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Kelvin waves (kelvons)—the distortion waves on vortex lines—play a key part in the relaxation of superfluid turbulence at low temperatures. We present a weak-turbulence theory of kelvons. We show that non-trivial kinetics arises only beyond the local-induction approximation and is governed by three-kelvon collisions; corresponding kinetic equation is derived. On the basis of the kinetic equation, we prove the existence of Kolmogorov cascade and find its spectrum. The qualitative analysis is corroborated by numeric study of the kinetic equation. The application of the results to the theory of superfluid turbulence is discussed.

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The distortion waves on a vortex filament—Kelvin waves (KW)—have been known for more than a century [1]. Superfluids with their topological (quantized) vorticity form a natural domain for KW [2]. Nowadays there is a strong interest to the non-linear aspects of KW associated with studying low-temperature superfluid turbulence of ^{4}He [3, 4, 5, 6, 7, 8, 9, 10], as well as vortex dynamics in ultra-cold atomic gases [11, 12].

The superfluid turbulence [2, 13] is a chaotic tangle of vortex lines. In the absence of the normal component ($T \to 0$ limit), KW play a crucial part in the vortex tangle relaxational dynamics. In contrast to a normal fluid, the quantization of the velocity circulation in a superfluid makes it impossible for the vortex line to relax by gradually slowing down. The only allowed way of relaxation is reducing the total line length. At $T = 0$ even this generic scenario becomes non-trivial, as the total line length is, to a very good approximation, a constant of motion. In the scenario proposed by one of us [3], the vortex line length—in the form of KW generated in the process of vortex line reconnections—*cascades* from the main length scale (typical interline separation, R_0) to essentially lower length scales; ultimately decaying into phonons, as it was pointed out by Vinen [4, 8].

A very specific feature of KW cascade is that the intrinsic vortex line dynamics in the local-induction approximation (LIA) (for an introduction, see, e.g., [2, 13]) controlled by the small parameter $1/\ln(R_0/\xi)$, with ξ the vortex core radius, is subject to a specific curvature-conservation constraint rendering it unable to support the cascade process [3] (see also below). Within LIA, an "external" ingredient of the vortex line dynamics—the vortex line crossings with subsequent reconnections—is required to push the KW cascade down towards arbitrarily small wavelengths. The most characteristic feature of this LIA scenario, distinguishing it from typical non-linear cascades, is the fragmentation of the vortex lines due to local self-crossings [3]; we will thus refer to this scenario as *fragmentational* scenario.

Experimentally, the main consequence of the existence of a cascade regime, no matter what is its microscopic nature, is independence of the relaxation time of superfluid turbulence on temperature in the $T \to 0$ limit. Davis *et al.* have observed such a regime set in in ^{4}He at $T < 70mK$ [5]. The numeric simulation of the vortex tangle decay at $T = 0$ performed by Tsubota *et al.* [6] within the framework of LIA has clearly revealed the cascade regime; this may be considered as a circumstantial [14] evidence of the proposed in [3] scenario.

A general question arises, however, of how far, in the wavenumber space, the structure of KW cascade is predetermined by LIA dynamics. At wavelengths $\lambda \ll R_0$ the non-local effects of the vortex line dynamics compete with LIA dynamics and can ultimately become the main driving force of the cascade. Moreover, given the specific spectrum of KW turbulence associated with the fragmentational scenario, where the amplitude of the turbulence is smaller than the wavelength only by a logarithmic factor [3], one concludes, that if non-local effects can in principle support the cascade, no matter how small is the corresponding contribution at the main wavelength scale R_0, there will inevitably be such a wavelength scale $\lambda_* \ll R_0$, where the fragmentational scenario will be replaced by a purely non-linear—to be referred as *pure*—scenario in which vortex line self-crossings play no role. The existence of the crossover between the two cascade scenarios has at least two important implications. First, the spectrum of sizes of vortex rings generated by decaying superfluid turbulence will have a lower cutoff $\sim \lambda_*$. Secondly, the spectrum of KW turbulence will be changed, which, in particular, is crucial for the cascade cutoff theory [8] where the characteristic wavelength $\lambda_{\rm ph}$ at which KW essentially decay into phonons is a function of the cascade spectrum.

A strong numeric evidence in favor of existence of pure KW cascade has been reported by Kivotides *et al.* [7]. Being very expensive numerically, this simulation was not able to accurately resolve the spectrum of KW turbulence. The data of recent simulation by Vinen *et al.* [10] seem to be more conclusive.

In this Letter, we propose a self-consistent (asymptotically exact in the limit of high wavenumbers) treatment for the pure KW cascade. More generally, we develop the KW kinetic theory in the regime of weak turbulence, where the smallness of non-linearities reduces the non-linear effects to scattering processes for the harmonic modes. We find that the leading elementary process responsible for the kinetics is the three-kelvon scattering. We demonstrate that the kinetics is *entirely* due to the spatially non-local interactions, the local contributions exactly cancelling each other. This cancellation sets the limit on the maximum power of the pure cascade: At the wavelength scale $\sim R_0$ the contribution of the kelvon-scattering processes to the KW cascade contains a small factor $1/\ln^2(R_0/\xi)$ as compared to the reconnection-induced part.

In terms of weak turbulence theory, KW cascade is a Kolmogorov cascade [15], associated with the transport of energy (closely related to the vortex line length in our case) in the wavenumber space. We establish the Kolmogorov spectrum and find the relation between the energy flux and the amplitude of KW turbulence. Finally, on the basis of our results we discuss the fragmentational-to-pure cascade crossover, which we predict to be rather extended (up to two decades) in the wavenumber space.

Hamiltonian. We employ the Hamiltonian representation of the vortex line motion [3], which is exact up to a certain geometric constraint: there should exist some axis z, with respect to which the position of the line can be specified in the parametric form $x = x(z)$, $y = y(z)$, where x and y are single-valued functions of the coordinate z. This perfectly suits our purposes, since we are interested in the wavelengths substantially smaller than R_0 and thus will treat the vortex line as a straight line with small-amplitude distortions. In terms of the complex canonic variable $w(z,t) = x(z,t) + iy(z,t)$, the Biot-Savart dynamic equation acquires the Hamiltonian form $i\dot{w} = \delta H[w]/\delta w^*$, with [3]

$$H = \frac{\kappa}{4\pi}\int dz_1 dz_2 \frac{1 + \mathrm{Re}\, w'^*(z_1)w'(z_2)}{\sqrt{(z_1-z_2)^2 + |w(z_1)-w(z_2)|^2}} \; , \quad (1)$$

where $\kappa = 2\pi\hbar/m$ is the circulation quantum (m is the particle mass). Our fundamental requirement that the amplitude of KW turbulence be small as compared to the wavelength is formulated as

$$\alpha(z_1, z_2) = \frac{|w(z_1) - w(z_2)|}{|z_1 - z_2|} \ll 1 \; . \quad (2)$$

This allows us to expand (1) in powers of α: $H = E_0 + H_0 + H_1 + H_2 + \ldots$ (E_0 is just a number and will be ignored). The terms that prove relevant are:

$$H_0 = \frac{\kappa}{8\pi}\int \frac{dz_1 dz_2}{|z_1 - z_2|}[2\mathrm{Re}\, w'^*(z_1)w'(z_2) - \alpha^2] \; , \quad (3)$$

$$H_1 = \frac{\kappa}{32\pi}\int \frac{dz_1 dz_2}{|z_1 - z_2|}[3\alpha^4 - 4\alpha^2 \mathrm{Re}\, w'^*(z_1)w'(z_2)] \; , \quad (4)$$

$$H_2 = \frac{\kappa}{64\pi}\int \frac{dz_1 dz_2}{|z_1 - z_2|}[6\alpha^4 \mathrm{Re}\, w'^*(z_1)w'(z_2) - 5\alpha^6] \; . \quad (5)$$

The Hamiltonian H_0 describes the linear properties of KW. It is diagonalized by the Fourier transformation $w(z) = L^{-1/2}\sum_k a_k e^{ikz}$ (L is the system size, periodic boundary conditions are assumed):

$$H_0 = \frac{\kappa}{4\pi}\sum_k \omega_k\, a_k^* a_k \; , \qquad \omega_k = (\kappa/4\pi)\, \ln(1/k\xi)\, k^2 \; , \quad (6)$$

yielding Kelvin's dispersion law ω_k .

Though the problem of KW cascade generated by decaying superfluid turbulence is purely classical, it is convenient to approach it quantum mechanically—by introducing KW quanta, kelvons. In accordance with the canonical quantization procedure, we understand a_k as the annihilation operator of the kelvon with momentum k and correspondingly treat $w(z)$ as a quantum field. A minor caveat is in order here. The Hamiltonian functional (1) is *proportional* to the energy—with the coefficient $\kappa\rho/2$, where ρ is the mass density—but not *equal* to it. This means that if one prefers to work with genuine Quantum Mechanics rather than a fake one (for our purposes, the latter is also enough), using true kelvon annihilation operators, $\hat{a}_k$, field operator $\hat{w}$, and Hamiltonian, $\hat{H}$, he should take into account proper dimensional coefficients: $\hat{H} = (\kappa\rho/2)H$, $\hat{w}(z) = \sqrt{2\hbar/\kappa\rho L}\sum_k \hat{a}_k e^{ikz}$. By choosing the units $\hbar = \kappa = 1$, $\rho = 2$, we ignore these coefficients until the final answers are obtained.

In the quantum approach, there naturally arises the notion of the number of kelvons. This number is *conserved* by the Hamiltonian (1) in view of its global $U(1)$ symmetry, $w \to e^{i\varphi}w$ (reflecting rotational symmetry of the problem). Another advantage of the quantum language in weak-turbulence problems is that the collision term of the kinetic equation immediately follows from the Golden Rule for corresponding elementary processes.

Kelvon scattering. As was mentioned above, Hamiltonian (1) implies only elastic scattering. The one-dimensional character of the problem in combination with the conservation of the momentum and energy suppresses the two-kelvon scattering channel: the process $(k_1, k_2) \to (k_3, k_4)$ is only possible if either $(k_3 = k_1,\ k_4 = k_2)$, or $(k_3 = k_2,\ k_4 = k_1)$ which does not lead to any kinetics. We thus conclude that the leading process in our case is the three-kelvon scattering. The processes involving four and more kelvons a much weaker due to the non-equality (2). The effective vertex, $V_{1,2,3}^{4,5,6}$, for the three-kelvon scattering process [subscripts (superscripts) stand for the initial (final) momenta of the three kelvons; we use a short-hand notation replacing each momentum k_j with its index j] consists of two different parts. The first part involves terms generated by the two-kelvon vertex, A (corresponding to the Hamiltonian H_1) in the second order of perturbation theory. [All these terms are similar to each other; we explicitly specify just one of them: $A_{1,2}^{4,7}\, G(\omega_7, k_7)\, A_{7,3}^{5,6}$. Here $G(\omega, k) = 1/(\omega - \omega_k)$ is the free-kelvon propagator, $\omega_7 = \omega_1 + \omega_2 - \omega_4$, $k_7 = k_1 + k_2 - k_4$.] The second part of the vertex V is the bare three-kelvon vertex, B, associated with the Hamiltonian H_2.

The explicit expressions for the bare vertices directly follow from (4) and (5): $A = (6D - E)/8\pi$, $D_{1,2,3}^{4,5,6} =$

$\int_0^L (\mathrm{d}x/x^5)[\,1 - C_{\{1\}} - C_{\{2\}} - C^{\{3\}} - C^{\{4\}} + C^{\{3\}}_{\{2\}} + C^{\{43\}} + C^{\{4\}}_{\{2\}}]$, $E^{4,5,6}_{1,2,3} = \int_0^L (\mathrm{d}x/x^3)[\,k_4 k_1(\,C^{\{4\}} + C_{\{1\}} - C^{\{43\}} - C^{\{4\}}_{\{2\}}\,) + k_3 k_1(\,C^{\{3\}} + C_{\{1\}} - C^{\{43\}} - C^{\{3\}}_{\{2\}}\,) + k_3 k_2(\,C^{\{3\}} + C_{\{2\}} - C^{\{43\}} - C^{\{3\}}_{\{1\}}\,) + k_4 k_2(\,C^{\{4\}} + C_{\{2\}} - C^{\{43\}} - C^{\{3\}}_{\{2\}}\,]$; $B = (3P - 5Q)/4\pi$, $P^{4,5,6}_{1,2,3} = \int_0^L (\mathrm{d}x/x^5)\;\; k_6 k_2 [C_{\{2\}} - C^{\{5\}}_{\{2\}} - C_{\{23\}} + C^{\{5\}}_{\{23\}} - C^{\{4\}}_{\{2\}} + C^{\{45\}}_{\{2\}} + C^{\{4\}}_{\{23\}} - C^{\{6\}}_{\{1\}} + C^{\{6\}} - C^{\{56\}} - C^{\{6\}}_{\{3\}} + C^{\{56\}}_{\{3\}} - C^{\{46\}} + C^{\{456\}} + C^{\{46\}}_{\{3\}} - C_{\{12\}}]$, $Q^{4,5,6}_{1,2,3} = \int_0^L (\mathrm{d}x/x^7)\;[1 - C^{\{4\}} - C_{\{1\}} + C^{\{4\}}_{\{1\}} - C^{\{6\}} + C^{\{46\}} + C^{\{6\}}_{\{1\}} - C^{\{46\}}_{\{1\}} - C^{\{5\}} + C^{\{45\}} + C^{\{5\}}_{\{1\}} - C^{\{45\}}_{\{1\}} + C^{\{65\}} - C^{\{456\}} - C^{\{56\}}_{\{1\}} + C_{\{23\}} - C_{\{3\}} + C^{\{4\}}_{\{3\}} + C_{\{13\}} - C^{\{4\}}_{\{13\}} + C^{\{6\}}_{\{3\}} - C^{\{46\}}_{\{3\}} - C^{\{6\}}_{\{13\}} + C^{\{5\}}_{\{2\}} + C^{\{5\}}_{\{3\}} - C^{\{45\}}_{\{3\}} - C^{\{5\}}_{\{13\}} + C^{\{6\}}_{\{2\}} - C^{\{65\}}_{\{3\}} + C_{\{12\}} + C^{\{4\}}_{\{2\}} - C_{\{2\}}]$. Here C's denote similar looking cosine functions: $C_{\{1\}} = \cos k_1 x$, $C^{\{4\}}_{\{1\}} = \cos(k_4 - k_1)x$, $C^{\{45\}}_{\{1\}} = \cos(k_4 + k_5 - k_1)x$, $C^{\{45\}}_{\{12\}} = \cos(k_4 + k_5 - k_1 - k_2)x$, and so forth. The integrals for bare vertices A and B logarithmically diverge at their lower limits and, generally speaking, one has to introduce a cut-off parameter $\sim \xi$. However, in the final answer for the effective vertex V the logarithmic in ξ terms cancel each other (it is important here to take into account that, in the propagator G, the frequency ω_k also diverges logarithmically with $\xi \to 0$), so that V is finite in the limit $\xi \to 0$. This fact is not just a mere coincidence. The small-x contributions to the integrals correspond to the local-induction dynamics, which cannot lead to any kinetics because of the extra constants of motion [3]. Cancellation of the LIA contributions at the level of the effective scattering amplitude is an explicit demonstration of this circumstance.

Kinetic equation. The kinetic equation is written in terms of averaged over the statistical ensemble kelvon occupation numbers $n_k = \langle a^\dagger_k a_k \rangle$:

$$\dot{n}_1 = \frac{1}{(3-1)!\,3!} \sum_{k_2,\ldots,k_6} \left(W^{1,2,3}_{4,5,6} - W^{4,5,6}_{1,2,3} \right) \,. \qquad (7)$$

Here $W^{4,5,6}_{1,2,3}$ is the probability per unit time for the elementary scattering event $(k_1,k_2,k_3) \to (k_4,k_5,k_6)$; combinatorial factor compensates multiple counting the same scattering event. For our interaction Hamiltonian $H_{\rm int} = \sum_{k_1,\ldots,k_6} \delta(\Delta k)\, \tilde{V}^{4,5,6}_{1,2,3}\, a^\dagger_6 a^\dagger_5 a^\dagger_4 a_3 a_2 a_1$ [where the vertex $\tilde{V}$ is obtained from V by symmetrization with respect to corresponding momenta permutations; $\delta(k)$ is understood discretely as $\delta_{k,0}$, and $\Delta k = k_1 + k_2 + k_3 - k_4 - k_5 - k_6$] the Golden Rule reads: $W^{4,5,6}_{1,2,3} = 2\pi |(3!)^2 \tilde{V}^{4,5,6}_{1,2,3}|^2 f^{4,5,6}_{1,2,3}\, \delta(\Delta\omega) \delta(\Delta k)$, where $f^{4,5,6}_{1,2,3} = n_1 n_2 n_3 (n_4+1)(n_5+1)(n_6+1)$, $\Delta\omega = \omega_1 + \omega_2 + \omega_3 - \omega_4 - \omega_5 - \omega_6$. Combinatorial factor $(3!)^2$ takes into account addition of equivalent amplitudes. The classical-field limit of the quantum kinetic equation (7) is obtained by retaining only the largest in occupation numbers terms:

$$\dot{n}_1 = 216\pi \sum_{k_2,\ldots,k_6} |\tilde{V}^{4,5,6}_{1,2,3}|^2\, \delta(\Delta\omega)\, \delta(\Delta k) \left(\tilde{f}^{1,2,3}_{4,5,6} - \tilde{f}^{4,5,6}_{1,2,3} \right) , \qquad (8)$$

where $\tilde{f}^{4,5,6}_{1,2,3} = n_1 n_2 n_3 (n_4 n_5 + n_4 n_6 + n_5 n_6)$.

Kolmogorov cascade. Kinetic equation (8) supports Kolmogorov energy cascade [15], provided two conditions are met: (i) the kinetic time is getting progressively smaller (vanishes) in the limit of large wavenumbers, (ii) the collision term is local in the wavenumber space—not to be confused with the local-induction approximation in the real space,—that is the relevant scattering events are only those where all the kelvon momenta are of the same order of magnitude. We make sure that both conditions are satisfied in our case: the condition (i) can be checked by a dimensional estimate, provided (ii) is true. The condition (ii) is verified numerically.

The cascade spectrum can be established by dimensional analysis of the kinetic equation. The estimate of Eq. (8) yields: $\dot{n}_k \sim k^5 \cdot |V|^2 \cdot \omega_k^{-1} \cdot k^{-1} \cdot n_k^5$, the factors go in the order of the appearance of corresponding terms in (8). At $k_1 \sim \ldots \sim k_6 \sim k$ we have $|V| \sim k^6$ and

$$\dot{n}_k \;\sim\; \omega_k^{-1}\, n_k^5\, k^{16} \,. \qquad (9)$$

The energy flux (per unit vortex line length), θ_k, at the momentum scale k is defined as

$$\theta_k \;=\; L^{-1} \sum_{k' < k} \omega_{k'}\, \dot{n}_{k'} \,, \qquad (10)$$

implying the estimate $\theta_k \sim k \dot{n}_k \omega_k$. Combined with (9), this yields $\theta_k \sim n_k^5 k^{17}$, and the cascade requirement that θ_k be actually k-independent leads to the spectrum:

$$n_k = A\, k^{-17/5} \,. \qquad (11)$$

The value of the parameter A in (11) corresponds to the value of the energy flux. An accurate relation between θ and A is (we restore all dimensional parameters):

$$\theta \approx 3 \cdot 10^{-4}\, \frac{\hbar^5 A^5}{\kappa^2 \rho^4} \,. \qquad (12)$$

The dimensionless coefficient in this formula has been established numerically (see below) with the error $\sim 75\%$. One should not be confused with the appearance of the Plank's constant in this relation—it is entirely due to the fact that we use quantum mechanical definition of n_k.

The exponent 17/5 is in a perfect agreement (within the error bars) with the spectrum observed in Ref. [10] ($\bar{\zeta}_k^2$ of Ref. [10] is proportional to our n_k); it fits the data much better than the exponent 3 used by the authors.

It is useful to express the KW cascade spectrum in terms of a geometric characteristic—typical amplitude, b_k, of the KW turbulence at the wavevector $\sim k$. By the definition of the field $\hat{w}(z)$ we have: $b_k^2 \sim L^{-1} \sum_{q \sim k} \langle \hat{a}^\dagger_q \hat{a}_q \rangle = L^{-1} \sum_{q \sim k} n_q \sim k\, n_k$. Hence,

$$b_k \propto A^{1/2} k^{-6/5} \,. \qquad (13)$$

One can convert (11) into the curvature spectrum. For the curvature $\mathbf{c}(\zeta) = \partial^2 \mathbf{s}/\partial\zeta^2$ [where $\mathbf{s}(\zeta)$ is the radius-vector of the curve as a function of the arc length ζ], the spectrum is defined as Fourier decomposition of the integral $I_c = \int |\mathbf{c}(\zeta)|^2 \, \mathrm{d}\zeta$. The smallness of α, Eq. (2), allows one to write $I_c \approx \int \mathrm{d}z \langle \hat{w}''^{\dagger}(z)\hat{w}''(z)\rangle = \sum_k k^4 \, n_k \propto \sum_k k^{3/5}$, arriving thus at the exponent 3/5.

Numerics. The aim of our numeric analysis is (i) to make sure that the collision term of the kinetic equation is local and (ii) to establish the value of the dimensionless coefficient in (12). The analysis is based on the following reasoning [16]. Consider a power-law distribution of occupation numbers, $n_k = A/k^\beta$, with the exponent β arbitrarily close, but not equal, to the cascade exponent $\beta_0 = 17/5$. Substitute this distribution in the collision term of the kinetic equation—right-hand side of (8). Given the scale invariance of the power-law distribution, the following alternative takes place. Case (1): collision integral converges for β's close to β_0, and, in accordance with a straightforward dimensional analysis, is equal to

$$\mathrm{Coll}([n_k = A/k^\beta], k) = C(\beta) A^5 \omega_k^{-1}/k^{5\beta-16} \; . \qquad (14)$$

Here $C(\beta)$ is a dimensionless function of β, such that $C(\beta_0) = 0$ since the cascade is a steady-state solution. Case (2): collision integral diverges for β close to β_0. The case (2) means that the collision term is non-local and the whole analysis in terms of Kolmogorov cascade is irrelevant. [Fortunately, our numerics show that we are dealing with the case (1).] Substituting (14) for $\dot{n}_k$ in (10), we obtain the expression $\theta_k(\beta) = (A^5/2\pi)\, k^{17-5\beta}\, C(\beta)/(17-5\beta)$. Taking the limit $\beta \to \beta_0$, we arrive at the k-independent flux $\theta = -C'(\beta_0)A^5/10\pi$. We use this formula to obtain the coefficient in (12) by calculating $C(\beta)$ and finding its derivative $C'(\beta_0)$. We simulated the collision integral by Monte Carlo method. The integrals D, E, P, Q, were calculated numerically. The slowing down of the simulation due to the latter circumstance is the main source of large relative error.

Superfluid turbulence. The main characteristic of the superfluid turbulence is the vortex line density, L (total line length per unit volume; note that $R_0 \sim 1/\sqrt{L}$.) The parameter A is related to L through the energy flux θ. Since (to a good accuracy) the energy is the total line length times $(\rho\kappa^2/4\pi)\ln(R_0/\xi)$, the same coefficient relates the energy flux to the line-length flux. The line-length flux, $\dot{L}/L$, is available from the simulation of Ref. [6]: $\dot{L}/L \approx 6\cdot 10^{-3}\kappa\, L\, \ln(1/\xi\sqrt{L})$. We thus get

$$(\hbar A/\kappa\rho)^5 \approx 2\, L\, \ln^2(1/\xi\sqrt{L}) \; . \qquad (15)$$

For the amplitude spectrum this yields:

$$b_k k \sim \left[(\sqrt{L}/k)\, \ln(1/\xi\sqrt{L})\right]^{1/5} \; . \qquad (16)$$

With this spectrum, Vinen's prediction for the cutoff momentum $k_{\mathrm{ph}} \sim \lambda_{\mathrm{ph}}^{-1}$, based on Eq. (2.24) of Ref. [8], should read (c is the sound velocity):

$$k_{\mathrm{ph}}/\sqrt{L} \sim \frac{\ln^{4/9}(1/\xi\sqrt{L})}{\ln^{10/9}(1/\xi k_{\mathrm{ph}})} \left[\frac{c}{\kappa\sqrt{L}}\right]^{5/6} \; . \qquad (17)$$

From (16) it is seen that the crossover from the fragmentational cascade regime to the pure one should be very slow. Indeed, to significantly suppress the fragmentational regime—that is to suppress local self-crossings of the vortex line—it is necessary to make the parameter $b_k k$ substantially smaller than unity. In view of the exponent 1/5, this requires increasing k by $\sim$ two orders of magnitude. In this extended crossover region, one has $b_k k \sim 1$, while for the applicability of the treatment developed in this Letter one needs $b_k k \ll 1$. Hence, a reliable description of the crossover from the fragmentational to the pure cascade seems to be impossible without a direct numeric simulation of vortex line dynamics.

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[14] Assuming the relevance of LIA at least at the length scales adjacent to R_0, as well as accepting the conservation-law argument of Ref. [3] about the impossibility of a cascade fragmentation of *simple* vortex ring into infinite number of arbitrarily small rings, one can replace the word "circumstantial" with "smoking gun": The simulation of Ref. [6] has explicitly revealed the fragmentation of the tangle into small rings.
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