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Abstract

We show how pairs of isothermic surfaces are given by curved flats in a pseudo Riemannian symmetric space and vice versa. Calapso's fourth order partial differential equation is derived and, using a solution of this equation, a Möbius invariant frame for an isothermic surface is built.

1 Introduction

These notes grew out of a series of discussions on a recent paper by J. Cieśliński, P. Goldstein and A. Sym [4]: these authors give a characterization of isothermic surfaces as "soliton surfaces" by introducing a spectral parameter. In trying to understand the geometric meaning of this spectral parameter, we observed some analogies with the theory of conformally flat hypersurfaces in a four-dimensional space form: Guichard's nets may be understood as a kind of analogue of isothermic parametrizations of Riemannian surfaces (cf.[7, no.3.4.1]), and so it seems natural to look for relations between the theory of isothermic surfaces in three-dimensional space forms and the theory of conformally flat hypersurfaces in four-dimensional space forms. Here we would like to present some results we found — especially the possibility of constructing isothermic surfaces using

2 Curved Flats

A curved flat is the natural generalization of a developable surface in Euclidean space: it is a submanifold $M \subset G/K$ of a (pseudo-Riemannian) symmetric space for which the curvature operator of G/K vanishes on $\bigwedge^2 TM$[1]. Thus, a curved flat may be thought of as the enveloping submanifold of a congruence of flats — totally geodesic submanifolds — of the symmetric space. Taking a regular parametrization $\gamma : M \to G/K$ of a curved flat and a framing $F : M \to G$ of this parametrization, the Maurer-Cartan form $\Phi = F^{-1}dF$

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[1] Thus M is *curvature isotropic* in the sense of [6]

of the framing has a natural decomposition $\Phi = \Phi_{\mathfrak{k}} + \Phi_{\mathfrak{p}}$ according to the symmetric decomposition[2] $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of the Lie algebra $\mathfrak{g}$. Now the condition for γ to parametrize a curved flat may be formulated as[3]

$$[[\Phi_{\mathfrak{p}} \wedge \Phi_{\mathfrak{p}}], \mathfrak{p}] \equiv \mathfrak{o} \ . \tag{1}$$

In case that G is semisimple, it is straightforward[4] to see that this is equivalent to

$$[\Phi_{\mathfrak{p}} \wedge \Phi_{\mathfrak{p}}] \equiv 0 \ . \tag{2}$$

To summarise, we have the

Definition of a curved flat: An immersion $\gamma : M \to G/K$ is said to parametrize a CURVED FLAT, if the $\mathfrak{p}$-part in the symmetric decomposition of the Maurer-Cartan form $F^{-1}dF = \Phi = \Phi_{\mathfrak{k}} + \Phi_{\mathfrak{p}}$ of a framing $F : M \to G$ of γ defines a congruence $p \mapsto \Phi_{\mathfrak{p}}|_p(T_pM)$ of abelian subalgebras of $\mathfrak{g}$.

At this point we should remark that curved flats naturally arise in one parameter families [6]: setting

$$\Phi_\lambda := \Phi_{\mathfrak{k}} + \lambda \Phi_{\mathfrak{p}} \tag{3}$$

the Maurer-Cartan equation $d\Phi_\lambda + \frac{1}{2}[\Phi_\lambda \wedge \Phi_\lambda] = 0$ for the loop $\lambda \mapsto \Phi_\lambda$ of forms splits into the three equations

$$\begin{array}{l} 0 = d\Phi_{\mathfrak{k}} + \frac{1}{2}[\Phi_{\mathfrak{k}} \wedge \Phi_{\mathfrak{k}}] \\ 0 = d\Phi_{\mathfrak{p}} + [\Phi_{\mathfrak{k}} \wedge \Phi_{\mathfrak{p}}] \\ 0 = [\Phi_{\mathfrak{p}} \wedge \Phi_{\mathfrak{p}}] \ , \end{array} \tag{4}$$

and hence the integrability of the loop $\lambda \mapsto \Phi_\lambda$ is equivalent to the forms Φ_λ being the Maurer-Cartan forms for some framings $F_\lambda : M \to G$ of curved flats $\gamma_\lambda : M \to G/K$. Thus integrable systems theory may be applied to produce examples.

Now we will consider the case leading to the theory of isothermic surfaces: let

$$G := O_1(5) \quad \text{and} \quad K := O(3) \times O_1(2) \ . \tag{5}$$

[2]Thus $\mathfrak{k}$ and $\mathfrak{p}$ are the $+1$ and -1-eigenspaces, respectively, of the involution fixing $\mathfrak{k}$ and so satisfy the characteristic conditions

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}, [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}.$$

[3]The product

$$[\Phi \wedge \Psi](v, w) := [\Phi(v), \Psi(w)] - [\Phi(w), \Psi(v)]$$

defines a symmetric product on the space of Lie algebra valued 1-forms with values in the space of Lie algebra valued 2-forms.

[4]In fact, $[\mathfrak{p}, \mathfrak{p}] \oplus \mathfrak{p}$ is an ideal of $\mathfrak{g}$ so that we have a decomposition $\mathfrak{g} = \mathfrak{k}' \oplus [\mathfrak{p}, \mathfrak{p}] \oplus \mathfrak{p}$ where $\mathfrak{k}'$ is a complementary ideal commuting with $[\mathfrak{p}, \mathfrak{p}] \oplus \mathfrak{p}$. Thus, if $\mathfrak{a} \subset \mathfrak{p}$ satisfies $[[\mathfrak{a}, \mathfrak{a}], \mathfrak{p}] = \mathfrak{o}$ we deduce that $[\mathfrak{a}, \mathfrak{a}]$ lies in the center of $\mathfrak{g}$ and so vanishes.

The coset space $G_+(5,3) = G/K$ of space-like 3-planes in the Minkowski space $\mathbb{R}^5_1$ becomes a six dimensional pseudo-Riemannian symmetric space of signature $(3,3)$ when endowed with the metric induced by the Killing form. We will consider two-dimensional curved flats

$$\gamma : M^2 \to G_+(5,3) \tag{6}$$

satisfying the regularity assumption that the metric on M^2 induced by γ is non-degenerate.

Fixing a pseudo orthonormal basis $(e_1, \ldots, e_5)$ of the Minkowski space $\mathbb{R}^5_1$ with

$$(\langle e_i, e_j \rangle)_{ij} = E_5 := \begin{pmatrix} I_3 & 0 \\ 0 & \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \end{pmatrix}, \tag{7}$$

we get the matrix representations

$$\begin{aligned} &O_1(5) = \{A \in Gl(5, \mathbb{R}) | A^t E_5 A = E_5\} \\ &\mathfrak{o}_1(5) = \{\mathfrak{X} \in \mathfrak{gl}(5, \mathbb{R}) | (\mathfrak{E}_5 \mathfrak{X}) + (\mathfrak{E}_5 \mathfrak{X})^{\mathfrak{t}} = \mathfrak{o}\} \ . \end{aligned} \tag{8}$$

The subalgebra $\mathfrak{k}$ and its complementary linear subspace $\mathfrak{p}$ in the symmetric decomposition of $\mathfrak{o}_1(5)$ are given by the $+1$- resp. -1-eigenspaces of the involutive automorphism $\mathrm{Ad}(Q) : \mathfrak{o}_1(5) \to \mathfrak{o}_1(5)$ with $Q = \left(\begin{smallmatrix} -I_3 & 0 \\ 0 & I_2 \end{smallmatrix} \right)$. Writing down the Maurer-Cartan form of a framing $F : M^2 \to O_1(5)$ of our curved flat $\gamma : M^2 \to G_+(5,3)$ with this notation we obtain

$$\begin{aligned} &F^{-1} dF = \Phi = \Phi_{\mathfrak{k}} + \Phi_{\mathfrak{p}} \quad \text{with} \\ &\Phi_{\mathfrak{k}} = \begin{pmatrix} \Omega & 0 \\ 0 & \nu \end{pmatrix} : \qquad TM \to \mathfrak{o}(3) \times \mathfrak{o}_1(2) \\ &\Phi_{\mathfrak{p}} = \begin{pmatrix} 0 & \eta \\ -E_2 \eta^t & 0 \end{pmatrix} : \quad TM \to \mathfrak{p} \ . \end{aligned} \tag{9}$$

The image of $\Phi_{\mathfrak{p}}$ at each $p \in M^2$ is a 2-dimensional abelian subspace of $\mathfrak{p}$ on which the Killing form is non-degenerate. One can show that there are precisely two K-orbits of maximal abelian subspaces of $\mathfrak{p}$: one consists of 3-dimensional subspaces which are isotropic for the Killing form while the other consists of 2-dimensional subspaces on which the Killing form has signature $(1,1)$. We therefore conclude that the images of each $\Phi_{\mathfrak{p}}$ are maximal abelian and K-conjugate and so we can put η into the standard form

$$\eta = \begin{pmatrix} \omega_1 & -\omega_1 \\ \omega_2 & \omega_2 \\ 0 & 0 \end{pmatrix} \tag{10}$$

by applying a gauge transformation $M \to K$.

Calculating the Maurer-Cartan equation using the ansatz

$$\Omega = \begin{pmatrix} 0 & \omega & -\psi_1 \\ -\omega & 0 & -\psi_2 \\ \psi_1 & \psi_2 & 0 \end{pmatrix} \quad \text{and} \quad \nu = \begin{pmatrix} \nu & 0 \\ 0 & -\nu \end{pmatrix} \tag{11}$$

together with η given by (10), we see that

$$d\omega_1 = d\omega_2 = 0 \ . \tag{12}$$

So we are given canonical coordinates $(x, y) : M \to I\!R^2$ by integrating[5] the forms ω_1 and ω_2. Moreover, since we also have $d\nu = 0$, we may set $\nu = -du$ for a suitable function $u \in C^\infty(M)$ — this gives us $\omega = u_y dx - u_x dy$, where u_x and u_y denote the partial derivatives of u in x- resp. y-directions. Finally, the equations $\psi_1 \wedge \omega_1 = 0$ and $\psi_2 \wedge \omega_2 = 0$ show that $\psi_1 = e^u k_1 dx$ and $\psi_2 = e^u k_2 dy$ for two functions $k_i \in C^\infty(M)$.

We now perform a final $O_1(2)$-gauge $\begin{pmatrix} I_3 & 0 \\ 0 & \begin{smallmatrix} e^u & 0 \\ 0 & e^{-u} \end{smallmatrix} \end{pmatrix} : M \to O(3) \times O_1(2)$ and insert the spectral parameter λ to obtain the Maurer-Cartan form discussed in (cf.[4]):

$$\Phi_\lambda = \begin{pmatrix} 0 & u_y dx - u_x dy & -e^u k_1 dx & \lambda e^u dx & -\lambda e^{-u} dx \\ -u_y dx + u_x dy & 0 & -e^u k_2 dy & \lambda e^u dy & \lambda e^{-u} dy \\ e^u k_1 dx & e^u k_2 dy & 0 & 0 & 0 \\ \lambda e^{-u} dx & -\lambda e^{-u} dy & 0 & 0 & 0 \\ -\lambda e^u dx & -\lambda e^u dy & 0 & 0 & 0 \end{pmatrix} . \tag{13}$$

We are now lead directly to the theory of

3 Isothermic Surfaces

In the context of Möbius geometry the three sphere S^3 is viewed as the projective light-cone $I\!PL^4$ in $I\!R^5_1$ while the Lorentzian sphere $\{v \in I\!R^5_1 | \langle v, v \rangle = 1\}$ should be interpreted as the space of (oriented) spheres in the three sphere[6] (cf.[1]). Now, denoting by

$$\begin{array}{l} n := Fe_3 : M \to S^5_1 = \{v \in I\!R^5_1 | \langle v, v \rangle = 1\} \\ f := Fe_4 : M \to L^4 = \{v \in I\!R^5_1 | \langle v, v \rangle = 0\} \\ \hat{f} := Fe_5 : M \to L^4 \end{array} \tag{14}$$

one of the sphere congruences resp. the two immersions given by our frame F, we see that

Theorem: *The sphere congruence n given by our curved flat is a Ribeaucour sphere congruence*[7]*, which is enveloped by two isothermic immersions f and $\hat{f}$* (cf.[1, p.362]): Since

$$\begin{array}{lll} \langle f, n \rangle = 0 & \text{and} & \langle df, n \rangle \equiv 0, \\ \langle \hat{f}, n \rangle = 0 & \text{and} & \langle d\hat{f}, n \rangle \equiv 0 \ , \end{array} \tag{15}$$

[5] Since our theory is local, all closed forms may be assumed to be exact.

[6] Or, equivalently, it may be interpreted as the space of (oriented) spheres and planes in Euclidean three space $I\!R^3$: the polar hyperplane to a vector v of the Lorentz sphere intersects the three sphere — thought of as the absolute quadric in projective four space — in a two sphere. Stereographic projection yields a sphere in $I\!R^3$ or, if the projection center lies on the sphere, a plane.

[7] The curvature lines on the two enveloping immersions correspond.

the immersions f and $\hat{f}$ do envelop the sphere congruence n and, since the bilinear forms

$$(16)\qquad \begin{array}{l}\langle df, dn\rangle = \lambda e^{2u}(k_1 dx^2 + k_2 dy^2),\\ \langle d\hat{f}, dn\rangle = \lambda(-k_1 dx^2 + k_2 dy^2)\end{array}$$

are diagonal with respect to the induced metrics

$$(17)\qquad \begin{array}{l}\langle df, df\rangle = \lambda^2 e^{2u}(dx^2 + dy^2),\\ \langle d\hat{f}, d\hat{f}\rangle = \lambda^2 e^{-2u}(dx^2 + dy^2)\ ,\end{array}$$

the two immersions f and $\hat{f}$ are isothermic[8].

It is quite difficult to calculate the first and second fundamental forms of these isothermic immersions, when they are projected to S^3 resp. $I\!R^3$, but applying a (constant) conformal change (constant $O_1(2)$-gauge)

$$(18)\qquad \begin{array}{lll} f \rightsquigarrow \frac{1}{\lambda} f & \text{and} & \hat{f} \rightsquigarrow \lambda \hat{f} \quad \text{or}\\ f \rightsquigarrow \lambda f & \text{and} & \hat{f} \rightsquigarrow \frac{1}{\lambda}\hat{f}\end{array}$$

and sending $\lambda \to 0$, $\hat{f}$ resp. f becomes a constant vector — $\Phi_{\lambda=0}e_5$ resp. $\Phi_{\lambda=0}e_4$ vanishes. This constant light-like vector may be interpreted as the point at infinity and we therefore obtain an isothermic immersion $f : M \to I\!R^3$ with first and second fundamental forms

$$(19)\qquad \begin{array}{l} I = e^{2u}(dx^2 + dy^2)\\ II = e^{2u}(k_1 dx^2 + k_2 dy^2)\end{array}$$

resp. its Euclidean dual surface $\hat{f} : M \to I\!R^3$ with first and second fundamental forms

$$(20)\qquad \begin{array}{l} \hat{I} = e^{-2u}(dx^2 + dy^2)\\ \hat{II} = -k_1 dx^2 + k_2 dy^2\ .\end{array}$$

We now recognise the remaining three equations from the Maurer-Cartan equation for Φ_λ

$$(21)\qquad \begin{array}{l} 0 = \Delta u + e^{2u} k_1 k_2\\ 0 = k_{1y} + (k_1 - k_2)u_y\\ 0 = k_{2x} - (k_1 - k_2)u_x\end{array}$$

[8] The bundle defined by span$\{n, f, \hat{f}\}$ over M is flat (cf.(13)) and so the map $p \mapsto d_p f(T_p M)$ defines a normal congruence of circles [5]: for each $p \in M$

$$t \mapsto f_t(p) := \frac{1}{\sqrt{2}}\sin t \cdot n(p) + \frac{1}{2}(1+\cos t)\cdot f(p) - \frac{1}{2}(1-\cos t)\cdot \hat{f}(p)$$

parametrizes the circle $(d_p f(T_p M))^\perp$ meeting the sphere $n(p)$ in $f(p)$ and $\hat{f}(p)$ orthogonal. Since n, f and $\hat{f}$ are parallel sections in this bundle, the maps $p \mapsto f_t(p)$ (which generically are not degenerate) parametrize the surfaces orthogonal to this congruence of circles.

In general the immersions f and $\hat{f} = f_\pi$ will be the only isothermic surfaces among the surfaces.

as the Gauß and Codazzi equations of the Euclidean immersion f resp. its dual $\hat{f}$ [9] [3], [4]. As a consequence, we can invert our construction and build a curved flat from an isothermic surface:

Theorem. *Given an isothermic surface $f : M^2 \to I\!R^3$ and its Euclidean dual surface $\hat{f} : M \to I\!R^3$ we get a curved flat $\gamma : M \to G_+(5,3)$ integrating the Maurer-Cartan form (13), which we are able to write down knowing the first and second fundamental forms of the immersions f and $\hat{f}$* [10].

Another way to obtain these two Euclidean immersions is presented in [4]. Applying Sym's formula to the associated family of frames $F = F(\lambda)$, we obtain a map

$$(\tfrac{\partial}{\partial\lambda}F)F^{-1}|_{\lambda=0} : M \to \mathfrak{p} \; ; \tag{22}$$

interpreting $\mathfrak{p}$ as two copies of Euclidean three space[11] $I\!R^3$ this map gives us the immersion f, and in the other copy of $I\!R^3$, its dual $\hat{f}$: this can be seen by looking at the differential

$$\begin{aligned} d(\tfrac{\partial}{\partial\lambda}F)F^{-1}|_{\lambda=0} &= F_0\Phi_{\mathfrak{p}}F_0^{-1} \\ &\cong H_3 \begin{pmatrix} e^u dx & -e^{-u}dx \\ e^u dy & e^{-u}dy \\ 0 & 0 \end{pmatrix} . \end{aligned} \tag{23}$$

Here $F_0 = \begin{pmatrix} H_3 & 0 \\ 0 & I_2 \end{pmatrix}$ solves the equation $F_0^{-1}dF_0 = \Phi_{\mathfrak{k}}$ and thus we may view $H_3 : M \to O(3)$ as a Euclidean framing of f resp. $\hat{f}$.

There is another possibility for producing isothermal surfaces in Euclidean space $I\!R^3$ (or S^3): that is, by using a solution of

4 Calapso's equation

To understand this, we write down the Maurer-Cartan form of a frame $F : M \to O_1(5)$, which is Möbius-invariantly connected to a given immersion: taking $f = Fe_4$ the (unique) isometric lift of the isothermic immersion and $n = Fe_3$ the central sphere congruence (conformal Gauß map) of the immersion, the frame is determined by the assumption of

[9] The Euclidean dual of an isothermic surface is obtained by integrating the closed 1-form $d\hat{f} := e^{-2u}(-f_x dx + f_y dy)$: see for example [2, p.14].

When the normal congruence of circles mentioned in footnote 8 is projected to Euclidean three space $I\!R^3$, we see that, in the limit $\lambda \to 0$, the circles become straight lines — circles meeting the collapsed surface $\hat{f}$ resp. f in the point at infinity — while the Ribeaucour sphere congruence enveloped by the two surfaces f and $\hat{f}$ becomes the congruence of tangent planes of f resp. $\hat{f}$.

[10] Since this construction depends on the conformal rather than the Euclidean geometry of the ambient space, we generally get a whole three parameter family of loops of curved flats from one isothermic surface: when viewing our given isothermic surface as a surface in the three sphere S^3, we may choose the point at infinity arbitrarily.

[11] Here the Euclidean metric is induced by the quadratic form $\frac{1}{2}\mathrm{tr}\Phi_{\mathfrak{p}}^t\Phi_{\mathfrak{p}}$ instead of the Killing form.

being an adapted frame (i.e. $Fe_1 = f_x$ and $Fe_2 = f_y$). The associated Maurer-Cartan form will be

$$\Phi = \begin{pmatrix} 0 & 0 & kdx & dx & \chi_1 \\ 0 & 0 & -kdy & dy & \chi_2 \\ -kdx & kdy & 0 & 0 & \tau \\ -\chi_1 & -\chi_2 & -\tau & 0 & 0 \\ -dx & -dy & 0 & 0 & 0 \end{pmatrix}, \tag{24}$$

k^2 being the conformal factor relating the metric induced by the central sphere congruence to the isometric one, and the 1-forms χ_1, χ_2 and τ to be determined. From the Maurer-Cartan equation for this form we learn that

$$\begin{aligned} \tau &= k_x dx - k_y dy \\ \chi_1 &= (\tfrac{1}{2}k^2 - u)dx - \tfrac{k_{xy}}{k}dy \\ \chi_2 &= -\tfrac{k_{xy}}{k}dx + (\tfrac{1}{2}k^2 + u)dy \ , \end{aligned} \tag{25}$$

where $u \in C^\infty(M)$ is a function satisfying the differential equation

$$du = -((\tfrac{k_{xy}}{k})_y + (k^2)_x)dx + ((\tfrac{k_{xy}}{k})_x + (k^2)_y)dy \tag{26}$$

— the integrability condition of this equation is a fourth order partial differential equation closely related to Calapso's original equation [3]:

$$0 = \Delta(\tfrac{k_{xy}}{k}) + 2(k^2)_{xy} \tag{27}$$

This shows, that

Theorem: *Any isothermic surface gives rise to a solution of Calapso's equation.*

Conversely, from a solution $k \in C^\infty(M)$ of Calapso's equation we can construct a Möbius invariant frame of an isothermic surface by integrating the Maurer-Cartan form (24), where the function u is a solution of (26).

Now, applying a conformal change $f \rightsquigarrow \frac{1}{k}f$ while fixing the central sphere congruence $n \rightsquigarrow n$, the Maurer-Cartan form of the associated frame becomes

$$\Phi = \begin{pmatrix} 0 & \omega & kdx & \frac{1}{k}dx & \chi_1 \\ -\omega & 0 & -kdy & \frac{1}{k}dy & \chi_2 \\ -kdx & kdy & 0 & 0 & 0 \\ -\chi_1 & -\chi_2 & 0 & 0 & 0 \\ -\frac{1}{k}dx & -\frac{1}{k}dy & 0 & 0 & 0 \end{pmatrix}, \tag{28}$$

where

$$\begin{aligned} \omega &= -\tfrac{k_y}{k}dx + \tfrac{k_x}{k}dy \\ \chi_1 &= k(\tfrac{k_{xx}}{k} - \tfrac{k_x^2+k_y^2}{2k^2} + \tfrac{1}{2}k^2 - u)dx \\ \chi_2 &= k(\tfrac{k_{yy}}{k} - \tfrac{k_x^2+k_y^2}{2k^2} + \tfrac{1}{2}k^2 + u)dy \end{aligned} \quad . \tag{29}$$

Here we see that the central sphere congruence of an isothermic surface is a Ribeaucour sphere congruence, which actually is a characterisation of isothermic surfaces (cf.[1, p.374]), and hence it has flat normal bundle as a codimension two surface in the Lorentz sphere S^4_1.

In general, the second enveloping surface of the central sphere congruence of an isothermic surface will not be an isothermic surface and it seems to be difficult to built a curved flat starting with it. But in a quite simple case this is possible:

5 Example

Starting with a surface of revolution

$$f(x,y) = (r(x)\cos y, r(x)\sin y, z(x)) \ , \tag{30}$$

the functions r and z satisfying the differential equation

$$r^2 = r'^2 + z'^2 \ , \tag{31}$$

i.e. the curve (r,z) being parametrized by arc length (thought of as a curve in the Poincaré half plane), we may write down the loop of Maurer-Cartan forms

$$\Phi_\lambda = \begin{pmatrix} 0 & -\frac{r'}{r}dy & -\frac{r'z''-r''z'}{r^2}dx & \lambda r dx & -\frac{\lambda}{r}dx \\ \frac{r'}{r}dy & 0 & -\frac{z'}{r}dy & \lambda r dy & \frac{\lambda}{r}dy \\ \frac{r'z''-r''z'}{r^2}dx & \frac{z'}{r}dy & 0 & 0 & 0 \\ \frac{\lambda}{r}dx & -\frac{\lambda}{r}dy & 0 & 0 & 0 \\ -\lambda r dx & -\lambda r dy & 0 & 0 & 0 \end{pmatrix} , \tag{32}$$

which gives us the immersion f and its dual $\hat{f}$ in the limit $\lambda \to 0$.

On the other hand, denoting by $H = \frac{1}{2}(\frac{z'}{r^2} + \frac{r'z''-r''z'}{r^3})$ the mean curvature of our surface of rotation, the central sphere congruence of f is $n + Hf$. The metric it induces has conformal factor k^2 (relative to the metric induced by f) given by

$$k = \frac{1}{2r^2}(rz' - r'z'' + r''z') \ . \tag{33}$$

Since $k_y \equiv 0$, this is obviously a solution of Calapso's equation and a function u solving (26) is $u = \lambda^2 - k^2$. So the Maurer-Cartan form (24) becomes

$$\Phi_\lambda = \begin{pmatrix} 0 & 0 & kdx & dx & (\frac{3}{2}k^2 - \lambda^2)dx \\ 0 & 0 & -kdy & dy & (-\frac{1}{2}k^2 + \lambda^2)dy \\ -kdx & kdy & 0 & 0 & k_x dx \\ -(\frac{3}{2}k^2 - \lambda^2)dx & (\frac{1}{2}k^2 - \lambda^2)dy & -k_x dx & 0 & 0 \\ -dx & -dy & 0 & 0 & 0 \end{pmatrix} . \tag{34}$$

A change $n \rightsquigarrow n + kf$ of the sphere congruence, enveloped by f, followed by an $O_1(2)$-gauge $f \rightsquigarrow \lambda f$ and $\hat{f} \rightsquigarrow \lambda^{-1}\hat{f}$ gives us the Maurer-Cartan form

$$\Phi_\lambda = \begin{pmatrix} 0 & 0 & 2kdx & \lambda dx & -\lambda dx \\ 0 & 0 & 0 & \lambda dy & \lambda dy \\ -2kdx & 0 & 0 & 0 & 0 \\ \lambda dx & -\lambda dy & 0 & 0 & 0 \\ -\lambda dx & -\lambda dy & 0 & 0 & 0 \end{pmatrix} . \tag{35}$$

of a curved flat, quite different from that coming from (32).

To understand the geometry of the two enveloping immersions $f = Fe_4$ and $\hat{f} = Fe_5$, we remark that the sphere congruence $n = Fe_3$ depends only on one variable and hence the two immersions parametrize a channel surface; moreover all spheres of the congruence are perpendicular to the fixed circle[12] given by $\mathrm{span}\{Fe_2, F(e_4 + e_5)\}$, which may be thought as an axis of rotation: the immersions f and $\hat{f}$ parametrize pieces of a surface of revolution[13], f and $\hat{f}$ being axisymmetric[14]. Taking now the limit $\lambda \to 0$, we obtain a cylinder resp. its dual, which is an (axial) reflection of the original cylinder.

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[12] We have $\Phi e_2 = -(e_4 + e_5)dy$ and $\Phi(e_4 + e_5) = 2e_2 dy$.

[13] The meridian curve is given by $\frac{1}{\sqrt{2}}(f - \hat{f})$ — which only depends on one variable — thought as a curve in the Poincaré half plane; its tangent field is given by Fe_1 and its unit normal field by $n = Fe_3$.

[14] The circles $\{F(p)e_1, F(p)e_2\}^\perp$ intersecting the sphere $n(p)$ orthogonally in $f(p)$ and $\hat{f}(p)$ all meet the axis (cf. footnote 8, page 5).