ON A NOTION OF MAPS BETWEEN ORBIFOLDS II. HOMOTOPY AND CW-COMPLEX

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ABSTRACT

This is the second of a series of papers which are devoted to a comprehensive theory of maps between orbifolds. In this paper, we develop a basic machinery for studying homotopy classes of such maps. It contains two parts: (1) the construction of a set of algebraic invariants – the homotopy groups, and (2) an analog of CW-complex theory. As a corollary of this machinery, the classical Whitehead theorem which asserts that a weak homotopy equivalence is a homotopy equivalence is extended to the orbifold category.

1. INTRODUCTION

In [3] we introduced a notion of maps between orbifolds, and established several basic results concerning the topological structure of the corresponding mapping spaces. This was in an attempt to initiate a comprehensive study of orbifolds, which was motivated by work of Dixon, Harvey, Vafa and Witten [9] on string theories of orbifolds. Particularly, the aforementioned theorems about the topological structure of mapping spaces had direct applications in the theory of pseudoholomorphic curves and Gromov-Witten invariants of symplectic orbifolds (cf. [7, 8, 4, 5, 6]). See the introduction of [3] for more detailed explanations on this aspect of the story.

The present paper is continuation of [3]. The main objective of this paper is to establish the corresponding homotopy theory for studying such maps between orbifolds. In particular, we have developed techniques of exhibiting a class of orbispaces (which include orbifolds) as results of gluing together some fundamental building blocks, ie. cells of various isotropy types. The corresponding algebraic invariants needed for describing such constructions are the sets of homotopy classes of the gluing maps. Thus as homotopy groups in this theory, we have studied in details the sets of homotopy classes of maps from spheres of various isotropy types into an orbispace. We remark that these homotopy groups form a strictly larger set of invariants than the usual homotopy groups of an orbifold (which by definition are the homotopy groups of the corresponding classifying spaces, cf. [11]). See Proposition 1.4 for more details in this regard.

In this paper, we shall mainly work with a more restricted class of maps rather than the general ones introduced in [3]. The details are presented in the following assumption throughout. We refer the reader to §2.2 of [3] for the basic definitions.

Convention In this paper, we shall only consider maps of orbispaces which are equivalence classes of groupoid homomorphisms $(\{f_\alpha\}, \{\rho_{\beta\alpha}\}) : \Gamma\{U_\alpha\} \rightarrow \Gamma\{U'_{\alpha'}\}$, where each $\rho_\alpha = \rho_{\alpha\alpha}$:

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$G_{U_\alpha} \rightarrow G_{U'_\alpha}$ is an injective homomorphism. As a result of this assumption, the mappings $\{\rho_{\beta\alpha}\}$ are partially injective in the sense that if $\rho_{\beta\alpha}(\xi_1) = \rho_{\beta\alpha}(\xi_2)$ and Domain $(\phi_{\xi_1}) \cap$ Domain $(\phi_{\xi_2}) \neq \emptyset$, then $\xi_1 = \xi_2$.

In studying homotopy classes of maps and homotopy types of orbispaces, we need to fix a base point in the orbispaces throughout the consideration, in which all maps have to preserve the pre-chosen base point structures. We shall give an introduction to this notion next, before we come to the detailed description of the homotopy classes of maps and homotopy types of orbispaces that are to be considered in this paper.

Let X be an orbispace, with its atlas of local charts denoted by $\mathcal{U} = \{U_i\}$. A base-point structure of X is a triple, denoted by $\underline{o} = (o, U_o, \hat{o})$, where $o \in X$, $U_o \in \mathcal{U}$ such that $o \in U_o$, and $\hat{o} \in \pi^{-1}_{U_o}(o) \subset \widehat{U_o}$. An orbispace X with a chosen base-point structure $\underline{o}$ will be denoted by $(X, \underline{o})$. We remark that for the special case where $X = Y/G$, with the action of G on Y being trivial, we have $X = Y$ and furthermore, a base-point structure $\underline{o} = (o, U_o, \hat{o})$ is completely determined by the base point o. We shall fix the notation $Y(G)$ for such an $X = Y/G$, and write $(Y(G), o)$ for $(X, \underline{o})$. The orbispace $Y(G)$ will be called a space of isotropy type G throughout. The most frequently used examples of this type of orbispaces are $S^k(G), D^k(G)$, ie. the k-sphere and k-cell of isotropy type G. Note that for the 1-cell and 0-cell of isotropy type G, we shall use different notations $I(G)$, B_G respectively.

With these notations being fixed throughout the paper, we now consider the maps from $(X, \underline{o})$ to $(X', \underline{o'})$, where $\underline{o} = (o, U_o, \hat{o})$ and $\underline{o'} = (o', U'_o, \hat{o'})$, which preserve the corresponding base-point structures. Let $\rho : G_{\hat{o}} \rightarrow G_{\hat{o'}}$ be any given injective homomorphism, where $G_{\hat{o}}$, $G_{\hat{o'}}$ are the subgroups of G_{U_o}, $G_{U'_o}$ that fix $\hat{o}$ and $\hat{o'}$ respectively, and let $\sigma = (\{f_\alpha\}, \{\rho_{\beta\alpha}\}) : \Gamma\{U_\alpha\} \rightarrow \Gamma\{U'_{\alpha'}\}$ be any groupoid homomorphism which satisfies the following conditions: (1) $U_o \in \{U_\alpha\}$, $U'_o \in \{U'_{\alpha'}\}$, and $U_o \mapsto U'_o$ under the correspondence $U_\alpha \mapsto U'_\alpha$, (2) $f_o : \widehat{U_o} \rightarrow \widehat{U'_o}$ satisfies $f_o(\hat{o}) = \hat{o'}$, and (3) $\rho_o|_{G_{\hat{o}}} = \rho$. Suppose $\tau = (\{f_a\}, \{\rho_{ba}\})$ is induced by σ via $\bar{\gamma} = (\theta, \{\xi_a\}, \{\xi'_a\})$ (cf. Lemma 2.2.4 in [3]), where Condition (1) above is satisfied for τ, and $\bar{\gamma}$ satisfies $\theta(o) = o$, $\xi_o = 1 \in T(U_o, U_o) = G_{U_o}$, and $\xi'_o = 1 \in T(U'_o, U'_o) = G_{U'_o}$, then one can easily verify that Conditions (2), (3) above are also satisfied for τ. In other words, the base-point structures $\underline{o}$ and $\underline{o'}$ are preserved under the process of taking equivalence classes in the sense described above. We will call such an equivalence class a map from $(X, \underline{o})$ to $(X', \underline{o'})$, and we denote the set of all such maps by $[(X, \underline{o}); (X', \underline{o'})]_\rho$.

Some variant or generalization of the preceding will also be considered. For example, suppose $(X, \underline{o})$ and $(X', \underline{o'})$ are orbispaces with pre-chosen base-point structures, where in the base-point structure $\underline{o} = (o, U_o, \hat{o})$, the group G_{U_o} acts trivially on $\widehat{U_o}$. In this case, we shall define maps from $(X, \underline{o})$ to $(X', \underline{o'})$ by further allowing, in the definition of equivalence relation, that U_o may be replaced by a $V_o \subset U_o$ where $o \in V_o$, for which the requirements $\xi_o = 1 \in G_{U_o} = G_{V_o}$ and $\xi'_o = 1 \in G_{U'_o}$ in $(\theta, \{\xi_a\}, \{\xi'_a\})$ still make sense. Correspondingly, this means that the base-point structure $\underline{o} = (o, U_o, \hat{o})$ is reduced to the ordinary base point o. We shall simply write o for $\underline{o}$. An important case for such an $(X, \underline{o})$ is $(Y(G), o)$.

On the other hand, given any X and X', we may fix a set of base-point structures $\{\underline{o}_i\}$ of X and a set of base-point structures $\{\underline{o'}_k\}$ of X', together with a correspondence $\underline{o}_i \mapsto \underline{o'}_k = \underline{o'}_{\theta(i)}$. In the same vein, we may define maps between these "multi-based" orbispaces, requiring that the given sets of base-point structures are preserved under these maps with respect to the given correspondence $\underline{o}_i \mapsto \underline{o'}_k = \underline{o'}_{\theta(i)}$.

Now we give some detailed descriptions about the homotopy classes of maps and homotopy types of orbispaces that will be considered in this paper.

Two maps $\Phi_1, \Phi_2 : X \rightarrow X'$ are said to be homotopic (and written $\Phi_1 \simeq \Phi_2$) if there exists a map $\Psi : X \times [a,b] \rightarrow X'$ such that Φ_1, Φ_2 are the restrictions of Ψ to the subspaces $X \times \{a\}$ and $X \times \{b\}$ respectively. The map Ψ is called a homotopy between Φ_1 and Φ_2. Let $\Psi_1 : X \times [a,b] \rightarrow X'$ be a homotopy between Φ_1 and Φ_2, and $\Psi_2 : X \times [b,c] \rightarrow X'$ be a homotopy between Φ_2 and Φ_3. Then there is a homotopy $\Psi_3 : X \times [a,c] \rightarrow X'$ between Φ_1 and Φ_3, whose restrictions to $X \times [a,b]$ and $X \times [b,c]$ are Ψ_1 and Ψ_2 respectively. A homomorphism representing Ψ_3 may be obtained as follows. Take a representing homomorphism σ_1 of Φ_1, and a representing homomorphism σ_2 of Φ_2. We may assume, by passing to an induced one, that near $X \times \{b\}$, σ_1 is given by $(\{F_{i,1}\}, \{\hat{\rho}_{ji,1}\}) : \Gamma\{U_i \times (b^i_-, b]\} \rightarrow \Gamma\{U'_{i'}\}$ and σ_2 is given by $(\{F_{i,2}\}, \{\hat{\rho}_{ji,2}\}) : \Gamma\{U_i \times [b, b^i_+)\} \rightarrow \Gamma\{U'_{i'}\}$ such that for each index i, $F_{i,1}|_{\widehat{U_i} \times \{b\}} = F_{i,2}|_{\widehat{U_i} \times \{b\}}$, and for each pair of indexes (i,j), $\hat{\rho}_{ji,1} = \hat{\rho}_{ji,2}$. (Note that $T(U_i \times (b^i_-, b], U_j \times (b^j_-, b]) = T(U_i, U_j) = T(U_i \times [b, b^i_+), U_j \times [b, b^j_+))$.) It is clear that such a pair (σ_1, σ_2) of homomorphisms can be patched together to form a homomorphism whose equivalence class is a map from $X \times [a,c]$ to X'. We define Ψ_3 to be the corresponding map. Note that although the homotopy Ψ_3 may not be uniquely determined, its existence implies that the homotopy relation is an equivalence relation on the set of maps $[X; X']$. The corresponding set of homotopy classes will be denoted by $[[X; X']]$.

Let $\Phi : Y \rightarrow X$ be any map. There are induced mappings $\Phi^* : [[X; X']] \rightarrow [[Y; X']]$ sending $[\Psi]$ to $[\Psi \circ \Phi]$, and $\Phi_* : [[X'; Y]] \rightarrow [[X'; X]]$ sending $[\Psi]$ to $[\Phi \circ \Psi]$. The mappings Φ^*, Φ_* depend only on the homotopy class of Φ. A map $\Phi : Y \rightarrow X$ is called a homotopy equivalence if there is a map $\Psi : X \rightarrow Y$ such that $\Phi \circ \Psi \simeq Id_X$ and $\Psi \circ \Phi \simeq Id_Y$. It is routine to check that $\Phi : Y \rightarrow X$ is a homotopy equivalence if and only if both mappings Φ^*, Φ_* are bijections for any orbispace X'.

Homotopy preserving given base-point structures can be defined in the same vein. Let $\underline{o} = (o, U_o, \hat{o})$, $\underline{o'} = (o', U'_o, \hat{o}')$ be any base-point structures on X and X' respectively. Two maps $\Phi_1, \Phi_2 \in [(X, \underline{o}); (X', \underline{o'})]_\rho$ are said to be homotopic (and written $\Phi_1 \simeq \Phi_2$) if there exists a homomorphism $\sigma = (\{F_i\}, \{\hat{\rho}_{ji}\}) : \Gamma\{U_i \times I_i\} \rightarrow \Gamma\{U'_{i'}\}$, where each I_i is a sub-interval of $[a,b]$, such that (1) $U_o \times [a,b] \in \{U_i \times I_i\}$, which corresponds to U'_o under $U_i \times I_i \mapsto U'_i$, (2) $F_o(\{\hat{o}\} \times [a,b]) = \hat{o}'$, (3) $\hat{\rho}_o|_{G_{\hat{o}}} = \rho$, and (4) Φ_1, Φ_2 are the equivalence classes of the restriction of σ to $X \times \{a\}$ and $X \times \{b\}$ respectively. The set of homotopy classes of elements in $[(X, \underline{o}); (X', \underline{o'})]_\rho$ will be denoted by $[[(X, \underline{o}); (X', \underline{o'})]]_\rho$. Given any $\Phi \in [(Y, \underline{p}); (X, \underline{o})]_\rho$, there are induced mappings $\Phi^* : [[(X, \underline{o}); (X', \underline{o'})]]_\eta \rightarrow [[(Y, \underline{p}); (X', \underline{o'})]]_{\eta \circ \rho}$ and $\Phi_* : [[(X', \underline{o'}); (Y, \underline{p})]]_\eta \rightarrow [[(X', \underline{o'}); (X, \underline{o})]]_{\rho \circ \eta}$, defined by $\Phi^*([\Psi]) = [\Psi \circ \Phi]$ and $\Phi_*([\Psi]) = [\Phi \circ \Psi]$, which depend only on the homotopy class of Φ.

Homotopy classes of maps between pairs of orbispaces (X, A) will also be considered, where A is a subspace of X. If $\underline{o} = (o, U_o, \hat{o})$ is a base-point structure of X such that $o \in A$, then there is a canonical base-point structure of A, denoted by $\underline{o}|_A = (o, V_o, \hat{o})$, where V_o is the connected component of $U_o \cap A$ that contains o. We will denote a pair with such a base-point structure by $(X, A, \underline{o})$. A map from (X, A) to (X', A') is an element $\Phi \in [X; X']$ such that $\Phi|_A \in [A; A']$, and a map from $(X, A, \underline{o})$ to $(X', A', \underline{o'})$ is an element $\Phi \in [(X, \underline{o}); (X', \underline{o'})]_\rho$ for some $\rho : G_{\hat{o}} \rightarrow G_{\hat{o}'}$, such that $\Phi|_A \in [(A, \underline{o}|_A); (A', \underline{o'}|_{A'})]_\rho$. The set of homotopy classes of the latter will be denoted by $[[(X, A, \underline{o}); (X', A', \underline{o'})]]_\rho$. Given any $\Phi \in [(Y, B, \underline{p}); (X, A, \underline{o})]_\rho$, there are induced mappings $\Phi^* : [[(X, A, \underline{o}); (X', A', \underline{o'})]]_\eta \to [[(Y, B, \underline{p}); (X', A', \underline{o'})]]_{\eta\circ\rho}$ and $\Phi_* : [[(X', A', \underline{o'}); (Y, B, \underline{p})]]_\eta \to [[(X', A', \underline{o'}); (X, A, \underline{o})]]_{\rho\circ\eta}$ defined by $\Phi^*([\Psi]) = [\Psi \circ \Phi]$, $\Phi_*([\Psi]) = [\Phi \circ \Psi]$. Again the mappings Φ^*, Φ_* depend only on the homotopy class of Φ.

With the preceding preparatory discussion, we now present the main results in this paper.

Definition 1.1 *For any base-point structure $\underline{o} = (o, U_o, \hat{o})$ and injective homomorphism $\rho : G \to G_{\hat{o}}$, we define*

$$\pi_k^{(G,\rho)}(X, \underline{o}) = [[(S^k(G), *); (X, \underline{o})]]_\rho, \ \forall k \geq 0,$$

and

$$\pi_{k+1}^{(G,\rho)}(X, A, \underline{o}) = [[(D^{k+1}(G), S^k(G), *); (X, A, \underline{o})]]_\rho, \ \forall k \geq 0.$$

For the special case when $G = \{1\}$, we simply write $\pi_k(X, \underline{o})$ and $\pi_k(X, A, \underline{o})$ instead.

Regarding the structure of these homotopy sets, we have

Theorem 1.2

(1) Algebraic structures: $\pi_k^{(G,\rho)}(X, \underline{o})$, $\pi_{k+1}^{(G,\rho)}(X, A, \underline{o})$ *are canonically identified with the π_{k-1} of $[(S^1(G), *); (X, \underline{o})]_\rho$ and the π_k of $[(I(G), S^0(G), *); (X, A, \underline{o})]_\rho$ respectively, hence have natural group structures for $k \geq 1$ which are Abelian when $k \geq 2$.*

(2) Functoriality: *For any $\Phi \in [(X, \underline{o}); (X', \underline{o'})]_\eta$* (resp. $\Phi \in [(X, A, \underline{o}); (X', A', \underline{o'})]_\eta$), *there are natural homomorphisms $\Phi_* : \pi_k^{(G,\rho)}(X, \underline{o}) \to \pi_k^{(G,\rho')}(X', \underline{o'})$* (resp. $\Phi_* : \pi_k^{(G,\rho)}(X, A, \underline{o}) \to \pi_k^{(G,\rho')}(X', A', \underline{o'})$) *with $\rho' = \eta \circ \rho$, which depend only on the homotopy class of Φ. For any (H, η) where $\eta : H \to G_{\hat{o}}$ is an injective homomorphism which factors through $\rho : G \to G_{\hat{o}}$ by $\iota : H \to G$, there are natural homomorphisms $\iota^* : \pi_k^{(G,\rho)}(X, \underline{o}) \to \pi_k^{(H,\eta)}(X, \underline{o})$ and $\iota^* : \pi_{k+1}^{(G,\rho)}(X, A, \underline{o}) \to \pi_{k+1}^{(H,\eta)}(X, A, \underline{o})$.*

(3) Exact sequence: *There exist natural homomorphisms*

$$\partial : \pi_{k+1}^{(G,\rho)}(X, A, \underline{o}) \to \pi_k^{(G,\rho)}(A, \underline{o}|_A), \ \forall k \geq 0$$

and a long exact sequence

$$\cdots \to \pi_{k+1}^{(G,\rho)}(X, A, \underline{o}) \xrightarrow{\partial} \pi_k^{(G,\rho)}(A, \underline{o}|_A) \xrightarrow{i_*} \pi_k^{(G,\rho)}(X, \underline{o}) \xrightarrow{j_*} \pi_k^{(G,\rho)}(X, A, \underline{o}) \xrightarrow{\partial} \cdots$$
$$\xrightarrow{j_*} \pi_1^{(G,\rho)}(X, A, \underline{o}) \xrightarrow{\partial} \pi_0^{(G,\rho)}(A, \underline{o}|_A) \xrightarrow{i_*} \pi_0^{(G,\rho)}(X, \underline{o}).$$

There are natural homomorphisms

$$C : \pi_1^{(G,\rho)}(X, \underline{o}) \to \mathrm{Aut}(\pi_k^{(G,\rho)}(X, \underline{o})), \ \forall k \geq 1$$

and

$$C' : \pi_1^{(G,\rho)}(A, \underline{o}|_A) \to \mathrm{Aut}(\pi_{k+1}^{(G,\rho)}(X, A, \underline{o})), \ \forall k \geq 1$$

which satisfy $C(z) \circ \partial = \partial \circ C'(z)$, $\forall z \in \pi_1^{(G,\rho)}(A, \underline{o}|_A)$.

Regarding the dependence of $\pi_k^{(G,\rho)}(X, \underline{o})$ and $\pi_{k+1}^{(G,\rho)}(X, A, \underline{o})$ on the base-point structure $\underline{o}$ and the data (G, ρ), we have

Proposition 1.3

(1) *Given any path* $u \in [(I(G),0,1);(X,\underline{o_1},\underline{o_2})]_{(\rho,\eta)}$, *there is an isomorphism, written* u_* : $\pi_k^{(G,\eta)}(X,\underline{o_2}) \rightarrow \pi_k^{(G,\rho)}(X,\underline{o_1})$, *whose inverse* u_*^{-1} *is the isomorphism associated to the inverse path* $\nu(u) \in [(I(G),0,1);(X,\underline{o_2},\underline{o_1})]_{(\eta,\rho)}$. *In fact, for any two paths* $u_1,u_2 \in [(I(G),0,1);(X,\underline{o_1},\underline{o_2})]_{(\rho,\eta)}$, $\nu(u_2)_* \circ (u_1)_* = C([\nu(u_2)\# u_1]) \in Aut(\pi_k^{(G,\eta)}(X,\underline{o_2}))$.

(2) *Given any path* $u \in [(I(G),0,1);(A,\underline{o_1}|_A,\underline{o_2}|_A)]_{(\rho,\eta)}$, *there is an associated mapping* $u_* : \pi_{k+1}^{(G,\eta)}(X,A,\underline{o_2}) \rightarrow \pi_{k+1}^{(G,\rho)}(X,A,\underline{o_1})$, *which is an isomorphism for* $k \geq 1$ *and a base point preserving bijection when* $k=0$. *Moreover, for any* $u_1,u_2 \in [(I(G),0,1);(A,\underline{o_1}|_A,\underline{o_2}|_A)]_{(\rho,\eta)}$, $\nu(u_2)_*\circ(u_1)_* = C'([\nu(u_2)\# u_1]) \in Aut(\pi_{k+1}^{(G,\eta)}(X,A,\underline{o_2}))$.

(3) *The isomorphism class of* $\pi_k^{(G,\rho)}(X,\underline{o})$ *or* $\pi_{k+1}^{(G,\rho)}(X,A,\underline{o})$ *depends only on the conjugacy class of the subgroup* $\rho(G) \subset G_{\hat{o}}$.

Regarding the nature of $\pi_k^{(G,\rho)}(X,\underline{o})$ and $\pi_{k+1}^{(G,\rho)}(X,A,\underline{o})$ in certain special cases, we have

Proposition 1.4

(1) $\pi_k(X,\underline{o})$, $\pi_{k+1}(X,A,\underline{o})$ *are naturally isomorphic to* $\pi_k(B\Gamma_X,*)$ *and* $\pi_{k+1}(B\Gamma_X,B\Gamma_A,*)$, *where* $B\Gamma_X, B\Gamma_A$ *are the classifying spaces of the defining groupoid for* X *and* A *respectively, cf. Haefliger* [10, 11].

(2) *When* $X=Y/G$ *is a global quotient, for any subgroup* $\rho : H \subset G$, *there are natural isomorphisms* $\pi_k^{(H,\rho)}(X,\underline{o}) \cong \pi_k(Y^H,\hat{o})$ *for all* $k \geq 2$, *and the natural exact sequence*

$$1 \rightarrow \pi_1(Y^H,\hat{o}) \rightarrow \pi_1^{(H,\rho)}(X,\underline{o}) \rightarrow C(H) \rightarrow \pi_0(Y^H,\hat{o}) \rightarrow \pi_0^{(H,\rho)}(X,\underline{o}),$$

where Y^H *is the fixed-point set of* H *and* $C(H)$ *is the centralizer of* H.

We remark that the homotopy groups $\{\pi_k(Y^H,\hat{o})\}$ of the various fixed-point sets appeared in the coefficient systems for the equivariant obstruction theory in Bredon [1]. In fact, the basic homotopy machinery developed in this paper laid the necessary foundation for extending the equivariant obstruction theory in Bredon [1] to the orbispace category.

This paper also contains a detailed, elementary treatment of the covering theory for orbispaces. Although the theory of coverings for étale topological groupoids is well-understood in principle (cf. [2], [12]), there is a number of results which will be used in the later constructions in this paper that are in a specific form and can not be found in the literature.

This roughly constitutes the first half of the paper, which is concerned with the basic properties of the algebraic invariants — the homotopy groups — for this homotopy theory. In the remaining second half, we develop an analog of the CW-complex theory for the orbispace category.

A fundamental construction in this part is given in the following

Proposition 1.5 *Let* Y *be a locally path-connected and semi-locally 1-connected orbispace* (cf. §2.4). *For any map* $\Phi : Y \rightarrow X$, *there is a canonical orbispace structure on the mapping cylinder* M_ϕ, *where* $\phi : Y \rightarrow X$ *is the induced map between underlying spaces, such that* (1) *there are natural embeddings* $i : Y \rightarrow M_\phi$, $j : X \rightarrow M_\phi$, *realizing the orbispaces* Y, X *as a subspace of* M_ϕ, *and* (2) $j : X \rightarrow M_\phi$ *is a strong deformation retract by a canonical retraction* $r : M_\phi \rightarrow X$ *which satisfies* $\Phi = r \circ i$.

The above 'mapping cylinder' construction is then applied to the special case where $Y = S^{k-1}(G)$. Under further conditions on both the map Φ and the orbispace X, we can collapse the subspace $S^{k-1}(G)$ in the mapping cylinder to a point. This procedure gives rise to a canonical orbispace structure on the mapping cone of the induced map $\phi : S^{k-1} \rightarrow X$, which is what we will refer to as 'attaching a k-cell of isotropy type G to X via Φ'.

For any $n \geq 0$, denote by $\mathcal{C}^n$ the set of orbispaces X, where there is a canonical filtration of subspaces

$$X_0 \subset X_1 \subset \cdots \subset X_n = X$$

such that X_0 is a finite subset and for each $1 \leq k \leq n$, X_k is resulted from attaching finitely many k-cells of various isotropy types to X_{k-1}. We let $\mathcal{C}$ be the union of $\mathcal{C}^n$ for all $n \geq 0$.

It turns out that this sub-category of orbispaces $\mathcal{C}$ is the right one for the main objective of this paper, ie., developing a machinery for studying homotopy classes of maps and homotopy types of orbispaces. In this regard, we have the following

Theorem 1.6

(1) *Suppose $\Phi : X \rightarrow X'$ is a weak homotopy equivalence and the mapping cylinder of Φ is defined. Then for any $Y \in \mathcal{C}$, the mapping $\Phi_* : [[Y; X]] \rightarrow [[Y; X']]$ is a bijection.*
(2) *For any $X, X' \in \mathcal{C}$, a map $\Phi : X \rightarrow X'$ is a homotopy equivalence if and only if it is a weak homotopy equivalence.*

Set $\pi_0^G(X) \equiv [[B_G; X]]$. Then in the preceding theorem, a map $\Phi : X \rightarrow X'$ is called a weak homotopy equivalence if $\Phi_* : \pi_0^G(X) \rightarrow \pi_0^G(X')$ is a bijection for all G, and $\Phi_* : \pi_k^{(G,\rho)}(X, \underline{o}) \rightarrow \pi_k^{(G,\rho')}(X', \underline{o'})$ is isomorphic for $k \geq 0$ for all possible data $\underline{o}, \underline{o'}, (G, \rho)$ and (G, ρ').

Although the sub-category $\mathcal{C}$ is favorable for the purpose of homotopy theory, it is not clear a priori that $\mathcal{C}$ will contain any geometrically interesting examples, such as compact smooth orbifolds. Next we describe a construction which will justify the sub-category $\mathcal{C}$ in this regard.

Let K be a finite CW-complex such that for any attaching map, if its image meets the interior of a cell, then it contains the whole cell. An arrow of K is an ordered pair (σ_1, σ_2) of cells in K where σ_2 is a face of σ_1. We denote arrows by lower case letters a, b, c, etc. Let $a = (\sigma_1, \sigma_2)$ be an arrow. Then σ_1 is called the initial point of a, denoted by $i(a)$, and σ_2 is called the terminal point of a, denoted by $t(a)$. Two arrows a, b are composable if $t(b) = i(a)$, and in this case, the composition of a, b, denoted by ab, is the arrow c uniquely determined by the conditions $i(c) = i(b)$, $t(c) = t(a)$. Composition of arrows is clearly associative.

With the preceding understood, a CW-complex of groups on K, denoted by $(K, G_\sigma, \psi_a, g_{a,b})$, is given by the following set of data:

(1) Each cell σ in K is associated with a group G_σ.
(2) Each arrow a of K is assigned with an injective homomorphism $\psi_a : G_{i(a)} \rightarrow G_{t(a)}$.
(3) Each pair of composable arrows a, b is assigned with an element $g_{a,b} \in G_{t(a)}$, such that

$$Ad(g_{a,b}) \circ \psi_{ab} = \psi_a \circ \psi_b.$$

Moreover, the following cocycle condition holds for any triple a, b, c of composable arrows

$$\psi_a(g_{b,c})g_{a,bc} = g_{a,b}g_{ab,c}.$$

Two CW-complexes of groups $(K, G_\sigma, \psi_a, g_{a,b})$, $(K, G_\sigma, \psi'_a, g'_{a,b})$ are said to be equivalent if there is a set $\{g_a \in G_{t(a)}\}$ such that

$$\psi'_a = Ad(g_a) \circ \psi_a, \;\; g'_{a,b} = g_a\psi_a(g_b)g_{a,b}g_{ab}^{-1}.$$

Finally, let $K_n \subset K$, $n \geq 0$, be the n-skeleton of K. Then any CW-complex of groups on K naturally induces one on K_n by restriction.

We remark that the notion 'CW-complex of groups' is a natural extension of the notion 'complex of groups' in Haefliger [12]. Moreover, the following result generalizes the one to one correspondence between equivalence classes of complexes of groups and isomorphism classes of orbihedra therein.

Proposition 1.7 *Let K be a finite CW-complex and X be its underlying space. To each equivalence class of CW-complexes of groups on K, there is associated an orbispace structure on X, called the geometric realization*[1]*, which gives a one to one correspondence, such that the orbispace $X \in \mathcal{C}$. Moreover, the geometric realizations of the restrictions of the CW-complex of groups to the n-skeletons of K provide the natural canonical filtration of subspaces for the geometric realization of the CW-complex of groups itself.*

Finally, recall the following fact about compact smooth orbifolds, cf. e.g. [13, 16]:

A compact smooth orbifold may be triangulated so that it becomes the geometric realization of a natural simplicial complex of groups on the resulting simplicial complex.

As a corollary, the classical Whitehead theorem is extended to the orbifold category.

Corollary *A map between compact smooth orbifolds is a homotopy equivalence if and only if it is a weak homotopy equivalence.*

The following is a glimpse at the organization of this paper.

§2.1 Space of guided loops.
§2.2 Generalities of homotopy groups.
§2.3 Exact sequence of a pair.
§2.4 The theory of coverings.
§3.1 Construction of mapping cylinders.
§3.2 Orbispaces via attaching cells of isotropy.
§3.3 CW-complex of groups and its geometric realization.

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[1] there is a minor difference, although conceptual in nature, between the notions 'geometric realization' and 'orbihedron', see §3.3 for more detailed comments.

2. Homotopy groups via guided loop spaces

2.1 Space of guided loops

The homotopy sets defined in Definition 1.1 will be studied through the space of 'guided' paths or loops in the orbispace. This subsection is devoted to a preliminary study of these path or loop spaces.

Recall that $I(G)$ denotes the 1-cell of isotropy type G, namely, the orbispace defined by the trivial action of G on the interval $I=[0,1]$. Let X be an orbispace, given with a pair of base-point structures $\underline{o_1}=(o_1,U_{o_1},\hat{o_1})$, $\underline{o_2}=(o_2,U_{o_2},\hat{o_2})$, and injective homomorphisms $\rho: G\rightarrow G_{\hat{o_1}}$, $\eta: G\rightarrow G_{\hat{o_2}}$. We consider the set of maps $[(I(G),0,1);(X,\underline{o_1},\underline{o_2})]_{(\rho,\eta)}$ from $(I(G),0,1)$ to $(X,\underline{o_1},\underline{o_2})$, which by definition are equivalence classes of groupoid homomorphisms $\sigma=(\{f_i\},\{\rho_{ji}\}):\Gamma\{I_i\}\rightarrow\Gamma\{U_{i'}\}$, where (1) $\{I_i\mid i=0,1,\cdots,n\}$ is a cover of I by sub-intervals such that $0\in I_0$, $1\in I_n$, $I_i\cap I_j\neq\emptyset$ iff $j=i$ or $i+1$, (2) denote by $U_i\in\{U_{i'}\}$ the local chart assigned to I_i, then $U_0=U_{o_1}$, $U_n=U_{o_2}$, (3) $f_0(0)=\hat{o_1}$, $f_n(1)=\hat{o_2}$, and (4) $\rho_0=\rho$ and $\rho_n=\eta$. Note that each mapping $\rho_{(i+1)i}:T(I_i,I_{i+1})\rightarrow T(U_i,U_{i+1})$ is completely determined by the image of $1\in G=T(I_i,I_{i+1})$ in $T(U_i,U_{i+1})$, we shall conveniently regard $\rho_{(i+1)i}$ as an element of $T(U_i,U_{i+1})$. On the other hand, the homomorphisms $\rho_i=\rho_{ii}:G\rightarrow G_{U_i}$ can be recovered inductively by $\rho_{i+1}=\lambda_{\rho_{(i+1)i}}\circ\rho_i$ with $\rho_0=\rho$ (cf. (2.1.3) in [3] for the definition of $\lambda_{\rho_{(i+1)i}}$). We observe that the image of $f_i: I_i\rightarrow\widehat{U_i}$ for each index i is forced to lie in the fixed-point set of $\rho_i(G)\subset G_{U_i}$. For this reason we call each element in $[(I(G),0,1);(X,\underline{o_1},\underline{o_2})]_{(\rho,\eta)}$ a (G,ρ,η)-guided path in $(X,\underline{o_1},\underline{o_2})$, or simply a guided path.

A canonical topology is given to $[(I(G),0,1);(X,\underline{o_1},\underline{o_2})]_{(\rho,\eta)}$ by the general construction in §3.2 of Part I of this series [3], as the orbispace $I(G)$ is trivially paracompact, locally compact and Hausdorff. For this purpose, we shall only consider the representatives $\sigma=(\{f_i\},\{\rho_{ji}\}):\Gamma\{I_i\}\rightarrow\Gamma\{U_{i'}\}$ which are admissible. In this case, it simply means that each $f_i: I_i\rightarrow\widehat{U_i}$ is extended over to the closure $\overline{I_i}$ of I_i. We recall that

$$\mathcal{O}_{\{\rho_{ji}\}}=\{\sigma\mid\sigma=(\{f_i\},\{\rho_{ji}\})\text{ is admissible},[\sigma]\in[(I(G),0,1);(X,\underline{o_1},\underline{o_2})]_{(\rho,\eta)}\}. \tag{2.1.1}$$

By Lemma 3.1.2 in [3], each $\mathcal{O}_{\{\rho_{ji}\}}$ is embedded into $[(I(G),0,1);(X,\underline{o_1},\underline{o_2})]_{(\rho,\eta)}$ via $\sigma\mapsto[\sigma]$. We consider the subsets

$$\mathcal{O}_{\{\rho_{ji}\}}(\{\underline{K_i}\},\{\underline{O_i}\})=\{\{f_i\}\in\mathcal{O}_{\{\rho_{ji}\}}\mid f_i(K_{i,s})\subset O_{i,s}, s\in\Lambda(i)\} \tag{2.1.2}$$

where $\underline{K_i}=\{K_{i,s}\mid s\in\Lambda(i)\}$ is a finite set of compact subsets of I_i, and $\underline{O_i}=\{O_{i,s}\mid s\in\Lambda(i)\}$ is a finite set of open subsets of $\widehat{U_i}$. The canonical topology on $[(I(G),0,1);(X,\underline{o_1},\underline{o_2})]_{(\rho,\eta)}$ is the one generated by the subsets in (2.1.2) for all possible data $\{\rho_{ji}\}$, $\{\underline{K_i}\}$ and $\{\underline{O_i}\}$. By Lemma 3.2.2 in [3], if $\{f_i\}\in\mathcal{O}_{\{\rho_{ji}\}}$ is equivalent to $\{f_k\}\in\mathcal{O}_{\{\rho_{lk}\}}$, then there is a local homeomorphism ϕ from an open neighborhood of $\{f_i\}$ in $\mathcal{O}_{\{\rho_{ji}\}}$ onto an open neighborhood of $\{f_k\}$ in $\mathcal{O}_{\{\rho_{lk}\}}$, where $\mathcal{O}_{\{\rho_{ji}\}}$, $\mathcal{O}_{\{\rho_{lk}\}}$ are given the topology generated by the subsets in (2.1.2). As a consequence, each $\mathcal{O}_{\{\rho_{ji}\}}$ is an open subset of $[(I(G),0,1);(X,\underline{o_1},\underline{o_2})]_{(\rho,\eta)}$ via the embedding $\sigma\mapsto[\sigma]$.

Now we shall present some standard constructions on the path spaces. First of all, given any (G,ρ,η)-guided path u, one obtains its inverse, denoted by $\nu(u)$, by pre-composing u by the map $\nu : (I(G), 0, 1) \rightarrow (I(G), 1, 0)$ sending $t \mapsto 1-t, \forall t \in I$ and $g \mapsto g, \forall g \in G$, which results in a (G, η, ρ)-guided path. Secondly, one may compose a (G, ρ, η)-guided path u_1 with a (G, η, ξ)-guided path u_2 to obtain a (G, ρ, ξ)-guided path $u_1 \# u_2$, which is done as follows. Pick representatives $\sigma_1 = (\{f_{i,1}\}, \{\rho_{ji}\}) : \Gamma\{I_i\} \rightarrow \Gamma\{U_{i'}\}$ of u_1, where $i = 0, 1, \cdots, n$, and $\sigma_2 = (\{f_{k,2}\}, \{\eta_{lk}\}) : \Gamma\{J_k\} \rightarrow \Gamma\{V_{k'}\}$ of u_2, where $k = 0, 1, \cdots, m$. Let a be the index running from 0 to $n+m$. We define intervals H_a by setting $H_a = I_a$ for $0 \leq a \leq n-1$, $H_n = I_n \cup J_0$, and $H_a = J_{a-n}$ for $n+1 \leq a \leq n+m$. We define local chart W_a on X by setting $W_a = U_a$ for $0 \leq a \leq n-1$, $W_n = U_{o_2}$, and $W_a = V_{a-n}$ for $n+1 \leq a \leq n+m$. We define f_a by setting $f_a = f_{a,1}$ for $0 \leq a \leq n-1$, $f_n = f_{n,1} \cup f_{0,2}$, and $f_a = f_{a-n,2}$ for $n+1 \leq a \leq n+m$. We define $\xi_{(a+1)a} = \rho_{(a+1)a}$ for $0 \leq a \leq n-1$, $\xi_{(n+1)n} = \eta_{10}$, $\xi_{(a+1)a} = \eta_{a+1-n,a-n}$ for $n+1 \leq a \leq n+m-1$. Then after the reparametrization $t \mapsto 2t$, $g \mapsto g$, we obtain a homomorphism $\sigma_1 \# \sigma_2 = (\{f_a\}, \{\xi_{ba}\}) : \Gamma\{H_a\} \rightarrow \Gamma\{W_{a'}\}$, whose equivalence class is a (G, ρ, ξ)-guided path. We define $u_1 \# u_2$ to be the equivalence class of $\sigma_1 \# \sigma_2$, which is clearly well-defined. Finally, given any $\Phi \in [(X, \underline{o_1}, \underline{o_2}); (X', \underline{o_1'}, \underline{o_2'})]_{(\eta_1, \eta_2)}$, one has the mapping $\Phi_\# : [(I(G), 0, 1); (X, \underline{o_1}, \underline{o_2})]_{(\rho_1, \rho_2)} \rightarrow [(I(G), 0, 1); (X', \underline{o_1'}, \underline{o_2'})]_{(\rho_1', \rho_2')}$ defined by $u \mapsto \Phi \circ u$, where $\rho_i' = \eta_i \circ \rho_i$ for $i = 1, 2$, and given any injective homomorphism $\iota : H \rightarrow G$, one has the mapping $\iota^\# : [(I(G), 0, 1); (X, \underline{o_1}, \underline{o_2})]_{(\rho, \eta)} \rightarrow [(I(H), 0, 1); (X, \underline{o_1}, \underline{o_2})]_{(\rho \circ \iota, \eta \circ \iota)}$ defined by pre-composing each guided path u by the map $(t, h) \mapsto (t, \iota(h)), \forall t \in I, h \in H$. The following lemma is straightforward.

Lemma 2.1.1 *The mappings $\nu, \#, \Phi_\#$ and $\iota^\#$ of path spaces are all continuous.*

Now we consider a special guided path space, the guided loop space $[(S^1(G), *); (X, \underline{o})]_\rho$. There is a natural base point in $[(S^1(G), *); (X, \underline{o})]_\rho$, i.e., the constant guided loop $\tilde{o}$ defined by $(t, g) \mapsto (\hat{o}, \rho(g)), \forall t \in S^1, g \in G$.

Lemma 2.1.2 *The based topological space $([(S^1(G), *); (X, \underline{o})]_\rho, \tilde{o})$ is an H-group with the homotopy associative multiplication $\#$ and the homotopy inverse ν. Moreover, the maps $\Phi_\#$, $\iota^\#$ are homomorphisms of H-groups.*

Proof We refer to [14] for the definition of H-group. Here we only sketch a proof that ν is a homotopy inverse, namely, both maps $\# \circ (\nu, Id)$, $\# \circ (Id, \nu) : ([(S^1(G), *); (X, \underline{o})]_\rho, \tilde{o}) \rightarrow ([S^1(G), *); (X, \underline{o})]_\rho, \tilde{o})$ are homopotic to the constant map into the base point $\tilde{o}$. The homotopy associativity of $\#$ can be proven in the same vein. The assertion on the maps $\Phi_\#$ and $\iota^\#$ are trivial. We leave the details to the reader.

First of all, we introduce some notations. For any $s \in [0, 1]$, we set

$$I_s = ([0, \frac{1}{2}(1-s)] \cup [\frac{1}{2}(1+s), 1]) / \{\frac{1}{2}(1-s) \sim \frac{1}{2}(1+s)\}.$$

Let $\beta_s : I \rightarrow I_s$, which are homeomorphisms for $s \neq 1$, be defined by

$$\beta_s(t) = \begin{cases} (1-s)t & 0 \leq t \leq \frac{1}{2} \\ (1-s)(t-1)+1 & \frac{1}{2} \leq t \leq 1. \end{cases} \tag{2.1.3}$$

For any homomorphism $\sigma = (\{f_i\}, \{\rho_{ji}\})$, we denote by $\nu(\sigma)$ the homomorphism obtained from σ by performing the reparametrization $t \mapsto 1-t, t \in I$.

We shall define a homotopy $F : ([(S^1(G),*);(X,\underline{o})]_\rho,\tilde{o})\times[0,1]\rightarrow([(S^1(G),*);(X,\underline{o})]_\rho,\tilde{o})$ between $\#\circ(Id,\nu)$ and the constant map into $\tilde{o}$. The construction of a homotopy between the map $\#\circ(\nu,Id)$ and the constant map into $\tilde{o}$ is completely parallel.

Given any guided loop u which is represented by a homomorphism σ, and any $s\in[0,1]$, we define the guided loop $F(u,s)$ as follows. Observe that for any $s\in[0,1]$, the restriction of the homomorphism $\sigma\#\nu(\sigma)$ to I_s is a homomorphism. We reparametrize it by β_s in (2.1.3), and denote it by $\sigma\#_s\nu(\sigma)$. If σ is replaced by an equivalent homomorphism, then $\sigma\#_s\nu(\sigma)$ will be changed to an equivalent one accordingly. Hence the (G,ρ)-guided loop defined by $\sigma\#_s\nu(\sigma)$ depends only on u and s. We define $F(u,s)=[\sigma\#_s\nu(\sigma)]$. Clearly, $F(u,0)=\#\circ(Id,\nu)(u)$ and $F(u,1)=\tilde{o}$ for all $u\in[(S^1(G),*);(X,\underline{o})]_\rho$.

It remains to verify that for any given $u_0\in[(S^1(G),*);(X,\underline{o})]_\rho$, $s_0\in[0,1]$, F is continuous at (u_0,s_0). We shall only give the details for the case when $s_0\neq 1$, the remaining case $s_0=1$ is easier and we leave it to the reader. Without loss of generality, we may assume that u_0 is represented by a homomorphism $\sigma_0=(\{f_{i,0}\},\{\rho_{ji}\}):\Gamma\{I_i\}\rightarrow\Gamma\{U_{i'}\}$, where $i=0,1,\cdots,n$, such that there exist an i_0 and a closed interval $[s_-,s_+]$ containing s_0 and satisfying $s_+\neq 1$, $1-s_0\in[1-s_+,1-s_-]\subset I_{i_0}\setminus\cup_{i\neq i_0}I_i$. Then for any $\sigma=\{f_i\}\in\mathcal{O}_{\{\rho_{ji}\}}$ and $s\in[s_-,s_+]$, we have $\sigma\#_s\nu(\sigma)$ given by $(\{f_{k,s}\},\{\xi_{lk}\}):\Gamma\{J_{k,s}\}\rightarrow\Gamma\{V_{k'}\}$, where $k=0,1,\cdots,2i_0$, and

$$J_{k,s}=\begin{cases}\{\beta_s^{-1}(\frac{1}{2}t)|t\in I_k\},\ 0\leq k\leq i_0-1\\ \{\beta_s^{-1}(\frac{1}{2}t)|t\in I_{i_0},t\leq 1-s,\ \text{or}\ 2-t\leq 1-s,2-t\in I_{i_0}\},\ k=i_0\\ \{\beta_s^{-1}(\frac{1}{2}t)|2-t\in I_{2i_0-k}\},\ i_0+1\leq k\leq 2i_0,\end{cases}\tag{2.1.4}$$

$$V_k=U_k,\ 0\leq k\leq i_0,\ \ V_k=U_{2i_0-k},\ i_0+1\leq k\leq 2i_0,\tag{2.1.5}$$

$$f_{k,s}(t)=\begin{cases}f_k(2\beta_s(t)),\ 0\leq k\leq i_0-1\\ \begin{cases}f_k(2\beta_s(t)),\ 2\beta_s(t)\leq 1-s,2\beta_s(t)\in I_{i_0}\\ f_k(2-2\beta_s(t)),\ 2-2\beta_s(t)\leq 1-s,2-2\beta_s(t)\in I_{i_0}\end{cases} & k=i_0\\ f_k(2-2\beta_s(t)),\ i_0+1\leq k\leq 2i_0\end{cases}\tag{2.1.6}$$

and

$$\xi_{k(k-1)}=\rho_{k(k-1)},\ 1\leq k\leq i_0,\ \ \xi_{k(k-1)}=\rho^{-1}_{(2i_0-k+1)(2i_0-k)},\ i_0+1\leq k\leq 2i_0.\tag{2.1.7}$$

Let $|s_+-s_-|\ll 0$ so that

$$J_{k,s_-}\cap J_{k+1,s_+}\neq\emptyset,\ 0\leq k\leq i_0-1,\ \ J_{k,s_+}\cap J_{k+1,s_-}\neq\emptyset,\ i_0\leq k\leq 2i_0.\tag{2.1.8}$$

We set $H_k=J_{k,s_-}\cap J_{k,s_+}$ for $0\leq k\leq 2i_0$. Then it is easy to see that $\{H_k\}$ is a cover of I by sub-intervals such that $H_l\cap H_k\neq\emptyset$ iff $l=k$ or $k+1$, and $H_k\subset J_{k,s}$ for any $0\leq k\leq 2i_0$ and $s\in[s_-,s_+]$. We define $g_{k,s}=f_{k,s}|_{H_k}$ for all $0\leq k\leq 2i_0$. Then $\sigma_s=(\{g_{k,s}\},\{\xi_{lk}\}):\Gamma\{H_k\}\rightarrow\Gamma\{V_{k'}\}$, where $k=0,1,\cdots,2i_0$, is equivalent to $\sigma\#_s\nu(\sigma)$. Observe that in the neighborhood $\mathcal{O}_{\{\rho_{ji}\}}\times[s_-,s_+]$ of (u_0,s_0), the map F is given by $(\sigma,s)\mapsto\sigma_s\in\mathcal{O}_{\{\xi_{lk}\}}$, which is clearly continuous with respect to the topology of $\mathcal{O}_{\{\rho_{ji}\}}$ and $\mathcal{O}_{\{\xi_{lk}\}}$. Hence F is continuous. $\square$

We end by introducing the guided relative loop space $[(I(G),S^0(G),0);(X,A,\underline{o})]_\rho$ for any pair $(X,A,\underline{o})$ and injective homomorphism $\rho:G\rightarrow G_{\hat{o}}$. Here $S^0=\{0,1\}$. Recall that an element of $[(I(G),S^0(G),0);(X,A,\underline{o})]_\rho$ is a map (a guided path) $u\in[(I(G),0);(X,\underline{o})]_\rho$ such that the restriction of u to the subspace $(S^0(G),0)$ is an element of $[(S^0(G),0);(A,\underline{o}|_A)]_\rho$.

A canonical topology can be given to $[(I(G), S^0(G), 0); (X, A, \underline{o})]_\rho$ along the general lines of §3.2 in Part I of this series [3]. For this purpose, we need to digress on the subspace structure of A in some details. Let $\mathcal{U} = \{U_i\}$ be the atlas of local charts on X and Γ be the defining groupoid for X with space of units $\bigsqcup_i \widehat{U_i}$. Denote by $\{V_\alpha\}$ the set of connected components of all $A \cap U_i$, $U_i \in \mathcal{U}$, and by $\widehat{V_\alpha}$ some fixed component of the inverse image of V_α in $\widehat{U_i}$ if V_α is a component of $A \cap U_i$. Then the orbispace structure on A is given by the restriction of Γ to $\bigsqcup_\alpha \widehat{V_\alpha}$. In terms of local charts, each $\widehat{V_\alpha}$ is acted upon by a group G_{V_α}, which is the subgroup of G_{U_i} that fixes the subset $\widehat{V_\alpha}$, with a map $\pi_{V_\alpha} = \pi_{U_i}|_{\widehat{V_\alpha}}$ which induces $\widehat{V_\alpha}/G_{V_\alpha} \cong V_\alpha$. Moreover, there are natural mappings $\rho_{\beta\alpha} : T(V_\alpha, V_\beta) \to T(U_i, U_j)$, where V_α, V_β are components of $A \cap U_i, A \cap U_j$, such that $\rho_\alpha = \rho_{\alpha\alpha} : G_{V_\alpha} \subset G_{U_i}$.

An element $u \in [(I(G), S^0(G), 0); (X, A, \underline{o})]_\rho$ is the equivalence class of $\sigma = (\{f_i\}, \{\xi_{ji}\}) : \Gamma\{I_i\} \to \Gamma\{U_{i'}\}$, where (1) $\{I_i \mid i = 0, 1, \cdots, n\}$ is a cover of I by sub-intervals such that $0 \in I_0$, $1 \in I_n$, $I_i \cap I_j \neq \emptyset$ iff $j = i$ or $i+1$, (2) denote by $U_i \in \{U_{i'}\}$ the local chart assigned to I_i, then $U_0 = U_o$, and there is a V_α which is a component of $A \cap U_n$, (3) $f_0(0) = \hat{o}$, $f_n(1) \in \widehat{V_\alpha}$, and (4) $\xi_0 = \rho$ and $\xi_n(G) \subset \rho_\alpha(G_{V_\alpha})$. Note that each mapping $\xi_{(i+1)i} : T(I_i, I_{i+1}) \to T(U_i, U_{i+1})$ is completely determined by the image of $1 \in G = T(I_i, I_{i+1})$ in $T(U_i, U_{i+1})$, we shall conveniently regard $\xi_{(i+1)i}$ as an element of $T(U_i, U_{i+1})$. On the other hand, the homomorphisms $\xi_i = \xi_{ii} : G \to G_{U_i}$ can be recovered inductively by $\xi_{i+1} = \lambda_{\xi_{(i+1)i}} \circ \xi_i$ with $\xi_0 = \rho$. We observe that the image of $f_i : I_i \to \widehat{U_i}$ for each index i is forced to lie in the fixed-point set of $\xi_i(G) \subset G_{U_i}$. Finally, the restriction of u to the subspace $(S^0(G), 0)$ is the map into $(A, \underline{o}|_A)$ defined by $(0, g) \mapsto (\hat{o}, \rho(g))$, $(1, g) \mapsto (f_n(1), \rho_\alpha^{-1} \circ \xi_n(g))$, $\forall g \in G$.

We introduce

$$\mathcal{O}_{(\{\xi_{ji}\}, V_\alpha)} = \{\sigma \mid \sigma = (\{f_i\}, \{\xi_{ji}\}) \text{ is admissible}, f_n(1) \in \widehat{V_\alpha}, \xi_n(G) \subset \rho_\alpha(G_{V_\alpha})\}. \tag{2.1.9}$$

By Lemma 3.1.2 in [3], $\mathcal{O}_{(\{\xi_{ji}\}, V_\alpha)}$ can be regarded as a subset of $[(I(G), S^0(G), 0); (X, A, \underline{o})]_\rho$ via $\sigma \mapsto [\sigma]$. We give each $\mathcal{O}_{(\{\xi_{ji}\}, V_\alpha)}$ a topology which is generated by

$$\mathcal{O}_{(\{\xi_{ji}\}, V_\alpha)}(\{\underline{K_i}\}, \{\underline{O_i}\}) = \{\{f_i\} \in \mathcal{O}_{(\{\xi_{ji}\}, V_\alpha)} \mid f_i(K_{i,s}) \subset O_{i,s}, \forall s \in \Lambda(i)\} \tag{2.1.10}$$

where $\underline{K_i} = \{K_{i,s} | s \in \Lambda(i)\}$ is any finite set of compact subsets of I_i, and $\underline{O_i} = \{O_{i,s} \mid s \in \Lambda(i)\}$ is any finite set of open subsets of $\widehat{U_i}$. Again by Lemma 3.2.2 in [3], for any $\{f_i\} \in \mathcal{O}_{(\{\xi_{ji}\}, V_\alpha)}$ and $\{f_k\} \in \mathcal{O}_{(\{\xi_{lk}\}, V_\beta)}$ which are equivalent, there is a local homeomorphism ϕ from an open neighborhood of $\{f_i\}$ in $\mathcal{O}_{(\{\xi_{ji}\}, V_\alpha)}$ onto an open neighborhood of $\{f_k\}$ in $\mathcal{O}_{(\{\xi_{lk}\}, V_\beta)}$. We give a topology to $[(I(G), S^0(G), 0); (X, A, \underline{o})]_\rho$ which is generated by the subsets in (2.1.10) for all possible data $\{\xi_{ji}\}, V_\alpha, \{\underline{K_i}\}$ and $\{\underline{O_i}\}$. However, because of the existence of the said local homeomorphisms $\{\phi\}$, each $\mathcal{O}_{(\{\xi_{ji}\}, V_\alpha)}$ is in fact an open subset of $[(I(G), S^0(G), 0); (X, A, \underline{o})]_\rho$.

There is a special element $\tilde{o} \in [(I(G), S^0(G), 0); (X, A, \underline{o})]_\rho$ which is the equivalence class of the constant guided relative loop, represented by any $\{f_i\} \in \mathcal{O}_{(\{\xi_{ji}\}, V_\alpha)}$ where each $f_i : I_i \to \widehat{U_i} = \widehat{U_o}$ has the point $\hat{o}$ as its image, each $\xi_{(i+1)i} = 1 \in G_{U_o} = T(U_i, U_j)$, and $V_\alpha = V_o$. We shall fix $\tilde{o}$ as the base point of the guided relative loop space.

We may compose a guided path with a guide relative loop. More precisely, Let $u_1 \in [(I(G), 0, 1); (X, \underline{o_1}, \underline{o_2})]_{(\rho,\eta)}$ be a guided path, and $u_2 \in [(I(G), S^0(G), 0); (X, A, \underline{o_2})]_\eta$ be a guided relative loop. Then the composition $u_1\#u_2$ is well-defined, which is an element in the guided relative loop space $[(I(G),S^0(G),0);(X,A,\underline{o_1})]_\rho$.

The mappings $\Phi_\# : [(I(G),S^0(G),0);(X,A,\underline{o})]_\rho \rightarrow [(I(G),S^0(G),0);(X',A',\underline{o'})]_{\eta\circ\rho}$ and $\iota^\# : [(I(G),S^0(G),0);(X,A,\underline{o})]_\rho \rightarrow [(I(H),S^0(H),0);(X,A,\underline{o})]_{\rho\circ\iota}$ are defined for any $\Phi \in [(X,A,\underline{o});(X',A',\underline{o'})]_\eta$ and injective homomorphism $\iota : H \rightarrow G$. The following lemma is straightforward.

Lemma 2.1.3 *The mappings $\#, \Phi_\#$ and $\iota^\#$ are all continuous.*

2.2 Generalities of homotopy groups

We first derive two preliminary lemmas.

Let (K,x_0) be a compact, locally connected topological space with base point $x_0 \in K$. The suspension of (K,x_0) is the based space $(SK,*)$, where

$$SK = (I\times K)/(\{0,1\}\times K)\cup(I\times\{x_0\}), \tag{2.2.1}$$

with the base point $* \in SK$ being the image of $(\{0,1\}\times K)\cup(I\times\{x_0\})$. The cone of (K,x_0) is the based space $(CK,*)$, where

$$CK = (I\times K)/(\{0\}\times K)\cup(I\times\{x_0\}), \tag{2.2.2}$$

with the base point $* \in CK$ being the image of $(\{0\}\times K)\cup(I\times\{x_0\})$. The based space (K,x_0) is canonically a subspace of $(CK,*)$ via the embedding $x\mapsto[1,x]$ where $[1,x]$ denotes the image of $(1,x)\in I\times K$ in CK.

Lemma 2.2.1

1. *The set of continuous maps from (K,x_0) into $([(S^1(G),*);(X,\underline{o})]_\rho,\tilde{o})$ is naturally identified with the set $[(SK(G),*);(X,\underline{o})]_\rho$.*
2. *The set of continuous maps from (K,x_0) into $([(I(G),S^0(G),0);(X,A,\underline{o})]_\rho,\tilde{o})$ is naturally identified with the set $[(CK(G),K(G),*);(X,A,\underline{o})]_\rho$.*

Proof (1) Given any continuous map $u:(K,x_0)\rightarrow([(S^1(G),*);(X,\underline{o})]_\rho,\tilde{o})$, we construct an element $\Phi_u \in [(SK(G),*);(X,\underline{o})]_\rho$ as follows. Since $u(K)\subset[(S^1(G),*);(X,\underline{o})]_\rho$ is a compact subset, there are finitely many open subsets

$$\{\mathcal{O}_\alpha|\mathcal{O}_\alpha=\mathcal{O}_{\{\rho_{ji,\alpha}\}},\alpha\in\Lambda,\#\Lambda<+\infty\} \tag{2.2.3}$$

such that $u(K)\subset\cup_{\alpha\in\Lambda}\mathcal{O}_\alpha$. Furthermore, we may require that (a) $u(x_0)=\tilde{o}$ is contained in $\mathcal{O}_{\alpha_0}$ where $U_{i,\alpha_0}=U_o$, $\rho_{ji,\alpha_0}=1\in G_{U_o}$ for all i,j, and (b) $\{I_{i,\alpha}\}=\{I_i\}$ is independent of α. The latter can be always achieved because $\#\Lambda<+\infty$.

We take a cover $\{O_a|a\in A\}$ of K, where each O_a is a connected open subset, together with $\theta:a\mapsto\alpha$ such that $u(O_a)\subset\mathcal{O}_{\theta(a)}$. Then for each $a\in A$, there is a set of data

$$(\{I_i\},\{U_{i,\theta(a)}\},\{f_{i,a}\},\{\rho_{ji,\theta(a)}\}), \tag{2.2.4}$$

where each $f_{i,a}:I_i\times O_a\rightarrow\widehat{U_{i,\theta(a)}}$ is continuous such that for any $x\in O_a$, the restriction of (2.2.4) to x is a homomorphism $\{f_{i,a}(\cdot,x)\}\in\mathcal{O}_{\theta(a)}$ representing $u(x)\in[(S^1(G),*);(X,\underline{o})]_\rho$.

Let $\{O_{ab,s} \mid s \in I_{ab}\}$ be the set of connected components of $O_a \cap O_b$, $a, b \in A$. For any $x \in O_{ab,s}$, since both $(\{I_i\}, \{U_{i,\theta(a)}\}, \{f_{i,a}(\cdot,x)\}, \{\rho_{ji,\theta(a)}\})$ and $(\{I_i\}, \{U_{i,\theta(b)}\}, \{f_{i,b}(\cdot,x)\}, \{\rho_{ji,\theta(b)}\})$ represent $u(x) \in [(S^1(G),*);(X,\underline{o})]_\rho$, we may apply Lemma 2.2.4 and then Lemma 3.1.2 of [3] to conclude that there exists a set of elements $\xi_i^{ba,s}(x) \in T(U_{i,\theta(a)}, U_{i,\theta(b)})$ such that

$$
\begin{aligned}
&f_{i,b}(\cdot,x) = \phi_{\xi_i^{ba,s}(x)} \circ f_{i,a}(\cdot,x), \ \forall i, \\
&\rho_{ji,\theta(b)} = \xi_j^{ba,s}(x) \circ \rho_{ji,\theta(a)} \circ \xi_i^{ba,s}(x)^{-1}(\mathbf{a}), \ \forall i,j,
\end{aligned}
\tag{2.2.5}
$$

where $\mathbf{a} \in \Lambda(\xi_i^{ba,s}(x)^{-1}, \rho_{ji,\theta(a)}, \xi_j^{ba,s}(x))$ is the element containing $f_{i,b}(I_i \cap I_j \times \{x\})$. Moreover, the elements $\xi_0^{ba,s}(x) \in T(U_{0,\theta(a)}, U_{0,\theta(b)})$ have to satisfy $\xi_0^{ba,s}(x) = 1 \in G_{U_o}$ (note that $U_{0,\alpha} = U_o$ for all $\alpha \in \Lambda$). Now the second equation in (2.2.5) implies inductively that each $\xi_i^{ba,s}(x)$ is constant in x, depending only on the connected component $O_{ab,s}$ that contains x. We denote the constant value by $\xi_i^{ba,s}$. It also follows from the second equation in (2.2.5) that if a connected component $\mathbf{b}$ of $O_{ab,s} \cap O_{bc,t}$ is contained in $O_{ac,r}$, then

$$
\xi_i^{cb,t} \circ \xi_i^{ba,s}(f_{i,a}(\mathbf{b})) = \xi_i^{ca,r}. \tag{2.2.6}
$$

It is easily seen that $\tau(\theta, \{O_a\}, \{\mathcal{O}_\alpha\}) = (\{f_{i,a}\}, \{\rho_{ji,\theta(b)} \circ \xi_i^{ba,s}\})$, where $\rho_{ji,\theta(b)} \circ \xi_i^{ba,s}$ is the element of $T(U_{i,\theta(a)}, U_{j,\theta(b)})$ determined by the component containing $f_{i,a}(I_j \cap I_i \times O_{ab,s})$, is a groupoid homomorphism from $\Gamma\{I_i \times O_a\}$ to $\Gamma\{U_{i',\alpha}\}$, whose equivalence class is an element of $[(SK(G),*);(X,\underline{o})]_\rho$. We define Φ_u to be the equivalence class of $\tau(\theta, \{O_a\}, \{\mathcal{O}_\alpha\})$.

It remains to verify that Φ_u is well-defined. For this purpose, let $\tau(\theta, \{O_a\}, \{\mathcal{O}_\alpha\})$ and $\tau(\theta', \{O_{a'}\}, \{\mathcal{O}_{\alpha'}\})$ be any two such homomorphisms. We shall construct a common induced homomorphism of them as follows. For any $x \in K$, we pick an α and an α' such that $u(x)$ is contained in $\mathcal{O}_\alpha \cap \mathcal{O}_{\alpha'}$. Then by Lemma 3.2.2 in [3], there is an open subset $\mathcal{O}_x$ of some $\mathcal{O}_{\{\rho_{lk,x}\}}$, together with open embeddings $\phi_\alpha : \mathcal{O}_x \to \mathcal{O}_\alpha$, $\phi_{\alpha'} : \mathcal{O}_x \to \mathcal{O}_{\alpha'}$, such that $u(x)$ is contained in $\mathcal{O}_x$ and each $\kappa \in \mathcal{O}_x$ is a common induced homomorphism of $\phi_\alpha(\kappa) \in \mathcal{O}_\alpha$ and $\phi_{\alpha'}(\kappa) \in \mathcal{O}_{\alpha'}$. Since $u(K)$ is compact in $[(S^1(G),*);(X,\underline{o})]_\rho$, we can cover it by finitely many such open subsets $\mathcal{O}_r = \mathcal{O}_{x_r}, r = 1, 2, \cdots, N$. Note that there are mappings $\imath : r \mapsto \alpha$, $\imath' : r \mapsto \alpha'$ such that $\mathcal{O}_r$ is mapped into $\mathcal{O}_{\imath(r)}$, $\mathcal{O}_{\imath'(r)}$ under the open embeddings $\phi_{\imath(r)}$, $\phi_{\imath'(r)}$ respectively. On the other hand, we take a cover $\{O_s\}$ of K by connected open subsets, together with mappings $\jmath : s \mapsto a$, $\jmath' : s \mapsto a'$ such that $O_s \subset O_{\jmath(s)} \cap O_{\jmath'(s)}$, and with a mapping $\bar\theta : s \mapsto r$ satisfying $u(O_s) \subset \mathcal{O}_{\bar\theta(s)}$ and $\imath \circ \bar\theta = \theta \circ \jmath$, $\imath' \circ \bar\theta = \theta' \circ \jmath'$. Then the set of data $(\bar\theta, \{O_s\}, \{\mathcal{O}_r\})$ gives rise to a homomorphism $\tau(\bar\theta, \{O_s\}, \{\mathcal{O}_r\})$, which is induced by both $\tau(\theta, \{O_a\}, \{\mathcal{O}_\alpha\})$ and $\tau(\theta', \{O_{a'}\}, \{\mathcal{O}_{\alpha'}\})$. Hence Φ_u is well-defined.

We denote the correspondence $u \mapsto \Phi_u$ by ϕ. Then the inverse of ϕ is obtained as follows. Given any $\Phi \in [(SK(G),*);(X,\underline{o})]_\rho$, we define a map $u_\Phi : (K, x_0) \to ([(S^1(G),*);(X,\underline{o})]_\rho, \tilde{o})$ by the rule that for any $x \in K$, $u_\Phi(x)$ is the restriction to the subspace $\{[t,x] | t \in I\}$ of $(SK(G),*)$, where $[t,x]$ denotes the image of $(t,x) \in I \times K$ in SK. It is easily seen that the map u_Φ is continuous, and the correspondence $\psi : \Phi \mapsto u_\Phi$ is the inverse of ϕ, namely, $u_{\Phi_u} = u$ and $\Phi_{u_\Phi} = \Phi$. Hence (1) of the lemma.

(2) We shall only sketch the proof here since it is completely parallel to the one above.

Given any continuous map $u : (K, x_0) \to ([(I(G), S^0(G), 0);(X, A, \underline{o})]_\rho, \tilde{o})$, we cover $u(K)$ by finitely many $\mathcal{O}_k, k = 0, 1, \cdots, m$, where $\mathcal{O}_k = \mathcal{O}_{(\{\xi_{ji,k}\}, V_{\alpha(k)})}$ with $i = 0, 1, \cdots, n$, such that $u(x_0) = \tilde{o}$ is contained in $\mathcal{O}_0$ with $U_{i,0} = U_o$, $\xi_{ji,0} = 1 \in G_{U_o}$ for all i, j. We then take a cover $\{O_a\}$ of K by connected open subsets, together with a mapping $\theta : a \mapsto k$ satisfying $u(O_a) \subset \mathcal{O}_{\theta(a)}$. Then for each index a, there is a set of data

$$(\{I_i\}, \{U_{i,\theta(a)}\}, V_{\alpha(\theta(a))}, \{f_{i,a}\}, \{\xi_{ji,\theta(a)}\}),\ i = 0, 1, \cdots, n, \tag{2.2.7}$$

where each $f_{i,a} : I_i \times O_a \to \widehat{U_{i,\theta(a)}}$ is continuous such that the restriction of (2.2.7) to each $x \in O_a$ is an element $\{f_{i,a}(\cdot, x)\} \in \mathcal{O}_{\theta(a)}$, which represents $u(x) \in [(I(G), S^0(G), 0); (X, A, \underline{o})]_\rho$. In particular, $f_{n,a}(1, x) \in \widehat{V_{\alpha(\theta(a))}}$ for all $x \in O_a$. We define $g_a : O_a \to \widehat{V_{\alpha(\theta(a))}}$ by $x \mapsto f_{n,a}(1, x)$.

For each connected component $O_{ab,s}$ of $O_a \cap O_b$, there exists a set of elements $\eta_i^{ba,s} \in T(U_{i,\theta(a)}, U_{i,\theta(b)})$, $i = 0, 1, \cdots, n$, where $\eta_0^{ba,s} = 1 \in G_{U_o}$ and $\eta_n^{ba,s} = \rho_{\alpha(\theta(b))\alpha(\theta(a))}(\zeta^{ba,s})$ for some $\zeta^{ba,s} \in T(V_{\alpha(\theta(a))}, V_{\alpha(\theta(b))})$, such that $g_a(O_{ab,s})$ is contained in Domain $(\phi_{\zeta^{ba,s}})$, and the following equations are satisfied:

$$\begin{aligned} &f_{i,b}(\cdot, x) = \phi_{\eta_i^{ba,s}} \circ f_{i,a}(\cdot, x),\ \forall x \in O_{ab,s}, \forall i, \\ &\xi_{ji,\theta(b)} = \eta_j^{ba,s} \circ \xi_{ji,\theta(a)} \circ (\eta_i^{ba,s})^{-1}\ \forall i, j, \end{aligned} \tag{2.2.8}$$

where $\eta_j^{ba,s} \circ \xi_{ji,\theta(a)} \circ (\eta_i^{ba,s})^{-1}$ is the element in $T(U_{i,\theta(b)}, U_{j,\theta(b)})$ that is determined by the component containing $f_{i,b}(I_j \cap I_i \times O_{ab,s})$. Analogous to (2.2.6), we have

$$\eta_i^{cb,t} \circ \eta_i^{ba,s}(f_{i,a}(\mathbf{b})) = \eta_i^{ca,r} \tag{2.2.9}$$

for any component $\mathbf{b}$ of $O_{ab,s} \cap O_{bc,t}$ which is contained in $O_{ac,r}$. Now $\tau(\theta, \{O_a\}, \{\mathcal{O}_k\}) = (\{f_{i,a}\}, \{\xi_{ji,\theta(b)} \circ \eta_i^{ba,s}\})$, where $\xi_{ji,\theta(b)} \circ \eta_i^{ba,s}$ is the element in $T(U_{i,\theta(a)}, U_{j,\theta(b)})$ determined by the component containing $f_{i,a}(I_j \cap I_i \times O_{ab,s})$, is a homomorphism from $\Gamma\{I_i \times O_a\}$ to $\Gamma\{U_{i',k}\}$. We define Φ_u to be the equivalence class of $\tau(\theta, \{O_a\}, \{\mathcal{O}_k\})$, which is well-defined by a similar argument as in (1) above, and is clearly an element of $[(CK(G), *); (X, \underline{o})]_\rho$.

We need to verify that the restriction of Φ_u to the subspace $(K(G), *)$ is an element of $[(K(G), *); (A, \underline{o}|_A)]_\rho$ so that $\Phi_u \in [(CK(G), K(G), *); (X, A, \underline{o})]_\rho$. Now the restriction of $\tau(\theta, \{O_a\}, \{O_k\})$ to $(K(G), *)$ is clearly the homomorphism $(\{g_a\}, \{\zeta^{ba,s}\}) : \Gamma\{O_a\} \to \Gamma\{V_{\alpha'(\theta(a))}\}$, which defines an element of $[(K(G), *); (A, \underline{o}|_A)]_\rho$. In order to justify that $(\{g_a\}, \{\zeta^{ba,s}\})$ is indeed a homomorphism, a number of equations needs to be verified: The first equation in (2.2.8) for the case $i = n$ implies that $g_b(x) = \phi_{\zeta^{ba,s}} \circ g_a(x)$, $\forall x \in O_{ab,s}$, and (2.2.9) for the case $i = n$ implies that $\zeta^{cb,t} \circ \zeta^{ba,s}(g_a(\mathbf{b})) = \zeta^{ca,r}$. Moreover, for each index a, recall that $\{\xi_{i,a} : G \to G_{U_{i,\theta(a)}}\}$ may be inductively determined by $\xi_{i+1,a} = \lambda_{\xi_{(i+1)i,\theta(a)}} \circ \xi_{i,a}$ with $\xi_{0,a} = \rho : G \to G_{U_o}$ and $\xi_{n,a}(G) \subset \rho_{\alpha(\theta(a))}(G_{V_{\alpha(\theta(a))}})$. We define $\zeta^a = \rho^{-1}_{\alpha(\theta(a))} \circ \xi_{n,a} : G \to G_{V_{\alpha(\theta(a))}}$. Then the second equation in (2.2.8) implies that $\zeta^b = \lambda_{\zeta^{ba,s}} \circ \zeta^a$.

The correspondence $\phi : u \mapsto \Phi_u$ is a bijection whose inverse ψ is defined as follows. For any $\Phi \in [(CK(G), K(G), *); (X, A, \underline{o})]_\rho$, let $u_\Phi(x)$ be the restriction of Φ to the subspace $\{[t, x] \mid t \in I\}$, which is an element of $[(I(G), S^0(G), 0); (X, A, \underline{o})]_\rho$. The mapping $x \mapsto u_\Phi(x)$ is easily seen to be continuous. The identities $u_{\Phi_u} = u$ and $\Phi_{u_\Phi} = \Phi$ are clear from the construction. Hence (2) of the lemma.

□

The second lemma is concerned with the special case when $X = Y/G$ is a global quotient.

We begin by fixing the notations. Let $\underline{o} = (o, U_o, \hat{o})$ be any base-point structure of X. We denote by U^o the connected component of X that contains o. The space $\widehat{U^o}$ is a connected component of Y that contains $\widehat{U_o}$, hence $\hat{o} \in \widehat{U_o}$ may be regarded as a point in $\widehat{U^o} \subset Y$.

Moreover, $T(U_o, U^o)$ is canonically identified with G_{U^o}. We assume $H \subset G$ is a subgroup and denote by $\rho : H \to G$ the inclusion. We set $Y^H = \{y \in Y \mid h \cdot y = y,\ \forall h \in H\}$ and $C(H) = \{g \in G \mid gh = hg,\ \forall h \in H\}$.

Let $P(Y^H, \hat{o}) = \{\gamma \mid \gamma : I \to Y^H$ continuous $, \gamma(0) = \hat{o}\}$ given with the usual "compact-open" topology, and let $P(Y^H, C(H), \hat{o}) = \{(\gamma, g) \mid \gamma \in P(Y^H, \hat{o}), g \in C(H),$ s.t. $\gamma(1) = g \cdot \hat{o}\}$ given with the relative topology as a subspace of $P(Y^H, \hat{o}) \times C(H)$. The space $P(Y^H, C(H), \hat{o})$ has a natural base point $\tilde{o} = (\gamma_o, 1)$ where γ_o is the constant map to $\hat{o}$.

Let $\Phi : (X, \underline{o}) \to (X', \underline{o'})$ be any map[2] defined by a pair $(f, \lambda) : (Y, G) \to (Y', G')$ where f is λ-equivariant, and $\hat{o'} = f(\hat{o})$ in the base-point structure $\underline{o'} = (o', U'_o, \hat{o'})$. For any subgroup $H \subset G$, we set $H' = \lambda(H)$ and denote the inclusion $H' \subset G'$ by ρ'. There is an induced map $P(f, \lambda) : (P(Y^H, C(H), \hat{o}), \tilde{o}) \to (P((Y')^{H'}, C(H'), \hat{o}'), \tilde{o}')$ defined by $(\gamma, g) \mapsto (f \circ \gamma, \lambda(g))$.

Lemma 2.2.2 *There exists a $\phi_X : ([(S^1(H), *); (X, \underline{o})]_\rho, \tilde{o}) \cong (P(Y^H, C(H), \hat{o}), \tilde{o})$ satisfying $P(f, \lambda) \circ \phi_X = \phi_{X'} \circ \Phi_\#$ for any map Φ defined by $(f, \lambda) : (Y, G) \to (Y', G')$.*

Proof First of all, we define $\phi_X : ([(S^1(H), *); (X, \underline{o})]_\rho, \tilde{o}) \to (P(Y^H, C(H), \hat{o}), \tilde{o})$ as follows. Let $u \in [(S^1(H), *); (X, \underline{o})]_\rho$ be any guided loop which is represented by $\sigma = \{f_i\} \in \mathcal{O}_{\{\xi_{ji}\}}$, $i = 0, 1, \cdots, n$. Since the image of u in the underlying space X lies in U^o, we may assume that $U_i = U^o$ for each $i \neq 0, n$ by replacing σ with an equivalent homomorphism. Consequently, each $\xi_{ji} \in T(U_i, U_j)$ is an element of G_{U^o}. We define a path $\gamma : I \to Y$ as follows: we set $\gamma = f_0$ on I_0, $\gamma = \phi_{\xi_{10}}^{-1} \circ f_1$ on I_1, $\cdots, \gamma = \phi_{\xi_{10}}^{-1} \circ \cdots \circ \phi_{\xi_{n(n-1)}}^{-1} \circ f_n$ on I_n, and we define $g = \xi_{10}^{-1} \circ \cdots \circ \xi_{n(n-1)}^{-1} \in G_{U^o} \subset G$. Clearly $\gamma(0) = \hat{o}$, $\gamma(t) \in Y^H, \forall t \in I$ and $\gamma(1) = g \cdot \hat{o}$. Moreover, $Ad(g) = \lambda_{\xi_{10}}^{-1} \circ \cdots \circ \lambda_{\xi_{n(n-1)}}^{-1}$ which satisfies $\rho = Ad(g)^{-1} \circ \rho$. Hence $Ad(g)(h) = h, \forall h \in H$ so that $g \in C(H)$. We define ϕ_X by setting $\phi_X(u) = (\gamma, g)$, which is clearly an element of $P(Y^H, C(H), \hat{o})$. It is easily seen that ϕ_X is well-defined, i.e., $\phi_X(u)$ is independent of the choice on σ. The map ϕ_X is continuous as well. Furthermore, ϕ_X is a base-point preserving map, and satisfies $P(f, \lambda) \circ \phi_X = \phi_{X'} \circ \Phi_\#$ for any map Φ defined by $(f, \lambda) : (Y, G) \to (Y', G')$.

It remains to show that ϕ_X is a homeomorphism. We construct a $\psi_X : P(Y^H, C(H), \hat{o}) \to [(S^1(H), *); (X, \underline{o})]_\rho$ as follows. Given any $(\gamma, g) \in P(Y^H, C(H), \hat{o})$, it is obvious that $\gamma(t) \in \widehat{U^o}$ for all $t \in I$, and $g \in G_{U^o}$ because $g \cdot \widehat{U^o} = \widehat{U^o}$. We define ψ_X by sending (γ, g) to the element in $[(S^1(H), *); (X, \underline{o})]_\rho$ which is the equivalence class of $\{f_i\} \in \mathcal{O}_{\{\xi_{ji}\}}$, $i = 0, 1, 2$, where $I_0 \subset \gamma^{-1}(\widehat{U_o})$, $I_1 = (0, 1)$, $I_2 \subset \gamma^{-1}(g \cdot \widehat{U_o})$, $U_0 = U_o$, $U_1 = U^o$, $U_2 = U_o$, $f_0 = \gamma|_{I_0}$, $f_1 = \gamma|_{I_1}$, $f_2 = g^{-1} \circ \gamma|_{I_2}$, and $\xi_{10} = 1$, $\xi_{21} = g^{-1}$. Clearly ψ_X is continuous, and $\phi_X \circ \psi_X = Id$, $\psi_X \circ \phi_X = Id$. Hence ϕ_X is a homeomorphism.

□

Now we present the first list of properties of the homotopy sets defined in Definition 1.1.

Proposition 2.2.3

(1) *There are natural group structures on $\pi_k^{(G,\rho)}(X, \underline{o})$ and $\pi_{k+1}^{(G,\rho)}(X, A, \underline{o})$ for $k \geq 1$ which are Abelian when $k \geq 2$.*

---

[2]here Φ is allowed to be a general map, i.e., Φ is not in the more restricted class specified by **Convention** in Introduction.

(2) *For any* $\Phi \in [(X,\underline{o});(X',\underline{o'})]_\eta$ *(resp.* $\Phi \in [(X,A,\underline{o});(X',A',\underline{o'})]_\eta$*), there are natural homomorphisms* $\Phi_* : \pi_k^{(G,\rho)}(X,\underline{o}) \to \pi_k^{(G,\rho')}(X',\underline{o'})$ *(resp.* $\Phi_* : \pi_k^{(G,\rho)}(X,A,\underline{o}) \to \pi_k^{(G,\rho')}(X',A',\underline{o'})$*) with* $\rho' = \eta\circ\rho$*, which depend only on the homotopy class of* Φ*. For any* (H,η) *where* $\eta : H \to G_{\hat{o}}$ *is an injective homomorphism which factors through* $\rho : G \to G_{\hat{o}}$ *by* $\iota : H \to G$*, there are natural homomorphisms* $\iota^* : \pi_k^{(G,\rho)}(X,\underline{o}) \to \pi_k^{(H,\eta)}(X,\underline{o})$ *and* $\iota^* : \pi_{k+1}^{(G,\rho)}(X,A,\underline{o}) \to \pi_{k+1}^{(H,\eta)}(X,A,\underline{o})$*.*

(3) *For the case* $G = \{1\}$*,* $\pi_k(X,\underline{o})$*,* $\pi_{k+1}(X,A,\underline{o})$ *are naturally isomorphic to* $\pi_k(B\Gamma_X,*)$ *and* $\pi_{k+1}(B\Gamma_X, B\Gamma_A,*)$*, where* $B\Gamma_X, B\Gamma_A$ *are the classifying spaces of the defining groupoid for* X *and* A *respectively.*

(4) *When* $X = Y/G$ *is a global quotient, for any subgroup* $\rho : H \subset G$*, there are natural isomorphisms* $\theta_X : \pi_k^{(H,\rho)}(X,\underline{o}) \cong \pi_k(Y^H,\hat{o})$ *for all* $k \geq 2$*, and the natural exact sequence*

$$1 \to \pi_1(Y^H,\hat{o}) \to \pi_1^{(H,\rho)}(X,\underline{o}) \to C(H) \to \pi_0(Y^H,\hat{o}) \to \pi_0^{(H,\rho)}(X,\underline{o}). \tag{2.2.10}$$

Proof (1) For any $k \geq 1$, we fix an identification $S^k = SS^{k-1}$ and $D^{k+1} = CS^k$. Then by Lemma 2.2.1, $\pi_k^{(G,\rho)}(X,\underline{o})$ and $\pi_{k+1}^{(G,\rho)}(X,A,\underline{o})$ are the π_{k-1} and π_k of $([(S^1(G),*);(X,\underline{o})]_\rho,\tilde{o})$ and $([(I(G),S^0(G),0);(X,A,\underline{o})]_\rho,\tilde{o})$ respectively. Hence they have natural group structures for $k \geq 1$ which are Abelian when $k \geq 2$, cf. Lemma 2.1.2.

(2) Straightforward.

(3) Note that a map from a topological space S (which is regarded as an orbispace trivially) to an orbispace X is an equivalence class of Γ_X-structures on S, where Γ_X is ranging in a certain set of étale topological groupoids which define the orbispace structure on X. It is easily seen that when a choice of Γ_X is fixed, a map from S to X can be naturally identified with a Γ_X-structure on S (cf. Lemma 3.1.2 in [3]). On the other hand, by a theorem of Haefliger in [10], the set of homotopy classes of Γ_X-structures on S is naturally in one to one correspondence with the set of homotopy classes of continuous maps from S into the classifying space $B\Gamma_X$ of Γ_X. The assertion follows easily.

(4) We consider the continuous map $\pi : (P(Y^H,C(H),\hat{o}),\tilde{o}) \to (C(H),1)$ by sending (γ,g) to g. The fiber at $1 \in C(H)$ is identified with the loop space $\Omega(Y^H,\hat{o})$ via the embedding $\Omega(Y^H,\hat{o}) \hookrightarrow P(Y^H,C(H),\hat{o})$ sending γ to $(\gamma,1)$. With Lemma 2.2.2, the assertion follows essentially from the fact that $\pi : P(Y^H,C(H),\hat{o}) \to C(H)$ is a fibration (note that $C(H)$ has a discrete topology), and that $\pi\circ\phi_X(u_1\#u_2) = \pi\circ\phi_X(u_1)\cdot\pi\circ\phi_X(u_2)$ for any guided loops $u_1,u_2 \in [(S^1(H),*);(X,\underline{o})]_\rho$. What remains is to examine the part $C(H) \to \pi_0(Y^H,\hat{o}) \to \pi_0^{(H,\rho)}(X,\underline{o})$ in (2.2.10).

The map $C(H) \to \pi_0(Y^H,\hat{o})$ is defined by sending $g \in C(H)$ to the class of $g\cdot\hat{o}$ in $\pi_0(Y^H,\hat{o})$, and the map $\pi_0(Y^H,\hat{o}) \to \pi_0^{(H,\rho)}(X,\underline{o})$ is defined by sending the class of $y \in Y^H$ to the class of $u_y \in [(S^0(H),0);(X,\underline{o})]_\rho$, where u_y is the map defined by $(f,\rho) : (S^0,H) \to (Y,G)$ where $f(0) = \hat{o}$, $f(1) = y$. The exactness of (2.2.10) at $C(H)$ is trivial, we shall focus on the remaining case at $\pi_0(Y^H,\hat{o})$. First of all, suppose $y \in Y^H$ is path-connected to $g\cdot\hat{o}$ in Y^H for some $g \in C(H)$ by a path γ. We set $z = g^{-1}\cdot y$. Then it is easily seen that $u_y = u_z$, which is homotopic to the base point in $\pi_0^{(H,\rho)}(X,\underline{o})$ via the guided path defined by $(g^{-1}\circ\gamma,\rho)$. On the other hand, suppose u_y is homotopic to the base point in $\pi_0^{(H,\rho)}(X,\underline{o})$.

Then the homotopy defines an element $\Phi \in [(I(H),0);(X,\underline{o})]_\rho$ such that the restriction of Φ to $S^0 = \partial I$ is u_y. We apply the construction in Lemma 2.2.2 to represent Φ by a pair $(\gamma,\rho) : (I,H) \to (Y,G)$ such that $\gamma(0) = \hat{o}$. Then there is a $g \in G$ such that $y = g \cdot \gamma(1)$ and $\rho = Ad(g)^{-1} \circ \rho$. The last equation implies that $g \in C(H)$. Moreover, y is path-connected to $g \cdot \hat{o}$ in Y^H through $g \circ \gamma$. Hence (2.2.10) is exact at $\pi_0(Y^H, \hat{o})$.

□

Now we turn our attention to the natural homomorphisms

$$C : \pi_1^{(G,\rho)}(X,\underline{o}) \to \mathrm{Aut}(\pi_k^{(G,\rho)}(X,\underline{o})),\ k \geq 1, \tag{2.2.11}$$

and

$$C' : \pi_1^{(G,\rho)}(A,\underline{o}|_A) \to \mathrm{Aut}(\pi_{k+1}^{(G,\rho)}(X,A,\underline{o})),\ k \geq 1. \tag{2.2.12}$$

The homomorphism in (2.2.11) is defined as follows. Let $u_0 \in [(S^1(G),*);(X,\underline{o})]_\rho$ be any guided loop. There is a continuous map $F_{u_0} : [(S^1(G),*);(X,\underline{o})]_\rho \to [(S^1(G),*);(X,\underline{o})]_\rho$ defined by $u \mapsto (u_0\#u)\#\nu(u_0)$. Moreover, if u_s, $s \in [0,1]$, is a path in $[(S^1(G),*);(X,\underline{o})]_\rho$, then the map $F : [(S^1(G),*);(X,\underline{o})]_\rho \times [0,1] \to [(S^1(G),*);(X,\underline{o})]_\rho$, where $F(u,s) = F_{u_s}(u)$, is continuous. We simply define (2.2.11) by setting

$$C([u]) = (F_u)_* : \pi_{k-1}([(S^1(G),*);(X,\underline{o})]_\rho,\tilde{o}) \to \pi_{k-1}([(S^1(G),*);(X,\underline{o})]_\rho, F_u(\tilde{o}))$$

and identifying $\pi_{k-1}([(S^1(G),*);(X,\underline{o})]_\rho, F_u(\tilde{o}))$ with $\pi_{k-1}([(S^1(G),*);(X,\underline{o})]_\rho,\tilde{o})$ by a canonically chosen path between $F_u(\tilde{o})$ and $\tilde{o}$ (cf. Lemma 2.1.2). The map $C([u])$ is a homomorphism because $([(S^1(G),*);(X,\underline{o})]_\rho,\tilde{o})$ is an H-group so that the multiplication in $\pi_{k-1}([(S^1(G),*);(X,\underline{o})]_\rho,\tilde{o})$ is also given by the homotopy associative multiplication $\#$. On the other hand, it is easy to see that $C([u]) \circ C([u']) = C([u\#u'])$ and $C([\tilde{o}]) = Id$. Hence C is a homomorphism from $\pi_1^{(G,\rho)}(X,\underline{o})$ into $\mathrm{Aut}(\pi_k^{(G,\rho)}(X,\underline{o}))$. By nature of construction, the homomorphism in (2.2.11) is natural with respect to the homomorphisms Φ_* and ι^* in Proposition 2.2.3 (2).

The homomorphism in (2.2.12) is defined as follows. Let $u \in [(S^1(G),*);(A,\underline{o}|_A)]_\rho$ be any guided loop. We consider a family of elements $u(s) \in [(I(G),S^0(G),0);(X,A,\underline{o})]_\rho$, $s \in [0,1]$, which is defined by restricting $u\#\tilde{o}$ to $[0,s]$ and then reparametrizing it by $t \mapsto ts, \forall t \in I$. Note that $u(0) = \tilde{o}$ and $u(1) = u\#\tilde{o}$. With $u(s)$, a continuous map F_u from the loop space $\Omega([(I(G),S^0(G),0);(X,A,\underline{o})]_\rho,\tilde{o})$ to itself is defined as follows. Given any loop $v : [0,1] \to [(I(G),S^0(G),0);(X,A,\underline{o})]_\rho$ where $v(0) = v(1) = \tilde{o}$, we define the loop $F_u(v)$ by

$$F_u(v)(s) = \begin{cases} u(3s) & 0 \leq s \leq \frac{1}{3} \\ u\#v(3s-1) & \frac{1}{3} \leq s \leq \frac{2}{3} \\ u(3-3s) & \frac{2}{3} \leq s \leq 1. \end{cases} \tag{2.2.13}$$

The map F_u is clearly continuous. Let v_0 be the constant loop into $\tilde{o}$, and set $v_0' = F_u(v_0)$. Then F_u induces a homomorphism

$$C_u' : \pi_k(\Omega([(I(G),S^0(G),0);(X,A,\underline{o})]_\rho,\tilde{o}),v_0) \to \pi_k(\Omega([(I(G),S^0(G),0);(X,A,\underline{o})]_\rho,\tilde{o}),v_0')$$

for $k \geq 0$. On the other hand, $v_0' = F_u(v_0)$ is canonically homotopic to v_0, so that C_u' gives rise to a homomorphism $C'(u) : \pi_{k+1}^{(G,\rho)}(X,A,\underline{o}) \to \pi_{k+1}^{(G,\rho)}(X,A,\underline{o})$ for $k \geq 1$. Finally, we observe that $C'(u)$ depends only on the homotopy class of u, and that $C'(u) \circ C'(u') = C'(u\#u')$, $C'(\tilde{o}) = Id$. Hence the homomorphism in (2.2.12).

We end this subsection with

Proof of Proposition 1.3

(1) Given any guided path $u \in [(I(G),0,1);(X,\underline{o_1},\underline{o_2})]_{(\rho,\eta)}$, the mapping $v \mapsto (u\#v)\#\nu(u)$ which is from $([(S^1(G),*);(X,\underline{o_2})]_\eta,\tilde{o_2})$ to $([(S^1(G),*);(X,\underline{o_1})]_\rho,(u\#\tilde{o_2})\#\nu(u))$ is continuous. The induced homomorphism between the homotopy groups is defined to be u_*, where we take a canonical path between $\tilde{o_1}$ and $(u\#\tilde{o_2})\#\nu(u)$ to identify the homotopy groups of $([(S^1(G),*);(X,\underline{o_1})]_\rho,\tilde{o_1})$ with the corresponding ones of $([(S^1(G),*);(X,\underline{o_1})]_\rho,(u\#\tilde{o_2})\#\nu(u))$. By the nature of definition, for any guided paths u_1,u_2, we have $\nu(u_2)_*\circ(u_1)_* = C([v(u_2)\#u_1])$, from which it follows that u_* is isomorphic.

(2) Given any guided path $u \in [(I(G),0,1);(A,\underline{o_1}|_A,\underline{o_2}|_A)]_{(\rho,\eta)}$, we define a continuous map $F_u : v \mapsto u\#v$ from $[(I(G),S^0(G),0);(X,A,\underline{o_2})]_\eta$ to $[(I(G),S^0(G),0);(X,A,\underline{o_1})]_\rho$, sending the base point $\tilde{o_2}$ to $u\#\tilde{o_2}$, which is canonically path-connected to the base point $\tilde{o_1}$ as follows. Consider the restriction of $u\#\tilde{o_2}$ to $[0,s]$, and then reparametrize it by $t \mapsto ts, \forall t \in I$. We obtain a path $u(s)$, $s \in [0,1]$, in $[(I(G),S^0(G),0);(X,A,\underline{o_1})]_\rho$ which satisfies $u(0)=\tilde{o_1}$ and $u(1)=u\#\tilde{o_2}$. With the canonical isomorphisms provided by the path $u(s)$, the map F_u induces a map u_* from the homotopy groups of $([(I(G),S^0(G),0);(X,A,\underline{o_2})]_\eta,\tilde{o_2})$ to the corresponding ones of $([(I(G),S^0(G),0);(X,A,\underline{o_1})]_\rho,\tilde{o_1})$, which is a homomorphism for $k \geq 1$ and base point preserving for $k=0$. By the nature of definition, for any guided paths u_1,u_2, we have $\nu(u_2)_*\circ(u_1)_* = C'([\nu(u_2)\#u_1])$ when $k\geq 1$, from which it follows that u_* is isomorphic. When $k=0$, one can easily check that $\nu(u)_*\circ(u)_*$ is the identity map. Hence u_* is a bijection when $k=0$.

(3) The existence of natural mappings ι^* in Proposition 2.2.3 (2) implies that as far as the isomorphism class of $\pi_k^{(G,\rho)}(X,\underline{o})$ or $\pi_{k+1}^{(G,\rho)}(X,A,\underline{o})$ is concerned, one may always assume that G is a subgroup of $G_{\hat{o}}$ and ρ is the inclusion $G\subset G_{\hat{o}}$. In order to see that for a different but conjugate subgroup, there is a canonical isomorphism intervening, we consider the following more general situation: Suppose $\underline{o_1}$, $\underline{o_2}$ are base-point structures such that $o_1=o_2=o$, and $G_1\subset G_{\hat{o_1}}$ and $G_2\subset G_{\hat{o_2}}$ such that there is a $\xi\in T(U_{o_1},U_{o_2})$ satisfying $\lambda_\xi(G_1)=G_2$. Then there is a guided path $u_\xi\in[(I(G_1),0,1);(X,\underline{o_1},\underline{o_2})]_{(\rho_1,\rho_2\circ\lambda_\xi)}$, or in the second case, in $[(I(G_1),0,1);(A,\underline{o_1}|_A,\underline{o_2}|_A)]_{(\rho_1,\rho_2\circ\lambda_\xi)}$, which is defined by $(\{f_0,f_1\},\{\xi\})$ where $f_0([0,\frac{2}{3}))=\hat{o_1}$ and $f_1((\frac{1}{3},1])=\hat{o_2}$. The isomorphism $(u_\xi)_*$ in (1) or (2) will then do.

$\square$

2.3 Exact sequence of a pair

Let $(X,A,\underline{o})$ be any pair, and $\rho : G\to G_{\hat{o}}$ be any injective homomorphism. Denote by $i : [(S^k(G),*);(A,\underline{o}|_A)]_\rho \to ([(S^k(G),*);(X,\underline{o})]_\rho$ the mapping induced by the inclusion $(A,\underline{o}|_A)\subset(X,\underline{o})$, and by $j : ([(S^1(G),*);(X,\underline{o})]_\rho,\tilde{o}) \to ([(I(G),S^0(G),0);(X,A,\underline{o})]_\rho,\tilde{o})$ the continuous map induced by the mapping $[(S^1(G),*);(X,\underline{o})]_\rho \to [(I(G),0);(X,\underline{o})]_\rho$ of forgetting the second base-point structure. Denote by i_* (for the case when $k=1$), j_* the corresponding homomorphisms between the homotopy groups. Then we have

Theorem 2.3.1 *Let $(X,A,\underline{o})$ be any pair, and $\rho : G\to G_{\hat{o}}$ be any injective homomorphism.*

(1) *For any $k \geq 0$, there exists a mapping*

$$\partial : \pi_{k+1}^{(G,\rho)}(X, A, \underline{o}) \to \pi_k^{(G,\rho)}(A, \underline{o}|_A) \tag{2.3.1}$$

which is a homomorphism for $k \geq 1$, and base point preserving for $k = 0$. Moreover, for any $z \in \pi_1^{(G,\rho)}(A, \underline{o}|_A)$, $\partial \circ C'(z) = C(z) \circ \partial$ holds where C and C' are given in (2.2.11) *and* (2.2.12)*, and for any map Φ between pairs, we have $\partial \circ \Phi_* = (\Phi|_A)_* \circ \partial$.*

(2) *There exists a natural long exact sequence*

$$\begin{aligned} \cdots \to \pi_{k+1}^{(G,\rho)}(X, A, \underline{o}) \xrightarrow{\partial} \pi_k^{(G,\rho)}(A, \underline{o}|_A) \xrightarrow{i_*} \pi_k^{(G,\rho)}(X, \underline{o}) \xrightarrow{j_*} \pi_k^{(G,\rho)}(X, A, \underline{o}) \xrightarrow{\partial} \cdots \\ \xrightarrow{j_*} \pi_1^{(G,\rho)}(X, A, \underline{o}) \xrightarrow{\partial} \pi_0^{(G,\rho)}(A, \underline{o}|_A) \xrightarrow{i_*} \pi_0^{(G,\rho)}(X, \underline{o}). \end{aligned} \tag{2.3.2}$$

Proof (1) The case $k = 0$. Given any guided relative loop $u \in [(I(G), S^0(G), 0); (X, A, \underline{o})]_\rho$, the restriction $u|_A \in [(S^0(G), 0); (A, \underline{o}|_A)]_\rho$, such that if u_1, u_2 are path-connected, so are $(u_1)|_A, (u_2)|_A$. We define $\partial[u] = [u|_A]$, which is clearly base point preserving and satisfies $\partial \circ \Phi_* = (\Phi|_A)_* \circ \partial$ for any map Φ between pairs.

To define (2.3.1) for $k \geq 1$, we regard $(S^k, *)$ as the suspension $(SS^{k-1}, *)$. Then for any continuous map $u : (S^k, *) \to ([(I(G), S^0(G), 0); (X, A, \underline{o})]_\rho, \tilde{o})$, we consider the corresponding map $\Phi_u \in [(CS^k(G), S^k(G), *); (X, A, \underline{o})]_\rho$ constructed in Lemma 2.2.1 (2). We define $\partial([u]) = [(\Phi_u)|_{S^k(G)}] \in \pi_k^{(G,\rho)}(A, \underline{o}|_A)$, which is also the homotopy class of the continuous map $\partial u = u_{(\Phi_u)|_{S^k(G)}} : (S^{k-1}, *) \to ([(S^1(G), *); (A, \underline{o}|_A)]_\rho, \widetilde{o|_A})$ constructed in Lemma 2.2.1 (1). To see that (2.3.1) is a homomorphism, we simply observe that $(SS^{k-1}, *)$ is an H-cogroup (cf. [14]) and the H-cogroup structure of $(SS^{k-1}, *)$, which defines the group structure of $\pi_{k+1}^{(G,\rho)}(X, A, \underline{o})$, corresponds to the H-group structure of $([(S^1(G), *); (A, \underline{O}|_A)]_\rho, \widetilde{o|_A})$ under the correspondence $u \mapsto \partial u$. Finally, it is straightforward from the definitions that $\partial \circ C'(z) = C(z) \circ \partial$ holds for any $z \in \pi_1^{(G,\rho)}(A, \underline{o}|_A)$, and $\partial \circ \Phi_* = (\Phi|_A)_* \circ \partial$ holds for any map Φ between pairs.

(2) We begin by showing that the composition of any two consecutive homomorphisms of (2.3.2) is zero. First of all, it is straightforward from the definition that $\partial \circ j_* = 0$. To see $j_* \circ i_* = 0$, suppose $u : (S^k, *) \to ([(S^1(G), *); (A, \underline{o}|_A)]_\rho, \widetilde{o|_A})$, $k \geq 0$, is any continuous map. We shall prove $j_* \circ i_*([u]) = 0$. We take a homomorphism $\tau(\theta, \{O_a\}, \{\mathcal{O}_\alpha\})$ constructed in the proof of Lemma 2.2.1 (1), which represents the map $\Phi_u \in [(SS^k(G), *); (A, \underline{o}|_A)]$ that corresponds to u in Lemma 2.2.1 (1). Now for any $s \in [0, 1]$, denote by κ_s the homomorphism which is obtained by rescaling the restriction of $\tau(\theta, \{O_a\}, \{\mathcal{O}_\alpha\})$ to $\{[t, x] | t \leq s, x \in S^k\}$ under $t \mapsto ts$. Each κ_s defines a continuous map $u_s : (S^k, *) \to ([(I(G), S^0(G), 0); (X, A, \underline{o})]_\rho, \tilde{o})$ such that $s \mapsto u_s$ is continuous. Moreover, we have $u_1 = j \circ i \circ u$ and $u_0(S^k) = \tilde{o}$. Hence $j_* \circ i_*([u]) = 0$. To see $i_* \circ \partial = 0$, we first look at the case $k \geq 1$. For any $u : (S^k, *) \to ([(I(G), S^0(G), 0); (X, A, \underline{o})]_\rho, \tilde{o})$, we consider the map $\Phi_u \in [(CS^k(G), S^k(G), *); (X, A, \underline{o})]_\rho$ constructed in Lemma 2.2.1 (2). We identify $(CS^k, *)$ with $(SCS^{k-1}, *)$, and then apply Lemma 2.2.1 (1) to obtain a continuous map $H = u_{\Phi_u} : (CS^{k-1}, *) \to ([(S^1(G), *); (X, \underline{o})]_\rho, \tilde{o})$. Now observe $H|_{(S^{k-1}, *)} = i \circ (\partial u)$. Hence $i_* \circ \partial([u]) = 0$ for any continuous map $u : (S^k, *) \to ([(I(G), S^0(G), 0); (X, A, \underline{o})]_\rho, \tilde{o})$. The case $k = 0$ is similar, where we start with a guided relative loop $u \in [(I(G), S^0(G), 0); (X, A, \underline{o})]_\rho$ instead of the map u, and replace Φ_u by the corresponding path $u \in [(I(G), 0); (X, \underline{o})]_\rho$ in the argument.

The exactness at $\pi_0^{(G,\rho)}(A,\underline{o}|_A)$: Let $u\in[(S^0(G),0),(A,\underline{o}|_A)]_\rho$ be any element such that $i_*([u])=[\tilde{o}]$. Then there is a homotopy between $i(u)$ and $\tilde{o}$, which may be interpreted as a guided path $\Phi\in[(I(G),0);(X,\underline{o})]_\rho$. Now $\Phi|_{S^0(G)}=u$ so that $\Phi\in[(I(G),S^0(G),0);(X,A,\underline{o})]_\rho$. Clearly $\partial([\Phi])=[u]$. Hence (2.3.2) is exact at $\pi_0^{(G,\rho)}(A,\underline{o}|_A)$.

The exactness at $\pi_1^{(G,\rho)}(X,A,\underline{o})$: Let $u\in[(I(G),S^0(G),0);(X,A,\underline{o})]_\rho$ be any guided relative loop such that $\partial([u])=[\widetilde{o|_A}]$. Then $u|_{(S^0(G),0)}$ is homotopic to $\widetilde{o|_A}$ in $[(S^0(G),0);(A,\underline{o}|_A)]_\rho$, which gives rise to an element $v\in[(I(G),0);(A,\underline{o}|_A)]_\rho$ satisfying $v|_{(S^0(G),0)}=u|_{(S^0(G),0)}$. We may 'compose' u with $i(\nu(v))$ to obtain a guided loop $u\#i(\nu(v))\in[(S^1(G),*);(X,\underline{o})]_\rho$ as follows. We pick a homomorphism $\{f_i\}\in\mathcal{O}_{(\{\xi_{ji}\},V_\alpha)}$ representing u and pick a homomorphism $\{h_k\}\in\mathcal{O}_{\{\eta_{lk}\}}$ representing $\nu(v)$ where each $\eta_{lk}\in T(V_{\alpha(k)},V_{\alpha(l)})$ for some correspondence $k\mapsto\alpha=\alpha(k)$. We compose $(\{f_i\},\{\xi_{ji}\})$ with $(\{h_k\},\{\rho_{\alpha(l)\alpha(k)}(\eta_{lk})\})$ to obtain a homomorphism τ, and define $u\#i(\nu(v))=[\tau]$. Note that $[\tau]$ may not be uniquely determined by u and v. But because of its specific form, τ provides a natural homotopy between $j(u\#i(\nu(v)))$ and u. Hence $j_*([u\#i(\nu(v))])=[u]$, and (2.3.2) is exact at $\pi_1^{(G,\rho)}(X,A,\underline{o})$.

The exactness at $\pi_k^{(G,\rho)}(X,\underline{o})$ with $k\geq 1$: Let $u:(S^{k-1},*)\rightarrow([(S^1(G),*);(X,\underline{o})]_\rho,\tilde{o})$ be any continuous map such that $[u]\in\ker\ j_*$. Then there is a homotopy $H:(CS^{k-1},*)\mapsto([(I(G),S^0(G),0);(X,A,\underline{o})]_\rho,\tilde{o})$ with $H|_{(S^{k-1},*)}=j\circ u$. As shown in Lemma 2.2.1 (2), H determines an element $\Phi_H\in[(CCS^{k-1}(G),CS^{k-1}(G),*);(X,A,\underline{o})]_\rho$, which is the equivalence class of a canonically constructed homomorphism $\tau=\tau(\theta,\{O_a\},\{\mathcal{O}_\alpha\})$. Let $[t_1,t_2,x],t_1\in I,t_2\in I,x\in S^{k-1}$ be the coordinates of CCS^{k-1}. Then we observe that the restriction of τ to $\{[t_1,1,x]|t_1\in I,x\in S^{k-1}\}$ represents the element $\Phi_u\in[(SS^{k-1}(G),*);(X,\underline{o})]_\rho$ determined by u in Lemma 2.2.1 (1), and the restriction of τ to $\{[1,t_2,x]|t_2\in I,x\in S^{k-1}\}$ represents $i((\Phi_H)|_{(CS^{k-1}(G),*)})\in[(CS^{k-1}(G),*);(X,\underline{o})]_\rho$. Moreover, $(\Phi_H)|_{(CS^{k-1}(G),*)}$ is in fact defined over $(SS^{k-1}(G),*)$ because its restriction to $(S^{k-1}(G),*)$ is $(\Phi_{j\circ u})|_{(S^{k-1}(G),*)}$, where $\Phi_{j\circ u}$ is the element in $[(CS^{k-1}(G),S^{k-1}(G),*);(X,A,\underline{o})]_\rho$ that is constructed in Lemma 2.2.1 (2) for $j\circ u$. $(\Phi_{j\circ u})|_{(S^{k-1}(G),*)}$ is obviously a constant map. Let $u_{(\Phi_H)|_{(CS^{k-1}(G),*)}}$, $u_{i((\Phi_H)|_{(CS^{k-1}(G),*)})}$ be the continuous maps defined in Lemma 2.2.1 (1) which correspond to $(\Phi_H)|_{(CS^{k-1}(G),*)}$ and $i((\Phi_H)|_{(CS^{k-1}(G),*)})$ respectively. Then the map $x\mapsto u(x)\#\nu(u_{i((\Phi_H)|_{(CS^{k-1}(G),*)})}(x))$, $x\in S^{k-1}$, is homotopic to the constant map $x\mapsto\tilde{o}$ via a homotopy $F(s):(S^{k-1},*)\rightarrow([(S^1(G),*);(X,\underline{o})]_\rho,\tilde{o})$, where $F(s)$ is defined by the restriction of τ to

$$\{[t_1,s,x]|t_1\leq s,x\in S^{k-1}\}\cup\{[s,t_2,x]|t_2\leq s,x\in S^{k-1}\}.$$

This implies that $[u]=[u_{i((\Phi_H)|_{(CS^{k-1}(G),*)})}]=[i(u_{(\Phi_H)|_{(CS^{k-1}(G),*)}})]=i_*([u_{(\Phi_H)|_{(CS^{k-1}(G),*)}}])$. Hence (2.3.2) is exact at $\pi_k^{(G,\rho)}(X,\underline{o})$.

The exactness at $\pi_k^{(G,\rho)}(A,\underline{o}|_A)$ with $k\geq 1$: Let $u:(S^{k-1},*)\rightarrow([(S^1(G),*);(A,\underline{o}|_A)]_\rho,\widetilde{o|_A})$ be any continuous map such that $[u]\in\ker\ i_*$. Then there is a homotopy $H:(CS^{k-1},*)\rightarrow([(S^1(G),*);(X,\underline{o})]_\rho,\tilde{o})$ with $H|_{(S^{k-1},*)}=i\circ u$. We shall repeat the construction in Lemma 2.2.1 (1) to the map H, but for the specific purpose here, we choose the cover $\{\mathcal{O}_\alpha\}$ in the construction of the homomorphism $\tau(\theta,\{O_a\},\{\mathcal{O}_\alpha\})$ as follows. First of all, we fix a homomorphism $\sigma=(\{i_\alpha\},\{\rho_{\beta\alpha}\})$ which represents the subspace inclusion $(A,\underline{o}|_A)\subset(X,\underline{o})$. Then we cover $u(S^{k-1})$ by finitely many $\{\mathcal{O}_s\}$, where each $\mathcal{O}_s$ has the form $\mathcal{O}_{\{\delta_{ji,s}\}}$. Consequently, $H(S^{k-1})=i\circ u(S^{k-1})$ is covered by $\{\sigma(\mathcal{O}_s)\}$, where $\sigma(\mathcal{O}_s)=\mathcal{O}_{\{\rho_{\alpha(s,j)\alpha(s,i)}(\delta_{ji,s})\}}$. Now we cover $H(CS^{k-1})$ by finitely many $\{\mathcal{O}_k\}$ such that $\{\sigma(\mathcal{O}_s)\} \subset \{\mathcal{O}_k\}$ and covers $H(S^{k-1})$. We proceed to construct the homomorphism $\tau(\theta, \{O_a\}, \{\mathcal{O}_k\})$, which defines the element $\Phi_H \in [(SCS^{k-1}(G), *); (X, \underline{o})]_\rho$ associated to H in Lemma 2.2.1 (1). Now we identify $(SCS^{k-1}, *)$ with $(CSS^{k-1}, *) = (CS^k, *)$. Then the special choice of the cover $\{\mathcal{O}_k\}$ in $\tau(\theta, \{O_a\}, \{\mathcal{O}_k\})$ implies that $\tau(\theta, \{O_a\}, \{\mathcal{O}_k\})$ also defines an element $\Psi \in [(CS^k(G), S^k(G), *); (X, A, \underline{o})]_\rho$, which is associated with a continuous map $u_\Psi : (S^k, *) \to ([(I(G), S^0(G), 0); (X, A, \underline{o})]_\rho, \tilde{o})$ in Lemma 2.2.1 (2), such that $\partial u_\Psi : (S^{k-1}, *) \to ([(S^1(G), *); (A, \underline{o}|_A)]_\rho, \widetilde{o|_A})$ equals u. Hence $[u] = \partial([u_\Psi])$ and (2.3.2) is exact at $\pi_k^{(G,\rho)}(A, \underline{o}|_A)$.

The exactness at $\pi_{k+1}^{(G,\rho)}(X, A, \underline{o})$, $k \geq 1$: Let $u : (S^k, *) \to ([(I(G), S^0(G), 0); (X, A, \underline{o})]_\rho, \tilde{o})$ be any continuous map such that $[u] \in \ker\ \partial$. Then there is a homotopy $H : (CS^{k-1}, *) \to ([(S^1(G), *); (A, \underline{o}|_A)]_\rho, \widetilde{o|_A})$ with $H|_{(S^{k-1}, *)} = \partial u$. We fix a homomorphism $\sigma = (\{i_\alpha\}, \{\rho_{\beta\alpha}\})$ which represents the subspace inclusion $(A, \underline{o}|_A) \subset (X, \underline{o})$, and cover $H(CS^{k-1})$ by finitely many $\{\mathcal{O}_s\}$ where each $\mathcal{O}_s = \mathcal{O}_{\{\xi_{ji,s}\}}$. Then $i \circ H(CS^{k-1})$ is covered by $\{\sigma(\mathcal{O}_s)\}$ where $\sigma(\mathcal{O}_s) = \mathcal{O}_{\{\rho_{\alpha(j,s)\alpha(i,s)}(\xi_{ji,s})\}}$. Now we construct a homomorphism $\tau_1 = \tau(\theta, \{O_a\}, \{\sigma(\mathcal{O}_s)\})$ which represents the element $\Phi_{i \circ H} \in [(SCS^{k-1}(G), *); (X, \underline{o})]_\rho$ associated to $i \circ H$ in Lemma 2.2.1 (1), and construct a homomorphism $\tau_2 = \tau(\theta', \{O_{a'}\}, \{\mathcal{O}_{\alpha'}\})$ which represents the element $\Phi_u \in [(CS^k(G), S^k(G), *); (X, A, \underline{o})]_\rho$ associated to u in Lemma 2.2.1 (2). By identifying $(SCS^{k-1}, *)$ with $(CSS^{k-1}, *) = (CS^k, *)$, we regard τ_1 as a homomorphism which defines a map from $(CS^k(G), *)$ to $(X, \underline{o})$. Since the restriction of τ_1 to the subspace $(S^k(G), *)$ represents $\Phi_{i \circ \partial u} = i \circ (\Phi_u)|_{(S^k, *)}$, we may arrange to join τ_2 with τ_1 along $(S^k(G), *)$ to define a homomorphism τ where $[\tau] \in [(SS^k(G), *); (X, \underline{o})]_\rho$, where we identify $(SS^k(G), *)$ with $(CS^k(G), *) \bigcup_{(S^k(G), *)} (CS^k(G), *)$. We then apply Lemma 2.2.1 (1) to $[\tau]$ to obtain a continuous map $u_{[\tau]} : (S^k, *) \to ([(S^1(G), *); (X, \underline{o})]_\rho, \tilde{o})$. The specific form of τ gives rise to a natural homotopy between $j \circ u_{[\tau]}$ and u. Hence $[u] = j_*([u_{[\tau]}])$ and (2.3.2) is exact at $\pi_{k+1}^{(G,\rho)}(X, A, \underline{o})$.

The proof of Theorem 2.3.1 is thus completed.

□

2.4 The theory of coverings

In this subsection we give a detailed, elementary presentation of the theory of coverings of orbispaces. We also derive the following useful corollary which was involved in the proof of the Arzela-Ascoli precompactness theorem for C^r maps between smooth orbifolds in §3.4 of [3]. Its proof is given at the end of this subsection.

Lemma 2.4.1 *Let $\Phi : X \to X'$ be a map*[3] *between global quotients, where $X = Y/G$, $X' = Y'/G'$, such that Y is connected and locally path-connected. Then Φ is defined by a pair $(f, \lambda) : (Y, G) \to (Y', G')$ where f is λ-equivariant if and only if there are base-point structures $\underline{o}$, $\underline{o'}$ such that under $\Phi_* : \pi_1(X, \underline{o}) \to \pi_1(X', \underline{o'})$, $\pi_1(Y, \hat{o})$ is mapped into*

[3] here Φ is allowed to be a general map, not restricted by **Convention** in Introduction.

$\pi_1(Y', \hat{o}')$, where $\pi_1(Y, \hat{o})$, $\pi_1(Y', \hat{o}')$ are regarded as subgroups of $\pi_1(X, \underline{o})$, $\pi_1(X', \underline{o'})$ via the exact sequence (2.2.10) *with $H = \{1\}$.*

The following definition generalizes the notion 'orbifold covering' in Thurston [15].

Definition 2.4.2 *Let $\Pi : Y \to X$ be a map of orbispaces and $\pi : Y \to X$ be the induced map between underlying spaces. We call Π a covering map, if there exists a homomorphism $(\{\pi_\alpha\}, \{\rho_{\beta\alpha}\}) : \Gamma\{V_\alpha\} \to \Gamma\{U_{\alpha'}\}$, $\alpha \in \Lambda$, whose equivalence class is the map Π, such that*

(a) *each $\pi_\alpha : \widehat{V_\alpha} \to \widehat{U_\alpha}$ is a ρ_α-equivariant homeomorphism,*
(b) *for any $U \in \{U_\alpha\}$, $\{V_\alpha | \alpha \in \Lambda(U)\}$ is the set of connected components of $\pi^{-1}(U)$, where $\Lambda(U)$ is the subset of Λ defined by $\Lambda(U) = \{\alpha \in \Lambda \mid U = U_\alpha\}$, and*
(c) *the map $\pi : Y \to X$ between underlying spaces is surjective.*

The orbispace Y together with the covering map $\Pi : Y \to X$ is called a covering space of X. Each element $U \in \{U_\alpha\}$, which all together form a cover of X by (c)*, is called an elementary neighborhood of X with respect to the covering map Π.*

Remark 2.4.3

(1) By passing to an induced homomorphism of $(\{\pi_\alpha\}, \{\rho_{\beta\alpha}\})$ if necessary, we may assume that a connected open subset of an elementary neighborhood is an elementary neighborhood.

(2) By the general assumption made in **Convention** in Introduction, each $\rho_\alpha : G_{V_\alpha} \to G_{U_\alpha}$ is injective, and each $\rho_{\beta\alpha} : T(V_\alpha, V_\beta) \to T(U_\alpha, U_\beta)$ is partially injective in the sense that if $\rho_{\beta\alpha}(\xi_1) = \rho_{\beta\alpha}(\xi_2)$ and Domain $(\phi_{\xi_1}) \cap$ Domain $(\phi_{\xi_2}) \neq \emptyset$, then $\xi_1 = \xi_2$. But in the case of a covering map, one can easily verify that $\rho_{\beta\alpha}$ is in fact injective.

(3) The composition of two covering maps is a covering map.

(4) Let $\underline{q} = (q, V_o, \hat{q})$, $\underline{p} = (p, U_o, \hat{p})$ be base-point structures of Y and X respectively. Given any covering map $\Pi : Y \to X$, we may always assume that $V_o \in \{V_\alpha\}$, $U_o \in \{U_{\alpha'}\}$, and $\pi_o : \widehat{V_o} \to \widehat{U_o}$ sends $\hat{q}$ to $\hat{p}$.

First of all, we investigate the path-lifting property of a covering map. To simplify the notation, we write $P(X, \underline{o_1}, \underline{o_2})$ for the path space $[(I, 0, 1); (X, \underline{o_1}, \underline{o_2})]$. Let $\Pi : (Y, \underline{q}) \to (X, \underline{p})$ be a covering map, represented by $(\{\pi_\alpha\}, \{\rho_{\beta\alpha}\})$ as described in Definition 2.4.2. Given any $\underline{p'} = (p', U, \hat{p}')$ where $U \in \{U_{\alpha'}\}$, and any connected component V_α of $\pi^{-1}(U)$, $(\{\pi_\alpha\}, \{\rho_{\beta\alpha}\})$ canonically determines a $\underline{q'} = (q', V_\alpha, \hat{q}')$ by setting $\hat{q}' = \pi_\alpha^{-1}(\hat{p}')$ and $q' = \pi_{V_\alpha}(\hat{q}') \in Y$. For any $g \in G_U$ and $\underline{p'} = (p', U, \hat{p}')$, we denote the base-point structure $(p', U, g \cdot \hat{p}')$ by $g \cdot \underline{p'}$. Note that there is a natural mapping from $P(X, \underline{p}, \underline{p'})$ to $P(X, \underline{p}, g \cdot \underline{p'})$, denoted by $u \mapsto g \cdot u$, which is defined by sending a representative $(\{f_i\}, \{\xi_{ji}\})$, $i = 0, 1, \cdots, n$, of u to $(\{f_i'\}, \{\xi_{ji}'\})$, $i = 0, 1, \cdots, n$, where $f_i' = f_i$ for $i \leq n-1$, $\xi'_{(i+1)i} = \xi_{(i+1)i}$ for $i \leq n-2$, and $f_n' = g \circ f_n$, $\xi'_{n(n-1)} = g \circ \xi_{n(n-1)}$.

Lemma 2.4.4 *Let $\Pi : (Y, \underline{q}) \to (X, \underline{p})$ be a covering map defined by $(\{\pi_\alpha\}, \{\rho_{\beta\alpha}\})$, and $\underline{p'} = (p', U, \hat{p}')$ where $U \in \{U_{\alpha'}\}$. For any path $u \in P(X, \underline{p}, \underline{p'})$, there is a unique component $V_{\alpha(u)}$ of $\pi^{-1}(U)$ and a unique left coset $\delta(u) \in G_U / \rho_{\alpha(u)}(G_{V_{\alpha(u)}})$, such that after fixing a representative $g \in G_U$ of $\delta(u)$, there is a unique path $\ell(u)_g \in P(Y, \underline{q}, \underline{q'}(g))$ where $\underline{q'}(g)$ is* *the base-point structure canonically associated to $g \cdot \underline{p'}$ and $V_{\alpha(u)}$ by $(\{\pi_\alpha\}, \{\rho_{\beta\alpha}\})$, such that $(\{\pi_\alpha\}, \{\rho_{\beta\alpha}\}) \circ \ell(u)_g = g \cdot u$. For any continuous family $u_s \in P(X, \underline{p}, \underline{p'})$, the component $V_{\alpha(u_s)}$ and the coset $\delta(u_s)$ are locally constant with respect to s. When the parameter space of s is connected, one can choose representatives of $\delta(u_s)$ independent of s, and in this case, the family of paths $\ell(u_s)_g$ is continuous in s, where g is any such a representative of $\delta(u_s)$.*

Proof Given any $u \in P(X, \underline{p}, \underline{p'})$, we pick a representative $\sigma = (\{f_i\}, \{\xi_{ji}\}) : \Gamma\{I_i\} \rightarrow \Gamma\{U_{i'}\}$, $i = 0, 1, \cdots, n$, where each $U_i \in \{U_{i'}\}$ is an elementary neighborhood of X with respect to the covering map Π. We shall construct a $\ell(\sigma) = (\{\ell(f_i)\}, \{\ell(\xi_{ji})\}) : \Gamma\{I_i\} \rightarrow \Gamma\{V_{i'}\}$, such that (1) each $V_i \in \{V_{i'}\}$ is a connected component of $\pi^{-1}(U_i)$, and (2) there exists $\{g_i \in G_{U_i} \mid g_0 = 1, i = 0, 1, \cdots, n\}$ satisfying

$$\pi_i \circ \ell(f_i) = g_i \circ f_i, \; \rho_{ji}(\ell(\xi_{ji})) = g_j \circ \xi_{ji} \circ g_i^{-1}, \tag{2.4.1}$$

where $\pi_i : \widehat{V_i} \rightarrow \widehat{U_i}$ and $\rho_{ji} : T(V_i, V_j) \rightarrow T(U_i, U_j)$ are given in $(\{\pi_\alpha\}, \{\rho_{\beta\alpha}\})$. $\ell(\sigma)$ is defined inductively as follows. For $i = 0$, we simply define $\ell(f_0) = \pi_o^{-1} \circ f_0 : I_0 \rightarrow \widehat{V_o}$, which clearly satisfies $\ell(f_0)(0) = \hat{q}$. Now suppose that $\ell(\xi_{k(k-1)})$ and $\ell(f_k)$ are defined for all $0 \leq k \leq i$. We shall define $\ell(\xi_{(i+1)i})$ and $\ell(f_{i+1})$ as follows. First of all, it follows from $\pi_i \circ \ell(f_i) = g_i \circ f_i$ that $\pi(\pi_{V_i}(\ell(f_i)(I_i \cap I_{i+1}))) \subset U_{i+1}$, hence by (b) of Definition 2.4.2, there is a unique connected component V_{i+1} of $\pi^{-1}(U_{i+1})$ which is determined by the condition $\pi_{V_i}(\ell(f_i)(I_i \cap I_{i+1})) \cap V_{i+1} \neq \emptyset$. Secondly, let V be the connected component of $V_i \cap V_{i+1}$ that contains $\pi_{V_i}(\ell(f_i)(I_i \cap I_{i+1}))$, and let W be the connected component of $U_i \cap U_{i+1}$ that contains $\pi(\pi_{V_i}(\ell(f_i)(I_i \cap I_{i+1}))) = \pi_{U_i}(f_i(I_i \cap I_{i+1}))$. Then $\rho_{(i+1)i}$ maps $T_V(V_i, V_{i+1})$ injectively into $T_W(U_i, U_{i+1})$, and $\xi_{(i+1)i}$ lies in $T_W(U_i, U_{i+1})$. We pick a $\eta_{(i+1)i} \in T_V(V_i, V_{i+1})$ such that $\ell(f_i)(I_i \cap I_{i+1}) \subset \text{Domain } (\phi_{\eta_{(i+1)i}})$. Then since $\pi_i \circ \ell(f_i) = g_i \circ f_i$, $\rho_{(i+1)i}(\eta_{(i+1)i})$ has the same domain with $\xi_{(i+1)i} \circ g_i^{-1}$, and therefore there exists a $g_{i+1} \in G_{U_{i+1}}$ such that $\rho_{(i+1)i}(\eta_{(i+1)i}) = g_{i+1} \circ \xi_{(i+1)i} \circ g_i^{-1}$. Finally, we define $\ell(f_{i+1}) = \pi_{i+1}^{-1} \circ (g_{i+1} \circ f_{i+1})$, $\ell(\xi_{(i+1)i}) = \eta_{(i+1)i}$. Then it is clear that $\ell(\xi_{(i+1)i}) \circ \ell(f_i) = \ell(f_{i+1})$ on $I_i \cap I_{i+1}$, and $\pi_{i+1} \circ \ell(f_{i+1}) = g_{i+1} \circ f_{i+1}$, $\rho_{(i+1)i}(\ell(\xi_{(i+1)i})) = g_{i+1} \circ \xi_{(i+1)i} \circ g_i^{-1}$. By induction, $\ell(\sigma)$ is defined with the claimed properties.

We observe that in each step of the above construction, V_i is uniquely determined, and if $\ell'(\sigma) = (\{\ell'(f_i)\}, \{\ell'(\xi_{ji})\})$ and $\{g'_i\}$ is another choice, then there exists a set $\{h_i \mid h_i \in G_{V_i}\}$ such that $g'_i = \rho_i(h_i) g_i$, and $\ell'(f_i) = h_i \circ \ell(f_i)$, $\ell'(\xi_{ji}) = h_j \circ \ell(\xi_{ji}) \circ h_i^{-1}$. In other words, $\ell'(\sigma)$ is conjugate to $\ell(\sigma)$.

Let $\tau = (\{f_k\}, \{\xi_{lk}\}) : \Gamma\{J_k\} \rightarrow \Gamma\{U_{k'}\}$, $k = 0, 1, \cdots, m$, be any induced homomorphism of σ, defined via $(\theta, \{\xi_k\}, \{\jmath_k\})$ where $\theta : \{0, 1, \cdots, m\} \rightarrow \{0, 1, \cdots, n\}$ satisfies $\theta(0) = 0$, $\theta(m) = n$, $\xi_k : J_k \rightarrow I_{\theta(k)}$ is the inclusion for all k, and $\jmath_k \in T(U_k, U_{\theta(k)})$ satisfies $\jmath_0 = 1$, $\jmath_m = 1$, such that

$$f_k = \jmath_k^{-1} \circ f_{\theta(k)}|_{J_k}, \; \xi_{lk} = \jmath_l^{-1} \circ \xi_{\theta(l)\theta(k)} \circ \jmath_k. \tag{2.4.2}$$

For any choice of $(\ell(\sigma), \{g_i\})$, $(\ell(\tau), \{g_k\})$ with (2.4.1) satisfied, if we set $\bar{\jmath}_k = g_{\theta(k)} \circ \jmath_k \circ g_k^{-1} \in T(U_k, U_{\theta(k)})$, then it is easy to check that the following hold:

$$\pi_k \circ \ell(f_k)) = \bar{\jmath}_k^{-1} \circ (\pi_{\theta(k)} \circ \ell(f_{\theta(k)})), \; \rho_{lk}(\ell(\xi_{lk})) = \bar{\jmath}_l^{-1} \circ \rho_{\theta(l)\theta(k)}(\ell(\xi_{\theta(l)\theta(k)})) \circ \bar{\jmath}_k \tag{2.4.3}$$

On the other hand, observe that $\bar{\jmath}_0 = 1$ and $V_0 = V_{\theta(0)} = V_o$, so that (2.4.3) implies inductively that $V_k \cap V_{\theta(k)} \neq \emptyset$, and $\bar{\jmath}_k = \rho_{\theta(k)k}(\imath_k)$ for a unique $\imath_k \in T(V_k, V_{\theta(k)})$ such that

$$\ell(f_k) = \imath_k^{-1} \circ \ell(f_{\theta(k)}),\ \ell(\xi_{lk}) = \imath_l^{-1} \circ \ell(\xi_{\theta(l)\theta(k)}) \circ \imath_k. \tag{2.4.4}$$

With the preceding understood, we now determine $V_{\alpha(u)}$, $\delta(u)$, and define $\ell(u)_g$ after having chosen a representative g of $\delta(u)$. First of all, observe that $V_m, V_{\theta(m)} = V_n$ are connected components of $\pi^{-1}(U)$, so that $V_m \cap V_{\theta(m)} \neq \emptyset$ implies $V_m = V_{\theta(m)}$. We define $V_{\alpha(u)} = V_m$. Secondly, $\imath_m \in T(V_m, V_{\theta(m)}) = G_{V_m}$ and $\rho_m(\imath_m) = \bar{\jmath}_m = g_{\theta(m)} \circ \jmath_m \circ g_m^{-1} = g_{\theta(m)} g_m^{-1}$ implies that the left coset of g_m in $G_U/\rho_{\alpha(u)}(G_{V_{\alpha(u)}})$ depends only on u. We define $\delta(u)$ to be the coset of g_m in $G_U/\rho_{\alpha(u)}(G_{V_{\alpha(u)}})$. Finally, if we fix a representative g of $\delta(u)$ for the g_m, g_n in the above consideration, then $\bar{\jmath}_m = 1$ and $\imath_m = 1$, so that $\ell(\sigma)$, $\ell(\tau)$ define the same element in $P(Y, \underline{q}, \underline{q}'(g))$, which is defined to be $\ell(u)_g$. By the nature of construction, $\Pi \circ \ell(u)_g = g \cdot u$, where Π is regarded as a map from $(Y, \underline{q}, \underline{q}'(g))$ to $(X, \underline{p}, g \cdot \underline{p}')$.

As for the uniqueness of $\ell(u)_g$, suppose $u_1, u_2 \in P(Y, \underline{q}, \underline{q}'(g))$ are two paths such that $\Pi \circ u_1 = \Pi \circ u_2$. Then in particular, the paths in the underlying space of X coincide, and hence there exist $\tau_1 = (\{f_{i,1}\}, \{\eta_{ji,1}\}) : \Gamma\{I_i\} \rightarrow \Gamma\{V_{i(1)}\}$, $\tau_2 = (\{f_{i,2}\}, \{\eta_{ji,2}\}) : \Gamma\{I_i\} \rightarrow \Gamma\{V_{i(2)}\}$ representing u_1, u_2 respectively, where for each i, $V_{i(1)}, V_{i(2)}$ are connected components of $\pi^{-1}(U_i)$ for some elementary neighborhood U_i. The compositions of τ_1, τ_2 with $(\{\pi_\alpha\}, \{\rho_{\beta\alpha}\})$ are equivalent, because they represent the same $\Pi \circ u_1 = \Pi \circ u_2$. Hence by Lemma 3.1.2 in [3], they must be conjugate, and furthermore, since they preserve the base-point structures, they are actually equal, which means

$$\pi_{i(1)} \circ f_{i,1} = \pi_{i(2)} \circ f_{i,2},\ \rho_{j(1)i(1)}(\eta_{ji,1}) = \rho_{j(2)i(2)}(\eta_{ji,2}). \tag{2.4.5}$$

We shall derive from (2.4.5) that $V_{i(1)} = V_{i(2)}$ for all i by induction. For $i = 0$, $V_{i(1)} = V_{i(2)} = V_o$, so that by (2.4.5), we obtain $f_{0,1} = f_{0,2}$ and $\eta_{10,1}$ and $\eta_{10,2}$ have the same domain. The latter then implies that $V_{i(1)} \cap V_{i(2)} \neq \emptyset$ for $i = 1$, or equivalently $V_{i(1)} = V_{i(2)}$ for $i = 1$. By induction $V_{i(1)} = V_{i(2)}$ for all i, with which (2.4.5) implies $\tau_1 = \tau_2$. Hence $u_1 = u_2$ and $\ell(u)_g$ is unique.

Finally, let u_s be any continuous family of paths. For any s_0, when s is sufficiently close to s_0, u_s is represented by a $\sigma_s = \{f_i(\cdot, s)\} \in \mathcal{O}_{\{\xi_{ji}\}}$, where each $f_i(t, s)$ is continuous in both variables. Thus when s is sufficiently close to s_0, $V_{\alpha(u_s)} = V_{\alpha(u_{s_0})}$, and $\delta(u_s) = \delta(u_{s_0})$. In other words, $V_{\alpha(u_s)}$ and $\delta(u_s)$ are locally constant in s. When the parameter space of s is connected, $V_{\alpha(u_s)}$ and $\delta(u_s)$ are constant so that we can choose representatives of $\delta(u_s)$ independent of s. The family of liftings $\ell(u_s)_g$ is continuous in s for any choice of such a representative g, because it can be locally represented by $\ell(\sigma_s)$ which is continuous in s.

□

As in the classical covering theory, we can deduce the following immediate corollary.

Corollary 2.4.5 *Let $\Pi : (Y, \underline{q}) \rightarrow (X, \underline{p})$ be a covering map. Then the induced homomorphism $\Pi_* : \pi_k(Y, \underline{q}) \rightarrow \pi_k(X, \underline{p})$ is isomorphic when $k \geq 2$ and injective when $k = 1$.*

We introduce the following definitions.

(1) An orbispace is locally path-connected if the associated étale topological groupoid is locally path-connected, or equivalently, each local chart $\widehat{U_i}$ is locally path-connected.

(2) An orbispace X is semi-locally 1-connected if for any point p, there is a local chart U_i containing p, such that the composition of homomorphisms $\pi_1(\widehat{U_i}, \hat{p}) \to \pi_1(U_i, \underline{p}) \to \pi_1(X, \underline{p})$ has trivial image for any base-point structure $\underline{p} = (p, U_i, \hat{p})$.
(3) A connected, locally path-connected covering space $\Pi : Y \to X$ is universal if $\pi_1(Y)$ is trivial.

The last definition is justified by the following lemma.

Lemma 2.4.6 *Let $\Pi : (Y, \underline{q}) \to (X, \underline{p})$ be a covering map. For any map*[4] *$\Phi : (Z, \underline{z}) \to (X, \underline{p})$ where Z is connected and locally path-connected, there exists a unique map $\ell(\Phi) : (Z, \underline{z}) \to (Y, \underline{q})$ such that $\Pi \circ \ell(\Phi) = \Phi$ if and only if $\Phi_*(\pi_1(Z, \underline{z})) \subset \Pi_*(\pi_1(Y, \underline{q}))$ in $\pi_1(X, \underline{p})$.*

Proof The 'only if' part is trivial as usual. We shall prove the 'if' part next.

We introduce some notations first. Let $(\{\pi_\alpha\}, \{\rho_{\beta\alpha}\})$ be a representative of Π as in Definition 2.4.2, which will be fixed throughout the proof. Let $\underline{q} = (q, V_o, \hat{q})$, $\underline{p} = (p, U_o, \hat{p})$ and $\underline{z} = (z, W_o, \hat{z})$ be the base-point structures.

We pick a representative of Φ, denoted by $\sigma = (\{\phi_i\}, \{\eta_{ji}\}) : \Gamma\{W_i\} \to \Gamma\{U_{i'}\}$, where we may assume that each $U_i \in \{U_{i'}\}$ is an elementary neighborhood. The strategy of the proof is to find a set $\{\delta_i \mid \delta_i \in G_{U_i}, \delta_o = 1 \in G_{U_o}\}$, such that $\delta(\sigma) = (\{\delta(\phi_i)\}, \{\delta(\eta_{ji})\})$, where $\delta(\phi_i) = \delta_i \circ \phi_i$, $\delta(\eta_{ji}) = \delta_j \circ \eta_{ji} \circ \delta_i^{-1}$, can be lifted to $(Y, \underline{q})$ through $(\{\pi_\alpha\}, \{\rho_{\beta\alpha}\})$ directly.

Such a set $\{\delta_i\}$ may be obtained as follows. For each i, we pick a point $z_i \in W_i$ and a $\hat{z}_i \in \widehat{W_i}$ such that $\pi_{W_i}(\hat{z}_i) = z_i$, with $z_o = z$ and $\hat{z}_o = \hat{z}$ where $z, \hat{z}$ are given in $\underline{z} = (z, W_o, \hat{z})$. This gives rise to a set of base-point structures $\underline{z_i} = (z_i, W_i, \hat{z}_i)$ of Z, and a corresponding set of base-point structures of X: $\underline{p_i} = (p_i, U_i, \hat{p_i})$ where each $p_i = \phi(z_i)$ and $\hat{p}_i = \phi_i(\hat{z}_i)$, with $\underline{z_o} = \underline{z}$, $\underline{p_o} = \underline{p}$. (Here $\phi : Z \to X$ is the induced map of Φ between underlying spaces.) Since Z is connected and locally path-connected, it follows that for each W_i, there exists a path $u_i \in P(Z, \underline{z}, \underline{z_i})$, and therefore a push-forward path $u_i' = \sigma \circ u_i \in P(X, \underline{p}, \underline{p_i})$. By Lemma 2.4.4, there exist a connected component $V_{\alpha(u_i')}$ of $\pi^{-1}(U_i)$ and a coset $\delta(u_i')$ such that after choosing a representative δ_i of $\delta(u_i')$, the path $\delta_i \cdot u_i'$ in $P(X, \underline{p}, \delta_i \cdot \underline{p_i})$ can be lifted to a path in Y through $(\{\pi_\alpha\}, \{\rho_{\beta\alpha}\})$. The upshot is that the component $V_{\alpha(u_i')}$ and the coset $\delta(u_i')$ are independent of the choice on the path u_i, because of the assumption that $\Phi_*(\pi_1(Z, \underline{z})) \subset \Pi_*(\pi_1(Y, \underline{q}))$ in $\pi_1(X, \underline{p})$. Now we set $V_i = V_{\alpha(u_i')}$ and choose a representative $\delta_i \in G_{U_i}$ of the coset $\delta(u_i')$ for each i. We define $\delta(\sigma)$ using the set $\{\delta_i\}$ thus obtained. Note that $\delta(\sigma)$ has the following property: for any path $u_i \in P(Z, \underline{z}, \underline{z_i})$, the push-forward path $\delta(\sigma) \circ u_i \in P(X, \underline{p}, \delta_i \cdot \underline{p_i})$ can be directly lifted to a path in $(Y, \underline{q}, \underline{q_i}(\delta_i))$ by $(\{\pi_\alpha\}, \{\rho_{\beta\alpha}\})$, where $\underline{q_i}(\delta_i) = (q_i, V_i, \hat{q}_i)$ is the base-point structure defined by $\hat{q}_i = \pi_i^{-1}(\delta_i \cdot \hat{p}_i)$ and $q_i = \pi_{V_i}(\hat{q}_i)$ (here $\pi_i : \widehat{V_i} \to \widehat{U_i}$ is the homeomorphism given in $(\{\pi_\alpha\}, \{\rho_{\beta\alpha}\})$).

We prove next that $\delta(\eta_{ji}) : T(W_i, W_j) \to T(U_i, U_j)$ has its image contained in $\rho_{ji}(T(V_i, V_j))$, in particular, each $\delta(\eta_i) = Ad(\delta_i) \circ \eta_i : G_{W_i} \to G_{U_i}$ has its image contained in $\rho_i(G_{V_i})$, so that we can define $\ell(\sigma) = (\{\ell(\phi_i)\}, \{\ell(\eta_{ji})\})$ by setting $\ell(\phi_i) = \pi_i^{-1} \circ \delta(\phi_i)$ and $\ell(\eta_{ji}) = \rho_{ji}^{-1} \circ \delta(\eta_{ji})$. The desired map $\ell(\Phi) : (Z, \underline{z}) \to (Y, \underline{q})$ is the equivalence class of $\ell(\sigma)$.

Given any $\xi \in T(W_i, W_j)$, we pick a point $\hat{z}' \in$ Domain (ϕ_ξ). Since Z is connected and locally path-connected, there is a path $u_i \in P(Z, \underline{z}, \underline{z_i})$, a path γ in $\widehat{W_i}$ joining $\hat{z}_i$ to $\hat{z}'$, and

[4] here both $\Phi, \ell(\Phi)$ are general maps, i.e., not restricted by **Convention** in Introduction.

a path γ' in $\widehat{W_j}$ joining $\phi_\xi(\hat{z}')$ to $\hat{z}_j$. Let $u_j = u_i \# \gamma \#_\xi \gamma'$ be the path in $P(Z, \underline{z}, \underline{z_j})$ which is defined by composing u_i with γ and then with γ' through ξ. The push-forward path $\delta(\sigma) \circ u_j \in P(X, \underline{p}, \delta_j \cdot \underline{p_j})$ can be lifted directly to $(Y, \underline{q}, \underline{q}_j(\delta_j))$ through $(\{\pi_\alpha\}, \{\rho_{\beta\alpha}\})$, which implies that $\delta(\eta_{ji})(\xi)$ lies in the image of ρ_{ji}.

Finally, we address the uniqueness of $\ell(\Phi)$. Let $\Psi_1, \Psi_2 : (Z, \underline{z}) \to (Y, \underline{q})$ be any maps such that $\Pi \circ \Psi_1 = \Pi \circ \Psi_2 = \Phi$. Since in particular the induced maps between the underlying spaces coincide, we may represent Ψ_1, Ψ_2 by $\tau_1 = (\{\psi_i^{(1)}\}, \{\theta_{ji}^{(1)}\}) : \Gamma\{W_i\} \to \Gamma\{V_{i(1)}\}$ and $\tau_2 = (\{\psi_i^{(2)}\}, \{\theta_{ji}^{(2)}\}) : \Gamma\{W_i\} \to \Gamma\{V_{i(2)}\}$ respectively, where $V_{i(1)}, V_{i(2)}$ are connected components of $\pi^{-1}(U_i)$ for some elementary neighborhood U_i. The compositions of τ_1, τ_2 with $(\{\pi_\alpha\}, \{\rho_{\beta\alpha}\})$ are equivalent, because they represent the same map Φ. Hence by Lemma 3.1.2 in [3], they are conjugate, and furthermore, because they also preserve the base-point structures and Z is connected, they are actually equal. If $V_{i(1)} = V_{i(2)}$ for all i, then it is easily seen that $\tau_1 = \tau_2$ so that $\Psi_1 = \Psi_2$. Suppose $V_{i(1)} \neq V_{i(2)}$ for some index i. We pick a $\underline{z_i} = (z_i, W_i, \hat{z}_i)$ and a path $u \in P(Z, \underline{z}, \underline{z_i})$. Then it is easily seen that $\Psi_1 \circ u$ and $\Psi_2 \circ u$ are two different liftings of $\Phi \circ u$, contradicting the uniqueness of path-lifting in Lemma 2.4.3. Hence $V_{i(1)} = V_{i(2)}$ for all i.

□

Next we discuss the existence of universal covering and deck transformations.

Proposition 2.4.7 *Let X be a connected, locally path-connected and semi-locally 1-connected orbispace. Fix a base-point structure $\underline{p} = (p, U_o, \hat{p})$ of X. Then for any subgroup H of $\pi_1(X, \underline{p})$, there is a connected covering space $\Pi : (Y, \underline{q}) \to (X, \underline{p})$, which is unique up to isomorphisms, such that $\Phi_*(\pi_1(Y, \underline{q})) = H$.*

Proof We first construct the underlying space of Y. To this end, we consider the path space $P(X, \underline{p}) = [(I, 0); (X, \underline{p})]$, which is given a natural topology by the general method described in §3.2 of [3]. We introduce an equivalence relation $\sim_H$ in $P(X, \underline{p})$ as follows. Given any $u_1, u_2 \in P(X, \underline{p})$, we define $u_1 \sim_H u_2$, if there are σ_1, σ_2 representing u_1, u_2 respectively, and there is a $\xi \in \sqcup_{i,j} T(U_i, U_j)$, where $\{U_i\}$ is the atlas of local charts on X, such that σ_1 may compose with $\nu(\sigma_2)$ via ξ to form a homomorphism $\sigma_1 \#_\xi \nu(\sigma_2)$, which defines a loop in $(X, \underline{p})$ whose homotopy class lies in H. We define the underlying space of Y to be $P(X, \underline{p})/\sim_H$, which is obviously path-connected, hence connected. As a notational convention, for any $u \in P(X, \underline{p})$, we denote its equivalence class under $\sim_H$ by $u_H \in Y$. There is a natural surjective continuous map $\pi : Y \to X$ defined by $u_H \mapsto u(1)$, where $u(1)$ is the terminal point of the path u in X. (Here the surjectivity of π relies on the fact that X is path-connected.) The space Y has a natural base point $q = \tilde{p}_H$, where $\tilde{p} \in P(X, \underline{p})$ is the constant path, defined by the constant map into $\hat{p} \in \widehat{U_o}$. Clearly $\pi(q) = p$.

For any point $y = u_H \in Y$, where u is represented by $\sigma = (\{\gamma_k\}, \{\xi_{lk}\}) : \Gamma\{I_k\} \to \Gamma\{U_{k'}\}$, $k = 0, 1, \cdots, n$, we set $U_\sigma = U_n$ and $\hat{\sigma} = \gamma_n(1) \in \widehat{U_\sigma}$. We may assume that the semi-locally 1-connectedness holds for U_σ without loss of generality. We define a map $\pi_\sigma : \widehat{U_\sigma} \to Y$ as follows. For each $z \in \widehat{U_\sigma}$, we connect $\hat{\sigma}$ to z by a path γ_z in $\widehat{U_\sigma}$ (the existence of γ_z is ensured by the locally path-connectedness of X), and define $\pi_\sigma(z) = [\sigma \# \gamma_z]_H$. The assumption that X is semi-locally 1-connected ensures that the map π_σ is well-defined, and the assumption that X is locally path-connected implies that π_σ is continuous. We set $V_\sigma = \pi_\sigma(\widehat{U_\sigma}) \subset Y$, which obviously satisfies $\pi(V_\sigma) = U_\sigma$.

We will show: (1) V_σ is a connected open neighborhood of y in Y. (2) There is a subgroup G_{V_σ} of G_{U_σ} such that $\widehat{U_\sigma}/G_{V_\sigma}$ is homeomorphic to V_σ under π_σ. (3) There is a set $\mathcal{T} = \{T(V_\sigma, V_\tau)\}$, where each $T(V_\sigma, V_\tau)$ is a subset of $T(U_\sigma, U_\tau)$. Together with the atlas of local charts $\{(\widehat{V_\sigma}, G_{V_\sigma}, \pi_\sigma)\}$, where $\widehat{V_\sigma} = \widehat{U_\sigma}$, it defines an orbispace structure on Y (cf. Proposition 2.1.1 in [3]). (4) The maps $\pi^\sigma : \widehat{V_\sigma} = \widehat{U_\sigma}$ and $\rho_{\tau\sigma} : T(V_\sigma, V_\tau) \subset T(U_\sigma, U_\tau)$ define a covering map $\Pi : Y \to X$. (5) We have $\Pi_*(\pi_1(Y, \underline{q})) = H$ where in $\underline{q} = (q, V_o, \hat{q})$, $V_o = \pi_{\sigma_q}(\widehat{U_o})$, and $\hat{q} = \hat{p} \in \widehat{U_o} = \widehat{V_o}$. Here σ_q is the canonical representative of q, the constant map into $\hat{p}$.

For (1), we first prove that V_σ is open. It suffices to show that the inverse image of V_σ under the projection $P(X, \underline{p}) \to Y$ is open in $P(X, \underline{p})$. To this end, we need to show that given any $u_0 \in P(X, \underline{p})$ with $(u_0)_H = \pi_\sigma(z_0)$ for some $z_0 \in \widehat{U_\sigma}$, and for any $u \in P(X, \underline{p})$ sufficiently close to u_0, there is a $z \in \widehat{U_\sigma}$ such that $\pi_\sigma(z) = u_H$. Let γ_{z_0} be a path in $\widehat{U_\sigma}$ connecting $\hat{\sigma}$ to z_0, such that $\pi_\sigma(z_0) = [\sigma \# \gamma_{z_0}]_H$. We take a $\tau_0 = \{f_{i,0}\} \in \mathcal{O}_{\{\xi_{ji}\}}$, $0 \le i \le m$, such that $[\tau_0] = u_0$, where without loss of generality, we assume that $U_m = U_\sigma$, $f_{m,0}(1) = z_0$, and the homotopy class of the loop $[\tau_0 \# \nu(\sigma \# \gamma_{z_0})]$ lies in H. Now we pick a $t_i \in I_i \cap I_{i+1}$ for each $i = 0, \cdots, m-1$, and for each $\tau = \{f_i\}$ in the open neighborhood $\mathcal{O}_{\{\xi_{ji}\}}$ of τ_0, we take a path γ_i in Domain $(\phi_{\xi_{(i+1)i}})$ running from $f_i(t_i)$ to $f_{i,0}(t_i)$ for each $i = 0, \cdots, m-1$, and take a path γ in $\widehat{U_\sigma}$ running from $f_m(1)$ to $f_{m,0}(1) = z_0$. We define $\tau' = \{f_i'\} \in \mathcal{O}_{\{\xi_{ji}\}}$ where for each $0 \le i \le m-1$, f_i' is obtained from pre-composing f_i by $\phi_{\xi_{i(i-1)}} \circ \nu(\gamma_{i-1})$ and post-composing f_i by γ_i, and f_m' is obtained from pre-composing f_m by $\phi_{\xi_{m(m-1)}} \circ \nu(\gamma_{m-1})$ and post-composing f_m by $\gamma \# \nu(\gamma)$. Then τ' is homotopic to τ so that $[\tau']_H = [\tau]_H$. On the other hand, by the semi-locally 1-connectedness of X, the loop $[\tau' \# \nu(\tau_0 \# \nu(\gamma))]$ is null-homotopic, so that $[\tau']_H = [\tau_0 \# \nu(\gamma)]_H$. Now observe that $\tau_0 \# \nu(\gamma) \# \nu(\sigma \# (\gamma_{z_0} \# \nu(\gamma)))$ is homotopic to $\tau_0 \# \nu(\sigma \# \gamma_{z_0})$, which defines a loop whose homotopy class lies in H. If we let $z = f_m(1)$, then we have $\pi_\sigma(z) = [\sigma \# (\gamma_{z_0} \# \nu(\gamma))]_H = [\tau_0 \# \nu(\gamma)]_H = [\tau']_H = [\tau]_H$. Hence V_σ is open in Y. Finally, V_σ is connected because $\widehat{U_\sigma}$ is and $V_\sigma = \pi_\sigma(\widehat{U_\sigma})$. This concludes the proof of (1).

For (2), we obtain the subgroup G_{V_σ} as follows. We denote by $\underline{\sigma}$ the base-point structure $(\pi(y), U_\sigma, \hat{\sigma})$, and consider the isomorphism $[\sigma]_* : \pi_1(X, \underline{\sigma}) \to \pi_1(X, \underline{p})$, where $[\sigma]$ is the path in $P(X, \underline{p}, \underline{\sigma})$ defined by σ, cf. Proposition 1.3 (1). Set $H_\sigma = [\sigma]_*^{-1}(H)$. Now for each $g \in G_{U_\sigma}$, we take a path γ_g in $\widehat{U}_\sigma$ connecting $\hat{\sigma}$ and $g \cdot \hat{\sigma}$. Then the pair (γ_g, g) determines an element $[(\gamma_g, g)]$ in $\pi_1(U_\sigma, \underline{\sigma})$, cf. Lemma 2.2.2, whose image $[g]$ under $\pi_1(U_\sigma, \underline{\sigma}) \to \pi_1(X, \underline{\sigma})$ is independent of the choice on the path γ_g by the semi-locally 1-connectedness of X. We simply put $G_{V_\sigma} = \{g \in G_{U_\sigma} \mid [g] \in H_\sigma\}$. The fact that $(\gamma_g, g) \# (\gamma_h, h) = (\gamma_{gh}, gh)$ for some path γ_{gh} connecting $\hat{\sigma}$ to $gh \cdot \hat{\sigma}$ in $\widehat{U}_\sigma$ shows that G_{V_σ} is a subgroup of G_{U_σ}. It remains to show that π_σ induces a homeomorphism between $\widehat{U_\sigma}/G_{V_\sigma}$ and V_σ. First of all, $\pi_\sigma : \widehat{U_\sigma} \to V_\sigma$ is G_{V_σ}-invariant. This can be seen as follows. Let g be any element in G_{V_σ}. Given any $z \in \widehat{U_\sigma}$, let γ_z be a path in $\widehat{U_\sigma}$ running from $\hat{\sigma}$ to z such that $\pi_\sigma(z) = [\sigma \# \gamma_z]_H$. Then $\pi_\sigma(g \cdot z) = [\sigma \# (\gamma_g \# g \circ \gamma_z)]_H$ because $\gamma_g \# g \circ \gamma_z$ is a path in $\widehat{U_\sigma}$ connecting $\hat{\sigma}$ and $g \cdot z$. Now it is easily seen that $\sigma \# (\gamma_g \# g \circ \gamma_z)$ may compose with $\nu(\sigma \# \gamma_z)$ through $g^{-1} \in G_{U_\sigma}$, and the resulting homomorphism represents a loop whose homotopy class is $[\sigma]_*([g]) \in H$. Hence $\pi_\sigma(g \cdot z) = \pi_\sigma(z)$. Secondly, suppose $\pi_\sigma(z) = \pi_\sigma(z')$. Take paths γ_z, $\gamma_{z'}$ connecting

$\hat{\sigma}$ to z and z' respectively. Then the assumption $\pi_\sigma(z) = \pi_\sigma(z')$ implies that there is some $g \in G_{U_\sigma}$ with $z' = g \cdot z$ such that $\sigma \# \gamma_z$ may compose with $\nu(\sigma \# \gamma_{z'})$ through g to form a homomorphism which defines a loop in X, whose homotopy class is an element $h \in H$. If we join $\hat{\sigma}$ and $g \cdot \hat{\sigma}$ by the path $\gamma_g = \gamma_{z'} \# \nu(g \circ \gamma_z)$, we see that the class $[(\gamma_g, g)]$ in $\pi_1(U_\sigma, \underline{\sigma})$ has its image $[g] = [\sigma]_*^{-1}(h^{-1}) \in H_\sigma$ under $\pi_1(U_\sigma, \underline{\sigma}) \to \pi_1(X, \underline{\sigma})$. Hence g lies in G_{V_σ}. From here it is easy to see that π_σ induces a homeomorphism between $\widehat{U_\sigma}/G_{V_\sigma}$ and V_σ.

For (3), suppose $(\widehat{V_{\sigma_1}}, G_{V_{\sigma_1}}, \pi_{V_{\sigma_1}})$, $(\widehat{V_{\sigma_2}}, G_{V_{\sigma_2}}, \pi_{V_{\sigma_2}})$ are two local charts constructed from σ_1, σ_2 respectively, such that $V_{\sigma_1} \cap V_{\sigma_2} \neq \emptyset$. The assumption $V_{\sigma_1} \cap V_{\sigma_2} \neq \emptyset$ implies that $U_{\sigma_1} \cap U_{\sigma_2} \neq \emptyset$ since $\pi(V_\sigma) = U_\sigma$. We shall define $T(V_{\sigma_1}, V_{\sigma_2})$ as a subset of $T(U_{\sigma_1}, U_{\sigma_2})$ as follows. Given any $\xi \in T(U_{\sigma_1}, U_{\sigma_2})$, if there exists a $z \in$ Domain (ϕ_ξ) such that the composition of $\sigma_1 \# \gamma_z$ with $\nu(\sigma_2 \# \gamma_{\phi_\xi(z)})$ through ξ defines a loop whose homotopy class lies in H, then we put ξ in $T(V_{\sigma_1}, V_{\sigma_2})$. Here γ_z, $\gamma_{\phi_\xi(z)}$ are paths in $\widehat{U_{\sigma_1}}$, $\widehat{U_{\sigma_2}}$ connecting $\hat{\sigma}_1$, $\hat{\sigma}_2$ to z, $\phi_\xi(z)$ respectively. Note that $T(V_\sigma, V_\sigma) = G_{V_\sigma}$ according to this definition. On the other hand, since Domain (ϕ_ξ) is path-connected, $\xi \in T(V_{\sigma_1}, V_{\sigma_2})$ implies that $\pi_{\sigma_1} = \pi_{\sigma_2} \circ \phi_\xi$ on Domain (ϕ_ξ) and that $\pi_{\sigma_1}(\text{Domain } (\phi_\xi))$ is a connected component of $V_{\sigma_1} \cap V_{\sigma_2}$. This allows us to assign ϕ_ξ to ξ even if ξ is regarded as an element of $T(V_{\sigma_1}, V_{\sigma_2})$. Finally, we observe that $\bigsqcup_{\sigma_1, \sigma_2} T(V_{\sigma_1}, V_{\sigma_2})$ is closed under taking inverse and composition as a subset of $\bigsqcup_{i,j} T(U_i, U_j)$. Hence by Proposition 2.1.1 in [3], it together with the atlas of local charts $\{(\widehat{V_\sigma}, G_{V_\sigma}, \pi_\sigma)\}$ defines an orbispace structure on Y. This concludes the proof of (3).

For (4), in order to show that $(\{\pi^\sigma\}, \{\rho_{\tau\sigma}\})$ defines a covering map $\Pi : Y \to X$, it suffices to verify (b) of Definition 2.4.2 for it. More concretely, we need to show that for any $U \in \{U_\sigma\}$, a connected component of $\pi^{-1}(U)$ is of the form V_σ. Let V be any connected component of $\pi^{-1}(U)$. Then for any $y \in V$, since $\pi(y) \in U$, there is a σ_y representing y, with a canonically constructed neighborhood V_{σ_y} satisfying $\widehat{V_{\sigma_y}} = \widehat{U}$. Since V_{σ_y} is connected and $\pi(V_{\sigma_y}) = U$, we have $V_{\sigma_y} \subset V$. Hence $V = \cup_{y \in V} V_{\sigma_y}$. On the other hand, it is easy to see that if V_σ and V_τ have non-empty intersection and $\pi(V_\sigma) = \pi(V_\tau)$, then $V_\sigma = V_\tau$. Hence $V = V_{\sigma_y}$ for any $y \in V$, and (b) is verified.

For (5), we first show that $H \subset \Pi_*(\pi_1(Y, \underline{q}))$. Let u be any loop in X whose homotopy class lies in H. We represent it by a homomorphism $\sigma \in \mathcal{O}_{\{\xi_{ji}\}}$, $\sigma : \Gamma\{I_i\} \to \Gamma\{U_{i'}\}$, $0 \leq i \leq n$. For each $i = 1, \cdots, n$, we pick a $t_i \in I_{i-1} \cap I_i$ and let τ_i be the restriction of σ on $[t, t_i]$. We let V_{σ_i} be the component of $\pi^{-1}(U_i)$ that contains $[\tau_i]_H \in Y$, denote by $z_i \in \widehat{V_{\sigma_i}}$ a point satisfying $\pi_{\sigma_i}(z_i) = [\tau_i]_H$, and denote by γ_{z_i} a path in $\widehat{V_{\sigma_i}}$ that connects $\hat{\sigma}_i$ to z_i. Then there exist $g_i \in G_{U_i}$, $i = 1, \cdots, n$, such that the composition of τ_i with $\nu(\sigma_i \# \gamma_{z_i})$ through $g_i \circ \xi_{i(i-1)}$ defines a loop whose homotopy class lies in H. This implies that $g_1 \circ \xi_{10} \in T(V_o, V_{\sigma_1})$, $g_i \circ \xi_{i(i-1)} \circ g_{i-1}^{-1} \in T(V_{\sigma_{i-1}}, V_{\sigma_i})$ for $i \geq 2$. Note that because the homotopy class of the loop u lies in H, $V_{\sigma_n} = V_o$ and g_n may be taken to be $1 \in G_{U_o}$. Thus according to Lemma 2.4.4, the loop u can be lifted to a loop in $(Y, \underline{q})$. Hence $H \subset \Pi_*(\pi_1(Y, \underline{q}))$. It remains to show that $\Pi_*(\pi_1(Y, \underline{q})) \subset H$. Given any $\tau \in \mathcal{O}_{\{\eta_{ji}\}}$, $i = 0, 1, \cdots, n$, where $\eta_{ji} \in T(V_{\sigma_i}, V_{\sigma_j})$ such that the equivalence class $[\tau] \in [(S^1, *); (Y, \underline{q})]$. Then there are, for $i = 0, \cdots, n-1$, points $z_i \in$ Domain $(\phi_{\eta_{(i+1)i}})$, $z_i' = \phi_{\eta_{(i+1)i}}(z_i)$, and paths γ_{z_i} in $\widehat{V_{\sigma_i}}$, $\gamma_{z_i'}$ in $\widehat{V_{\sigma_{i+1}}}$, which connect $\hat{\sigma}_i$, $\hat{\sigma}_{i+1}$ to z_i, z_i' respectively, such that the composition of $\sigma_i \# \gamma_{z_i}$ with $\nu(\sigma_{i+1} \# \gamma_{z_i'})$ through

$\eta_{(i+1)i}$ defines a loop whose homotopy class is an element $h_i \in H$. By the semi-locally 1-connectedness of X and the path-connectedness of each Domain $(\phi_{\eta_{(i+1)i}})$, we see that $\Pi \circ [\tau]$ is a loop whose homotopy class equals $h_0 \cdots h_{n-1} \in H$. Hence $\Pi_*(\pi_1(Y, \underline{q})) \subset H$.

$\square$

We shall derive a short exact sequence, which relates the π_1 of an orbispace with the group of deck transformations of its universal covering, whose existence was established in the preceding proposition. By definition, a deck transformation of a covering space $\Pi : Y \to X$ is an isomorphism of orbispaces $\Phi : Y \to Y$ such that $\Pi \circ \Phi = \Pi$. The group of deck transformations of $\Pi : Y \to X$ is denoted by Deck (Π).

Let $\Pi : (Y, \underline{q}) \to (X, \underline{p})$ be a covering map with a representative $(\{\pi_\alpha\}, \{\rho_{\beta\alpha}\})$ fixed throughout. For each α, we set $K_{U_\alpha} = \{g \in G_{U_\alpha} \mid g \cdot x = x, \forall x \in \widehat{U_\alpha}\}$. We consider

$$K_\Pi = \{\{g_\alpha\} \mid g_\alpha \in K_{U_\alpha} \text{ and } \rho_{\beta\alpha}(\xi) = g_\beta \circ \rho_{\beta\alpha}(\xi) \circ g_\alpha^{-1} \; \forall \xi \in T(V_\alpha, V_\beta)\}, \tag{2.4.6}$$

which is a group under the multiplication $\{g_\alpha\}\{h_\alpha\} = \{g_\alpha h_\alpha\}$. The subgroup

$$C_\Pi = \{\{g_\alpha\} \in K_\Pi \mid g_\alpha \in \rho_\alpha(G_{V_\alpha})\} \tag{2.4.7}$$

is contained in the center of K_Π, hence is a normal subgroup of K_Π.

Let $\tilde{p}$ and $\tilde{q}$ be the constant map from I into $\hat{p}$ and $\hat{q}$ respectively. For any $g \in K_{U_o}$, $h \in K_{V_o}$, we set $[g] = [(\tilde{p}, g)] \in \pi_1(X, \underline{p})$, $[h] = [(\tilde{q}, h)] \in \pi_1(Y, \underline{q})$. Clearly $\Pi_*([h]) = [\rho_o(h)]$. For any $\{g_\alpha\} \in K_\Pi$, we observe that $[g_o]$ actually lies in the center of $\Pi_*(\pi_1(Y, \underline{q}))$ in $\pi_1(X, \underline{p})$, where g_o is the component of $\{g_\alpha\}$ corresponding to U_o. To see this, let $u \in \pi_1(Y, \underline{q})$ be an element which is represented by $\{f_i\} \in \mathcal{O}_{\{\xi_{ji}\}}$, $0 \leq i \leq n$. Then $\Pi_*(u)$ is represented by $\{\pi_i \circ \gamma_i\} \in \mathcal{O}_{\{\rho_{ji}(\xi_{ji})\}}$, and $[g_o]^{-1} \# \Pi_*(u) \# [g_o]$ is represented by $\{\pi_i \circ \gamma_i\} \in \mathcal{O}_{\{\eta_{ji}\}}$, where $\eta_{10} = \rho_{10}(\xi_{10}) \circ g_o$, $\eta_{ji} = \rho_{ji}(\xi_{ji})$ for $2 \leq j \leq n-1$, and $\eta_{n(n-1)} = g_o^{-1} \circ \rho_{n(n-1)}(\xi_{n(n-1)})$. Observe that if we let g_i be the component of $\{g_\alpha\}$ corresponding to U_i, then $\rho_{ji}(\xi_{ji}) = g_j \circ \rho_{ji}(\xi_{ji}) \circ g_i^{-1}$ are satisfied, which imply that $g_1^{-1} \circ \eta_{10} = \rho_{10}(\xi_{10})$, $g_j^{-1} \circ \eta_{ji} \circ g_i = \rho_{ji}(\xi_{ji})$ for $2 \leq j \leq n-1$, and $\eta_{n(n-1)} \circ g_n = \rho_{n(n-1)}(\xi_{n(n-1)})$. In other words, $(\{\pi_i \circ \gamma_i\}, \{\eta_{ji}\})$ is conjugate to $(\{\pi_i \circ \gamma_i\}, \{\rho_{ji}(\xi_{ji})\})$. Hence $[g_o]^{-1} \# \Pi_*(u) \# [g_o] = \Pi_*(u)$.

Thus there is a homomorphism

$$\Xi_{p,q} : K_\Pi / C_\Pi \to N(\Pi_*(\pi_1(Y, \underline{q}))) / \Pi_*(\pi_1(Y, \underline{q})), \tag{2.4.8}$$

induced by $\{g_\alpha\} \mapsto [g_o]$, where $N(\Pi_*(\pi_1(Y, \underline{q})))$ is the normalizer of $\Pi_*(\pi_1(Y, \underline{q}))$ in $\pi_1(X, \underline{p})$. Note that $\Xi_{p,q}$ is injective when Y is connected. This is because for any $\{g_\alpha\} \in K_\Pi$, $g_o \in \rho_o(G_{V_o})$ implies $g_\alpha \in \rho_\alpha(G_{V_\alpha})$ by $\rho_{\beta\alpha}(\xi) = g_\beta \circ \rho_{\beta\alpha}(\xi) \circ g_\alpha^{-1} \; \forall \xi \in T(V_\alpha, V_\beta)$ in (2.4.6), and by the connectedness of Y.

Proposition 2.4.8 *Let $\Pi : (Y, \underline{q}) \to (X, \underline{p})$ be a connected, locally path-connected covering space of X. Set $H = \Pi_*(\pi_1(Y, \underline{q}))$ and denote by $N(H)$ the normalizer of H in $\pi_1(X, \underline{p})$. Then there is a homomorphism $\Theta_{p,q} : N(H)/H \to$ Deck (Π), such that $\Xi_{p,q}$ and $\Theta_{p,q}$ fit into a short exact sequence*

$$1 \to K_\Pi / C_\Pi \xrightarrow{\Xi_{p,q}} N(H)/H \xrightarrow{\Theta_{p,q}} \textit{Deck}\ (\Pi) \to 1. \tag{2.4.9}$$

Moreover, if $\Pi : (Y, \underline{q}) \to (X, \underline{p})$, $\Pi' : (Y', \underline{q'}) \to (X', \underline{p'})$ are connected, locally path-connected covering spaces, where $\Pi_*(\pi_1(Y, \underline{q}))$, $\Pi'_*(\pi_1(Y', \underline{q'}))$ are normal subgroups, and there are maps[5]

[5]here Φ, Ψ are general maps, i.e., not in the restricted class specified by **Convention** in Introduction.

$\Phi : (X, \underline{p}) \to (X', \underline{p'})$, $\Psi : (Y, \underline{q}) \to (Y', \underline{q'})$ *such that* $\Pi' \circ \Psi = \Phi \circ \Pi$, *then we have*

$$\Psi \circ \Theta_{p,q}(z) = \Theta_{p',q'}(z') \circ \Psi, \ \forall z \in \pi_1(X, \underline{p})/\Pi_*(\pi_1(Y, \underline{q})), \tag{2.4.10}$$

where in (2.4.10), z' *is the image of* z *in* $\pi_1(X', \underline{p'})/\Pi'_*(\pi_1(Y', \underline{q'}))$ *under* Φ_*, *and* Ψ *is regarded as an element of* $[Y; Y']$.

Proof Let u be a loop in $(X, \underline{p})$ such that its homotopy class $[u] \in N(H)$. By Lemma 2.4.4, we associate u with a pair $(V_{\alpha(u)}, \delta(u))$, where $V_{\alpha(u)}$ is a connected component of $\pi^{-1}(U_o)$ and $\delta(u)$ is a left coset in $G_{U_o}/\rho_{\alpha(u)}(G_{V_{\alpha(u)}})$, such that for any chosen representative g of $\delta(u)$, there is a unique path $\ell(u)_g \in P(Y, \underline{q}, \underline{q}(g))$ satisfying $\Pi \circ \ell(u)_g = g \cdot u$. We consider the covering maps $\Pi^{(1)} = \Pi : (Y, \underline{q}) \to (X, \underline{p})$ and $\Pi^{(2)} : (Y, \underline{q}(g)) \to (X, \underline{p})$. The assumption that $[u] \in N(H)$ implies that $\Pi^{(1)}_*(\pi_1(Y, \underline{q})) = \Pi^{(2)}_*(\pi_1(Y, \underline{q}(g)))$ in $\pi_1(X, \underline{p})$. Hence by Lemma 2.4.6, there is a unique map $\Phi_u : (Y, \underline{q}) \to (Y, \underline{q}(g))$ such that $\Pi^{(2)} \circ \Phi_u = \Pi^{(1)}$. If we pick a different representative g' of $\delta(u)$, where $g' = \rho_{\alpha(u)}(h)g$ for some $h \in G_{V_{\alpha(u)}}$, then $\underline{q}(g)$ is changed to $\underline{q}(g') = h \cdot \underline{q}(g)$. This implies that the corresponding map $\Phi_u : Y \to Y$ is independent of the choice of the representative g. As for the dependence on $(\{\pi_\alpha\}, \{\rho_{\beta\alpha}\})$, we go back to the proof of Lemma 2.4.4 and observe that in the construction of $\ell(\sigma)$, if we change $(\{\pi_\alpha\}, \{\rho_{\beta\alpha}\})$ by conjugation, then the set $\{g_i\}$ in (2.4.1) will change accordingly such that overall, the orbit of $\underline{q'}(g)$ under the action of $G_{V_{\alpha(u)}}$ remains the same. On the other hand, by Remark 2.4.3 (1), two different choices of $(\{\pi_\alpha\}, \{\rho_{\beta\alpha}\})$ differ essentially by a conjugation after we replace σ by an appropriate induced one. Hence $\Phi_u : Y \to Y$ is also independent of the choice on $(\{\pi_\alpha\}, \{\rho_{\beta\alpha}\})$. It is obvious that Φ_u depends only on the class of $[u]$ in $N(H)/H$, and is a deck transformation of $\Pi : Y \to X$. The map $\Theta_{p,q} : N(H)/H \to \text{Deck }(\Pi)$ is defined by $[u] \mapsto \Phi_u$.

In order to verify (2.4.10), we pick representatives σ, τ for Φ and Ψ respectively, and fix $(\{\pi_\alpha\}, \{\rho_{\beta\alpha}\})$, $(\{\pi'_{\alpha'}\}, \{\rho'_{\beta'\alpha'}\})$ for Π, Π', such that

$$\sigma \circ (\{\pi_\alpha\}, \{\rho_{\beta\alpha}\}) = (\{\pi'_{\alpha'}\}, \{\rho'_{\beta'\alpha'}\}) \circ \tau. \tag{2.4.11}$$

Given any $z \in \pi_1(X, \underline{p})/\Pi_*(\pi_1(Y, \underline{q}))$, we represent z by a loop u in $(X, \underline{p})$. We pick a representative g of $\delta(u)$ (w.r.t $(\{\pi_\alpha\}, \{\rho_{\beta\alpha}\})$), and then use σ to determine a representative g' of $\delta(\Phi \circ u)$ (w.r.t $(\{\pi'_{\alpha'}\}, \{\rho'_{\beta'\alpha'}\})$), such that τ preserves $\underline{q}(g)$ and $\underline{q'}(g')$ (because of (2.4.11)). Now we consider the problem of factoring the map $\Phi \circ \Pi : (Y, \underline{q}) \to (X', \underline{p'})$ through the covering map $(Y', \underline{q'}(g')) \to (X', \underline{p'})$. There are apparently two solutions: $\Psi \circ \Theta_{p,q}(z)$ and $\Theta_{p',q'}(z') \circ \Psi$ (in their appropriate based versions). By the uniqueness in Lemma 2.4.6, these two solutions must be the same. Hence (2.4.10).

Next we verify that $\Theta_{p,q}$ is a homomorphism. Let u_1, u_2 be two loops in $(X, \underline{p})$ such that both $[u_1], [u_2] \in N(H)$. We pick representatives g_1 of $\delta(u_1)$, g_2 of $\delta(u_2)$, and g of $\delta(u_1 \# u_2)$. Then by the uniqueness of path-lifting in Lemma 2.4.4, the following holds for the based version $\Theta_{p,q}([u_1]) : (Y, \underline{q}, \underline{q}(g_2)) \to (Y, \underline{q}(g_1), \underline{q}(g))$,

$$\ell(u_1)_{g_1} \# (\Theta_{p,q}([u_1]) \circ \ell(u_2)_{g_2}) = \ell(u_1 \# u_2)_g. \tag{2.4.12}$$

In particular, there is a based version

$$\Theta_{p,q}([u_1]) : (Y, \underline{q}(g_2)) \to (Y, \underline{q}(g)). \tag{2.4.13}$$

Now by the uniqueness in Lemma 2.4.6, the composition of $\Theta_{p,q}([u_2]) : (Y, \underline{q}) \to (Y, \underline{q}(g_2))$ with (2.4.13) must be equal to $\Theta_{p,q}([u_1\#u_2]) : (Y, \underline{q}) \to (Y, \underline{q}(g))$. The corresponding maps satisfy

$$\Theta_{p,q}([u_1\#u_2]) = \Theta_{p,q}([u_1]) \circ \Theta_{p,q}([u_2]), \tag{2.4.14}$$

which shows that $\Theta_{p,q}$ is a homomorphism.[6]

It remains to verify (2.4.9). As for the surjectivity of $\Theta_{p,q}$, given any $\Phi \in \text{Deck}\ (\Pi)$, we take a path $u \in P(Y, \underline{q}, \underline{q'})$ where $\underline{q'}$ is the image of $\underline{q}$ under a choice of based versions of Φ. Then $\Pi \circ u$ determines a loop v in $(X, \underline{p})$, and

$$\Pi_*(\pi_1(Y, \underline{q})) = \Pi_*(\pi_1(Y, \underline{q'})) = \Pi_*(\nu(u)_*(\pi_1(Y, \underline{q}))) = [v]^{-1} \cdot \Pi_*(\pi_1(Y, \underline{q})) \cdot [v], \tag{2.4.15}$$

which implies that $[v] \in N(H)$. The uniqueness in Lemma 2.4.6 then asserts that $\Theta_{p,q}([v]) = \Phi$. Hence $\Theta_{p,q}$ is surjective.

Finally, we determine the kernel of $\Theta_{p,q}$. Suppose $\Theta_{p,q}([u]) = 1$ for some loop u in $(X, \underline{p})$. Then from the construction in Lemma 2.4.6 and by the definition of $\Theta_{p,q}$ above, we see that $\Theta_{p,q}([u])$ is defined by $\ell((\{\pi_\alpha\}, \{\rho_{\beta\alpha}\}))$ as constructed in Lemma 2.4.6, where there exists a set $\{g_\alpha \mid g_\alpha \in G_{U_\alpha}\}$, such that $\ell((\{\pi_\alpha\}, \{\rho_{\beta\alpha}\})) = (\{f_\alpha\}, \{\eta_{\beta\alpha}\}) : \Gamma\{V_\alpha\} \to \Gamma\{V_\alpha\}$ with $f_\alpha = \pi_\alpha^{-1} \circ g_\alpha \circ \pi_\alpha$, $\eta_{\beta\alpha}(\xi) = \rho_{\beta\alpha}^{-1}(g_\beta \circ \rho_{\beta\alpha}(\xi) \circ g_\alpha^{-1})$. Moreover, a different choice of $\{g_\alpha\}$ has the form $\{\rho_\alpha(h_\alpha) g_\alpha\}$ for some $h_\alpha \in G_{V_\alpha}$. Now the assumption $\Theta_{p,q}([u]) = 1$ implies that $\ell((\{\pi_\alpha\}, \{\rho_{\beta\alpha}\}))$ is conjugate to the identity, and because of this, we can actually choose a set $\{g_\alpha\}$ such that $\ell((\{\pi_\alpha\}, \{\rho_{\beta\alpha}\}))$ is the identity. It then follows easily that $\{g_\alpha\} \in K_\Pi$. Furthermore, one is ready to check that $u\#(\tilde{p}, g_o^{-1})$, where g_o is the component of $\{g_\alpha\}$ that corresponds to U_o, can be lifted to a loop in $(Y, \underline{q})$, hence $[u] = [g_o] \bmod \Pi_*(\pi_1(Y, \underline{q}))$. This exactly means that $\ker\ \Theta_{p,q} \subset \text{Im}\ \Xi_{p,q}$. It remains to show that $\text{Im}\ \Xi_{p,q} \subset \ker\ \Theta_{p,q}$, so that $\ker\ \Theta_{p,q} = \text{Im}\ \Xi_{p,q}$ and (2.4.9) is verified. To this end, let $\{g_\alpha\} \in K_\Pi$ be any element, we consider $\Theta_{p,q}([u])$ where $[u] = [g_o]$. It is clear that the component $V_{\alpha(u)}$ associated to u is V_o, and g_o can be chosen to serve as a representative g of $\delta(u)$. With $g = g_o$, we see that $\underline{q}(g)$ in the definition of $\Theta_{p,q}([u])$ is actually $\underline{q}$ since $g_o \in K_{U_o}$. By the uniqueness in Lemma 2.4.6, we conclude that $\Theta_{p,q}([u])$ is the identity map. Hence $\text{Im}\ \Xi_{p,q} \subset \ker\ \Theta_{p,q}$, and the proof of Proposition 2.4.8 is completed.

$\square$

Proof of Lemma 2.4.1

The 'only if' part follows from the fact that (2.2.10) is natural with respect to (f, λ). We shall prove the 'if' part next.

First of all, we observe that for any global quotient $X = Y/G$, there is a natural covering map $\Pi : Y \to X$ defined by $(Id, 1) : (Y, \{1\}) \to (Y, G)$, with the orbit map $\pi : Y \to Y/G = X$ being the induced map between underlying spaces. Moreover, for any base-point structure $\underline{o} = (o, U_o, \hat{o})$, the injective homomorphism $\Pi_* : \pi_1(Y, \hat{o}) \to \pi_1(X, \underline{o})$ coincides with the one in (2.2.10).

To define the map $f : Y \to Y'$, we denote $\Pi' : Y' \to X'$ the canonical covering map for X'. Then the assumption for the "if" part of the lemma can be rephrased as that there are base-point structures $\underline{o}$, $\underline{o'}$, such that $(\Phi \circ \Pi)_*(\pi_1(Y, \hat{o})) \subset \Pi'_*(\pi_1(Y', \hat{o}'))$. By Lemma 2.4.6, there is a unique map $f : (Y, \hat{o}) \to (Y', \hat{o}')$ such that $\Phi \circ \Pi = \Pi' \circ f$.

[6] note that $\Theta_{p,q}$ induces a left-action of $N(H)/H$ on Y.

As for the homomorphism $\lambda : G \rightarrow G'$, we observe that $\pi_1(X, \underline{o}) \rightarrow G$ in (2.2.10) is surjective since Y is path-connected, hence by the assumption that $\Phi_*(\pi_1(Y, \hat{o})) \subset \pi_1(Y', \hat{o}')$, Φ_* induces a homomorphism $\lambda : G \rightarrow G'$.

It is easily seen that the action of $\pi_1(X, \underline{o})/\pi_1(Y, \hat{o})$ on Y through deck transformations coincides with the action of G on Y under the isomorphism $\pi_1(X, \underline{o})/\pi_1(Y, \hat{o}) \cong G$ given by (2.2.10). Hence f is λ-equivariant by virtue of (2.4.10). Note that here we need not to assume that Y' is connected, locally path-connected because the action of $\pi_1(X', \underline{o'})/\pi_1(Y', \hat{o}')$ on Y' through deck transformations is given a priori by the homomorphism $\pi_1(X', \underline{o'})/\pi_1(Y', \hat{o}') \rightarrow G'$ in (2.2.10) and the action G' on Y'.

It remains to show that Φ is defined by (f, λ). To this end, we first observe that an induced homomorphism $(\{f_a\}, \{\lambda_{ba}\}) : \Gamma\{U_a\} \rightarrow \Gamma\{U'_{a'}\}$ of (f, λ) has the form $f_a = \delta_a^{-1} \circ f|_{\widehat{U_a}}$ and $\lambda_{ba}(g) = \delta_b^{-1}\lambda(g)\delta_a$ for a set $\{\delta_a \in G'\}$, where each $\widehat{U_a}$ is an open subset of Y, and $g \in T(U_a, U_b) \subset G$. Secondly, we observe that a more concrete description of a representative of the canonical covering map $\Pi : Y \rightarrow X$, which is induced by $(Id, 1) : (Y, \{1\}) \rightarrow (Y, G)$, may be given as

$$(\{V_{a,i}|i \in I_a\}, \{U_a\}, \{(\pi_{a,i}, \delta_{a,i})\}, \{\rho^s_{(b,j)(a,i)}\}), \tag{2.4.16}$$

where (1) $\{U_a\}$ is a cover of X, (2) for each a, $\{V_{a,i}|i \in I_a\}$ is the set of connected components of $\pi^{-1}(U_a)$ in Y, with V_{a,i_0} taken to be $\widehat{U_a}$ for some index i_0, (3) $\pi_{a,i} : V_{a,i} \rightarrow \widehat{U_a}$ is the homeomorphism induced by an element $\delta_{a,i} \in G$, and (4) for any $V_{a,i}$, $V_{b,j}$ with $V_{a,i} \cap V_{b,j} \neq \emptyset$, there is a set $\{\rho^s_{(b,j)(a,i)} \in T(U_a, U_b)\}$ labeled by the set of components of $V_{a,i} \cap V_{b,j}$, denoted by $\{s\}$, where each $\rho^s_{(b,j)(a,i)} = \delta_{b,j}\delta_{a,i}^{-1} \in G$. For simplicity, we shall require $\delta_{a,i_0} = 1$ for each index a without loss of generality.

Now according to the construction in Lemma 2.4.6, the map $f : (Y, \hat{o}) \rightarrow (Y', \hat{o}')$ is defined as follows. First of all, we fix a representative κ' of $\Pi' : (Y', \hat{o}') \rightarrow (X', \underline{o}')$ as described in (2.4.16). Then we can choose a representative κ for $\Pi : (Y, \hat{o}) \rightarrow (X, \underline{o})$ as in (2.4.16), and a representative σ of $\Phi : (X, \underline{o}) \rightarrow (X', \underline{o}')$, such that $\sigma \circ \kappa$ can be lifted by κ' to Y', which is defined to be f. In particular, $\sigma \circ \kappa = \kappa' \circ f$. To fix the notations, we write $\sigma = (\{f_a\}, \{\lambda_{ba}\})$. We denote the component of $(\pi')^{-1}(U'_a)$ that contains $f(V_{a,i})$ by $V'_{a,i}$ for any $V_{a,i}$, and denote by $\delta'_{a,i}$ the element of G' that induces the homeomorphism $\pi'_{a,i} : V'_{a,i} \rightarrow \widehat{U'_a}$. Note that V'_{a,i_0}, which is the component containing $f(V_{a,i_0}) = f(\widehat{U_a})$, is not necessarily $\widehat{U'_a}$.

With these notational conventions understood, we can easily see that $f_a = \delta'_{a,i_0} \circ f|_{\widehat{U_a}}$ as a consequence of $\sigma \circ \kappa = \kappa' \circ f$ and the assumption $\delta_{a,i_0} = 1$. We set $\delta_a = (\delta'_{a,i_0})^{-1}$. Then to show that σ is induced by (f, λ), we only need to check

$$\lambda_{ba}(g) = \delta_b^{-1}\lambda(g)\delta_a. \tag{2.4.17}$$

It suffices to check (2.4.17) for two special cases: (a) $g = \rho^s_{(b,j)(a,i)}$ for any indexes $i \in I_a, j \in I_b$, and (b) the indexes $a = b$ and $g \in G_{U_a}$.

For case (a) where $g = \rho^s_{(b,j)(a,i)}$ for some indexes $i \in I_a, j \in I_b$, we shall prove the relation $(\delta'_{a,i_0})^{-1}\delta'_{a,i} = \lambda(\delta_{a,i})$ for any index a and $i \in I_a$, which implies case (a) because $\rho^s_{(b,j)(a,i)} = \delta_{b,j}\delta_{a,i}^{-1}$, and $\lambda_{ba}(\rho^s_{(b,j)(a,i)}) = \delta'_{b,j}(\delta'_{a,i})^{-1}$ as a consequence of $\sigma \circ \kappa = \kappa' \circ f$. To see $(\delta'_{a,i_0})^{-1}\delta'_{a,i} = \lambda(\delta_{a,i})$, we pick paths γ_0, γ in Y satisfying $\gamma_0(0) = \gamma(0) = \hat{o}$, $\gamma_0(1) \in \widehat{U_a}$, and $\gamma(1) \in V_{a,i}$ with $\gamma_0(1) = \delta_{a,i} \cdot \gamma(1)$. Then $\kappa \circ \gamma_0$ may compose with $\nu(\kappa \circ \gamma)$ to define a loop u in

$(X, \underline{o})$. The lifting of u to $(Y, \hat{o})$ by κ is easily seen to be the path $\gamma_0 \# \nu(\delta_{a,i} \circ \gamma)$ whose terminal point is $\delta_{a,i} \cdot \hat{o}$. Hence the image of $[u]$ under the homomorphism $\pi_1(X, \underline{o}) \rightarrow G$ in (2.2.10) is $\delta_{a,i}$. On the other hand, the push-forward $\sigma \circ u$ is lifted to $(f \circ \gamma_0) \# \nu((\delta'_{a,i_0})^{-1} \delta'_{a,i} \circ (f \circ \gamma))$ by κ', whose terminal point is $(\delta'_{a,i_0})^{-1} \delta'_{a,i} \cdot \hat{o}'$. Hence the image of $\Phi_*([u])$ under $\pi_1(X', \underline{o'}) \rightarrow G'$ in (2.2.10) is $(\delta'_{a,i_0})^{-1} \delta'_{a,i}$. This gives $(\delta'_{a,i_0})^{-1} \delta'_{a,i} = \lambda(\delta_{a,i})$ by the definition of λ.

For case (b) where the indexes $a = b$ and $g \in G_{U_a}$, we pick a path γ_0 in Y with $\gamma_0(0) = \hat{o}$, $\gamma_0(1) \in \widehat{U_a}$, and pick a path γ in $\widehat{U_a}$ connecting $\gamma_0(1)$ to $g \cdot \gamma_0(1)$. Note that if we set $x = \pi(\gamma_0(1))$, $\underline{x} = (x, U_a, \gamma_0(1))$, then (γ, g) defines a loop v in $(X, \underline{x})$ by Lemma 2.2.2. We set $u = (\kappa \circ \gamma_0) \# v \# \nu(\kappa \circ \gamma_0)$, which is a loop in $\Omega(X, \underline{o})$. The lifting of u by κ is $\gamma_0 \# \gamma \# \nu(g \circ \gamma_0)$, whose terminal point is $g \cdot \hat{o}$. Hence the image of $[u]$ under the homomorphism $\pi_1(X, \underline{o}) \rightarrow G$ in (2.2.10) is g. On the other hand, the push-forward $\sigma \circ u$ is lifted by κ' to the path $\ell(\sigma \circ u) = (f \circ \gamma_0) \# \gamma' \# \nu(Ad((\delta'_{a,i_0})^{-1})(\lambda_a(g)) \circ (f \circ \gamma_0))$ for some path γ' connecting $f(\gamma_0(1))$ to $Ad((\delta'_{a,i_0})^{-1})(\lambda_a(g)) \cdot f(\gamma_0(1))$. The terminal point of $\ell(\sigma \circ u)$ is $Ad((\delta'_{a,i_0})^{-1})(\lambda_a(g)) \cdot \hat{o}'$, which implies that the image of $\Phi_*([u])$ under $\pi_1(X', \underline{o'}) \rightarrow G'$ in (2.2.10) is $Ad((\delta'_{a,i_0})^{-1})(\lambda_a(g))$. Hence $\lambda(g) = Ad((\delta'_{a,i_0})^{-1})(\lambda_a(g))$, from which (2.4.17) for case (b) can be easily deduced.

$\square$

3. An analog of CW-complex theory

3.1 Construction of mapping cylinders

In this subsection we extend the mapping cylinder construction in the ordinary homotopy theory (cf. e.g. [14]) to the orbispace category. Recall that for any map $f : Y \rightarrow X$ between topological spaces, the mapping cylinder of f, denoted by M_f, is the topological space obtained by identifying $(y, 1)$ with $f(y)$ in the disjoint union $Y \times I \bigsqcup X$ for all $y \in Y$. We set $[y, t]$ for the image of $(y, t) \in Y \times I \bigsqcup X$ in M_f. There are embeddings $i : Y \rightarrow M_f$, $j : X \rightarrow M_f$ given by $i(y) = [y, 0]$ and $j(x) = x$, realizing Y, X as subspaces of M_f. The space X is a strong deformation retract of M_f, with the canonical retraction $r : M_f \rightarrow X$ where $r([y, t]) = f(y)$ and $r(x) = x$. The homotopy $H : M_f \times [0, 1] \rightarrow M_f$ between $j \circ r$ and Id_{M_f} is given by $H([y, t], s) = [y, 1 - (1 - t)s]$ and $H(x, s) = x$. Finally, given any homotopy $F : Y \times [0, 1] \rightarrow X$ between $f_1, f_2 : Y \rightarrow X$, there are canonical maps $\phi_F : M_{f_1} \rightarrow M_{f_2}$, $\psi_F : M_{f_2} \rightarrow M_{f_1}$, such that $\psi_F \circ \phi_F$ and $\phi_F \circ \psi_F$ are canonically homotopic to $Id_{M_{f_1}}$ and $Id_{M_{f_2}}$ respectively, relative to X and Y. For instance, we may define ϕ_F, ψ_F by

$$\text{(3.1.1)} \quad \phi_F([y,t]) = \begin{cases} [y,t] & 0 \leq t \leq \frac{1}{2} \\ F(y, 2-2t) & \frac{1}{2} \leq t \leq 1 \end{cases} \quad \psi_F([y,t]) = \begin{cases} [y,t] & 0 \leq t \leq \frac{1}{2} \\ F(y, 2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

$$\phi_F(x) = x \qquad\qquad \psi_F(x) = x$$

The canonical homotopy between $\psi_F \circ \phi_F$ and $Id_{M_{f_1}}$ may be taken to be $H : M_{f_1} \times [0,1] \rightarrow M_{f_1}$ where

$$H([y,t],s) = \begin{cases} [y,(4-3s)t], & 0 \le t \le \frac{1}{4-3s} \\ F(y,(4-3s)t-1), & \frac{1}{4-3s} \le t \le \frac{2-s}{4-3s} \\ F(y,\frac{1}{2}(4-3s)(1-t)), & \frac{2-s}{4-3s} \le t \le 1 \end{cases} \tag{3.1.2}$$
$$H(x,s) = x.$$

In the same vein, we obtain the canonical homotopy between $\phi_F \circ \psi_F$ and $Id_{M_{f_2}}$.

We derive two technical lemmas first.

Lemma 3.1.1 *Let $\Phi : X \rightarrow X'$ be a map, where $X' = Y'/G'$ is a global quotient, and X is connected, locally path-connected and semi-locally 1-connected. Then there exists a connected, locally path-connected space Y with a discrete group action of G, such that $X = Y/G$. Moreover, $\Phi : X \rightarrow X'$ is represented by a pair $(f,\lambda) : (Y,G) \rightarrow (Y',G')$ where f is λ-equivariant. Such a (Y,G) is uniquely determined up to an isomorphism, and (f,λ) is unique up to conjugation by an element $g \in G'$, i.e., $(f,\lambda) \mapsto (g \circ f, Ad(g) \circ \lambda)$.*

Proof We take a based version of Φ with respect to some base-point structures $\underline{o}$, $\underline{o'}$ of X, X' respectively. Since X is connected, locally path-connected and semi-locally 1-connected, there is a covering space $\Pi : (Y,\underline{q}) \rightarrow (X,\underline{o})$ such that $\Pi_*(\pi_1(Y,\underline{q})) = (\Phi_*)^{-1}(\pi_1(Y',\hat{o}'))$ by Proposition 2.4.7. Moreover, by Lemma 2.4.6, there is a map $\Psi : (Y,\underline{q}) \rightarrow (Y',\hat{o}')$ satisfying $\Phi \circ \Pi = \Pi' \circ \Psi$, where $\Pi' : (Y',\hat{o}') \rightarrow (X',\underline{o'})$ is the canonical covering map associated to the global quotient $X' = Y'/G'$.

Now recall that Φ satisfies the assumption in **Convention** in Introduction, so does the map $\Psi : (Y,\underline{q}) \rightarrow (Y',\hat{o}')$. This implies that Y is actually a topological space. We rename Ψ by f, and set $G = \pi_1(X,\underline{o})/\Pi_*(\pi_1(Y,\underline{q}))$. Then G acts on Y through deck transformations, and there is an injective homomorphism $\lambda : G \rightarrow G'$ such that $f : Y \rightarrow Y'$ is λ-equivariant.

We next verify that X is isomorphic to the global quotient Y/G and $\Pi : Y \rightarrow X$ is isomorphic to the canonical covering map. To this end, we pick a representative of $\Pi : (Y,\underline{q}) \rightarrow (X,\underline{o})$, which can be described in the following form:

$$(\{V_{\alpha,i} | i \in I_\alpha\}, \{U_\alpha\}, \{\pi_{\alpha,i}\}, \{\rho^s_{(\beta,j)(\alpha,i)}\}), \tag{3.1.3}$$

where (1) $\{U_\alpha\}$ is a cover of X by local charts, (2) for each index α, $\{V_{\alpha,i} | i \in I_\alpha\}$ is the set of connected components of $\pi^{-1}(U_\alpha) \subset Y$, (3) each $\pi_{\alpha,i} : V_{\alpha,i} \rightarrow \widehat{U_\alpha}$ is a homeomorphism, (4) for any $V_{\alpha,i}, V_{\beta,j}$ with $V_{\alpha,i} \cap V_{\beta,j} \neq \emptyset$, there is a set $\{\rho^s_{(\beta,j)(\alpha,i)} \in T(U_\alpha, U_\beta)\}$ labeled by the set $\{V^s_{(\alpha,i)(\beta,j)}\}$ of connected components of $V_{\alpha,i} \cap V_{\beta,j}$, such that the following equations are satisfied,

$$\pi_{\beta,j} = \rho^s_{(\beta,j)(\alpha,i)} \circ \pi_{\alpha,i} \text{ on } V^s_{(\alpha,i)(\beta,j)}, \text{ and } \rho^t_{(\gamma,k)(\beta,j)} \circ \rho^s_{(\beta,j)(\alpha,i)}(\pi_{\alpha,i}(\mathbf{a})) = \rho^r_{(\gamma,k)(\alpha,i)}, \tag{3.1.4}$$

where $\mathbf{a}$ is any connected component of $V^s_{(\alpha,i)(\beta,j)} \cap V^t_{(\beta,j)(\gamma,k)}$ that is contained in $V^r_{(\alpha,i)(\gamma,k)}$.

We claim: (1) Set $G_{\alpha,i} = \{g \in G \mid g \cdot V_{\alpha,i} = V_{\alpha,i}\}$, then there is an isomorphism $\lambda_{\alpha,i} : G_{\alpha,i} \rightarrow G_{U_\alpha}$ such that $\pi_{\alpha,i}$ is $\lambda_{\alpha,i}$-equivariant. (2) For any $V_{\alpha,i}, V_{\alpha,j}$, there is a $g^\alpha_{ji} \in G$ satisfying $\pi_{\alpha,j} \circ g^\alpha_{ji} = \pi_{\alpha,i}$ and $g^\alpha_{ki} = g^\alpha_{kj} g^\alpha_{ji}$. To sketch a proof, we pick, for each $V_{\alpha,i}$, a path γ^α_i in Y satisfying $\gamma^\alpha_i(0) = \hat{q}$, $\gamma^\alpha_i(1) \in V_{\alpha,i}$ and $\pi_{\alpha,i}(\gamma^\alpha_i(1)) = \pi_{\alpha,j}(\gamma^\alpha_j(1))$. We introduce base-point structures $\underline{x_\alpha} = (x_\alpha, U_\alpha, \hat{x}_\alpha)$ where $x_\alpha = \pi(\gamma^\alpha_i(1))$, $\hat{x}_\alpha = \pi_{\alpha,i}(\gamma^\alpha_i(1))$, and denote by $\bar{\gamma_i}^\alpha$ the push-down path $\Pi \circ \gamma_i^\alpha \in P(X, \underline{o}, \underline{x_\alpha})$. Then the isomorphism $\lambda_{\alpha,i} : G_{\alpha,i} \rightarrow G_{U_\alpha}$ is defined as follows. For any $g \in G_{U_\alpha}$, we pick a path γ in $\widehat{U_\alpha}$ connecting $\hat{x}_\alpha$ to $g \cdot \hat{x}_\alpha$, and denote by γ_g the loop in $(X, \underline{x_\alpha})$ defined by (γ, g). The inverse of $\lambda_{\alpha,i}$ is defined by sending g to the image of $[\bar{\gamma_i}^\alpha \# \gamma_g \# \nu(\bar{\gamma_i}^\alpha)] \in \pi_1(X, \underline{o})$ under the homomorphism $\pi_1(X, \underline{o}) \rightarrow G = \pi_1(X, \underline{o})/\Pi_*(\pi_1(Y, \underline{q}))$. As for the set of elements $\{g_{ji}^\alpha\}$, we define g_{ji}^α to be the image of $[\bar{\gamma_j}^\alpha \# \nu(\bar{\gamma_i}^\alpha)] \in \pi_1(X, \underline{o})$ under the homomorphism $\pi_1(X, \underline{o}) \rightarrow G$. With these definitions, the verification of the claim is straightforward, hence we leave it to the reader.

For each α, we fix an index $i_0 \in I_\alpha$, and set $W_\alpha = V_{\alpha,i_0}/G_{\alpha,i_0}$. Then an isomorphism between X and Y/G is defined by $\sigma = (\{\pi_{\alpha,i_0}^{-1}\}, \{\eta_{\beta\alpha}\}) : \Gamma\{U_\alpha\} \rightarrow \Gamma\{W_\alpha\}$, where $\eta_{\beta\alpha}$ is the mapping uniquely determined by the conditions $\eta_{\alpha\alpha} = \lambda_{\alpha,i_0}^{-1}$ and $\eta_{\beta\alpha}(\rho^s_{(\beta,j)(\alpha,i)}) = g^\beta_{j_0 j} g^\alpha_{i i_0}$. Under this isomorphism, the covering map $\Pi : Y \rightarrow X$ is isomorphic to the canonical one associated to the global quotient Y/G.

Finally, by Lemma 2.4.1, $\Phi : X \rightarrow X'$ is represented by $(f, \lambda) : (Y, G) \rightarrow (Y', G')$. By Lemma 3.1.2 in [3], (f, λ) is uniquely determined up to conjugation. To see that (Y, G) is uniquely determined up to an isomorphism, we simply observe that the injectivity of $\lambda : G \rightarrow G'$ implies $\pi_1(Y, \hat{o}) = (\Phi_*)^{-1}(\pi_1(Y', \hat{o}'))$ for appropriate base-point structures $\underline{o}$, $\underline{o'}$ of X, X'. In other words, $\pi_1(Y, \hat{o})$ is uniquely determined, hence so is the covering space Y up to an isomorphism by Lemma 2.4.6.

□

Let Y be a locally path-connected, semi-locally 1-connected orbispace. For any map $\Phi : Y \rightarrow X$, we consider the set $\mathcal{V}(\Phi)$ of connected open subsets V of Y such that $V \subset \phi^{-1}(U)$, where $\phi : Y \rightarrow X$ is the induced map of Φ between underlying spaces, and $U \in \mathcal{U}$ is a local chart on X. We apply the preceding lemma to $\Phi|_V : V \rightarrow U$, so that for each $V \in \mathcal{V}(\Phi)$, there is a $(\widetilde{V}, G^V)$ such that the subspace V is isomorphic to the global quotient $\widetilde{V}/G^V$, under which $\Phi|_V$ is represented by a pair $(f_V, \lambda_V) : (\widetilde{V}, G^V) \rightarrow (\widehat{U}, G_U)$ where f_V is λ_V-equivariant. Set $\pi^V : \widetilde{V} \rightarrow \widetilde{V}/G^V = V$. We remark that

(1) $(\widetilde{V}, G^V, \pi^V)$ depends only on V. This is because, as we have seen in the proof of Lemma 3.1.1, the isomorphism class of $(\widetilde{V}, G^V)$ is determined by $(\Phi|_V)_*^{-1}(\pi_1(\widehat{U}))$, and on the other hand, for any $U_1 \subset U_2$, $(\Phi|_V)_*^{-1}(\pi_1(\widehat{U_1})) = (\Phi|_V)_*^{-1}(\pi_1(\widehat{U_2}))$. The last identity follows from the fact that the inclusion $U_1 \subset U_2$ induces an injective homomorphism from G_{U_1} to G_{U_2} so that by (2.2.10), the inverse image of $\pi_1(\widehat{U_2})$ is $\pi_1(\widehat{U_1})$ under $\pi_1(U_1) \rightarrow \pi_1(U_2)$.

(2) If $V_1, V_2 \in \mathcal{V}(\Phi)$ such that $V_1 \subset V_2$, then $(\widetilde{V_1}, G^{V_1}, \pi^{V_1})$ is isomorphic to an induced one from $(\widetilde{V_2}, G^{V_2}, \pi^{V_2})$. This is because if we denote by $I : V_1 \rightarrow V_2$ the inclusion as a subspace, then $\Phi|_{V_1} = \Phi|_{V_2} \circ I$, and hence $I_*^{-1} \circ (\Phi|_{V_2})_*^{-1}(\pi_1(\widehat{U})) = (\Phi|_{V_1})_*^{-1}(\pi_1(\widehat{U}))$ where $V_1 \subset V_2 \subset \phi^{-1}(U)$, so that $I_*^{-1}(\pi_1(\widetilde{V_2})) = \pi_1(\widetilde{V_1})$ under $I_* : \pi_1(V_1) \rightarrow \pi_1(V_2)$. By Lemma 2.4.1, I is represented by a pair $(f, \lambda) : (\widetilde{V_1}, G^{V_1}) \rightarrow (\widetilde{V_2}, G^{V_2})$ where λ is injective and f is a λ-equivariant open embedding.

(3) When V is a local chart on Y such that $\Phi|_V$ is represented by a pair $(f, \rho) : (\widehat{V}, G_V) \rightarrow (\widehat{U}, G_U)$, then $(\widehat{V}, G_V, \pi_V)$ is isomorphic to $(\widetilde{V}, G^V, \pi^V)$. For instance, suppose Φ is represented by $(\{f_\alpha\}, \{\rho_{\beta\alpha}\}) : \Gamma\{V_\alpha\} \rightarrow \Gamma\{U_{\alpha'}\}$, then if a local chart $V \subset V_\alpha$ for some index α, then $V \in \mathcal{V}(\Phi)$ and $(\widehat{V}, G_V, \pi_V)$ is isomorphic to $(\widetilde{V}, G^V, \pi^V)$.

Lemma 3.1.2 *There is a set $\mathcal{T}(\Phi) = \{T(V_1,V_2) \mid V_1,V_2 \in \mathcal{V}(\Phi),\ s.t.\ V_1 \cap V_2 \neq \emptyset\}$, which together with the atlas of local charts $\{(\widetilde{V}, G^V, \pi^V) \mid V \in \mathcal{V}(\Phi)\}$ defines an orbispace structure on Y that is equivalent to the original one. Moreover, given any cover $\{V_\alpha\} \subset \mathcal{V}(\Phi)$ of Y, where $V_\alpha \subset \phi^{-1}(U_\alpha)$ for some local chart U_α on X, there is a homomorphism $(\{f_\alpha\},\{\lambda_{\beta\alpha}\}) : \Gamma\{V_\alpha\} \to \Gamma\{U_\alpha\}$ whose equivalence class is the given map Φ, in which $(f_\alpha,\lambda_\alpha) = (f_{V_\alpha},\lambda_{V_\alpha}) : (\widetilde{V_\alpha}, G^{V_\alpha}) \to (\widehat{U_\alpha}, G_{U_\alpha})$.*

Proof For any $V \in \mathcal{V}(\Phi)$, the subspace structure of $V \subset Y$ is given by an atlas of local charts $\mathcal{N}(V) = \{(\widehat{W}, G_W, \pi_W)\}$ and a set $\mathcal{T}(V) = \{T(W_1,W_2)\}$, where $W \subset V$ is a local chart on Y. Since the subspace V is isomorphic to $\widetilde{V}/G^V$, we may further require that for each $W \in \mathcal{N}(V)$, $W \in \mathcal{V}(\Phi)$ and $(\widetilde{W}, G^W, \pi^W) = (\widehat{W}, G_W, \pi_W)$. Moreover, a set $T(W,V)$ is defined, such that each $\xi \in T(W,V)$ is associated with a $(\phi_\xi,\lambda_\xi) : (\widehat{W}, G_W) \to (\widetilde{V}, G^V)$ where λ_ξ is injective and ϕ_ξ is a λ_ξ-equivariant open embedding, and for any $g \in G^V$, $g \circ \xi \in T(W,V)$ and $(\phi_{g\circ\xi},\lambda_{g\circ\xi}) = (\phi_g, Ad(g)) \circ (\phi_\xi,\lambda_\xi)$.

Let $V_1,V_2 \in \mathcal{V}(\Phi)$ such that $V_1 \cap V_2 \neq \emptyset$. The set $T(V_1,V_2)$ is defined as follows. First of all, let V be any connected component of $V_1 \cap V_2$, and $W \in \mathcal{N}(V_1) \cap \mathcal{N}(V_2)$, $W \subset V$. To each $(\xi_1,\xi_2) \in T(W,V_1) \times T(W,V_2)$, we assign a pair $\chi_V^W(\xi_1,\xi_2) = (f,\lambda)$, where $f : \widetilde{Z_1} \to \widetilde{Z_2}$ is a homeomorphism from a connected component of $(\pi^{V_1})^{-1}(V) \subset \widetilde{V_1}$ to a connected component of $(\pi^{V_2})^{-1}(V) \subset \widetilde{V_2}$ satisfying $\pi^{V_1} = \pi^{V_2} \circ f$, and $\lambda : G_{\widetilde{Z_1}} \to G_{\widetilde{Z_2}}$ is an isomorphism from the subgroup of G^{V_1} fixing $\widetilde{Z_1}$ to the subgroup of G^{V_2} fixing $\widetilde{Z_2}$, such that f is λ-equivariant. To this end, for $i = 1,2$, we let $\widetilde{Z_i}$ be the connected component of $(\pi^{V_i})^{-1}(V) \subset \widetilde{V_i}$ that contains Range (ϕ_{ξ_i}). Suppose $V \subset \phi^{-1}(U)$ for some local chart U on X. We pick base-point structures $\underline{q} = (q, W, \hat{q})$ and $\underline{p} = (p, U, \hat{p})$ such that $\Phi : (Y,\underline{q}) \to (X,\underline{p})$. Now for $i = 1,2$ we set $\hat{q}_i = \phi_{\xi_i}(\hat{q})$, and denote by $\Pi_i : (\widetilde{Z_i}, \hat{q}_i) \to (V,\underline{q})$ the canonical covering map associated to the isomorphism $V \cong \widetilde{Z_i}/G_{\widetilde{Z_i}}$ which is defined by $\phi_{\xi_i}^{-1}$ at the base-point structures. Then we have $(\Pi_1)_*(\pi_1(\widetilde{Z_1},\hat{q_1})) = (\Phi|_V)_*^{-1}(\pi_1(\widehat{U},\hat{p})) = (\Pi_2)_*(\pi_1(\widetilde{Z_2},\hat{q_2}))$ in $\pi_1(V,\underline{q})$. By Lemma 2.4.6, there is a unique $f : (\widetilde{Z_1},\hat{q_1}) \to (\widetilde{Z_2},\hat{q_2})$ satisfying $\Pi_1 = \Pi_2 \circ f$. Furthermore, by Proposition 2.4.8, there is a natural isomorphism $\lambda : G_{\widetilde{Z_1}} \to G_{\widetilde{Z_2}}$ such that f is λ-equivariant. We define $\chi_V^W(\xi_1,\xi_2) = (f,\lambda)$. It is easy to see that (1) $\chi_V^W(\xi_1,\xi_2)$ is independent of the choices on $U, \underline{q}$ and $\underline{p}$, (2) $\chi_V^W(\xi_1,\xi_2) = (\phi_{\xi_2},\lambda_{\xi_2}) \circ (\phi_{\xi_1}^{-1},\lambda_{\xi_1}^{-1})$ when restricted to the domain of the latter, and (3) $\chi_V^W(\xi_1,\xi_2) = \chi_V^W(g\circ\xi_1, \lambda(g)\circ\xi_2)$, $\forall g \in G_{\widetilde{Z_1}}$.

Secondly, in $\bigsqcup_{W\in\mathcal{N}(V_1)\cap\mathcal{N}(V_2), W\subset V} T(W,V_1)\times T(W,V_2)$ we introduce an equivalence relation $\sim$ generated as follows: (a) for any $(\xi_1,\xi_2), (\eta_1,\eta_2) \in T(W,V_1)\times T(W,V_2)$, $(\xi_1,\xi_2) \sim (\eta_1,\eta_2)$ if $\eta_1 = g\circ\xi_1, \eta_2 = \lambda(g)\circ\xi_2$ for some $g \in$ Domain (λ), where λ is given in $\chi_V^W(\xi_1,\xi_2) = (f,\lambda)$, and (b) for any $(\xi_1,\xi_2) \in T(W,V_1)\times T(W,V_2)$, $(\eta_1,\eta_2) \in T(W',V_1)\times T(W',V_2)$ where $W' \subset W$, $(\xi_1,\xi_2) \sim (\eta_1,\eta_2)$ if $\eta_1 = \xi_1\circ\epsilon, \eta_2 = \xi_2\circ\epsilon$ for some $\epsilon \in T(W',W)$. We define $T_V(V_1,V_2) = \bigsqcup_{W\in\mathcal{N}(V_1)\cap\mathcal{N}(V_2), W\subset V} T(W,V_1)\times T(W,V_2)/\sim$, and for each $\xi = [(\xi_1,\xi_2)]$ where $(\xi_1,\xi_2) \in T(W,V_1)\times T(W,V_2)$, define $(\phi_\xi,\lambda_\xi) = \chi_V^W(\xi_1,\xi_2)$.

Finally, we let $T(V_1,V_2)$ be the disjoint union of $T_V(V_1,V_2)$ for all components V of $V_1\cap V_2$. Note that with this definition, $T(V,V)$, $\forall V \in \mathcal{V}(\Phi)$, is naturally identified with G^V.

The composition is defined as follows. Let $\xi \in T(V_1,V_2)$, $\eta \in T(V_2,V_3)$, and $x \in \phi_\xi^{-1}(\text{Domain }(\phi_\eta))$. We pick a $W \in \mathcal{N}(V_1)\cap\mathcal{N}(V_2)\cap\mathcal{N}(V_3)$ such that $\pi^{V_1}(x) \in W$. Then we can write $\xi = [(\xi_1, \xi_2)]$ for some $(\xi_1, \xi_2) \in T(W, V_1) \times T(W, V_2)$, $\eta = [(\eta_2, \eta_3)]$ for some $(\eta_2, \eta_3) \in T(W, V_2) \times T(W, V_3)$ such that Range (ϕ_{ξ_2}) = Range (ϕ_{η_2}). The last condition allows us to modify (ξ_1, ξ_2) without changing its class $[(\xi_1, \xi_2)]$ but to further satisfy $\xi_2 = \eta_2$. With these arranged, we define $\eta \circ \xi(x) = [(\xi_1, \eta_3)]$. As for the inverse, we simply define $\xi^{-1} = [(\xi_2, \xi_1)]$ if $\xi = [(\xi_1, \xi_2)]$.

The verification that $\mathcal{V}(\Phi) = \{(\widetilde{V}, G^V, \pi^V)\}$, $\mathcal{T}(\Phi) = \{T(V_1, V_2)\}$ satisfy the conditions in Proposition 2.1.1 of [3] is straightforward, which is left to the reader. Thus they define an orbispace structure on Y by Proposition 2.1.1 of [3], which is clearly 'equivalent' to the original one on Y in the sense of Remark 2.1.2 (5) in [3].

Finally, given any cover $\{V_\alpha\} \subset \mathcal{V}(\Phi)$ of Y, where $V_\alpha \subset \phi^{-1}(U_\alpha)$ for some local chart U_α on X, the existence of a homomorphism $(\{f_\alpha\}, \{\lambda_{\beta\alpha}\}) : \Gamma\{V_\alpha\} \to \Gamma\{U_\alpha\}$, whose equivalence class is the given map Φ and in which $(f_\alpha, \lambda_\alpha) = (f_{V_\alpha}, \lambda_{V_\alpha}) : (\widetilde{V_\alpha}, G^{V_\alpha}) \to (\widehat{U_\alpha}, G_{U_\alpha})$, is the content of Lemma 3.1.3 in [3].

□

Let Γ^0 be an orbispace structure on X, which is given by the data $\{(\widehat{U}_i^0, G^0_{U_i}, \pi^0_{U_i})\}$ and $\mathcal{T}^0 = \{T^0(U_i, U_j)\}$ as described in Proposition 2.1.1 of [3]. Suppose for each U_i, there is a triple $(\widehat{U_i}, G_{U_i}, \pi_{U_i})$, where $\widehat{U_i}$ is not connected in general, such that $\pi_{U_i} : \widehat{U_i} \to U_i$ induces a homeomorphism $\widehat{U_i}/G_{U_i} \cong U_i$, and $(\widehat{U}_i^0, G^0_{U_i}, \pi^0_{U_i})$ is obtained from $(\widehat{U_i}, G_{U_i}, \pi_{U_i})$ by restricting to a connected component $\widehat{U}_i^0$ of $\widehat{U_i}$. Then there is canonically an orbispace structure Γ on X, with $\Gamma^0 \subset \Gamma$ being an equivalence, and $\{(\widehat{U_i}, G_{U_i}, \pi_{U_i})\}$ being the atlas of local charts. As for the set $\mathcal{T} = \{T(U_i, U_j)\}$ of Γ, each $T(U_i, U_j)$ is the orbit space of $G_{U_i} \times T^0(U_i, U_j) \times G_{U_j}$ modulo the action $h \cdot (g_i, \xi_0, g_j) = (hg_i, \xi_0, \lambda_{\xi_0}(h) g_j)$, $\forall h \in$ Domain (ϕ_{ξ_0}). (It is instructive to think (g_i, ξ_0, g_j) as $g_j^{-1} \circ \xi_0 \circ g_i$.)

Moreover, suppose $(\{f^0_\alpha\}, \{\rho^0_{\beta\alpha}\}) : \Gamma^0\{U_\alpha\} \to \Gamma\{U'_{\alpha'}\}$ is a homomorphism of groupoids, where each $\rho^0_\alpha : G^0_{U_\alpha} \to G_{U'_\alpha}$ is assumed to be injective. Then by replacing Γ^0 with Γ on X, we may canonically replace $(\{f^0_\alpha\}, \{\rho^0_{\beta\alpha}\})$ with a homomorphism $(\{f_\alpha\}, \{\rho_{\beta\alpha}\}) : \Gamma\{U_\alpha\} \to \Gamma\{U'_{\alpha'}\}$ such that (1) each $\rho_\alpha : G_{U_\alpha} \to G_{U'_\alpha}$ is isomorphic, and (2) $f^0_\alpha, \rho^0_{\beta\alpha}$ are obtained by restricting $f_\alpha, \rho_{\beta\alpha}$ to $\widehat{U}^0_\alpha, T^0(U_\alpha, U_\beta)$ respectively. It is done as follows. We define $\widehat{U_\alpha} = (G_{U'_\alpha} \times \widehat{U}^0_\alpha)/G^0_{U_\alpha}$ where the action is given by $g^0 \cdot (g', x) = (g' \rho^0_\alpha(g^0)^{-1}, g^0 \cdot x)$, we define $G_{U_\alpha} = G_{U'_\alpha}$ with a left action on $\widehat{U_\alpha}$ induced by $g \cdot (g', x) = (gg', x)$, and we define $\pi_{U_\alpha} : \widehat{U_\alpha} \to U_\alpha$ by $[(g', x)] \mapsto \pi^0_{U_\alpha}(x)$, which induces $\widehat{U_\alpha}/G_{U_\alpha} \cong U_\alpha$. Note that $(\widehat{U}^0_\alpha, G^0_{U_\alpha}, \pi^0_{U_\alpha})$ is obtained from $(\widehat{U_\alpha}, G_{U_\alpha}, \pi_{U_\alpha})$ under the mapping $(x, g^0) \mapsto ([(1, x)], \rho^0_\alpha(g^0))$. The pair of maps $(f_\alpha, \rho_\alpha) : (\widehat{U_\alpha}, G_{U_\alpha}) \to (\widehat{U'_\alpha}, G_{U'_\alpha})$ is defined by $f_\alpha([(g', x)]) = g' \cdot f^0_\alpha(x)$ and $\rho_\alpha = Id$. Its restriction to $(\widehat{U}^0_\alpha, G^0_{U_\alpha})$ is clearly $(f^0_\alpha, \rho^0_\alpha)$. The mappings $\rho_{\beta\alpha} : T(U_\alpha, U_\beta) \to T(U'_\alpha, U'_\beta)$ are defined by $\rho_{\beta\alpha}(g_\beta^{-1} \circ \xi_0 \circ g_\alpha) = \rho_\beta(g_\beta)^{-1} \circ \rho^0_{\beta\alpha}(\xi_0) \circ \rho_\alpha(g_\alpha) = g_\beta^{-1} \circ \rho^0_{\beta\alpha}(\xi_0) \circ g_\alpha$, which clearly satisfy $\rho_{\beta\alpha}|_{T^0(U_\alpha, U_\beta)} = \rho^0_{\beta\alpha}$. One can check directly that $(\{f_\alpha\}, \{\rho_{\beta\alpha}\})$ is indeed a homomorphism. We leave the details to the reader.

Proof of Proposition 1.5

By Lemma 3.1.2, we may assume that the original orbispace structure on Y is given by $(\mathcal{V}(\Phi), \mathcal{T}(\Phi))$ for simplicity without loss of generality.

Let $\{V_\alpha\}$ be the subset of $\mathcal{V}(\Phi)$ which consists of connected components of $\phi^{-1}(U)$ for all local chart U on X. For each index α, we pick a U_α such that V_α is a component of $\phi^{-1}(U_\alpha)$. By Lemma 3.1.2, there is a homomorphism, denoted by $(\{f^0_\alpha\},\{\rho^0_{\beta\alpha}\}):\Gamma^0\{V_\alpha\}\rightarrow\Gamma\{U_\alpha\}$, whose equivalence class is the given map Φ. We further apply the trick described above to replace $(\{f^0_\alpha\},\{\rho^0_{\beta\alpha}\})$ by a homomorphism $(\{f_\alpha\},\{\rho_{\beta\alpha}\}):\Gamma\{V_\alpha\}\rightarrow\Gamma\{U_\alpha\}$ such that each ρ_α is an isomorphism and $(\{f^0_\alpha\},\{\rho^0_{\beta\alpha}\})$ is the restriction of $(\{f_\alpha\},\{\rho_{\beta\alpha}\})$ to the sub-groupoid $\Gamma^0\{V_\alpha\}$ of $\Gamma\{V_\alpha\}$.

Note that for each index α there may be more than one U such that V_α is a connected component of $\phi^{-1}(U)$. We shall modify $(\{f_\alpha\},\{\rho_{\beta\alpha}\})$ by adding all such U to $\{U_\alpha\}$ and by allowing the V_α's in $\{V_\alpha\}$ to repeat. The resulting homomorphism may be expressed as

$$(\{V_{a,i}|i\in I_a\},\{U_a\},\{f_{a,i}\},\{\rho_{(b,j)(a,i)}\}) \tag{3.1.5}$$

where $\{U_a\}$ is the set of local charts on X such that $\phi^{-1}(U_a)\neq\emptyset$, and $\{V_{a,i}|i\in I_a\}$ is the set of connected components of $\phi^{-1}(U_a)$. Note that $V_{a,i},V_{b,j}$ may be identical even if $a\neq b$. Finally, we wish to emphasize that each $\rho_{(a,i)}:G_{V_{a,i}}\rightarrow G_{U_a}$ is an isomorphism.

With these preparations, we shall construct a canonical orbispace structure on the mapping cylinder M_ϕ as follows.

First of all, we specify the atlas of local charts on M_ϕ. Observe that the atlas of local charts on X is the disjoint union of $\{U_a\}$ with a set $\{U_i\}$ where each $\phi^{-1}(U_i)=\emptyset$. If we set $\phi_a=\phi|_{\phi^{-1}(U_a)}$, then it is readily seen that $\mathcal{W}=\{M_{\phi_a}\}\cup\{U_i\}$ is a cover of the mapping cylinder M_ϕ. To each $W\in\mathcal{W}$, we assign a $(\widehat{W},G_W,\pi_W)$ such that $\pi_W:\widehat{W}\rightarrow M_\phi$ induces a homeomorphism $\widehat{W}/G_W\cong W$ as follows. If $W=U_i$ for some index i, we simply put $(\widehat{W},G_W,\pi_W)=(\widehat{U_i},G_{U_i},\pi_{U_i})$. If $W=M_{\phi_a}$ for some index a, we define $\widehat{W}=M_{f_a}$, where $f_a=\sqcup_{i\in I_a}f_{a,i}:\bigsqcup_{i\in I_a}\widehat{V_{a,i}}\rightarrow\widehat{U_a}$ (cf. (3.1.5)), and define $G_W=G_{U_a}$ with the action on $\widehat{W}$ given by the extension of the one on $\widehat{U_a}$ by $g\cdot[y,t]=[\rho^{-1}_{(a,i)}(g)\cdot y,t]$, $\forall g\in G_{U_a},[y,t]\in M_{f_{a,i}}\subset\widehat{W}$, and define π_W by $[y,t]\mapsto[\pi_{V_{a,i}}(y),t],\forall[y,t]\in M_{f_{a,i}}$ and $x\mapsto\pi_{U_a}(x),\forall x\in\widehat{U_a}$.

Secondly, we define the set $\mathcal{T}=\{T(W_1,W_2)\mid W_1,W_2\in\mathcal{W}$, s.t. $W_1\cap W_2\neq\emptyset\}$. We first look at the most complicated case where $W_1=M_{\phi_a}$, $W_2=M_{\phi_b}$ for some indexes a,b. In this case it is important to observe that the set of connected components of $U_a\cap U_b$ is a subset $\{U_c\mid c\in I_{a,b}\}\subset\{U_a\}$, and correspondingly the set of connected components of $W_1\cap W_2$ is $\{W_c=M_{\phi_c}\mid c\in I_{a,b}\}$. As a set, we shall define $T_{W_c}(W_1,W_2)=T_{U_c}(U_a,U_b)$, and define $T(W_1,W_2)=\bigsqcup_{c\in I_{a,b}}T_{W_c}(W_1,W_2)=\bigsqcup_{c\in I_{a,b}}T_{U_c}(U_a,U_b)=T(U_a,U_b)$. However, each $\xi\in T_{W_c}(W_1,W_2)$ is assigned with a pair $(\bar{\phi}_\xi,\bar{\lambda}_\xi)$ as follows. First, we regard $\xi\in T_{U_c}(U_a,U_b)$ and write $\xi=\xi_2\circ\xi_1^{-1}$ for some $\xi_1\in T(U_c,U_a)$, $\xi_2\in T(U_c,U_b)$. Second, set $\{\eta_\alpha\mid\alpha\in I_{\xi_1}\}=\{\eta\in T(V_{c,s},V_{a,i})\mid s\in I_c,i\in I_a,$ s.t. $\rho_{(a,i)(c,s)}(\eta)=\xi_1\}$. It is a routine exercise to check that (1) both $\eta_\alpha\mapsto$ Domain (ϕ_{η_α}), $\eta_\alpha\mapsto$ Range (ϕ_{η_α}) are bijections, and (2) $\bigsqcup_{\alpha\in I_{\xi_1}}$ Domain $(\phi_{\eta_\alpha})=\bigsqcup_{s\in I_c}\widehat{V_{c,s}}$ and $\bigsqcup_{\alpha\in I_{\xi_1}}$ Range $(\phi_{\eta_\alpha})=f_a^{-1}($Domain $(\phi_{\xi_1}))\subset\bigsqcup_{i\in I_a}\widehat{V_{a,i}}$. It then follows that $\sqcup_{\alpha\in I_{\xi_1}}\phi_{\eta_\alpha}:\bigsqcup_{s\in I_c}\widehat{V_{c,s}}\rightarrow f_a^{-1}($Domain $(\phi_{\xi_1}))$ is a homeomorphism which satisfies $f_a\circ(\sqcup_{\alpha\in I_\xi}\phi_{\eta_\alpha})=\phi_{\xi_1}\circ f_c$. We define $\bar{\phi}_{\xi_1}:M_{f_c}\rightarrow M_{f_a}$ to be the corresponding open embedding between the mapping cylinders. Clearly the range of $\bar{\phi}_{\xi_1}$ is a connected component of $\pi^{-1}_{W_1}(W_c)$, and $\bar{\phi}_{\xi_1}$ is equivariant with respect to $\lambda_{\xi_1}:G_{U_c}=G_{W_c}\rightarrow G_{U_a}=G_{W_1}$. Similarly, one has an open embedding $\bar{\phi}_{\xi_2}:M_{f_c}\rightarrow M_{f_b}$ which is λ_{ξ_2}-equivariant. Finally, we define

$(\bar{\phi}_\xi, \bar{\lambda}_\xi) = (\bar{\phi}_{\xi_2}, \lambda_{\xi_2}) \circ (\bar{\phi}_{\xi_1}^{-1}, \lambda_{\xi_1}^{-1})$, which is independent of the choices on ξ_1, ξ_2. The definition for the remaining cases is obvious, so we leave the details to the reader.

Finally, the composition and inverse for the elements in $\mathcal{T}$ are to be inherited directly from those in the orbispace structure on X under the natural identification described above. By Proposition 2.1.1 in [3], $(\mathcal{W}, \mathcal{T})$ defines an orbispace structure on M_ϕ.

It remains to define the maps of orbispaces $i : Y \to M_\phi$, $j : X \to M_\phi$ and $r : M_\phi \to X$ with the claimed properties. First, $i : Y \to M_\phi$ is defined by the homomorphism

(3.1.6) $$(\{V_{a,i} | i \in I_a\}, \{M_{\phi_a}\}, \{i_{a,i}\}, \{\rho_{(b,j)(a,i)}\})$$

where $i_{a,i} : \widehat{V_{a,i}} \to M_{f_a}$ is $y \mapsto [y, 0]$. It realizes Y as a subspace of M_ϕ essentially because by Lemma 3.1.2, the orbispace structure given by $\mathcal{V}(\Phi), \mathcal{T}(\Phi)$ is equivalent to the original one on Y in the sense of Remark 2.1.2 (5) in [3]. Second, $j : X \to M_\phi$ is defined by $(\{j_\alpha\}, \{\delta_{\beta\alpha}\}) : \Gamma\{U_\alpha\} \to \{W_\alpha\}$, which is clearly a subspace, where (a) $W_\alpha = U_i$ if $U_\alpha = U_i$, $W_\alpha = M_{\phi_a}$ if $U_\alpha = U_a$, (b) $j_\alpha = Id$ if $U_\alpha = U_i$ and $j_\alpha = j_a : \widehat{U_a} \to M_{f_a}$ if $U_\alpha = U_a$, and (c) each $\delta_{\beta\alpha}$ is the identity map under the natural identification.

Last, the retraction $r : M_\phi \to X$ is defined by $(\{r_\alpha\}, \{\delta_{\beta\alpha}\}) : \Gamma\{W_\alpha\} \to \Gamma\{U_\alpha\}$, where (a) $U_\alpha = U_i$ if $W_\alpha = U_i$, $U_\alpha = U_a$ if $W_\alpha = M_{\phi_a}$, (b) $r_\alpha = Id$ if $W_\alpha = U_i$ and $r_\alpha = r_a : M_{f_a} \to \widehat{U_a}$ is the usual retraction if $W_\alpha = M_{\phi_a}$, and (c) each $\delta_{\beta\alpha}$ is the identity map. Clearly $r \circ j = Id_X$. On the other hand, there is a canonical homotopy between $j \circ r$ and Id_{M_ϕ}, defined by $(\{H_\alpha\}, \{\delta_{\beta\alpha}\}) : \Gamma\{W_\alpha \times [0,1]\} \to \Gamma\{U_\alpha\}$, where $H_\alpha = Id$ if $W_\alpha = U_i$, and H_α is the usual homotopy between $j_a \circ r_a$ and $Id_{M_{f_a}}$ if $W_\alpha = M_{\phi_a}$. Hence $j : X \to M_\phi$ is a strong deformation retract by $r : M_\phi \to X$. Finally, we note that $\Phi = r \circ i$.

□

3.2 Orbispaces via attaching cells of isotropy

In this subsection we apply the mapping cylinder construction in the preceding subsection to a special case where $Y = S^{k-1}(G)$, $k \geq 1$, and make the meaning of 'attaching a k-cell of isotropy type G to an orbispace' mathematically precise. To this end, we have to impose further conditions on both the orbispace X and the attaching map $\Phi : S^{k-1}(G) \to X$, in order to deal with two additional issues that are involved in the process.

The first one is how to construct an orbispace by collapsing a subspace to a point. Suppose $A \subset X$ is a closed, connected subspace of an orbispace X. We denote by X/A the topological space obtained by collapsing A in the underlying space of X to a point and by $* \in X/A$ the image of A under the canonical projection $X \to X/A$.

Lemma 3.2.1 *Suppose A is further contained in a local chart on X. Then there is a canonical orbispace structure on X/A, and a map of orbispaces $\pi : X \to X/A$ covering the canonical projection $X \to X/A$ between the underlying spaces, such that the restriction of π to the open subspace $X \setminus A$ is an isomorphism of orbispaces onto $(X/A) \setminus \{*\}$.*

Proof Let $\mathcal{U}_1$ be the set of local charts U on X such that $U \cap A = \emptyset$, and $\mathcal{U}_2$ be the set of local charts U on X such that $A \subset U$. By the assumption, $\mathcal{U}_2 \neq \emptyset$.

We shall define a canonical orbispace structure on X/A, where the atlas of local charts $\mathcal{V}$ consists of connected open subsets V such that either $V \in \mathcal{U}_1$ as a subset of X, or $V = U/A$ for some $U \in \mathcal{U}_2$. In the former case, we let $(\widehat{V}, G_V, \pi_V)$ be the one in the orbispace structure on X, while in the latter case, we let $\widehat{V}$ be the space obtained by collapsing each connected component of $\pi_U^{-1}(A)$ in $\widehat{U}$ to a point, let $G_V = G_U$ with the induced action on $\widehat{V}$, and let $\pi_V : \widehat{V} \to V$ be the map induced by π_U. The set $\mathcal{T} = \{T(V_1, V_2) \mid V_1, V_2 \in \mathcal{V}, \text{ s.t. } V_1 \cap V_2 \neq \emptyset\}$ is identical to the one in the orbispace structure on X along with the assignment $\xi \mapsto \phi_\xi$ and the composition and inverse, except for the case when $\xi \in T_V(V_1, V_2) = T_U(U_1, U_2)$, where $V_i = U_i/A$, $i = 1, 2$, and $V = U/A$, we instead assign ξ with $\bar{\phi}_\xi$, whose domain is the space obtained by collapsing each connected component of $\pi_{U_1}^{-1}(A)$ in Domain (ϕ_ξ) to a point, whose range is the space obtained by collapsing each connected component of $\pi_{U_2}^{-1}(A)$ in Range (ϕ_ξ) to a point, and $\bar{\phi}_\xi$ is the map induced by ϕ_ξ. It is clear that $\mathcal{V}, \mathcal{T}$ define an orbispace structure on X/A.

The map $\pi : X \to X/A$ is defined by $(\{\pi_\alpha\}, \{\delta_{\beta\alpha}\}) : \Gamma\{U_\alpha\} \to \Gamma\{V_\alpha\}$, where (a) $\{U_\alpha\} = \mathcal{U}_1 \cup \mathcal{U}_2$, which is a cover of X since A is closed, and $\{V_\alpha\} = \mathcal{V}$, (b) $\pi_\alpha = Id$ if $U_\alpha \in \mathcal{U}_1$, and π_α is the canonical projection if $U_\alpha \in \mathcal{U}_2$, and (c) each $\delta_{\beta\alpha}$ is the identity map, which is clearly an isomorphism onto $(X/A) \setminus \{*\}$ when restricted to the open subspace $X \setminus A$.

□

In order to address the second preparatory issue, we note that there is a natural map of orbispaces $k : Y \times I \to M_\phi$, which is defined by the homomorphism

$$(\{V_{a,i} \times I | i \in I_a\}, \{M_{\phi_a}\}, \{k_{a,i}\}, \{\rho_{(b,j)(a,i)}\}), \tag{3.2.1}$$

where $k_{a,i} : \widehat{V_{a,i}} \times I \to M_{f_a}$ is $(y, t) \mapsto [y, t]$, cf. (3.1.5), (3.1.6).

Lemma 3.2.2 *Let $\Phi : Y \to X$ be any map of orbispaces whose mapping cylinder M_ϕ is defined. Suppose both Y, X are compact and Hausdorff, and for any $V \in \mathcal{V}(\Phi)$, $\widetilde{V}$ is locally compact, Hausdorff, and the map $\pi^V : \widetilde{V} \to V$ is proper. Then for any maps of orbispaces $\Psi : X \to X'$, $\Upsilon : Y \times I \to X'$ such that $\Psi \circ \Phi = \Upsilon|_{Y \times \{1\}}$, there is a map $\Xi : M_\phi \to X'$ which satisfies $\Xi \circ j = \Psi$ and $\Xi \circ k = \Upsilon$.*

Proof We represent $\Psi : X \to X'$ by a homomorphism $\tau = (\{\psi_\alpha\}, \{\lambda_{\beta\alpha}\}) : \Gamma\{U_\alpha\} \to \Gamma\{U'_{\alpha'}\}$, where $\{U_\alpha\}$ is a finite cover of X such that each U_α is admissible as defined in §3.2 of [3], and each ψ_α can be extended over the closure of $\widehat{U_\alpha}$. This is possible because X is compact and Hausdorff. For each U_α, let $\{V_{\alpha,i} \mid i \in I_\alpha\}$ be the set of connected components of $\phi^{-1}(U_\alpha)$. We assign U_α to each $V_{\alpha,i}$, then by Lemma 3.1.2, there is a homomorphism $\sigma = (\{f_{\alpha,i}\}, \{\rho_{(\beta,j)(\alpha,i)}\}) : \Gamma\{V_{\alpha,i}\} \to \Gamma\{U_\alpha\}$, whose equivalence class is the map $\Phi : Y \to X$. The composition $\epsilon = (\{\varphi_{\alpha,i}\}, \{\delta_{(\beta,j)(\alpha,i)}\})$, where $\varphi_{\alpha,i} = \psi_\alpha \circ f_{\alpha,i}$ and $\delta_{(\beta,j)(\alpha,i)} = \lambda_{\beta\alpha} \circ \rho_{(\beta,j)(\alpha,i)}$, is a representative of $\Psi \circ \Phi$, which is admissible as defined in §3.2 of [3] by the assumptions made on Y, $\mathcal{V}(\Phi)$ and τ.

On the other hand, we represent $\Upsilon : Y \times I \to X'$ by a homomorphism $\kappa = (\{h_a\}, \{\eta_{ba}\}) : \Gamma\{V_a \times I_a\} \to \Gamma\{U'_{a'}\}$, where $a \in \Lambda$, which may be made admissible as defined in §3.2 of [3] by the assumptions we have on Y and $\mathcal{V}(\Phi)$. We may require $\#\Lambda < \infty$ since Y is compact. Set $\Lambda_0 = \{a \in \Lambda | 1 \in I_a\}$, and we assume without loss of generality that $\overline{I_a} \subset [0, 1)$, $\forall a \in \Lambda \setminus \Lambda_0$. Note that $\kappa(1) = (\{h_a(\cdot, 1)\}, \{\eta_{ba}\}) : \Gamma\{V_a\} \to \Gamma\{U'_{a'}\}$, where $a \in \Lambda_0$, also represents $\Upsilon|_{Y \times \{1\}} = \Psi \circ \Phi$. By passing to an induced homomorphism of κ, we may assume that $\kappa(1)$ is induced by ϵ via some $\bar{\gamma} = (\theta, \{\xi_a\}, \{\xi'_a\})$. By Lemma 3.2.2 in [3], there is a local homeomorphism $\phi_{\bar{\gamma}}$ from an open neighborhood of $\kappa(1)$ onto an open neighborhood of ϵ, sending $\kappa(1)$ to ϵ. Since Λ is finite and $\overline{I_a} \subset [0,1)$, $\forall a \in \Lambda \setminus \Lambda_0$, there is a $t_0 \in [0,1)$, $t_0 \in I \setminus \overline{I_a}, \forall a \in \Lambda \setminus \Lambda_0$, such that for any $t \in (t_0, 1]$, $\kappa(t) = (\{h_a(\cdot, t)\}, \{\eta_{ba}\}) : \Gamma\{V_a\} \to \Gamma\{U'_{a'}\}$, where $a \in \Lambda_0$, lies in the domain of $\phi_{\bar{\gamma}}$. We write $\phi_{\bar{\gamma}}(\kappa(t)) = \epsilon(t) = (\{\varphi^{(t)}_{\alpha,i}\}, \{\delta_{(\beta,j)(\alpha,i)}\})$, $t \in (t_0, 1]$. Note that $\varphi^{(1)}_{\alpha,i} = \psi_\alpha \circ f_{\alpha,i}$.

Introduce the following notation: Let f be a continuous map. For any $t_0 \in [0,1)$, we denote by $M_f(t_0)$ the open subset $M_f \setminus \{[y,t] \mid t \in [0,t_0]\}$ in the mapping cylinder M_f of f. Then there is a homomorphism $\varsigma = (\{\chi_\alpha\}, \{\lambda_{\beta\alpha}\}) : \Gamma\{W_\alpha\} \to \Gamma\{U'_{\alpha'}\}$, where either $W_\alpha = U_\alpha$ and $\chi_\alpha = \psi_\alpha$, or $W_\alpha = M_{\phi_\alpha}(t_0)$ and $\chi_\alpha = \psi_\alpha$ on $j_\alpha(\widehat{U_\alpha}) \subset M_{f_\alpha}(t_0) = \widehat{W_\alpha}$, $\chi_\alpha([y,t]) = \varphi^{(t)}_{\alpha,i}(y)$, $\forall (y,t) \in \widehat{V_{\alpha,i}} \times (t_0, 1], i \in I_\alpha$. On the other hand, consider the restriction of κ to $Y \times (I \setminus \{1\})$, which is still denoted by $\kappa = (\{h_a\}, \{\eta_{ba}\}) : \Gamma\{V_a \times I_a\} \to \Gamma\{U'_{a'}\}$ for the sake of simplicity. We can join ς and κ via $\bar{\gamma} = (\theta, \{\xi_a\}, \{\xi'_a\})$ to construct a homomorphism $\varsigma \cup_{\bar{\gamma}} \kappa$ as follows. We add a set of mappings $\lambda_{\alpha a} : T(V_a \times I_a, W_\alpha) \to T(U'_a, U'_\alpha)$, where $(V_a \times I_a) \cap W_\alpha \neq \emptyset$, to ς, κ. Note that any $\xi \in T(V_a \times I_a, W_\alpha)$ may be regarded as an element of $T(V_a, V_{\alpha,i})$ for some $i \in I_\alpha$. In this case we define (cf. (3.2.8) in §3.2 of [3])

$$\lambda_{\alpha a}(\xi) = \delta_{(\alpha,i)\theta(a)}(\xi \circ \xi_a^{-1}) \circ \xi'_a(x), \ \forall x \in h_a(\text{Domain } (\phi_\xi)). \tag{3.2.2}$$

As seen in the proof of Lemma 3.2.2 in [3], $\varsigma \cup_{\bar{\gamma}} \kappa$ is indeed a homomorphism, whose equivalence class is a map from the orbispace M_ϕ to X'. We define $\Xi = [\varsigma \cup_{\bar{\gamma}} \kappa]$, which clearly satisfies $\Xi \circ j = \Psi$ and $\Xi \circ k = \Upsilon$.

□

We remark that the map Ξ in the preceding lemma may not be uniquely determined, but the ambiguity is caused only by the different choices of $\gamma = [\bar{\gamma}] \in \Gamma_{\epsilon\kappa(1)}$ (cf. Lemma 3.1.1 in [3]). If we work with the based version, then $\#\Gamma_{\epsilon\kappa(1)} = 1$ when Y is connected, and Ξ is uniquely determined in this case.

Now we are ready to describe the precise meaning of attaching a k-cell $D^k(G)$ of isotropy type G to an orbispace X via a map $\Phi : S^{k-1}(G) \to X$. First of all, we list the additional conditions we need to impose on the orbispace X and the attaching map $\Phi : S^{k-1}(G) \to X$, where $k \geq 1$:

- X is compact and Hausdorff.
- The orbispace structure on S^{k-1} given by $\mathcal{V}(\Phi), \mathcal{T}(\Phi)$ is contained in the standard one on $S^{k-1}(G)$.

Note that the last condition means that for each $V \in \mathcal{V}(\Phi)$, $\widetilde{V} = V$, $G^V = G$ which acts on $\widetilde{V}$ trivially, and $\pi^V : \widetilde{V} \to V$ is the identity map. Moreover, each $T(V_1, V_2) \in \mathcal{T}(\Phi)$ is a disjoint union of a number of copies of G which is naturally labeled by the set of connected components of $V_1 \cap V_2$, and for each element $\xi \in T(V_1, V_2)$, ϕ_ξ is the identity map, and $\lambda_\xi = Ad(\xi) : G \to G$.

Recall that the mapping cone of a continuous map $\phi : Y \to X$, denoted by C_ϕ, is the space $M_\phi / i(Y)$ obtained by collapsing the subspace $i : Y \to M_\phi$ to a point. When $Y = S^{k-1}$, C_ϕ is the result of "attaching a k-cell D^k to X via the map $\phi : S^{k-1} = \partial D^k \to X$". The interior of the k-cell in C_ϕ is the image of the open subset $D_0 = \{[y,t] \mid y \in S^{k-1}, t \in [0,1)\} \subset M_\phi$ under the canonical projection $M_\phi \to C_\phi = M_\phi / i(S^{k-1})$.

Proposition 3.2.3 *With the preceding understood, we assert that*

(1) *There is a canonical orbispace structure on C_ϕ such that* (a) *there is a canonical map of orbispaces $\pi : M_\phi \rightarrow C_\phi$ which is an open embedding when restricted to $M_\phi \setminus i(S^{k-1}(G))$, and* (b) *the interior of the attached k-cell in C_ϕ is an open k-cell of isotropy type G as a subspace of the orbispace C_ϕ.*
(2) *The homotopy type of the orbispace C_ϕ depends only on the homotopy class of the attaching map Φ.*
(3) *The canonical embedding of the orbispace X into C_ϕ induced by $j : X \rightarrow M_\phi$ is a cofibration, i.e., it has the homotopy extension property with respect to all orbispaces.*
(4) *A map $\Psi : X \rightarrow X'$ can be extended over the attached k-cell of isotropy type G to C_ϕ iff $\Psi \circ \Phi : S^{k-1}(G) \rightarrow X'$ is null-homotopic, and such an extension of Ψ is given by a null-homotopy of $\Psi \circ \Phi$. Moreover, suppose F is a homotopy between Ψ_1 and Ψ_2, which are extended to C_ϕ by the null-homotopies h_1, h_2 of $\Psi_1 \circ \Phi$, $\Psi_2 \circ \Phi$ respectively. Then F can be extended to $C_\phi \times [0,1]$ to a homotopy between the corresponding extensions of Ψ_1 and Ψ_2 iff the null-homotopies h_1, h_2 are homotopic via a homotopy whose restriction to $S^{k-1}(G) \times [0,1]$ is $F \circ (\Phi \times Id)$.*

Proof (1) The canonical orbispace structure on C_ϕ will be the one obtained from the canonical orbispace structure on the mapping cylinder M_ϕ by collapsing the subspace $i(S^{k-1}(G))$ to a point. To this end, we need to verify the hypothesis in Lemma 3.2.1. The case when $k = 1$ requires a separate, but similar argument because $i(S^{k-1}(G))$ is not connected. We shall only consider the cases when $k \geq 2$, the remaining case is left to the reader for simplicity.

In order to apply Lemma 3.2.1, it suffices to show that we may add the local chart $(\widehat{D_0}, G_{D_0}, \pi_{D_0}) = (D_0, G, \pi_0)$, where $D_0 = \{[y,t] \mid y \in S^{k-1}, t \in [0,1)\} \subset M_\phi$, G acts on D_0 trivially, and $\pi_0 : D_0 \rightarrow D_0$ is the identity map, to the canonical orbispace structure on M_ϕ constructed in Proposition 1.5, so that with respect to the new orbispace structure on M_ϕ, which contains the original one hence equivalent, the subspace $i(S^{k-1}(G))$ is contained in a local chart, i.e., $(\widehat{D_0}, G_{D_0}, \pi_{D_0}) = (D_0, G, \pi_0)$. The condition that the orbispace structure on S^{k-1} given by $\mathcal{V}(\Phi), \mathcal{T}(\Phi)$ is contained in the standard one on $S^{k-1}(G)$ guarantees this. We shall continue to use the notations in the proof of Proposition 1.5 concerning the canonical orbispace structure on M_ϕ.

More concretely, we need to define a set $\{T(D_0, W) \mid W \in \mathcal{W}, \text{ s.t. } D_0 \cap W \neq \emptyset\}$, and add it to the canonical orbispace structure on M_ϕ along with $(\widehat{D_0}, G_{D_0}, \pi_{D_0})$. Note that if $D_0 \cap W \neq \emptyset$, then W must be $W_a = M_{\phi_a}$ for some index a, where M_{ϕ_a} is the mapping cylinder of $\phi_a = \phi|_{\phi^{-1}(U_a)} : \sqcup_{i \in I_a} V_{a,i} \rightarrow U_a$, and the set of connected components of $D_0 \cap W_a$ is $\{W_{a,i} = V_{a,i} \times [0,1) | i \in I_a\}$. We define $T(D_0, W_a) = \sqcup_{i \in I_a} T_{W_{a,i}}(D_0, W_a)$ where $T_{W_{a,i}}(D_0, W_a) = G_{U_a}$, and assign each $\xi \in T_{W_{a,i}}(D_0, W_a)$ with a pair (ϕ_ξ, λ_ξ) as follows. Note that the assumption that the orbispace structure on S^{k-1} given by $\mathcal{V}(\Phi), \mathcal{T}(\Phi)$ is contained in the standard one on $S^{k-1}(G)$ implies that the inverse image of $W_{a,i}$ in $\widehat{W_a}$ is $(G_{U_a}/\rho_{(a,i)}(G)) \times W_{a,i}$, where $G_{U_a}/\rho_{(a,i)}(G)$ is the set of right cosets. The map ϕ_ξ is the homeomorphism sending $x \in W_{a,i}$ to $([\xi], x) \in (G_{U_a}/\rho_{(a,i)}(G)) \times W_{a,i}$, and $\lambda_\xi = Ad(\xi) \circ \rho_{(a,i)} : G \rightarrow Ad(\xi)(\rho_{(a,i)}(G))$. It is easily seen that this will give rise to an orbispace structure on M_ϕ, containing the original one. Now the subspace $i(S^{k-1}(G))$ is closed, connected, and is contained in the local chart $(\widehat{D_0}, G_{D_0}, \pi_{D_0})$. Hence by Lemma 3.2.1, we can collapse $i(S^{k-1}(G))$ to a point, and a canonical orbispace structure is resulted on $C_\phi = M_\phi / i(S^{k-1})$, with a canonical map of orbispaces $\pi : M_\phi \to C_\phi$ whose restriction to $M_\phi \setminus i(S^{k-1})$ is an isomorphism of orbispaces onto $C_\phi \setminus \{*\}$. The attached k-cell in C_ϕ is the image of $\overline{D_0} = \{[y,t] \mid y \in S^{k-1}, t \in I\} \subset M_\phi$ under the projection $M_\phi \to C_\phi = M_\phi/i(S^{k-1})$. Hence its interior is $D = D_0/i(S^{k-1})$. Clearly, as a subspace of the orbispace C_ϕ, it is $D(G)$, the open k-cell of isotropy type G, cf. Lemma 3.2.1.

(2) This is obtained by applying Lemma 3.2.2 along with (3.1.1), (3.1.2). Note that the hypothesis in Lemma 3.2.2 is met by the assumptions made in the present proposition.

(3) By definition the embedding of orbispace $i : X \to C_\phi$ is called a cofibration, where i is $j : X \to M_\phi$ followed by $\pi : M_\phi \to C_\phi$, if the following is true: given any orbispace X' and any map $\Psi : C_\phi \to X'$, if $H : X \times [0,1] \to X'$ is a homotopy from its restriction $\Psi|_X$ to another map from X to X', then there is a homotopy $F : C_\phi \times [0,1] \to X'$ extending H, such that $F|_{C_\phi \times \{0\}} = \Psi$. Consequently, $\Psi : C_\phi \to X'$ is homotopic through F to another map from C_ϕ to X', i.e., $F|_{C_\phi \times \{1\}}$, extending that for the restriction $\Psi|_X : X \to X'$.

We apply Lemma 3.2.2 to the mapping cylinder of $\Phi \times Id : S^{k-1}(G) \times [0,1] \to X \times [0,1]$, with the map $H : X \times [0,1] \to X'$ and a map $\Upsilon : S^{k-1}(G) \times [0,1] \times I \to X'$ constructed as follows. Let $R : [0,1] \times I \to [0,1] \times \{1\} \cup \{0\} \times I$ be a retraction which sends $[0,1] \times \{0\}$ to $\{0\} \times \{0\}$. We define

$$\Upsilon = (H \circ (\Phi \times Id) \cup \Psi \circ \pi) \circ (Id \times R),$$

which satisfies $\Upsilon|_{S^{k-1}(G)\times[0,1]\times\{1\}} = H \circ (\Phi \times Id)$ and $\Upsilon|_{S^{k-1}(G)\times\{0\}\times I} = \Psi \circ \pi|_{S^{k-1}(G)\times I}$, where $\pi : M_\phi \to C_\phi$, and $\Upsilon|_{S^{k-1}(G)\times[0,1]\times\{0\}} = \Psi \circ \pi|_{i(S^{k-1}(G))}$. Let $\Xi : M_{\phi\times Id} \to X'$ be the resulting map. Then the last property of Υ implies that Ξ factors through $\pi \times Id : M_{\phi\times Id} = M_\phi \times [0,1] \to C_\phi \times [0,1]$, which results in the homotopy $F : C_\phi \times [0,1] \to X'$ with the claimed properties.

(4) Straightforward application of Lemma 3.2.2.

□

We will say that the orbispace C_ϕ is obtained by attaching a k-cell of isotropy type G to X via the map Φ.

Now we consider a subcategory $\mathcal{C}$ of the category of orbispaces introduced in Part I of this series [3], which is the union of $\mathcal{C}^n$, $n \geq 0$. Each orbispace $X \in \mathcal{C}^n$ admits a canonical filtration of subspaces

$$X_0 \subset X_1 \subset \cdots \subset X_n = X, \tag{3.2.3}$$

where X_0 is a set of finitely many points in X, and for each $k = 1, 2, \cdots, n$, X_k is obtained by attaching to X_{k-1} finitely many k-cells of various isotropy type. We shall call each subspace X_k in (3.2.3), $0 \leq k \leq n$, the k-skeleton of X.

Note that in general it is not clear that for any $X_1, X_2 \in \mathcal{C}$, the product $X_1 \times X_2$ is still an object in $\mathcal{C}$. However, let us be content for now with the observation that for any $X \in \mathcal{C}^n$, the product $Z = X \times I \in \mathcal{C}^{n+1}$. In fact, let

$$X_0 \subset X_1 \subset \cdots \subset X_n = X$$

be the filtration of skeletons for X. Then correspondingly

$$Z_0 \subset Z_1 \subset \cdots \subset Z_{n+1} = Z,$$

where $Z_0 = X_0 \times \{0\} \cup X_0 \times \{1\}$, $Z_k = X_k \times \{0\} \cup X_k \times \{1\} \cup X_{k-1} \times I$ for $1 \le k \le n$, and $Z_{n+1} = X_n \times I$, is the canonical filtration of skeletons for $Z = X \times I$.

The remaining of this subsection is occupied by the study of some fundamental homotopy properties of the objects in $\mathcal{C}$. We begin by observing

Lemma 3.2.4 *For $k \ge 1$, an element $\Phi \in [(D^k(G), S^{k-1}(G), *); (X, A, \underline{\varrho})]_\rho$ defines the trivial element in $\pi_k^{(G,\rho)}(X, A, \underline{\varrho}) = [[(D^k(G), S^{k-1}(G), *); (X, A, \underline{\varrho})]]_\rho$ iff there is a homotopy H between Φ and a Φ' as elements of $[(D^k(G), *); (X, \underline{\varrho})]_\rho$, such that $H|_{S^{k-1}(G)\times\{t\}} = \Phi|_{S^{k-1}(G)}$, $\forall t \in [0,1]$, and $\Phi' \in [(D^k(G), *); (A, \underline{\varrho}|_A)]_\rho$.*

Proof Suppose there is such a homotopy H. Then $H|_{S^{k-1}(G)\times\{t\}} = \Phi|_{S^{k-1}(G)}$ and $\Phi \in [(D^k(G), S^{k-1}(G), *); (X, A, \underline{\varrho})]_\rho$ imply that H is also a homotopy between Φ and Φ' as elements of $[(D^k(G), S^{k-1}(G), *); (X, A, \underline{\varrho})]_\rho$. On the other hand, $\Phi' \in [(D^k(G), *); (A, \underline{\varrho}|_A)]_\rho$ implies that Φ' defines the trivial element in $\pi_k^{(G,\rho)}(X, A, \underline{\varrho})$, hence so does Φ.

Conversely, suppose $\Phi \in [(D^k(G), S^{k-1}(G), *); (X, A, \underline{\varrho})]_\rho$ defines the trivial element in $\pi_k^{(G,\rho)}(X, A, \underline{\varrho})$. Then there exists an $F \in [(CD^k(G), CS^{k-1}(G), *); (X, A, \underline{\varrho})]_\rho$ such that $F|_{D^k(G)} = \Phi$. Denote by C the subset $\{(x,t) \mid ||x|| \le 1-t\}$ of $D^k \times [0,1]$. Then the map $(x,t) \mapsto ((1-t)x, t)$ from $D^k \times [0,1]$ onto C factors through CD^k, which defines an isomorphism $\Psi : CD^k(G) \to C(G)$. On the other hand, there is a map $\Upsilon : D^k(G) \times [0,1] \to C(G)$ defined by $(x,t) \mapsto (x,t), \forall (x,t) \in C$, $(x,t) \mapsto (x, 1-||x||), \forall (x,t) \in (D^k \times [0,1]) \setminus C$, and $Id : G \to G$. We simply let $H = F \circ \Psi^{-1} \circ \Upsilon$, which clearly provides the desired homotopy between Φ and $\Phi' = H|_{D^k(G)\times\{1\}}$, i.e, $\Phi' \in [(D^k(G), *); (A, \underline{\varrho}|_A)]_\rho$, and H satisfies $H|_{S^{k-1}(G)\times\{t\}} = \Phi|_{S^{k-1}(G)}$, $\forall t \in [0,1]$.

□

Recall that $\pi_0^G(X) = [[B_G; X]]$ where B_G denotes the 0-cell of isotropy type G. We shall say that a pair (Y, B), where B is a subspace of Y via $i : B \to Y$, is 0-connected, if $i_* : \pi_0^G(B) \to \pi_0^G(Y)$ is a bijection for all G. For $n \ge 1$, we say that (Y, B) is n-connected if it is 0-connected and for all possible $\underline{\varrho}$ and (G, ρ), $\pi_k^{(G,\rho)}(Y, B, \underline{\varrho})$ is trivial for $1 \le k \le n$.

Lemma 3.2.5 *Suppose (Y, B) is a n-connected pair and $X \in \mathcal{C}^m$ where $m \le n$. Then any map $\Phi : X \to Y$ is homotopic to a map $\Phi' : X \to B$. Moreover, if a subspace $A \subset X$ is a union of cells and $\Phi|_A \in [A; B]$, we may even arrange to have $\Phi|_A = \Phi'|_A$.*

Proof Let $X_0 \subset X_1 \subset \cdots \subset X_m = X$ be the canonical filtration of X. Given any map $\Phi : X \to Y$, the restriction to X_0, $\Phi|_{X_0}$, is a map from a finite disjoint union of 0-cells of various isotropy type into Y. Since $i_* : \pi_0^G(B) \to \pi_0^G(Y)$ is a bijection for all G, $\Phi|_{X_0}$ is homotopic to a map $\Psi_0 : X_0 \to B$ as maps into Y. By Proposition 3.2.3 (3), $X_0 \subset X$ is a cofibration, hence this homotopy can be extended to X, so that Φ is homotopic to a map $\Phi_0' : X \to Y$ such that $(\Phi_0')|_{X_0} : X_0 \to B$. Now consider the restriction of Φ_0' to any of the 1-cells of various isotropy type in X_1 that are attached to X_0. There are $\underline{\varrho}$, (G, ρ) such that it defines an element in $\pi_1^{(G,\rho)}(Y, B, \underline{\varrho})$, which is trivial because (Y, B) is 1-connected. By Lemma 3.2.4 and Proposition 3.2.3 (4), the restriction of Φ_0' to X_1 is homotopic to a map $\Psi_1 : X_1 \to B$. Again because $X_1 \subset X$ is a cofibration, there is a homotopy between Φ_0' and a $\Phi_1' : X \to Y$ such that the restriction of Φ_1' to X_1 maps into B. The lemma follows by repeating this process under the condition $m \leq n$. It is clear that if the restriction of Φ to a cell in X is a map into B, one may keep it unchanged in the above process.

$\square$

Proposition 3.2.6 *Let $X \in \mathcal{C}$. For any $n \geq 0$, the pair (X, X_n) is n-connected.*

Proof It suffices to show that for all G, $i_* : \pi_0^G(X_0) \to \pi_0^G(X)$ is surjective, and for all possible $\underline{o}$, (G,ρ), $\pi_k^{(G,\rho)}(X, X_n, \underline{o})$ is trivial, $1 \leq k \leq n$.

The key to the proof is the following fact by Proposition 3.2.3 (1): Denote by $*$ the cone point in C_ϕ, i.e., the image of $i(S^{k-1})$ under $\pi : M_\phi \to C_\phi$. Since when restricted to $M_\phi \setminus i(S^{k-1})$, π is an isomorphism onto $C_\phi \setminus \{*\}$, there is a strong deformation retraction which shrinks the attached k-cell of isotropy type G to its boundary after a point in the interior is removed.

Given any map $\Phi : B_G \to X$, if its image lies in the interior of an attached k-cell in X_k, $k \geq 1$, then for dimensional reason it is in the complement of another interior point. By the said strong deformation retraction, Φ is homotopic to another map $\Psi : B_G \to X$ whose image lies in X_{k-1}. By induction, Φ is homotopic to a map in X_0, hence $i_* : \pi_0^G(X_0) \to \pi_0^G(X)$ is surjective.

Similarly, let $\Phi \in [(D^k(G), S^{k-1}(G), *); (X, X_n, \underline{o})]_\rho$, $1 \leq k \leq n$, whose image meets the interior of an attached m-cell $e^m = D^m(H)$, $m \geq n+1$. In this case, we need to employ Theorem 1.4 (2) in [3] to approximate Φ by a smooth map first. More concretely, let D_1, D_2 be the closed ball of radius $\frac{1}{3}, \frac{2}{3}$ centered at the cone point, and let $W_1 = \phi^{-1}(D_1), W_2 = \phi^{-1}(D_2)$ where ϕ is the induced map of Φ between underlying spaces. By Theorem 1.4 (2) in [3], there is a smooth map $\Upsilon : W_2 \to e^m$ such that $||\Upsilon - \Phi|_{W_2}||_{C^0} < \frac{1}{100}$. Moreover, by Theorem 1.4 (1), the difference $\Upsilon - \Phi|_{W_2}$ can be regarded as a C^0 section of a C^0 orbifold vector bundle over W_2, where $\Phi|_{W_2}$ is identified with the zero section. Let β be a smooth function on W_2 compactly supported in the interior W_2, such that $|\beta| \leq 1$ and $\beta = 1$ on W_1. Then the map $\Phi' : D^k(G) \to X$, where $\Phi' = \Phi + \beta(\Upsilon - \Phi|_{W_2})$, is homotopic to Φ relative to $S^{k-1}(G)$, and the image of Φ' will miss a point z in the open ball of radius $\frac{1}{4}$ centered at the cone point. The reason for the latter: on the interior W_1, Φ' is smooth so that it will miss z by transversality (represent $\Phi'|_{W_1}$ by a $(\{f_i\}, \{\rho_{ji}\})$ and then apply transversality argument to each f_i); on the complement, $||\Phi' - \Phi||_{C^0} < \frac{1}{100}$ so that it will be in the complement of the ball of radius $\frac{1}{4}$ centered at the cone point. By the strong deformation retraction mentioned earlier, Φ' is homotopic relative to $S^{k-1}(G)$ to a map whose image will miss the entire interior of the m-cell e^m. By induction, Φ is homotopic relative to $S^{k-1}(G)$ to a map into X_n. By Lemma 3.2.4, $\pi_k^{(G,\rho)}(X, X_n, \underline{o})$ is trivial for $1 \leq k \leq n$.

$\square$

Corollary 3.2.7 *Let $X, X' \in \mathcal{C}$. Then any map $\Phi : X \to X'$ is homotopic to a map $\Psi : X \to X'$, which is 'cellular' in the sense that $\Psi|_{X_n} \in [X_n; X'_n]$ for any $n \geq 0$. Moreover, if a subspace $A \subset X$ is a union of cells and $\Phi|_{A \cap X_n} \in [A \cap X_n; X'_n]$ for any $n \geq 0$, one may even arrange to have $\Phi|_A = \Psi|_A$.*

Proof of Theorem 1.6

(1) Let $i : X \to M_\phi$, $j : X' \to M_\phi$ be the canonical embeddings into the mapping cylinder which realize X, X' as a subspace, and let $r : M_\phi \to X'$ be the canonical strong deformation retraction, which satisfies $r \circ i = \Phi$. By the exact sequence (2.3.2) and the assumption that Φ_* is a weak homotopy equivalence, the pair (M_ϕ, X) is n-connected for any $n \geq 0$.

Now let $Y \in \mathcal{C}$ be any element. We first show that $\Phi_* : [[Y; X]] \to [[Y; X']]$ is surjective. In other words, for any map $\Psi' : Y \to X'$, we will find a map $\Psi : Y \to X$ such that $\Phi \circ \Psi \cong \Psi'$. This is done by applying Lemma 3.2.5 to the map $j \circ \Psi' : Y \to M_\phi$, which gives a map $\Psi : Y \to X$ satisfying $i \circ \Psi \cong j \circ \Psi'$. Composing both sides with $r : M_\phi \to X'$, we obtain $\Phi \circ \Psi = r \circ i \circ \Psi \cong r \circ j \circ \Psi' = \Psi'$.

As for the injectivity of Φ_*, we apply the above argument with Y replaced by $Z = Y \times I$, which is also in $\mathcal{C}$. More precisely, suppose $\Psi_1, \Psi_2 : Y \to X$ are any two maps such that $\Phi \circ \Psi_1 \cong \Phi \circ \Psi_2$ via a homotopy $H : Y \times I \to X'$. Since $j \circ r \cong Id_{M_\phi}$, we have $j \circ \Phi \circ \Psi_l = j \circ r \circ i \circ \Psi_l \cong i \circ \Psi_l$ for $l = 1, 2$. Combining with $j \circ H$, we obtain a homotopy $F' : Y \times I \to M_\phi$ between $i \circ \Psi_1$ and $i \circ \Psi_2$. On the other hand, suppose $Y \in \mathcal{C}^n$, then $Z = Y \times I \in \mathcal{C}^{n+1}$, with the canonical filtration of skeletons $Z_0 \subset Z_1 \subset \cdots \subset Z_{n+1} = Z$, where $Z_0 = Y_0 \times \{0\} \cup Y_0 \times \{1\}$, $Z_k = Y_k \times \{0\} \cup Y_k \times \{1\} \cup Y_{k-1} \times I$ for $1 \leq k \leq n$, and $Z_{n+1} = Y_n \times I$. Clearly, the subspace $Y \times \{0\} \cup Y \times \{1\} \subset Z$ is a union of cells. Furthermore, $F'|_{Y \times \{0\}} = i \circ \Psi_1 \in [Y; X]$ and $F'|_{Y \times \{1\}} = i \circ \Psi_2 \in [Y; X]$. Hence by Lemma 3.2.5, F' is homotopic to an $F : Y \times I \to X$ satisfying $F'|_{Y \times \{0\} \cup Y \times \{1\}} = F|_{Y \times \{0\} \cup Y \times \{1\}}$. It is easily seen that $\Psi_1 \cong \Psi_2$ via F. Thus $\Phi_* : [[Y; X]] \to [[Y; X']]$ is injective, and hence a bijection.

(2) Suppose $\Phi : X \to X'$ is a homotopy equivalence. We need to show that (1) for all G, $\Phi_* : \pi_0^G(X) \to \pi_0^G(X')$ is a bijection, which is trivial because $\pi_0^G(X) = [[B_G; X]]$, and (2) for all possible data $\underline{o}, \underline{o'}, (G, \rho)$ and (G, ρ'), $\Phi_* : \pi_k^{(G,\rho)}(X, \underline{o}) \to \pi_k^{(G,\rho')}(X', \underline{o'})$ is isomorphic for all $k \geq 0$. Here possible $\underline{o}, \underline{o'}$ are meant to be those with respect to which Φ has a based version $\Phi \in [(X, \underline{o}); (X', \underline{o'})]_\eta$ for some injective homomorphism $\eta : G_{\hat{o}} \to G_{\hat{o}'}$. For instance, this is the case when Φ can be represented by a homomorphism which also defines an element in $[(X, \underline{o}); (X', \underline{o'})]_\eta$. Note that not all base-point structures are being considered here. But in light of Proposition 1.3 (3), no generality is lost. Now let $\Phi_0, \Phi_1 : X \to X'$ be any maps which have based versions $\Phi_0 \in [(X, \underline{o}); (X', \underline{o'}_0)]_{\eta_0}$ and $\Phi_1 \in [(X, \underline{o}); (X', \underline{o'}_1)]_{\eta_1}$ for some $\underline{o}, \underline{o'}_0$ and $\underline{o'}_1$. Furthermore, Φ_0, Φ_1 are homotopic through a homotopy F. Then there is a guided path $u \in [(I(G_{\hat{o}}), 0, 1); (X', \underline{o'}_0, \underline{o'}_1)]_{(\eta_0, \eta_1)}$, which is defined by the restriction of F to $\{o\} \times I$. We claim that $(\Phi_0)_* = u_* \circ (\Phi_1)_* : \pi_k^{(G,\rho)}(X, \underline{o}) \to \pi_k^{(G,\eta_0 \circ \rho)}(X', \underline{o'}_0)$, where u_* is the isomorphism associated to the guided path u, cf. Proposition 1.3 (1). It follows easily from the claim that homotopy equivalence implies weak homotopy equivalence. To prove the claim, we observe that $(u_t)_* \circ (\Phi_t)_* : \pi_k^{(G,\rho)}(X, \underline{o}) \to \pi_k^{(G,\eta_0 \circ \rho)}(X', \underline{o'}_0)$ is locally constant in t, where $\Phi_t = F|_{X \times \{t\}} \in [(X, \underline{o}); (X', \underline{o'}_t)]_{\eta_t}$ and $u_t \in [(I(G_{\hat{o}}), 0, 1); (X', \underline{o'}_0, \underline{o'}_t)]_{(\eta_0, \eta_t)}$, which is defined by the restriction of F to $\{o\} \times [0, t]$. Clearly, $(u_0)_* \circ (\Phi_0)_* = (\Phi_0)_*$ and $(u_1)_* \circ (\Phi_1)_* = u_* \circ (\Phi_1)_*$. Hence the claim.

Conversely, suppose $\Phi : X \to X'$ is a weak homotopy equivalence. We need to find a homotopy inverse $\Psi : X' \to X$ of Φ. First of all, since any orbispace in $\mathcal{C}$ is locally path-connected and semi-locally 1-connected, the mapping cylinder M_ϕ of $\Phi : X \to X'$ is defined. Hence by (1) above, $\Phi_* : [[X'; X]] \to [[X'; X']]$ is a bijection. In particular, there is a map $\Psi : X' \to X$ such that $\Phi \circ \Psi \cong Id_{X'} \in [X'; X']$. On the other hand, note that $\Phi \circ \Psi \cong Id_{X'}$ implies $\Psi_* = \Phi_*^{-1}$ so that Ψ is also a weak homotopy equivalence. Thus there is a map

$\Upsilon : X \to X'$ such that $\Psi \circ \Upsilon \cong Id_X$. But $\Upsilon \cong \Phi \circ \Psi \circ \Upsilon \cong \Phi$. Hence Ψ is a homotopy inverse of Φ, and $\Phi : X \to X'$ is a homotopy equivalence.

□

3.3 CW-complex of groups and its geometric realization

This final subsection is concerned with a subcategory of $\mathcal{C}$, where the objects have much more pleasant geometrical properties. Most importantly, this subcategory is large enough that all compact smooth orbifolds are contained in it. In a certain sense, this subcategory of $\mathcal{C}$ consists of those orbispaces which are the geometric analogs of finite CW-complexes. We denote this subcategory of $\mathcal{C}$ by $\mathcal{G}$.

We give a description of $\mathcal{G}$ first. For the purpose here, we need to impose an additional condition on the general CW-complexes, for instance, as defined in [14], to which we refer the reader for the basic definitions and properties of CW-complexes. The condition is: for each attaching map, if its image meets the interior of a cell, it contains the whole cell. Note that simplicial complexes satisfy this condition. Immediate consequences of this assumption include that every face of a cell is an immediate face, and that each cell is a disjoint union of the interiors of finitely many cells.

An open CW-complex is an open subset of a CW-complex which is a disjoint union of the interiors of a subset of cells. Note that under the additional assumption here, the closure of an open CW-complex is also the smallest sub-complex containing the open CW-complex. We will assume that every open CW-complex is associated with a CW-complex which contains it as the interior, and any map between open CW-complexes is the restriction of a map between the closures. An open CW-complex is called an open sub-complex of a CW-complex K if the associated closure is the closure in K.

For any cell σ, the star of σ, denoted by $\mathrm{St}(\sigma)$, is the smallest open sub-complex that contains the interior of σ. It is also the disjoint union of the interiors of those cells which have σ as a face. In the case of simplicial complex, this definition coincides with the usual one. We remark that the star of a cell is connected, and the underlying space of a CW-complex is locally connected.

Group actions on a CW-complex are required to satisfy the following conditions: (1) the image of a cell is a cell, (2) if an interior point of a cell is fixed, the whole cell must be fixed. Let G be a discrete group acting on a CW-complex K. Then K/G is naturally a CW-complex, and the orbit map $\pi : K \to K/G$ is cellular.

Definition 3.3.1 *The subcategory $\mathcal{G}$ consists of orbispaces X where*

(1) *X is the underlying space of a finite CW-complex K,*

(2) *for each cell $\sigma \in K$, $St(\sigma)$ is a local chart on X, such that in $(\widehat{St(\sigma)}, G_{St(\sigma)}, \pi_{St(\sigma)})$, $\widehat{St(\sigma)}$ is an open CW-complex, $\hat{\sigma} = \pi^{-1}_{St(\sigma)}(\sigma)$ is a cell in the associated closure of $\widehat{St(\sigma)}$, and $St(\hat{\sigma}) = \widehat{St(\sigma)}$.*

For the sake of simplicity, we denote $G_{St(\sigma)}$ by G_σ and $\pi_{St(\sigma)}$ by π_σ. We call G_σ the isotropy group of the cell σ. Note that if $X, X' \in \mathcal{G}$, the product $X \times X'$ is also in $\mathcal{G}$.

The objects of $\mathcal{G}$ are closely related to the notion 'CW-complex of groups' described in Introduction. More precisely, to each $X \in \mathcal{G}$, which is the underlying space of a finite CW-complex K, one can associate an equivalence class of CW-complexes of groups on K as follows.

(1) Each cell $\sigma \in K$ is associated with its isotropy group G_σ.
(2) For each arrow a, note that $\mathrm{St}(i(a)) \subset \mathrm{St}(t(a))$ because $t(a)$ is a face of $i(a)$. Assign the arrow a with $\psi_a : G_{i(a)} \rightarrow G_{t(a)}$, where ψ_a is given by λ_{ξ_a} for a fixed choice of $\xi_a \in T(\mathrm{St}(i(a)), \mathrm{St}(t(a)))$.
(3) To each pair of composable arrows a, b, an element $g_{a,b} \in G_{t(a)}$ is assigned, which is the unique element satisfying $\xi_a \circ \xi_b = g_{a,b} \circ \xi_{ab}$. The equation

$$Ad(g_{a,b}) \circ \psi_{ab} = \psi_a \circ \psi_b \tag{3.3.1}$$

follows immediately from the definition. The cocycle condition for a triple of composable arrows a, b, c

$$\psi_a(g_{b,c})g_{a,bc} = g_{a,b}g_{ab,c} \tag{3.3.2}$$

is a consequence of the associativity of composition in the groupoid.

One can easily check that a different choice of $\{\xi_a\}$ will result in an equivalent CW-complex of groups on K. The CW-complex of groups thus obtained is called associated to the orbispace $X \in \mathcal{G}$, and the orbispace $X \in \mathcal{G}$ is called the geometric realization of the associated CW-complex of groups.

We remark that the relationship between CW-complex of groups and the geometric realization is analogous to that between complex of groups and the associated orbihedron in Haefliger [12]. In fact, when the underlying CW-complex is simplicial, a CW-complex of groups is simply a complex of groups, and the corresponding geometric realization is isomorphic as an orbispace to the associated orbihedron. However, there is a minor difference between these two concepts which lies in the fact that in the definition of an orbihedron in Haefliger [12], the star of a cell is not chosen to be the one in the simplicial complex, but rather in the barycentric subdivision of it. For details, see [12].

The central result of this subsection is $\mathcal{G} \subset \mathcal{C}$.

Let $X \in \mathcal{G}$, with the finite CW-complex structure K, and $\Phi : S^{k-1}(G) \rightarrow X$ be any map, such that the induced map $\phi : S^{k-1} \rightarrow X$ is an attaching map so that the mapping cone C_ϕ supports a canonical finite CW-complex L obtained by attaching a k-cell to K via ϕ. Recall that there is a canonical orbispace structure on S^{k-1} defined by $\mathcal{V}(\Phi), \mathcal{T}(\Phi)$ as constructed in Lemma 3.1.2, which is equivalent to the standard one on $S^{k-1}(G)$. In order to attach a k-cell of isotropy type G to X via Φ, we assume further that the orbispace structure $(\mathcal{V}(\Phi), \mathcal{T}(\Phi))$ is contained in the standard one on $S^{k-1}(G)$, cf. Proposition 3.2.3. In the present case, in order to ensure that the orbispace C_ϕ constructed in Proposition 3.2.3 is an object of the subcategory $\mathcal{G}$, we need to further impose two additional conditions:

- Let $(\{V_{\sigma,i} | i \in I_\sigma\}, \{\mathrm{St}(\sigma)\}, \{f_{\sigma,i}\}, \{\rho_{(\tau,j)(\sigma,i)}\})$ be the canonical representative of Φ constructed in Lemma 3.1.2, where $\sigma \in K$, and $\phi^{-1}(\mathrm{St}(\sigma)) = \sqcup_{i \in I_\sigma} V_{\sigma,i}$. (cf. (3.1.5) also.) We further assume that each $\rho_{(\tau,j)(\sigma,i)} : G \rightarrow T(\mathrm{St}(\sigma), \mathrm{St}(\tau))$ is independent of the indexes $i \in I_\sigma, j \in I_\tau$.

- For any cell $\sigma \subset \phi(S^{k-1})$, we require that the map $\pi_\sigma : \widehat{\mathrm{St}(\sigma)} \rightarrow \mathrm{St}(\sigma)$, which is defined over the associated closures of the open CW-complexes by our earlier assumption, is one to one when restricted to the closure of $\bigcup_{i \in I_\sigma} f_{\sigma,i}(\widehat{V_{\sigma,i}})$.

Lemma 3.3.2 *With the preceding understood, the orbispace C_ϕ constructed in Proposition 3.2.3 is canonically an object of the subcategory $\mathcal{G}$.*

Proof First of all, we recall the following notation: the open subset $M_f \setminus \{[y,t] | t \in [0,t_0]\}$ of a mapping cylinder M_f is denoted by $M_f(t_0)$.

Now observe that in Proposition 3.2.3 (1), a local chart on C_ϕ is either a local chart W on the open subspace $C_\phi \setminus \{*\}$ where $\{*\}$ is the cone point, or the interior D of the attached k-cell of isotropy type G. If $W \cap D \neq \emptyset$, W must be $M_{\phi_\sigma}(0)$ for some cell $\sigma \in K$, where M_{ϕ_σ} is the mapping cylinder of $\phi_\sigma = \phi|_{\sqcup_{i \in I_\sigma} V_{\sigma,i}}$. Moreover, $\widehat{W} = M_{f_\sigma}(0)$ where f_σ is a map into $\widehat{\mathrm{St}(\sigma)}$ defined as follows. Domain $(f_\sigma) = \bigsqcup_{i \in I_\sigma} (G_\sigma \times \widehat{V_{\sigma,i}})/\rho_{(\sigma,i)}(G)$, where the action is given by $\rho_{(\sigma,i)}(g) \cdot (g', x) = (g' \rho_{(\sigma,i)}(g)^{-1}, g \cdot x)$, $\forall g \in G$, and f_σ is defined by $f_\sigma([(g',x)]) = g' \cdot f_{\sigma,i}(x)$, $\forall g' \in G_\sigma, x \in \widehat{V_{\sigma,i}}, i \in I_\sigma$. With this understood, the action of $G_W = G_\sigma$ on $\widehat{W}$ is given by $h \cdot [(g',x)] = [(hg',x)]$. Now by the first imposed condition, $\rho_{(\sigma,i)} = \rho_\sigma : G \rightarrow G_\sigma$ is independent of the index $i \in I_\sigma$. This allows us to define $D_\sigma = M_{\phi_\sigma}(0) \cup D$, $\widehat{D_\sigma} = M_{f_\sigma}(0) \cup (G_\sigma \times D)/\rho_\sigma(G)$, and $G_{D_\sigma} = G_\sigma$ with the natural action by multiplication from the left. Again the first imposed condition that each $\rho_{(\tau,j)(\sigma,i)} : G \rightarrow T(\mathrm{St}(\sigma), \mathrm{St}(\tau))$ is independent of the indexes $i \in I_\sigma, j \in I_\tau$ allows us to define naturally a set $\{T(D_\sigma, D_\tau)\}$, such that a set of new local charts $\{(\widehat{D_\sigma}, G_{D_\sigma}, \pi_{D_\sigma})\}$ may be added consistently to the canonical orbispace structure on C_ϕ.

Next we prove that with the modified orbispace structure which is equivalent to the original one, C_ϕ belongs to $\mathcal{G}$. Recall that the finite CW-complex structure L on C_ϕ is the one obtained by attaching the k-cell $\overline{D}$ to K via ϕ. Thus the following is true for L: (1) For any $\sigma \in L$ such that $\sigma \neq \overline{D}$ and σ is not a face of $\overline{D}$, the star of σ in L is $\mathrm{St}(\sigma)$, the star of σ in K. (2) If $\sigma = \overline{D}$, then the star of σ in L is the interior of σ. (3) If $\sigma \subset \phi(S^{k-1})$, then the star of σ in L is $\mathrm{St}(\sigma) \sqcup D = D_\sigma$. It is clear that in order to show that C_ϕ belongs to $\mathcal{G}$, it suffices to verify that $\widehat{D_\sigma}$, which is $\widehat{\mathrm{St}(\sigma)} \sqcup (G_\sigma/\rho_\sigma(G)) \times D$, has a natural open CW-complex structure, and that it is the star of $\hat{\sigma} = \pi_\sigma^{-1}(\sigma)$ in the associated closure. To this end, we observe that the closure of D_σ in L is $\overline{\mathrm{St}(\sigma)} \sqcup (\overline{D} \setminus \overline{\mathrm{St}(\sigma)})$. The second imposed condition then implies that

$$\text{(closure of } \widehat{\mathrm{St}(\sigma)}) \sqcup (G_\sigma/\rho_\sigma(G)) \times (\overline{D} \setminus \overline{\mathrm{St}(\sigma)}) \tag{3.3.3}$$

is naturally a CW-complex, which contains $\widehat{D_\sigma}$ as the interior, and admits a natural action of G_σ extending that of G_σ on the interior, such that the orbit space is the closure of D_σ in L. Hence $\widehat{D_\sigma}$ has a natural open CW-complex structure. Finally, it is easily seen that $\widehat{D_\sigma}$ is the star of $\hat{\sigma} = \pi_\sigma^{-1}(\sigma)$ in the associated closure (3.3.3).

□

Now we are ready for a proof of Proposition 1.7.

Proposition 3.3.3

(1) *Let $(K, G_\sigma, \psi_a, g_{a,b})$ be a CW-complex of groups, and X be the underlying space of K. Then there is a canonical orbispace structure on X such that the orbispace X belongs to $\mathcal{G}$, and the associated CW-complex of groups is $(K, G_\sigma, \psi_a, g_{a,b})$. Moreover, the orbispace X also belongs to $\mathcal{C}$ with the canonical filtration of skeletons*

$$X_0 \subset X_1 \subset \cdots \subset X_n = X,$$

where X_k, $0 \leq k \leq n$, is the underlying space of the k-skeleton of K, such that the orbispace structure on X_k is the one canonically determined by the restriction of $(K, G_\sigma, \psi_a, g_{a,b})$ to the k-skeleton of K.

(2) *The orbispace structure of an object in $\mathcal{G}$ is uniquely determined by the equivalence class of the associated CW-complexes of groups.*

Proof (1) The statement is true when the dimension of K is zero. We shall prove that if it is true when the dimension is n, it is also true when the dimension is $n+1$.

First of all, by the induction assumption, there is a canonical orbispace structure on X_n such that the orbispace $X_n \in \mathcal{G}$ and the associated CW-complex of groups is $(K_n, G_\sigma, \psi_a, g_{a,b})$. In particular, the latter means that there are $\xi_a \in T(\mathrm{St}(i(a)), \mathrm{St}(t(a)))$ satisfying $\xi_a \circ \xi_b = g_{a,b} \circ \xi_{ab}$ and $\lambda_{\xi_a} = \psi_a$. We also observe, from the proof of Lemma 3.3.2, that $\pi_\sigma|_{\hat{\sigma}}$ is a homeomorphism onto σ for any cell σ.

In order to construct an orbispace structure on X_{n+1} such that $X_{n+1} \in \mathcal{G}$, we shall define, for each (n+1)-cell e, a map $\Phi_e : S^n(G_e) \to X_n$ whose induced map $\phi_e : S^n \to X_n$ is the attaching map for the (n+1)-cell e, such that the hypothesis in Lemma 3.3.2 holds for Φ_e. To this end, we set $\eta_a = g_{a,d}^{-1} \circ \xi_a$ for any arrow a such that both $i(a), t(a)$ are contained in the image of the attaching map ϕ_e of e, where d is the arrow $(e, i(a))$. It is easy to check that $\eta_a \circ \eta_b = \eta_{ab}$ holds for any composable arrows a, b. Now for any cell $\sigma \subset \mathrm{Im}\ \phi_e$, we define a map $u_\sigma : \mathrm{Im}\ \phi_e \cap \mathrm{St}(\sigma) \to \widehat{\mathrm{St}(\sigma)}$ by setting $u_\sigma|_{\sigma^0} = \pi_\sigma^{-1}|_{\sigma^0}$ and $u_\sigma|_{\tau^0} = \phi_{\eta_a} \circ \pi_\tau^{-1}|_{\tau^0}$ for any cell τ such that $\tau^0 \subset \mathrm{Im}\ \phi_e \cap \mathrm{St}(\sigma)$ and $\tau \neq \sigma$, where σ^0, τ^0 denote the interior of the corresponding cell, and the arrow $a = (\tau, \sigma)$. We note that (1) $\pi_\sigma \circ u_\sigma = Id$, (2) $\phi_{\eta_a} \circ u_{i(a)} = u_{t(a)}|_{Dom(u_{i(a)})}$ for any arrow a such that both $i(a), t(a)$ are contained in $\mathrm{Im}\ \phi_e$, and (3) the image of u_σ lies in the fixed-point set of $\psi_d(G_e) \subset G_\sigma$ where d is the arrow (e, σ). Furthermore, it follows from (2) above that each u_σ is continuous.

For each cell $\sigma \subset \mathrm{Im}\ \phi_e$, let $\{V_{\sigma,i} | i \in I_\sigma\}$ be the set of connected components of $\phi_e^{-1}(\mathrm{St}(\sigma))$. We define $f_{\sigma,i} = u_\sigma \circ \phi_e|_{V_{\sigma,i}}$ (with $\widehat{V_{\sigma,i}} = V_{\sigma,i}$ understood), $\rho_{(\sigma,i)} = \psi_d$ where $d = (e, \sigma)$, and for any σ, τ such that τ is a face of σ, define $\rho_{(\tau,j)(\sigma,i)}$ by $g \mapsto \eta_a \circ \psi_d(g)$, $\forall g \in G_e$, where $a = (\sigma, \tau)$ and $d = (e, \sigma)$. One can easily check, using the properties of $\{u_\sigma\}$ and $\{\eta_a\}$ established in the preceding paragraph, that $(\{f_{\sigma,i}\}, \{\rho_{(\tau,j)(\sigma,i)}\})$ is a homomorphism of groupoids. The equivalence class of $(\{f_{\sigma,i}\}, \{\rho_{(\tau,j)(\sigma,i)}\})$ is defined to be Φ_e, which satisfies the hypothesis in Lemma 3.3.2 by the nature of construction. By Lemma 3.3.2, there is a canonical orbispace structure on X_{n+1} such that $X_{n+1} \in \mathcal{G}$.

To see that the associated CW-complex of groups is $(K_{n+1}, G_\sigma, \psi_a, g_{a,b})$, we need to find a $\xi_d \in T(\mathrm{St}(e), \mathrm{St}(\sigma))$ for each arrow $d = (e, \sigma)$, where e is a (n+1)-cell, such that $\lambda_{\xi_d} = \psi_d$, and for any arrow a satisfying $i(a) = t(d)$, $\xi_a \circ \xi_d = g_{a,d} \circ \xi_{ad}$. Observe that, according to Lemma 3.3.2, the inverse image of e^0 (the interior of e) in $\widehat{\mathrm{St}(\sigma)}$ is $(G_\sigma / \psi_d(G_e)) \times e^0$, and $T(\mathrm{St}(e), \mathrm{St}(\sigma)) = G_\sigma$ with $\lambda_\xi = Ad(\xi) \circ \psi_d : G_e \to G_\sigma$. With this understood, $\xi_d = 1 \in G_\sigma$

will work for the purpose here. This completes the induction step, and hence (1) of the proposition.

(2) The proof goes by induction on the dimension of K. Assume the statement is true when the dimension is n. Then it is easy to see that the induction step boils down to the verification that for each (n+1)-cell e and any arrow $d = (e, \sigma)$, the attaching maps of $\pi_\sigma^{-1}(e)$ in $\widehat{\mathrm{St}(\sigma)}$ are canonically determined by the CW-complex of groups. For this we simply observe: When $d = (e, \sigma)$ is primitive, i.e., there is no τ such that d is the composition of (e, τ) with (τ, σ), the attaching maps are unique. When $d = ad'$ where d' is primitive, the attaching maps are determined by $g_{a,d'}^{-1} \circ \xi_a$ from those of $\pi_{t(d')}^{-1}(e)$ in $\widehat{\mathrm{St}(t(d'))}$.

$\square$

The preceding proposition clearly established the one to one correspondence between equivalence classes of CW-complexes of groups and isomorphism classes of orbispaces in $\mathcal{G}$, and in particular, the inclusion $\mathcal{G} \subset \mathcal{C}$.

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