^+(\theta)f_m + \sum_{m=n}^{\infty} \psi_n^-(\theta)\psi_m^+(\theta)f_m \right)^2.
$$

We may assume that $n \geq 1$ for definiteness and split

$$
\sum_{m=-\infty}^{n-1} \psi_n^-(\theta)f_m = \sum_{m=0}^{n-1} \psi_n^-(\theta)f_m + \sum_{m=-\infty}^{n-1} e^{im\theta} f_m + \sum_{m=-\infty}^{n-1} e^{-im\theta} (\Psi_m^--1)f_m := I_1 + I_2 + I_3
$$

and

$$
\sum_{m=n}^{\infty} \psi_n^+(\theta)f_m = \sum_{m=n}^{\infty} e^{-im\theta} f_m + \sum_{m=n}^{\infty} e^{-im\theta} (\Psi_m^+-1)f_m := I_4 + I_5
$$

We are using the scattering theory from [13] to claim that

$$
\sup_{\theta \in [-\theta_0, \theta_0]} \left( \|\psi^+(\theta)\|_{l^\infty(\mathbb{Z}_+)} + \|\psi^-(\theta)\|_{l^\infty(\mathbb{Z}_-)} \right) < \infty,
$$

where $\langle n \rangle = (1 + n^2)^{1/2}$. Then, we have

$$
|I_1| \leq \left( \sum_{m=0}^{n-1} |\Psi_m^-|^2 \right)^{1/2} \left( \sum_{m=0}^{n-1} |f_m|^2 \right)^{1/2} \leq C_1 \langle n \rangle^{3/2} \|f\|_{l^2},
$$

$$
|I_3| \leq \left( \sum_{m=-\infty}^{1} |\Psi_m^- - 1|^2 \right)^{1/2} \left( \sum_{m=-\infty}^{1} |f_m|^2 \right)^{1/2} \leq C_3 \sum_{k=-\infty}^{m} |m-k||V_k|| \|f\|_{l^2},
$$

$$
|I_5| \leq \left( \sum_{m=n}^{\infty} |\Psi_m^+ - 1|^2 \right)^{1/2} \left( \sum_{m=n}^{\infty} |f_m|^2 \right)^{1/2} \leq C_5 \sum_{l=m}^{\infty} |k-m||V_k|| \|f\|_{l^2},
$$

for some $C_1, C_3, C_5 > 0$. We note that

$$
\left\| \sum_{k=-\infty}^{m} |m-k||V_k|| \right\|_{l^2_+(\mathbb{Z}_-)} \leq \sum_{k=-\infty}^{m} |m-k||V_k|| \leq C_4 \|V\|_{l^2_+(\mathbb{Z}_-)}.
$$
for some $C_4 > 0$. Therefore, the brackets in $I_3$ and $I_5$ are bounded if $V \in L^1_{\sigma}$ for $\sigma > \frac{5}{2}$. Since $I_2$ and $I_4$ are given by the discrete Fourier transform, Parseval’s equality implies that

$$\int_{-\pi}^{\pi} (I_2^2 + I_4^2) \, d\theta \leq C_2 \|f\|_{L^2}^2,$$

for some $C_2 > 0$. Using now the fact that $|W(\theta)| \geq W_0$ and $|\sin \theta| \leq C_0$ uniformly in $[-\theta_0, \theta_0]$, the support of $\chi_0(\theta)$, and using the property (36), we obtain

$$\|\chi_0 R_1^2(\omega) \|_{L^2_\omega(0,4)}^2 \leq C \left( 1 + \langle n \rangle^2 + \langle n \rangle^3 \right) \|f\|_{L^2}^2,$$

which gives (29).

References


