2009

PERFECT FORMS OVER TOTALLY REAL NUMBER FIELDS

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Abstract. A rational positive-definite quadratic form is *perfect* if it can be reconstructed from the knowledge of its minimal nonzero value m and the finite set of integral vectors v such that $f(v) = m$. This concept was introduced by Voronoï and later generalized by Koecher to arbitrary number fields. One knows that up to a natural "change of variables" equivalence, there are only finitely many perfect forms, and given an initial perfect form one knows how to explicitly compute all perfect forms up to equivalence. In this paper we investigate perfect forms over totally real number fields. Our main result explains how to find an initial perfect form for any such field. We also compute the inequivalent binary perfect forms over real quadratic fields $\mathbb{Q}(\sqrt{d})$ with $d \leq 66$.

1. Introduction

Let f be a positive-definite rational quadratic form in n variables. Let $m(f)$ be the minimal nonzero value attained by f on $\mathbb{Z}^n$, and let $M(f)$ be the set of vectors v such that $f(v) = m(f)$. Voronoï defined f to be *perfect* if f is reconstructible from the knowledge of $m(f)$ and $M(f)$ [Vor08]. Voronoï's theory was later extended by Koecher to a much more general setting that includes quadratic forms over arbitrary number fields F [Koe60]. Koecher also generalized a fundamental result of Voronoï, which says that modulo a natural $\mathrm{GL}_n(\mathcal{O})$-equivalence, where $\mathcal{O}$ is the ring of integers of F, there are only finitely many n-ary perfect forms. Moreover, there is an explicit algorithm to determine the set of inequivalent perfect forms, given the input of an initial perfect form [Vor08, Gun99].

Voronoï proved that the quadratic form A_n is perfect for all n, and using this was able to classify n-ary rational perfect forms for $n \leq 5$. In this paper, we consider totally real fields F and explain how to construct an initial perfect form. Rather than trying to give a closed form expression of such a form, we show how to use the geometry of symmetric spaces and modular symbols to find an initial perfect form. A key role is played by the notion of *lattices of E-type* [Kit93]. For F real quadratic and $n = 2$, we carry out our construction explicitly to compute all inequivalent binary perfect forms for $F = \mathbb{Q}(\sqrt{d})$, $d \leq 66$. These results complement work of Leibak and Ong [Lei08, Lei05, Ong86].

Our main interest in Voronoï and Koecher's results is that they provide topological models for computing the cohomology of subgroups of $\mathrm{GL}_n(\mathcal{O})$, where F is

Date: August 18, 2009.

1991 *Mathematics Subject Classification.* 11H55, 11E10, 11F67, 11F75.

Key words and phrases. Perfect forms, modular symbols, well-rounded retract, Eisenstein cocycle.

The first named author wishes to thank the National Science Foundation for support of this research through NSF grant DMS-0801214.

any number field. This cohomology gives a concrete realization of certain automorphic forms that conjecturally have deep connections with arithmetic geometry. The cohomology of subgroups of $\mathrm{GL}_n(\mathbb{Z})$, for instance, has a (well known) relationship with holomorphic modular forms when $n = 2$, and for higher n has connections with K-theory, multiple zeta values, and Galois representations [AM92,EVGS02,Gon01]. The cohomology of subgroups of $\mathrm{GL}_2(\mathcal{O})$, when F is totally real, is related to Hilbert modular forms [Fre90]. However, computing these models for any example, a prerequisite for using them to explicitly compute cohomology, is a nontrivial problem as soon as $\mathcal{O} \neq \mathbb{Z}$ or $n > 2$. Our work is a first step towards more cohomology computations for F totally real, computations that we plan to pursue in the future.

Acknowledgements. We thank Avner Ash, Farshid Hajir, and Mark McConnell for helpful conversations and their interest in our work.

2. Preliminaries

Let F be a totally real number field of degree m with ring of integers $\mathcal{O}$. Let $\iota = (\iota_1, \ldots, \iota_m)$ denote the m embeddings $F \to \mathbb{R}$. For $z \in F$, let z^k denote $\iota_k(z)$. We extend this notation to other F-objects. For example, if $A = \left[a_{ij}\right]$ is a matrix with entries in F, then A^k denotes the real matrix $A^k = \left[a_{ij}^k\right]$. An element $z \in F$ is called totally positive if $z^k > 0$ for each k. We write $z \gg 0$ if z is totally positive.

2.1. **n-ary quadratic forms over F.** An *n-ary quadratic form over F* is a map $f : \mathcal{O}^n \to \mathbb{Q}$ of the form

$$f(x_1, \ldots, x_n) = \mathrm{Tr}_{F/\mathbb{Q}}\Bigg(\sum_{1 \le i,j \le n} a_{ij} x_i x_j\Bigg), \quad \text{where } a_{ij} \in F. \tag{1}$$

Our main object of study will be *positive-definite* n-ary quadratic forms over F. Specifically, since $\mathcal{O}^n \simeq \mathbb{Z}^{nm}$, f can be viewed as a quadratic form on $\mathbb{Z}^{nm}$. We require that under this identification f is a positive-definite quadratic form on $\mathbb{Z}^{nm}$. Equivalently, we can use the m embeddings $\iota\colon F \to \mathbb{R}^m$ to view f as a m-tuple $(f^1, \ldots, f^m)$ of real quadratic forms of $\mathbb{Z}^n$. It is easy to check that a quadratic form over F is positive-definite if and only if each f^i is positive-definite.

2.2. **Minimal vectors.** There is a value $m(f)$ associated to each positive-definite quadratic form f, called the *minimum* of f, given by

$$m(f) = \min_{v \in \mathcal{O}^n \smallsetminus \{0\}} f(v).$$

Definition 2.1. A vector $v \in \mathcal{O}^n \smallsetminus \{0\}$ is a *minimal vector for f* if $f(v) = m(f)$. The set of minimal vectors is denoted $M(f)$.

Note that v is a minimal vector for f if and only if $-v$ is as well. In our considerations the distinction between v and $-v$ will be irrelevant, and so we abuse notation and let $M(f)$ denote a set of representatives for the minimal vectors modulo $\{\pm 1\}$.

2.3. **Perfect forms.** For most quadratic forms, knowledge of the set $M(f)$ is not enough to reconstruct f. A simple example is provided by the one-parameter family of rational quadratic forms

$$f_\lambda(x, y) = x^2 + \lambda xy + y^2, \quad \lambda \in (-1, 1) \cap \mathbb{Q},$$

all of which are easily seen to satisfy $M(f_\lambda) = \{e_1, e_2\}$, where the e_i are the standard basis vectors of $\mathbb{Z}^2$. On the other hand the rational binary form

$$g(x, y) = x^2 + xy + y^2$$

is reconstructible from the data of $\{M(g), m(g)\}$ (cf. §3.2), which equals

$$\bigl\{\{e_1, e_2, e_1 - e_2\}, 1\bigr\}.$$

We formalize this notion, due to Voronoï for rational quadratic forms, with the following definition:

Definition 2.2 ([Koe60, §3.1])**.** A positive-definite quadratic form f over F is said to be *perfect* if f is uniquely determined by its minimum value $m(f)$ and its minimal vectors $M(f)$. That is, given the data $\{M(f), m(f)\}$, the system of linear equations

$$\bigl\{\mathrm{Tr}_{F/\mathbb{Q}}(v^t X v) = m(f)\bigr\}_{v \in M(f)} \tag{2}$$

has a unique solution.

We warn the reader that there are other notions of perfection for quadratic forms over number fields in the literature, notably in the work of Icaza [Ica97] and Coulangeon [Cou01]. All notions involve the reconstruction of f from its minimal vectors, but these authors use the norm where we have used the trace in the evaluation (1) of a form on a vector in $\mathcal{O}^n$. Moreover, Coulangeon uses a larger group to define equivalence of forms.

3. Positive lattices

3.1. **Lattices of E-type.** For these results we follow [Kit93].

Definition 3.1. Let $V/\mathbb{Q}$ be a vector space with positive-definite quadratic form ϕ. A lattice $L \subset V$ is a *positive lattice for* ϕ if for a $\mathbb{Z}$-basis $\mathcal{B} = \{e_1, \ldots, e_n\}$ of L, the associated symmetric matrix for ϕ in $\mathcal{B}$-coordinates has rational entries.

We will denote a positive lattice for ϕ by pair (L, ϕ). As before, one can define minimal vectors and the minimum for a positive lattice (L, ϕ). We denote these $M(\phi)$ and $m(\phi)$, with L understood.

Given two positive lattices (L_1, ϕ_1) and (L_2, ϕ_2), with $L_1 \subset V_1$ and $L_2 \subset V_2$, one can construct a new positive lattice $(L_1 \otimes L_2, \phi_1 \otimes \phi_2)$. Specifically, let B_1 denote the symmetric bilinear form giving rise to ϕ_1, and let B_2 denote the symmetric bilinear form giving rise to ϕ_2. We define a symmetric bilinear form B on $V_1 \otimes V_2$ by first defining

$$B(v \otimes w, \hat{v} \otimes \hat{w}) = B_1(v, \hat{v}) B_2(w, \hat{w})$$

on simple tensors and then by linearly extending to all of $V_1 \otimes V_2$. Then one has a positive-definite quadratic form $\phi = \phi_1 \otimes \phi_2$ on $V_1 \otimes V_2$ given by $\phi(x) = B(x, x)$. Note that by construction, we have $\phi(v \otimes w) = \phi_1(v)\phi_2(w)$.

Definition 3.2. A positive lattice (L, ϕ) is of *E-type* if

$$M(\phi \otimes \phi') \subset \{u \otimes v \mid u \in L, v \in L'\}$$

for every positive lattice (L', ϕ').

In other words, a lattice is E-type if, whenever it is tensored with another positive lattice, the minimal vectors of the tensor product decompose as simple tensors. Positive lattices of E-type are particularly well-behaved with respect to tensor product:

Proposition 3.3 ([Kit93, Lemma 7.1.1])**.** *Let (L, ϕ) and (L', ϕ') be positive lattices. If (L, ϕ) is of E-type, then*

(i) $m(\phi \otimes \phi') = m(\phi)m(\phi')$, *and*
(ii) $M(\phi \otimes \phi') = \{x \otimes y \mid x \in M(\phi),\ y \in M(\phi')\}$.

3.2. **The form A_n.** We now give an example that will be important in the sequel. Let A_n be the rational quadratic form

$$A_n(x_1, \dots, x_n) = \sum_{1 \le i \le j \le n} x_i x_j.$$

It is easy to check, as was first done by Voronoï [Vor08], that A_n is perfect. One computes

$$m(A_n) = 1 \quad \text{and} \quad M(A_n) = \{e_i\} \cup \{e_i - e_j\}.$$

According to [Kit93, Theorem 7.1.2], $(\mathbb{Z}^n, A_n)$ is of E-type.

3.3. **Application to n-ary quadratic forms over F.** Fix $\alpha \in F$ totally positive. Consider the n-ary quadratic form over F given by

$$f_\alpha(x_1, \dots x_n) = \mathrm{Tr}_{F/\mathbb{Q}}\left(\alpha A_n(x_1, \dots, x_n)\right). \tag{3}$$

Lemma 3.4. *We have*

$$(\mathcal{O}^n, f_\alpha) = (\mathcal{O} \otimes \mathbb{Z}^n, \phi_\alpha \otimes A_n),$$

where $\phi_\alpha(x) = \mathrm{Tr}_{F/\mathbb{Q}}(\alpha x^2)$.

Proof. It is clear that as $\mathbb{Z}$-modules, we have $\mathcal{O}^n \simeq \mathcal{O} \otimes \mathbb{Z}^n$. Thus we want to show that under this isomorphism, the quadratic form f_α on $\mathcal{O}^n$ is taken to the quadratic form $\phi_\alpha \otimes A_n$ on $\mathcal{O} \otimes \mathbb{Z}^n$. We do this by explicit computation. Let $a \in \mathcal{O}$ and let $\mathbf{x} = \sum x_i e_i \in \mathbb{Z}^n$. Then we have

$$\begin{aligned}
(\phi_\alpha \otimes A_n)(a \otimes \mathbf{x}) &= \phi_\alpha(a) A_n(\mathbf{x}) \\
&= \mathrm{Tr}_{F/\mathbb{Q}}\left(\alpha a^2\right) \sum_{1 \le i \le j \le n} x_i x_j \\
&= \mathrm{Tr}_{F/\mathbb{Q}}\Big(\alpha \sum_{1 \le i \le j \le n} a^2 x_i x_j\Big) \\
&= \mathrm{Tr}_{F/\mathbb{Q}}\left(\alpha A_n(a\mathbf{x})\right) \\
&= f_\alpha(a\mathbf{x}),
\end{aligned}$$

which completes the proof. □

Theorem 3.5. *Let f_α be as in* (3)*. Then there exist nonzero $\eta_1, \dots, \eta_r \in \mathcal{O}$ such that*

(i) *the minimum of f_α is*

$$m(f_\alpha) = m(\phi_\alpha) = \mathrm{Tr}_{F/\mathbb{Q}}(\alpha \eta_i^2), \quad \textit{and}$$

(ii) *the minimal vectors of* f_α *are*

$$M(f_\alpha) = \bigcup_{1\leq k\leq r} \{\eta_k e_i\} \cup \{\eta_k(e_i - e_j)\}.$$

Proof. Since $(\mathbb{Z}^n, A_n)$ is of E-type, the result follows from Proposition 3.3 and Lemma 3.4 by taking $\{\eta_i\}$ to be the minimal vectors for ϕ_α. □

4. The geodesic action, the well-rounded retract, and the Eisenstein cocycle

In this section we present three tools that play a key role in the proof of Theorem 5.3. The *geodesic action* [BS73] is an action of certain tori on locally symmetric spaces. The *well-rounded retract* [Ash84] is a deformation retract of certain locally symmetric spaces. The *Eisenstein cocycle* [Scz93, GS03] is a cohomology class for $\mathrm{SL}_m(\mathbb{Z})$ that gives a cohomological interpretation of special values of the partial zeta functions of totally real number fields of degree m.

4.1. **Geodesic action.** Let G be a semisimple connected Lie group, let K be a maximal compact subgroup, and let X be the symmetric space G/K. Fix a basepoint $x \in X$. This choice of basepoint determines a Cartan involution θ_x. For a parabolic subgroup $P \subset G$, the Levi quotient is $L_P = P/N_P$, where N_P is the unipotent radical of P. Let A_P denote the (real points of) the maximal $\mathbb{Q}$-split torus in the center of L_P, and let $A_{P,x}$ denote the unique lift of A_P to P that is stable under the Cartan involution θ_x.

Since P acts transitively on X, every point $z \in X$ can be written as $z = p\cdot x$ for some $p \in P$. Then Borel–Serre define the geodesic action of A_P on X by

$$a \circ z = (p\widetilde{a}) \cdot x,$$

where $\widetilde{a}$ is the lift of a to $A_{P,x}$. This action is independent of the choice of basepoint x, justifying the notation. Note that at the basepoint x, the geodesic action of A_P agrees with the ordinary action of its lift $A_{P,x}$.

4.2. **Well-rounded retract.** Now let $G = \mathrm{SL}_m(\mathbb{R})$, $K = \mathrm{SO}(m)$. The space X is naturally isomorphic to the space of m-ary positive-definite real quadratic forms modulo homotheties. Indeed, this follows easily from the Cholesky decomposition from computational linear algebra: if S is a symmetric positive-definite matrix of determinant 1, then there exists a matrix $g \in G$ such that $gg^t = S$.

Let $W \subset X$ be the subset consisting of all forms whose minimal vectors span $\mathbb{R}^m$. Then Ash proved that W is an $\mathrm{SL}_m(\mathbb{Z})$-equivariant deformation retract of X. Moreover W naturally has the structure of a cell complex with polytopal cells, and $\mathrm{SL}_m(\mathbb{Z})$, and thus any finite-index subgroup $\Gamma \subset \mathrm{SL}_m(\mathbb{Z})$, act cellularly on W with finitely many orbits. The retract can be used to compute the cohomology of Γ for certain $\mathbb{Z}\Gamma$-modules in the following way. Let M be a $\mathbb{Z}\Gamma$-module attached to a rational representation of $\mathrm{SL}_m(\mathbb{Q})$ and let $\widetilde{M}$ be the associated local coefficient system on $\Gamma\backslash X$. Then we have isomorphisms

$$H^*(\Gamma; M) \simeq H^*(\Gamma\backslash X; \widetilde{M}) \simeq H^*(\Gamma\backslash W; \widetilde{M}).$$

4.3. **Eisenstein cocycle.** As before let $\mathcal{O}$ be the ring of integers in a totally real number field of degree m. Let $\mathfrak{b}, \mathfrak{f} \subset \mathcal{O}$ be relatively prime ideals. The partial zeta function $\zeta(\mathfrak{b}, \mathfrak{f}; s)$ attached to the ray class $\mathfrak{b} \pmod{\mathfrak{f}}$ is defined by the analytic continuation of the Dirichlet series

$$\zeta(\mathfrak{b}, \mathfrak{f}; s) = \sum_{\mathfrak{a}} \mathrm{N}(\mathfrak{a})^{-s}, \quad \mathrm{Re}(s) > 1,$$

where the sum is taken over all integral ideals $\mathfrak{a}$ such that $\mathfrak{a}\mathfrak{b}^{-1}$ is principal with a totally positive generator in the coset $1 + \mathfrak{f}\mathfrak{b}^{-1}$. By the theorem of Klingen–Siegel [Sie70], the special values $\zeta(\mathfrak{b}, \mathfrak{f}; 1-k)$, where $k \in \mathbb{Z}_{>0}$, are rational.

The special values have a cohomological interpretation. Let U be the group of totally positive units in the coset $1 + \mathfrak{f}$. Sczech constructed a sequence of rational cocycles $\eta(\mathfrak{b}, \mathfrak{f}, k) \in H^{m-1}(U; \mathbb{Q})$ which give the numbers $\zeta(\mathfrak{b}, \mathfrak{f}; 1-k)$ by evaluation on the fundamental cycle in $H_{m-1}(U; \mathbb{Z})$. To construct the cocycles $\eta(\mathfrak{b}, \mathfrak{f}, k)$, one specializes a "universal" cocycle $\Psi \in H^{m-1}(\mathrm{SL}_m(\mathbb{Z}); M)$, where M is a certain module. After choosing $\mathfrak{b}, \mathfrak{f}, k$, one plugs U-invariant parameters into M to obtain a module M_k, which is $\mathbb{C}$ with a nontrivial $\mathrm{SL}_m(\mathbb{Z})$-action, and a class $\Psi(\mathfrak{b}, \mathfrak{f}, k) \in H^{m-1}(\mathrm{SL}_m(\mathbb{Z}); M_k)$. Then $\eta(\mathfrak{b}, \mathfrak{f}, k)$ is obtained by restriction, after realizing U as a subgroup of $\mathrm{SL}_m(\mathbb{Z})$ via a regular representation; note that M_k restricted to U is trivial.

5. Scaled trace forms and the main result

We now return to perfect forms. The quadratic form ϕ_α from Lemma 3.4 will play an imporant role in Theorem 5.3, so we give it a name:

Definition 5.1. For $\alpha \in F$, the *scaled trace form associated to* α is the map

$$\phi_\alpha : \mathcal{O} \to \mathbb{Q}$$

given by $\phi_\alpha(\eta) = \mathrm{Tr}_{F/\mathbb{Q}}(\alpha\eta^2)$.

For the remainder of the paper, fix a $\mathbb{Z}$-basis $\mathcal{B} = \{\omega_1, \dots, \omega_m\}$ of $\mathcal{O}$. Then for $\mathbf{x} = \sum x_i\omega_i \in \mathcal{O}$ with $x_i \in \mathbb{Z}$, we have

$$\phi_\alpha(\mathbf{x}) = \sum_{1 \le i,j \le m} \mathrm{Tr}_{F/\mathbb{Q}}(\alpha\omega_i\omega_j)x_i x_j.$$

In particular, fixing $\mathcal{B}$ allows us to view the form ϕ_α as an m-ary quadratic form $[\phi_\alpha]_\mathcal{B}$ over $\mathbb{Q}$.

Let V denote the $m(m+1)/2$-dimensional $\mathbb{R}$-vector space of m-ary quadratic forms over $\mathbb{R}$. Note that for $\alpha, \beta \in F$, we have

$$[\phi_{\alpha+\beta}]_\mathcal{B} = [\phi_\alpha]_\mathcal{B} + [\phi_\beta]_\mathcal{B}.$$

In particular the image of $F \otimes \mathbb{R}$ in V is an m-dimensional subspace. The form ϕ_α is positive-definite if $\alpha \gg 0$. Let $C \subseteq V$ denote the real cone of positive-definite m-ary quadratic forms, and let $C^+_\mathcal{B}$ denote the subcone corresponding to the totally positive scaled trace forms. More precisely, let

$$C^+_\mathcal{B} = \mathrm{Cone}(\{[\phi_\alpha]_\mathcal{B} \mid \alpha \gg 0\}) \otimes \mathbb{R}.$$

Let X be the global symmetric space G/K, where $G = \mathrm{SL}_m(\mathbb{R})$ and $K = \mathrm{SO}(m)$. Recall (§4.2) that we have an isomorphism $X \simeq C/\mathbb{R}_{>0}$, where $\mathbb{R}_{>0}$ acts on C by homotheties. We denote by $X^+_\mathcal{B}$ the image of $C^+_\mathcal{B}$ in X under the projection $C \to X$.

Lemma 5.2. *There exists an $\mathbb{R}$-split torus $A \subset G$ and a point $x_{\mathcal{B}} \in X$ such that*
$$X_{\mathcal{B}}^+ = \{a \circ x_{\mathcal{B}} \mid a \in A\},$$
where $\circ$ is the geodesic action of A on X (§4.1).

Proof. We begin with some computations in C. For $\alpha \in F$ totally positive, let $S(\alpha) = \big[S(\alpha)_{ij}\big]$ denote the positive-definite symmetric $m \times m$ matrix corresponding to $[\phi_\alpha]_{\mathcal{B}}$. Thus
$$S(\alpha)_{ij} = \mathrm{Tr}_{F/\mathbb{Q}}(\alpha\omega_i\omega_j) = \sum_{1\leq k\leq m} \alpha^k \omega_i^k \omega_j^k.$$
We can write $S(\alpha)$ as $g(\alpha)g(\alpha)^t$, where $g(\alpha)$ is given by
$$g(\alpha)_{ij} = \sqrt{\alpha^j}\omega_i^j.$$
This implies $g(\alpha) = \Omega a$, where
$$\Omega_{ij} = \omega_i^j \quad \text{and} \quad a = \mathrm{diag}(\sqrt{\alpha^1}, \ldots, \sqrt{\alpha^m}). \tag{4}$$
From these considerations it is clear that the cone $C_{\mathcal{B}}^+ \subset C$ is given by the set of matrices of the form Ωa, where a is allowed to vary over *all* positive real diagonal matrices, not just those of the special form on the right of (4).

Now we pass to X by modding out by homotheties. Let $x_{\mathcal{B}}$ be the image of the point $[\phi_1]_{\mathcal{B}}$. Let Υ be the unique positive multiple of Ω such that $\det(\Upsilon) = 1$; this also maps onto $x_{\mathcal{B}}$. Then the subset of C given by
$$\{\Upsilon a \mid a = \mathrm{diag}(a_1, \ldots, a_m), a_k \in \mathbb{R}_{>0}, a_1 \cdots a_m = 1\}$$
maps diffeomorphically onto $X_{\mathcal{B}}^+ \subset X$. Let $P_\infty \subset G$ be the parabolic subgroup of upper-triangular matrices, and let $A_\infty \subset P_\infty$ be the diagonal subgroup. Let x denote the point of X fixed by K. Note that $X_{\mathcal{B}}^+$ is precisely a translate of the submanifold defined by the geodesic action of A_∞ on x:
$$X_{\mathcal{B}}^+ = \Upsilon \cdot \{a \circ x \mid a \in A_\infty\}.$$
By "transport de structure" we can express this at the basepoint, that is $x_{\mathcal{B}} = \Upsilon \cdot x$
$$X_{\mathcal{B}} = \{b \circ x_{\mathcal{B}} \mid b \in \Upsilon A_\infty \Upsilon^{-1}\}.$$
More precisely,
$$(\Upsilon a) \cdot x = (\Upsilon a \Upsilon^{-1})\Upsilon \cdot x = (\Upsilon a \Upsilon^{-1}) \cdot x_{\mathcal{B}}. \tag{5}$$

Now $\Upsilon a \Upsilon^{-1} \in \Upsilon A_\infty \Upsilon^{-1}$, and (5) is exactly the geodesic action of the element $\Upsilon a \Upsilon^{-1}$ on the point $x_{\mathcal{B}}$. Thus we may take $A = \Upsilon A_\infty \Upsilon^{-1}$, and the lemma follows. □

By Theorem 3.5, to find a perfect n-ary form over F, one can look for scaled trace forms with many linearly independent minimal vectors. Specifically, a scaled trace form $f_\alpha = \phi_\alpha \otimes A_n$ is perfect if
$$\big\{\mathrm{Tr}_{F/\mathbb{Q}}(v^t X v) = m(f_\alpha)\big\}_{v \in M(f_\alpha)} \tag{6}$$
defines $mn(n+1)/2$ linearly independent conditions on the space of quadratic forms. Since A_n is perfect over $\mathbb{Q}$, the minimal vectors of A_n define $n(n+1)/2$ linearly independent conditions. Thus by Theorem 3.5, if ϕ_α has m linearly independent minimal vectors, then (6) will impose $mn(n+1)/2$ linearly independent relations in the space of quadratic forms over F, and f_α will be perfect. We now prove our main result, which asserts that such an α can always be found:

Theorem 5.3. *There exists $\alpha \in F$ totally positive such that ϕ_α has m linearly independent minimal vectors, and thus*

$$f_\alpha(x_1,\dots,x_n) = \mathrm{Tr}_{F/\mathbb{Q}}\left(\alpha A_n(x_1,\dots,x_n)\right)$$

is a perfect form over F.

Proof. We must show that we can find $\alpha \gg 0$ such that $[\phi_\alpha]_{\mathcal{B}}$ is well-rounded. By Lemma 5.2, we know that a choice of basis $\mathcal{B}$ gives rise to a point $x_{\mathcal{B}}$ and a maximal $\mathbb{R}$-split torus A such that $X_{\mathcal{B}}^+ = \{a \circ x_{\mathcal{B}} \mid a \in A\}$. We will show that $X_{\mathcal{B}}^+ \cap W \neq \emptyset$, where W is the retract for $\Gamma = \mathrm{SL}_m(\mathbb{Z})$ (§4.2), and that the inverse image in C of any point in $X_{\mathcal{B}}^+ \cap W$ is a ray containing an F-rational point ϕ_α. This will prove the theorem.

To show $X_{\mathcal{B}}^+ \cap W \neq \emptyset$ we use the Eisenstein cocycle (§4.3). Let $\mathfrak{f} = \mathcal{O}$, so that $\zeta(\mathfrak{b},\mathfrak{f};s)$ is the Dedekind zeta function $\zeta_F(s)$, and U is the group of totally positive units. Abbreviate $\Psi(\mathfrak{b},\mathfrak{f},k)$ (respectively, $\eta(\mathfrak{b},\mathfrak{f},k)$) to $\Psi(k)$ (resp., $\eta(k)$). Using the regular representation attached to the basis $\mathcal{B}$, we have an injection $i\colon U \to \Gamma$. Let M_k' be the module dual to M_k. Since M_k and M_k' are trivial after restriction to U, we obtain induced maps $i_*\colon H_{m-1}(U;\mathbb{C}) \to H_{m-1}(\Gamma;M_k')$ and $i^*\colon H^{m-1}(\Gamma;M_k) \to H^{m-1}(U;\mathbb{C})$.

Let $\langle\ ,\ \rangle_\star$ the pairing between H^{m-1} and H_{m-1}, where $\star$ is either U or Γ, and where the target is $\mathbb{C} \simeq M_k \otimes_\Gamma M_k'$. Let $\xi \in H_{m-1}(U;\mathbb{C})$ be the fundamental class. Then

$$\langle i_*(\xi), \Psi(k)\rangle_\Gamma = \langle \xi, i^*(\Psi(k))\rangle_U = \langle \xi, \eta(k)\rangle_U = \zeta_F(1-k).$$

Since the special values $\zeta_F(1-k)$ do not vanish identically, the class $i_*(\xi)$ pairs nontrivially with $\Psi(k)$ for some k. But it is easy to see that $i_*(\xi)$ is the same as the class of $X_{\mathcal{B}}^+ \pmod{\Gamma}$ in the homology of the quotient $\Gamma\backslash X$. If $X_{\mathcal{B}}^+ \cap W$ were empty, then by the discussion in §4.2 the class of $X_{\mathcal{B}}^+ \pmod{\Gamma}$ would pair trivially with all cohomology classes for all coefficient modules, which is a contradiction. Thus $X_{\mathcal{B}}^+ \cap W \neq \emptyset$.

To finish we must prove that the ray above an intersection point in $X_{\mathcal{B}}^+ \cap W$ contains a form ϕ_α with $\alpha \in F$. This follows easily since W is cut out by linear equations with $\mathbb{Q}$-coefficients and from the explicit form of the cone $C_{\mathcal{B}}^+$. This completes the proof of the theorem. □

6. Real quadratic fields

6.1. **Preliminaries.** Let d be a square-free positive integer, and $\mathcal{O}$ be the ring of integers in the real quadratic field $F = \mathbb{Q}(\sqrt{d})$. Then $\mathcal{O}$ is a $\mathbb{Z}$-lattice in $\mathbb{R}^2$, generated by 1 and ω, where $\omega = (1+\sqrt{d})/2$ if $d \equiv 1 \pmod 4$ and $\omega = \sqrt{d}$ otherwise. The discriminant D equals d if $d \equiv 1 \pmod 4$ and equals $4d$ otherwise.

6.2. **Scaled trace forms.** Let C be the cone of positive-definite binary quadratic forms. Modding out by homotheties, we can identify $C/\mathbb{R}_{>0}$ with the upper half-plane $\mathfrak{H}$. One such identification is given by

$$x + iy \longmapsto \begin{bmatrix} 1 & -x \\ -x & x^2+y^2 \end{bmatrix}. \tag{7}$$

Fixing a $\mathbb{Z}$-basis $\mathcal{B} = \{1,\omega\}$ for $\mathcal{O}$, we consider the subcone $C_{\mathcal{B}}^+ \subset C$ of totally positive scaled trace forms as described in §5.

By Lemma 5.2, it follows that the image of $C_{\mathcal{B}}^{+}$ is a geodesic in $\mathfrak{H}$. Considering (7) and our choice of basis, we see that $C_{\mathcal{B}}^{+}$ corresponds to

$$\left\{ \begin{bmatrix} \mathrm{Tr}_{F/\mathbb{Q}}(\alpha) & \mathrm{Tr}_{F/\mathbb{Q}}(\alpha\omega) \\ \mathrm{Tr}_{F/\mathbb{Q}}(\alpha\omega) & \mathrm{Tr}_{F/\mathbb{Q}}(\alpha\omega^2) \end{bmatrix} \;\middle|\; \alpha \gg 0 \right\} \otimes \mathbb{R}.$$

On $\mathfrak{H}$ this becomes the geodesic $X_{\mathcal{B}}^{+}$ defined by

$$(x+\frac{1}{2})^2 + y^2 = \frac{d}{4} \quad \text{if } d \equiv 1 \pmod 4,$$
$$x^2 + y^2 = d \quad \text{otherwise}.$$

The well-rounded retract $W \subset \mathfrak{H}$ is the infinite trivalent tree shown in Figure 6.2. The crenellation comes from arcs of circles of the form $(x-n)^2 + y^2 = 1$, where $n \in \mathbb{Z}$. One can compute a point $x_0 + iy_0$ of the intersection of W and the geodesic corresponding to $C_{\mathcal{B}}^{+}$. Let

$$X(n) = \begin{cases} \dfrac{4n^2+d-5}{4+8n} & \text{if } d \equiv 1 \pmod 4, \\ \dfrac{n^2+d-1}{2n} & \text{otherwise.} \end{cases}$$

Then $x_0 = \min_{n\in\mathbb{Z}} |X(n)|$, and y_0 can be explicitly computed from x_0. Specifically, let $\widetilde{n}$ be a non-negative integer such that $X(\widetilde{n}) = x_0$. Then y_0 satisfies

$$(x_0 - \widetilde{n})^2 + y_0^2 = 1.$$

The corresponding scaled trace form is ϕ_α, where

$$\alpha = \begin{cases} \dfrac{d-(2x_0+1)\sqrt{d}}{2d} & \text{if } d \equiv 1 \pmod 4, \\ \dfrac{d-x_0\sqrt{d}}{2d} & \text{otherwise.} \end{cases}$$

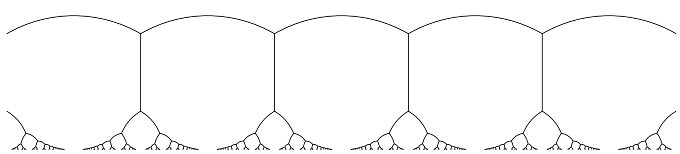

Figure 1. The well-rounded retract $W \subset \mathfrak{H}$.

We summarize the results of this computation below.

Proposition 6.1. *Let α and $\widetilde{n}$ be as above. Let $\eta = \widetilde{n} + \omega$. Then the scaled trace form ϕ_α is minimized at $\{\pm 1, \pm\eta\}$.*

6.3. **Binary perfect quadratic forms.** Given a binary perfect form as an initial input, there is an algorithm to compute the $\mathrm{GL}_2(\mathcal{O})$-equivalence classes of binary perfect forms over F [Gun99]. This was done in [Ong86] for $F = \mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{3})$, $\mathbb{Q}(\sqrt{5})$ and in [Lei05] for $F = \mathbb{Q}(\sqrt{6})$. Using Proposition 6.1, we compute the $\mathrm{GL}_2(\mathcal{O})$-equivalence classes of these forms over $F = \mathbb{Q}(\sqrt{d})$ for square-free $d \leq 66$. The computations were done using `Magma` [BCP97] and `PORTA` [CL]. The number of $\mathrm{GL}_2(\mathcal{O})$-classes of perfect forms is given in Table 1. Figure 2 shows a plot of the data (D, N_D) from Table 1.

D	h_D	N_D	D	h_D	N_D	D	h_D	N_D	D	h_D	N_D
5	1	2	37	1	97	76	1	404	168	2	918
8	1	2	40	2	52	88	1	645	172	1	2683
12	1	3	41	1	144	92	1	343	184	1	4306
13	1	9	44	1	66	104	2	326	188	1	1358
17	1	34	53	1	93	120	2	524	204	2	1992
21	1	16	56	1	144	124	1	1473	220	2	2397
24	1	22	57	1	515	136	2	1135	232	2	3042
28	1	30	60	2	89	140	2	487	236	1	2977
29	1	39	61	1	206	152	1	995	248	1	2360
33	1	149	65	2	301	156	2	941	264	2	3139

TABLE 1. $\mathrm{GL}_2(\mathcal{O})$-classes of perfect binary quadratic forms over real quadratic fields. The discriminant is D, the class number of $\mathbb{Q}(\sqrt{D})$ is h_D, and the number of inequivalent forms is N_D.

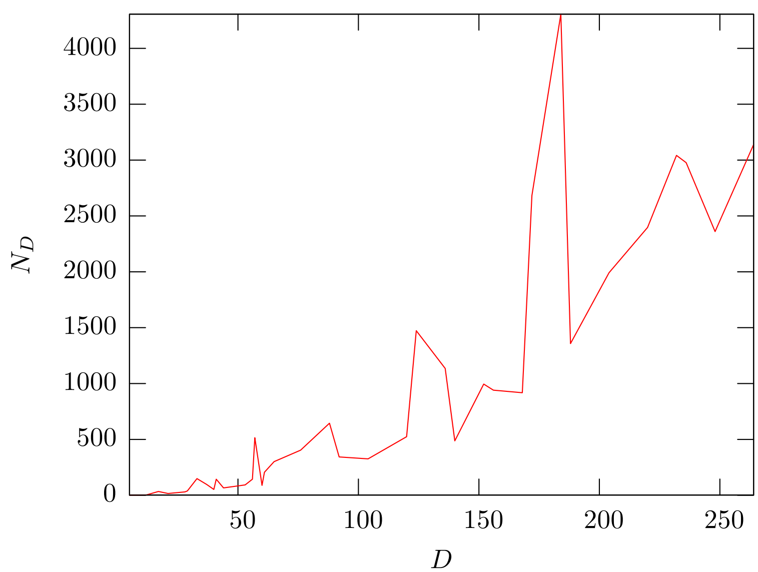

FIGURE 2. Perfect forms by discriminant

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