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Existence, Stability, and Dynamics of Bright Vortices in the Cubic-Quintic Nonlinear Schrödinger Equation

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Abstract

We study the existence and azimuthal modulational stability of vortices in the two-dimensional (2D) cubic-quintic nonlinear Schrödinger (CQNLS) equation. We use a variational approximation (VA) based on an asymptotically derived ansatz, seeding the result as an initial condition into a numerical optimization routine. Previously known existence bounds for the vortices are recovered by means of this approach. We study the azimuthal modulational stability of the vortices by freezing the radial direction of the Lagrangian functional of the CQNLS, in order to derive a quasi-1D azimuthal equation of motion. A stability analysis is then done in the Fourier space of the azimuthal modes, and the results are analyzed using both the asymptotic variational ansatz and numerically-exact vortices. For unstable vortices, predictions are given for the growth rates of the most unstable azimuthal mode. We also give predictions for the critical value of the frequency, above which all vortices are azimuthally stable. Our predictions are compared to earlier works and corroborated by full 2D simulations. We then briefly study the collisional dynamics between stable vortices of different topological charges.

Key words: Nonlinear optics, nonlinear Schrödinger equation, vortices, modulational instability

\textsuperscript{1}URL: http://nlds.sdsu.edu/
1 Introduction

The cubic-quintic nonlinear Schrödinger (CQNLS) equation is used to model a variety of physical settings. These include the light propagation in optical media such as non-Kerr crystals [17], chalcogenide glasses [29,30], organic materials [33], colloids [28], dye solutions [12], and ferroelectrics [14]. It has also been predicted that this complex nonlinearity can be synthesized by means of a cascading mechanism [10].

The key feature of the CQNLS model is the competition of the focusing (cubic) and defocusing (quintic) nonlinear terms. This allows for the existence of stable multidimensional structures which are definitely unstable in the focusing cubic nonlinear Schrödinger (NLS) equation. In particular, the CQNLS equation supports vortices in the two-dimensional (2D) and three-dimensional (3D) geometry. These are ring-shaped structures which carry a rotational angular phase. The respective integer winding number of the phase structure in a vortex is referred to as its topological charge, \( m \). An example of a localized vortex is depicted in Fig. 1.

![Figure 1](image)

Fig. 1. An example of a numerically found two-dimensional vortex of charge \( m = 5 \) with frequency \( \Omega = 0.16 \) in the CQNLS equation. The “height” of the profiles represents the value of the wave function. The gray profile displays the squared absolute value, while the dark (blue) and light (red) meshes are the real and imaginary parts, respectively.

It is well known that the cubic NLS equation can support “dark” (delocalized) and “bright” (localized) vortices for the defocusing and focusing cases, respectively. Dark vortices of unitary charge are stable, while higher-order ones are unstable (unless stabilized by an external potential [26,24,15,22]). On the other hand, “bright” vortex solutions are always azimuthally unstable in the framework of the cubic NLS equation or its counterpart with the saturable nonlinearity [11], their core breaking down in the azimuthal direction into a few fragments [6,21]. An example of the azimuthal breakup of an unstable vortex is displayed in Fig. 2.

In contrast to this, the CQNLS equation can support azimuthally stable bright vortices [27,31,20], in addition to unstable ones. Moreover, 3D solitons with embedded vorticity \( m = 1 \) also have their stability region in the CQNLS model [23] (see also reviews [8,21]). Azimuthally stable vortices may find their potential applications in the design of all-optical data-processing schemes. In that respect, knowing the growth rates of unstable modes is also important because if the rates are small enough, then the vortices may be considered as practically stable ones, as they will not exhibit an observable instability over the relevant propagation distance.

In this paper we investigate the azimuthal modulational stability (AMS) of vortex solutions in
the 2D CQNLS equation. As mentioned above, there have already been numerous studies on this topic \cite{27,31,8,20,24}. In each case, the analysis aimed to find a critical value ($\Omega_{st}$) of the intrinsic frequency of the vortex, above which all vortices (with a given charge) are stable. In Ref. \cite{27}, 2D azimuthally stable vortices of charge $m = 1$ were shown to exist. It was found that the slope of the vortex profile at the origin peaked at a specific value of $\Omega$, which was considered as the critical frequency, $\Omega_{st}(m = 1) \approx 0.145$. They had also developed a variational approach (VA) and found a similar result by means of this method. They then performed full 2D simulations and found that vortices with a frequency greater than their critical one were indeed stable, while those below the threshold were not. Works \cite{31,8,20} took a different approach. They introduced a small perturbation of the 2D vortex and solved the resulting eigenvalue problem numerically. In this way, the critical frequencies were found as $\Omega_{st}(m = 1) \approx 0.16$ and $\Omega_{st}(m = 2) \approx 0.17$. In Ref. \cite{24}, a 2D perturbation was considered too. Through an extensive analysis, the problem was transformed into finding of numerical zeros of a respective Evans functions for a set of ordinary differential equations (ODEs). Finally, in Ref. \cite{25}, using Gagliardo-Nirenberg and Hölder inequalities together with Pohozaev identities, it was shown that the solutions to the cubic-quintic NLS possess an upper cutoff value. In this way, the existence of azimuthally stable vortices of all integer values of $m$ was predicted (in Refs. \cite{31} and \cite{27}, the stability regions for $m \geq 3$ were not found, as they are extremely small). The predictions for the critical frequencies reported in Ref. \cite{24}, which are somewhat different from those in Refs. \cite{31} and \cite{27}, are shown in Table 2.

We feel that an additional study of the AMS of vortices in the CQNLS equation is justified for two reasons. Firstly, the previous studies demonstrate some disagreement in the predictions of the critical frequency. Therefore, since we will use a different approach, our results could be of use in comparing the previous findings. Secondly, although in Ref. \cite{27} 2D simulations were performed for the vortices, they were only done for $m = 1$, and further simulations later showed that vortices thought to be stable in those simulations were in fact unstable in longer simulations \cite{31}. Therefore, additional numerical examination is needed for the comparison with the various AMS predictions. In fact, our numerical results will illustrate that the predictions of Ref. \cite{24} are the most accurate.
the critical frequency are less accurate when compared to the simulations. An additional important result of the analysis is that our analytic expression produced by the VA for the vortex solution profiles is extremely close to the true profile. This analytic profile can be used to derive predictions of the AMS, as well as to confirm the existence bounds of the vortex solutions.

The paper is organized as follows. In Sec. 2 we briefly describe the CQNLS equation and the general form of steady-state vortex solutions. Then, in Sec. 3, we derive an approximate analytic description for the vortex profiles by first finding a one-parameter asymptotic profile and then using the VA to find required parameters. We also study the existence bounds of the vortices. In Sec. 4 we refine our variational ansatz into numerically-exact vortex profiles, using a nonlinear optimization routine. We then compare the true profiles with our variational ansatz to show its accuracy. In Sec. 5 we formulate a quasi-1D equation of motion for the dynamics along the azimuthal direction of the vortices. A linear stability analysis is performed to find stability criteria and growth rates of unstable modes. Analytic predictions are then obtained using our variational ansatz. In Sec. 6 we present full 2D simulations of vortices and respective results for the stability. These results are compared to our predictions, as well as to the predictions produced by previous studies. In Sec. 7, we explore collisions between stable vortices. Finally, in Sec. 8, we summarize our findings and give some concluding remarks.

2 The Cubic-Quintic Nonlinear Schrödinger Equation and Vortex Solutions

The scaled form of CQNLS equation for wave function $\Psi$ (the amplitude of the electromagnetic wave in an optical medium) is well known:

$$i\Psi_t + \nabla^2 \Psi + |\Psi|^2 \Psi - |\Psi|^4 \Psi = 0,$$

(1)

where $\nabla^2$ is the two-dimensional Laplacian, and the last two terms represent the focusing cubic and defocusing quintic nonlinearities (in optics, evolution variable $t$ is not time, but rather the propagation distance). This combination of the nonlinearities implies that the equation may give rise to “bright” vortices, built on top of the zero background. In the polar coordinates, i.e., the natural coordinate system for the study of vortices, the Laplacian takes on the well-known form:

$$\nabla^2 \Psi = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \Psi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \Psi}{\partial \theta^2}.$$  

(2)

A steady-state vortex is described in general as

$$\Psi(r, \theta, t) = f(r) A(\theta, t),$$

(3)

where a real function $f(r)$ is a steady state radial profile, and

$$A(\theta, t) = e^{i(m\theta + \Omega t)},$$

(4)

where $m$ is the topological charge, and $\Omega$ is the frequency. Finding profiles $f(r)$ for different values of $m$ and $\Omega$ is not trivial, no analytical solution being available for it. In addition, $f(r)$ does not exist for certain values of $\Omega$, as seen in Sec. 3.3. We begin our study by finding an approximate analytic expression for $f(r)$ and studying its existence bounds.
3 Approximate Analytical Steady-State Vortex Profiles

In this section, we develop an approximate analytical solution for the vortex profile in the CQNLS equation. Our profile turns out to be very close to the true solution (as will be shown in Sec. 4.2), and therefore can be used to predict existence regions and make analytical predictions for the AMS.

3.1 Asymptotic Vortex Profile in the Two-Dimensional CQNLS Equation

We start by inserting the general form of a vortex as described in Eq. (3) into the 2D CQNLS equation (1), which yields the following ODE for the radial profile of the vortex with charge $m$:

$$- \left( \Omega + \frac{m^2}{r^2} \right) f(r) + \frac{1}{r} \frac{d}{dr} \left( r \frac{df}{dr} \right) + f^3(r) - f^5(r) = 0. \tag{5}$$

If we assume that the vortex has a large radius, then the region of interest in the evolution equation has $r \gg 1$, in which case variable $r$ may be approximately replaced by a constant (denoted $r_c$), which we take to be a point where the radial profile attains its maximum. Since we assume $r_c \gg 1$, we neglect the $1/r_c$ term in the Laplacian. The consequences of eliminating this term is that the approximate profiles that we will derive will be less accurate for smaller values of $r_c$ as will be discussed in Sec. 4.2. With this large radius assumption, Eq. (5) becomes

$$- \Omega^* f(r) + \frac{d^2 f}{dr^2} + f^3(r) - f^5(r) = 0, \tag{6}$$

where we have defined

$$\Omega^* = \Omega + \frac{m^2}{r_c^2}. \tag{7}$$

Assuming $r_c \gg 1$, we can treat Eq. (6) as if $r \in (-\infty, \infty)$, in which case it is known to have a closed-form analytic solution [7],

$$f_{\text{asy}}^2(r) = \frac{4\Omega^*}{1 + \sqrt{1 - (16/3)\Omega^* \cosh \left(2\sqrt{\Omega^*} (r - r_c)\right)}}. \tag{8}$$

It is apparent that the solution does not exist for some values of $\Omega^*$, hence $\Omega^*$ is bounded by the existence region, $\Omega^* \in [0, \Omega_{1D}^{\text{max}}]$ where $\Omega_{1D}^{\text{max}} = 3/16 = 0.1875$. At point $\Omega = \Omega_{1D}^{\text{max}}$, the solution is simply a constant, $f(r) = f_0 = \sqrt{3}/4$. Solution (8) describes a sech-shaped curve which flattens out as $\Omega^* \to \Omega_{1D}^{\text{max}}$. The family of solutions parameterized by $\Omega^*$ is depicted in Fig. 3.

What is needed now, is an expression for the radius of the profile, $r_c$, and the shifted frequency, $\Omega^*$. Finding one of the two automatically gets us the other from the expression in Eq. (7). To find these parameters, we use the following VA.
Fig. 3. Steady-state solitary solutions to Eq. (6) for various values of $\Omega^*$. Profiles for $\Omega^* \in [0.05, 0.1, 0.15, 0.18, 0.1874, 0.1874999, 0.1875]$ are shown from bottom to top. We see that as $\Omega^*$ increases, the profile flattens out. At the maximum value of $\Omega$, it becomes a constant.

3.2 Variational Approach

In order to apply the VA to finding approximate solutions for the radial profile of the vortex, we start by inserting the general vortex solution (3) into the Lagrangian density of the CQNLS equation,

$$L = \frac{i}{2} (\Psi \Psi^*_t - \Psi^* \Psi_t) + |\Psi_r|^2 + \frac{1}{r^2} |\Psi_\theta|^2 - \frac{1}{2} |\Psi|^4 + \frac{1}{3} |\Psi|^6, \quad (9)$$

which yields

$$L(r, \theta) = \left( \Omega + \frac{m^2}{r^2} \right) f^2(r) + \left( \frac{df}{dr} \right)^2 - \frac{1}{2} f^4(r) + \frac{1}{3} f^6(r). \quad (10)$$

Since $f(r)$ is a steady state, the radial integrals over $f(r)$ in the Lagrangian depend only on $m$ and $\Omega$, hence we can write the Lagrangian as

$$L = 2\pi \left( \Omega C_1 + m^2 C_3 + C_2 - \frac{1}{2} C_4 + \frac{1}{3} C_5 \right), \quad (11)$$

where

$$C_1 = \int_0^\infty f^2(r) r \, dr, \quad C_2 = \int_0^\infty \left( \frac{df}{dr} \right)^2 r \, dr, \quad C_3 = \int_0^\infty \frac{1}{r^2} f^2(r) r \, dr, \quad (12)$$

$$C_4 = \int_0^\infty f^4(r) r \, dr, \quad C_5 = \int_0^\infty f^6(r) r \, dr.$$

We use our asymptotic approximate profile (8) as an ansatz, treating $\Omega^*$ and $r_c$ as variational parameters. Inserting Eq. (8) into the integral terms in the Lagrangian, and using the large-
radius approximation yields \[5\]

\[C_1 \approx 2\sqrt{3} T r_c, \quad C_2 \approx r_c \left[\frac{3}{9} \sqrt{16\Omega^* - \sqrt{3} T \left(\frac{3}{16} - \Omega^*\right)}\right],\tag{13}\]

\[C_3 \approx \frac{2\sqrt{3} T}{r_c}, \quad C_4 \approx -3 r_c \left[\sqrt{16\Omega^* - \frac{3}{2} T}\right],\]

\[C_5 \approx -r_c \left[\frac{27}{8} \sqrt{16\Omega^* - \sqrt{3} T\left(\frac{27}{16} - 3\Omega^*\right)}\right],\]

where

\[T = \arctanh \left[\sqrt{\frac{3}{16\Omega^*}} - \sqrt{\frac{3}{16\Omega^*} - 1}\right],\tag{14}\]

The respective Euler-Lagrange equations \(\partial L/\partial \Omega^* = 0\) and \(\partial L/\partial r_c = 0\) yield the following system of coupled algebraic equations for \(\Omega^*\) and \(r_{cva}\):

\[r_{cva} = m \left[\Omega - \frac{3}{16} + \frac{1}{2T} \sqrt{3 \frac{16\Omega^*}{16\Omega^* - 1}}\right]^{-1/2},\tag{15}\]

\[\Omega^* = \Omega + \frac{m^2}{(r_{cva})^2},\tag{16}\]

where Eq. (16) is tantamount to the definition of \(\Omega^*\) in Eq. (7). Eliminating \(r_{cva}\) from these equations one obtains that \(\Omega^*\) is, for a chosen value of \(\Omega\), the solution to the following equation,

\[\Omega = G(\Omega^*) \equiv \frac{\Omega^*}{2} + \frac{3}{32} - \frac{1}{4T} \sqrt{3 \frac{16\Omega^*}{16\Omega^* - 1}}.\tag{17}\]

This yields the final form of the VA vortex radial profile:

\[f_{cva}^2(r) = \frac{4\Omega^*}{1 + \sqrt{1 - (16/3)\Omega^* \cosh \left[2\sqrt{16\Omega^* \left(r - r_{cva}\right)}\right]}},\tag{18}\]

where the vortex radius is

\[r_{cva} = m \frac{1}{\sqrt{1 - G(\Omega^*)}}.\tag{19}\]

We note that since Eq. (17) is transcendental, we need to use a numerical root-finder to solve for the variational parameters, for which we use a simple bisection method with a tolerance of \(10^{-15}\).

In Fig. 4 we show \(r_{cva}\) versus \(\Omega\) and \(\Omega^*\) for \(m = 5\), and the relationship of Eq. (17). We see that, with the increases of \(\Omega\) (and \(\Omega^*\)), the radius of the vortex starts from very large values (infinite at \(\Omega = 0\)), decreases until it reaches a minimum value, and then increases rapidly (it becomes infinite once again at \(\Omega = \Omega_{\max}^{2D}\)). We also see that the relationship between \(\Omega\) and \(\Omega^*\) does not depend on charge \(m\), and that \(\Omega \rightarrow 0.1875\) as \(\Omega^* \rightarrow 0.1875\) (the apparent gap near the right edge of the plot is due to the sensitivity of the relationship which we will now discuss).
3.3 Existence Bounds for Two-Dimensional Vortex Profiles

From numerical investigations performed in Refs. [27] and [9], the existence border the two-dimensional CQNLS equation vortex solution was found to be \( \Omega_{2D}^{\max} \approx 0.180 \). Later, it was shown analytically in Ref. [24] that

\[
\Omega_{2D}^{\max} = \Omega_{1D}^{\max} = 3/16 = 0.1875.
\]  

(20)

The numerical discrepancy can possibly be explained by noticing the sensitivity of the relationship between \( \Omega \) and \( \Omega^* \) produced by the VA in the form of Eq. (17). From Eqs. (17) and (14) we see that as \( \Omega^* \to 0.1875, \) \( T \to \infty, \) and so \( \Omega = G(\Omega^*) \to \Omega^*/2 + 3/32 = 0.1875. \) This confirms that \( \Omega_{2D}^{\max} = \Omega_{1D}^{\max} = 0.1875. \) However, as we have seen in the right panel of Fig. 4, the relationship seems to be extremely sensitive for larger values of \( \Omega \) (see the inset in the right panel of Fig. 4). This sensitivity can be observed in Table 1 where some values of \( \Omega^* \) and their corresponding \( G(\Omega^*) = \Omega \) are shown. As is clearly seen, one quickly approaches the limit of the machine precision in \( \Omega^* \) as \( \Omega \to 0.1875. \) From a functional point of view, this can also be understood from the logarithmic divergence of \( T(\Omega^*), \) since \( \text{arctanh}(x) = (1/2) \log[(1 + x)/(1 - x)], \) as the the relevant maximum value is approached. It is not surprising then that numerical estimates of \( \Omega_{2D}^{\max} \) (obtained by means of a shooting method looking for profiles at different values of

<table>
<thead>
<tr>
<th>( \Omega^* )</th>
<th>( \Omega = G(\Omega^*) )</th>
</tr>
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<tbody>
<tr>
<td>0.1874</td>
<td>0.1664...</td>
</tr>
<tr>
<td>0.187499</td>
<td>0.1736...</td>
</tr>
<tr>
<td>0.18749999</td>
<td>0.1783...</td>
</tr>
<tr>
<td>0.187499999</td>
<td>0.1806...</td>
</tr>
<tr>
<td>0.187499999999</td>
<td>0.1823...</td>
</tr>
</tbody>
</table>

Table 1
Evaluation of \( \Omega = G(\Omega^*) \) near \( \Omega = 0.1875. \) We show four significant digits in \( G(\Omega^*). \) The extreme sensitivity of the relationship is easily observed.
\( \Omega \) gave lower estimates of the existence bound. In Sec. 4.1, we confirm that the existence bound of \( \Omega_{\text{max}}^{2D} \) is greater than the numerical estimates of Refs. [27] and [9] by finding numerically exact vortex profiles for \( \Omega > 0.180 \). Now that we here discussed the analytical approximations to the vortex profile, we proceed to numerically refine them into true (up to a prescribed numerical accuracy) solutions of the relevant ordinary differential equation [5].

4 Numerically-Exact Steady State Vortex Profiles

4.1 Numerical Steady-State Profile Results

To find numerically-exact profiles of the solutions, we use a modified Gauss-Newton (GN) optimization routine with a tolerance between \( 10^{-8} \) and \( 10^{-4} \) [6]. In Fig. 5, we show some radial profiles computed by dint of our GN routine for various values of \( \Omega \), with charge \( m = 3 \). We show some profiles with \( \Omega > 0.18 \), including one with \( \Omega \) as high as \( \Omega = 0.187 \), which is consistent with the existence bound of \( \Omega_{\text{max}}^{2D} = 0.1875 \), as reported in Ref. [24] and predicted by our variational approach. We see that as \( \Omega \) grows, the profiles flatten out. The further increase in \( \Omega \) pushes the profile farther from \( r = 0 \), but does change its width or height.

Fig. 5. Steady-state vortex radial profiles computed using the GN routine for \( m = 3 \) and \( \Omega \in [0.01, 0.18] \), with \( \Delta \Omega = 0.01 \), as well as for \( \Omega = 0.184 \) and \( \Omega = 0.187 \). For all the profiles, we used our variational ansatz as an initial condition (not shown here). The radial direction was discretized with a grid spacing of \( \Delta r = 1 \) and use a stopping tolerance of \( 10^{-5} \) in the GN routine. We see that as \( \Omega \) grows, the profiles flatten out due to the increase of the vortex’ radius.
### 4.2 Comparison Between the Variational Approximation and Numerically-Exact GN Vortex Profiles

Here we show the accuracy of our VA ansatz by comparing it to the numerically exact solutions computed with the GN routine. In Fig. 6 we compare the radius $r_c$ as given by the VA and GN profiles for $m = 5$. We observe that the values of the radius provided by the VA and GN are extremely close, allowing one to use the VA radius as an accurate prediction of the true radius of a vortex, that could be useful in experimental situations.

![Fig. 6. The vortex radius $r_c$ versus $\Omega$ for the variational profile ($r_{c}^{va}$, solid line) and the GN profile ($r_{c}^{gn}$, crosses) for $m = 5$. The VA and GN profiles are extremely close for all the possible $\Omega$ values.](image)

To get an even better idea of how close the VA is to the GN profile overall, in Fig. 7 we compare the relative sum of squared errors between our VA profiles and the numerically exact GN profiles for different charges and different values of $\Omega$. We also plot a series of profiles for the VA and GN for $\Omega = 0.15$. We see that, as $m$ increases, the error between the VA and the GN profiles decreases. This is understandable in that we had neglected the $1/r_c$ term in the Laplacian in formulating the VA ansatz, and since lower charge vortices have lower central radii, the discrepancy for lower charges between the VA ansatz and the numerically exact solution are expected. The total error is observed to be quite low overall, showing that our VA profile provides for an extremely close description of the true solution, especially so for larger values of $m$. Since our VA provides an accurate profile, we can justifiably use it to analyze the AMS, see Sec. 5.3.

### 5 Azimuthal Modulational Stability: Theory

Now that steady-state vortices in the 2D CQNLS equation are available, we can study their azimuthal modulational stability (AMS). To this end, we apply the methodology utilized in a previous work for the cubic NLS equation [6]. We derive an azimuthal equation of motion
by assuming a frozen radial profile, and then perform a perturbation analysis to predict the stability of azimuthal perturbation modes.

We should note that, in Ref. [6], numerically computed eigenvectors of the vortex solutions were seen to have slight coupling in the radial and azimuthal directions, and therefore assuming perfect separability of the radial and azimuthal directions for the evolution of a vortex is not entirely accurate, and can lead to discrepancies between the AMS predictions and the numerical results. Nonetheless, the coupling between radial and azimuthal directions seems to decrease as one increases charge \( m \). Thus predictions made assuming the separability (between radial and azimuthal directions) increase in their accuracy for higher charges. This increase in accuracy with charge can be observed in the numerical results presented in Sec. [6].

5.1 The Azimuthal Equation of Motion

To derive the azimuthal equation of motion, we first insert the separable ansatz (3) into the Lagrangian density of the CQNLS equation (9) to yield:

\[
\mathcal{L} = f^2(r) \left( \frac{i}{2} (AA_t^* - A^*A_t) + \left( \frac{df}{dr} \right)^2 |A|^2 + \frac{1}{r^2} f^2(r) |A_\theta|^2 - f^4(r) \frac{1}{2} |A|^4 + f^6(r) \frac{1}{3} |A|^6 \right).
\]

To compute the Lagrangian, one must integrate its density over all space. Since our radial profile is steady-state and uncoupled to the azimuthal direction, the integration over the radial direction causes all terms containing \( f(r) \) to become constants. Thus we can essentially remove the radial direction and are left with a quasi-1D (in \( \theta \)) Lagrangian that can be used to derive the equation of motion for \( A(\theta, t) \):

\[
L_{1D} = \int_0^{2\pi} \mathcal{L}_{1D} \, d\theta,
\]
\[ \mathcal{L}_{1D} \equiv \frac{i}{2} C_1 (AA^* - A^*A_t) + C_2 |A|^2 + C_3 |A_\theta|^2 - \frac{s_1}{2} C_4 |A|^4 - \frac{s_2}{3} C_5 |A|^6, \]  

(21)

where \( C_j, j = 1, ..., 5 \), are the radial integrals of the steady-state profile \( f(r) \) as defined in Eq. (12). Evaluating the variational derivative of the action functional \[19\], and applying the rescalings of \( A \rightarrow A \exp(-iC_2 t/C_1) \) and \( t \rightarrow C_3 t/C_1 \), yields the evolution equation for \( A(\theta,t) \):

\[ iA_t = -A_{\theta\theta} - \frac{C_4}{C_3} C_3 |A|^2 A + \frac{C_5}{C_3} |A|^4 A. \]  

(22)

We now have a description of how a radially-frozen, azimuthal time-dependent solution will evolve in the CQNLS equation.

5.2 Stability Analysis

To study the AMS of vortex solutions to the CQNLS equation, we performed a perturbation analysis for the azimuthal equation of motion (22), with the objective to compute the growth rates of a small complex modulated perturbation. We begin with an azimuthal “plane wave” solution (whose amplitude has been absorbed into \( f(r) \)) perturbed by a complex, time-dependent perturbation:

\[ A(\theta, t) = (1 + u(\theta, t) + iv(\theta, t)) e^{i(m\theta + \Omega t)}, \]  

(23)

where \(|u|, |v| \ll 1\). Inserting this into Eq. (22) and separating the result into real and imaginary parts, we get a system of coupled partial differential equations describing the evolution of the perturbation \( u(\theta, t) \) and \( v(\theta, t) \), which read (after rescaling \( t \rightarrow t + \theta/(2m) \)):

\[ u_t = -v_{\theta\theta} + \text{[hot]}, \]

\[ v_t = u_{\theta\theta} + \left( \frac{2C_4 - 4C_5}{C_3} \right) u + \text{[hot]}, \]  

(24)

where \([\text{hot}]\) refers to higher order terms (which do not concern us as we are performing the linear stability analysis).

In order to study the AMS of the vortices, we obtain amplitude equations for the azimuthal modes by expanding \( u \) and \( v \) in the discrete Fourier series. The amplitudes of the perturbation for any specific azimuthal mode, \( K \), are given by the transforms:

\[ \hat{u}(K, t) = \int_{0}^{2\pi} u(\theta, t) e^{iK\theta} d\theta, \quad \hat{v}(K, t) = \int_{0}^{2\pi} v(\theta, t) e^{iK\theta} d\theta. \]  

(25)

Applying these transforms to Eq. (24) yields two coupled nonlinear ODEs describing the dynamics for the amplitudes of \( u \) and \( v \) for each mode. The linearized ODEs in a matrix form are

\[ \frac{d}{dt} \begin{bmatrix} \hat{u} \\ \hat{v} \end{bmatrix} = \begin{bmatrix} 0 & K^2 \\ \left( \frac{2C_4 - 4C_5}{C_3} - K^2 \right) & 0 \end{bmatrix} \begin{bmatrix} \hat{u} \\ \hat{v} \end{bmatrix}. \]  

(26)
The growth rates for each azimuthal mode $K$ are simply the eigenvalues of Eq. (26) which, taking into account the rescaling of time, are:

$$\lambda_{1,2} = \pm \frac{C_3}{C_1} \sqrt{K^2 (K_{\text{crit}}^2 - K^2)},$$

(27)

where $K_{\text{crit}}$ is the critical value of $K$, below which all modes are stable:

$$K_{\text{crit}} \equiv \sqrt{\frac{2C_4 - 4C_5}{C_3}}.$$ 

(28)

For the azimuthal moditionally unstable vortices, it may be useful to know the maximum growth rate and the mode that exhibits it. This is because, in the experiment, even an unstable vortex may be “stable enough” for the practical use, if the maximum perturbation growth rate is small. These can be found simply from Eq. (27) by equating its derivative to zero and solving for $K$, which reveals the fastest growing perturbation mode (and subsequently, the prediction of the number of fragments that the unstable vortices will break up into):

$$K_{\text{max}} = \sqrt{\frac{C_4 - 2C_5}{C_3}} = \frac{1}{\sqrt{2}} K_{\text{crit}}.$$ 

(29)

We can then insert this value into Eq. (28), which yields a maximum growth rate,

$$\lambda_{\text{max}} = \frac{C_4 - 2C_5}{C_1} = \frac{C_3}{C_1} \frac{1}{2} (K_{\text{crit}})^2.$$ 

(30)

For a vortex to be azimuthally stable for all modes, $K_{\text{crit}}$ needs to be either less than 1 or purely imaginary. These two criteria for stability are:

$$K_{\text{crit}} \in \Im : C_4 - 2C_5 < 0,$$ 

(31)

and

$$|K_{\text{crit}}| < 1 : C_4 - 2C_5 - \frac{C_3}{2} < 0.$$ 

(32)

Since $C_3$ is positive, the first condition is stricter than the second one. Although the relevant condition for the azimuthal periodic problem corresponds to the second one, we keep the first condition (relevant to the infinite line) because it leads to an estimation of the critical frequency that is independent of the charge $m$ (see below). It should be noted that the constants in these conditions depend on $m$ and $\Omega$, as these two parameters determine the shape of the radial profiles. Therefore, it is possible to use Eqs. (31) and (32) to determine the critical value of frequency $\Omega_{\text{st}}(m)$, above which all vortices are azimuthally stable.

5.3 Analytical Stability Predictions Using the VA Profile

As we saw in Sec. 4.2, our VA ansatz is a useful approximation to the true radial profiles. As such, we can use the VA to formulate expressions for the constants in Eq. (13), and thereby obtain analytical predictions for the AMS. For studying the AMS of unstable vortices, we see
from Sec. 5.2 that the expressions for $K_{\text{crit}}$ and $C_3/C_1$ are all that is required, since the other results can be expressed in terms of these two expressions. The VA expression for $K_{\text{crit}}$ yields

$$K_{\text{crit}}^{\text{va}} = \frac{m}{\sqrt{\Omega^* - G(\Omega^*)}} \sqrt{6\Omega^* - \frac{15}{8} + \frac{5\sqrt{3\Omega^*}}{4T}},$$

(33)

while that for $C_3/C_1$ is

$$\left(\frac{C_3}{C_1}\right)_{\text{va}} = \frac{\Omega^* - G(\Omega^*)}{m^2}.$$

(34)

Fig. 8. Left: The critical value of frequency $\Omega_{\text{st}}^{\text{va}}(m)$ versus topological charge $m$ according to the VA predictions. We see how as $m$ increases, $\Omega_{\text{st}}^{\text{va}}(m) \to \Omega_{\text{st}}^{\text{va}} < \Omega_{\text{max}}^{2D}$ implying that there exist azimuthally stable vortices for all values of $m$. Right: The rate of the convergence between $\Omega_{\text{st}}^{\text{va}}(m)$ and $\Omega_{\text{st}}^{\text{va}}$ for increasing $m$. We see that the convergence rate is proportional to $1/m^2$. We plot a $1/m^2$ line starting from our $m = 1$ computed value for comparison.

The VA ansatz can also be used to predict the critical value of $\Omega^*$ above which all vortices are modulationally stable. Furthermore, because the criteria given by Eq. (31) and (32) are difficult to make predictions with numerically computed constants, in Sec. 6 we use our VA predictions of the critical frequency to compare with numerical simulations. By inserting the constants defined in Eq. (13) into the stability criteria, and setting each criteria equal to zero, we get the following expressions to compute the critical frequency:

$$K_{\text{crit}} = 0 : \quad 6\Omega^* - \frac{15}{8} + \frac{5\sqrt{3\Omega^*}}{4T} = 0;$$

(35)

$$K_{\text{crit}} = 1 : \quad \frac{1}{\sqrt{\Omega^* - G(\Omega^*)}} \sqrt{6\Omega^* - \frac{15}{8} + \frac{5\sqrt{3\Omega^*}}{4T}} - \frac{1}{m} = 0,$$

(36)

where $T$ is defined as in Eq. (14), and, as before, $\Omega = G(\Omega^*)$. We see that, as $m \to \infty$, the criterion given by Eq. (36) tends to coincide with Eq. (35). Thus, the two criteria based on Eqs. (35) and (36) give different critical values of $\Omega^*$, and hence $\Omega$ – one which is dependent on charge $m$, and the other which is independent of $m$, representing an upper bound on the critical
value for all charges. In general, we denote the charge-dependent critical value as $\Omega_{st}(m)$, and the charge-independent one as $\Omega_{st^*}$.

Solving Eq. (35) for $\Omega^*$ by means of a root finder and inserting the result into Eq. (17) yield

$$\Omega_{va}^{st^*} = 0.144320424.$$  \hspace{1cm} (37)

Since this value is less than $\Omega_{2D}^{max}$, this result predicts that azimuthally stable vortices exist for all values of $m$, as long as $\Omega > \Omega_{va}^{st^*}$. We also solve for $\Omega$ using the criterion of Eq. (36) for various values of $m$. The results are shown in Fig. 8. It can be seen how the critical complex frequency $\Omega_{va}^{st}(m)$ increases in value with the charge and eventually converges to $\Omega_{va}^{st^*}$. Thus, the stability window in $\Omega$ is higher for lower charges. In Ref. [24], it was also concluded that azimuthally stable vortices exist for all $m$, and that lower charges have a larger stability window. However, according Ref. [24], $\Omega_{st^*} = \Omega_{2D}^{max}$, hence the stability window shrinks towards 0 as $m \to \infty$. The estimate given in that work is that the window shrinks proportional to $1/m^2$. This would preclude the hope of experimentally achieving a high-charge stable vortex, as the stability window would be too small.

According to our VA predictions, the value of $\Omega_{st}$ does increase as $m$ increases (and as shown in Fig. 8 also does so proportional to $1/m^2$ compared with our constant $\Omega_{st^*}$). However, since our predicted value of $\Omega_{st^*}$ is low, there is a substantial stability window for all charges $m$ which remains constant for $m \to \infty$. If correct, this would open the door for achieving very high charge vortices experimentally. However, as will be shown in Sec. 6 it appears that the predictions of Ref. [24] are more accurate, in this respect, and the stability window indeed tends to zero at high values of $m$.

6 Azimuthal Modulational Stability: Numerical Results

Here we report the numerical predictions and results for AMS in vortices. For our 2D simulations, we used a finite-difference scheme with a second-order central differencing in space and fourth-order Runge-Kutta in time [13]. We use both a polar-coordinate grid and a Cartesian one. The polar grid makes computing the growth of individual perturbation modes easier, and thus we use it to test our predictions for azimuthally unstable vortices. However, the polar grid forces one to use smaller time steps in the finite-difference scheme than one would need for an equivalent Cartesian grid. Therefore, for testing the critical complex frequency $\Omega_{st}(m)$, which requires running multiple long-time simulations, we use the Cartesian grid.

6.1 Unstable Vortices

In Fig. 9 we show the results for the AMS of vortices with charges $m = 1, \ldots, 5$ and $\Omega \in [0.03, 0.14]$. In general, we see good agreement for $m > 2$, however for high values of $\Omega$ our predictions become too low in each case. This implies that our predictions for $\Omega_{st}(m)$ will not be precise. To test this, we ran long-time simulations of randomly perturbed vortices.

For $m = 1, 2, 3$ we were able to notice the transition from unstable to stable vortices easily, but for $m > 3$ we found it very difficult to find a stable solution. This is because we observe a snake-like instability which deforms and breaks up the vortex into asymmetric irregular fragments.
An example of this is shown in Fig. 10. Obviously, this snake-like instability is not captured by our AMS study and, we believe, is the principal culprit of the discrepancy between the analytic and numerical growth rates presented in Fig. 9.

6.2 Stable Vortices

To determine the values of $\Omega_{st}(m)$, we simulated the evolution of vortices with $\Omega$ taken near the predicted values of $\Omega_{st}(m)$, perturbed by a small uniformly distributed random noise in the azimuthal direction. Our aim was to find the value $\Omega_1$ that results in an unstable vortex (and to observe the actual breakup), and then show that the vortex solution for $\Omega_1 + 0.001$ is stable, by simulating its evolution for an extremely long time, compared to the time necessary to show the full breakup in the former case. We did this for various charges $m$. In Fig. 11, we show the initial and final states produced by this analysis for $m = 1$. Since this effect hinders our ability to simulate the vortices for extended time periods, we are not able to make precise AMS predictions for vortices with $m > 3$. However, since this effect is dynamically distinct from the azimuthal breakup, we can still give an approximate estimate of the critical frequency for higher charges. In Table 2 we display our VA predictions and simulation results for $m = 1, ..., 5$. We also show the predictions from previous studies for comparison. We see that most of the predictions for $m = 1$ are close to our numerical result. For the other charges, the predictions made by Ref. [24] are most accurate ones.

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<tr>
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Table 2
Comparison of the analytical predictions and numerical results for $\Omega_{st}(m)$. The VA predictions are labeled ‘VA’, and the numerical results are labeled “NUM”. Numerical predictions in parentheses are those that were hard to fix due to the emergence of a snake-like instability (not comprised by our stability analysis), and may therefore be less accurate than the others. We also show predictions made by previous studies. When no value has been computed or reported, we label the entry as “NC”. It seems that the predictions made in Ref. [24] are the most accurate compared to our simulations.

We have seen in Sec. 4.2 that our VA ansatz is extremely close to the numerically-exact vortex profiles. However, our AMS predictions made with the VA are as accurate (although the scaling does match the scaling reported in Ref. [24]). Therefore, we conclude that the approximation of the vortex as a completely separable entity consisting of radial and azimuthal parts (as discussed in Sec. 5) is likely to be the cause of the discrepancy.
7 Dynamics of Collisions of Stable Vortices

Here we study the dynamics of collisions of stable vortices in the CQNLS. In Ref. [27], such collisions of vortices of unit charge were studied. It was shown that both elastic and destructive collisions could be observed depending on the phase difference between colliding vortices as well as on their velocity. Here we expand this study to include vortices of charge $m = 2$.

After numerous simulations, we have found that for small enough collision velocities, whether a collision is elastic or destructive is solely determined by the phase difference at the point of contact. This fact seems to be independent of the charge of the vortices involved in the collision. In Fig. 12 we display four different cases that illustrate the main features of the collisions between vortices of different charges. In the first two columns of Fig. 12 we show a collision of a vortex of charge $m = 2$ and one of charge $m = 1$, with and without a $\pi$-phase difference, respectively. It can be seen that since the vortex with charge $m = 2$ has the opposite phase on the collision side, with respect to the collision side of the vortex of charge $m = 1$, the interaction is repulsive since it locally emulates the interaction of two out-of-phase bright solitons [18], [1]. Therefore, if the velocity is small enough the mutual repulsion will win over the momentum imprinted towards the collision. On the other hand, when the vortex with charge $m = 2$ is phase-shifted by $\pi$ (or similarly, if the charge $m = 2$ vortex were to be placed on the opposite side of the charge $m = 1$ vortex), the adjacent phases of the respective sides are in-phase. Thus, in this case, the local scenario corresponds to the interaction of two in-phase bright solitons which is attractive in nature [18], [1] and thus the vortices mutually attract. This results in the merger of the two vortices and the eventual breakup into fragments.

Clearly, even if the phase is opposite at the point of contact (i.e., the repulsive force acts between vortices), if the velocity of collision is too high, then the vortices have enough momentum to allow them to merge and eventually break up as well. This case is depicted in the third column of Fig. 12.

The collisions between slow enough vortices with opposite phases of the sides at the point of contact seem to be quite robust as it is illustrated in the fourth column of Fig. 12. In this panel we depict a “billiard”-type example, in which one vortex collides with another, which in turn collides with a third vortex. All collisions in this case are elastic because the relative phases have been chosen such that the colliding sides are always out-of-phase providing for the necessary repulsion. It is worth noting that, although these collisions at low velocities seem to be elastic, there clearly exist internal breathing modes of the vortices that are excited during the collisions. Therefore, in reality there must be a small fraction of the collisional energy that is transferred to these internal modes and thus the collisions are not completely elastic. A more in-depth analysis of the excitation of these internal modes and the degree of the elasticity of the collisions falls outside of the scope of the present work.

On a general note, it is worth contrasting the interactions between bright vortices and between dark vortices. On the one hand, dark vortices are supported by a non-zero background that carries the phase information at a long range, making them interact at long distances through the phase gradient that they induce in the background field [28,16]. On the other hand, bright vortices are supported on a zero background and, therefore, they behave more like localized particles and only interact when their respective sides are close enough.
8 Conclusions

In this work, we have studied the existence of vortices in the cubic-quintic nonlinear Schrödinger equation both numerically and analytically. The latter has been done through a variational approach that has been shown to be increasingly more accurate as the vortex charge increases. In addition, we have considered the azimuthal modulational stability of vortices, using the method described in Ref. [6]. By assuming a steady-state radial profile, we were able to derive a quasi-one-dimensional equation of motion for the azimuthal direction dynamics. Examining the stability (in Fourier space) of the resulting azimuthal equation, we were able to predict the azimuthal stability of the vortices. We then ran full two-dimensional simulations to compare them to the predictions and also briefly considered vortex collisions.

For azimuthally unstable vortices, our predictions of the maximum growth rate are fairly accurate over a wide range of the vortex’ intrinsic frequency (especially for vortices with charges greater than 2). This result could be potentially useful in relevant applications. In studying stable vortices, our predictions of the critical frequency are not precise for charges greater than 1. Based on numerous simulations, we believe that this discrepancy is due to the assumption (required by our method of the analysis) that the dynamics of the vortex are separable into radial and azimuthal parts. After comparing our numerical results for the critical frequency with previously published ones, we find that they most closely corroborate the predictions reported in Ref. [24].

We also briefly investigated the dynamics of stable vortices undergoing collisions. We found that the critical factor as to whether the vortices merge and break up or undergo elastic collisions is the phase difference at the point of contact and their relative velocities. It would be interesting to perform a more detailed study of the threshold velocities necessary for the elastic collisions, as a function of the relative phase between the colliding sides of the vortices. This study falls outside of the scope of the present manuscript and will be presented elsewhere.

References

Fig. 9. Theoretical predictions and numerical results for the growth rates of the azimuthal mode exhibiting the maximum growth rate for azimuthally unstable vortices of charges $m = 1, \ldots, 5$ (left to right, top to bottom) in the CQNLS equation. For each value of $\Omega \in [0.03, 0.14]$, with the step of $\Delta \Omega = 0.01$, we predicted the growth rates (shown as circles), using our numerically-exact profile for the computation of integrals in the Lagrangian. We then ran 2D simulations of the vortex and recorded the average growth rate (shown as squares). We set the discretized radial spacing to $\Delta r = 1$, azimuthal spacing to $\Delta \theta = (2\pi)/(20 \max[m, K_{\max}, 2])$, and time step to $\Delta t = 0.001$. Overall, the predictions match the numerical results very well.
Fig. 10. Example of shape distortion for a vortex of charge $m = 4$ and $\Omega = 0.18$, perturbed with a random perturbation of size $\epsilon = 0.05$. Pictured is the squared absolute value of the wave function. We have set $\Delta r = 2$, $r_{\text{max}} = 120$ and $\Delta t = 0.6$. The initial vortex shape becomes deformed and then breaks up irregularly.

Fig. 11. An example of numerically determining $\Omega_{\text{st}}(m)$ for $m = 1$. In this case the evolution of the vortex with $\Omega_1 = 0.146$ (top left) was simulated with a random-noise perturbation in the azimuthal direction, until it broke up into fragments due to the azimuthal instability (top right). We then simulated the evolution of a vortex with $\Omega_2 = \Omega_1 + 0.001 = 0.147$ (bottom left) for “long enough” to make it sure that the vortex is stable — in this example, up to $t = 50,000$ (bottom right). The drift in the center position of the vortex is due to the momentum imparted to it from the random perturbations. We then conclude that $\Omega_{\text{st}}(m = 1) \in [0.146, 0.147]$. In this example, we set the grid spacing to be $\Delta x = \Delta y = 1$, perturbation amplitude to $\epsilon = 0.05$, and time step to $\Delta t = 0.2$. 
Fig. 12. Collisions between stable vortices. All vortices have frequency of $\Omega = 0.17$. First column: an elastic collision between a moving vortex of charge $m = 2$ and a stationary vortex of charge $m = 1$. The charge $m = 2$ vortex is given the initial velocity of $0.015$. The vortices have no relative phase difference between them, which corresponds to colliding sides that are out-of-phase and thus repulsive. Therefore, due to the low relative momentum between the vortices, they undergo the elastic collision. Second column: same as in the first column, but with a $\pi$ phase difference between vortices. The $\pi$ phase difference between these vortices corresponds to the vortices having colliding sides that are in-phase. This, in turn, provides a mutual attraction and thus the vortices attract, merge and eventually break up into fragments. Third column: same as in the first column, but for a larger collisional velocity, $0.15$. The mutual repulsion between the in-phase vortices is not enough to counterbalance the high collisional momentum, and the vortices merge and break up into fragments. Fourth column: a charge $m = 1$ vortex with velocity $0.015$ undergoing an elastic collision with an $m = 2$ vortex with a $\pi$ phase difference between them, which in turn collides with a $\pi$ phase-shifted vortex of charge $m = 1$ in another elastic collision. This phase arrangement corresponds to vortices colliding with sides that are mutually out-of-phase and thus repulsive. This shows the robustness of the vortices.