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Curved Flats and Isothermic Surfaces

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Abstract

We show how pairs of isothermic surfaces are given by curved flats in a pseudo-Riemannian symmetric space and vice versa. Calapso's fourth order partial differential equation is derived and, using a solution of this equation, a Möbius invariant frame for an isothermic surface is built.

1 Introduction

These notes grew out of a series of discussions on a recent paper by J. Cieśliński, P. Goldstein and A. Sym [4]: these authors give a characterization of isothermic surfaces as "soliton surfaces" by introducing a spectral parameter. In trying to understand the geometric meaning of this spectral parameter, we observed some analogies with the theory of conformally flat hypersurfaces in a four-dimensional space form: Guichard's nets may be understood as a kind of analogue of isothermic parametrizations of Riemannian surfaces (cf.[7, no.3.4.1]), and so it seems natural to look for relations between the theory of isothermic surfaces in three-dimensional space forms and the theory of conformally flat hypersurfaces in four-dimensional space forms. Here we would like to present some results we found — especially the possibility of constructing isothermic surfaces using

2 Curved Flats

A curved flat is the natural generalization of a developable surface in Euclidean space: it is a submanifold $M \subset G/K$ of a (pseudo-Riemannian) symmetric space for which the curvature operator of G/K vanishes on $\wedge^2 TM^1$. Thus, a curved flat may be thought of as the enveloping submanifold of a congruence of flats — totally geodesic submanifolds — of the symmetric space. Taking a regular parametrization $\gamma : M \rightarrow G/K$ of a curved flat and a framing $F : M \rightarrow G$ of this parametrization, the Maurer-Cartan form $\Phi = F^{-1}dF$

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¹Thus M is *curvature isotropic* in the sense of [6]

of the framing has a natural decomposition $\Phi = \Phi_{\mathfrak{k}} + \Phi_{\mathfrak{p}}$ according to the symmetric decomposition² $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of the Lie algebra \mathfrak{g} . Now the condition for γ to parametrize a curved flat may be formulated as³

$$(1) \quad [[\Phi_{\mathfrak{p}} \wedge \Phi_{\mathfrak{p}}], \mathfrak{p}] \equiv \mathfrak{o} .$$

In case that G is semisimple, it is straightforward⁴ to see that this is equivalent to

$$(2) \quad [\Phi_{\mathfrak{p}} \wedge \Phi_{\mathfrak{p}}] \equiv 0 .$$

To summarise, we have the

Definition of a curved flat: An immersion $\gamma : M \rightarrow G/K$ is said to parametrize a CURVED FLAT, if the \mathfrak{p} -part in the symmetric decomposition of the Maurer-Cartan form $F^{-1}dF = \Phi = \Phi_{\mathfrak{k}} + \Phi_{\mathfrak{p}}$ of a framing $F : M \rightarrow G$ of γ defines a congruence $p \mapsto \Phi_{\mathfrak{p}}|_p(T_pM)$ of abelian subalgebras of \mathfrak{g} .

At this point we should remark that curved flats naturally arise in one parameter families [6]: setting

$$(3) \quad \Phi_{\lambda} := \Phi_{\mathfrak{k}} + \lambda \Phi_{\mathfrak{p}}$$

the Maurer-Cartan equation $d\Phi_{\lambda} + \frac{1}{2}[\Phi_{\lambda} \wedge \Phi_{\lambda}] = 0$ for the loop $\lambda \mapsto \Phi_{\lambda}$ of forms splits into the three equations

$$(4) \quad \begin{aligned} 0 &= d\Phi_{\mathfrak{k}} + \frac{1}{2}[\Phi_{\mathfrak{k}} \wedge \Phi_{\mathfrak{k}}] \\ 0 &= d\Phi_{\mathfrak{p}} + [\Phi_{\mathfrak{k}} \wedge \Phi_{\mathfrak{p}}] \\ 0 &= [\Phi_{\mathfrak{p}} \wedge \Phi_{\mathfrak{p}}] , \end{aligned}$$

and hence the integrability of the loop $\lambda \mapsto \Phi_{\lambda}$ is equivalent to the forms Φ_{λ} being the Maurer-Cartan forms for some framings $F_{\lambda} : M \rightarrow G$ of curved flats $\gamma_{\lambda} : M \rightarrow G/K$. Thus integrable systems theory may be applied to produce examples.

Now we will consider the case leading to the theory of isothermic surfaces: let

$$(5) \quad G := O_1(5) \quad \text{and} \quad K := O(3) \times O_1(2) .$$

²Thus \mathfrak{k} and \mathfrak{p} are the +1 and -1-eigenspaces, respectively, of the involution fixing \mathfrak{k} and so satisfy the characteristic conditions

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}, [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k} .$$

³The product

$$[\Phi \wedge \Psi](v, w) := [\Phi(v), \Psi(w)] - [\Phi(w), \Psi(v)]$$

defines a symmetric product on the space of Lie algebra valued 1-forms with values in the space of Lie algebra valued 2-forms.

⁴In fact, $[\mathfrak{p}, \mathfrak{p}] \oplus \mathfrak{p}$ is an ideal of \mathfrak{g} so that we have a decomposition $\mathfrak{g} = \mathfrak{k}' \oplus [\mathfrak{p}, \mathfrak{p}] \oplus \mathfrak{p}$ where \mathfrak{k}' is a complementary ideal commuting with $[\mathfrak{p}, \mathfrak{p}] \oplus \mathfrak{p}$. Thus, if $\mathfrak{a} \subset \mathfrak{p}$ satisfies $[[\mathfrak{a}, \mathfrak{a}], \mathfrak{p}] = \mathfrak{o}$ we deduce that $[\mathfrak{a}, \mathfrak{a}]$ lies in the center of \mathfrak{g} and so vanishes.

The coset space $G_+(5, 3) = G/K$ of space-like 3-planes in the Minkowski space \mathbb{R}_1^5 becomes a six dimensional pseudo-Riemannian symmetric space of signature $(3, 3)$ when endowed with the metric induced by the Killing form. We will consider two-dimensional curved flats

$$(6) \quad \gamma : M^2 \rightarrow G_+(5, 3)$$

satisfying the regularity assumption that the metric on M^2 induced by γ is non-degenerate.

Fixing a pseudo orthonormal basis (e_1, \dots, e_5) of the Minkowski space \mathbb{R}_1^5 with

$$(7) \quad (\langle e_i, e_j \rangle)_{ij} = E_5 := \begin{pmatrix} I_3 & 0 \\ 0 & \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \end{pmatrix},$$

we get the matrix representations

$$(8) \quad \begin{aligned} O_1(5) &= \{A \in Gl(5, \mathbb{R}) \mid A^t E_5 A = E_5\} \\ \mathfrak{o}_1(5) &= \{\mathfrak{X} \in \mathfrak{gl}(5, \mathbb{R}) \mid (\mathfrak{E}_5 \mathfrak{X}) + (\mathfrak{E}_5 \mathfrak{X})^t = \mathfrak{o}\}. \end{aligned}$$

The subalgebra \mathfrak{k} and its complementary linear subspace \mathfrak{p} in the symmetric decomposition of $\mathfrak{o}_1(5)$ are given by the +1- resp. -1-eigenspaces of the involutive automorphism $\text{Ad}(Q) : \mathfrak{o}_1(5) \rightarrow \mathfrak{o}_1(5)$ with $Q = \begin{pmatrix} -I_3 & 0 \\ 0 & I_2 \end{pmatrix}$. Writing down the Maurer-Cartan form of a framing $F : M^2 \rightarrow O_1(5)$ of our curved flat $\gamma : M^2 \rightarrow G_+(5, 3)$ with this notation we obtain

$$(9) \quad \begin{aligned} F^{-1}dF &= \Phi = \Phi_{\mathfrak{k}} + \Phi_{\mathfrak{p}} \quad \text{with} \\ \Phi_{\mathfrak{k}} &= \begin{pmatrix} \Omega & 0 \\ 0 & \nu \end{pmatrix} : TM \rightarrow \mathfrak{o}(3) \times \mathfrak{o}_1(2) \\ \Phi_{\mathfrak{p}} &= \begin{pmatrix} 0 & \eta \\ -E_2 \eta^t & 0 \end{pmatrix} : TM \rightarrow \mathfrak{p}. \end{aligned}$$

The image of $\Phi_{\mathfrak{p}}$ at each $p \in M^2$ is a 2-dimensional abelian subspace of \mathfrak{p} on which the Killing form is non-degenerate. One can show that there are precisely two K -orbits of maximal abelian subspaces of \mathfrak{p} : one consists of 3-dimensional subspaces which are isotropic for the Killing form while the other consists of 2-dimensional subspaces on which the Killing form has signature $(1, 1)$. We therefore conclude that the images of each $\Phi_{\mathfrak{p}}$ are maximal abelian and K -conjugate and so we can put η into the standard form

$$(10) \quad \eta = \begin{pmatrix} \omega_1 & -\omega_1 \\ \omega_2 & \omega_2 \\ 0 & 0 \end{pmatrix}$$

by applying a gauge transformation $M \rightarrow K$.

Calculating the Maurer-Cartan equation using the ansatz

$$(11) \quad \Omega = \begin{pmatrix} 0 & \omega & -\psi_1 \\ -\omega & 0 & -\psi_2 \\ \psi_1 & \psi_2 & 0 \end{pmatrix} \quad \text{and} \quad \nu = \begin{pmatrix} \nu & 0 \\ 0 & -\nu \end{pmatrix}$$

together with η given by (10), we see that

$$(12) \quad d\omega_1 = d\omega_2 = 0 .$$

So we are given canonical coordinates $(x, y) : M \rightarrow \mathbb{R}^2$ by integrating⁵ the forms ω_1 and ω_2 . Moreover, since we also have $d\nu = 0$, we may set $\nu = -du$ for a suitable function $u \in C^\infty(M)$ — this gives us $\omega = u_y dx - u_x dy$, where u_x and u_y denote the partial derivatives of u in x - resp. y -directions. Finally, the equations $\psi_1 \wedge \omega_1 = 0$ and $\psi_2 \wedge \omega_2 = 0$ show that $\psi_1 = e^u k_1 dx$ and $\psi_2 = e^u k_2 dy$ for two functions $k_i \in C^\infty(M)$.

We now perform a final $O_1(2)$ -gauge $\begin{pmatrix} I_3 & 0 \\ 0 & e^u & 0 \\ & 0 & e^{-u} \end{pmatrix} : M \rightarrow O(3) \times O_1(2)$ and insert the spectral parameter λ to obtain the Maurer-Cartan form discussed in (cf.[4]):

$$(13) \quad \Phi_\lambda = \begin{pmatrix} 0 & u_y dx - u_x dy & -e^u k_1 dx & \lambda e^u dx & -\lambda e^{-u} dx \\ -u_y dx + u_x dy & 0 & -e^u k_2 dy & \lambda e^u dy & \lambda e^{-u} dy \\ e^u k_1 dx & e^u k_2 dy & 0 & 0 & 0 \\ \lambda e^{-u} dx & -\lambda e^{-u} dy & 0 & 0 & 0 \\ -\lambda e^u dx & -\lambda e^u dy & 0 & 0 & 0 \end{pmatrix} .$$

We are now lead directly to the theory of

3 Isothermic Surfaces

In the context of Möbius geometry the three sphere S^3 is viewed as the projective light-cone IPL^4 in \mathbb{R}_1^5 while the Lorentzian sphere $\{v \in \mathbb{R}_1^5 | \langle v, v \rangle = 1\}$ should be interpreted as the space of (oriented) spheres in the three sphere⁶ (cf.[1]). Now, denoting by

$$(14) \quad \begin{aligned} n &:= Fe_3 : M \rightarrow S_1^5 = \{v \in \mathbb{R}_1^5 | \langle v, v \rangle = 1\} \\ f &:= Fe_4 : M \rightarrow L^4 = \{v \in \mathbb{R}_1^5 | \langle v, v \rangle = 0\} \\ \hat{f} &:= Fe_5 : M \rightarrow L^4 \end{aligned}$$

one of the sphere congruences resp. the two immersions given by our frame F , we see that

Theorem: *The sphere congruence n given by our curved flat is a Ribeaucour sphere congruence⁷, which is enveloped by two isothermic immersions f and \hat{f} (cf.[1, p.362]):*

Since

$$(15) \quad \begin{aligned} \langle f, n \rangle &= 0 & \text{and} & & \langle df, n \rangle &\equiv 0, \\ \langle \hat{f}, n \rangle &= 0 & \text{and} & & \langle d\hat{f}, n \rangle &\equiv 0, \end{aligned}$$

⁵Since our theory is local, all closed forms may be assumed to be exact.

⁶Or, equivalently, it may be interpreted as the space of (oriented) spheres and planes in Euclidean three space \mathbb{R}^3 : the polar hyperplane to a vector v of the Lorentz sphere intersects the three sphere — thought of as the absolute quadric in projective four space — in a two sphere. Stereographic projection yields a sphere in \mathbb{R}^3 or, if the projection center lies on the sphere, a plane.

⁷The curvature lines on the two enveloping immersions correspond.

the immersions f and \hat{f} do envelop the sphere congruence n and, since the bilinear forms

$$(16) \quad \begin{aligned} \langle df, dn \rangle &= \lambda e^{2u} (k_1 dx^2 + k_2 dy^2), \\ \langle d\hat{f}, dn \rangle &= \lambda (-k_1 dx^2 + k_2 dy^2) \end{aligned}$$

are diagonal with respect to the induced metrics

$$(17) \quad \begin{aligned} \langle df, df \rangle &= \lambda^2 e^{2u} (dx^2 + dy^2), \\ \langle d\hat{f}, d\hat{f} \rangle &= \lambda^2 e^{-2u} (dx^2 + dy^2), \end{aligned}$$

the two immersions f and \hat{f} are isothermic⁸.

It is quite difficult to calculate the first and second fundamental forms of these isothermic immersions, when they are projected to S^3 resp. \mathbb{R}^3 , but applying a (constant) conformal change (constant $O_1(2)$ -gauge)

$$(18) \quad \begin{aligned} f &\rightsquigarrow \frac{1}{\lambda} f & \text{and} & \quad \hat{f} \rightsquigarrow \lambda \hat{f} & \text{or} \\ f &\rightsquigarrow \lambda f & \text{and} & \quad \hat{f} \rightsquigarrow \frac{1}{\lambda} \hat{f} \end{aligned}$$

and sending $\lambda \rightarrow 0$, \hat{f} resp. f becomes a constant vector — $\Phi_{\lambda=0} e_5$ resp. $\Phi_{\lambda=0} e_4$ vanishes. This constant light-like vector may be interpreted as the point at infinity and we therefore obtain an isothermic immersion $f : M \rightarrow \mathbb{R}^3$ with first and second fundamental forms

$$(19) \quad \begin{aligned} I &= e^{2u} (dx^2 + dy^2) \\ II &= e^{2u} (k_1 dx^2 + k_2 dy^2) \end{aligned}$$

resp. its Euclidean dual surface $\hat{f} : M \rightarrow \mathbb{R}^3$ with first and second fundamental forms

$$(20) \quad \begin{aligned} \hat{I} &= e^{-2u} (dx^2 + dy^2) \\ \hat{II} &= -k_1 dx^2 + k_2 dy^2. \end{aligned}$$

We now recognise the remaining three equations from the Maurer-Cartan equation for Φ_λ

$$(21) \quad \begin{aligned} 0 &= \Delta u + e^{2u} k_1 k_2 \\ 0 &= k_{1y} + (k_1 - k_2) u_y \\ 0 &= k_{2x} - (k_1 - k_2) u_x \end{aligned}$$

⁸The bundle defined by $\text{span}\{n, f, \hat{f}\}$ over M is flat (cf.(13)) and so the map $p \mapsto d_p f(T_p M)$ defines a normal congruence of circles [5]: for each $p \in M$

$$t \mapsto f_t(p) := \frac{1}{\sqrt{2}} \sin t \cdot n(p) + \frac{1}{2} (1 + \cos t) \cdot f(p) - \frac{1}{2} (1 - \cos t) \cdot \hat{f}(p)$$

parametrizes the circle $(d_p f(T_p M))^\perp$ meeting the sphere $n(p)$ in $f(p)$ and $\hat{f}(p)$ orthogonal. Since n , f and \hat{f} are parallel sections in this bundle, the maps $p \mapsto f_t(p)$ (which generically are not degenerate) parametrize the surfaces orthogonal to this congruence of circles.

In general the immersions f and $\hat{f} = f_\pi$ will be the only isothermic surfaces among the surfaces.

as the Gauß and Codazzi equations of the Euclidean immersion f resp. its dual \hat{f} ⁹ [3], [4]. As a consequence, we can invert our construction and build a curved flat from an isothermic surface:

Theorem. *Given an isothermic surface $f : M^2 \rightarrow \mathbb{R}^3$ and its Euclidean dual surface $\hat{f} : M \rightarrow \mathbb{R}^3$ we get a curved flat $\gamma : M \rightarrow G_+(5, 3)$ integrating the Maurer-Cartan form (13), which we are able to write down knowing the first and second fundamental forms of the immersions f and \hat{f} ¹⁰.*

Another way to obtain these two Euclidean immersions is presented in [4]. Applying Sym's formula to the associated family of frames $F = F(\lambda)$, we obtain a map

$$(22) \quad \left(\frac{\partial}{\partial \lambda} F\right)F^{-1}\Big|_{\lambda=0} : M \rightarrow \mathfrak{p} ;$$

interpreting \mathfrak{p} as two copies of Euclidean three space¹¹ \mathbb{R}^3 this map gives us the immersion f , and in the other copy of \mathbb{R}^3 , its dual \hat{f} : this can be seen by looking at the differential

$$(23) \quad \begin{aligned} d\left(\frac{\partial}{\partial \lambda} F\right)F^{-1}\Big|_{\lambda=0} &= F_0 \Phi_{\mathfrak{p}} F_0^{-1} \\ &\cong H_3 \begin{pmatrix} e^u dx & -e^{-u} dx \\ e^u dy & e^{-u} dy \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Here $F_0 = \begin{pmatrix} H_3 & 0 \\ 0 & I_2 \end{pmatrix}$ solves the equation $F_0^{-1} dF_0 = \Phi_{\mathfrak{t}}$ and thus we may view $H_3 : M \rightarrow O(3)$ as a Euclidean framing of f resp. \hat{f} .

There is another possibility for producing isothermal surfaces in Euclidean space \mathbb{R}^3 (or S^3): that is, by using a solution of

4 Calapso's equation

To understand this, we write down the Maurer-Cartan form of a frame $F : M \rightarrow O_1(5)$, which is Möbius-invariantly connected to a given immersion: taking $f = Fe_4$ the (unique) isometric lift of the isothermic immersion and $n = Fe_3$ the central sphere congruence (conformal Gauß map) of the immersion, the frame is determined by the assumption of

⁹The Euclidean dual of an isothermic surface is obtained by integrating the closed 1-form $d\hat{f} := e^{-2u}(-f_x dx + f_y dy)$: see for example [2, p.14].

When the normal congruence of circles mentioned in footnote 8 is projected to Euclidean three space \mathbb{R}^3 , we see that, in the limit $\lambda \rightarrow 0$, the circles become straight lines — circles meeting the collapsed surface \hat{f} resp. f in the point at infinity — while the Ribeaucour sphere congruence enveloped by the two surfaces f and \hat{f} becomes the congruence of tangent planes of f resp. \hat{f} .

¹⁰Since this construction depends on the conformal rather than the Euclidean geometry of the ambient space, we generally get a whole three parameter family of loops of curved flats from one isothermic surface: when viewing our given isothermic surface as a surface in the three sphere S^3 , we may choose the point at infinity arbitrarily.

¹¹Here the Euclidean metric is induced by the quadratic form $\frac{1}{2}\text{tr}\Phi_{\mathfrak{p}}^t \Phi_{\mathfrak{p}}$ instead of the Killing form.

being an adapted frame (i.e. $Fe_1 = f_x$ and $Fe_2 = f_y$). The associated Maurer-Cartan form will be

$$(24) \quad \Phi = \begin{pmatrix} 0 & 0 & kdx & dx & \chi_1 \\ 0 & 0 & -kdy & dy & \chi_2 \\ -kdx & kdy & 0 & 0 & \tau \\ -\chi_1 & -\chi_2 & -\tau & 0 & 0 \\ -dx & -dy & 0 & 0 & 0 \end{pmatrix},$$

k^2 being the conformal factor relating the metric induced by the central sphere congruence to the isometric one, and the 1-forms χ_1 , χ_2 and τ to be determined. From the Maurer-Cartan equation for this form we learn that

$$(25) \quad \begin{aligned} \tau &= k_x dx - k_y dy \\ \chi_1 &= \left(\frac{1}{2}k^2 - u\right)dx - \frac{k_{xy}}{k} dy \\ \chi_2 &= -\frac{k_{xy}}{k} dx + \left(\frac{1}{2}k^2 + u\right)dy, \end{aligned}$$

where $u \in C^\infty(M)$ is a function satisfying the differential equation

$$(26) \quad du = -\left(\left(\frac{k_{xy}}{k}\right)_y + (k^2)_x\right)dx + \left(\left(\frac{k_{xy}}{k}\right)_x + (k^2)_y\right)dy$$

— the integrability condition of this equation is a fourth order partial differential equation closely related to Calapso's original equation [3]:

$$(27) \quad 0 = \Delta\left(\frac{k_{xy}}{k}\right) + 2(k^2)_{xy}$$

This shows, that

Theorem: *Any isothermic surface gives rise to a solution of Calapso's equation.*

Conversely, from a solution $k \in C^\infty(M)$ of Calapso's equation we can construct a Möbius invariant frame of an isothermic surface by integrating the Maurer-Cartan form (24), where the function u is a solution of (26).

Now, applying a conformal change $f \rightsquigarrow \frac{1}{k}f$ while fixing the central sphere congruence $n \rightsquigarrow n$, the Maurer-Cartan form of the associated frame becomes

$$(28) \quad \Phi = \begin{pmatrix} 0 & \omega & kdx & \frac{1}{k}dx & \chi_1 \\ -\omega & 0 & -kdy & \frac{1}{k}dy & \chi_2 \\ -kdx & kdy & 0 & 0 & 0 \\ -\chi_1 & -\chi_2 & 0 & 0 & 0 \\ -\frac{1}{k}dx & -\frac{1}{k}dy & 0 & 0 & 0 \end{pmatrix},$$

where

$$(29) \quad \begin{aligned} \omega &= -\frac{k_y}{k}dx + \frac{k_x}{k}dy \\ \chi_1 &= k\left(\frac{k_{xx}}{k} - \frac{k_x^2 + k_y^2}{2k^2}\right) + \frac{1}{2}k^2 - u)dx \quad . \\ \chi_2 &= k\left(\frac{k_{yy}}{k} - \frac{k_x^2 + k_y^2}{2k^2}\right) + \frac{1}{2}k^2 + u)dy \end{aligned}$$

Here we see that the central sphere congruence of an isothermic surface is a Ribeaucour sphere congruence, which actually is a characterisation of isothermic surfaces (cf.[1,

p.374]), and hence it has flat normal bundle as a codimension two surface in the Lorentz sphere S_1^4 .

In general, the second enveloping surface of the central sphere congruence of an isothermic surface will not be an isothermic surface and it seems to be difficult to build a curved flat starting with it. But in a quite simple case this is possible:

5 Example

Starting with a surface of revolution

$$(30) \quad f(x, y) = (r(x) \cos y, r(x) \sin y, z(x)) ,$$

the functions r and z satisfying the differential equation

$$(31) \quad r^2 = r'^2 + z'^2 ,$$

i.e. the curve (r, z) being parametrized by arc length (thought of as a curve in the Poincaré half plane), we may write down the loop of Maurer-Cartan forms

$$(32) \quad \Phi_\lambda = \begin{pmatrix} 0 & -\frac{z'}{r} dy & -\frac{r'z''-r''z'}{r^2} dx & \lambda r dx & -\frac{\lambda}{r} dx \\ \frac{r'}{r} dy & 0 & -\frac{z'}{r} dy & \lambda r dy & \frac{\lambda}{r} dy \\ \frac{r'z''-r''z'}{r^2} dx & \frac{z'}{r} dy & 0 & 0 & 0 \\ \frac{\lambda}{r} dx & -\frac{\lambda}{r} dy & 0 & 0 & 0 \\ -\lambda r dx & -\lambda r dy & 0 & 0 & 0 \end{pmatrix} ,$$

which gives us the immersion f and its dual \hat{f} in the limit $\lambda \rightarrow 0$.

On the other hand, denoting by $H = \frac{1}{2}(\frac{z'}{r^2} + \frac{r'z''-r''z'}{r^3})$ the mean curvature of our surface of rotation, the central sphere congruence of f is $n + Hf$. The metric it induces has conformal factor k^2 (relative to the metric induced by f) given by

$$(33) \quad k = \frac{1}{2r^2}(rz' - r'z'' + r''z') .$$

Since $k_y \equiv 0$, this is obviously a solution of Calapso's equation and a function u solving (26) is $u = \lambda^2 - k^2$. So the Maurer-Cartan form (24) becomes

$$(34) \quad \Phi_\lambda = \begin{pmatrix} 0 & 0 & k dx & dx & (\frac{3}{2}k^2 - \lambda^2) dx \\ 0 & 0 & -k dy & dy & (-\frac{1}{2}k^2 + \lambda^2) dy \\ -k dx & k dy & 0 & 0 & k_x dx \\ -(\frac{3}{2}k^2 - \lambda^2) dx & (\frac{1}{2}k^2 - \lambda^2) dy & -k_x dx & 0 & 0 \\ -dx & -dy & 0 & 0 & 0 \end{pmatrix} .$$

A change $n \rightsquigarrow n + kf$ of the sphere congruence, enveloped by f , followed by an $O_1(2)$ -gauge $f \rightsquigarrow \lambda f$ and $\hat{f} \rightsquigarrow \lambda^{-1} \hat{f}$ gives us the Maurer-Cartan form

$$(35) \quad \Phi_\lambda = \begin{pmatrix} 0 & 0 & 2k dx & \lambda dx & -\lambda dx \\ 0 & 0 & 0 & \lambda dy & \lambda dy \\ -2k dx & 0 & 0 & 0 & 0 \\ \lambda dx & -\lambda dy & 0 & 0 & 0 \\ -\lambda dx & -\lambda dy & 0 & 0 & 0 \end{pmatrix} .$$

of a curved flat, quite different from that coming from (32).

To understand the geometry of the two enveloping immersions $f = Fe_4$ and $\hat{f} = Fe_5$, we remark that the sphere congruence $n = Fe_3$ depends only on one variable and hence the two immersions parametrize a channel surface; moreover all spheres of the congruence are perpendicular to the fixed circle¹² given by $\text{span}\{Fe_2, F(e_4 + e_5)\}$, which may be thought as an axis of rotation: the immersions f and \hat{f} parametrize pieces of a surface of revolution¹³, f and \hat{f} being axisymmetric¹⁴. Taking now the limit $\lambda \rightarrow 0$, we obtain a cylinder resp. its dual, which is an (axial) reflection of the original cylinder.

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¹²We have $\Phi e_2 = -(e_4 + e_5)dy$ and $\Phi(e_4 + e_5) = 2e_2dy$.

¹³The meridian curve is given by $\frac{1}{\sqrt{2}}(f - \hat{f})$ — which only depends on one variable — thought as a curve in the Poincaré half plane; its tangent field is given by Fe_1 and its unit normal field by $n = Fe_3$.

¹⁴The circles $\{F(p)e_1, F(p)e_2\}^\perp$ intersecting the sphere $n(p)$ orthogonally in $f(p)$ and $\hat{f}(p)$ all meet the axis (cf. footnote 8, page 5).