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# PROPERTIES OF SINGULAR SCHUBERT VARIETIES

A Dissertation Presented

by

JENNIFER KOONZ

Submitted to the Graduate School of the  
University of Massachusetts Amherst in partial fulfillment  
of the requirements for the degree of

DOCTOR OF PHILOSOPHY

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Department of Mathematics and Statistics

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JENNIFER KOONZ

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# ABSTRACT

PROPERTIES OF SINGULAR SCHUBERT VARIETIES

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This thesis deals with the study of Schubert varieties, which are subsets of flag varieties indexed by elements of Weyl groups. We start by defining Lascoux elements in the Hecke algebra, and showing that they coincide with the Kazhdan-Lusztig basis elements in certain cases. We then construct a resolution  $(Z_w, \pi)$  of the Schubert variety  $X_w$  for which  $R\pi_*(\mathbb{C}[\ell(w)])$  is a sheaf on  $X_w$  whose expression in the Hecke algebra is closely related to the Lascoux element. We also define two new polynomials which coincide with the intersection cohomology Poincaré polynomial in certain cases. In the final chapter, we discuss some interesting combinatorial results concerning Bell and Catalan numbers which arose throughout the course of this work.

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# CHAPTER 1

## INTRODUCTION

Let  $G$  be a simple linear algebraic group over the complex numbers. Let  $B$  be a Borel subgroup in  $G$ , and let  $T$  be a maximal torus in  $B$ . The  $T$ -fixed points of the action of  $T$  on the flag variety  $G/B$ , denoted  $e_w$ , correspond bijectively with the elements of the Weyl group  $W = N_G(T)/T$ . The *Schubert variety*  $X_w$  is the closure of the  $B$ -orbit of  $e_w$ . Since Schubert varieties are indexed by elements of the Weyl group of  $G$ , many geometric properties of Schubert varieties can be determined by studying the combinatorial properties of the corresponding Weyl group elements.

For example, when  $G = SL(n)$  is the group of all  $n \times n$  matrices over  $\mathbb{C}$  with determinant 1, we can take  $B$  to be the set of all upper triangular matrices in  $G$ , and  $T$  to be the set of diagonal matrices in  $G$ . Then the normalizer  $N_G(T)$  of  $T$  in  $G$  is the set of all matrices in  $G$  which have exactly one nonzero entry in each row and column. From this we can see that  $W = N_G(T)/T \cong S_n$ , the symmetric group on  $n$  letters. Let  $V = \mathbb{C}^n$  with the standard basis  $e_1, e_2, \dots, e_n$ . A (complete) flag  $F$  in  $\mathbb{C}^n$  is a sequence of subspaces  $\{0\} = V_0 \subset V_1 \subset V_2 \subset \dots \subset V_n = \mathbb{C}^n$  where  $\dim(V_i) = i$  for each  $1 \leq i \leq n$ . The flag variety  $\mathcal{F}_n$  has as points flags in  $\mathbb{C}^n$ . Any flag  $F = (V_0 \subset V_1 \subset \dots \subset V_n) \in \mathcal{F}_n$  can be represented by an invertible  $n \times n$  matrix  $A$ , where the first column of  $A$  spans  $V_1$ , and in general

the first  $i$  columns of  $A$  span  $V_i$ . Identifying flags with matrices in this way, we can see that  $G = SL(n)$  acts on  $\mathcal{F}_n$  transitively by matrix multiplication, and that two matrices  $A_1$  and  $A_2$  represent the same flag in  $\mathcal{F}_n$  if and only if  $A_1 = A_2b$  for some  $b \in B$ . Hence  $\mathcal{F}_n \cong G/B$ . The case of  $G = SL(n)$  and  $W = S_n$  is referred to as type  $A_{n-1}$  (or type  $A$ ) using the Coxeter-Killing classification of simple algebraic groups.

## 1.1 Outline

We have two main goals for this thesis, which are closely related to each other. We wish to compute and study the Kazhdan-Lusztig basis elements of the Hecke algebra (which are equivalent in definition to Kazhdan-Lusztig polynomials) and the intersection cohomology Poincaré polynomials of Schubert varieties (which can be computed using Kazhdan-Lusztig polynomials) in a combinatorial and efficient manner. The Kazhdan-Lusztig basis elements were developed by Kazhdan and Lusztig in [18] for the purpose of satisfying certain desirable properties.

In Chapter 2, we will provide notation, terminology, and results from the existing literature which we will assume and reference throughout the rest of the document.

In Chapter 3, we seek to develop a new combinatorial method for computing the Kazhdan-Lusztig basis elements  $C_w$  of the Hecke algebra  $\mathcal{H}$ . The definition of these elements  $C_w$ , which are equivalent to the definition of Kazhdan-Lusztig polynomials, was developed by Kazhdan and Lusztig in 1979 [18]. Kazhdan-Lusztig basis elements, and thus Kazhdan-Lusztig polynomials, are difficult to

compute in general. Simpler methods for these computations have been developed in limited cases (e.g. see [22] and [6], for example). Lascoux described an efficient way to factor and compute the element  $C_w$  when  $w \in S_n$  corresponds to a rationally smooth Schubert variety [22]. We show that by generalizing Lascoux's factorization in a natural way, we can define what we call Lascoux elements  $L_w$  associated to fixed reduced expressions of elements  $w \in W$ . We show that for any rationally smooth element  $w \in S_n$ , there exists a reduced expression for which  $L_w$  coincides with the basis element  $C_w$ . We will then prove that a similar factorization of  $C_w$  holds for certain non-rationally smooth elements in  $S_n$  as well.

In Chapter 4, we reprove and extend results of Ryan [27] and Wolper [34], who showed that if  $w \in S_n$  corresponds to a rationally smooth Schubert variety  $X_w$ , then  $X_w$  is isomorphic to an iterated fibration  $X_w = F_0 \rightarrow F_1 \rightarrow \cdots \rightarrow F_r$  such that each fiber  $F_i/F_{i+1}$ , as well as the final space  $F_r$ , is isomorphic to a Grassmannian. In particular, for any  $w$  belonging to a general Weyl group  $W$ , we construct a resolution  $(Z_w, \pi)$  of  $X_w$  and show that this resolution is an iterated fibration of partial flag varieties. These resolutions are deeply connected to the Lascoux elements of Chapter 3, for  $R\pi_*(\mathbb{C}[\ell(w)])$  is a sheaf on  $X_w$  whose expression in the Hecke algebra is essentially the Lascoux element. When  $W = S_n$ , this resolution can be reinterpreted combinatorially, and using combinatorial methods, we are able to recover the results of Ryan and Wolper in a very explicit way.

In Chapters 5 and 6, we combinatorially develop new polynomials which are relatively simple and efficient to compute. These polynomials, called the inversion polynomial  $\mathcal{N}_w(q)$  and the closure polynomial  $\mathcal{M}_w(q)$ , both coincide with the ordinary Poincaré polynomial  $\mathcal{P}_w(q)$  when  $X_w$  is a rationally smooth Schu-

bert variety of type  $A$ , and both also coincide with the intersection cohomology Poincaré polynomial in some non-rationally smooth type  $A$  cases as well.

One unintended but mathematically relevant consequence of these techniques led us to the study of Bell and Catalan sequences, which have arisen in many unexpected and unrelated ways in combinatorics. A sequence of positive integers  $(a_m, a_{m-1}, \dots, a_1)$  is called a Bell sequence if  $a_1 = 1$  and if for each  $1 < i \leq m$ , we have  $a_i \leq 1 + \max\{a_j : 1 \leq j < i\}$ . A Bell sequence is called a Catalan sequence if it satisfies the stricter condition that  $a_i \leq 1 + a_{i-1}$  for each  $1 \leq i < m$ . These sequences are so named because the number of Catalan sequences with entries from  $\{1, 2, \dots, n\}$  is  $C_n$ , the  $n$ -th Catalan number, and the number of Bell sequences with entries from  $\{1, 2, \dots, n\}$  is the  $n$ -th Bell number [15]. In Chapter 7, we will elaborate on properties of Bell and Catalan sequences discovered in the course of our investigation of Schubert varieties.

# CHAPTER 2

## BACKGROUND MATERIAL

In this chapter, we will review definitions, facts, and results from the literature which will be assumed throughout the rest of the thesis.

### 2.1 Root Systems and Weyl groups

A (*crystallographic*) root system  $\Phi$  of rank  $n$  is a collection of vectors in Euclidean space  $\mathbb{R}^n$  which satisfy the following four properties.

1. We have  $\text{span}(\Phi) = \mathbb{R}^n$ .
2. The only scalar multiples of a vector  $\alpha \in \Phi$  that belong to  $\Phi$  are  $\pm\alpha$ .
3. Let  $\alpha \in \Phi$  and let  $s_\alpha : \mathbb{R}^n \rightarrow \mathbb{R}^n$  denote reflection over the hyperplane orthogonal to  $\alpha$ . Then  $s_\alpha(\Phi) = \Phi$ .
4. If  $\alpha_1, \alpha_2 \in \Phi$ , then the projection of  $\alpha_1$  onto  $\text{span}(\alpha_2)$  is a half-integral multiple of  $\alpha_2$ .

The elements of  $\Phi$  are called *roots*. We can fix a set of *positive roots*  $\Phi^+ \subset \Phi$  to be any subset which satisfies the conditions that

- for any  $\alpha \in \Phi$ , exactly one of  $\alpha, -\alpha$  lie in  $\Phi^+$ , and
- If  $\alpha_1, \alpha_2 \in \Phi^+$  and  $\alpha_1 + \alpha_2 \in \Phi$ , then  $\alpha_1 + \alpha_2 \in \Phi^+$ .

The roots belonging to the set  $\Phi^- := \Phi \setminus \Phi^+$  are called *negative roots*. We write  $\alpha \succ 0$  if  $\alpha \in \Phi^+$  and  $\alpha \prec 0$  if  $\alpha \in \Phi^-$ . Consistent with this notation, for any  $\beta, \gamma \in \mathbb{R}^n$ , we write  $\beta \prec \gamma$  if  $\gamma - \beta$  is a sum of nonnegative roots [16].

Once  $\Phi^+$  is fixed, there exists a unique basis  $\Delta = \{\alpha_1, \alpha_2, \dots, \alpha_n\} \subset \Phi^+$  for  $\mathbb{R}^n$  which satisfies the condition that for any  $\beta \in \Phi$ , we can express  $\beta$  as an integral sum  $\beta = \sum_{\alpha \in \Delta} c_\alpha \alpha$  where all of the coefficients  $c_\alpha$  are either nonnegative or nonpositive. Such a basis always exists (see [16]). Then for any  $\beta = \sum c_\alpha \alpha \in \Phi$ , this expression of  $\beta$  is unique, and we define the *height* of  $\beta$  to be the sum of the coefficients  $\sum c_\alpha$ . We call the elements of  $\Delta$  *simple roots*.

For each  $\alpha_i \in \Delta$ , let  $s_i$  denote reflection over the hyperplane orthogonal to  $\alpha_i$ . The reflection group  $W$  generated by the reflections  $s_1, \dots, s_n$  is called the *Weyl group* associated to  $\Phi$ , and the generators  $s_i$  corresponding to the simple roots  $\alpha_i$  are called *simple reflections*.

**Example 2.1.** *The Weyl group  $W$  associated to the root system of type  $A_n$  can be described explicitly. Let  $e_1, \dots, e_{n+1}$  denote the standard basis of  $\mathbb{R}^{n+1}$ . Then the roots are given by  $\Phi = \{e_i - e_j : 1 \leq i \neq j \leq n + 1\}$ , the positive roots are given by  $\Phi^+ = \{e_i - e_j : 1 \leq i < j \leq n + 1\}$ , and the simple roots are given by  $\Delta = \{\alpha_i := e_i - e_{i+1} : 1 \leq i \leq n\}$ . The Weyl group of type  $A_n$  is isomorphic to the symmetric group  $S_{n+1}$  as follows. Each simple reflection  $s_i$  of  $W(A_n)$  corresponds to the generator  $(i, i + 1)$  of  $S_{n+1}$ , the involution which permutes the indices  $i$  and  $i + 1$ .*

Let  $w \in W$ . Then  $w$  is some composition of the simple reflections  $s_1, \dots, s_n$ . The *length*  $\ell(w)$  of  $w$  is defined to be the length of a shortest word in these gen-

erators representing  $w$ . An expression for  $w$  of minimal length  $\ell(w)$  is called a *reduced word* for  $w$  and may not be unique.

Let  $W$  be a Weyl group generated by a collection  $S$  of simple reflections. For any subset  $I \subset S$ , let  $W_I$  denote the Weyl group generated by  $I$ . Then  $W_I$  is called a (*standard*) *parabolic subgroup* of  $W$  [17].

For two elements  $u, w \in W$ , we say that  $u \leq w$  if a substring of some reduced word for  $w$  is an expression of  $u$ . Under this ordering, called the *Bruhat-Chevalley ordering*, the group  $W$  acquires the structure of a partially ordered set.

For any  $w \in W$ , the *right descent set* of  $w$ , denoted  $D_R(w)$ , is the set of all simple reflections  $s$  for which  $ws < s$ . (One can similarly define the left descent set of any  $w \in W$ ).

Let  $w \in W$  and fix a reduced word  $\underline{w} = s_{i_1} s_{i_2} \cdots s_{i_\ell}$  of  $w$ . Let

$$N(w) := \{\beta \in \Phi^+ : w^{-1}(\beta) \prec 0\}$$

be the collection of all positive roots which are sent negative by  $w^{-1}$ , called the *inversion set* of  $w$ . Throughout this text, we will identify the set  $N(w)$  with the set

$$\{\alpha_{i_1}, s_{i_1}(\alpha_{i_2}), s_{i_1} s_{i_2}(\alpha_{i_3}), \dots, s_{i_1} s_{i_2} \cdots s_{i_{(\ell-1)}}(\alpha_{i_\ell})\}$$

(see [31]). Associated to the reduced word  $\underline{w}$ , we can consider  $N(w)$  to be an ordered set  $N(\underline{w}) = \{\beta_1, \beta_2, \dots, \beta_{\ell(w)}\}$  where  $\beta_1 = \alpha_{i_1}$  and for  $1 < j \leq \ell(w)$ , we have  $\beta_j = s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j})$ .

## 2.2 Algebraic Groups and Schubert Varieties

In this thesis we will work with a simple linear algebraic group  $G$  defined over the complex numbers. Let  $B$  be a Borel subgroup in  $G$  and let  $T$  be a maximal torus in  $B$ . Let  $\Phi$  be the abstract root system defined by  $T$  and  $\Phi^+ \subset \Phi$  the



set of positive roots determined by  $B$ . We identify the Weyl group  $W$  of  $\Phi$  from the previous section with  $N_G(T)/T$ , where  $N_G(T)$  denotes the normalizer in  $G$  of  $T$ .

The fixed points of the action of  $T$  on the flag variety  $G/B$ , denoted  $e_w$ , correspond bijectively with the elements of the Weyl group  $W = N_G(T)/T$ . We are now in a position to define Schubert varieties, the central objects of the thesis.

**Definition 2.2.** *For any  $w \in W$ , the Schubert variety  $X_w$  is the closure of the  $B$ -orbit of  $e_w$ .*

Since Schubert varieties are indexed by elements of the Weyl group of  $G$ , many geometric properties of Schubert varieties can be determined by studying the combinatorial properties of the corresponding Weyl group elements. For example, for  $u \leq w$  in  $W$ , the Schubert variety  $X_w$  contains the Schubert variety  $X_u$  [10]. In fact, many take this to be the starting definition of the Bruhat-Chevalley ordering. It is well-known that for  $w \in W$ , we have  $\dim(X_w) = \ell(w)$ .

The Schubert varieties  $X_w$  are projective varieties, which can be singular. This thesis is concerned with understanding the singularities of  $X_w$ . In particular a major outcome of the thesis is the introduction and study of certain new resolutions of  $X_w$ , which have appeared in some special cases in type  $A$  and which generalize the Bott-Samelson resolutions, discussed in the next section.

In the Bott-Samelson resolution of  $X_w$  the resolving object  $Z_w$  is a smooth projective variety, which can be presented as an iterated  $\mathbb{P}^1$  fibration. The new resolutions generalize the Bott-Samelson resolution, but the resolving smooth projective object is an iterated  $P/Q$  fibration, where  $P$  and  $Q$  are a standard parabolic subgroups of  $G$  with  $Q \subset P$  a maximal (proper) subgroup of  $P$ . Recall that  $P$  is a *standard parabolic subgroup* of  $G$  if  $B \subset P$ . Any such  $P$  corresponds to

a unique subset  $I$  of the simple roots  $\Delta \subset \Phi^+$ .

### 2.3 The Bott-Samelson Resolution

We now describe the Bott-Samelson resolution of  $X_w$ , which depends on a choice of reduced expression for  $w$ . The Bott-Samelson resolutions have played an important role in understanding the singularities of  $X_w$ . The following definitions and facts can all be found in [9].

Fix a reduced expression  $s_{i_1}s_{i_2}\cdots s_{i_r}$  for  $w \in W$ . Define  $Z_w := (P_{i_1} \times^B P_{i_2} \times^B \cdots \times^B P_{i_r})/B$  where

- $P_i$  is the minimal parabolic subgroup of  $GL(n)$  generated by the Borel subgroup  $B$  and  $s_i$ , and
- $(P_{i_1} \times^B \cdots \times^B P_{i_r})/B = (P_{i_1} \times \cdots \times P_{i_r})/ \sim$  where  $\sim$  is the equivalence relation arising from the  $B^r$  action given by

$$(b_1, b_2, \dots, b_r)(g_1, g_2, \dots, g_r) = (g_1 b_1^{-1}, b_1 g_2 b_2^{-1}, \dots, b_{r-1} g_r b_r^{-1}).$$

The Schubert variety  $X_w$  can be expressed as

$$X_w = P_{i_1} X_{s_{i_1}w} = \cdots = P_{i_1} P_{i_2} \cdots P_{i_{r-1}} X_{s_{i_r}} = P_{i_1} \cdots P_{i_r}/B.$$

The Bott-Samelson resolution is then given by  $\pi : Z_w \rightarrow X_w$  where

$$\pi : (g_1, g_2, \dots, g_r)B^r \mapsto g_1 g_2 \cdots g_r B.$$

We will now describe how  $Z_w$  is a sequence of fibrations where the fibers are all isomorphic to  $\mathbb{P}^1$ . Define  $\varphi_{r-1} : (P_{i_1} \times^B \cdots \times^B P_{i_r})/B \rightarrow (P_{i_1} \times^B \cdots \times^B P_{i_{r-1}})/B$

by  $\varphi_{r-1} : (g_1, g_2, \dots, g_r)B^r \mapsto (g_1, g_2, \dots, g_{r-1})B^{r-1}$ . This is a fibration with fiber  $P_{i_r}/B \cong \mathbb{P}^1$ . So we have

$$\begin{array}{ccc}
\mathbb{P}^1 \cong P_{i_r}/B & \longrightarrow & (P_{i_1} \times^B \dots \times^B P_{i_r})/B \\
& & \downarrow \\
\mathbb{P}^1 \cong P_{i_{r-1}}/B & \longrightarrow & (P_{i_1} \times^B \dots \times^B P_{i_{r-1}})/B \\
& & \downarrow \\
& & \vdots \\
& & \downarrow \\
\mathbb{P}^1 \cong P_{i_2}/B & \longrightarrow & (P_{i_1} \times^B P_{i_2})/B \\
& & \downarrow \\
& & P_{i_1}/B \cong \mathbb{P}^1
\end{array}$$

Later we will see the connection between the Bott-Samelson resolution of  $X_w$  and certain elements in the Hecke algebra of  $W$ .

## 2.4 Hecke Algebra and Kazhdan-Lusztig Elements

Let  $\mathcal{H}$  denote the Hecke algebra associated to  $W$  over the ring  $\mathbb{Z}[q, q^{-1}]$ . Let  $\{T_w\}_{w \in W}$  denote the standard basis of  $\mathcal{H}$ , normalized so that

$$T_s T_w = \begin{cases} T_{sw} & \text{if } sw > w \\ (q - q^{-1})T_w + T_{sw} & \text{if } sw < w \end{cases}$$

for any simple reflection  $s$  and any element  $w$  in  $W$ . The *Kazhdan-Lusztig basis elements* of  $\mathcal{H}$ , developed in [18], are defined to be the unique elements

$$\sum_{x \leq w} f_x(q, q^{-1})T_x \in \mathcal{H}$$

such that the coefficients  $f_x(q, q^{-1})$  are all nonzero polynomials with no constant term, and such that  $C_w$  is fixed under the involution  $\bar{\cdot} : \mathcal{H} \rightarrow \mathcal{H}$  defined by  $\overline{T_x} = T_{x^{-1}}^{-1}$  and  $\bar{q} = q^{-1}$ .

There exists a family of polynomials  $\{P_{x,w}(q) : x \leq w \in W\} \subseteq \mathbb{Z}[q]$  for which

$$C_w = \sum_{x \leq w} (-q)^{\ell(w) - \ell(x)} P_{x,w}(q^{-2}) T_x$$

for all  $w \in W$  ([18]). The polynomials  $P_{x,w}(q)$  are known as the *Kazhdan-Lusztig* polynomials, and they are completely characterized by the following three characteristics:

1.  $P_{x,w}(q) = 0$  whenever  $x \not\leq w$  in the Bruhat-Chevalley order.
2.  $P_{x,w}(q) = 1$  whenever  $x = w$ .
3.  $\deg(P_{x,w}(q)) \leq \frac{1}{2}(\ell(x, w) - 1)$  whenever  $x < w$ .

**Theorem 2.3.** [18] *The Kazhdan-Lusztig polynomials  $P_{x,w}(q)$  satisfy the following recursive formula. Let  $x \leq w$  and suppose  $s$  is a simple reflection such that  $ws < s$ . Then*

$$P_{x,w}(q) = q^c P_{x,ws}(q) + q^{1-c} P_{xs,ws}(q) - \sum_{\substack{x \leq z < ws \\ zs < z}} \mu(z, ws) q^{\frac{1}{2}l(z,w)} P_{x,z}(q) \quad (2.1)$$

where  $c = 1$  if  $xs < x$ ,  $c = 0$  if  $xs > x$ , and  $\mu(z, ws)$  is the coefficient of  $q^{\frac{1}{2}(\ell(z,ws)-1)}$  in  $P_{z,ws}(q)$ .

In the next section we will see a topological interpretation of the Kazhdan-Lusztig polynomials.

## 2.5 Poincaré Polynomials and Rational Smoothness

For a complex algebraic variety  $X$ , the Poincaré polynomial of  $X$  is given by

$$\mathcal{P}_X(q) = \sum_{i \geq 0} \dim_{\mathbb{C}}(H^i(X))q^i$$

where  $H^i(X)$  is the singular homology of  $X$ , viewed in its analytic topology. If  $X_w$  is a Schubert variety, we define  $\mathcal{P}_w(q^2) = \mathcal{P}_{X_w}(q)$ , and then

$$\mathcal{P}_w(q) = \sum_{x \leq w} q^{\ell(x)}$$

where the sum is over all elements  $x \leq w$  in the Bruhat-Chevalley order on  $W$ .

One can also consider the Poincaré polynomials arising from other types of homology. In particular, this thesis will often deal with the intersection cohomology Poincaré polynomial

$$\mathcal{I}_X(q) = \sum_{i \geq 0} \dim_{\mathbb{C}}(IH^i(X))q^i.$$

As was the case for the ordinary Poincaré polynomial  $\mathcal{P}_X(q)$ , we define  $\mathcal{I}_w(q^2) = \mathcal{I}_{X_w}(q)$ . Then the polynomial  $\mathcal{I}_w(q)$  has a combinatorial description, discovered by Kazhdan and Lusztig [18], described as follows. For  $w \in W$  and  $x \leq w$  in the Bruhat-Chevalley order on  $W$ , let  $P_{x,w}$  denote the Kazhdan-Lusztig polynomial indexed by  $x$  and  $w$ . The Poincaré polynomial for the full intersection cohomology for  $X_w$  is then given by

$$\mathcal{I}_w(q) = \sum_{u \leq w} P_{u,w}(q)q^{\ell(u)}.$$

We will have occasion to focus on those Schubert varieties which are smooth, but also those which satisfy a weaker notion of being rationally smooth.

**Definition 2.4.** *For any irreducible complex algebraic variety  $X$  and for any point  $x \in X$ , let  $H_x^*(X) = H^*(X, X - \{x\})$  be the cohomology with support in  $\{x\}$ . We say that*

$X$  is rationally smooth if

$$H_x^m(X) \cong \begin{cases} 0, & m \neq 2 \dim(X) \\ \mathbb{Q}, & m = 2 \dim(X) \end{cases}$$

for all points  $x \in X$ .

Since intersection cohomology satisfies Poincaré duality, the intersection cohomology polynomial  $\mathcal{I}_X(q)$  is always symmetric. In general, ordinary cohomology and  $\mathcal{P}_X(q)$  do not have these properties. In fact, it was shown by McCrory [24] that a complex projective variety  $X$  is rationally smooth if and only if its ordinary cohomology satisfies Poincaré duality. This is equivalent to the statement that  $X$  is rationally smooth if and only if  $\mathcal{P}_X(q) = \mathcal{I}_X(q)$  (see [14], and [3] Chapter 6).

## 2.6 Bott-Samelson Elements

Each Bott-Samelson resolution determines an element in the Hecke algebra, following Springer's result [30]. Fix a reduced expression  $s_{i_1} s_{i_2} \cdots s_{i_r}$  for  $w \in W$ . Let  $\pi : Z_w \rightarrow X_w$  be the corresponding Bott-Samelson resolution. We use the notation of Williamson. For an element  $\mathcal{E}$  in the bounded derived category of  $B$ -equivariant constructible sheaves on  $G/B$  define  $ch(\mathcal{E}) \in \mathcal{H}$  by

$$ch(\mathcal{E}) = \sum_{w \in W} \left( \sum_{i \geq 0} \dim H^i(\mathcal{E}_w) q^{\ell(w)+i} \right) T_w.$$

**Theorem 2.5.** ([30], Theorem 2.8)

$$ch(R\pi_*(\mathbb{C}[\ell(w)])) = (T_{s_{i_1}} + q^{-1}) \cdots (T_{s_{i_r}} + q^{-1})$$

In particular if  $\pi$  is a small resolution, then the right-hand side is the previously defined  $C_w$  after replacing  $q$  by  $-q^{-1}$ .

This thesis is about generalizing both sides of the equation in the theorem.

## 2.7 Pattern Avoidance

We now recall some results about pattern avoidance, especially in type  $A$ .

For any  $w \in S_n$ , we often express  $w$  in one-line notation rather than as a composition of simple reflections, depending on our purposes. The one-line expression for any  $w \in S_n$  is given by

$$w(1) w(2) \cdots w(n).$$

For example, if  $w = s_2 s_1 s_3 s_2 \in S_4$ , then the one-line expression for  $w$  is 3412. If the one-line expression for  $w \in S_n$  is  $w_1 w_2 \cdots w_n$ , we refer to each value  $w_i$  as the entry corresponding to the index  $i$ . So for  $w = 35214 \in S_5$ , the entry 5 occurs at index 2.

**Definition 2.6.** *An element  $w \in S_n$  is said to contain the pattern  $v \in S_k$  if  $w$ , when expressed in one-line notation, contains a subword of length  $k$  whose entries are in the same relative order as the entries of  $v$ . If  $w$  does not contain the pattern  $v$ , we say that  $w$  avoids  $v$ .*

For example, the element  $42513 \in S_5$  contains the pattern 3412, which appears in  $w$  as the subword 4513, and  $w$  avoids the pattern 123 because there are no indices  $i < j < k$  such that  $w_i < w_j < w_k$ .

Many geometric properties of a Schubert variety  $X_w$  are equivalent to combinatorial statements about pattern containment and avoidance. Perhaps most famously, we have the following.

**Theorem 2.7.** [21] *For any  $w \in S_n$ , the singular locus of the Schubert variety  $X_w$  is nonempty if and only if  $w$  contains one of two specific patterns: 3412 and 4231.*

Theorem 2.7 has been widely used to study the properties of singular Schubert varieties. In Chapter 3, we will explore and expand upon a result of Lascoux which says that if  $w \in S_n$  avoids the patterns 3412 and 4231, the associated Kazhdan-Lusztig basis element  $C_w$  can be factored into a product of terms of the form  $(T_{s_i} - f(q))$  where  $f$  is a rational function.

In [2], Billey and Braden extend the notion of pattern avoidance to apply to all Weyl groups. They show that if  $W'$  is any subgroup of  $W$  conjugate to a standard parabolic subgroup, then there exists a unique map  $\phi : W \rightarrow W'$ , called the *pattern map* for  $W'$ , such that  $\phi$  is  $W'$ -equivariant, and if  $\phi(w) \leq' \phi(uw)$  for some  $w \in W$  and  $u \in W'$ , then  $w \leq uw$ , where  $\leq'$  denotes the restriction of the Bruhat-Chevalley ordering on  $W$  to  $W'$ . For  $w \in W$  and  $v \in W'$ , we say that  $w$  *contains the pattern  $v$*  if  $\phi(w) = v$ , and that  $w$  *avoids the pattern  $v$*  otherwise. When  $W$  is of type  $A$ , this definition of pattern containment agrees with the previous definition. Extending upon Theorem 2.7, Billey has determined the list of patterns which are avoided precisely when  $X_w$  is smooth/rationally smooth for types  $B$ ,  $D$  and  $E$  [1].

The notion of flattening is closely related to pattern avoidance, and will be used occasionally throughout this thesis when convenient.

**Definition 2.8.** *Let  $0 \leq k \leq n$ , let  $Z = \{i_1 < i_2 < \dots < i_k\} \subset \{1, 2, \dots, n\}$ . The flattening map  $fl_Z : S_n \rightarrow S_k$  maps an element  $x \in S_n$  to the element of  $S_k$  whose entries are in the same relative order as the entries  $x(i_1) x(i_2) \dots x(i_k)$ . When we mean to emphasize entries rather than indices, we may write  $fl(x(i_1) x(i_2) \dots x(i_k))$  instead of  $fl_{i_1, \dots, i_k}(x)$ .*



For example,  $\text{fl}_{\{3,5,6\}}(153462) = \text{fl}(362) = 231 \in S_3$ . Pattern containment is closely related to flattening, as an element  $w \in S_n$  contains the pattern  $v \in S_k$  if and only if there exist indices  $i_1 < i_2 < \dots < i_k$  such that  $\text{fl}_{\{i_1, \dots, i_k\}}(w) = v$ . For instance, an element  $x \in S_n$  contains a 3412 pattern if there exist  $1 \leq i < j < k < l \leq n$  such that  $\text{fl}_{\{i,j,k,l\}}(x) = 3412$ .

**Definition 2.9.** Given a set  $Z = \{i_1 < i_2 < \dots < i_k\} \subset \{1, 2, \dots, n\}$  and an element  $x \in S_n$ , the unflattening map  $\text{unfl}_Z^x : S_k \rightarrow S_n$  maps an element  $u \in S_k$  to the element  $y \in S_n$  for which  $\text{fl}_Z(y) = u$  and  $x(a) = y(a)$  for all  $a \in [n] \setminus Z$ .

For example, we have  $\text{unfl}_{\{3,5,6\}}^{136245}([231]) = 135264$ .

The following two results regarding nonsingular Schubert varieties of type  $A$  are due to Gasharov. The first provides an elegant means of describing the elements  $w \in S_n$  corresponding to nonsingular Schubert varieties, and it will be used often throughout the thesis. The second result provides a generalization of the factorization of the Poincaré polynomial for the full flag variety due to Kostant [20] and Macdonald [23].

**Theorem/Definition 2.10.** [12] Let  $w \in S_n$  be an element which avoids the patterns 3412 and 4231. Let  $d = w^{-1}(n)$  and let  $c = w(n)$ . The element  $w$  falls into one or both of the following cases.

$$\text{Case 1: } w(d) > w(d+1) > \dots > w(n)$$

$$\text{Case 2: } w^{-1}(c) > w^{-1}(c+1) > \dots > w^{-1}(n)$$

If  $w$  is in Case 1, define  $w' = w \setminus n$  and define  $m = n - d$ . If  $w$  is in Case 2, define  $w' \in S_{n-1}$  to be the element whose entries are in the same relative order as  $w \setminus c$ , and define  $m = n - c$ . (In terms of flattening, we have  $w' = \text{fl}(w \setminus n)$  if  $w$  is in Case 1, and  $w' = \text{fl}(w \setminus c)$  if  $w$  is in Case 2).

**Definition 2.11.** For any integer  $a$ , the  $q$ -number associated to  $a$  is the polynomial

$$[a]_q := q^{a-1} + q^{a-2} + \cdots + q + 1.$$

**Theorem 2.12.** [12] Suppose  $w$  corresponds to a nonsingular Schubert variety, and let  $w'$  and  $m$  be defined as in Theorem/Definition 2.10. The element  $w'$  also corresponds to a nonsingular Schubert variety, and the ordinary Poincaré polynomial  $\mathcal{P}_w(q)$  satisfies the following recursive factorization formula:

$$\mathcal{P}_w(q) = [m + 1]_q \mathcal{P}_{w'}(q).$$

## CHAPTER 3

### Modified Height and Lascoux Elements

In this chapter, we will give a modified definition of root height, and use it to define new elements in an extension of the Hecke algebra, which will be called Lascoux elements. These Lascoux elements were developed to generalize a construction given by Lascoux in [22]. We will show that Lascoux elements are similar to the Kazhdan-Lusztig basis elements of the Hecke algebra in quite a few ways.

#### 3.1 Preliminaries

For any  $w \in W$ , we can modify the usual definition of root height for any root  $\beta \in N(w)$  in the following manner.

**Definition 3.1.** *Let  $w \in W$ . The modified height  $ht_w(\beta)$  of any root  $\beta \in N(w)$  is the largest integer  $h$  for which  $\beta$  can be expressed as a sum of  $h$  roots in  $N(w)$ .*

Let  $w \in W$  and consider any decomposition of a fixed reduced word  $\underline{w}$  of  $w$  into two factors:  $\underline{w} = \underline{w}'(s_{i_1}s_{i_2} \cdots s_{i_r})$ . Then the associated inversion set of  $w$  is ordered in the following way (see Section 2.1).

$$N(\underline{w}) = \{N(\underline{w}'), w'(\alpha_{i_1}), w's_{i_1}(\alpha_{i_2}), \dots, w's_{i_1}s_{i_2} \cdots s_{i_{r-1}}(\alpha_{i_r})\}$$

Since  $N(w') \subset N(w)$ , we can compare  $\text{ht}_{w'}(\beta)$  and  $\text{ht}_w(\beta)$  for any  $\beta \in N(w')$ .

**Definition 3.2.** Let  $w \in W$  and consider a reduced expression of  $w$  of the form  $\underline{w} = w'(s_{i_1} s_{i_2} \cdots s_{i_r})$ . We say that heights are preserved with respect to  $\underline{w}'$  (or that heights are preserved with respect to this expression) if  $\text{ht}_{w'}(\beta) = \text{ht}_w(\beta)$  for all  $\beta \in N(w')$ .

**Definition 3.3.** Let  $w \in W$  with fixed reduced expression  $\underline{w} = s_{i_1} s_{i_2} \cdots s_{i_r}$ . Let  $N(\underline{w}) = \{\beta_1, \beta_2, \dots, \beta_r\}$  be the associated ordered inversion set of  $w$ . The height sequence associated to  $\underline{w}$  is simply the sequence of integers

$$\text{hts}(\underline{w}) = (\text{ht}_w(\beta_1), \text{ht}_w(\beta_2), \dots, \text{ht}_w(\beta_r)).$$

Connecting these two definitions, we can see that the reduced expression

$$\underline{w} = \underline{w}'(s_{i_1} s_{i_2} \cdots s_{i_r})$$

is height-preserving if  $\text{hts}(\underline{w}) = (\text{hts}(\underline{w}'), \dots)$ .

We will now provide some general results on modified height.

**Proposition 3.4.** Let  $w \in W$  and consider a reduced expression of  $w$  of the form  $\underline{w} = s_{i_1} s_{i_2} \cdots s_{i_r}$ . Fix the reduced expression  $\underline{w}^{-1} = s_{i_r} \cdots s_{i_2} s_{i_1}$  of  $w^{-1}$ . If  $\text{hts}(\underline{w}) = (a_1, a_2, \dots, a_r)$ , then  $\text{hts}(\underline{w}^{-1}) = (a_r, \dots, a_2, a_1)$ .

*Proof.* Say  $N(\underline{w}) = \{\beta_1, \dots, \beta_r\}$  where  $\beta_j = s_{i_1} s_{i_2} \cdots s_{i_{j-1}}(\alpha_{i_j})$  for each  $1 \leq j \leq r$ . Then  $N(\underline{w}^{-1}) = \{\beta'_r, \dots, \beta'_1\}$  where each  $\beta'_j = s_{i_r} s_{i_{r-1}} \cdots s_{i_{j+1}}(\alpha_{i_j})$ . Suppose there exist indices  $m, n, k$  such that  $\beta_m = \beta_n + \beta_k$ , i.e.

$$s_{i_1} \cdots s_{i_{m-1}}(\alpha_{i_m}) = s_{i_1} \cdots s_{i_{n-1}}(\alpha_{i_n}) + s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}).$$

Applying  $w^{-1}$  to both sides, we have

$$\begin{aligned} s_{i_r} \cdots s_{i_{m+1}} s_{i_m}(\alpha_{i_m}) &= s_{i_r} \cdots s_{i_{n+1}} s_{i_n}(\alpha_{i_n}) + s_{i_r} \cdots s_{i_{k+1}} s_{i_k}(\alpha_{i_k}), \\ \text{i.e. } s_{i_r} \cdots s_{i_{m+1}}(\alpha_{i_m}) &= s_{i_r} \cdots s_{i_{n+1}}(\alpha_{i_n}) + s_{i_r} \cdots s_{i_{k+1}}(\alpha_{i_k}), \\ \text{i.e. } \beta'_m &= \beta'_n + \beta'_k. \end{aligned}$$

Thus every linear relation for the roots  $\beta_j$  holds for the roots  $\beta'_j$ , and the desired conclusion follows.  $\square$

**Lemma 3.5.** *Fix a reduced expression  $\underline{w}$  of  $w \in W$  and let  $\text{hts}(w) = (a_1, a_2, \dots, a_r)$  be the associated height sequence. Then  $a_1 = a_r = 1$ .*

*Proof.* Say  $\underline{w} = s_{i_1} s_{i_2} \cdots s_{i_r}$  and  $N(\underline{w}) = \{\beta_1, \dots, \beta_r\}$ . Clearly  $\beta_1 = \alpha_{i_1}$  has modified height 1, since the modified height of any inversion vector is less than or equal to its height as a root.

Let  $u = w s_{i_r} = s_{i_1} s_{i_2} \cdots s_{i_{r-1}}$ . Then  $N(u) = \{\beta_1, \dots, \beta_{r-1}\}$  and  $\beta_r = u(\alpha_{i_r})$ . Suppose for a contradiction that  $\beta_r$  has modified height  $> 1$  in  $N(w)$ . Then there exist  $\beta_i, \beta_j \in N(w)$  such that  $\beta_r = \beta_i + \beta_j$ . Then  $\beta_i, \beta_j \in N(u)$ , so applying  $u^{-1}$  to both sides of this equation, we get  $\alpha_{i_r} = u^{-1}(\beta_i) + u^{-1}(\beta_j)$ . This is a contradiction because we have a positive root  $\alpha_{i_r}$  on the left side, and since  $\beta_i, \beta_j \in N(u)$ , we know that  $u^{-1}(\beta_i)$  and  $u^{-1}(\beta_j)$  are negative roots.  $\square$

The following result for  $S_n$  illustrates that modified height can be reinterpreted in a natural way with respect to one-line notation.

**Proposition 3.6.** *Let  $w \in S_n$ . Let  $e_p - e_q \in N(w)$  with modified height  $\text{ht}_w(e_p - e_q) = m$ . Then the longest decreasing subword in the one-line expression of  $w$  between the entries  $q$  and  $p$  consists of  $m + 1$  entries, as does the longest subword of the one-line expression of  $w^{-1}$  between entry  $w^{-1}(p)$  and entry  $w^{-1}(q)$ .*

*Proof.* The two statements of the proposition are equivalent, and we will prove the second. First note that  $e_p - e_q \in N(w)$  if and only if  $w^{-1}(p) > w^{-1}(q)$ . If  $\text{ht}_w(e_p - e_q) = m$ , then we can express  $e_p - e_q$  as the sum of  $m$  distinct roots in  $N(w)$ :

$$e_p - e_q = (e_p - e_{i_1}) + (e_{i_1} - e_{i_2}) + \cdots + (e_{i_{m-1}} - e_q).$$

By the above, we have  $w^{-1}(p) > w^{-1}(i_1) > \cdots > w^{-1}(i_{m-1}) > w^{-1}(q)$ , so  $w^{-1}$  contains a decreasing subword between indices  $p$  and  $q$  of length  $m + 1$ . Thus the longest decreasing subword of the one-line expression of  $w^{-1}$  starting at index  $p$  and ending at index  $q$  consists of at least  $m + 1$  entries.

Suppose the one-line expression for  $w^{-1}$  contains a decreasing subword

$$w^{-1}(p) w^{-1}(j_1) w^{-1}(j_2) \cdots w^{-1}(j_m) w^{-1}(q)$$

of length  $m + 2$  between indices  $p$  and  $q$ . Then  $e_p - e_{j_1}, e_{j_1} - e_{j_2}, \dots, e_{j_m} - e_q$  is a collection of  $m + 1$  roots in  $N(w)$  and

$$e_p - e_q = (e_p - e_{j_1}) + (e_{j_1} - e_{j_2}) + \cdots + (e_{j_m} - e_q),$$

so  $\text{ht}_w(e_p - e_q) \geq m + 1$ , a contradiction.  $\square$

**Example 3.7.** Let  $w = 7326541 \in S_7$ . Then  $\text{ht}_w(e_1 - e_7) = 4$  because

$$e_1 - e_7 = (e_1 - e_4) + (e_4 - e_5) + (e_5 - e_6) + (e_6 - e_7)$$

and this expression of  $e_1 - e_7$  as the sum of distinct roots in  $N(w)$  has the maximal number of terms possible. Alternatively, from the one-line notation for  $w$ , we can see that the decreasing subwords of  $w$  between the entries 7 and 1 are given by

$$71, 731, 721, 7321, 761, 751, 741, 7651, 7641, 7541, \text{ and } 76541.$$

The longest of these is 76541 which has 5 entries, so we know that the modified height of  $e_1 - e_7$  with respect to  $w$  is 4.

## 3.2 Lascoux Elements

We start this section by providing an algorithm which will be fundamental to the rest of this chapter and to other chapters to come. The algorithm, which

provides a method for computing a particular type of reduced expression for an element  $w \in W$ , will allow us to define Lascoux elements, the main object of study in this chapter.

**Algorithm 3.8.** *Let  $w \in W$ , let  $R_1$  be a nontrivial subset of  $D_R(w)$ , and let  $W_{R_1}$  denote the Weyl subgroup of  $W$  generated by the reflections in  $R_1$ . Let  $T_1$  be a maximal proper subset of  $R_1$  and let  $W_{T_1}$  be the maximal parabolic subgroup of  $W_{R_1}$  generated by  $T_1$ . Then let  $w_0$  denote the longest element of  $W_{R_1}$ , let  $x_1$  be the unique element of minimal length in  $W_{T_1}w_0$ , and define  $w_1 = wx_1^{-1}$ .*

*Repeat this process with  $w_1$  in place of  $w$  to obtain elements  $x_2$  and  $w_2 = w_1x_2^{-1}$ . Continue on in this fashion until arriving at some  $w_m = e$ .*

*This process provides a reduced expression for  $w$  given by*

$$\underline{w} = \underline{x}_m \cdots \underline{x}_2 \underline{x}_1.$$

In general, Algorithm 3.8 does not produce a unique expression of  $w \in W$ . At each step of the algorithm, a choice of  $R_i$  is made and a choice of  $T_i \subset R_i$  is made. In Examples 3.13 and 3.14, we will see how Algorithm 3.8 can be used to compute different reduced expressions for the same element.

Note that if the subsets  $T_i$  in the algorithm are all taken to be trivial, then each  $x_i$  is equal to a single simple reflection and the resulting reduced expression has the form  $\underline{w} = (s_{i_1}) \cdots (s_{i_r})$ . This reduced expression has corresponding Bott-Samelson resolution  $(\mathbf{P}_1 \times^B \mathbf{P}_2 \times^B \cdots \times^B \mathbf{P}_r)/B$ , where each  $\mathbf{P}_i$  is the parabolic subgroup of  $G$  indexed by the reflections of  $R_i$ . In Chapter 4, we will use Algorithm 3.8 to generalize the Bott-Samelson resolution in such a way as to accommodate the cases where the  $T_i$  are not all taken to be trivial.

The following notation will be used throughout the rest of this chapter.

**Notation 3.9.** • For any integer  $a$ , let  $[a] := \frac{q^a - q^{-a}}{q - q^{-1}}$  (this is not to be confused with the polynomial  $[a]_q = \frac{q^a - 1}{q - 1}$ ).

- If  $s_i$  is any simple generator of  $W$ , let  $T_i := T_{s_i}$  be the standard Hecke algebra basis element associated to  $s_i$ .

Using Algorithm 3.8, we can now define Lascoux elements, the main objects of study in this chapter.

**Definition 3.10.** Let  $w \in W$ . Apply the first step of Algorithm 3.8 to  $w$  to produce a reduced expression  $\underline{w} = \underline{w}_1(s_{i_1} s_{i_2} \cdots s_{i_r})$ , and say  $N(w) = \{N(\underline{w}_1), \beta_1, \beta_2, \dots, \beta_r\}$ . Continue applying the algorithm until  $\underline{w}$  is completely factored. We recursively define the Lascoux element associated to the reduced expression  $\underline{w}$  to be

$$L_{\underline{w}} = L_{\underline{w}_1} \left( T_{i_1} - \frac{q^{ht_w(\beta_1)}}{[ht_w(\beta_1)]} \right) \left( T_{i_2} - \frac{q^{ht_w(\beta_2)}}{[ht_w(\beta_2)]} \right) \cdots \left( T_{i_r} - \frac{q^{ht_w(\beta_r)}}{[ht_w(\beta_r)]} \right)$$

where  $L_e = 1$ . Note that in general the coefficients of  $L_{\underline{w}}$  lie in the ring  $\mathbb{Q}(q)$ , rather than  $\mathbb{Z}[q, q^{-1}]$ , and thus  $L_{\underline{w}}$  lies in an extension of the Hecke algebra.

**Proposition 3.11.** Let  $w \in W$  and let  $\underline{w} = \underline{w}_1 \underline{x}$  be a reduced expression for  $w$  obtained by applying Algorithm 3.8. Then the associated Lascoux element  $L_{\underline{w}}$  is independent of the choice of reduced expression  $\underline{x}$  of  $x$ .

The proof of this proposition, which is a subject of joint work with Eric Sommers, will be published separately.

The following result illustrates a key way in which the Lascoux elements are similar to the Kazhdan-Lusztig basis elements.

**Lemma 3.12.** Let  $w \in W$  and use Algorithm 3.8 to fix a reduced expression  $\underline{w}$  of  $w$ . Then  $\overline{L_{\underline{w}}} = L_{\underline{w}}$ .



*Proof.* Observe that for any simple reflection  $s$  in  $W$ , we have  $T_s^{-1} = T_s - (q - q^{-1})$ .

Thus, for any simple reflection  $s$  and any non-zero integer  $a$ , we have

$$\begin{aligned}
\overline{\left(T_s - \frac{q^a}{[a]}\right)} &= T_s^{-1} - \frac{q^{-a}(q - q^{-1})}{(q^a - q^{-a})} && \text{(by the definitions of } [a], \overline{T_s}, \overline{q}\text{)} \\
&= T_s - (q - q^{-1}) \left(1 + \frac{q^{-a}}{q^a - q^{-a}}\right) && \text{(by the definition of } T_s^{-1}\text{)} \\
&= T_s - (q - q^{-1}) \left(\frac{q^a}{q^a - q^{-a}}\right) && \text{(rewriting)} \\
&= T_s - \frac{q^a}{[a]}.
\end{aligned}$$

The desired conclusion follows immediately.  $\square$

Let  $w \in W$  and let  $\underline{w}$  be a fixed reduced expression of  $w$  obtained via Algorithm 3.8. Since  $L_{\underline{w}}$  is bar-invariant, we can see that  $L_{\underline{w}} = C_w$  if and only if the coefficients of  $L_{\underline{w}}$  are all polynomials with zero constant term.

The bar-invariance of the factor  $\left(T_i - \frac{q^a}{[a]}\right)$  was known previously by Lascoux [22]. We will elaborate on Lascoux's work in Section 3.3 below.

We will now provide a string of examples which will serve not only to provide a sense of familiarity with Algorithm 3.8 and the computation of Lascoux elements, but also to illustrate a fundamental observation which has guided the direction of much of our work.

**Example 3.13.** Let  $w = 53241 \in S_5$ . We will use Algorithm 3.8 to produce a reduced expression  $\underline{w}$  for  $w$ , and compute the associated Lascoux element  $L_{\underline{w}}$ . We start the algorithm by observing that  $D_R(w) = \{s_1, s_2, s_4\}$  and letting  $R_1$  be the entire set  $D_R(w)$ .

- Let  $T_1 = \{s_1, s_2\}$ . Then  $x_1 = s_4$ ,  $w_1 = 53214$ , and  $D_R(w_1) = \{s_1, s_2, s_3\}$ . Let  $R_2 = D_R(w_1)$ .
- Let  $T_2 = \{s_1, s_2\}$ . Then  $x_2 = s_3s_2s_1$ ,  $w_2 = 32154$ , and  $D_R(w_2) = \{s_1, s_2, s_4\}$ . Let  $R_3 = D_R(w_2)$ .

- Let  $T_3 = \{s_1, s_2\}$ . Then  $x_3 = s_4$ ,  $w_3 = 32145$ , and  $D_R(w_3) = \{s_1, s_2\}$ . Let  $R_3 = D_R(w_3)$ .
- Let  $T_4 = \{s_1\}$ . Then  $x_4 = s_2s_1$ ,  $w_4 = 21345$ , and  $D_R(w_4) = \{s_1\}$ . Let  $R_3 = D_R(w_4)$ .
- Let  $T_5 = \emptyset$ . Then  $x_5 = s_1$ , and  $w_5 = 12345 = e$ .

This process factors  $w$  into the reduced word

$$\underline{w} = x_5x_4x_3x_2x_1 = (s_1)(s_2s_1)(s_4)(s_3s_2s_1)(s_4).$$

Heights are preserved at every stage of this factorization, which we can see immediately by aligning the height sequences of the  $w_j$  vertically as below.

$$\begin{aligned} hts(\underline{w}) &= (1, 2, 1, 1, 3, 2, 1, 1) \\ hts(\underline{w}_1) &= (1, 2, 1, 1, 3, 2, 1) \\ hts(\underline{w}_2) &= (1, 2, 1, 1) \\ hts(\underline{w}_3) &= (1, 2, 1) \\ hts(\underline{w}_4) &= (1) \end{aligned}$$

We then have

$$\begin{aligned} L_{\underline{w}} &= L_{\underline{w}_1} (T_4 - q) \\ &= L_{\underline{w}_2} \left( T_3 - \frac{q^3}{[3]} \right) \left( T_2 - \frac{q^2}{[2]} \right) (T_1 - q) (T_4 - q) \\ &\quad \vdots \\ &= (T_1 - q) \left( T_2 - \frac{q^2}{[2]} \right) (T_1 - q) (T_4 - q) \left( T_3 - \frac{q^3}{[3]} \right) \left( T_2 - \frac{q^2}{[2]} \right) (T_1 - q) (T_4 - q) \end{aligned}$$

One could now compute  $C_w$  and check directly that  $L_{\underline{w}} = C_w$ , but a direct computation is in fact unnecessary in this case. Indeed, by computing only  $L_{\underline{w}}$ , and verifying

that the coefficients are all polynomials with zero constant term, we can conclude immediately from Lemma 3.12 that  $L_{\underline{w}} = C_w$ , since bar-invariance and these conditions on coefficients are the three defining characteristics of the Kazhdan-Lusztig elements.

**Example 3.14.** Let  $w = 53241 \in S_5$  as in Example 3.13. In this example, we will use Algorithm 3.8 to obtain a reduced expression  $\underline{w}$  for which heights are not preserved. As before, we have  $D_R(w) = \{s_1, s_2, s_4\}$ . Let  $R_1 = D_R(w)$ .

- Let  $T_1 = \{s_1, s_4\}$ . Then  $x_1 = s_2s_1$ ,  $w_1 = 32541$ , and  $D_R(w_1) = \{s_1, s_3, s_4\}$ . Let  $R_2 = D_R(w_1)$ .
- Let  $T_2 = \{s_1, s_3\}$ . Then  $x_2 = s_4s_3$ ,  $w_2 = 32415$ , and  $D_R(w_2) = \{s_1, s_3\}$ . Let  $R_3 = D_R(w_2)$ .
- Let  $T_3 = \{s_1\}$ . Then  $x_3 = s_3$ ,  $w_3 = 32145$ , and  $D_R(w_3) = \{s_1, s_2\}$ . Let  $R_4 = D_R(w_3)$ .
- Let  $T_4 = \{s_1\}$ . Then  $x_4 = s_2s_1$ ,  $w_4 = 21345$ , and  $D_R(w_4) = \{s_1\}$ . Let  $R_5 = D_R(w_4)$ .
- Let  $T_5 = \emptyset$ . Then  $x_5 = s_1$ , and  $w_5 = 12345 = e$ .

This process factors  $w$  into the reduced word

$$\underline{w} = x_5x_4x_3x_2x_1 = (s_1)(s_2s_1)(s_3)(s_4s_3)(s_2s_1).$$

Heights are not preserved with respect to this factorization. Indeed, we have  $\text{hts}(\underline{w}) = (1, 2, 1, 1, 3, 1, 2, 1)$  and  $\text{hts}(w_1) = (1, 2, 1, 1, 2, 1)$ . The corresponding Lascoux element is

$$L_{\underline{w}} = (T_1 - q) \left( T_2 - \frac{q^2}{[2]} \right) (T_1 - q) (T_3 - q) \left( T_4 - \frac{q^2}{[2]} \right) (T_3 - q) \left( T_2 - \frac{q^2}{[2]} \right) (T_1 - q)$$

which is not equal to  $C_w$ . Indeed, the coefficient of  $T_1T_2T_1T_4$  in  $L_{\underline{w}}$  is  $q^4 + q^2 + 1$ , which has nonzero constant term, and thus cannot appear in the expression for  $C_w$ .

In the next example, we encounter an element for which no reduced expression preserves heights.

**Example 3.15.** Let  $w = 45312 \in S_5$ . In this example, we will show that there is no reduced expression of  $w$  following Algorithm 3.8 which preserves heights. Our choices of  $R_1$  are  $\{s_2, s_3\}$ ,  $\{s_2\}$  or  $\{s_3\}$ . If we take  $R_1 = \{s_2\}$ , then we must take  $T_1 = \emptyset$ , which gives us  $x_1 = s_2$  and  $w_1 = 43512$ . Then from Proposition 3.6, we can immediately see that heights are not preserved in this step since the longest decreasing subword between 5 and 2 consists of three entries in  $w$  and only two entries in  $w_1$ . Similarly we can verify that taking  $R_1 = \{s_3\}$  will not preserve heights. So we must take  $R_1 = \{s_2, s_3\}$ . We can then take  $T_1 = \{s_2\}$  or  $T_1 = \{s_3\}$ . If we take  $T_1 = \{s_2\}$ , then we will have  $w_1 = 43152$ , where the longest decreasing subword between 5 and 2 again consists of only two entries. If we take  $T_1 = \{s_3\}$ , then we will have  $w_1 = 41532$ . Then the longest decreasing subword between 4 and 1 consists of three entries in  $w$  and only two entries in  $w_1$ . So with either choice of  $T_1$ , heights are not preserved.

This shows that there is no way to preserve heights even at the first step of the algorithm. One can check that for this element  $w$ , there is also no reduced expression  $\underline{w}$  of  $w$  for which  $L_{\underline{w}} = C_w$ .

We are very interested in understanding the conditions under which  $L_{\underline{w}} = C_w$  for some reduced expression  $\underline{w}$  of  $w \in W$ . Experimentation has led us to believe that preserving heights is a necessary condition for Lascoux elements and Kazhdan-Lusztig elements to be equal.

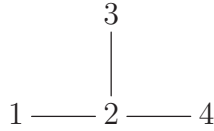
**Conjecture 3.16.** Let  $w \in W$  and let  $\underline{w}$  be a reduced expression for  $w$  obtained via Algorithm 3.8. If  $L_{\underline{w}} = C_w$ , then heights were preserved at each step of the algorithm.

**Proposition 3.17.** Let  $w \in W$  and let  $\underline{w}$  be a reduced expression for  $w$  obtained via Algorithm 3.8. The coefficients of  $L_{\underline{w}}$  are Laurent polynomials.

Proposition 3.17 indicates that Lascoux elements always have at least two of the three defining properties of Kazhdan-Lusztig elements, namely they are bar-invariant and their coefficients are all Laurent polynomials. The third missing property is for all coefficients to be polynomials with no constant term. The proof of Proposition 3.17 will be given for some special cases in the following sections, and Conjecture 3.16 and Proposition 3.17 are both subjects of future work with Eric Sommers.

The following examples show that the condition of height preservation is not sufficient to have  $L_w = C_w$  in general. The elements  $w$  in Examples 3.18 and 3.19 also appear in work of Williamson and Braden on intersection cohomology complexes on flag varieties [33] (see Section 4.3 for further discussion).

**Example 3.18.** We will consider  $D_4$  to have the Dynkin diagram shown in Figure 1. In



**Figure 1. Dynkin Diagram of Type  $D_4$**

other words, we will consider the Weyl group of  $D_4$  to be generated by the simple reflections  $s_1, s_2, s_3, s_4$  where  $s_2$  does not commute with any of the other three and  $s_1, s_3, s_4$  all commute with each other.

Let  $w$  denote the element  $s_2s_1s_2s_3s_4s_2s_1 \in W(D_4)$ , which corresponds to a singular Schubert variety. We will follow Algorithm 3.8 to produce a reduced word for  $w$ . We have  $D_R(w) = \{s_1, s_2\}$ . Let  $R_1 = D_R(w)$ .

- Let  $T_1 = \{s_1\}$ . Then  $x_1 = s_2s_1$ ,  $w_1 = s_1s_2s_4s_3s_1$ , and  $D_R(w_1) = \{s_1, s_3, s_4\}$ . Let  $R_2 = D_R(w_1)$ .

- Let  $T_2 = \{s_1, s_3\}$ . Then  $x_2 = s_4$ ,  $w_2 = s_1s_2s_3s_1$ , and  $D_R(w_2) = \{s_1, s_3\}$ . Let  $R_3 = D_R(w_2)$ .
- Let  $T_3 = \{s_1\}$ . Then  $x_3 = s_3$ ,  $w_3 = s_1s_2s_1$ , and  $D_R(w_3) = \{s_1, s_2\}$ . Let  $R_4 = D_R(w_3)$ .
- Let  $T_4 = \{s_1\}$ . Then  $x_4 = s_2s_1$ ,  $w_4 = s_1$ , and  $D_R(w_4) = \{s_1\}$ . Let  $R_5 = D_R(w_4)$ .
- Let  $T_5 = \emptyset$ . Then  $x_5 = s_1$  and  $w_5 = e$ .

This process produces the reduced word  $\underline{w} = (s_1)(s_2s_1)(s_3)(s_4)(s_2s_1)$  where  $hts(\underline{w}) = (1, 2, 1, 1, 1, 2, 1)$ . Although heights were preserved in this factorization at every step of the algorithm, we have  $L_{\underline{w}} \neq C_w$ . Indeed, the coefficient of  $T_1T_2T_1$  in  $L_{\underline{w}}$  is  $(q^4 + q^2 + 1)$ , which could not possibly appear in the expression for  $C_w$  since it is a polynomial with nonzero constant term.

**Example 3.19.** Let  $w = 84567123 \in S_8$ . We can apply Algorithm 3.8 to produce the reduced word

$$\underline{w} = (s_3)(s_2)(s_4)(s_7)(s_3)(s_5)(s_6s_5)(s_4s_5)(s_3)(s_7)(s_1)(s_6)(s_2s_3)(s_4)(s_5)(s_1)$$

which has associated height sequence

$$hts(\underline{w}) = (1, 1, 1, 1, 1, 1, 2, 1, 2, 1, 1, 1, 1, 1, 2, 1, 1, 1, 1).$$

As in Example 3.18, heights are preserved at every stage of this factorization, and yet  $L_{\underline{w}} \neq C_w$ , which can be verified without computing  $C_w$  by noting that for the elements  $u = s_7s_3s_4s_5s_6s_4s_5s_4s_1s_2s_1$  and  $v = s_6s_7s_6s_2s_3s_4s_5s_2s_3s_2s_1$ , the coefficients of  $T_u$  and  $T_v$  in  $L_{\underline{w}}$  are both  $(q^8 + 7q^6 + 13q^4 + 7q^2 + 1)$ , which is a polynomial with nonzero constant term.

Observe that if heights are preserved throughout the factorization of Algorithm 3.8, we get

$$L_{\underline{w}} = \left( T_{i_1} - \frac{q^{\text{ht}_w(\beta_1)}}{[\text{ht}_w(\beta_1)]} \right) \left( T_{i_2} - \frac{q^{\text{ht}_w(\beta_2)}}{[\text{ht}_w(\beta_2)]} \right) \cdots \left( T_{i_r} - \frac{q^{\text{ht}_w(\beta_r)}}{[\text{ht}_w(\beta_r)]} \right)$$

where  $N(\underline{w}) = \{\beta_1, \dots, \beta_r\}$  is the ordered inversion set associated to the reduced expression  $\underline{w}$ . However, computing such a product without respect to the algorithm will often produce an element with rational (non-polynomial) coefficients, as in Example 3.20.

**Example 3.20.** Let  $w = 53241$  with reduced expression  $\underline{w} = (s_1)(s_2s_1)(s_3)(s_4s_3)(s_2s_1)$  as in Example 3.14. Then  $\text{hts}(\underline{w}) = (1, 2, 1, 1, 3, 1, 2, 1)$ , and we can consider the following product, which does not give the Lascoux element  $L_{\underline{w}}$ .

$$(T_1 - q) \left( T_2 - \frac{q^2}{[2]} \right) (T_1 - q) (T_3 - q) \left( T_4 - \frac{q^3}{[3]} \right) (T_3 - q) \left( T_2 - \frac{q^2}{[2]} \right) (T_1 - q)$$

In this product, we are simply taking the factors to be  $\left( T_i - \frac{q^a}{[a]} \right)$  where  $i$  is an index of a simple reflection in  $\underline{w}$  and  $a$  is the corresponding height in  $\text{hts}(\underline{w})$ . The coefficient of  $T_1T_2T_1T_3T_2T_1$  in this product is  $(q^6 + q^4)/(q^4 + q^2 + 1)$ , which is clearly not a Laurent polynomial. From this we can see that in general, the Lascoux element  $L_{\underline{w}}$  depends on more than the reduced expression  $\underline{w}$ .

Suppose  $\underline{w}$  is a reduced expression for some  $w \in W$  obtained via Algorithm 3.8, for which  $L_{\underline{w}} \neq C_w$ . Then there exists some element  $u \leq w$  of maximal length for which the coefficient of  $T_u$  in  $L_{\underline{w}}$  has terms of degree  $\leq 0$ . Since all Lascoux elements are bar-invariant and have Laurent polynomial coefficients, we can choose a bar-invariant Laurent polynomial  $c(q)$  such that the element  $L_{\underline{w}} - c(q)L_{\underline{u}}$  will have a polynomial coefficient of  $T_u$  with no constant term. We can repeat this process to eventually obtain an expression which satisfies all of

the characteristic properties of Kazhdan-Lusztig elements and must therefore equal  $C_w$ . This provides a new and potentially efficient way to calculate the Kazhdan-Lusztig basis elements and thus the Kazhdan-Lusztig polynomials. This process is related to applying the decomposition theorem to the resolutions we introduce in Chapter 4 and is analogous to the procedure used by Springer [30] for the Bott-Samelson resolutions (see also Polo's paper [26]).

**Example 3.21.** *Let  $w = 53241 \in S_5$  with  $\underline{w} = (s_1)(s_2s_1)(s_3)(s_4s_3)(s_2s_1)$ . As we saw in Example 3.14, the coefficient of  $T_1T_2T_1T_4$  in  $L_{\underline{w}}$  is  $(q^4 + q^2 + 1)$ , and because of the constant term 1, we know immediately that  $L_{\underline{w}} \neq C_w$ . However, for any reduced expression  $\underline{u}$  of  $u = s_1s_2s_1s_4$  obtained via Algorithm 3.8, we can easily verify that  $L_{\underline{w}} - L_{\underline{u}}$  has no coefficients with degree  $\leq 0$ , and so  $L_{\underline{w}} - L_{\underline{u}} = C_w$ .*

The main goals of this chapter are to explore properties of  $L_{\underline{w}}$  for various reduced expressions, investigate cases for which  $L_{\underline{w}}$  might equal  $C_w$ , and to consider the consequences of such an equality.

### 3.3 Previous Work

Lascoux has shown that for any element  $w \in S_n$  which avoids the patterns 3412 and 4231, there exists a reduced expression  $\underline{w}$  of  $w$  for which the corresponding product  $\prod \left( T_i - \frac{q^a}{[a]} \right)$  equals the Kazhdan-Lusztig element  $C_w$ .

**Proposition 3.22.** [22] *Let  $w \in S_n$ .*

(1) *If there exists an integer  $k$  for which  $n = w(k) > w(k+1) > \dots > w(n)$ , then*

$$C_w = C_{w \setminus n} \left( T_{n-1} - \frac{q^{n-k}}{[n-k]} \right) \cdots \left( T_{k+1} - \frac{q^2}{[2]} \right) \left( T_k - \frac{q^1}{[1]} \right).$$



(2) If there exists an integer  $k$  for which  $n = w^{-1}(k) > w^{-1}(k+1) > \dots > w^{-1}(n)$ , then

$$C_w = \left(T_k - \frac{q^1}{[1]}\right) \left(T_{k+1} - \frac{q^2}{[2]}\right) \cdots \left(T_{n-1} - \frac{q^{n-k}}{[n-k]}\right) C_{(w^{-1} \setminus n)^{-1}}.$$

It was shown by Gasharov in [12] (see Proposition 2.10), that if  $w \in S_n$  corresponds to a nonsingular Schubert variety  $X_w$ , then  $w$  will satisfy at least one of the hypotheses of (1) and (2), and that the smaller length element  $w \setminus n$  or  $(w^{-1} \setminus n)^{-1}$  would also correspond to a nonsingular Schubert variety. This allows Proposition 3.22 to be applied inductively until the Kazhdan-Lusztig basis element  $C_w$  is completely factored into terms of the form  $\left(T_i - \frac{q^a}{[a]}\right)$ .

Proposition 3.22 extended previous work which showed that these factorizations hold when  $w \in S_n$  is the longest element [11]. It was soon after showed by Kirillov and Lascoux that if  $w \in S_n$  corresponds to a Schubert subvariety of a Grassmann variety, then  $C_w$  factors into terms of the form  $\left(T_i - \frac{q^a}{[a]}\right)$  [19].

### 3.4 Properties of Modified Height

**Lemma 3.23.** *Let  $\underline{w}'$  be a fixed reduced word in  $S_n$  and suppose  $w \in S_n$  can be factored into a reduced word of the form  $\underline{w} = \underline{w}' s_{k+h} \cdots s_{k+1} s_k \in S_n$  with associated height sequence  $hts(\underline{w}) = (hts(\underline{w}'), h+1, h, \dots, 2, 1)$ . Then  $s_{k+i} \in D_R(w')$  for each  $i \in \{0, 1, \dots, h-1\}$ .*

*Proof.* Recall that for any simple reflection  $s_j$  and any element  $u \in S_n$ , we have  $s_j \in D_R(u)$  if and only if  $u(j) > u(j+1)$ . We therefore wish to show that  $w'(k) > w'(k+1) > \dots > w'(k+i+1)$ . Let  $0 \leq i < h$  and suppose for a contradiction that  $w'(k+i) < w'(k+i+1)$ . By the hypothesis, we know that  $ht_w(e_{w'(k+i)} -$

$e_{w'(k+h+1)} = i + 1$  and  $\text{ht}_w(e_{w'(k+i+1)} - e_{w'(k+h+1)}) = i + 2$ . In other words, when  $w^{-1}$  is expressed in one-line notation, then longest decreasing sequence between the index  $w'(k+i)$  and the index  $w'(k+h+1)$  is shorter than the longest decreasing sequence between the index  $w'(k+i+1)$  and the index  $w'(k+h+1)$ . Also, the assumption  $w'(k+i) < w'(k+i+1)$  indicates that when  $w^{-1}$  is expressed in one-line notation, the entry  $w^{-1}(w'(k+i)) = k+i+1$  appears to the left of the entry  $w^{-1}(w'(k+i+1)) = k+i+2$ . Both of these entries necessarily appear to the left of the entry  $w^{-1}(w'(k+h+1)) = k$ . However, since  $k+i+1$  and  $k+i+2$  are consecutive numbers, it is impossible for  $w^{-1}$  to satisfy the two conditions that (a)  $k+i+1$  appears to the left of  $k+i+2$ , and (b) the longest decreasing sequence between  $k+i+1$  and  $k+h+1$  is shorter than the longest decreasing sequence between  $k+i+2$  and  $k+h+1$ . Thus, we have arrived at a contradiction.  $\square$

**Definition 3.24.** Let  $w \in S_n$ . Consider any maximal decreasing subword of adjacent entries  $w_j w_{j+1} \cdots w_{j+r}$  of the one-line expression of  $w$ . A shift by  $N$  applied to this decreasing subword is the reordering of it to obtain the subword

$$w_{j+r-N+1} w_{j+r-N+2} \cdots w_{j+r} w_i w_{j+1} \cdots w_{j+r-N}.$$

In other words, if  $w'$  is the element obtained by applying a shift by  $N$  to a particular decreasing subword  $w_j w_{j+1} \cdots w_{j+r}$  of  $w$ , then the one-line expression for  $w'$  is obtained from the one-line expression for  $w$  by switching the decreasing subword  $w_j \cdots w_{j+N-1}$  (the first  $N$  entries in the given subword) with the subword  $w_{j+N} \cdots w_{j+r}$  (the remaining part of the subword). As an expression in simple reflections, we can express  $w$  as  $w'x$  where

$$x = (s_{j+r-N} \cdots s_{j+1} s_j)(s_{j+r-N+1} \cdots s_{j+2} s_{j+1}) \cdots (s_{j+r-1} \cdots s_{j+N} s_{j+N-1}).$$

We will say that a shift *preserves heights* if heights are preserved with respect to the corresponding factorization of  $w$ .

**Example 3.25.** Consider the subword 6431 in the element  $w = 7564312 \in S_7$ . Applying a shift by 1 to this subword would produce the element  $7543162 \in S_7$ , and applying a shift by 2 to this subword would produce the element  $7531642 \in S_7$ .

**Observation 3.26.** Let  $w \in S_n$  and suppose  $w_j w_{j+1} \cdots w_{j+r}$  is an adjacent decreasing subword of the one-line expression of  $w$ . Let  $w' \in S_n$  be the element obtained from  $w$  by shifting this decreasing subword by  $N$ . By Proposition 3.6, we can see that this shift is height-preserving if the following two conditions hold.

1. For all  $p < j$ , the length of the longest decreasing subword from the entry  $w_p$  to any one of the entries  $w_j, w_{j+1}, \dots, w_{j+r}$  is the same in  $w$  as in  $w'$ .
2. For all  $q > j + r$ , the length of the longest decreasing subword from any one of the entries  $w_j, \dots, w_{j+r}$  to the entry  $w_q$  is the same in  $w$  as in  $w'$ .

**Example 3.27.** Let  $w = 35421$  and consider the decreasing sequence 5421 in  $w$ . A shift by 3 of this subword would not preserve heights, because the longest decreasing sequence between the entries 3 and 1 in  $w$  involves three entries (3, 2, 1), while the longest decreasing sequence between 3 and 1 in 31542 involves only two entries (3, 1). However, a shift by 2 of this subword, which results in the element  $32154 \in S_5$ , is height-preserving.

**Lemma 3.28.** Suppose  $w \in S_n$  avoids the patterns 45312 and 4231. Let  $w(d) = n$  and let  $w(d) w(d+1) \cdots w(d+k)$  be the maximal adjacent decreasing subword of  $w$  starting with  $n$ .

1. Then there exists some  $p \in \{1, 2, \dots, k\}$  such that for all  $i < d$  we have  $w(i) < w(d+p-1)$ , and for all  $i > d+k$ , we have  $w(i) > w(d+p)$ .
2. It follows immediately that heights are preserved under a shift by  $p$  applied to this subword.

3. Suppose additionally that  $w$  avoids the pattern 3412. For the value of  $p$  found in (1), let  $w'$  denote the element obtained from  $w$  by applying a shift by  $p$  to the decreasing subword  $w(d)w(d+1)\cdots w(d+k)$ . Then  $w'$  belongs to a maximal parabolic subgroup isomorphic to  $S_{d+p-1} \times S_{n-(d+p-1)}$ .

*Proof.* 1. If  $d+k = n$ , then for all  $i < d$  we have  $w(i) < w(d) = w(d+k-k)$  and there are no entries  $w(i)$  with  $i > d+k$ . We can therefore assume that  $d+k < n$ . Then since this decreasing sequence is maximal, we know that  $w(d+k+1) > w(d+k)$ . This fact alone implies that there exists some  $p \in \{1, \dots, k\}$  such that  $w(d+p-1) > w(d+k+1) > w(d+p)$ . Fix such a  $p$ . Then if  $w(i) > w(d+p-1)$  for any  $i < d$ , the element  $w$  would contain the subword  $w(i)w(d)w(d+p-1)w(d+k)w(d+k+1) \sim 45312$ . And if  $w(i) < w(d+p)$  for any  $i > d+k$ , we would have  $w(d+p-1)w(d+p)w(d+k+1)w(i) \sim 4231$ . Thus, the claim is satisfied for this choice of  $p$ .

3. From the above and by the construction of  $w'$ , it is clear that we have  $w'(i) < w'(j)$  for all  $i \leq d+p-1$  and all  $j > d+p-1$ . Let  $J = \{s_1, \dots, s_{n-1}\} \setminus \{s_{d+p-1}\}$ . Then  $w'$  belongs to the subgroup of  $S_n$  generated by the simple reflections in  $J$ . By standard facts (see [7]), we know that this subgroup is isomorphic to  $S_{d+p-1} \times S_{n-(d+p-1)}$ .

□

### 3.5 Identities in the Nonsingular Case

In this section we will prove certain identities with a focus on elements  $C_w$  and  $L_{\underline{w}}$  where  $w \in W$  corresponds to a nonsingular Schubert variety. This section reproves Proposition 3.22.

Lemma 3.29 below was originally shown by Kazhdan and Lusztig in their seminal 1979 paper “Representations of Coxeter Groups and Hecke Algebras”. We will reprove this result using our terminology.

**Lemma 3.29.** [18] *Let  $w \in W$  be any element other than the identity element. Let  $s_1$  be a simple reflection in the right descent set of  $w$  and let  $s_2$  be a simple reflection in the left descent set of  $w$  (in other words, suppose  $ws_1 < w$  and  $s_2w < w$ ). Then*

$$C_w T_{s_1} = (-q^{-1})C_w = T_{s_2} C_w.$$

*Proof.* For any element  $v \leq w$ , we have  $vs_1 \leq w$  and  $P_{v,w}(q) = P_{vs_1,w}(q)$ , since  $s_1 \in D_R(w)$  (see [17] Chapters 5 and 7). Let  $v \leq w$  and assume  $vs_1 < v$ . Let  $\ell$  denote  $\ell(w) - \ell(v)$ . We will compute the coefficients of  $T_v$  and  $T_{vs_1}$  in the product  $C_w T_{s_1}$ . Both terms are obtained from the expression

$$(P_{v,w}(q^{-2})(-q)^{\ell(v,w)}T_v + P_{vs_1,w}(q^{-2})(-q)^{\ell(vs_1,w)}T_{vs_1})T_{s_1}.$$

Observe that the factor

$$(P_{v,w}(q^{-2})(-q)^{\ell(v,w)}T_v + P_{vs_1,w}(q^{-2})(-q)^{\ell(vs_1,w)}T_{vs_1})$$

can be re-expressed as  $P_{v,w}(q^{-2})(-q)^\ell(T_v - qT_{vs_1})$ . Also observe that

$$(T_v - qT_{vs_1})T_{s_1} = (-q^{-1})T_v + (1 + q)T_{vs_1}.$$

We therefore have

$$\begin{aligned} & (P_{v,w}(q^{-2})(-q)^{\ell(v,w)}T_v + P_{vs_1,w}(q^{-2})(-q)^{\ell(vs_1,w)}T_{vs_1})T_{s_1} \\ &= P_{v,w}(q^{-2})(-q)^\ell(T_v - qT_{vs_1})T_{s_1} \\ &= P_{v,w}(q^{-2})(-q)^\ell((-q^{-1})T_v + (1 + q)T_{vs_1}). \end{aligned}$$

By factoring out  $(-q^{-1})$  from the entire expression and then simplifying, we have

$$(P_{v,w}(q^{-2})(-q)^{\ell(v,w)}T_v + P_{vs_1,w}(q^{-2})(-q)^{\ell(vs_1,w)}T_{vs_1})T_{s_1}$$

$$\begin{aligned}
&= (-q^{-1})(P_{v,w}(q^{-2})(-q)^\ell T_v + P_{v,w}(q^{-2})(-q)^{\ell+1} T_{vs_1}) \\
&= (-q^{-1})(P_{v,w}(q^{-2})(-q)^{\ell(v,w)} T_v + P_{vs_1,w}(q^{-2})(-q)^{\ell(vs_1,w)} T_{vs_1}).
\end{aligned}$$

Hence  $C_w T_{s_1} = (-q^{-1})C_w$ . An analogous computation proves that  $T_{s_2} C_w = (-q^{-1})C_w$ .  $\square$

If  $w \in S_n$  corresponds to a nonsingular Schubert variety, then by Theorem 2.7 and Theorem/Definition 2.10, at least one of Propositions 3.30 and 3.31 will hold.

Recall that if  $w \in S_n$ , then for any  $1 \leq k \leq n$ , the element  $\text{fl}(w \setminus k) \in S_{n-1}$  is an element which permutes the indices 1 through  $n-1$ , and thus lies in  $S_n$  via the standard embedding of  $S_{n-1} \hookrightarrow S_n$ .

**Proposition 3.30.** *Let  $w \in S_n$  and suppose  $n = w(k) > w(k+1) > \dots > w(n)$  for some  $1 \leq k \leq n$ . Then  $w = (w')_{s_{n-1}s_{n-2}\dots s_k} \in S_n$  where  $w' = \text{fl}(w \setminus n) \in S_{n-1}$ , and*

$$\begin{aligned}
C_{w'} &\left(T_{n-1} - \frac{q^{n-k}}{[n-k]}\right) \left(T_{n-2} - \frac{q^{n-k-1}}{[n-k-1]}\right) \dots \left(T_{k+1} - \frac{q^2}{[2]}\right) \left(T_k - \frac{q^1}{[1]}\right) \\
&= C_{w'}(T_{n-1}T_{n-2}\dots T_k + (-q)^1 T_{n-1}T_{n-2}\dots T_{k+1} + \dots + (-q)^{n-k-1} T_{n-1} + (-q)^{n-k}).
\end{aligned}$$

*Proof.* Since  $n = w(k) > w(k+1) > \dots > w(n)$ , we have  $s_k, s_{k+1}, \dots, s_{n-1} \in D_R(w)$ . Observe that for any integer  $r \geq 1$ , we have

$$q^{-1} - \frac{q^r}{[r]} = \frac{-q^{-1}(q^r - q^{-r}) - q^r(q - q^{-1})}{q^r - q^{-r}} = -\frac{(q^{r+1} - q^{-(r+1)})}{q^r - q^{-r}}. \quad (3.1)$$

We will proceed by comparing coefficients on the left and right. First consider terms of the form  $C_{w'} f(q, q^{-1})T_v$  where  $f$  is some rational function of  $q, q^{-1}$  and  $v$  is a subword of  $s_{n-2}s_{n-3}\dots s_k$ . On the left side of the equation, these terms are obtained by the expression

$$\begin{aligned}
&C_{w'} \left(-\frac{q^{n-k}}{[n-k]}\right) \left(T_{n-2} - \frac{q^{n-k-1}}{[n-k-1]}\right) \dots \left(T_k - \frac{q^1}{[1]}\right) \\
&= C_{w'} \left(-\frac{q^{n-k}}{[n-k]}\right) \left(-q^{-1} - \frac{q^{n-k-1}}{[n-k-1]}\right) \dots \left(-q^{-1} - \frac{q^1}{[1]}\right)
\end{aligned}$$

where the equality follows from Lemma 3.29. By Equation (3.1), the above is equal to

$$C_{w'} \left( \frac{-q^{n-k}(q - q^{-1})}{q^{n-k} - q^{-(n-k)}} \right) \left( \frac{-(q^{n-k} - q^{-(n-k)})}{q^{n-k-1} - q^{-(n-k-1)}} \right) \cdots \left( \frac{-(q^2 - q^{-2})}{q^1 - q^{-1}} \right).$$

This expression simplifies to  $C_{w'}(-q)^{n-k}$ , which is precisely the term on the right side which does not involve  $T_{n-1}$ .

Let  $1 \leq j < n - k$ . We will now compute the terms on the right and the left of the form  $C_{w'}f(q, q^{-1})T_v$  where  $f$  is some rational function and  $v = s_{n-1}s_{n-2} \cdots s_{n-j}u$  and  $u$  is a subword of  $s_{n-j-2} \cdots s_k$  (i.e.  $v$  is a subword of  $s_{n-1}s_{n-2} \cdots s_k$  which involves  $s_{n-1}, s_{n-2}, \dots, s_{n-j}$  but does not involve  $s_{n-j-1}$ ). On the left side, these terms are obtained from the expression

$$C_{w'}T_{n-1}T_{n-2} \cdots T_{n-j} \left( -\frac{q^{n-k-j}}{[n-k-j]} \right) \left( T_{n-j-2} - \frac{q^{n-k-j-1}}{[n-k-j-1]} \right) \cdots \left( T_k - \frac{q^1}{[1]} \right)$$

which, using Lemma 3.29 and Equation (3.1), simplifies to become

$$C_{w'}(-q)^{n-k-j}T_{n-1, \dots, n-j}.$$

This is precisely the term on the right side which involves  $T_{n-1}T_{n-2} \cdots T_{n-j}$ .

The only remaining term to be considered is the term involving  $T_{n-1, \dots, k}$ . On both sides, this term is simply  $C_{w'}T_{n-1}T_{n-2} \cdots T_k$ . We have now verified term-by-term that the left side of the equation is equal to the right side.  $\square$

Analogous reasoning, or the application of Proposition 3.30 to  $w^{-1}$ , proves the following.

**Proposition 3.31.** *Let  $w \in S_n$  and suppose  $n = w^{-1}(k) > w^{-1}(k+1) > \cdots > w^{-1}(n)$  for some  $1 \leq k \leq n$ . Then  $w = s_k s_{k+1} \cdots s_{n-1}(w') \in S_n$  where  $w' = fl((w^{-1} \setminus n)^{-1}) \in S_{n-1}$ , and we have*

$$\begin{aligned} & \left( T_k - \frac{q^1}{[1]} \right) \left( T_{k+1} - \frac{q^2}{[2]} \right) \cdots \left( T_{n-1} - \frac{q^{n-k}}{[n-k]} \right) C_{w'} \\ &= (T_{n-1}T_{n-2} \cdots T_k + (-q)^1 T_{n-1}T_{n-2} \cdots T_{k+1} + \cdots + (-q)^{n-k-1} T_{n-1} + (-q)^{n-k}) C_{w'}. \end{aligned}$$

**Corollary 3.32.** [22] Suppose  $w \in S_n$  corresponds to a nonsingular Schubert variety.

If  $w$  belongs to Case 1 of Theorem/Definition 2.10, then

$$\begin{aligned} C_w &= C_{w'} \left( T_{n-1} - \frac{q^{n-d}}{[n-d]} \right) \left( T_{n-2} - \frac{q^{n-k-1}}{[n-d-1]} \right) \cdots \left( T_{d+1} - \frac{q^2}{[2]} \right) \left( T_d - \frac{q^1}{[1]} \right) \\ &= C_{w'} (T_{n-1} T_{n-2} \cdots T_d + (-q)^1 T_{n-1} T_{n-2} \cdots T_{d+1} + \cdots + (-q)^{n-d-1} T_{n-1} + (-q)^{n-d}). \end{aligned}$$

If  $w$  belongs to Case 2 of Theorem/Definition 2.10, we have

$$\begin{aligned} C_w &= \left( T_c - \frac{q^1}{[1]} \right) \left( T_{c+1} - \frac{q^2}{[2]} \right) \cdots \left( T_{n-1} - \frac{q^{n-c}}{[n-c]} \right) C_{w'} \\ &= (T_{n-1} T_{n-2} \cdots T_c + (-q)^1 T_{n-1} T_{n-2} \cdots T_{c+1} + \cdots + (-q)^{n-c-1} T_{n-1} + (-q)^{n-c}) C_{w'}. \end{aligned}$$

*Proof.* Suppose  $w$  belongs to Case 1. Then by Proposition 3.30, we know that

$$\begin{aligned} &C_{w'} \left( T_{n-1} - \frac{q^{n-d}}{[n-d]} \right) \left( T_{n-2} - \frac{q^{n-d-1}}{[n-d-1]} \right) \cdots \left( T_{d+1} - \frac{q^2}{[2]} \right) \left( T_d - \frac{q^1}{[1]} \right) \\ &= C_{w'} (T_{n-1} T_{n-2} \cdots T_d + (-q)^1 T_{n-1} T_{n-2} \cdots T_{d+1} + \cdots + (-q)^{n-d-1} T_{n-1} + (-q)^{n-d}). \end{aligned}$$

Note that the left side of this equation is bar-invariant and has dominant term  $T_w$ .

Note also that each coefficient on the right side of this equation is a polynomial in  $q$  with no constant term. Any expression with these properties is necessarily equal to  $C_w$ .

Similarly, if  $w$  belongs to Case 2, then by Proposition 3.31 we have

$$\begin{aligned} &\left( T_c - \frac{q^1}{[1]} \right) \left( T_{c+1} - \frac{q^2}{[2]} \right) \cdots \left( T_{n-1} - \frac{q^{n-c}}{[n-c]} \right) C_{w'} \\ &= (T_{n-1} T_{n-2} \cdots T_c + (-q)^1 T_{n-1} T_{n-2} \cdots T_{c+1} + \cdots + (-q)^{n-c-1} T_{n-1} + (-q)^{n-c}) C_{w'}. \end{aligned}$$

As before, the left side of this equation is bar-invariant and has dominant term  $T_w$ , and each coefficient on the right side of this equation is a polynomial with no constant term. Hence these expressions must be equal to  $C_w$ .  $\square$

Since every nonsingular element  $w \in S_n$  necessarily belongs to either Case 1 or Case 2 in Theorem/Definition 2.10, we immediately can conclude that for any nonsingular  $w \in S_n$ , there exists a product of factors of the form  $\left( T_i - \frac{q^a}{[a]} \right)$  which equals  $C_w$ .



### 3.6 Identities in the Singular Case

In this section we will prove certain identities with a focus on elements  $C_w$  and  $L_{\underline{w}}$  where  $w \in W$  corresponds to a singular Schubert variety.

**Lemma 3.33.** *Suppose  $\underline{w} = \underline{w}'s_{k+h} \cdots s_{k+1}s_k \in S_n$  has associated height sequence  $hts(\underline{w}) = \{hts(\underline{w}'), h+1, h, \dots, 2, 1\}$ . Then we obtain the following generalization of Proposition 3.30:*

$$\begin{aligned} C_{w'} \left( T_{k+h} - \frac{q^{h+1}}{[h+1]} \right) \cdots (T_k - q) \\ = C_{w'} (T_{k+h} \cdots T_k - qT_{k+h} \cdots T_{k+1} + \cdots + (-q)^h T_{k+h} + (-q)^{h+1}). \end{aligned}$$

*Proof.* We will compare coefficients on the left and right sides of the equation above. First consider terms of the form  $C_{w'} f(q, q^{-1}) T_v$  where  $f$  is some rational function of  $q, q^{-1}$  and  $v$  is a subword of  $s_{k+h-1}s_{k+h-2} \cdots s_k$ . On the left side of the equation, these terms are obtained by the expression

$$\begin{aligned} C_{w'} \left( -\frac{q^{h+1}}{[h+1]} \right) \left( T_{k+h-1} - \frac{q^h}{[h]} \right) \cdots \left( T_k - \frac{q^1}{[1]} \right) \\ = C_{w'} \left( -\frac{q^{h+1}}{[h+1]} \right) \left( -q^{-1} - \frac{q^h}{[h]} \right) \cdots \left( -q^{-1} - \frac{q^1}{[1]} \right) \\ = C_{w'} \left( \frac{-q^{h+1}(q - q^{-1})}{q^{h+1} - q^{-(h+1)}} \right) \left( \frac{-(q^{h+1} - q^{-(h+1)})}{q^h - q^{-(h)}} \right) \cdots \\ \cdots \left( \frac{-(q^3 - q^{-(3)})}{q^2 - q^{-(2)}} \right) \left( \frac{-(q^2 - q^{-(2)})}{q^1 - q^{-(1)}} \right) \\ = C_{w'} (-q)^{h+1} \end{aligned}$$

which is precisely the term on the right side which does not involve  $T_{k+h}$ .

Let  $1 \leq j < h$ . We will now compute the terms on the right and the left of the form  $C_{w'} f(q, q^{-1}) T_v$  where  $f$  is a rational function and  $v = s_{k+h}s_{k+h-1} \cdots s_{k+j+1}u$  and  $u$  is a subword of  $s_{k+j-1} \cdots s_k$  (i.e.  $v$  is a subword of  $s_{k+h}s_{k+h-1} \cdots s_k$  which involves  $s_{k+h}, s_{k+h-1}, \dots, s_{k+j+1}$  but does not involve  $s_{k+j}$ ). On the left side, these terms are obtained by the expression

$$\begin{aligned}
& C_{w'} T_{k+h} T_{k+h-1} \cdots T_{k+j+1} \left( -\frac{q^{j+1}}{[j+1]} \right) \left( T_{k+j-1} - \frac{q^j}{[j]} \right) \cdots \left( T_k - \frac{q^1}{[1]} \right) \\
&= C_{w'} \left( -\frac{q^{j+1}}{[j+1]} \right) \left( T_{k+j-1} - \frac{q^j}{[j]} \right) \cdots \left( T_k - \frac{q^1}{[1]} \right) T_{k+h} T_{k+h-1} \cdots T_{k+j+1} \\
&= C_{w'} \left( -\frac{q^{j+1}}{[j+1]} \right) \left( -q^{-1} - \frac{q^j}{[j]} \right) \cdots \left( -q^{-1} - \frac{q^1}{[1]} \right) T_{k+h} \cdots T_{k+j+1} \\
&= C_{w'} \left( \frac{-q^{j+1}(q - q^{-1})}{q^{j+1} - q^{-(j+1)}} \right) \left( \frac{-(q^{j+1} - q^{-(j+1)})}{q^j - q^{-(j)}} \right) \cdots \\
&\quad \cdots \left( \frac{-(q^3 - q^{-(3)})}{q^2 - q^{-(2)}} \right) \left( \frac{-(q^2 - q^{-(2)})}{q^1 - q^{-(1)}} \right) T_{k+h} \cdots T_{k+j+1} \\
&= C_{w'} (-q)^{j+1} T_{k+h} \cdots T_{k+j+1}
\end{aligned}$$

which is precisely the term on the right side which involves  $T_{k+h} \cdots T_{k+j+1}$ .

The terms involving  $T_{k+h} \cdots T_{k+1}$  on the right and left are both clearly given by  $C_{w'}(-q)T_{k+h} \cdots T_{k+1}$ .

The only remaining term to be considered is the term involving  $T_{k+h} \cdots T_k$ . On both sides, this term is simply  $C_{w'} T_{k+h} \cdots T_k$ . We have now verified term-by-term that the left side of the equation is equal to the right side.  $\square$

Consider the equation of Lemma 3.33. Note that the left side of this equation is bar-invariant, the right side satisfies the conditions that each coefficient is a Laurent polynomial in  $q$ , and the dominant term on either side is  $T_w$ , which are all properties satisfied by the Kazhdan-Lusztig basis element  $C_w$  for  $w$ . It turns out that the coefficients of Lascoux elements occasionally have nonzero terms of degree  $\leq 0$  (see the examples of Section 4.3). When this is not the case, the Lascoux elements are the Kazhdan-Lusztig basis elements.

### 3.7 Connection to Intersection Cohomology

For any  $w \in W$ , we will show that under the transformation  $T_s \mapsto -1/q$ , the Kazhdan-Lusztig basis element  $C_w$  specializes to the polynomial  $\mathcal{I}_w(q)$ .

**Definition 3.34.** Define a ring homomorphism  $F : \mathcal{H} \rightarrow \mathbb{Z}[q, q^{-1}]$  by  $F(T_{s_i}) = \frac{-1}{q}$ . So in general  $F(T_u) = \left(\frac{-1}{q}\right)^{\ell(u)}$ .

**Lemma 3.35.** For any  $w \in W$ , we have  $F(C_w) = \left(\frac{-1}{q}\right)^{\ell(w)} \mathcal{I}_w(q^2)$ .

*Proof.* First note that since  $\mathcal{I}_w(q)$  is symmetric, we have

$$\mathcal{I}_w(q) = q^{\ell(w)} \mathcal{I}_w(q^{-1}) = q^{\ell(w)} \sum_{u \leq w} q^{-\ell(u)} P_{u,w}(q^{-1}) = \sum_{u \leq w} q^{\ell(w)-\ell(u)} P_{u,w}(q^{-1}).$$

Using this fact, we then have

$$\begin{aligned} F(C_w) &= \sum_{u \leq w} (-q)^{\ell(w)-\ell(u)} P_{u,w}(q^{-2}) \left(\frac{-1}{q}\right)^{\ell(w)} \\ &= \left(\frac{-1}{q}\right)^{\ell(w)} \sum_{u \leq w} (q^2)^{\ell(w)-\ell(u)} P_{u,w}(q^{-2}) \\ &= \left(\frac{-1}{q}\right)^{\ell(w)} \mathcal{I}_w(q^2). \end{aligned}$$

□

Lemma 3.35 was originally shown by Lascoux in [22]. Note that if  $w \in W$  has a reduced expression for which  $L_{\underline{w}} = C_w$ , then by computing  $F(L_{\underline{w}}) = \mathcal{I}_w(q)$ , we obtain a factorization of  $\mathcal{I}_w(q)$  into  $q$ -numbers. We will revisit this relationship again in Chapter 5.

### 3.8 Future Work

Let  $W$  denote a general Weyl group. In joint work with Eric Sommers, we hope to provide a proof of the following statement.

**Claim 3.36.** Lemma 3.33 has a generalization to all maximal parabolic subgroups, in Weyl groups of type  $A$  and in other types as well. This implies that the coefficients of the Lascoux elements  $L_w$  are all Laurent polynomials.

## CHAPTER 4

### A RESOLUTION OF SCHUBERT VARIETIES

Let  $W$  be a Weyl group and let  $w \in W$ . In this chapter, we will describe a particular method for constructing a smooth variety  $Z_{w^{-1}}$  which will turn out not only to be a resolution of the Schubert variety  $X_{w^{-1}}$ , but which can also be described as an iterated fibration with fibers isomorphic to partial flag varieties. We will see that the resolution  $Z_{w^{-1}}$  is closely related to the Lascoux elements of Chapter 3.

This resolution  $Z_{w^{-1}}$ , is a generalization of the well-known Bott-Samelson resolution. Zelevinsky defined these resolutions in certain type  $A$  cases [35], and this work was generalized by Sankaran and Vanchinathan in [28]. These resolutions have been used to prove important results on Schubert varieties and Kazhdan-Lusztig polynomials (see [26] and [8]).

#### 4.1 Defining the variety $Z_{w^{-1}}$

Let  $w \in W$ . Use Algorithm 3.8 to construct subsets  $R_i$  and  $T_i$ ,  $1 \leq i \leq m$ , and obtain an associated reduced expression  $\underline{w}$  for  $w$ . Let  $\mathbf{P}_i$  be the parabolic subgroup of  $G$  indexed by  $R_i$  for each  $1 \leq i \leq m$ , and similarly define parabolic

subgroups  $\mathbf{Q}_i$  to be those indexed by the sets  $T_i$ . We define a variety  $Z_{w^{-1}}$  as

$$Z_{w^{-1}} = \mathbf{P}_1 \times^{\mathbf{Q}_1} \mathbf{P}_2 \times^{\mathbf{Q}_2} \dots \times^{\mathbf{Q}_{m-1}} \mathbf{P}_m / \mathbf{Q}_m.$$

Here, we have

$$\mathbf{P}_1 \times^{\mathbf{Q}_1} \mathbf{P}_2 \times^{\mathbf{Q}_2} \dots \times^{\mathbf{Q}_{m-1}} \mathbf{P}_m / \mathbf{Q}_m \equiv (\mathbf{P}_1 \times \mathbf{P}_2 \times \dots \times \mathbf{P}_m) / \sim$$

where  $\sim$  is the equivalence relation arising from the action of  $\mathbf{Q}_1 \times \mathbf{Q}_2 \times \dots \times \mathbf{Q}_m$  on  $\mathbf{P}_1 \times \mathbf{P}_2 \times \dots \times \mathbf{P}_m$  given by

$$(q_1, q_2, \dots, q_m) : (a_1, a_2, \dots, a_m) \mapsto (a_1 q_1, q_1^{-1} a_2 q_2, \dots, q_{m-1}^{-1} a_m q_m).$$

Define  $\pi : Z_{w^{-1}} \rightarrow G/B$  by  $\pi(a_1, a_2, \dots, a_m) = a_1 a_2 \dots a_m B$ . Then  $\pi : Z_{w^{-1}} \rightarrow \text{im}(\pi)$  is a proper  $\mathbf{P}_1$ -equivariant birational map with  $\mathbf{P}_1$ -stable image. We have  $\text{im}(\pi) = X_{w^{-1}}$ , and since  $\ell(w) = \ell(w') + \ell(x)$ , we can conclude that  $\pi$  is injective over the open Schubert cell  $X_{w^{-1}}^\circ$ . Thus  $(Z_{w^{-1}}, \pi)$  is a resolution of the Schubert variety  $X_{w^{-1}}$ .

The resolution  $Z_{w^{-1}}$  is a generalization of the Bott-Samelson resolution. In the Bott-Samelson construction, each  $\mathbf{P}_i$  is taken to be indexed by a single element of  $D_R(w')$ , rather than a subset of elements, and each  $\mathbf{Q}_i$  is taken simply to be the Borel subgroup  $B$ .

Define  $\phi : Z_{w^{-1}} \rightarrow \mathbf{P}_1 \times^{\mathbf{Q}_1} \dots \times^{\mathbf{Q}_{m-2}} \mathbf{P}_{m-1} / \mathbf{Q}_{m-1}$  by

$$\phi(a_1, a_2, \dots, a_m) = (a_1, a_2, \dots, a_{m-1}).$$

Then  $\phi$  is a fibration with associated fiber  $\mathbf{P}_m / \mathbf{Q}_m$ . When  $W = W(A_n)$ , the fiber  $\mathbf{P}_m / \mathbf{Q}_m$  will be isomorphic to a Grassmannian. In general, the fiber  $\mathbf{P}_m / \mathbf{Q}_m$  will be isomorphic to a minimal generalized flag variety for a simple group  $G$ . In this way, we can see that  $Z_{w^{-1}}$  is an iterated fibration with fibers  $\mathbf{P}_m / \mathbf{Q}_m, \dots, \mathbf{P}_1 / \mathbf{Q}_1$

isomorphic to partial flag varieties:

$$\begin{array}{ccc}
\mathbf{P}_m/\mathbf{Q}_m & \longrightarrow & \mathbf{P}_1 \times^{\mathbf{Q}_1} \cdots \times^{\mathbf{Q}_{m-1}} \mathbf{P}_m/\mathbf{Q}_m \\
& & \downarrow \\
\mathbf{P}_{m-1}/\mathbf{Q}_{m-1} & \longrightarrow & \mathbf{P}_1 \times^{\mathbf{Q}_1} \cdots \times^{\mathbf{Q}_{m-2}} \mathbf{P}_{m-1}/\mathbf{Q}_{m-1} \\
& & \downarrow \\
& & \vdots \\
& & \downarrow \\
& & \mathbf{P}_1/\mathbf{Q}_1
\end{array}$$

The resolution  $(Z_{w^{-1}}, \pi)$  of  $X_{w^{-1}}$  is closely related to the Lascoux elements of Chapter 3, which were constructed using the same algorithm. When  $(Z_{w^{-1}}, \pi)$  is the Bott-Samelson, recall that  $ch(R\pi_*(\mathbb{C}[\ell(w)]))$  produces the Kazhdan-Lusztig basis element  $L_{w^{-1}}$  in the Hecke algebra under the substitution  $q \mapsto -q^{-1}$  (see Theorem 2.5). More generally, we have the following.

**Proposition 4.1.** *When the reduced word  $\underline{w}$  is produced using the Algorithm 3.8, then  $R\pi_*(\mathbb{C}[\ell(w)])$  is a sheaf on the Schubert variety  $X_{w^{-1}}$  (after the substitution  $q \mapsto -q^{-1}$ ) whose expression in the Hecke algebra is exactly  $L_{\underline{w}}$ .*

The proof is the subject of future joint work with Eric Sommers.

**Example 4.2.** *Let  $w = 53241$  as in Examples 3.13 and 3.14. Recall that the reduced word  $\underline{w}_1 = (s_1)(s_2s_1)(s_3)(s_4s_3)(s_2s_1)$  used in Example 3.14 did not preserve heights, and we saw that  $L_{\underline{w}_1} \neq C_{w_1}$ . The resolution associated with this reduced word is given by*

$$Z_{w^{-1}} = P_{124} \times^{P_{14}} P_{134} \times^{P_{13}} P_{13} \times^{P_1} P_{12} \times^{P_1} P_1/B.$$

*The reduced word  $\underline{w}_2 = (s_1)(s_2s_1)(s_4)(s_3s_2s_1)(s_4)$  used in Example 3.13 did preserve heights, and we saw that  $L_{\underline{w}_2} = C_{w_2}$ . The resolution associated with  $\underline{w}_2$  is given by*

$$Z_{w^{-1}} = P_{124} \times^{P_{12}} P_{123} \times^{P_{12}} P_{124} \times^{P_{12}} P_{12} \times^{P_1} P_1/B.$$

## 4.2 Applications in Type A

For  $w \in S_n$ , the processes discussed in the previous section can be reinterpreted in terms of pattern avoidance to yield some interesting results. In particular, Lemma 3.28 (3) has a geometric implication as we will now show.

**Proposition 4.3.** *Suppose  $w \in S_n$  avoids the patterns 3412 and 4231 (equivalently, suppose  $w \in S_n$  corresponds to a nonsingular Schubert variety). Then there exists an isomorphism between a smooth space  $Z_{w^{-1}}$  (the same  $Z_{w^{-1}}$  described in the last section), which can be decomposed as an iterated fibration of Grassmannians, and the Schubert variety  $X_{w^{-1}}$ .*

*Proof.* Let  $w(d)w(d+1)\cdots w(d+k)$  be the maximal adjacent decreasing sequence in the one-line notation for  $w$  starting with  $w(d) = n$ . Let  $m$  denote the integer between 1 and  $k$  described in Lemma 3.28. Let  $w'$  be the element obtained from applying the height-preserving shift by  $m$  to this decreasing subword. Then  $w'$  fixes the index  $d+m-1$ , and so  $w'$  belongs to the subgroup of  $W$  generated by the reflections  $\{s_1, \dots, s_n\} \setminus \{s_{d+m-1}\}$ , which is isomorphic to  $S_{d+m-1} \times S_{n-(d+m-1)}$  (see Lemma 3.28). We have  $Z_{w^{-1}} = P_{d,d+1,\dots,d+k-1} \times P_{d,\dots,d+m-2,d+m,\dots,d+k-1} Z_{(w')^{-1}}$  and  $\dim(Z_{w^{-1}}) = \dim(X_{w^{-1}})$ . Let  $\mathbf{P}_1 = P_{d,d+1,\dots,d+k-1}$  and  $\mathbf{Q}_1 = P_{d,\dots,d+m-2,d+m,\dots,d+k-1}$ .

We will now show that  $\pi : Z_{w^{-1}} \rightarrow X_{w^{-1}}$  is injective, and hence bijective. Since Schubert varieties are normal, it then follows that  $\pi$  is an isomorphism of varieties.

Since  $w'$  also avoids the required patterns 3412 and 4231, we can continue factoring  $Z_{(w')^{-1}}$  as above, obtaining

$$Z_{w^{-1}} = \mathbf{P}_1 \times^{\mathbf{Q}_1} \mathbf{P}_2 \times^{\mathbf{Q}_2} \cdots \times^{\mathbf{Q}_{r-1}} \mathbf{P}_r / B.$$

Suppose  $\pi(a_1, \dots, a_r) = \pi(c_1, \dots, c_r)$  for some  $(a_1, \dots, a_r), (c_1, \dots, c_r) \in Z_{w^{-1}}$ .

Then the element  $y_r := a_r^{-1} \cdots a_1^{-1} c_1 \cdots c_r \in B$ . Let  $y_1 = a_1^{-1} c_1 \in \mathbf{P}_1$ . By Lemma 3.28 (3), we know that  $y_1 = a_2 a_3 \cdots a_r \cdot y_r \cdot c_r^{-1} \cdots c_3^{-1} c_2^{-1}$  belongs to the parabolic subgroup  $P_{d, \dots, d+m-2, d+m, \dots, d+k-1}$  of  $G$  corresponding to the subgroup  $S_{d+m-1} \times S_{n-(d+m-1)}$  of  $W$ , since

$$\mathbf{P}_i, \mathbf{Q}_i \subset P_{d, \dots, d+m-2, d+m, \dots, d+k-1}$$

for all  $2 \leq i \leq r$ . Thus  $y_1$  lies in the intersection of these two parabolic subgroups, so  $y_1 \in \mathbf{Q}_1$ .

By induction, we can apply the same process as above to show that

$$\begin{aligned} y_2 &:= a_2^{-1} a_1^{-1} b_1 b_2 \in \mathbf{Q}_2 \\ &\vdots \\ y_{r-1} &:= a_{r-1}^{-1} \cdots a_1^{-1} b_1 \cdots b_{r-1} \in \mathbf{Q}_{r-1}. \end{aligned}$$

Then we have  $(y_1, \dots, y_r) \in \mathbf{Q}_1 \times \mathbf{Q}_2 \times \cdots \times \mathbf{Q}_{r-1} \times B$ , and

$$(a_1 y_1, y_1^{-1} a_2 y_2, y_2^{-1} a_3 y_3, \dots, y_{r-1}^{-1} a_r y_r) = (c_1, c_2, \dots, c_r)$$

so  $(a_1, \dots, a_r)$  and  $(c_1, \dots, c_r)$  belong to the same equivalence class in  $Z_{w^{-1}}$ . Thus  $\pi$  is injective.  $\square$

**Corollary 4.4.** *Suppose  $w \in S_n$  avoids the patterns 3412 and 4231 (equivalently suppose the Schubert variety  $X_w$  is nonsingular). Then the Poincaré polynomial of the Schubert variety  $X_w$  factors into a product of symmetric polynomials, each of which are Poincaré polynomials indexed by elements in a maximal parabolic quotient  $W/W_J$ .*

Proposition 4.3 reproves results of Wolper [34] and Ryan [27], who have shown that any nonsingular Schubert variety of type  $A$  can be realized as an iterated sequence of fibrations ending in a Grassmannian, for which all fibers are



isomorphic to Grassmannians. Gasharov and Reiner extended this result to all classical Weyl groups [13]. Billey and Postnikov then showed that even in the exceptional Weyl groups, the Poincaré polynomial of any smooth Schubert variety factors as a product of symmetric polynomials each of which are Poincaré polynomials indexed by elements in a maximal parabolic quotient  $W/W_J$  [4].

### 4.3 Future Work

When Algorithm 3.8 is applied to some element  $w \in W$  to produce a reduced expression  $\underline{w}$  for which heights are preserved, we often have  $L_{\underline{w}} = C_w$ , but this is not necessarily the case. We know that  $L_{\underline{w}} = C_w$  if and only if the resolution  $(Z_w, \pi)$  is small. In joint work with Eric Sommers, we are currently investigating the problem of determining when these resolutions are small. This will build upon previous work of Zelevinsky [35], who determined an explicit small resolution on Grassmann Schubert varieties in type  $A$ , and Sankaran and Vanchinathan [28], who extended Zelevinsky's construction to types  $C$  and  $D$ , and it will also build upon the work of Billey and Warrington [5], who showed 321-hexagon avoiding permutations in  $S_n$  are precisely the elements of  $S_n$  corresponding to Schubert varieties with small Bott-Samelson resolutions.

Suppose Algorithm 3.8 is applied to some element  $w \in W$  to produce a reduced expression  $\underline{w}$  for which heights are preserved, but that  $L_{\underline{w}} \neq C_w$  (see Examples 3.18 and 3.19). All such elements that we have found so far coincide exactly with the elements discovered by Williamson and Braden in [33] for which the intersection cohomology complexes have torsion in their stalks or costalks. This connection is also being explored.

# CHAPTER 5

## THE INVERSION POLYNOMIAL

In this chapter, we will define a new polynomial, called the inversion polynomial, using the Lascoux elements developed in Chapter 3. The inversion polynomial  $\mathcal{N}_w(q)$  will equal the intersection cohomology polynomial  $\mathcal{I}_w(q)$  whenever  $L_{\underline{w}} = C_w$ , regardless of the reduced expression chosen for  $w$ . This polynomial will have a natural factorization into a product of  $q$ -numbers, which will allow us to define its exponents and study them in comparison with those of  $\mathcal{I}_w(q)$ . At the end of the chapter, we will analyze  $\mathcal{N}_w(q)$  and compare it to  $\mathcal{I}_w(q)$  in a manner which is independent of the Lascoux elements  $L_w$ .

### 5.1 Definition

In Section 3.7, we defined a function  $F : \mathcal{H} \rightarrow \mathbb{Z}[q, q^{-1}]$  by  $F(T_{s_i}) = \frac{-1}{q}$ , and saw that we can recover the intersection cohomology Poincaré polynomial for any  $w \in W$  by applying  $F$  to its associated Kazhdan-Lusztig basis element  $C_w$ . Specifically, Lemma 3.35 shows that  $F(C_w) = \left(\frac{-1}{q}\right)^{\ell(w)} \mathcal{I}_w(q^2)$ . When  $L_{\underline{w}} = C_w$  for some reduced expression  $\underline{w}$  of  $w \in W$ , this specialization produces a particular product of polynomials, which will be the main topic of study in this chapter.

**Lemma 5.1.** *Let  $w \in W$ , and suppose  $(\lambda_1, \lambda_2, \dots)$  forms a partition of  $\ell(w)$ , where  $\lambda_i := \#\{\beta \in N(w) : ht_w(\beta) = i\}$ . Then regardless of reduced expression, the product  $q^{\ell(w)}F(L_{\underline{w}})$  is a polynomial.*

*Proof.* Fix a reduced expression  $\underline{w} = s_{i_1}s_{i_2} \cdots s_{i_r}$  and consider the associated ordered inversion set of  $w$  given by  $N(w) = \{\beta_1, \beta_2, \dots, \beta_r\}$ . Then the Lascoux element associated with this reduced expression is defined to be

$$L_{\underline{w}} = \left( T_{s_{i_1}} - \frac{q^{ht_w(\beta_1)}}{[ht_w(\beta_1)]} \right) \cdots \left( T_{s_{i_r}} - \frac{q^{ht_w(\beta_r)}}{[ht_w(\beta_r)]} \right).$$

For brevity, let  $h_j := ht_w(\beta_j)$  and write  $L_{\underline{w}} = \prod_{j=1}^r \left( T_j - \frac{q^{h_j}}{[h_j]} \right)$ . Then we have  $F(L_{\underline{w}}) = \prod_{j=1}^r \left( \frac{-1}{q} - \frac{q^{h_j}}{[h_j]} \right)$ . Since  $[a] := \frac{q^a - q^{-a}}{q - q^{-1}}$  for any integer  $a$ , we can rewrite this expression to obtain

$$F(L_{\underline{w}}) = \left( \frac{-1}{q} \right)^r \prod_{j=1}^r \frac{[h_j + 1]_{q^2}}{[(h_j - 1) + 1]_{q^2}}$$

where each term  $[h + 1]_{q^2}$  is simply the  $q^2$ -number  $(q^2)^h + (q^2)^{h-1} + \cdots + (q^2) + 1$ .

We can now expand this product and cancel terms:

$$\begin{aligned} F(L_{\underline{w}}) &= \left( \frac{-1}{q} \right)^r \left( [1 + 1]_{q^2}^{\lambda_{i_1}} \frac{[2 + 1]_{q^2}^{\lambda_{i_2}}}{[1 + 1]_{q^2}^{\lambda_{i_2}}} \cdots \frac{[r + 1]_{q^2}^{\lambda_{i_r}}}{[(r - 1) + 1]_{q^2}^{\lambda_{i_r}}} \right) \\ &= \left( \frac{-1}{q} \right)^r \left( [1 + 1]_{q^2}^{\lambda_{i_1} - \lambda_{i_2}} [2 + 1]_{q^2}^{\lambda_{i_2} - \lambda_{i_3}} \cdots [(r - 1) + 1]_{q^2}^{\lambda_{i_{r-1}} - \lambda_{i_r}} [r + 1]_{q^2}^{\lambda_{i_r}} \right). \end{aligned}$$

Since  $r = \ell(w)$ , the desired conclusion follows.  $\square$

**Definition 5.2.** *Let  $w \in W$  and suppose  $(\lambda_1, \lambda_2, \dots)$  forms a partition of  $\ell(w)$ , where  $\lambda_i := \#\{\beta \in N(w) : ht_w(\beta) = i\}$ . Define the inversion polynomial  $\mathcal{N}_w(q)$  to be the unique polynomial for which  $F(L_w) = (-1/q)^{\ell(w)}\mathcal{N}_w(q^2)$ . (We write  $L_w$  rather than  $L_{\underline{w}}$  to emphasize that  $F(L_{\underline{w}})$  is independent of reduced expression).*

By Lemma 3.35 and Definition 5.2, the following is clear.

**Proposition 5.3.** *Let  $w \in W$ . We have  $\mathcal{N}_w(q) = \mathcal{I}_w(q)$  whenever  $w$  has a reduced expression  $\underline{w}$  for which  $L_{\underline{w}} = C_w$ .*

## 5.2 Exponents of $\mathcal{N}_w(q)$

When a polynomial has the form  $\prod_i [a_i + 1]_q$ , the values  $a_i$  are often called the *exponents* of the polynomial. Inversion polynomials are defined so as to always have a factorization as a product of  $q$ -numbers, and so it is natural to study their exponents. Doing so will allow us to describe the inversion polynomials in a manner that is independent of Lascoux elements.

For any  $w \in W$ , the exponents of  $\mathcal{N}_w(q)$  can be computed in the following way. Let  $\lambda_i := \#\{\beta \in N(w) : \text{ht}_w(\beta) = i\}$ . Suppose  $(\lambda_1, \lambda_2, \dots)$  forms a partition of  $\ell(w)$ . (In Example 5.6, we will encounter an element for which the sequence  $(\lambda_i)$  does not form a partition of  $\ell(w)$ ). Let  $(m_1, m_2, \dots)$  be the partition conjugate to  $(\lambda_i)$ . Then the exponents of  $\mathcal{N}_w(q)$  are precisely the values  $m_i$ ; i.e. the inversion polynomial for  $w$  is given by

$$\mathcal{N}_w(q) = \prod_{i \geq 1} [m_i + 1]_q.$$

**Example 5.4.** Let  $w = 3421 = s_2 s_1 s_2 s_3 s_2 \in S_4$ . Then  $w$  corresponds to a nonsingular Schubert variety. The positive roots sent negative by  $w^{-1}$ , along with their heights relative to  $N(w)$ , are given below.

$\beta \in N(w) :$	$e_2 - e_3$	$e_2 - e_4$	$e_3 - e_4$	$e_1 - e_4$	$e_1 - e_3$
$\text{ht}_w(\beta) :$	1	2	1	2	1

where, for example,  $\text{ht}_w(e_1 - e_4) = 2$  because  $e_1 - e_4 = (e_1 - e_3) + (e_3 - e_4)$ . From this we can see that  $(\lambda_1, \lambda_2) = (3, 2)$ , which has conjugate partition  $(\widetilde{\lambda}_i) = (2, 2, 1)$ . We therefore have

$$\mathcal{N}_w(q) = [2 + 1]_q^2 [1 + 1]_q = 1 + 3q + 5q^2 + 5q^3 + 3q^4 + q^5.$$

For this element, we have  $\mathcal{N}_w(q) = \mathcal{P}_w(q)$ .

**Example 5.5.** Let  $w = 53412 = s_2s_1s_3s_2s_4s_3s_2s_1 \in S_5$ , which corresponds to a singular Schubert variety (in fact, the element 53412 contains both patterns 3412 and 4231). The positive roots sent negative by  $w^{-1}$ , along with their heights relative to  $N(w)$ , are given below.

$\beta \in N(w) :$	$e_2 - e_3$	$e_1 - e_3$	$e_2 - e_4$	$e_1 - e_4$	$e_2 - e_5$	$e_1 - e_5$	$e_4 - e_5$	$e_3 - e_5$
$ht_w(\beta) :$	1	1	1	1	2	2	1	1

From this we can see that  $(\lambda_1, \lambda_2) = (6, 2)$ , which has conjugate partition  $(\widetilde{\lambda}_i) = (2, 2, 1, 1, 1, 1)$ . We therefore have

$$\mathcal{N}_w(q) = [2 + 1]_q^2 [1 + 1]_q^4 = 1 + 6q + 17q^2 + 30q^3 + 36q^4 + 30q^5 + 17q^6 + 6q^7 + q^8.$$

For this element, we have  $\mathcal{N}_w(q) \neq \mathcal{P}_w(q)$ , but  $\mathcal{N}_w(q) = \mathcal{I}_w(q)$ .

**Example 5.6.** Let  $w = 564123 = s_3s_2s_1s_4s_3s_2s_1s_5s_4s_3s_2 \in S_6$ , which also corresponds to a singular Schubert variety. The roots sent negative by  $w^{-1}$ , along with their heights relative to  $N(w)$ , are given below.

$\beta \in N(w) :$	$e_3 - e_4$	$e_2 - e_4$	$e_1 - e_4$	$e_3 - e_5$	$e_2 - e_5$	$e_1 - e_5$
$ht_w(\beta) :$	1	1	1	2	2	2

$\beta \in N(w) :$	$e_4 - e_5$	$e_3 - e_6$	$e_2 - e_6$	$e_1 - e_6$	$e_4 - e_6$
$ht_w(\beta) :$	1	2	2	2	1

From this we can see that  $(\lambda_1, \lambda_2) = (5, 6)$ , which is not a partition of  $\ell(w)$ , and so  $\mathcal{N}_w(q)$  is not defined.

### 5.3 Properties of the Inversion Polynomial

In this section, we will study  $\mathcal{N}_w(q)$  independent of  $L_w$ . Since it is possible in theory for  $\mathcal{N}_w(q)$  to equal  $\mathcal{I}_w(q)$  (or for  $\mathcal{N}_w(q)$  to have other interesting properties) in cases where there is no reduced expression for which  $L_{\underline{w}} = C_w$ , it is useful to be able to study properties of  $\mathcal{N}_w(q)$  as a polynomial in its own right.

### 5.3.1 The Nonsingular Case in Type A

When  $w \in S_n$  corresponds to a nonsingular Schubert variety, we know from Chapter 3 that  $C_w = L_{\underline{w}}$  for some reduced expression  $\underline{w}$  of  $w$ , and thus by Proposition 5.3, we have  $\mathcal{N}_w(q) = \mathcal{I}_w(q)$  (and hence  $\mathcal{N}_w(q) = \mathcal{P}_w(q)$ ). In this section, we will analyze  $\mathcal{N}_w(q)$  independently of the Hecke algebra to attempt to arrive at the same result.

Note that from Theorem 2.12, we can see that for nonsingular elements  $w \in S_n$ , both the ordinary Poincaré polynomial  $\mathcal{P}_w(q)$  and the inversion polynomial  $\mathcal{N}_w(q)$  factor into  $q$ -numbers. We will now show the much stronger statement that in fact we have  $\mathcal{N}_w(q) = \mathcal{P}_w(q)$  in this case.

**Proposition 5.7.** *Suppose  $w \in S_n$  corresponds to a nonsingular Schubert variety. Then  $\mathcal{N}_w(q) = \mathcal{P}_w(q)$  (and hence  $\mathcal{N}_w(q) = \mathcal{I}_w(q)$ ).*

*Proof.* Since  $w$  avoids 3412 and 4231, we know that  $w$  belongs to one of the two cases described in Theorem/Definition 2.10. Suppose  $w$  belongs to the first case, i.e. suppose  $n = w(d) > w(d+1) > \dots > w(n)$ . Then  $m := n-d$ , and  $w' := \text{fl}(w \setminus n)$  can be considered in one-line notation as  $w$  with the entry  $n$  replaced by a blank space. This allows us to consider  $N((w')^{-1})$  as a subset of  $N(w^{-1})$ . For example, if  $w = 31542$ , then we can consider  $w' = 31\_42$ , so  $N((w')^{-1}) = \{e_1 - e_2, e_1 - e_5, e_4 - e_5\}$  is a subset of  $N(w^{-1}) = \{e_1 - e_2, e_1 - e_5, e_3 - e_4, e_3 - e_5, e_4 - e_5\}$ .

Let  $e_i - e_j \in N((w')^{-1})$  with  $\text{ht}_{w'}(e_i - e_j) = h'$ . Let  $h = \text{ht}_w(e_i - e_j)$ . Since every decreasing sequence of entries between index  $i$  and index  $j$  in the one-line expression for  $w'$  also occurs in the one-line expression for  $w$ , it is clear that  $h' \leq h$ . Since neither  $i$  nor  $j$  can be equal to  $d$ , and since  $w(d) = n$ , any decreasing sequence of entries between index  $i$  and index  $j$  in  $w$  cannot involve index  $d$ , and therefore must also occur in  $w'$ . Thus  $h' = h$ .

We will use this fact to describe the modified heights of all roots in  $N(w^{-1})$ .

Note that

$$N(w^{-1}) = N((w')^{-1}) \sqcup \{e_{n-1} - e_n, e_{n-2} - e_n, \dots, e_d - e_n\}$$

where any  $e_j - e_n$  has modified height  $\text{ht}_w(e_j - e_n) = n - j$  and each root  $e_i - e_j \in N((w')^{-1})$  has  $\text{ht}_{w'}(e_i - e_j) = \text{ht}_w(e_i - e_j)$ . This allows us to compute the inversion polynomial of  $w$ :

$$\mathcal{N}_w(q) = [m + 1]_q \mathcal{N}_{w'}(q).$$

By Theorem 2.12, we know that  $\mathcal{P}_w(q)$  satisfies the same recurrence relation, so since  $\mathcal{N}_{id}(q) = 1 = \mathcal{P}_{id}(q)$ , we have  $\mathcal{N}_w(q) = \mathcal{P}_w(q)$  in this case.

Now suppose  $w$  belongs to Case 2 of Theorem/Definition 2.10. Then  $n = w^{-1}(c) > w^{-1}(c + 1) > \dots > w^{-1}(n)$ ,  $m := n - c$ , and  $w' := \text{fl}(w \setminus c)$ . Then we can consider  $w'$  to be an element of  $S_n$  by defining  $w'(n) = n$ . For example, if  $w = 4132 \in S_4$ , then we will consider  $w' = 1324 \in S_4$  instead of  $132 \in S_3$ . As before, this allows us to directly see  $N((w')^{-1})$  as a subset of  $N(w^{-1})$ . If  $e_i - e_j \in N((w')^{-1})$  with  $\text{ht}_{w'}(e_i - e_j) = h'$  and  $\text{ht}_w(e_i - e_j) = h$ , then  $j \neq n$ , and so it is again clear that  $h = h'$ . We have

$$N(w^{-1}) = N((w')^{-1}) \sqcup \{e_{w^{-1}(c+1)} - e_n, e_{w^{-1}(c+2)} - e_n, \dots, e_{w^{-1}(c+m)} - e_n\}$$

where each  $\text{ht}_w(e_{w^{-1}(c+j)} - e_n) = j$ , and for any  $e_i - e_j \in N((w')^{-1})$ , we have  $\text{ht}_w(e_i - e_j) = \text{ht}_{w'}(e_i - e_j)$ . Hence we again have  $\mathcal{N}_w(q) = [m + 1]_q \mathcal{N}_{w'}(q)$ , and so inductively we again have  $\mathcal{N}_w(q) = \mathcal{P}_w(q)$ .  $\square$

### 5.3.2 The Singular Case in Type A

Since  $\mathcal{N}_w(q)$  and  $\mathcal{I}_w(q)$  are both symmetric polynomials, it is obvious that neither is equal to the ordinary Poincaré polynomial  $\mathcal{P}_w(q)$  when  $w$  corresponds

to a singular Schubert variety. In this subsection, we will isolate some conditions under which  $w$  corresponds to a singular Schubert variety and  $\mathcal{N}_w(q) = \mathcal{I}_w(q)$ .

**Lemma 5.8.** *Suppose  $w \in S_n$  avoids the pattern 45312 but not 3412. Then there exist indices  $1 \leq i < j < k \leq n$  such that  $\text{fl}_{\{i,j,j+1,k\}}(w) = 3412$ . In other words, there exists a 3412 pattern in  $w$  in which the middle two entries of the pattern are adjacent.*

*Proof.* Let  $w(i)w(j)w(k)w(l)$  denote a 3412 pattern in  $w$  for which  $k - j$  is minimal. Suppose for a contradiction that  $k \neq j + 1$ . If  $w(j + 1) < w(k)$ , then  $\text{fl}_{\{i,j,j+1,l\}} = 3412$ , which contradicts the minimality of  $k - j$ . Similarly if  $w(j + 1) > w(i)$ , then  $\text{fl}_{\{i,j+1,k,l\}} = 3412$ , again contradicting the minimality of  $k - j$ . Finally, if  $w(k) < w(j + 1) < w(i)$ , then  $\text{fl}_{\{i,j,j+1,k,l\}} = 45312$ , contradicting the hypothesis.  $\square$

Let  $w \in S_n$  be an element which avoids the patterns 4231, 45312, 45213 and 35412, but not 3412. This means that for any 3412 pattern  $w(i)w(j)w(k)w(l)$  in  $w$ , the root  $e_j - e_k$  is not a linear combination of the other roots in  $N(w^{-1})$ . By Lemma 5.8, we can find indices  $i, j, k$  such that  $\text{fl}_{\{i,j,j+1,k\}}(w) = 3412$ . Let  $s = s_j$  (so  $\text{fl}_{\{i,j,j+1,k\}}(ws) = 3142$ ). Throughout the rest of this subsection, we will consider  $w$  and  $s$  to be fixed.

**Lemma 5.9.** *The element  $ws$  will also avoid the patterns 4231, 45312, 45213, and 35412.*

*Proof.* Observe that since  $ws$  is obtained from  $w$  by switching two adjacent entries, if  $ws$  contains a pattern which  $w$  does not, then this pattern must utilize both the entries  $w(j)$  and  $w(j + 1)$ .

First suppose that  $ws$  contains the pattern 4231. Then there must exist indices  $p < j$  and  $q > j + 1$  such that  $\text{fl}(w(p)w(j + 1)w(j)w(q)) = 4231$ . However, if  $p < i$ , then  $\text{fl}(w(p)w(i)w(j)w(j + 1)) = 4231$ , and if  $p > i$ , then  $\text{fl}(w(i)w(p)w(j)w(j + 1)w(k)) = 35412$ , both of which contradict the assumptions on  $w$ .



Suppose  $ws$  contains the pattern 45312. Then either there exist indices  $p < q < r$  all greater than  $j + 1$  such that  $\text{fl}(w(j + 1)w(j)w(p)w(q)w(r)) = 45312$ , or else there exist indices  $p < q < r$  all less than  $j$  such that  $\text{fl}(w(p)w(q)w(r)w(j + 1)w(j)) = 45312$ . In the former case, we have  $\text{fl}(w(i)w(j)w(p)w(q)w(r)) = 45312$ , and in the latter case, we have  $\text{fl}(w(p)w(q)w(r)w(j + 1)w(k)) = 45312$ . In both cases, we contradict the assumption that  $w$  avoids the pattern 45312.

Suppose  $ws$  contains the pattern 45213. Then either there exist indices  $p < q < r$  all greater than  $j + 1$  such that  $\text{fl}(w(j + 1)w(j)w(p)w(q)w(r)) = 45213$ , or there exist indices  $p < q < r$  all less than  $j$  such that  $\text{fl}(w(p)w(q)w(r)w(j + 1)w(j)) = 45213$ . In the first case, we have  $\text{fl}(w(i)w(j)w(p)w(q)w(r)) = 45213$ , and in the second we have  $\text{fl}(w(p)w(q)w(r)w(j + 1)w(k)) = 45312$ . Either way, this contradicts the assumptions on  $w$ .

Finally, suppose  $ws$  contains the pattern 35412. Then either there exist indices  $p < q < r$  all greater than  $j + 1$  such that  $\text{fl}(w(j + 1)w(j)w(p)w(q)w(r)) = 35412$ , or there exist indices  $p < q < r$  all less than  $j$  such that  $\text{fl}(w(p)w(q)w(r)w(j + 1)w(j)) = 35412$ . In the first case, we have  $\text{fl}(w(i)w(j)w(j + 1)w(p)w(q)) = 45312$ , and in the second case, we have  $\text{fl}(w(p)w(q)w(r)w(j + 1)w(k)) = 45312$ . Either way, we contradict the assumptions.  $\square$

**Proposition 5.10.** *We have  $\mathcal{N}_w(q) = (q + 1)\mathcal{N}_{ws}(q)$ .*

*Proof.* It is clear that  $N(w^{-1}) = N((ws)^{-1}) \sqcup \{e_j - e_{j+1}\}$ , and  $\text{ht}_w(e_j - e_{j+1}) = 1$ . We will now show that for any root  $e_p - e_q \in N((ws)^{-1})$ , we have  $\text{ht}_{ws}(e_p - e_q) = \text{ht}_w(e_p - e_q)$ .

Let  $e_p - e_q \in N((ws)^{-1})$  with  $h' = \text{ht}_{ws}(e_p - e_q)$  and  $h = \text{ht}_w(e_p - e_q)$ . Clearly  $h' \leq h$  since every subword of the one-line expression for  $w'$  is also a subword of the one-line expression for  $w$ . Suppose for a contradiction that  $h' \leq h$ . This

implies that the longest decreasing subword between index  $p$  and index  $q$  in  $w$  necessarily involves both the entries  $w(j)$  and  $w(j + 1)$ . In particular, we have  $p \leq j < j + 1 \leq q$  with  $w(p) \geq w(j) > w(j + 1) \geq w(q)$ , and either  $p \leq j$  or  $j + 1 \leq q$  or both. In what follows, let

$$w(p) w(a_1) \cdots w(a_s) w(j) w(j + 1) w(b_1) \cdots w(b_t) w(q)$$

denote the longest decreasing subword between index  $p$  and index  $q$  in the one-line expression for  $w$ .

Suppose  $p \leq j$ . If  $a_s < i$ , then  $w(a_s)$  occurs to the left of  $w(i)$  in the one-line expression of  $w$ . Then  $w(a_s) > w(j)$  implies  $w(a_s) > w(i)$ , so since we know that  $w(i) > w(j + 1)$  as well, we can simply replace  $w(j)$  with  $w(i)$  in the decreasing subword above to obtain a decreasing subword of the same length between index  $p$  and index  $q$  which does not use both  $w(j)$  and  $w(j + 1)$ . However, this contradicts the reasoning above. We must therefore have  $a_s > i$ . But in this case, we have  $i < a_s < j$  and  $\text{fl}_{\{i, a_s, j, j+1, k\}}(w) = 35412$ , which contradicts the hypothesis.

Suppose instead that  $q \geq j + 1$ . We will proceed in a similar fashion. If  $b_1 > k$ , then  $w(j + 1)$  can be replaced with  $w(b_1)$  in the decreasing subword above to obtain a decreasing subword of the same length between index  $p$  and index  $q$  which does not use both  $w(j)$  and  $w(j + 1)$ . Otherwise, we have  $j + 1 < b_1 < k$  and  $\text{fl}_{\{i, j, j+1, b_1, k\}}(w) = 45213$ , a contradiction.

We have now shown that for any  $e_p - e_q \in N((ws)^{-1})$ , we have  $\text{ht}_{ws}(e_p - e_q) = \text{ht}_w(e_p - e_q)$ . It follows that  $\mathcal{N}_w(q) = (q + 1)\mathcal{N}_{ws}(q)$ .  $\square$

We will now shift our attention to computing the polynomial  $\mathcal{I}_w(q)$ .

**Lemma 5.11.** *Let  $z < ws$  with  $\ell(z, ws) = 1$ . Then  $zs \not\prec z$ .*

*Proof.* Since  $z < ws$  with  $\ell(z) = \ell(ws) - 1$ , we must have  $z = (ws)t$  for some transposition  $t$ . In particular, the permutation  $z$  is obtained from  $ws$  by switching two entries.

Suppose for a contradiction that  $zs < z$ . Then  $z(j) > z(j+1)$ . Since  $ws(j) < ws(j+1)$ , either  $z$  is obtained from  $ws$  by switching  $ws(j) = w(j+1)$  with an entry  $A > w(j)$ , or else  $z$  is obtained from  $ws$  by switching  $ws(j+1) = w(j)$  with an entry  $B < w(j+1)$ .

In the former case, since  $\ell(z) < \ell(ws)$ , the entry  $A$  must occur to the left of  $w(j+1)$  in the one-line expression of  $ws$ . However, if  $A$  occurs to the left of  $w(i)$ , then  $\text{fl}(Aw(i)w(j)w(j+1)) = 4231$ , and if  $A$  occurs to the right of  $w(i)$ , then  $\text{fl}(w(i)Aw(j)w(j+1)w(k)) = 35412$ . Both contradict the assumptions on  $w$ .

Similarly, if  $z$  is obtained from  $ws$  by switching  $w(j)$  with an entry  $B < w(j+1)$ , then since  $\ell(z) < \ell(ws)$ , the entry  $B$  must occur to the right of  $w(j)$  in  $ws$ . If  $B$  occurs to the right of  $w(k)$  in the one-line expression of  $ws$ , then  $\text{fl}(w(i)w(j+1)w(k)B) = 4231$ , and if  $B$  occurs to the left of  $w(k)$ , then  $\text{fl}(w(i)w(j)w(j+1)Bw(k)) = 45213$ . Again, both options contradict the assumptions.  $\square$

Throughout the rest of this section, we will often refer to elements which satisfy the following condition.

**Definition 5.12.** We will say that a pair of elements  $u \leq v$  in  $W$  satisfies the Submaximal Degree Condition if  $\deg(P_{u,v}(q)) \leq \frac{1}{2}(\ell(u, v) - 1)$ .

**Lemma 5.13.** Assume that for all  $z < ws$  with  $\ell(z, ws) > 1$ , the pair  $(z, ws)$  satisfies the Submaximal Degree Condition. Then for any  $x \leq w$ , we have  $P_{x,w}(q) = q^c P_{x,ws}(q) + q^{1-c} P_{xs,ws}(q)$ ; i.e. the sum on the right side of Equation (2.1) is 0.

*Proof.* Suppose  $x < ws$  and that there is some  $z$  for which  $x \leq z < ws$  and  $zs < z$ . If the term of the sum in Equation (2.1) corresponding to  $z$  is nonzero,

then  $\ell(z, ws)$  must be odd. By Lemma 5.11, we know that if  $\ell(z, ws) = 1$ , then  $zs \not\leq z$ , and thus  $z$  cannot contribute a term to the sum. And if  $\ell(z, ws) > 1$ , then since  $(z, ws)$  satisfies the Submaximal Degree Condition, we have  $\mu(z, ws) = 0$ , and thus  $z$  does not contribute a term to the sum.  $\square$

**Conjecture 5.14.** *For all elements  $x < ws$  with  $xs < ws$ , we have  $P_{x,ws}(q) = P_{xs,ws}(q)$ .*

**Proposition 5.15.** *Assume that for all  $z < ws$  with  $\ell(z, ws) > 1$ , the pair  $(z, ws)$  satisfies the Submaximal Degree Condition. Assume also that Conjecture 5.14 holds. Then  $\mathcal{I}_w(q) = (q + 1)\mathcal{I}_{ws}(q)$ .*

*Proof.* Note that for all  $x \leq w$ , we have  $P_{x,w}(q) = P_{xs,w}(q)$ . This would be true just from the fact that  $ws < w$  ([7]), but it is easy enough to verify at this point:

$$P_{x,w}(q) = qP_{x,ws}(q) + P_{xs,ws}(q) = P_{xs,w}(q).$$

Using this fact, it is straightforward to compute  $P_{x,w}(q)$  for each  $x \leq w$ . For example, consider the elements  $x < w$  such that  $x, xs$  both correspond to singular points in  $X_{ws}$ . Then  $P_{x,w}(q) = qP_{x,ws}(q) + P_{x,ws}(q) = (q + 1)P_{x,ws}(q)$  by the conjecture. Continuing on in this way and then computing the sum  $\mathcal{I}_w(q) = \sum_{x \leq w} P_{x,w}(q)q^{\ell(x)}$ , one can verify that

$$\sum_{x \leq w} P_{x,w}(q)q^{\ell(x)} = (q + 1) \sum_{x \leq ws} P_{x,ws}(q)q^{\ell(x)}$$

or in other words,  $\mathcal{I}_w(q) = (q + 1)\mathcal{I}_{ws}(q)$ .  $\square$

**Proposition 5.16.** *Assume that for all  $z < ws$  with  $\ell(z, ws) > 1$ , the pair  $(z, ws)$  satisfies the Submaximal Degree Condition. Assume also that Conjecture 5.14 holds. Then  $\mathcal{N}_w(q) = \mathcal{I}_w(q)$ .*

*Proof.* By results 5.10 - 5.15 above, we have

$$\mathcal{N}_w(q) = (q + 1)\mathcal{N}_{ws}(q) \text{ and } \mathcal{I}_w(q) = (q + 1)\mathcal{I}_{ws}(q),$$

where  $ws$  avoids the required patterns 4231, 45312, 45213, and 35412. This process can then be iterated, allowing us to obtain the formulas

$$\mathcal{N}_w(q) = (q+1)^r \mathcal{N}_{ws_1 s_2 \dots s_r}(q) \text{ and } \mathcal{I}_w(q) = (q+1)^r \mathcal{I}_{ws_1 s_2 \dots s_r}(q)$$

where the element  $ws_1 \dots s_r$  corresponds to a nonsingular Schubert variety. By 5.7, we know that  $\mathcal{N}_{ws_1 s_2 \dots s_r}(q) = \mathcal{P}_{ws_1 s_2 \dots s_r}(q) = \mathcal{I}_{ws_1 s_2 \dots s_r}(q)$ . Hence  $\mathcal{N}_w(q) = \mathcal{I}_w(q)$ . □

## CHAPTER 6

### THE CLOSURE POLYNOMIAL

In 2008, Oh, Postnikov, and Yoo described a root theoretic method for determining the ordinary Poincaré polynomial  $\mathcal{P}_w(q)$  when  $w \in S_n$  corresponds to a nonsingular Schubert variety [25]. They defined a polynomial  $\mathcal{R}_w(q)$ , called the distance enumerating polynomial of  $w$ , which like the intersection cohomology polynomial  $\mathcal{I}_w(q)$  is always palindromic and coincides with the ordinary Poincaré polynomial  $\mathcal{P}_w(q)$  when the Schubert variety  $X_w$  is nonsingular. In our quest to develop an efficient combinatorial method for determining the intersection cohomology Poincaré polynomial in as many cases as possible, we define in this chapter another new polynomial, called the closure polynomial  $\mathcal{M}_w(q)$ . The closure polynomial is a generalization of the distance enumerating polynomial in the sense that it coincides with  $\mathcal{R}_w(q)$  (and hence  $\mathcal{P}_w(q)$  and  $\mathcal{I}_w(q)$ ) when  $w \in S_n$  corresponds to a nonsingular Schubert variety, and it coincides with  $\mathcal{I}_w(q)$  in many singular cases as well. One advantage of  $\mathcal{M}_w(q)$  is that it can be computed for any element  $w$  of any Weyl group.

## 6.1 Hyperplane Arrangements

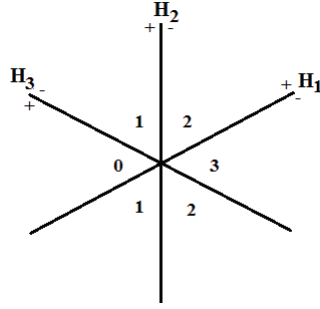
For any  $w \in S_n$ , the inversion set  $N(w^{-1})$  gives rise to the inversion hyperplane arrangement of  $w$ , denoted  $\mathcal{A}_w$ , which consists of all hyperplanes  $x_i - x_j = 0$  in  $\mathbb{R}^n$  for which  $e_i - e_j \in N(w^{-1})$ .

**Definition 6.1.** *The distance enumerating polynomial  $\mathcal{R}_w(q) = \sum_r q^{d(r_0, r)}$  is the generating function that counts regions  $r$  of the arrangement  $\mathcal{A}_w$  according to the distance  $d(r_0, r)$  from a fixed initial region  $r_0$ .*

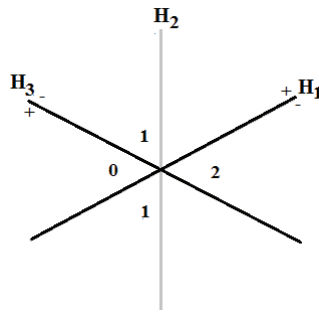
The polynomial  $\mathcal{R}_w(q)$  is always palindromic, and for a certain fixed initial region, it has been shown that when  $X_w$  is nonsingular, we have  $\mathcal{R}_w = \mathcal{P}_w$  [25].

When  $n$  is small ( $n \leq 4$ ), we can easily visualize hyperplane arrangements. For  $W = S_3$ , for instance, there are three possible hyperplanes which might appear in an arrangement. Let  $H_1$  denote the plane defined by  $x_1 - x_2 = 0$ , let  $H_2$  denote the plane defined by  $x_2 - x_3 = 0$ , and let  $H_3$  denote the plane defined by  $x_1 - x_3 = 0$ . The intersection of any two of these planes is the line given by  $x_1 = x_2 = x_3$ . Since the three planes intersect in a common line, we can visualize them as three lines intersecting in a common point. This allows us to draw them in  $\mathbb{R}^2$ . All three hyperplanes will appear in the hyperplane arrangement for the longest element  $w_0 = 321$ , and the hyperplanes  $H_1$  and  $H_3$  will appear in the hyperplane arrangement for  $w = 312$ . See Figures 2 and 3.

For each hyperplane in  $\mathcal{A}(w)$ , choose one side to be positive and one to be negative. Let the fundamental region of the hyperplane be the region which is positive with respect to each hyperplane. Then the distance enumerating polynomial can be computed by simply letting the coefficient of  $q^m$  be the number of regions which are distance  $m$  away from the fundamental region. As we can see from Figures 2 and 3, we have  $\mathcal{R}_{[321]}(q) = 1 + 2q + 2q^2 + q^3$ , and  $\mathcal{R}_{[312]} = 1 + 2q + q^2$ .



**Figure 2.**  $\mathcal{A}_{[321]}: \mathcal{R}_{[321]}(q) = 1 + 2q + 2q^2 + q^3$



**Figure 3.**  $\mathcal{A}_{[312]}: \mathcal{R}_{[321]}(q) = 1 + 2q + q^2$

For  $W = S_4$ , there are six possible hyperplanes, which we will denote as follows.

$$H_1 = \{(x_1, x_2, x_3, x_4) : x_1 - x_2 = 0\}$$

$$H_2 = \{(x_1, x_2, x_3, x_4) : x_1 - x_3 = 0\}$$

$$H_3 = \{(x_1, x_2, x_3, x_4) : x_1 - x_4 = 0\}$$

$$H_4 = \{(x_1, x_2, x_3, x_4) : x_2 - x_3 = 0\}$$

$$H_5 = \{(x_1, x_2, x_3, x_4) : x_2 - x_4 = 0\}$$

$$H_6 = \{(x_1, x_2, x_3, x_4) : x_3 - x_4 = 0\}$$



These hyperplanes have seven distinct lines of intersection:

$$L_1 = \{(s, s, s, -3s) : s \in \mathbb{R}\} = H_1 \cap H_2 = H_1 \cap H_4 = H_2 \cap H_4$$

$$L_2 = \{(s, -3s, s, s) : s \in \mathbb{R}\} = H_2 \cap H_3 = H_2 \cap H_6 = H_3 \cap H_6$$

$$L_3 = \{(s, s, -3s, s) : s \in \mathbb{R}\} = H_1 \cap H_3 = H_1 \cap H_5 = H_3 \cap H_5$$

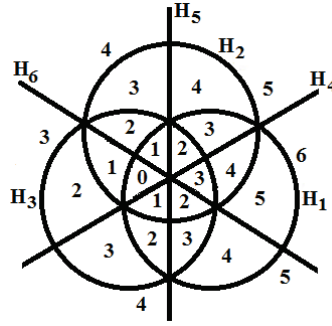
$$L_4 = \{(-3s, s, s, s) : s \in \mathbb{R}\} = H_4 \cap H_5 = H_4 \cap H_6 = H_5 \cap H_6$$

$$L_5 = \{(s, s, -s, -s) : s \in \mathbb{R}\} = H_1 \cap H_6$$

$$L_6 = \{(s, s, -s, -s) : s \in \mathbb{R}\} = H_3 \cap H_4$$

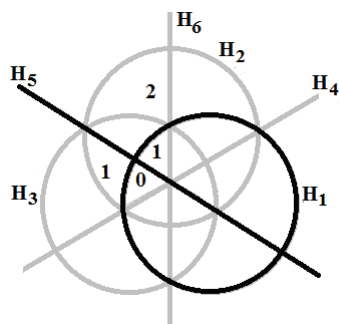
$$L_7 = \{(s, s, -s, -s) : s \in \mathbb{R}\} = H_2 \cap H_5$$

Let  $N$  be one point on  $L_4$ . Let  $\mathbb{S}$  denote the three dimensional sphere through  $N$  and  $-N$  with center at the origin. Each line  $L_i$  intersects  $\mathbb{S}$  at two points. Projecting stereographically from  $N$ , we obtain the arrangement shown in Figure 4.

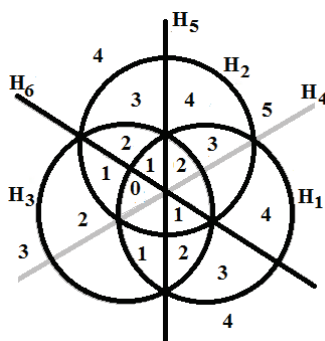


**Figure 4.**  $\mathcal{A}_{[4321]}: \mathcal{R}_{[4321]}(q) = 1 + 3q + 5q^2 + 6q^3 + 5q^4 + 3q^5 + q^6$

**Example 6.2.** For the elements  $4321, 2143, 4231 \in S_4$ , the associated hyperplane arrangements and distance enumerating polynomials are given in Figure 4, Figure 5, and Figure 6, respectively. In these diagrams, we take the positive sides of the hyperplanes  $H_1, H_2$ , and  $H_3$  to be the inner sides, and we take  $H_4, H_5$ , and  $H_6$  to be oriented as  $H_1, H_2$ , and  $H_3$  were, respectively, in Figures 2 and 3.



**Figure 5.**  $\mathcal{A}_{[2143]}: \mathcal{R}_{[2143]}(q) = 1 + 2q + q^2$



**Figure 6.**  $\mathcal{A}_{[4231]}: \mathcal{R}_{[4231]}(q) = 1 + 4q + 4q^2 + 4q^3 + 4q^4 + q^5$

## 6.2 Definition of the Closure Polynomial

A central goal of this thesis has been to develop a polynomial combinatorially and efficiently which coincides with the intersection cohomology Poincaré polynomial in as many non-rationally smooth instances as possible. One candidate for this, which we will now discuss, is called the closure polynomial and was heavily inspired by the distance enumerating polynomial  $\mathcal{R}_w(q)$  described above.

**Definition 6.3.** *We say that a set  $S \subset N(w)$  is  $N(w)$ -closed if whenever  $\alpha, \beta \in S$  and  $\alpha + \beta \in N(w)$ , we have  $\alpha + \beta \in S$ . Define the collection of all  $\mathcal{M}$ -allowable sets to be*

$$\mathcal{M}(w) := \{S \subset N(w) : \text{both } S \text{ and } N(w) \setminus S \text{ are } N(w)\text{-closed}\}.$$

The closure polynomial for  $w$  is then defined to be

$$\mathcal{M}_w(q) = \sum_{S \subset \mathcal{M}(w)} q^{|S|}.$$

One can easily confirm that (like  $\mathcal{I}_w(q)$ ) the polynomial  $\mathcal{M}_w(q)$  is always palindromic, and always satisfies the relation  $\mathcal{M}_w(q) = \mathcal{M}_{w^{-1}}(q)$ .

Our definition of  $\mathcal{M}$ -sets was motivated by Tymoczko's work in [32], where an analog of the  $\mathcal{M}$ -sets is used to compute the ordinary Poincaré polynomial of regular nilpotent Hessenberg varieties. In [29], Sommers and Tymoczko used modified heights to factor the Poincaré polynomials of regular nilpotent Hessenberg varieties into products of  $q$ -numbers. We will see similar factorizations of  $\mathcal{M}_w(q)$  in Propositions 6.6 and 6.8.

**Example 6.4.** Let  $w = 4312$ , which corresponds to a nonsingular Schubert variety. Then  $w^{-1} = 3421$ . We have

$$N(w) = \{e_1 - e_3, e_1 - e_4, e_2 - e_3, e_2 - e_4, e_3 - e_4\}.$$

The  $\mathcal{M}$ -allowable sets for  $w$  are given below, arranged according to their cardinalities.

size 0 :  $\emptyset$

size 1 :  $\{e_1 - e_3\}, \{e_2 - e_3\}, \{e_3 - e_4\}$

size 2 :  $\{e_1 - e_3, e_2 - e_3\}, \{e_1 - e_3, e_1 - e_4\}, \{e_1 - e_4, e_3 - e_4\},$   
 $\{e_2 - e_3, e_2 - e_4\}, \{e_2 - e_4, e_3 - e_4\}$

size 3 : the complements in  $N(w)$  of all  $\mathcal{M}$ -allowable sets of size 2

size 4 : the complements in  $N(w)$  of all  $\mathcal{M}$ -allowable sets of size 1

size 5 :  $N(w) =$  the complements in  $N(w)$  of all  $\mathcal{M}$ -allowable sets of size 0

Hence, we have  $\mathcal{M}_w(q) = 1 + 3q + 5q^2 + 5q^3 + 3q^4 + q^5$ . In this example, we have  $\mathcal{M}_w(q) = \mathcal{R}_w(q) = \mathcal{P}_w(q)$ .

**Example 6.5.** Let  $w = 3412$ , which corresponds to a singular Schubert variety. Then  $w^{-1} = 3412$  also, and we have  $N(w) = \{e_1 - e_3, e_1 - e_4, e_2 - e_3, e_2 - e_4\}$ . There are no nontrivial linear relations satisfied by the roots in  $N(w)$ , and so every subset of  $N(w)$  is an  $\mathcal{M}$ -allowable set. Hence, we have  $\mathcal{M}_w(q) = 1 + 4q + 6q^2 + 4q^4 + q^5$ . In this example, we have  $\mathcal{M}_w(q) \neq \mathcal{R}_w(q)$ , and  $\mathcal{M}_w(q) \neq \mathcal{P}_w(q)$ , but  $\mathcal{M}_w(q) = \mathcal{I}_w(q)$ .

Examples 6.4 and 6.5 above illustrate a fact we shall prove in Proposition 6.6: if  $w \in S_n$  corresponds to a nonsingular Schubert variety, then  $\mathcal{M}_w(q)$  coincides with  $\mathcal{R}_w(q) = \mathcal{P}_w(q) = \mathcal{I}_w(q)$ . We will also soon see that in many cases where  $w \in S_n$  corresponds to a singular Schubert variety, we still have  $\mathcal{M}_w(q) = \mathcal{I}_w(q)$ . Thus, the polynomial  $\mathcal{M}_w(q)$  is an extension of  $\mathcal{R}_w(q)$  to a polynomial which coincides with  $\mathcal{I}_w(q)$  in certain non-rationally smooth cases.

### 6.3 Results on Nonsingular Schubert Varieties of Type A

**Proposition 6.6.** Suppose  $w \in S_n$  corresponds to a nonsingular Schubert variety, and let  $m$  and  $w'$  be defined as in Theorem/Definition 2.10. Then the polynomial  $\mathcal{M}_w(q)$  satisfies the recursion relation

$$\mathcal{M}_w(q) = [m + 1]_q \mathcal{M}_{w'}(q).$$

Since  $\mathcal{M}_{id}(q) = 1 = \mathcal{P}_{id}(q)$ , it follows that  $\mathcal{M}_w(q) = \mathcal{P}_w(q)$  for all rationally smooth elements  $w \in S_n$ .

*Proof.* Since  $w$  corresponds to a nonsingular variety, it must belong to one of the two cases described in Theorem/Definition 2.10. Assume  $w$  belongs to Case 1. Then  $n = w(d) > w(d+1) > \cdots > w(n)$ ,  $m = n - d$ , and  $w'$  is the element of  $S_{n-1}$  obtained by deleting the entry  $n$  from the one-line expression of  $w$ .

We will abuse notation and consider  $N((w')^{-1}) \subset N(w^{-1})$  in the following way. If  $e_i - e_j \in N((w')^{-1})$  for some  $1 \leq i < j < n$ , then  $\widetilde{e_i - e_j} \in N(w^{-1})$  where

$$\widetilde{e_i - e_j} = \begin{cases} e_i - e_j & \text{if } j < d \\ e_i - e_{j+1} & \text{if } i < d \leq j \\ e_{i+1} - e_{j+1} & \text{if } d \leq i < j \end{cases} .$$

In other words, since the entries of  $w'$  are in the same relative order in  $w$ , every inversion of  $w'$  corresponds to an inversion in  $w$ . Observe that since every linear relation satisfied by roots in  $N((w')^{-1})$  is also satisfied by the corresponding roots in  $N(w^{-1})$ , we in fact have  $\mathcal{M}(w') \subset \mathcal{M}(w)$ .

Suppose there is some relation  $(e_i - e_j) + (e_j - e_k) = e_i - e_k$  satisfied by three roots in  $N(w^{-1})$ . Then at most one of  $i, j, k$  can be equal to  $d$ , which means that either one or three of these roots belongs to  $N((w')^{-1})$ . In fact, note that  $N(w^{-1}) \setminus N((w')^{-1}) = \{e_d - e_k : d < k \leq n\}$ . One immediate consequence of this is the fact that  $\#N(w^{-1}) - \#N((w')^{-1}) = n - d = m$ .

We will now illustrate why for any  $0 \leq k \leq m$ , there is exactly one set in  $\mathcal{M}(w)$  consisting only of elements from  $N(w^{-1}) \setminus N((w')^{-1})$  of size  $k$ . Indeed, it is clear that

$$\emptyset, \{e_d - e_{d+1}\}, \{e_d - e_{d+1}, e_d - e_{d+1}\}, \dots, \{e_d - e_{d+1}, \dots, e_d - e_n\} \in \mathcal{M}(w).$$

Suppose for a contradiction that there is some set  $S \in \mathcal{M}(w)$  consisting only of elements in  $N(w^{-1}) \setminus N((w')^{-1})$  which does not appear in this list. Then we can find  $j < k$  such that  $e_d - e_{d+k} \in S$  and  $e_d - e_{d+j} \notin S$ . However, this contradicts the assumption that  $S$  is  $\mathcal{M}$ -allowable, since these roots satisfy the relation  $(e_d - e_{d+j}) + (e_{d+j} - e_{d+k}) = e_d - e_{d+k}$ . Let  $\mathcal{M}(w/w')$  denote the collection of these  $\mathcal{M}$ -allowable sets.

Now, let  $\mathcal{M}' := \{A \cup B : A \in \mathcal{M}((w')^{-1}), B \in \mathcal{M}(w/w')\}$ . We will now endeavor to show that for any integer  $k$ , the number of sets in  $\mathcal{M}'$  of size  $k$  is equal to the number of sets in  $\mathcal{M}((w^{-1})$  of size  $k$ . We will do this by constructing a bijection of sets  $F : \mathcal{M}(w^{-1}) \rightarrow \mathcal{M}'$ .

Let  $S \in \mathcal{M}(w^{-1})$ . Let  $S_1 = S \cap (N(w^{-1}) \setminus N((w')^{-1}))$  and let  $S_2 = S \setminus S_1$ . Note that  $S_2 \in \mathcal{M}((w')^{-1})$ . Say  $\#S_1 = k$ . Let  $S'_1 = \{e_d - e_{d+1}, \dots, e_d - e_{d+k}\}$ . Using this notation, define  $F : \mathcal{M}(w^{-1}) \rightarrow \mathcal{M}'$  by  $F(S) = S'$  where  $S' := S'_1 \cup S_2$ . To see that  $F$  is injective, suppose  $F(S) = F(T)$  where  $S = S_1 \cup S_2$ ,  $T = T_1 \cup T_2$ ,  $S_1, T_1 \subset N(w^{-1}) \setminus N((w')^{-1})$ , and  $S_2, T_2 \in \mathcal{M}((w')^{-1})$ . Then we must have  $S_2 = T_2$  and  $\#S_1 = \#T_1$ . Assume for a contradiction that  $S_1 \neq T_1$ . Then we can find some root  $e_d - e_{d+j} \in S_1$  such that  $e_d - e_{d+j} \notin T_1$ , and we can find some root  $e_d - e_{d+k} \in T_1$  such that  $e_d - e_{d+k} \notin S_1$ . Assume without loss of generality that  $j < k$ . Then  $e_{d+j} - e_{d+k} \in N(w^{-1})$ , so since  $T \in \mathcal{M}(w^{-1})$ , we must have  $e_{d+j} - e_{d+k} \in T$ , so  $e_{d+j} - e_{d+k} \in T_2 = S_2$ . But then  $e_d - e_{d+j}, e_{d+j} - e_{d+k} \in S$  and  $e_d - e_{d+k} \notin S$ , which contradicts the assumption that  $S \in \mathcal{M}(w^{-1})$ . Hence  $F$  is injective.

To prove that  $F$  is surjective, we will construct for it a right inverse  $G : \mathcal{M}' \rightarrow \mathcal{M}(w^{-1})$ . Let  $A \cup B \in \mathcal{M}'$ , so that  $A \in \mathcal{M}((w')^{-1})$  and  $B \in \mathcal{M}(w/w')$ . Say  $\#B = k$ , i.e. suppose  $B = \{e_d - e_{d+1}, \dots, e_d - e_{d+k}\}$ . We will construct  $G(A \cup B)$  algorithmically. Let  $S_0 = A$  and let  $L_0 \subset S_0$  be the set of all roots  $e_h - e_k \in S_0$  such that  $h > d$ .

- Let  $i_1 \in \{1, 2, \dots, m\}$  be minimal such that there is no element in  $L_0$  of the form  $e_{d+i_1} - e_j$  for any  $j$ . Then let  $S_1 = S_0 \cup \{e_d - e_{d+i_1}\}$  and let  $L_1$  be the set obtained from  $L_0$  by deleting any elements of the form  $e_h - e_{d+i_1}$ . Note that we have ensured  $S_1 \in \mathcal{M}(w)$ .

- Let  $i_2 \in \{1, 2, \dots, m\}$  be minimal such that there is no element in  $L_1$  of the form  $e_{d+i_2} - e_j$  for any  $j$ . Then let  $S_2 = S_1 \cup \{e_d - e_{d+i_2}\}$  and let  $L_2$  be the set obtained from  $L_1$  by deleting any elements of the form  $e_h - e_{d+i_2}$ .

⋮

Continuing on in this fashion, we obtain sets  $S_0, \dots, S_m$ , which can be shown inductively to belong to  $\mathcal{M}(w)$ . Note that each  $S_i$  is of the form  $A_i \cup B_i$  where  $A_i \subset N(w^{-1}) \setminus N((w')^{-1})$  is a set of size  $i$ . We will define  $G(A \cup B) = S_i$ . One easily verifies that  $F \circ G$  is the identity on  $\mathcal{M}'$ . Hence  $F$  is a bijection.

For each integer  $k$ , let  $b_k$  denote the number of sets in  $\mathcal{M}(w/w')$  of size  $k$  and let  $a_k$  denote the number of sets in  $\mathcal{M}(w')$  of size  $k$ . In other words, we have  $b_k = 1$  for each  $0 \leq k \leq m$  and  $b_k = 0$  otherwise, and the closure polynomial of  $w'$  is given by  $\mathcal{M}_{w'}(q) = a_0 + a_1q + \dots + a_\ell(w')q^\ell(w')$ . Define  $a_i = 0$  for any  $i < 0$  or  $i > \ell(w')$ . Then the number of sets in  $\mathcal{M}(w)$  of size  $h$  for any integer  $h$  is precisely the number of sets in  $\mathcal{M}'$  of size  $h$ :

$$a_h \cdot b_0 + a_{h-1} \cdot b_1 + \dots + a_{h-m} \cdot b_m = a_h + a_{h-1} + \dots + a_{h-m}.$$

In other words, we have

$$\begin{aligned} \mathcal{M}_w(q) &= \mathcal{M}_{w'}(q) + q\mathcal{M}_{w'}(q) + \dots + q^m \mathcal{M}_{w'}(q) \\ &= [m+1]_q \mathcal{M}_{w'}(q). \end{aligned}$$

We have now proved the desired claim for those elements  $w$  which belong to Case 1 in Theorem/Definition 2.10. This entire proof needs only a slight modification to be applicable to those elements belonging to Case 2. If  $w$  belongs to Case 2, then  $n = w^{-1}(c) > w^{-1}(c+1) > \dots > w^{-1}(n)$ ,  $m = n - c$ , and  $w'$  is obtained from the one-line expression of  $w$  by deleting the  $n$ -th entry and flattening. In

this case, it is even more straightforward to see that  $N((w')^{-1}) \subset N(w^{-1})$ : if  $e_i - e_j \in N((w')^{-1})$ , then  $e_i - e_j \in N(w^{-1})$ . To complete the proof, simply replace every instance of an element of the form  $e_d - e_{d+j}$  with the element  $e_{w^{-1}(c+j)} - e_n$ .

□

## 6.4 Results on Singular Schubert Varieties of Type A

In this section, we will show that we in fact have  $\mathcal{M}_w(q) = \mathcal{I}_w(q)$  in at least all of the cases for which we have shown that  $\mathcal{N}_w(q) = \mathcal{I}_w(q)$  (see Chapter 5). Let  $w \in S_n$  and assume  $w$  avoids the patterns 4231, 45312, 45213, and 35412, but not the pattern 3412.

**Lemma 6.7.** *Then for any 3412 pattern  $w(i)w(j)w(k)w(l)$  in  $w$ , the root  $e_j - e_k$  is not a linear combination of the other vectors in  $N(w^{-1})$ .*

*Proof.* Let  $w(i)w(j)w(k)w(l)$  be a 3412 pattern in the one-line expression for  $w$ , and suppose for a contradiction that  $e_j - e_k$  can be expressed as a linear combination of the other vectors in  $N(w^{-1})$ . Then one of the following three cases must hold: there exists an entry  $w(m) > w(j)$  which occurs to the left of  $w(j)$ , there exists an entry  $w(m)$  between  $w(j)$  and  $w(k)$  with  $w(j) > w(m) > w(k)$ , or there exists an entry  $w(m) < w(k)$  to the right of  $w(k)$ . In the first case, if  $w(m)$  occurs to the left of  $w(i)$ , then  $w$  contains the 4231 pattern  $w(m)w(i)w(j)w(k)$ , and if  $w(m)$  occurs between  $w(i)$  and  $w(j)$ , then  $w$  contains the 35412 pattern  $w(i)w(m)w(j)w(k)w(l)$ . In the second case, the subword  $w(i)w(j)w(m)w(k)w(l)$  must be an instance of one of the patterns 45213, 45312, or 35412. And in the third case, the one-line expression of  $w$  will contain either the 45213 pattern  $w(i)w(j)w(k)w(m)w(l)$  or the 4231 pattern  $w(j)w(k)w(l)w(m)$ . □



Under these conditions on  $w$ , we know by Lemma 5.8 that we can find indices  $i, j, k$  such that  $\text{fl}_{\{i,j,j+1,k\}}(w) = 3412$ . Let  $s = s_j$  (so  $\text{fl}_{\{i,j,j+1,k\}}(ws) = 3142$ ). Throughout the rest of this section, we will consider  $w$  and  $s$  to be fixed.

**Proposition 6.8.** *With  $w$  and  $s$  as above, we have  $\mathcal{M}_w(q) = (q + 1)\mathcal{M}_{ws}(q)$ .*

*Proof.* Note that for any root  $e_p - e_q \in N((ws)^{-1})$ , we have  $p < q$  with  $ws(p) > ws(q)$ . Then we necessarily have  $ws(p)$  appearing to the left of  $ws(q)$  in the one-line expression of  $w$ , so  $w^{-1}(ws(p)) < w^{-1}(ws(q))$ . In other words, we have  $s(p) < s(q)$  and  $w(s(p)) > w(s(q))$ , so  $e_{s(p)} - e_{s(q)} \in N(w^{-1})$ .

Let  $s(N((ws)^{-1}))$  denote the set  $\{e_{s(p)} - e_{s(q)} : e_p - e_q \in N((ws)^{-1})\}$ . Note that if  $p < q < r$  are indices such that  $e_p - e_q, e_q - e_r \in N((ws)^{-1})$ , then  $e_{s(p)} - e_{s(q)}, e_{s(q)} - e_{s(r)} \in s(N((ws)^{-1}))$  with  $(e_{s(p)} - e_{s(q)}) + (e_{s(q)} - e_{s(r)}) = e_{s(p)} - e_{s(r)}$ . Thus for any integer  $C$ , the  $\mathcal{M}$ -allowable sets in  $N((ws)^{-1})$  of size  $C$  are in bijection with the  $\mathcal{M}$ -allowable sets in  $s(N((ws)^{-1}))$  of size  $C$ .

Since  $e_j - e_{j+1}$  is the only inversion of  $w$  which does not correspond to an inversion of  $ws$  in the above way, we have  $N(w^{-1}) = s(N((ws)^{-1})) \cup \{e_j - e_{j+1}\}$ . It follows that an  $\mathcal{M}$ -allowable set  $S \subset N(w^{-1})$  of size  $C$  is either an  $\mathcal{M}$ -allowable set in  $s(N((ws)^{-1}))$  of size  $C$ , or it is the union of  $\{e_j - e_{j+1}\}$  with an  $\mathcal{M}$ -allowable set in  $s(N((ws)^{-1}))$  of size  $C - 1$ .

Hence if  $\mathcal{M}_{ws}(q) = b_0 + b_1q + \cdots + b_{\ell(w)-1}q^{\ell(w)-1}$ , then  $\mathcal{M}_w(q) = a_0 + a_1q + \cdots + a_{\ell(w)}q^{\ell(w)}$  where  $a_i = b_{i-1} + b_i$  for each  $0 \leq i \leq \ell(w)$  (and where  $b_{-1} := 0$  and  $b_{\ell(w)} := 0$ ). In other words, we have  $\mathcal{M}_w(q) = (q + 1)\mathcal{M}_{ws}(q)$ .  $\square$

**Observation 6.9.** *Under these conditions on  $w$  and  $s$ , Proposition 6.8 implies that  $\mathcal{M}_w(q) = \mathcal{N}_w(q)$ , since  $\mathcal{M}_w(q)$  and  $\mathcal{N}_w(q)$  satisfy the same recursive factorization and since  $\mathcal{M}_{id}(q) = 1 = \mathcal{N}_{id}(q)$ . It follows that  $\mathcal{M}_w(q) = \mathcal{I}_w(q)$  in at least all of the cases where we have shown that  $\mathcal{N}_w(q) = \mathcal{I}_w(q)$  in Chapter 5.*

## 6.5 Future Work

The closure polynomial  $\mathcal{M}_w(q)$  was developed before the discovery of the inversion polynomial  $\mathcal{N}_w(q)$ , and was abandoned at that point because it seemed from preliminary data that  $\mathcal{N}_w(q)$  coincides with  $\mathcal{I}_w(q)$  in more cases than does  $\mathcal{M}_w(q)$ . For example, for  $w$  equal to the singular element  $4231 \in S_4$ , we have the following polynomials.

$$\begin{aligned}\mathcal{P}_w(q) &= (1+q)^2(1+q+2q^2+q^3) \\ \mathcal{I}_w(q) &= (1+q)^3(1+q+q^2) \\ \mathcal{M}_w(q) &= (1+q)(1+3q+q^2+3q^3+q^4) \\ \mathcal{N}_w(q) &= (1+q)^3(1+q+q^2) = \mathcal{I}_w(q)\end{aligned}$$

Another advantage of  $\mathcal{N}_w(q)$  over  $\mathcal{M}_w(q)$  is that it is generally more straightforward to work with (compare the proof of Proposition 6.8 to the proof of Proposition 5.10). However, both of these polynomials were developed for the purpose of computing  $\mathcal{I}_w(q)$  outside of type  $A$ . Thus, work on either polynomial is unfinished, and both should be studied for other Weyl groups. It may turn out that one or the other provides a better estimation of  $\mathcal{I}_w(q)$  in general, or that one or both provides other useful information which is interesting independent of intersection cohomology.

# CHAPTER 7

## HEIGHT SEQUENCES

Let  $W$  be a Weyl group and let  $w \in W$ . Consider a factorization of a reduced word for  $w$  of the form  $\underline{w} = (\underline{w}')x$  where  $\underline{w}'$  is a reduced expression for an element belonging to a maximal parabolic subgroup of  $W$  and  $x$  is a minimal coset representative of that subgroup in  $W$ . Since this expression of  $w$  is reduced, the associated ordered inversion set of  $w$  has the form

$$N(\underline{w}) = \{N(\underline{w}'), \beta_1, \beta_2, \dots, \beta_{\ell(x)}\}.$$

Suppose  $\text{hts}(\underline{w}) = (\text{hts}(\underline{w}'), b_1, b_2, \dots, b_{\ell(x)})$  is the associated height sequence for  $\underline{w}$ .

**Definition 7.1.** *With notation as above, we call the subsequence  $(b_1, b_2, \dots, b_{\ell(x)})$  of  $\text{hts}(\underline{w})$  the ending height sequence of  $w$  with respect to this factorization.*

In this chapter, we will describe a method for factoring any  $w \in W$  uniquely as a product of an element of a fixed maximal parabolic subgroup and a coset representative of that subgroup as described above. We will then seek to classify the ending height sequences of elements  $w \in S_n$  with respect to this factorization.

The results of this chapter were discovered throughout the course of our investigations into the Kazhdan-Lusztig basis elements  $C_w$  of the Hecke algebra

and the intersection cohomology Poincaré polynomials  $\mathcal{I}_w(q)$ . This diversion became an interesting side project, with applications to combinatorics independent from the study of Schubert varieties.

## 7.1 Minimal Coset Representatives and Reduced Expressions

Let  $W$  be a Weyl group generated by the reflections  $\langle s_1, s_2, \dots, s_n \rangle$ .

**Definition 7.2.** *For any subgroup  $W'$  of  $W$ , a minimal coset representative of  $W'$  in  $W$  is a coset representative of minimal length.*

In this section, we will discuss a method of uniquely factoring an element of  $W$  relative to a fixed maximal parabolic subgroup of  $W$ . In particular, we will outline a method for factoring any  $w \in W$  into the form  $w'x$  where  $w'$  is an element of a fixed maximal parabolic subgroup  $W'$  of  $W$ , and  $x$  is a minimal right coset representative of  $W'$  in  $W$ .

**Lemma 7.3.** *Let  $W = W(A_n)$ , let  $W'$  denote the parabolic subgroup  $\langle s_1, s_2, \dots, s_{n-1} \rangle$  of  $W$ , and let  $e \in W$  denote the identity element. Then  $\{e, s_n, s_n s_{n-1}, \dots, s_n s_{n-1} \cdots s_1\}$  is a complete set of right coset representatives of  $W'$  in  $W$ .*

*Proof.* We will explain why

$$W = \{w'x : w' \in W' \text{ and } x \in \{e, s_n, s_n s_{n-1}, \dots, s_n s_{n-1} \cdots s_1\}\}.$$

The set on the right is clearly contained in  $W$ . And the cardinality of the set on the right is equal to

$$(\#W') \cdot (\#\{e, s_n, s_n s_{n-1}, \dots, s_n s_{n-1} \cdots s_1\}) = (n!)(n+1) = (n+1)! = \#W.$$

□

Thus, for any  $w \in W(A_n)$ , we can factor  $w$  into a reduced word of the form  $w = w' s_n s_{n-1} \cdot s_k$  where  $w' \in W(A_{n-1})$ . Inductively factoring an element  $w \in W$  in this way produces the lexicographic reduced expression for  $w$ .

**Example 7.4.** Let  $w = 265314 \in W(A_5)$ . Then the inductive factorization described above produces the following reduced expression for  $w$ .

$$\begin{aligned}
w &= [25314] \cdot s_5 s_4 s_3 s_2 \\
&= [2314] \cdot s_4 s_3 s_2 \cdot s_5 s_4 s_3 s_2 \\
&= [231] \cdot e \cdot s_4 s_3 s_2 \cdot s_5 s_4 s_3 s_2 \\
&= [21] \cdot s_2 \cdot e \cdot s_4 s_3 s_2 \cdot s_5 s_4 s_3 s_2 \\
&= s_1 \cdot s_2 \cdot e \cdot s_4 s_3 s_2 \cdot s_5 s_4 s_3 s_2.
\end{aligned}$$

Suppose now that we instead take  $W'$  to be the subgroup  $W' = \langle s_2, s_3, \dots, s_n \rangle$  in  $W$ . Then we similarly have

$$W = \{w'x : w' \in W' \text{ and } x \in \{e, s_1, s_1 s_2, \dots, s_1 s_2 \cdots s_n\}\}.$$

Inductively factoring an element  $w \in W$  in this way produces what we will refer to as the *reverse lexicographic* reduced expression for  $w$ .

**Example 7.5.** Using  $w = 265314$  as in the last example, the reverse lexicographic reduced expression for  $w$  is obtained from the following inductive process.

$$\begin{aligned}
w &= [126534] \cdot s_1 s_2 s_3 s_4 \\
&= [126534] \cdot e \cdot s_1 s_2 s_3 s_4 \\
&= [123654] \cdot s_3 s_4 \cdot e \cdot s_1 s_2 s_3 s_4 \\
&= [123465] \cdot s_4 s_5 \cdot s_3 s_4 \cdot e \cdot s_1 s_2 s_3 s_4 \\
&= s_5 \cdot s_4 s_5 \cdot s_3 s_4 \cdot e \cdot s_1 s_2 s_3 s_4.
\end{aligned}$$

Henceforth, we will always choose to factor  $w \in W$  into its lexicographic reduced expression, and if  $W(A_n)$  is generated by the simple reflections  $s_1, \dots, s_n$ , we will consider the maximal parabolic subgroup  $W(A_{n-1})$  to be generated by the subset  $s_1, s_2, \dots, s_{n-1}$ .

We now wish to determine the order of the elements of  $N(\underline{w})$  if  $w \in W(A_n)$  is expressed lexicographically. First, note that for any  $w' \in W(A_{n-1})$  and for any  $j < n + 1$ , we have

$$w'(e_j - e_{n+1}) = e_{w'(j)} - e_{w'(n+1)} = e_{w'(j)} - e_{n+1}.$$

**Proposition 7.6.** *Let  $w \in W(A_n)$  with fixed reduced expression  $\underline{w} = \underline{w}'\underline{x} \in W(A_n)$ , where  $w'$  is an element of the parabolic subgroup  $W(A_{n-1})$  and  $\underline{x}$  is a coset representative of  $W(A_{n-1})$  in  $W(A_n)$  of the form  $\underline{x} = s_n s_{n-1} \cdots s_k$  for some  $1 \leq k \leq n$ . Then  $N(\underline{w})$  is ordered in the following manner.*

$$N(\underline{w}) = \{N(\underline{w}'), e_{w'(n)} - e_{n+1}, e_{w'(n-1)} - e_{n+1}, \dots, e_{w'(1)} - e_{n+1}\}.$$

*Proof.* Recall that for any  $1 \leq j \leq n$ , we have  $s_j(\alpha_i) = \alpha_i + \alpha_j$  if  $i = j \pm 1$ ,  $s_j(\alpha_i) = -\alpha_i$  if  $i = j$ , and  $s_j(\alpha_i) = \alpha_i$  otherwise. We can now explicitly describe the ordered set  $N(\underline{w})$ :

$$\begin{aligned} N(\underline{w}) &= \{N(\underline{w}'), w'(\alpha_n), w's_n(\alpha_{n-1}), \dots, w's_n \cdots s_2(\alpha_1)\} \\ &= \{N(\underline{w}'), w'(\alpha_n), w'(\alpha_{n-1} + \alpha_n), \dots, w'(\alpha_1 + \cdots + \alpha_n)\} \\ &= \{N(\underline{w}'), w'(e_n - e_{n+1}), w'(e_{n-1} - e_{n+1}), \dots, w'(e_1 - e_{n+1})\} \\ &= \{N(\underline{w}'), e_{w'(n)} - e_{n+1}, e_{w'(n-1)} - e_{n+1}, \dots, e_{w'(1)} - e_{n+1}\}. \end{aligned}$$

□

## 7.2 Classifying Ending Height Sequences in Type A

Let  $W = W(A_n)$ . Throughout this section, we will only consider the lexicographic reduced expression of any element  $w \in W$ . We will focus on two particular types of ending height sequences: Bell sequences and Catalan sequences.

**Definition 7.7.** A sequence of positive integers  $(a_m, a_{m-1}, \dots, a_1)$  is called a Bell sequence if  $a_1 = 1$  and if for each  $1 < i \leq m$ , we have  $a_i \leq 1 + \max\{a_j : 1 \leq j < i\}$ . A Bell sequence is called a Catalan sequence if it satisfies the stricter condition that  $a_i \leq 1 + a_{i-1}$  for each  $1 \leq i < m$ .

These sequences are so named because the number of Catalan sequences on the numbers  $\{1, 2, \dots, n\}$  is  $C_n$ , the  $n$ -th Catalan number. And similarly the number of Bell sequences on the numbers  $\{1, 2, \dots, n\}$  is the  $n$ -th Bell number [15].

**Example 7.8.** The lexicographic reduced expression of  $w = 265314 \in W(A_5)$  is  $w = (s_1s_2s_4s_3s_2)s_5s_4s_3s_2$ , where  $w' = s_1s_2s_4s_3s_2 \in W(A_4)$  and  $x = s_5s_4s_3s_2$  is the minimal coset representative of  $W(A_4)$  in  $W(A_5)$ . The associated inversion set is ordered in the following way:

$v \in N(\underline{w}) :$	$e_1 - e_2$	$e_1 - e_3$	$e_4 - e_5$	$e_1 - e_5$	$e_3 - e_5$	$e_4 - e_6$	$e_1 - e_6$	$e_3 - e_6$	$e_5 - e_6$
$ht_w(v) :$	1	1	1	2	1	2	3	2	1

Thus, the ending height sequence of  $w$  is the sequence  $(2, 3, 2, 1)$ . This ending height sequence is Catalan.

Now consider the element  $w = 53241 \in W(A_4)$ , which has lexicographic reduced expression  $(s_1s_2s_1s_3)s_4s_3s_2s_1$ . Associated to this reduced expression, we have

$N(\underline{w}) :$	$e_1 - e_2$	$e_1 - e_3$	$e_2 - e_3$	$e_1 - e_4$	$e_1 - e_5$	$e_4 - e_5$	$e_2 - e_5$	$e_3 - e_5$
$ht :$	1	2	1	1	3	1	2	1

The ending height sequence for 53241 is therefore  $(3, 1, 2, 1)$ , which is Bell but not Catalan.

### 7.2.1 Bell Ending Height Sequences

In this section, we will show that every ending height sequence for  $w \in W$  (expressed lexicographically) is a Bell sequence, and every Bell sequence of length  $n$  is obtained as the ending height sequence for some  $w \in W$  (expressed lexicographically).

**Proposition 7.9.** *The ending height sequence of any  $w \in W(A_n)$ , when expressed lexicographically, is a Bell sequence.*

*Proof.* Let  $w \in W(A_n)$  and express  $w$  lexicographically. So  $w$  is of the form  $\underline{w} = (\underline{w}')s_n s_{n-1} \cdots s_k$  where  $w' \in W(A_{n-1})$ . Then we know that, as an ordered set, we have

$$\begin{aligned} N(\underline{w}) &= \{N(\underline{w}'), w'(\alpha_n), w's_n(\alpha_{n-1}), \dots, w's_n s_{n-1} \cdots s_{k+1}(\alpha_k)\} \\ &= \{N(\underline{w}'), w'(\alpha_n), w'(\alpha_{n-1} + \alpha_n), \dots, w'(\alpha_k + \alpha_{k+1} + \cdots + \alpha_n)\}. \end{aligned}$$

For notational purposes, define  $\gamma_j := w'(\alpha_j + \alpha_{j+1} + \cdots + \alpha_n)$  for each  $k \leq j \leq n$ .

For each  $j \in \{k+1, k+2, \dots, n\}$ , define  $h_j = \max\{\text{ht}_w(\gamma_i) : k \leq i \leq j-1\}$  and define  $h_k = 0$ . We will show that  $\text{ht}_w(\gamma_j) \leq 1 + h_j$  for all  $j \in \{k, k+1, \dots, n\}$ , from which it follows immediately that the ending height sequence of  $w$  is a Bell sequence. Fix an integer  $j \in \{k, k+1, \dots, n\}$ .

We will first show that it is impossible to express  $\gamma_j$  as a sum of distinct roots in  $N(w') \cup \{\gamma_{j+1}, \gamma_{j+2}, \dots, \gamma_n\}$ . Suppose for a contradiction that we can find roots  $\beta_1, \dots, \beta_m \in N(w') \cup \{\gamma_{j+1}, \gamma_{j+2}, \dots, \gamma_n\}$  such that  $\gamma_j = \beta_1 + \cdots + \beta_m$ . Let  $u =$



$w' s_n s_{n-1} \cdots s_{j+1}$ . Then  $\beta_1, \dots, \beta_m \in N(u)$ , so we have

$$u^{-1}(\gamma_j) = u^{-1}(\beta_1 + \cdots + \beta_m) = u^{-1}(\beta_1) + \cdots + u^{-1}(\beta_m) \prec 0.$$

On the other hand, since  $\gamma_j = u(\alpha_j)$ , we have  $u^{-1}(\gamma_j) = \alpha_j \succ 0$ , a clear contradiction.

Thus, any expression of  $\gamma_j$  as a sum of distinct elements of  $N(w)$  must involve at least one of the roots  $\gamma_k, \gamma_{k+1}, \dots, \gamma_{j-1}$ . Let  $\gamma_m \in \{\gamma_k, \dots, \gamma_{j-1}\}$  be of maximal modified height with respect to  $N(w)$  such that  $\gamma_j = \alpha + \gamma_m$  for some  $\alpha \in N(w)$ . We claim that  $\text{ht}_w(\gamma_j) = \text{ht}_w(\alpha) + \text{ht}_w(\gamma_m) = 1 + \text{ht}_w(\gamma_m)$ , from which it follows immediately that  $\text{ht}_w(\gamma_j) \leq 1 + h_j$ . Since  $\gamma_j = \alpha + \gamma_m$ , we have

$$\text{ht}_w(\gamma_j) \geq \text{ht}_w(\alpha) + \text{ht}_w(\gamma_m) \geq 1 + \text{ht}_w(\gamma_m).$$

Suppose for a contradiction that  $\text{ht}_w(\gamma_j) \geq 1 + \text{ht}_w(\gamma_m)$ . Let  $\gamma_j = \beta_1 + \cdots + \beta_{\text{ht}_w(\gamma_j)}$  be a longest expression of  $\gamma_j$  as a sum of distinct roots in  $N(w)$ . Then each  $\beta_i$  has  $\text{ht}_w(\beta_i) = 1$  (otherwise a longer expression for  $\gamma_j$  could be found). Then  $\beta_2 + \cdots + \beta_{\text{ht}_w(\gamma_j)} \in N(w)$  with  $\text{ht}_w(\beta_2 + \cdots + \beta_{\text{ht}_w(\gamma_j)}) = \text{ht}_w(\gamma_j) - 1 \geq \text{ht}_w(\gamma_m)$ , which contradicts the maximality of  $\gamma_m$ . Hence, we must have  $\text{ht}_w(\gamma_j) = 1 + \text{ht}_w(\gamma_m) \leq 1 + h_j$ .  $\square$

We now prove the other main result of this subsection.

**Proposition 7.10.** *Every Bell sequence of length  $n$  is the ending height sequence of some  $w \in W(A_n)$  expressed in its lexicographic reduced notation.*

*Proof.* Let  $\mathbf{a} = (a_n, a_{n-1}, \dots, a_1)$  be a Bell sequence of length  $n$ . We will show by induction that there exists some  $w \in W(A_n)$  of the form  $(w') s_n s_{n-1} \cdots s_1$ , with  $w' \in W(A_{n-1})$ , such that the ending height sequence of  $w$  with respect to this factorization is  $\mathbf{a}$ . This statement is certainly true when  $n = 2$ . Indeed,

the element  $s_2 s_1 \in W(A_2)$  has ending height sequence  $(1, 1)$  and the element  $(s_1) s_2 s_1 \in W(A_2)$  has ending height sequence  $(2, 1)$ .

Assume that for any Bell sequence of length  $n - 1$ , there exists some  $w'' \in W(A_{n-2})$  such that the ending height sequence of  $(w'') s_{n-1} s_{n-2} \cdots s_1$  is precisely that Bell sequence. Then we can find an element  $v \in W(A_{n-1})$  and an element  $v' \in W(A_{n-2})$  such that  $v = (v') s_{n-1} \cdots s_1$  and  $v$  has ending height sequence  $(a_{n-1}, a_{n-2}, \dots, a_1)$ .

Define elements  $w_1, w_2, \dots, w_n \in W(A_n)$  as follows.

$$\begin{aligned} w_1 &= s_1 s_2 \cdots s_{n-1} (v') s_n s_{n-1} \cdots s_1 \\ &\vdots \\ w_{n-1} &= s_{n-1} (v') s_n s_{n-1} \cdots s_1 \\ w_n &= (v') s_n s_{n-1} \cdots s_1 \end{aligned}$$

We will now explore properties of these elements  $w_1, \dots, w_n$  and show that at least one of them has the ending height sequence  $\mathbf{a}$ .

Firstly, we clearly have  $v^{-1}(n) = 1$ , and so the one-line expression for  $v$  has the form  $v^{-1} = Y_1 Y_2 \cdots Y_{n-1} 1$ . Computing the elements  $w_1^{-1}, \dots, w_n^{-1}$  in terms of the entries of  $v^{-1}$ , we have.

$$\begin{aligned} w_1^{-1} &= (n+1) Y_1 Y_2 \cdots Y_{n-2} Y_{n-1} 1 \\ w_2^{-1} &= Y_1 (n+1) Y_2 \cdots Y_{n-2} Y_{n-1} 1 \\ &\vdots \\ w_{n-1}^{-1} &= Y_1 Y_2 Y_3 \cdots (n+1) Y_{n-1} 1 \\ w_n^{-1} &= Y_1 Y_2 Y_3 \cdots Y_{n-1} (n+1) 1 \end{aligned}$$

Observe also that, as ordered sets, the final  $n$  elements of  $N(\underline{w}_i)$  are precisely

$$s_i \cdots s_{n-1} v'(\alpha_n), s_i \cdots s_{n-1} v'(\alpha_{n-1} + \alpha_n), \dots, s_i \cdots s_{n-1} v'(\alpha_1 + \alpha_n)$$

where the first element is

$$s_i \cdots s_{n-1} v'(\alpha_n) = s_i \cdots s_{n-1}(\alpha_n) = \alpha_i + \alpha_{i+1} + \cdots + \alpha_n.$$

For any  $1 \leq j \leq n-1$  and any  $1 \leq i \leq n$ , let  $y_i^j = s_i \cdots s_{n-1} v'(\alpha_j + \cdots + \alpha_n)$ . Then from Proposition 3.6, it is clear that

$$\begin{aligned} \text{ht}_{w_i}(y_i^j) &\equiv \text{ht}_{w_i}(e_{s_i \cdots s_{n-1} v'(j)} - e_{n+1}) \\ &= 1 + \text{the length of the longest decreasing subword of } w_i^{-1} \\ &\quad \text{between index } s_i \cdots s_{n-1} v'(j) \text{ and index } n+1 \\ &= 1 + \text{the length of the longest decreasing subword of } v^{-1} \\ &\quad \text{between index } v'(j) \text{ and index } n \\ &= \text{ht}_v(e_{v'(j)} - e_{n+1}) \\ &\equiv a_j. \end{aligned}$$

It follows that each  $w_i$  has ending height sequence of the form  $(b_i, a_{n-1}, \dots, a_2, a_1)$  for some  $b_i \geq 1$ . We will now compute each  $b_i$ .

For each  $i$ , we have  $b_i = \text{ht}_{w_i}(e_{s_i \cdots s_{n-1} v'(1)} - e_{n+1})$ , i.e.  $b_i$  is the longest decreasing subword of the one-line expression of  $w_i^{-1}$  between the entry  $n+1$  and the entry 1. From observing the one-line expressions of each  $w_i^{-1}$  above, we can now easily conclude that  $b_n = 1$ , and for each  $1 \leq i < n$  we have  $b_{i+1} \leq b_i \leq 1 + b_{i+1}$ . Finally, note that  $b_1$  is equal to 1+ the length of the longest subword of  $w_1^{-1}$  between the entry  $n+1$ , which occurs in index 1, and the entry 1, which occurs in index  $n+1$ . This means that  $b_1$  is equal to 1 more than the length of the longest decreasing subword of  $v^{-1}$ , i.e.

$$b_1 = 1 + \max\{a_{n-1}, a_{n-2}, \dots, a_1\}.$$

To summarize, we have shown that for any integer value  $b$  between 1 and  $1 + \max\{a_{n-1}, \dots, a_1\}$ , the sequence  $(b, a_{n-1}, a_{n-2}, \dots, a_1)$  is obtained as the end-

ing height sequence of  $w_i$  for some  $1 \leq i \leq n$ . In particular, the sequence  $(a_n, a_{n-1}, \dots, a_1)$  is obtained as the ending height sequence of one of the elements  $w_i$ .

□

From these results, we can conclude:

**Corollary 7.11.** *A sequence of numbers  $\mathbf{a}$  is an ending height sequence of an element  $w \in W$  (with respect to its lexicographic reduced expression) if and only if  $\mathbf{a}$  is a Bell sequence.*

**Example 7.12.** *Consider the Bell sequence  $\mathbf{a} = (3, 3, 2, 1)$  of length 4. We will follow the construction of the proof of Proposition 7.10 to find an element  $w \in W(A_4)$  with this ending height sequence.*

*First we must find an element  $v \in W(A_3)$  (possibly through an iterative process) which has the ending height sequence  $(3, 2, 1)$ . One choice is the longest element  $v = (s_1 s_2 s_1) s_3 s_2 s_1$ . With notation as in the proof, we have  $v' = s_1 s_2 s_1$ , and we define the elements  $w_1, w_2, w_3, w_4 \in W(A_4)$  as follows.*

$$\begin{aligned} w_1 &= s_1 s_2 s_3 (v') s_4 s_3 s_2 s_1 = 54321 \Rightarrow w_1^{-1} = 54321 \\ w_2 &= s_2 s_3 (v') s_4 s_3 s_2 s_1 = 54312 \Rightarrow w_2^{-1} = 45321 \\ w_3 &= s_3 (v') s_4 s_3 s_2 s_1 = 54213 \Rightarrow w_3^{-1} = 43521 \\ w_4 &= (v') s_4 s_3 s_2 s_1 = 53214 \Rightarrow w_4^{-1} = 43251 \end{aligned}$$

*Each of these expressions is reduced, and the associated ending height sequences are*

given by:

$$w_1 : (4, 3, 2, 1)$$

$$w_2 : (3, 3, 2, 1)$$

$$w_3 : (2, 3, 2, 1)$$

$$w_4 : (1, 3, 2, 1).$$

Thus  $w_2$  is an element in  $W(A_4)$  with ending height sequence  $\mathbf{a}$ .

## 7.2.2 Catalan Ending Height Sequences

In this subsection, we will connect Catalan ending height sequences to the notion of pattern avoidance. In particular, we will show that if  $w \in W(A_n)$  avoids the pattern 53241, then the ending height sequence of  $w$  with respect to its lexicographic reduced expression is a Catalan sequence, and conversely that every Catalan sequence arises as the ending height sequence of an element in  $W(A_n)$  avoiding the pattern 53241.

**Proposition 7.13.** *Suppose  $w \in W(A_n)$  (with fixed lexicographic reduced expression) has a non-Catalan ending height sequence. Then  $w$  contains the pattern 53241.*

*Proof.* The element  $w$  expressed lexicographically has the form  $\underline{w} = \underline{w'}x$  where  $w' \in W(A_{n-1})$  and  $x$  is a coset representative of  $W(A_{n-1})$  in  $W$  of the form  $\underline{x} = s_n s_{n-1} \cdots s_k$ . Note that if  $x = e$ , the ending height sequence of  $w$  has length 0 and is trivially Catalan. We will assume therefore that  $1 \leq k \leq n$ .

We know that we can write

$$N(\underline{w}) = \{N(\underline{w'}), e_{w'(n)} - e_{n+1}, e_{w'(n-1)} - e_{n+1}, \dots, e_{w'(k)} - e_{n+1}\}.$$

For each  $1 \leq i \leq n$ , let  $h_i$  denote the modified height of the inversion  $e_{w'(i)} - e_{n+1}$  in  $N(w)$ , so that the ending height sequence of  $w$  is given by  $(h_n, h_{n-1}, \dots, h_k)$ . Since this sequence is non-Catalan by assumption, there is some  $j \in \{k, k + 1, \dots, n\}$  for which  $h_j \geq 2 + h_{j-1}$ . By Proposition 3.6 we know that the one-line notation expression of  $w^{-1}$  must contain a decreasing subword of length  $3 + h_{j-1}$  of the form

$$[r_1, r_2, r_3, \dots, r_{2+h_{j-1}}, r_{3+h_{j-1}}]$$

where  $r_1 = w^{-1}(w'(j))$  and  $r_{3+h_{j-1}} = w^{-1}(n+1)$ . Note that for any  $k < p \leq n+1$ , we have  $w(p) = w's_n \cdots s_1(p) = w'(p-1)$ , so  $w^{-1}(w'(p-1)) = p$ . In particular, we have  $r_1 = w^{-1}(w'(j)) = j+1$  and we also have  $w^{-1}(w'(j-1)) = j$ .

Now, if  $j$  occurs anywhere before  $r_3$  in  $w^{-1}$ , then  $w^{-1}$  would contain the subword

$$[j, r_3, \dots, r_{2+h_{j-1}}, r_{3+h_{j-1}}]$$

which would imply that  $h_{j-1} \geq 1 + h_{j-1}$ , a clear contradiction. So  $j$  occurs somewhere after  $r_3$  and before  $1$ , which means that  $w^{-1}$  contains the 53241 pattern occurring in the subword  $[(j+1), r_2, r_3, j, r_{3+h_{j-1}}]$ . Then  $w$  contains the pattern  $(53241)^{-1} = 53241$  as well.  $\square$

**Proposition 7.14.** *Every Catalan sequence of length  $n$  arises as the ending height sequence of an element in  $W(A_n)$  which avoids the pattern 53241.*

*Proof.* Let  $\mathbf{a} = (a_n, a_{n-1}, \dots, a_1)$  be any Catalan sequence of length  $n$ . Then the sequence of length  $n-1$  given by  $(a_{n-1}, a_{n-2}, \dots, a_1)$  is certainly a Bell sequence, so by the proof of Proposition 7.10, we can find some  $v \in W(A_{n-1})$  with fixed lexicographic reduced expression  $v = v's_{n-1}s_{n-2} \cdots s_1$ ,  $v' \in W(A_{n-2})$ , such that

$v$  has ending height sequence  $(a_{n-1}, a_{n-2}, \dots, a_1)$ . Define

$$w_n := (v')s_n s_{n-1} \cdots s_1, \text{ and}$$

$$w_i := s_i s_{i+1} \cdots s_{n-1} (v')s_n s_{n-1} \cdots s_1 \text{ for each } 1 \leq i \leq n-1.$$

Denote the ending height sequence of  $w_i$  as  $(b_i, a_{n-1}, \dots, a_1)$  for each  $1 \leq i \leq n$ .

Assume by induction that  $v$  avoids the pattern 53241, and suppose for a contradiction that  $w_i$  contains the pattern 53241. Since 53241 is an involution, this means that  $v^{-1}$  also avoids 53241 and  $w_i^{-1}$  also contains 53241.

Observe that

$$(w_i)^{-1}(j) = \begin{cases} v^{-1}(j) & \text{if } j < i \\ n+1 & \text{if } j = i \\ v^{-1}(j-1) & \text{if } j > i \end{cases}$$

and thus, in one-line notation, we can write

$$\begin{aligned} v^{-1} &= [v^{-1}(1), v^{-1}(2), \dots, v^{-1}(n-1), 1, n+1] \\ w_n^{-1} &= [v^{-1}(1), v^{-1}(2), \dots, v^{-1}(n-1), n+1, 1] \\ w_{n-1}^{-1} &= [v^{-1}(1), v^{-1}(2), \dots, n+1, v^{-1}(n-1), 1] \\ &\vdots \\ w_2^{-1} &= [v^{-1}(1), n+1, v^{-1}(2), \dots, v^{-1}(n-1), 1] \\ w_1^{-1} &= [n+1, v^{-1}(1), v^{-1}(2), \dots, v^{-1}(n-1), 1]. \end{aligned}$$

(Here we can see that the entry  $(n+1)$  moves from the upper right corner of the diagram down to the lower left corner of the diagram).

For notational brevity, let  $y = s_i s_{i+1} \cdots s_{n-1}$ . Since  $w_i = yv' s_n s_{n-1} \cdots s_1$ , the final  $n$  elements of the set  $N(\underline{w_i})$  (ordered according to the fixed reduced expression of  $w_i$ ) are precisely

$$\{e_{yv'(n)} - e_{n+1}, e_{yv'(n-1)} - e_{n+1}, \dots, e_{yv'(1)} - e_{n+1}\}.$$

As sets, we have

$$\{e_1 - e_{n+1}, e_2 - e_{n+1}, \dots, e_n - e_{n+1}\} = \{e_{yv'(1)} - e_{n+1}, e_{yv'(2)} - e_{n+1}, \dots, e_{yv'(n)} - e_{n+1}\}.$$

Note that

$$\begin{aligned} a_n &= \mathbf{ht}(e_{y(v'(n))} - e_{n+1}) = \mathbf{ht}(e_i - e_{n+1}), \\ a_{n-1} &= \mathbf{ht}(e_{y(v'(n-1))} - e_{n+1}), \\ &\vdots \\ a_1 &= \mathbf{ht}(e_{y(v'(1))} - e_{n+1}) \end{aligned}$$

and note also that

$$y(v'(n-1)) = s_i s_{i+1} \cdots s_{n-1}(v'(n-1)) = \begin{cases} 1 + v'(n-1) & \text{if } v'(n-1) \geq i \\ v'(n-1) & \text{if } v'(n-1) < i \end{cases}.$$

We will proceed by considering the three cases  $v'(n-1) < i$ ,  $v'(n-1) = i$ , and  $v'(n-1) > i$ . In each case, we will find a contradiction of the hypotheses, allowing us to conclude the desired claim.

**Case 1:** Suppose  $v'(n-1) < i$ . We have  $w_i^{-1}(v'(n-1)) = v^{-1}(v'(n-1)) = n$ . So in the one-line expression for  $w_i^{-1}$ , the entry  $n$  appears to the left of  $n+1 = w_i^{-1}(i)$ . If  $(n+1)r_2r_3r_41$  is a subword of  $w_i^{-1}$  in the same relative order as 53241, then the pattern  $n r_2 r_3 r_4 1$  is also an occurrence of the pattern 53241 which appears in both  $w_i^{-1}$  and  $v^{-1}$ . This contradicts the assumption that  $v$  and  $v^{-1}$  avoids the pattern 53241.

**Case 2:** Suppose  $v'(n-1) = i$ . Then we have

$$a_{n-1} = \mathbf{ht}_{w_i}(e_{1+v'(n-1)} - e_{n+1}) = \mathbf{ht}_{w_i}(e_{i+1} - e_{n+1})$$

where  $w_i^{-1}(i+1) = v^{-1}(i) = v^{-1}(v'(n-1)) = s_1 s_2 \cdots s_{n-1}(v')^{-1}(v'(n-1)) = n$ .

This implies that  $w_i^{-1}$  contains a 653241 pattern in the subword

$$w_i^{-1}(i) w_i^{-1}(i+1) r_2 r_3 r_4 1 = (n+1)(n) r_2 r_3 r_4 1.$$



However, this means that  $v^{-1}$  contains the pattern 53241 in the subword  $(n) r_2 r_3 r_4 1$ , a contradiction.

**Case 3:** Suppose  $v'(n-1) > i$ . Then  $a_{n-1} = \text{ht}_{w_i}(e_{1+v'(n-1)} - e_{n+1})$ , where

$$w_i^{-1}(1 + v'(n-1)) = v^{-1}(v'(n-1)) = s_1 \cdots s_{n-1}(v')^{-1}(v'(n-1)) = n.$$

Since  $n$  appears to the right of  $w^{-1}(i) = n+1$  in the one-line expression of  $w_i^{-1}$ , we know that the longest decreasing subword of  $w_i^{-1}$  between the entries  $n+1$  and 1 is strictly longer than the longest decreasing subword of  $w_i^{-1}$  between  $n$  and 1. In other words, we must have  $a_n^i \geq a_{n-1}$ , so by the hypothesis that  $a$  is a Catalan sequence, we have  $a_n^i = a_{n-1} + 1$ .

Now consider the element  $w_{i+1}^{-1}$ . This element has ending height sequence  $(a_n^{i+1}, a_{n-1}, \dots, a_1)$ . First suppose that  $v'(n-1) \geq i+1$ . Then  $a_{n-1} = \text{ht}_{w_{i+1}}(e_{1+v'(n-1)} - e_{n+1})$  by the same reasoning as for  $w_i^{-1}$  above, and we have  $w_{i+1}^{-1}(1 + v'(n-1)) = v^{-1}(v'(n-1)) = n$ . So as was the case for  $w_i^{-1}$  above, we have  $a_n^{i+1} \geq a_{n-1}$ , and so  $a_n^{i+1} = 1 + a_{n-1} = a_n^i$ . This contradicts the maximality of  $i$ .

Then  $v'(n-1) = i+1$ . We have  $a_{n-1} = \text{ht}_{w_{i+1}}(e_{1+v'(n-1)} - e_{n+1}) = \text{ht}_{w_{i+1}}(e_{i+2} - e_{n+1})$ . Since  $n+1 = w_{i+1}^{-1}(i+1) > w_{i+1}^{-1}(i+2)$ , we know that the longest decreasing subword of  $w_{i+1}^{-1}$  between  $n+1$  and 1 is strictly longer than the longest decreasing subword in  $w_{i+1}^{-1}$  between  $w_{i+1}^{-1}(i+2)$  and 1. This implies that  $\text{ht}_{w_{i+1}}(e_{i+1} - e_{n+1}) \geq \text{ht}_{w_{i+1}}(e_{i+2} - e_{n+1})$ , so  $a_n^{i+1} \geq a_{n-1}$ . This again results in the equality  $a_n^{i+1} = 1 + a_{n-1} = a_n^i$ , which contradicts the maximality of  $i$ .

Since every case results in a contradiction of the hypotheses, we can conclude that  $w_i^{-1}$  (and thus  $w_i$ ) avoids the pattern 53241.

□

# APPENDIX A

## DESCRIPTION OF SOFTWARE DEVELOPED

Throughout the course of this work, we found it beneficial to develop programs using the programming language Python and the Python-based software system Sage to do the following.

- Compute the distance enumerating polynomial  $\mathcal{R}_w(q)$  in type  $A$ .
- Compute the closure polynomial  $\mathcal{M}_w(q)$  for  $w$  belonging to a general Weyl group.
- Compute the heights associated with a reduced word for  $w$  when  $w$  belongs to a general Weyl group.
- Compute the inversion polynomial  $\mathcal{N}_w(q)$  for  $w$  belonging to a general Weyl group.
- Compute the Lascoux element  $L_{\underline{w}}$  when  $\underline{w}$  is any fixed reduced word of an element  $w$  belonging to a general Weyl group.

The source code for any of these programs is available upon request.

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