Equivariant homology and K-theory of affine Grassmannians and Toda lattices

EQUIVARIANT (K-)HOMOLOGY OF AFFINE GRASSMANNIAN AND TODA LATTICE

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1. Introduction

1.1. Let G be an almost simple complex algebraic group, and let Gr_G be its affine Grassmannian. Recall that if we set $\mathbf{O} = \mathbb{C}[[\mathsf{t}]],\ \mathbf{F} = \mathbb{C}((\mathsf{t}))$, then $\mathrm{Gr}_G = G(\mathbf{F})/G(\mathbf{O})$.

It is well-known that the subgroup ΩK of polynomial loops into a maximal compact subgroup $K \subset G$ projects isomorphically to Gr_G; thus Gr_G acquires the structure of a topological group. An algebro-geometric counterpart of this structure is provided by *the convolution diagram* $G(\mathbf{F}) \times_{G(\mathbf{O})} \mathrm{Gr}_G \to \mathrm{Gr}_G$.

It allows one to define the *convolution* of two $G(\mathbf{O})$ equivariant geometric objects (such as sheaves, or constrictible functions) on Gr_G. A famous example of such a structure is the category of $G(\mathbf{O})$ equivariant perverse sheaves on Gr ("Satake category" in the terminology of Beilinson and Drinfeld); this is a semi-simple abelian category, and convolution provides it with a symmetric monoidal structure. By results of [10], [19], [2] this category is identified with the category of (algebraic) representations of the Langlands dual group.

The starting point for the present work was the observation that a similar definition works in another setting, yielding a monoidal structure on the category of $G(\mathbf{O})$ equivariant *perverse coherent sheaves* on Gr (the "coherent Satake category"). The latter is a non-semisimple artinian abelian category, the heart of the middle perversity t-structure on the derived category of $G(\mathbf{O})$ equivariant coherent sheaves on Gr_G; existence of this t-structure is due to the fact that dimensions of all $G(\mathbf{O})$-orbits inside a given component of Gr_G are of the same parity, cf. [3]. The resulting monoidal category turns out to be non-symmetric, though its Grothendieck ring $K^{G(\mathbf{O})}(\mathrm{Gr}_G)$ is commutative. One of the results of this paper is a computation of this ring. Along with $K^{G(\mathbf{O})}(\mathrm{Gr}_G)$ we compute its "graded version", the ring $H^{G(\mathbf{O})}(\mathrm{Gr})$ of equivariant homology of Gr, where the algebra structure is again provided by convolution.[1] (The ring $H_\bullet^{G(\mathbf{O})}(\mathrm{Gr}_G)$ was essentially computed by Dale Peterson [20], cf. also [15].)

To describe the answer, let $\check{G}$ be the Langlands dual group to G, and let $\check{\mathfrak{g}}$ be its Lie algebra. Consider the *universal centralizers* $\mathfrak{Z}^{\check{G}}_{\check{\mathfrak{g}}}$ and $\mathfrak{Z}^{\check{G}}_{\check{G}}$: if we denote by $C_{\check{G},\check{\mathfrak{g}}} \subset \check{G} \times \check{\mathfrak{g}}$ (resp. $C_{\check{G},\check{G}} \subset \check{G} \times \check{G}$) the locally closed subvariety formed by all the pairs (g, x) such that $Ad_g(x) = x$ and x is regular (resp. all the pairs (g_1, g_2) such that $Ad_{g_1} g_2 = g_2$ and

[1] The two rings are related via the Chern character homomorphism from $K^{G(\mathbf{O})}(\mathrm{Gr})$ to the completion of $H^{G(\mathbf{O})}(\mathrm{Gr})$.

g_2 is regular), then $\mathfrak{Z}^{\check{G}}_{\check{\mathfrak{g}}}$ (resp. $\mathfrak{Z}^{\check{G}}_{\check{G}}$) is the categorical quotient $C_{\check{G},\check{\mathfrak{g}}}//\check{G}$ (resp. $C_{\check{G},\check{G}}//\check{G}$) with respect to the diagonal adjoint action of $\check{G}$.

We identify Spec $\left(H^{G(\mathbf{O})}_\bullet(\mathrm{Gr}_G)\right)$ with $\mathfrak{Z}^{\check{G}}_{\check{\mathfrak{g}}}$. Also, we identify Spec $\left(K^{G(\mathbf{O})}(\mathrm{Gr}_G)\right)$ with *a variant* of $\mathfrak{Z}^{\check{G}}_{\check{G}}$ (the isomorphism Spec $\left(K^{G(\mathbf{O})}(\mathrm{Gr}_G)\right) \simeq \mathfrak{Z}^{\check{G}}_{\check{G}}$ holds true iff G is of type E_8).

Notice that $\mathfrak{Z}^{\check{G}}_{\check{\mathfrak{g}}}$ inherits a canonical symplectic structure as a hamiltonian reduction of the cotangent bundle $\mathsf{T}^*\check{G}$. Also, $\mathfrak{Z}^{\check{G}}_{\check{G}}$ inherits a canonical Poisson structure as a q-Hamiltonian reduction of the q-Hamiltonian $\check{G}$-space *internal fusion double* $\mathbf{D}(\check{G})$ (see [1]); this Poisson structure is in fact symplectic iff $\check{G}$ is simply connected (that is, G is adjoint).

The corresponding Poisson structures on $K^{G(\mathbf{O})}(\mathrm{Gr}_G)$, $H^{G(\mathbf{O})}(\mathrm{Gr}_G)$ come from a deformation of these commutative algebras to non-commutative algebras $H^{G(\mathbf{O})\ltimes\mathbb{G}_m}_\bullet(\mathrm{Gr}_G)$ (resp. $K^{G(\mathbf{O})\ltimes\mathbb{G}_m}(\mathrm{Gr}_G)$); here $\mathbb{G}_m$ acts on Gr_G by loop rotation. We conjecture that the non-commutative algebra $H^{G(\mathbf{O})\ltimes\mathbb{G}_m}_\bullet(\mathrm{Gr}_G)$ can also be obtained from the ring of differential operators on $\check{G}$ by quantum Hamiltonian reduction.

The space $\mathfrak{Z}^{\check{G}}_{\check{\mathfrak{g}}}$ contains an open piece $\mathfrak{Z}(\check{G})$ which for $\check{G}$ adjoint (that is, for G simply connected) is a complexification of the Kostant's phase space of the classical Toda lattice ([14], Theorem 2.6). We remark in passing that Toda lattice also appears in the (apparently related) computations by Givental, Kim and others of quantum cohomology of flag varietites (see e.g. [13]).

Our computation should be compared with (and is to a large extent inspired by) [10] where equivariant cohomology $H_{G(\mathbf{O})}(\mathrm{Gr}_G)$ were computed[2] in terms of the $\check{G}$. (The precise relation between the two computations is spelled out in Remark 2.13).

The second main object considered in the paper is another derived category of coherent sheaves with a convolution monoidal structure, namely the derived category $D^bCoh^{G(\mathbf{O})}_{\Lambda_G}(T^*\mathrm{Gr})$ of $G(\mathbf{O})$-equivariant coherent sheaves on the cotangent bundle of Gr_G supported on the union Λ_G of conormal bundles to the $G(\mathbf{O})$-orbits (the definition of involved objects requires extra work since Gr_G is infinite dimensional). (In this case we do not find a t-structure compatible with convolution, so all we get is a monoidal triangulated category). Notice that the singular support of a $G(\mathbf{O})$-equivariant D-module on Gr_G is an object of $Coh^{G(\mathbf{O})}_{\Lambda_G}(T^*\mathrm{Gr})$, thus this category can be considered a "classical limit" of the (derived) Satake category. We compute the Grothendieck ring of $D^bCoh^{G(\mathbf{O})}_{\Lambda_G}(T^*\mathrm{Gr})$ identifying its spectrum with $(T\times\check{T})/W$, where $T\subset G$, and $\check{T}\subset\check{G}$ are Cartan subgroups. This is a singular variety birationally equivalent to Spec $\left(K^{G(\mathbf{O})}(\mathrm{Gr}_G)\right)$. Unlike the latter, the former remains unchanged if we replace G by $\check{G}$. This motivates a conjecture that the corresponding triangulated monoidal categories for G and $\check{G}$ are equivalent. The conjecture is compatible with a "classical

[2]Another description for $H_{G(\mathbf{O})}(\mathrm{Gr}_G)$ is provided by a general result of [16]; in fact, its extension from [17] gives also an answer for $K^{G(\mathbf{O})}(\mathrm{Gr}_G)$, and a similar technique can be applied to compute $H^{G(\mathbf{O})}(\mathrm{Gr}_G)$. However, this form of the answer does not make the relation to the (dual) group geometry explicit.

limit" of the geometric Langlands conjecture of Beilinson and Drinfeld (see 7.9 below for a more precise statement of the conjecture).

Finally, we remark that the convolution of $G(\mathbf{O})$-equivariant perverse coherent sheaves is closely related to the *fusion product* of $G(\mathbf{O})$-modules introduced by B. Feigin[3] [6] (see Section 8). In fact, our desire to understand the category $\mathcal{P}^{G(\mathbf{O})}(\mathrm{Gr}_G)$, and the work [6] of B. Feigin and S. Loktev, was one of the motivations for the present work.

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2. Notations and statements of the results

2.1. **Kostant slices.** G is an almost simple algebraic group with the Lie algebra $\mathfrak{g}$. We choose a principal $\mathfrak{sl}_2$ triple (e,h,f) in $\mathfrak{g}$. Let $\phi:\ \mathfrak{sl}_2\to\mathfrak{g}$ (resp. $\Phi:\ SL_2\to G$) be the corresponding homomorphism. We denote by e_G (resp. f_G) the image $\Phi\begin{pmatrix}1&1\\0&1\end{pmatrix}$ (resp. $\Phi\begin{pmatrix}1&0\\1&1\end{pmatrix}$). We denote by $\mathfrak{z}(e)$ the centralizer of e in $\mathfrak{g}$, and by $Z(e)$ (resp. $Z^0(e)$) the centralizer of e (equivalently, of e_G) in G (resp. its neutral connected component). We denote by $\Sigma_\mathfrak{g}\subset\mathfrak{g}$ (resp. $\Sigma_G\subset G$) the *Kostant slice* $\mathfrak{z}(e)+f$ (resp. $Z^0(e)\cdot f_G$). It is known that $\Sigma_\mathfrak{g}\subset\mathfrak{g}^{reg}$ (resp. $\Sigma_G\subset G^{reg}$), and the projection to the categorical quotient $\Sigma_\mathfrak{g}\hookrightarrow\mathfrak{g}\twoheadrightarrow\mathfrak{g}//Ad_G=\mathfrak{t}/W$ induces an isomorphism $\Sigma_\mathfrak{g}\simeq\mathfrak{t}/W$. Similarly, if G is simply connected, the projection to the categorical quotient $\Sigma_G\hookrightarrow G\twoheadrightarrow G//Ad_G=T/W$ induces an isomorphism $\Sigma_G\simeq T/W$.

2.2. **The universal centralizers.** We consider the locally closed subvariety $C_{\mathfrak{g},\mathfrak{g}}\subset\mathfrak{g}\times\mathfrak{g}$ (resp. $C_{\mathfrak{g},G}\subset\mathfrak{g}\times G$, $C_{G,\mathfrak{g}}\subset G\times\mathfrak{g}$, $C_{G,G}\subset G\times G$) formed by all the pairs (x_1,x_2) such that $[x_1,x_2]=0$ and x_2 is regular (resp. all the pairs (x,g) such that $Ad_g(x)=x$ and g is regular; all the pairs (g,x) such that $Ad_g(x)=x$ and x is regular; all the pairs (g_1,g_2) such that $Ad_{g_1}(g_2)=g_2$ and g_2 is regular). The categorical quotients with respect to the diagonal adjoint action of G are denoted respectively $C_{\mathfrak{g},\mathfrak{g}}//G=\mathfrak{Z}^\mathfrak{g}_\mathfrak{g}$, $C_{\mathfrak{g},G}//G=\mathfrak{Z}^\mathfrak{g}_G$, $C_{G,\mathfrak{g}}//G=\mathfrak{Z}^G_\mathfrak{g}$, $C_{G,G}//G=\mathfrak{Z}^G_G$. The projections to the second (regular) factor are denoted by $\varpi:\ \mathfrak{Z}^\mathfrak{g}_\mathfrak{g}\to\mathfrak{g}^{reg}//G=\mathfrak{t}/W$, $\varpi:\ \mathfrak{Z}^\mathfrak{g}_G\to G^{reg}//G=T/W$, $\varpi:\ \mathfrak{Z}^G_\mathfrak{g}\to\mathfrak{g}^{reg}//G=\mathfrak{t}/W$, $\varpi:\ \mathfrak{Z}^G_G\to G^{reg}//G=T/W$. In all the four cases ϖ is flat.

We consider the restrictions of our centralizer varieties to the Kostant slices: $C^\Sigma_{\mathfrak{g},\mathfrak{g}}=C_{\mathfrak{g},\mathfrak{g}}\cap(\mathfrak{g}\times\Sigma_\mathfrak{g})$, $C^\Sigma_{\mathfrak{g},G}=C_{\mathfrak{g},G}\cap(\mathfrak{g}\times\Sigma_G)$, $C^\Sigma_{G,\mathfrak{g}}=C_{G,\mathfrak{g}}\cap(G\times\Sigma_\mathfrak{g})$, $C^\Sigma_{G,G}=C_{G,G}\cap(G\times\Sigma_G)$.

[3] The relation between convolution and fusion was known to B. Feigin since 1997.

Then the locally closed embedding $C^\Sigma_{\mathfrak{g},\mathfrak{g}} \hookrightarrow C_{\mathfrak{g},\mathfrak{g}} \twoheadrightarrow \mathfrak{Z}^{\mathfrak{g}}_{\mathfrak{g}}$ induces an isomorphism $C^\Sigma_{\mathfrak{g},\mathfrak{g}} \simeq \mathfrak{Z}^{\mathfrak{g}}_{\mathfrak{g}}$. Similarly, we have isomorphisms $C^\Sigma_{G,\mathfrak{g}} \simeq \mathfrak{Z}^G_{\mathfrak{g}}$ and (for simply connected G) $C^\Sigma_{\mathfrak{g},G} \simeq \mathfrak{Z}^{\mathfrak{g}}_G$, $C^\Sigma_{G,G} \simeq \mathfrak{Z}^G_G$.

Thus both $\mathfrak{Z}^{\mathfrak{g}}_{\mathfrak{g}} \to \mathfrak{t}/W$ and $\mathfrak{Z}^{\mathfrak{g}}_G \to T/W$ (for simply connected G) are the sheaves of abelian Lie algebras, while both $\mathfrak{Z}^G_{\mathfrak{g}} \to \mathfrak{t}/W$ and $\mathfrak{Z}^G_G \to T/W$ (for simply connected G) are the sheaves of abelian Lie groups.

2.3. **Isogenies.** The center $Z(G)$ acts naturally on $\mathfrak{Z}^{\mathfrak{g}}_G$ (resp. $\mathfrak{Z}^G_{\mathfrak{g}}$) by $z(x,g) = (x,zg)$ (resp. $z(g,x) = (zg,x)$). The center $Z(G)$ acts on $\mathfrak{Z}^G_G$ on both sides: $z_1(g_1,g_2)z_2 = (z_1g_1, z_2g_2)$. Let $\tilde{G}$ denote the universal cover of G. Then the fundamental group $\pi_1(G)$ is embedded into $Z(\tilde{G})$, and we have $\mathfrak{Z}^{\mathfrak{g}}_G = \pi_1(G)\backslash\mathfrak{Z}^{\mathfrak{g}}_{\tilde{G}}$, $\mathfrak{Z}^G_{\mathfrak{g}} = \pi_1(G)\backslash\mathfrak{Z}^{\tilde{G}}_{\mathfrak{g}}$, $\mathfrak{Z}^G_G = \pi_1(G)\backslash\mathfrak{Z}^{\tilde{G}}_{\tilde{G}}/\pi_1(G)$.

2.4. **Symplectic structures.** We fix an invariant identification $\mathfrak{g} \simeq \mathfrak{g}^*$, hence $\mathfrak{t} \simeq \mathfrak{t}^*$. Then $\mathfrak{g} \times \mathfrak{g}$ gets identified with $\mathfrak{g} \times \mathfrak{g}^* = \mathsf{T}^*\mathfrak{g}$ (the cotangent bundle), and $G \times \mathfrak{g}$ gets identified with $G \times \mathfrak{g}^* = \mathsf{T}^*G$. After this $\mathfrak{Z}^{\mathfrak{g}}_{\mathfrak{g}}$ (resp. $\mathfrak{Z}^G_{\mathfrak{g}}$) can be viewed as a hamiltonian reduction of $\mathsf{T}^*\mathfrak{g}$ (resp. T^*G); thus it inherits a canonical symplectic structure.

Identifying $\mathfrak{g} \times G$ with $\mathfrak{g}^* \times G = \mathsf{T}^*G$ we can view $\mathfrak{Z}^{\mathfrak{g}}_G$ as a hamiltonian reduction of T^*G as well; thus it inherits a canonical Poisson structure. Note that $\mathfrak{Z}^{\mathfrak{g}}_G$ is smooth and symplectic iff G is simply connected. We have symplectic isomorphisms $\mathfrak{Z}^{\mathfrak{g}}_{\mathfrak{g}} \simeq \mathsf{T}^*(\mathfrak{t}/W)$, and (in case G is simply connected) $\mathfrak{Z}^{\mathfrak{g}}_G \simeq \mathsf{T}^*(T/W)$.

Note that $\mathfrak{Z}^{\mathfrak{g}}_G$ and $\mathfrak{Z}^G_{\mathfrak{g}}$ share a common open piece $\mathsf{Z}(G)$ formed by the classes of pairs (g,x) where both g and x are regular. The canonical symplectic structures agree on $\mathfrak{Z}^{\mathfrak{g}}_G \supset \mathsf{Z}(G) \subset \mathfrak{Z}^G_{\mathfrak{g}}$. Note also that for adjoint G the space $\mathsf{Z}(G)$ contains (a complexification of) the Kostant's phase space $\mathfrak{Z}(G)$ of the classical Toda lattice [14], and the embedding $\mathfrak{Z}(G) \hookrightarrow \mathfrak{Z}^G_{\mathfrak{g}}$ is given by the Theorem 2.6 of *loc. cit.*

A. Alexeev, A. Malkin and E. Meinrenken introduced in [1] Example 6.1 the q-Hamiltonian G-space *internal fusion double* $\mathbf{D}(G)$. Its q-Hamiltonian reduction is $\mathfrak{Z}^G_G$, so it inherits a canonical Poisson structure. For a simply connected G the space $\mathfrak{Z}^G_G$ is smooth and symplectic.

2.5. **Affine blow-ups.** The set of roots of G (resp. $\check{G}$) is denoted by R (resp. $\check{R}$). We will view $\alpha \in R$ (resp. $\check{\alpha} \in \check{R}$) as a homomorphism $\mathfrak{t} \to \mathbb{C}$ (resp. $\check{\mathfrak{t}} \to \mathbb{C}$) or as a homomorphism $T \to \mathbb{C}^*$ (resp. $\check{T} \to \mathbb{C}^*$) depending on a context. Also, for a root $\alpha \in R$ we denote by ${}^1\alpha$ (resp. ${}^2\alpha$) the linear function on $\mathfrak{t} \times \mathfrak{t}$ obtained as a composition of α with the projection on the first (resp. second) factor.

We consider the following affine blow-up of $\mathfrak{t} \times \mathfrak{t}$ at the diagonal walls: $\overset{\bullet}{\mathfrak{B}}{}^{\mathfrak{g}}_{\mathfrak{g}} = \operatorname{Spec}(\mathbb{C}[\mathfrak{t} \times \mathfrak{t}, \frac{{}^1\alpha}{{}^2\alpha}, \alpha \in R])$. We also set $\overset{\bullet}{\mathfrak{B}}{}^G_{\mathfrak{g}} = \operatorname{Spec}(\mathbb{C}[\mathfrak{t} \times T, \frac{{}^1\alpha}{{}^2\alpha - 1}, \alpha \in R])$; $\overset{\bullet}{\mathfrak{B}}{}^G_G = \operatorname{Spec}(\mathbb{C}[T \times T, \frac{{}^1\alpha - 1}{{}^2\alpha - 1}, \alpha \in R])$, $\overset{\bullet}{\mathfrak{B}}{}^{\check{G}}_G = \operatorname{Spec}(\mathbb{C}[\check{T} \times T, \frac{{}^1\check{\alpha} - 1}{{}^2\alpha - 1}, \alpha \in R])$; and let $\mathfrak{B}^{\mathfrak{g}}_{\mathfrak{g}} = \overset{\bullet}{\mathfrak{B}}{}^{\mathfrak{g}}_{\mathfrak{g}}/W$, $\mathfrak{B}^G_G = \overset{\bullet}{\mathfrak{B}}{}^G_G/W$, $\mathfrak{B}^{\check{G}}_G = \overset{\bullet}{\mathfrak{B}}{}^{\check{G}}_G/W$ (thus $\mathfrak{B}^{\mathfrak{g}}_{\mathfrak{g}} = \operatorname{Spec}(\mathbb{C}[\mathfrak{t} \times \mathfrak{t}, \frac{{}^1\alpha}{{}^2\alpha}, \alpha \in R]^W)$, etc.). We denote by ϖ the projection of $\mathfrak{B}$ to the second factor; thus we have $\varpi:\ \mathfrak{B}^{\mathfrak{g}}_{\mathfrak{g}} \to \mathfrak{t}/W$, $\mathfrak{B}^{\mathfrak{g}}_G \to T/W$, $\mathfrak{B}^G_{\mathfrak{g}} \to \mathfrak{t}/W$, $\mathfrak{B}^G_G \to T/W$, $\mathfrak{B}^{\check{G}}_G \to T/W$.

2.6. **Poisson structures.** We have the canonical trivializations of the tangent bundles $\mathsf{T}(\mathfrak{t}\times\mathfrak{t}) = (\mathfrak{t}\times\mathfrak{t})\times(\mathfrak{t}\times\mathfrak{t})$, $\mathsf{T}(\mathfrak{t}\times T) = (\mathfrak{t}\times T)\times(\mathfrak{t}\times\mathfrak{t})$, $\mathsf{T}(T\times\mathfrak{t}) = (T\times\mathfrak{t})\times(\mathfrak{t}\times\mathfrak{t})$, $\mathsf{T}(T\times T) = (T\times T)\times(\mathfrak{t}\times\mathfrak{t})$, $\mathsf{T}(T\times\check{T}) = (T\times\check{T})\times(\mathfrak{t}\times\check{\mathfrak{t}})$. Making use of the identification $\check{\mathfrak{t}} = \mathfrak{t}^*\simeq\mathfrak{t}$ we obtain the W-invariant symplectic structures on the above varieties. Thus the above affine blow-ups carry the rational Poisson structures (regular off the discriminants $\mathbf{D}\subset\mathfrak{B}$).

Proposition 2.7. *The Poisson structure on $\mathfrak{B}^{\mathfrak{g}}_{\mathfrak{g}}-\mathbf{D}$ (resp. $\mathfrak{B}^{\mathfrak{g}}_{G}-\mathbf{D}$, $\mathfrak{B}^{G}_{\mathfrak{g}}-\mathbf{D}$, $\mathfrak{B}^{G}_{G}-\mathbf{D}$, $\mathfrak{B}^{\check{G}}_{G}-\mathbf{D}$) extends to the global Poisson structure; it is a symplectic structure if the corresponding variety is smooth.*

Proposition 2.8. *We are in the setup of 2.5.*

a) ϖ is flat if G is simply connected;

b) There are natural identifications $\mathfrak{B}^{\mathfrak{g}}_{\mathfrak{g}}\simeq\mathfrak{Z}^{\mathfrak{g}}_{\mathfrak{g}}$, $\mathfrak{B}^{\mathfrak{g}}_{G}\simeq\mathfrak{Z}^{\mathfrak{g}}_{G}$, $\mathfrak{B}^{G}_{\mathfrak{g}}\simeq\mathfrak{Z}^{G}_{\mathfrak{g}}$, $\mathfrak{B}^{G}_{G}\simeq\mathfrak{Z}^{G}_{G}$ commuting with ϖ.

c) If G is simply laced and adjoint, we have an identification $\mathfrak{B}^{G}_{\check{G}}\simeq Z(\check{G})\backslash\mathfrak{Z}^{\check{G}}_{\check{G}}$ commuting with ϖ;

d) If G is simply laced and simply connected, we have an identification $\mathfrak{B}^{G}_{\check{G}}\simeq\mathfrak{Z}^{G}_{G}/Z(G)$ commuting with ϖ;

e) The above identifications respect the Poisson structures.

2.9. **Flat group sheaves.** We consider the functor $\mathfrak{F}^{\mathfrak{g}}_{\mathfrak{g}}$ on the category $\mathrm{Flat}_{\mathfrak{t}/W}$ of schemes flat over $\mathfrak{t}/W$ to the category of sets, sending a test scheme S to the set of W-invariant morphisms $\left(\mathrm{Mor}(S\times_{\mathfrak{t}/W}\mathfrak{t},\mathfrak{t})\right)^W$. Similarly, we consider the functor $\mathfrak{F}^{\mathfrak{g}}_{G}$ on the category $\mathrm{Flat}_{T/W}$ sending a test scheme S to the set of W-invariant morphisms $\left(\mathrm{Mor}(S\times_{T/W}T,\mathfrak{t})\right)^W$. Also, we consider the functor $\mathfrak{F}^{G}_{\mathfrak{g}}$ on the category $\mathrm{Flat}_{\mathfrak{t}/W}$ sending a test scheme S to the set of W-invariant morphisms $\left(\mathrm{Mor}(S\times_{\mathfrak{t}/W}\mathfrak{t},T)\right)^W_0\subset\left(\mathrm{Mor}(S\times_{\mathfrak{t}/W}\mathfrak{t},T)\right)^W$ subject to the condition (cf. [5] 4.2)

$$\alpha\left(f(\alpha^{-1}(0))\right)=1\ \forall\ \alpha\in R. \tag{1}$$

(note that the W-invariance condition automatically implies $\alpha\left(f(\alpha^{-1}(0))\right)=\pm1\ \forall\ \alpha\in R$.)

Furthermore, we consider the functor $\mathfrak{F}^{G}_{G}$ on the category $\mathrm{Flat}_{T/W}$ sending a test scheme S to the set of W-invariant morphisms $\left(\mathrm{Mor}(S\times_{T/W}T,T)\right)^W_0\subset\left(\mathrm{Mor}(S\times_{T/W}T,T)\right)^W$ subject to the condition

$$\alpha\left(f(\alpha^{-1}(1))\right)=1\ \forall\ \alpha\in R. \tag{2}$$

(note that the W-invariance condition automatically implies $\alpha\left(f(\alpha^{-1}(1))\right)=\pm1\ \forall\ \alpha\in R$.)

Finally, we consider the functor $\mathfrak{F}^{\check{G}}_{G}$ on the category $\mathrm{Flat}_{T/W}$ sending a test scheme S to the set of W-invariant morphisms $\left(\mathrm{Mor}(S\times_{T/W}T,\check{T})\right)^W_0\subset\left(\mathrm{Mor}(S\times_{T/W}T,\check{T})\right)^W$ subject to the condition

$$\check{\alpha}\left(f(\alpha^{-1}(1))\right)=1\ \forall\ \alpha\in R. \tag{3}$$

(note that the W-invariance condition automatically implies $\check{\alpha}\left(f(\alpha^{-1}(1))\right) = \pm 1\ \forall\, \alpha \in R$.)

The following Proposition is a generalization of [5] 11.6.

Proposition 2.10. *Assume that G is simply connected. The functor $\mathfrak{F}^{\mathfrak{g}}_{\mathfrak{g}}$ (resp. $\mathfrak{F}^{\mathfrak{g}}_{G}$, $\mathfrak{F}^{G}_{\mathfrak{g}}$, $\mathfrak{F}^{G}_{G}$, $\mathfrak{F}^{\check{G}}_{G}$) is representable by the scheme $\mathfrak{B}^{\mathfrak{g}}_{\mathfrak{g}}$ (resp. $\mathfrak{B}^{\mathfrak{g}}_{G}$, $\mathfrak{B}^{G}_{\mathfrak{g}}$, $\mathfrak{B}^{G}_{G}$, $\mathfrak{B}^{\check{G}}_{G}$).*

2.11. **Equivariant Borel-Moore Homology.** For the definition of convolution in equivariant Borel-Moore Homology we refer the reader to [4] 2.7, 8.3 or [18] Chapter 2.

We have $H^{G(\mathbf{O})}_{\bullet}(pt) = H^{\bullet}_{G(\mathbf{O})}(pt) = \mathbb{C}[\mathfrak{t}/W]$, and $H^{G(\mathbf{O})\ltimes\mathbb{G}_m}_{\bullet}(pt) = H^{\bullet}_{G(\mathbf{O})\ltimes\mathbb{G}_m}(pt) = \mathbb{C}[\mathfrak{t}/W][\hbar]$ where $\hbar$ is the generator of $H^2_{\mathbb{G}_m}(pt)$. We will consider the $\mathbb{C}[\mathfrak{t}/W]$-algebra (resp. $\mathbb{C}[\mathfrak{t}/W][\hbar]$-algebra) (with respect to convolution) $H^{G(\mathbf{O})}_{\bullet}(\mathrm{Gr}_G)$ (resp. $H^{G(\mathbf{O})\ltimes\mathbb{G}_m}_{\bullet}(\mathrm{Gr}_G)$). Note that setting $\hbar = 0$ in $H^{G(\mathbf{O})\ltimes\mathbb{G}_m}_{\bullet}(\mathrm{Gr}_G)$ we obtain $H^{G(\mathbf{O})}_{\bullet}(\mathrm{Gr}_G)$; indeed for any group H, a space X with an $H \times \mathbb{G}_m$ action, and an $H \times \mathbb{G}_m$-equivariant complex $\mathcal{F}$ on X we have a long exact sequence $\cdots \to H^{i-2}_{H\times\mathbb{G}_m}(X,\mathcal{F}) \xrightarrow{\hbar} H^{i}_{H\times\mathbb{G}_m}(X,\mathcal{F}) \to H^{i}_{H}(X,\mathcal{F}) \to H^{i-1}_{H\times\mathbb{G}_m}(X,\mathcal{F}) \to \ldots$ coming from the principal $\mathbb{G}_m$-bundle $E(H\times\mathbb{G}_m)\times_H X \to E(H\times\mathbb{G}_m)\times_{H\times\mathbb{G}_m} X$; if the space of $H\times\mathbb{G}_m$-equivariant cohomology is $\hbar$-torsion free, then we get $H^{\bullet}_{H}(X,\mathcal{F}) = H^{\bullet}(X,\mathcal{F})|_{\hbar=0}$.

Theorem 2.12. *a) The algebra $H^{G(\mathbf{O})}_{\bullet}(\mathrm{Gr}_G)$ is commutative;*

b) Its spectrum together with the projection onto $\mathfrak{t}/W = \check{\mathfrak{t}}/W$ is naturally isomorphic to $\mathfrak{Z}^{\check{G}}_{\check{\mathfrak{g}}} \xrightarrow{\varpi} \check{\mathfrak{t}}/W$;

c) The Poisson structure on $H^{G(\mathbf{O})}_{\bullet}(\mathrm{Gr}_G)$ arising from the $\hbar$-deformation $H^{G(\mathbf{O})\ltimes\mathbb{G}_m}_{\bullet}(\mathrm{Gr}_G)$, corresponds under the above identification to the Poisson structure of 2.4 on $\mathfrak{Z}^{\check{G}}_{\check{\mathfrak{g}}}$.

Remark 2.13. The equivariant cohomology ring $H^{\bullet}_{G(\mathbf{O})}(\mathrm{Gr}_G,\mathbb{C}) = H^{\bullet}_{G(\mathbf{O})}(\mathrm{Gr}_G)$ was computed by V. Ginzburg [10]. More precisely, the projection to the second (regular) factor $\mathfrak{Z}^{\check{\mathfrak{g}}}_{\check{\mathfrak{g}}} \to \check{\mathfrak{g}}^{reg}/\!/\check{G} = \check{\mathfrak{t}}/W$ makes $\mathfrak{Z}^{\check{\mathfrak{g}}}_{\check{\mathfrak{g}}}$ a sheaf of abelian Lie algebras. V. Ginzburg identifies $H^{\bullet}_{G(\mathbf{O})}(\mathrm{Gr}_G)$ with the global sections of the relative universal enveloping algebra $U_{\check{\mathfrak{t}}/W}\left(\mathfrak{Z}^{\check{\mathfrak{g}}}_{\check{\mathfrak{g}}}\right)$. One can easily check that this result is compatible with our Theorem 2.12(b) as follows. For a group scheme A over a base S one has a natural pairing $U(\mathfrak{a}) \times \mathcal{O}(A) \to \mathcal{O}(S)$ where $U(\mathfrak{a})$ is the enveloping (over $\mathcal{O}(S)$) of the Lie algebra of A; the pairing sends (ξ, f) to $\xi(f)$ restricted to the identity of A. On the other hand, for a compact (or ind-compact) H-space X we have a pairing $H^{\bullet}_{H}(X) \times H^{H}_{\bullet}(X) \to H^{\bullet}_{H}(pt)$ induced by the action of cohomology on homology, and the push-forward map in Borel-Moore homology $H^{H}_{\bullet}(X) \to H^{\bullet}_{H}(pt)$. The isomorphisms of [10] and of Theorem 2.12 take the first pairing into the second one.

2.14. **Equivariant K-theory.** For the definition of convolution in equivariant K-theory we refer the reader to Chapter 5 of [4].

We have $K^{G(\mathbf{O})}(pt) = \mathbb{C}[T/W]$, and $K^{G(\mathbf{O})\ltimes\mathbb{G}_m}(pt) = \mathbb{C}[T/W][q^{\pm 1}]$. We will consider the $\mathbb{C}[T/W]$-algebra (resp. $\mathbb{C}[T/W][q^{\pm 1}]$-algebra) (with respect to convolution)

$K^{G(\mathbf{O})}(\mathrm{Gr}_G)$ (resp. $K^{G(\mathbf{O})\ltimes\mathbb{G}_m}(\mathrm{Gr}_G)$). Note that setting $q=1$ in $K^{G(\mathbf{O})\ltimes\mathbb{G}_m}(\mathrm{Gr}_G)$ we obtain $K^{G(\mathbf{O})}(\mathrm{Gr}_G)$.

Theorem 2.15. *a) The algebra $K^{G(\mathbf{O})}(\mathrm{Gr}_G)$ is commutative;*

b) Its spectrum together with the projection onto T/W is naturally isomorphic to $\mathfrak{B}_G^{\check{G}} \overset{\varpi}{\to} T/W$;

c) The Poisson structure on $K^{G(\mathbf{O})}(\mathrm{Gr}_G)$ arising from the q-deformation $K^{G(\mathbf{O})\ltimes\mathbb{G}_m}(\mathrm{Gr}_G)$, corresponds under the above identification to the Poisson structure of 2.7 on $\mathfrak{B}_G^{\check{G}}$ in case the latter variety is smooth, i.e. G is simply connected.

3. Calculations in rank 1

In this section $G=SL_2$, and $\check{G}=PGL_2$. The Weyl group $W=\mathbb{Z}/2\mathbb{Z}$, the Cartan torus $T=\mathbb{G}_m=\mathbb{C}^*$ with a coordinate z, and the only simple root $\alpha(z)=z^2$. The dual torus $\check{T}=\mathbb{G}_m=\mathbb{C}^*$ with a coordinate t, and $\check{\alpha}(t)=t$. The Cartan Lie algebra $\mathfrak{t}=\mathbb{C}$ with a coordinate $x=\alpha(x)$. We fix a $\sqrt{-1}$.

3.1. **$\mathfrak{Z}_G^G$ and $\mathfrak{B}_G^G$.** We choose the standard $\mathfrak{sl}_2$-triple $e=\begin{pmatrix}0&1\\0&0\end{pmatrix}$, $h=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$, $f=\begin{pmatrix}0&0\\1&0\end{pmatrix}$. Then the Kostant slice $\Sigma_G=\left\{\begin{pmatrix}a-1&a-2\\1&1\end{pmatrix},\ a\in\mathbb{C}\right\}$. One checks that a matrix $\begin{pmatrix}x_{11}&x_{12}\\x_{21}&x_{22}\end{pmatrix}$ commutes with $\begin{pmatrix}a-1&a-2\\1&1\end{pmatrix}$ iff $\begin{pmatrix}x_{11}&x_{12}\\x_{21}&x_{22}\end{pmatrix}=\sqrt{-1}\begin{pmatrix}(1-a)c+b&(2-a)c\\-c&b-c\end{pmatrix}$ for $b,c\in\mathbb{C}$. Then the condition $\det=1$ reads as

$$1=abc-b^2-c^2. \tag{4}$$

Thus, $\mathfrak{Z}_G^G$ is identified with a hypersurface $\mathcal{S}$ in $\mathbb{A}^3$ given by the equation (4). The left (resp. right) multiplication by $-1\in Z(SL_2)$ is an involution $\imath$ (resp. $\jmath$) on $\mathcal{S}$ given by $\imath(a,b,c)=(a,-b,-c)$ (resp. $\jmath(a,b,c)=(-a,b,-c)$). Hence, $\mathfrak{Z}_{\check{G}}^{\check{G}}=\imath\backslash\mathcal{S}/\jmath$.

Generically, we can diagonalize two commuting matrices simultaneously, that is, there is $g\in SL_2$ such that $g\sqrt{-1}\begin{pmatrix}(1-a)c+b&(2-a)c\\-c&b-c\end{pmatrix}g^{-1}=\begin{pmatrix}y&0\\0&y^{-1}\end{pmatrix}$ and $g\begin{pmatrix}a-1&a-2\\1&1\end{pmatrix}g^{-1}=\begin{pmatrix}z&0\\0&z^{-1}\end{pmatrix}$ for some $y,z\in\mathbb{G}_m=\mathbb{C}^*=T$ defined up to simultaneous inversion. Then we have

$$a=z+z^{-1},\ b=\frac{-\sqrt{-1}}{2}\left(y+y^{-1}+\frac{(y-y^{-1})(z+z^{-1})}{z-z^{-1}}\right),\ c=-\sqrt{-1}\frac{y-y^{-1}}{z-z^{-1}}. \tag{5}$$

We conclude that $\mathbb{C}[\mathcal{S}]=\mathbb{C}[y^{\pm1},z^{\pm1},\frac{y-y^{-1}}{z-z^{-1}}]^W$ where the nontrivial element $w\in W$ acts by $w(y,z)=(y^{-1},z^{-1})$. We can rewrite $\mathbb{C}[y^{\pm1},z^{\pm1},\frac{y-y^{-1}}{z-z^{-1}}]^W$ as $\mathbb{C}[y^{\pm1},z^{\pm1},\frac{y^2-1}{z^2-1}]^W$ to manifest its coincidence with $\mathbb{C}[\mathfrak{B}_G^G]$. All in all, we have $\mathfrak{B}_G^G\simeq\mathcal{S}\simeq\mathfrak{Z}_G^G$. Since we can identify $\check{T}$ with $T/Z(G)$, the identifications $\mathfrak{B}^G_{\check{G}} \simeq \mathcal{S}/\jmath$, $\mathfrak{B}^{\check{G}}_G \simeq \imath\backslash\mathcal{S}$, $\mathfrak{B}^{\check{G}}_{\check{G}} \simeq \imath\backslash\mathcal{S}/\jmath \simeq \mathfrak{Z}^{\check{G}}_{\check{G}}$ follow immediately.

3.2. $\mathfrak{Z}^G_{\mathfrak{g}}$ and $\mathfrak{B}^G_{\mathfrak{g}}$. The Kostant slice $\Sigma_{\mathfrak{g}} = \left\{\begin{pmatrix} 0 & \delta \\ 1 & 0 \end{pmatrix}, \ \delta \in \mathbb{C}\right\}$. One checks that a matrix $\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$ commutes with $\begin{pmatrix} 0 & \delta \\ 1 & 0 \end{pmatrix}$ iff $\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} \xi & \delta\eta \\ \eta & \xi \end{pmatrix}$ for $\xi, \eta \in \mathbb{C}$. Then the condition $\det = 1$ reads as

$$1 = \xi^2 - \delta\eta^2. \tag{6}$$

Thus, $\mathfrak{Z}^G_{\mathfrak{g}}$ is identified with a hypersurface $\mathcal{S}'$ in $\mathbb{A}^3$ given by the equation (6). The action of $-1 \in Z(SL_2)$ is an involution $\imath$ on $\mathcal{S}'$ given by $\imath(\delta, \xi, \eta) = (\delta, -\xi, -\eta)$. Hence, $\mathfrak{Z}^{\check{G}}_{\check{\mathfrak{g}}} = \imath\backslash\mathcal{S}'$.

Generically, we can diagonalize two commuting matrices simultaneously, that is, there is $g \in SL_2$ such that $g\begin{pmatrix} \xi & \delta\eta \\ \eta & \xi \end{pmatrix}g^{-1} = \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix}$ and $g\begin{pmatrix} 0 & \delta \\ 1 & 0 \end{pmatrix}g^{-1} = \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix}$ for some $y \in \mathbb{G}_m = \mathbb{C}^* = T$, $x \in \mathbb{C} = \mathfrak{t}$, defined up to $(y, x) \mapsto (y^{-1}, -x)$. Then we have

$$\delta = x^2, \ \xi = \frac{y + y^{-1}}{2}, \ \eta = \frac{y - y^{-1}}{2x}.$$

We conclude that $\mathbb{C}[\mathcal{S}'] = \mathbb{C}[y^{\pm 1}, x, \frac{y - y^{-1}}{x}]^W$ where the nontrivial element $w \in W$ acts by $w(y, x) = (y^{-1}, -x)$. We can rewrite $\mathbb{C}[y^{\pm 1}, x, \frac{y-y^{-1}}{x}]^W$ as $\mathbb{C}[y^{\pm 1}, x, \frac{y^2-1}{x}]^W$ to manifest its coincidence with $\mathbb{C}[\mathfrak{B}^G_{\mathfrak{g}}]$. All in all, we have $\mathfrak{B}^G_{\mathfrak{g}} \simeq \mathcal{S}' \simeq \mathfrak{Z}^G_{\mathfrak{g}}$. Since we can identify $\check{T}$ with $T/Z(G)$, the identification $\mathfrak{B}^{\check{G}}_{\check{\mathfrak{g}}} \simeq \imath\backslash\mathcal{S}' \simeq \mathfrak{Z}^{\check{G}}_{\check{\mathfrak{g}}}$ follows immediately.

3.3. $\mathfrak{Z}^{\mathfrak{g}}_G$ and $\mathfrak{B}^{\mathfrak{g}}_G$. Recall the Kostant slice $\Sigma_G = \left\{\begin{pmatrix} a-1 & a-2 \\ 1 & 1 \end{pmatrix}, \ a \in \mathbb{C}\right\}$. One checks that a traceless matrix $\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & -x_{11} \end{pmatrix}$ commutes with $\begin{pmatrix} a-1 & a-2 \\ 1 & 1 \end{pmatrix}$ iff $\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & -x_{11} \end{pmatrix} = \zeta\begin{pmatrix} 2-a & 4-2a \\ -2 & a-2 \end{pmatrix}$ for $\zeta \in \mathbb{C}$.

Thus, $\mathfrak{Z}^{\mathfrak{g}}_G$ is identified with $\mathbb{A}^2$ with coordinates a, ζ. The action of $-1 \in Z(SL_2)$ is an involution $\jmath$ on $\mathbb{A}^2$ given by $\jmath(a, \zeta) = (-a, -\zeta)$. Hence, $\mathfrak{Z}^{\check{\mathfrak{g}}}_{\check{G}} = \mathbb{A}^2/\jmath$.

Generically, we can diagonalize two commuting matrices simultaneously, that is, there is $g \in SL_2$ such that $g\zeta\begin{pmatrix} 2-a & 4-2a \\ -2 & a-2 \end{pmatrix}g^{-1} = \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix}$ and $g\begin{pmatrix} a-1 & a-2 \\ 1 & 1 \end{pmatrix}g^{-1} = \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}$ for some $x \in \mathbb{C} = \mathfrak{t}$, $z \in \mathbb{G}_m = \mathbb{C}^* = T$ defined up to $(x, z) \mapsto (-x, z^{-1})$. Then we have

$$a = z + z^{-1}, \ \zeta = \frac{x}{z - z^{-1}}.$$

We conclude that $\mathbb{C}[\mathbb{A}^2] = \mathbb{C}[x, z^{\pm 1}, \frac{x}{z-z^{-1}}]^W$ where the nontrivial element $w \in W$ acts by $w(x,z) = (-x, z^{-1})$. We can rewrite $\mathbb{C}[x, z^{\pm 1}, \frac{x}{z-z^{-1}}]^W$ as $\mathbb{C}[x, z^{\pm 1}, \frac{x}{z^2-1}]^W$ to manifest its coincidence with $\mathbb{C}[\mathfrak{B}^{\mathfrak{g}}_G]$. All in all, we have $\mathfrak{B}^{\mathfrak{g}}_G \simeq \mathbb{A}^2 \simeq \mathfrak{Z}^{\mathfrak{g}}_G$. Since we can identify $\check{T}$ with $T/Z(G)$, the identification $\mathfrak{B}^{\check{\mathfrak{g}}}_{\check{G}} \simeq \mathbb{A}^2/\jmath \simeq \mathfrak{Z}^{\check{\mathfrak{g}}}_{\check{G}}$ follows immediately.

3.4. **$\mathfrak{Z}^{\mathfrak{g}}_{\mathfrak{g}}$ and $\mathfrak{B}^{\mathfrak{g}}_{\mathfrak{g}}$.** Recall the Kostant slice $\Sigma_{\mathfrak{g}} = \left\{ \begin{pmatrix} 0 & \delta \\ 1 & 0 \end{pmatrix},\ \delta \in \mathbb{C} \right\}$. One checks that a traceless matrix $\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & -x_{11} \end{pmatrix}$ commutes with $\begin{pmatrix} 0 & \delta \\ 1 & 0 \end{pmatrix}$ iff $\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & -x_{11} \end{pmatrix} = \begin{pmatrix} 0 & \delta\theta \\ \theta & 0 \end{pmatrix}$ for $\theta \in \mathbb{C}$. Thus, $\mathfrak{Z}^{\mathfrak{g}}_{\mathfrak{g}}$ is identified with $\mathbb{A}^2$ with coordinates δ, θ.

Generically, we can diagonalize two commuting matrices simultaneously, that is, there is $g \in SL_2$ such that $g \begin{pmatrix} 0 & \delta\theta \\ \theta & 0 \end{pmatrix} g^{-1} = \begin{pmatrix} u & 0 \\ 0 & -u \end{pmatrix}$ and $g \begin{pmatrix} 0 & \delta \\ 1 & 0 \end{pmatrix} g^{-1} = \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix}$ for some $u, x \in \mathbb{C} = \mathfrak{t}$, defined up to $(u,x) \mapsto (-u,-x)$. Then we have

$$\delta = x^2,\ \theta = \frac{u}{x}.$$

We conclude that $\mathbb{C}[\mathbb{A}^2] = \mathbb{C}[u, x, \frac{u}{x}]^W$ where the nontrivial element $w \in W$ acts by $w(u,x) = (-u,-x)$. Hence we get an identification $\mathfrak{B}^{\mathfrak{g}}_{\mathfrak{g}} \simeq \mathbb{A}^2 \simeq \mathfrak{Z}^{\mathfrak{g}}_{\mathfrak{g}}$.

3.5. **$\mathfrak{B}^G_{\mathfrak{g}}$ and $\mathfrak{F}^G_{\mathfrak{g}}$.** Recall the setup of Proposition 2.10. We will prove that the functor $\mathfrak{F}^G_{\mathfrak{g}}$ is representable by the scheme $\mathfrak{B}^G_{\mathfrak{g}}$; the other parts of the Proposition are proved absolutely similarly, as well as the Proposition for G replaced by $\check{G}$. For a scheme S flat over $\mathfrak{t}/W$ we will denote by $S_{\mathfrak{t}}$ the cartesian product $S \times_{\mathfrak{t}/W} \mathfrak{t}$. Our usual coordinate x on $\mathfrak{t}$ gives rise to the same named function on $S_{\mathfrak{t}}$. The nontrivial element $w \in W$ acts by the involution of $S_{\mathfrak{t}}$. Finally, we denote by $(\mathfrak{B}^G_{\mathfrak{g}})_{S_{\mathfrak{t}}}$ the affine blow-up of $S_{\mathfrak{t}} \times T$, that is $S_{\mathfrak{t}} \times_{\mathfrak{t}} \mathfrak{B}^G_{\mathfrak{g}}$. Clearly, w acts as an involution of $(\mathfrak{B}^G_{\mathfrak{g}})_{S_{\mathfrak{t}}}$.

Note that the condition (1) is void in the case under consideration. Given a w-equivariant morphism $f:\ S_{\mathfrak{t}} \to T = \mathbb{G}_m$ we see that $f^2 - 1$ is divisible by x, hence f lifts uniquely to a section $\hat{f}$ of $(\mathfrak{B}^G_{\mathfrak{g}})_{S_{\mathfrak{t}}}$ over $S_{\mathfrak{t}}$. Evidently, $\hat{f}$ is w-invariant. If we consider $\hat{f}$ as a closed subscheme of $(\mathfrak{B}^G_{\mathfrak{g}})_{S_{\mathfrak{t}}}$, then $\hat{f}/W$ is a closed subscheme of $(\mathfrak{B}^G_{\mathfrak{g}})_{S_{\mathfrak{t}}}/W = S \times_{\mathfrak{t}/W} \mathfrak{B}^G_{\mathfrak{g}}$ which is the graph of a morphism $\tilde{f}:\ S \to \mathfrak{B}^G_{\mathfrak{g}}$.

Conversely, given a morphism $\tilde{f}:\ S \to \mathfrak{B}^G_{\mathfrak{g}}$ we consider its graph $\Gamma_{\tilde{f}}$ as a closed subscheme of $S \times_{\mathfrak{t}/W} \mathfrak{B}^G_{\mathfrak{g}}$, and then the cartesian product $\Gamma_{\tilde{f}} \times_{S \times_{\mathfrak{t}/W} \mathfrak{B}^G_{\mathfrak{g}}} (\mathfrak{B}^G_{\mathfrak{g}})_{S_{\mathfrak{t}}}$ is a section $\hat{f}$ of $(\mathfrak{B}^G_{\mathfrak{g}})_{S_{\mathfrak{t}}}$ over $S_{\mathfrak{t}}$. Evidently, $\hat{f}$ gives rise to a w-equivariant function $f:\ S_{\mathfrak{t}} \to T$.

3.6. **A basis in equivariant K-theory.** We recall a few standard facts about the affine Grassmannians Gr_G and $\mathrm{Gr}_{\check{G}}$. The $G(\mathbf{O})$-orbits (equivalently, $\check{G}(\mathbf{O})$-orbits) on $\mathrm{Gr}_{\check{G}}$ are numbered by nonnegative integers and denoted by $\mathrm{Gr}_{\check{G},n}$, $n \in \mathbb{N}$. The orbits $\mathrm{Gr}_{\check{G},2n}$, $n \in \mathbb{N}$, form a connected component of $\mathrm{Gr}_{\check{G}}$ equal to Gr_G. The open embedding of an orbit into its closure will be denoted by $j_n:\ \mathrm{Gr}_{\check{G},n} \hookrightarrow \overline{\mathrm{Gr}}_{\check{G},n}$ or simply by j if no confusion is likely. We have $\dim \mathrm{Gr}_{\check{G},n} = n$; in particular, $\mathrm{Gr}_{\check{G},0}$ is a point.

We have $K^{G(\mathbf{O})}(\mathrm{Gr}_{\check{G},0}) = \mathrm{Rep}(G)$ with a basis $\mathbf{v}(n),\ n \in \mathbb{N}$, formed by the classes of irreducible G-modules $\mathcal{V}(n)$. Also, $K^{\check{G}(\mathbf{O})}(\mathrm{Gr}_{\check{G},0}) = \mathrm{Rep}(\check{G}) \subset \mathrm{Rep}(G)$ has a basis $\mathbf{v}(2n),\ n \in \mathbb{N}$.

For $m > 0$ the $G(\mathbf{O})$-equivariant line bundles in $\mathrm{Gr}_{\check{G},m}$ are numbered by integers and denoted by $\mathcal{L}(n)_m$. Among them, the $\check{G}(\mathbf{O})$-equivariant line bundles are exactly $\mathcal{L}(2n)_m,\ n \in \mathbb{Z}$. We define $\mathcal{V}(n)_m$ as $j_*\mathcal{L}(n)_m[\frac{m}{2}]$, that is, the (nonderived) direct image to the orbit closure placed in the homological degree $-\frac{m}{2}$. Note that since the complement $\overline{\mathrm{Gr}}_{\check{G},m} - \mathrm{Gr}_{\check{G},m}$ has codimension 2, the above direct image is a coherent sheaf. The degree shift will become clear later. The class $[\mathcal{L}(n)_m]$ in $K^{G(\mathbf{O})}(\mathrm{Gr}_{\check{G}})$ will be denoted by $\mathbf{v}(n)_m$. Thus, it is natural to denote $\mathbf{v}(n)$ above by $\mathbf{v}(n)_0$; we will keep both names.

The collection $\{\mathbf{v}(n)_m :\ n \in \mathbb{N} \text{ if } m = 0;\ n \in \mathbb{Z} \text{ if } m \in \mathbb{N} - 0\}$ forms a basis in $K^{G(\mathbf{O})}(\mathrm{Gr}_{\check{G}})$. Among this collection, all the $\mathbf{v}(n)_m$ with n even (resp. m even) form a basis in $K^{\check{G}(\mathbf{O})}(\mathrm{Gr}_{\check{G}})$ (resp. $K^{G(\mathbf{O})}(\mathrm{Gr}_G)$).

3.7. **Convolution: commutativity.** In this subsection G is an arbitrary semisimple group. We prove 2.15 (a). We refer the reader to [7] for the basics of Beilinson-Drinfeld Grassmannian. Recall that $\mathrm{Gr}_G^{BD} \xrightarrow{\pi} \mathbb{A}^1$ is a flat ind-scheme such that $\pi^{-1}(\mathbb{A}^1 - 0) = (\mathbb{A}^1 - 0) \times \mathrm{Gr}_G \times \mathrm{Gr}_G$, while $\pi^{-1}(0) = \mathrm{Gr}_G$. We also have the deformed convolution diagram $\mathrm{Gr}_G^{BD,conv} \xrightarrow{\Pi} \mathrm{Gr}_G^{BD}$ such that Π is an isomorphism over $\mathbb{A}^1 - 0$, while over $0 \in \mathbb{A}^1$ our Π is the usual convolution diagram $G(\mathbf{F}) \times_{G(\mathbf{O})} \mathrm{Gr}_G \xrightarrow{\Pi_0} \mathrm{Gr}_G$.

Given two $G(\mathbf{O})$-equivariant complexes of coherent sheaves $\mathcal{A}, \mathcal{B}$ on Gr_G, we can form their "deformed convolution" complex $\mathcal{A}\tilde{\star}\mathcal{B}$ on $\mathrm{Gr}_G^{BD,conv}$ such that over $\mathbb{A}^1 - 0$ it is isomorphic to $\mathcal{O}_{\mathbb{A}^1-0} \boxtimes \mathcal{A} \boxtimes \mathcal{B}$, while over $0 \in \mathbb{A}^1$ it is isomorphic to the usual twisted product $\mathcal{A} \ltimes \mathcal{B}$ on the convolution diagram $G(\mathbf{F}) \times_{G(\mathbf{O})} \mathrm{Gr}_G$. In addition, if $\mathcal{A}, \mathcal{B}$ are coherent sheaves, then $\mathcal{A}\tilde{\star}\mathcal{B}$ is flat over $\mathbb{A}^1$. It implies that in the K-group the class $[\mathcal{A} \ltimes \mathcal{B}]$ is the *specialization* (see [4] 5.3) of the class $[\mathcal{O}_{\mathbb{A}^1-0} \boxtimes \mathcal{A} \boxtimes \mathcal{B}]$ in the family $\mathrm{Gr}_G^{BD,conv} \xrightarrow{\pi\circ\Pi} \mathbb{A}^1$, and also the class $[\mathcal{A} \star \mathcal{B}] = [\Pi_{0*}(\mathcal{A} \ltimes \mathcal{B})]$ is the specialization of the class $[\mathcal{O}_{\mathbb{A}^1-0} \boxtimes \mathcal{A} \boxtimes \mathcal{B}]$ in the family $\mathrm{Gr}_G^{BD} \xrightarrow{\pi} \mathbb{A}^1$. Hence the desired commutativity.

3.8. **Convolution: relations.** We return to the setup of 3.6. Note that $\mathrm{Gr}_{\check{G},1} \simeq \mathbb{P}^1$, and $\mathcal{V}(n)_1$ is the line bundle $\mathcal{O}(n)$ on $\mathbb{P}^1$. The twisted product $\mathcal{V}(n)_1 \ltimes \mathcal{V}(l)_1$ is the line bundle $\mathcal{O}(n,l)$ on the 2-dimensional subvariety $\mathcal{H}_2 \subset \check{G}(\mathbf{F}) \times_{\check{G}(\mathbf{O})} \mathrm{Gr}_{\check{G}}$ isomorphic to the Hirzebruch surface $\mathbb{P}(\mathcal{O}(2) \oplus \mathcal{O})$ over $\mathbb{P}^1$. The projection $\Pi_0 :\ \mathcal{H}_2 \to \mathrm{Gr}_{\check{G},2}$ is the contraction of the -2-section $\mathbb{P}^1 \hookrightarrow \mathcal{H}_2$.

Now it is easy to compute $\mathbf{v}(n)_1 \star \mathbf{v}(n)_1 = \mathbf{v}(2n)_2,\ \mathbf{v}(1)_1 \star \mathbf{v}(-1)_1 = \mathbf{v}(0)_2 + 1$. Taking into account the evident relation $\mathbf{v}(1)_0 \star \mathbf{v}(0)_1 = \mathbf{v}(1)_1 + \mathbf{v}(-1)_1$ we arrive at

$$\mathbf{v}(1)_0 \star \mathbf{v}(0)_1 \star \mathbf{v}(1)_1 = \mathbf{v}(1)_1 \star \mathbf{v}(1)_1 + \mathbf{v}(0)_1 \star \mathbf{v}(0)_1 + 1. \tag{7}$$

A moment of reflection shows that $K^{G(\mathbf{O})}(\mathrm{Gr}_G)$ is generated as algebra by $\mathbf{v}(1)_0,\ \mathbf{v}(0)_2 = \mathbf{v}(0)_1 \star \mathbf{v}(0)_1,\ \mathbf{v}(2)_2 = \mathbf{v}(1)_1 \star \mathbf{v}(1)_1,\ \mathbf{v}(1)_2 = \mathbf{v}(1)_1 \star \mathbf{v}(0)_1$ (one has to use that $\mathbf{v}(k)_{2l} \star \mathbf{v}(n)_{2m} = \mathbf{v}(k+n)_{2l+2m}$ plus the terms supported on the smaller orbits). Similarly, $K^{\check{G}(\mathbf{O})}(\mathrm{Gr}_{\check{G}})$ is generated as algebra by $\mathbf{v}(2)_0=\mathbf{v}(1)_0\star\mathbf{v}(1)_0-1,\ \mathbf{v}(0)_1,\ \mathbf{v}(2)_2=\mathbf{v}(1)_1\star\mathbf{v}(1)_1,\ \mathbf{v}(2)_1=\mathbf{v}(1)_1\star\mathbf{v}(1)_0-\mathbf{v}(0)_1$.

Note that both algebras $K^{G(\mathbf{O})}(\mathrm{Gr}_G)$ and $K^{\check{G}(\mathbf{O})}(\mathrm{Gr}_{\check{G}})$ lie in the vector space $K^{G(\mathbf{O})}(\mathrm{Gr}_{\check{G}})$, and their intersection is the common subalgebra $K^{\check{G}(\mathbf{O})}(\mathrm{Gr}_G)$. The tensor product algebra $K^{G(\mathbf{O})}(\mathrm{Gr}_G)\otimes_{K^{\check{G}(\mathbf{O})}(\mathrm{Gr}_G)}K^{\check{G}(\mathbf{O})}(\mathrm{Gr}_{\check{G}})$ can be identified as a vector space with $K^{G(\mathbf{O})}(\mathrm{Gr}_{\check{G}})$, and then it is generated by the three basic elements $\mathbf{v}(1)_0,\mathbf{v}(0)_1,\mathbf{v}(1)_1$ subject to the only relation (7).

The comparison of equations (7) and (4) shows that the assignment $a\mapsto\mathbf{v}(1)_0,\ b\mapsto\mathbf{v}(0)_1,\ c\mapsto\mathbf{v}(1)_1$ establishes an isomorphism $\mathbb{C}[\mathcal{S}]\simeq K^{G(\mathbf{O})}(\mathrm{Gr}_{\check{G}})$. It identifies the spectrum of $K^{G(\mathbf{O})}(\mathrm{Gr}_G)$ with $\imath\backslash\mathcal{S}\simeq\mathfrak{B}_G^{\check{G}}$, and the spectrum of $K^{\check{G}(\mathbf{O})}(\mathrm{Gr}_{\check{G}})$ with $\mathcal{S}/\jmath\simeq\mathfrak{B}_{\check{G}}^G$.

3.9. **Iwahori-equivariant K-theory.** Let $I\subset G(\mathbf{O})$ be the Iwahori subgroup. The space $K^I(\mathrm{Gr}_G)=K^T(\mathrm{Gr}_G)=K^{T(\mathbf{O})}(\mathrm{Gr}_G)=K(T(\mathbf{O})\backslash G(\mathbf{F})/G(\mathbf{O}))$ is equipped with the two commuting actions: $K(T(\mathbf{O})\backslash T(\mathbf{F})/T(\mathbf{O}))$ acts by convolutions on the left, and $K^G(\mathrm{Gr}_G)=K^{G(\mathbf{O})}(\mathrm{Gr}_G)=K(G(\mathbf{O})\backslash G(\mathbf{F})/G(\mathbf{O}))$ acts by convolutions on the right. Also, W acts on $K^T(\mathrm{Gr}_G)$ commuting with the right action of $K^G(\mathrm{Gr}_G)$. Clearly, the algebra $K(T(\mathbf{O})\backslash T(\mathbf{F})/T(\mathbf{O}))$ is isomorphic to $\mathbb{C}[\check{T}\times T]$. The action of W on $K^T(\mathrm{Gr}_G)$ normalizes the action of $K(T(\mathbf{O})\backslash T(\mathbf{F})/T(\mathbf{O}))$ and induces the natural (diagonal) action of W on $\mathbb{C}[\check{T}\times T]$.

Our aim in this subsection is to identify the $K(T(\mathbf{O})\backslash T(\mathbf{F})/T(\mathbf{O}))\ltimes W-K^G(\mathrm{Gr}_G)$-bimodule $K^T(\mathrm{Gr}_G)$ with the $\mathbb{C}[\check{T}\times T]\ltimes W-\mathbb{C}[\mathfrak{B}_G^{\check{G}}]$-bimodule $\mathbb{C}[\dot{\mathfrak{B}}_G^{\check{G}}]$ (and similarly for G replaced by $\check{G}$). As in 3.8, it suffices to identify the $K(T(\mathbf{O})\backslash\check{T}(\mathbf{F})/\check{T}(\mathbf{O}))\ltimes W-K^G(\mathrm{Gr}_{\check{G}})$-bimodule $K^T(\mathrm{Gr}_{\check{G}})$ with the $\mathbb{C}[T\times T]\ltimes W-\mathbb{C}[\mathfrak{B}_G^G]$-bimodule $\mathbb{C}[\dot{\mathfrak{B}}_G^G]$.

Note that $K^G(\mathrm{Gr}_{\check{G}})\subset K^T(\mathrm{Gr}_{\check{G}})$, and the $K^G(\mathrm{Gr}_{\check{G}})$-module $K^T(\mathrm{Gr}_{\check{G}})$ is free of rank 2 with the generators $1,z$ where z is the generator of $K^T(pt)=\mathbb{C}[T]$ (so that, e.g. $\mathbf{v}(1)_0=z+z^{-1}$). Furthermore, $\mathbb{C}[y^{\pm1},z^{\pm1}]=\mathbb{C}[T\times T]=K(T(\mathbf{O})\backslash\check{T}(\mathbf{F})/\check{T}(\mathbf{O}))\subset K^T(\mathrm{Gr}_{\check{G}})$, and one can check

$$y^{-1}=\sqrt{-1}(\mathbf{u}_0-\mathbf{u}_2),\ y=\sqrt{-1}(\mathbf{v}(0)_1-\mathbf{v}(2)_1+\mathbf{u}_2-\mathbf{u}_0)\tag{8}$$

where $\mathbf{u}_0\in K^T(\mathrm{Gr}_{\check{G}})$ (resp. $\mathbf{u}_2$) is the class of the irreducible skyscraper sheaf supported at the one-point Iwahori orbit in $\mathrm{Gr}_{\check{G},1}=\mathbb{P}^1$ with the trivial action of T (resp. with the action of T given by z^2), and placed in the homological degree $-\frac{1}{2}$. Hence

$$y+y^{-1}=\sqrt{-1}(2\mathbf{v}(0)_1-\mathbf{v}(1)_0\star\mathbf{v}(1)_1),\ y-y^{-1}=\sqrt{-1}(z-z^{-1})\mathbf{v}(1)_1.\tag{9}$$

Comparing (9) with (5) we get the desired identification of the $K(T(\mathbf{O})\backslash\check{T}(\mathbf{F})/\check{T}(\mathbf{O}))\ltimes W-K^G(\mathrm{Gr}_{\check{G}})$-bimodule $K^T(\mathrm{Gr}_{\check{G}})$ with the $\mathbb{C}[y^{\pm1},z^{\pm1}]\ltimes W-\mathbb{C}[y^{\pm1},z^{\pm1},\frac{y-y^{-1}}{z-z^{-1}}]^W$-bimodule $\mathbb{C}[y^{\pm1},z^{\pm1},\frac{y-y^{-1}}{z-z^{-1}}]$.

3.10. **Borel-Moore Homology.** For an arbitrary semisimple G one proves the commutativity of $H_\bullet^{G(\mathbf{O})}(\mathrm{Gr}_G)$ (Theorem 2.12 a) exactly as in 3.7 using the Beilinson-Drinfeld Grassmannian and the *specialization* in Borel-Moore Homology (see [4] 2.6.30).

For $\check{G} = PGL_2$, let us denote by $\delta \in H^4_{\check{G}(\mathbf{O})}(pt, \mathbb{Z}) = H_4^{\check{G}(\mathbf{O})}(pt, \mathbb{Z})$ the generator of the equivariant (co)homology. Furthermore, we denote by η (resp. ξ) the generator of $H_{-2}^{\check{G}(\mathbf{O})}(\mathrm{Gr}_{\check{G},1}, \mathbb{Z})$ (resp. the generator of $H_0^{\check{G}(\mathbf{O})}(\mathrm{Gr}_{\check{G},1}, \mathbb{Z})$). Then it is easy to see that δ, ξ, η generate $H_\bullet^{\check{G}(\mathbf{O})}(\mathrm{Gr}_{\check{G}})$ (while $\delta, \xi^2, \eta^2, \xi\eta$ generate the subalgebra $H_\bullet^{G(\mathbf{O})}(\mathrm{Gr}_G)$), and we claim that

$$1 = \xi^2 - \delta\eta^2. \tag{10}$$

In effect, this is an equality in $H_0^{\check{G}(\mathbf{O})}(\mathrm{Gr}_{\check{G},2})$. Since $\mathrm{Gr}_{\check{G},2}$ is rationally smooth, $H_0^{\check{G}(\mathbf{O})}(\mathrm{Gr}_{\check{G},2}) = H^4_{\check{G}(\mathbf{O})}(\mathrm{Gr}_{\check{G},2})$. Let us denote by $\mathbf{B}\,\mathrm{Gr}_{\check{G},2} \xrightarrow{p} \mathbf{B}\check{G}(\mathbf{O})$ the associated fibre bundle over the classifying space of $\check{G}(\mathbf{O})$ with the fiber $\mathrm{Gr}_{\check{G},2}$. Then $1 \in H^4_{\check{G}(\mathbf{O})}(\mathrm{Gr}_{\check{G},2}) = H^4(\mathbf{B}\,\mathrm{Gr}_{\check{G},2})$ is the Poincaré dual class of the codimension 2 cycle $\mathbf{B}\check{G}(\mathbf{O}) = \mathbf{B}\,\mathrm{Gr}_{\check{G},0} \hookrightarrow \mathbf{B}\,\mathrm{Gr}_{\check{G},2}$, and $\delta\eta^2 = p^*\delta$.

Recall the convolution morphism $\Pi_0 : \mathcal{H}_2 \to \mathrm{Gr}_{\check{G},2}$ of 3.8. This is a morphism of $\check{G}(\mathbf{O})$-varieties, and we denote by $\Pi_0 : \mathbf{B}\mathcal{H}_2 \to \mathbf{B}\,\mathrm{Gr}_{\check{G},2}$ the corresponding morphism of associated fibre bundles. Note that (additively) $H^\bullet(\mathbf{B}\mathcal{H}_2) = H^\bullet(\mathbf{B}\,\mathrm{Gr}_{\check{G},1}) \otimes_{H^\bullet(\mathbf{B}\check{G}(\mathbf{O}))} H^\bullet(\mathbf{B}\,\mathrm{Gr}_{\check{G},1})$. Recall that ξ is the generator of $H_0^{\check{G}(\mathbf{O})}(\mathrm{Gr}_{\check{G},1}) = H^2_{\check{G}(\mathbf{O})}(\mathrm{Gr}_{\check{G},1}) = H^2(\mathbf{B}\,\mathrm{Gr}_{\check{G},1})$. Finally, we have $\xi^2 = \Pi_{0*}(\xi \otimes \xi)$. Now (10) follows easily.

Comparing the sizes of $H_\bullet^{\check{G}(\mathbf{O})}(\mathrm{Gr}_{\check{G}})$ and $\mathbb{C}[\delta, \xi, \eta]/(\xi^2 - \delta\eta^2 - 1)$ we conclude that they are isomorphic. The comparison with the equation (6) establishes an isomorphism $H_\bullet^{\check{G}(\mathbf{O})}(\mathrm{Gr}_{\check{G}}) \simeq \mathbb{C}[\mathcal{S}']$, and identifies the spectrum of $H_\bullet^{\check{G}(\mathbf{O})}(\mathrm{Gr}_{\check{G}})$ with $\mathcal{S}' \simeq \mathfrak{Z}_{\mathfrak{g}}^G$, and the spectrum of $H_\bullet^{G(\mathbf{O})}(\mathrm{Gr}_G)$ with $\imath\backslash\mathcal{S}' \simeq \mathfrak{Z}_{\check{\mathfrak{g}}}^{\check{G}}$.

4. Centralizers and blow-ups

The aim of this section is a proof of Proposition 2.8. We will consider $\mathfrak{B}_G^G$ and $\mathfrak{Z}_G^G$, the other cases being similar. Till the further notice G is assumed simply connected.

Lemma 4.1. *$\varpi : \mathfrak{B}_G^G \to T/W$ is flat.*

Proof It suffices to prove that the first projection of $\dot{\mathfrak{B}}_G^G$ to T is smooth (recall that $\dot{\mathfrak{B}}_G^G$ is defined as $\mathrm{Spec}(\mathbb{C}[T \times T, \frac{{}^1\alpha - 1}{{}^2\alpha - 1}, \alpha \in R]))$. In effect, then $\mathbb{C}[T \times T, \frac{{}^1\alpha-1}{{}^2\alpha-1}, \alpha \in R]$ is a flat $\mathbb{C}[T]$-module; hence it is a flat $\mathbb{C}[T]^W$-module (since $\mathbb{C}[T]$ is free over $\mathbb{C}[T]^W$, see [21]). Finally, $\mathbb{C}[T \times T, \frac{{}^1\alpha-1}{{}^2\alpha-1}, \alpha \in R]^W$ is a direct summand of a flat $\mathbb{C}[T]^W$-module $\mathbb{C}[T \times T, \frac{{}^1\alpha-1}{{}^2\alpha-1}, \alpha \in R]$; hence it is flat.

The affine blow-up $\dot{\mathfrak{B}}_G^G$ is the result of the following successive blow up of $T \times T$. We choose an ordering $\alpha_1, \ldots, \alpha_\nu$ of the set of positive roots R^+. We define $\mathfrak{B}_1$ as the blow up of $T\times T$ at the diagonal wall ${}^1\alpha_1 = {}^2\alpha_1 = 1$ with the proper preimage of the divisor ${}^1\alpha_1 = 1$ removed. We define $\mathfrak{B}_2$ as the blow up of $\mathfrak{B}_1$ at the proper transform of the diagonal wall ${}^1\alpha_2 = {}^2\alpha_2 = 1$ with the proper preimage of the divisor ${}^1\alpha_2 = 1$ removed. Going on like this we construct $\mathfrak{B}_\nu$; evidently, it coincides with $\dot{\mathfrak{B}}^G_G$.

Note that at each step the center of the blow-up is smooth over the corresponding wall ${}^2\alpha_i = 1$ in T by the following Claim. Thus the desired flatness assertion follows inductively from the

Claim. Let $p:\ X\to Y$ be a smooth morphism of smooth varieties; let $X'\subset X$ be a subvariety such that $Y' = f(X')\subset Y$ is a smooth hypersurface, and $p:\ X'\to Y'$ is also smooth. Then the blow-up $\mathrm{Bl}_{X'}X$ with the proper preimage of the divisor $p^{-1}(p(X'))$ removed is smooth over Y.

The smoothness is checked in the formal neighbourhoods of points by direct calculation in coordinates. This completes the proof of the lemma.

4.2. **The simultaneous resolution.** Recall that $\{(g,B):\ g\in B\} = \dot{G}\xrightarrow{p} G$ is the Grothendieck simultaneous resolution; here B is a Borel subgroup, and $p(g,B) = g$. We also have the projection $\varrho:\ \dot{G}\to T$ to the abstract Cartan, which we identify with T; namely, $\varrho(g,B) = g \pmod{\mathrm{rad}(B)}$. The preimage $p^{-1}(\Sigma_G)\subset\dot{G}$ is identified with T by ϱ. We denote by $\dot{\mathfrak{Z}}^G_G\subset G\times\dot{G}$ the subset of triples (g_1,g_2,B) such that $Ad_{g_1} = g_2$ and $(g_2,B)\in p^{-1}(\Sigma_G)$. Note that necessarily $g_1\in B$ (as well as $g_2\in B$); hence we have the projections $\varrho_1,\varrho_2:\ \dot{\mathfrak{Z}}^G_G\to T$; namely, $\varrho_i(g_1,g_2,B) = g_i \pmod{\mathrm{rad}(B)}$.

The natural projection $\dot{\mathfrak{Z}}^G_G\to\mathfrak{Z}^G_G$ (forgetting B) is a Galois W-covering. Finally, $\varrho_2:\ \dot{\mathfrak{Z}}^G_G\to T$ is flat.

4.3. **The proof of Proposition 2.8.** In order to identify $\mathfrak{Z}^G_G$ and $\mathfrak{B}^G_G$ it suffices to identify their Galois W-coverings $\dot{\mathfrak{Z}}^G_G\to T$ and $\dot{\mathfrak{B}}^G_G\to T$ in an equivariant way. Let $\mathbf{D}\subset T$ denote the discriminant, so that $T-\mathbf{D} = T^{reg}$. Let $\Delta\in\mathbb{C}[T]^W$ denote the product $\prod_{\alpha\in R}(\alpha-1)$, so that $\mathbf{D}$ is the divisor cut out by Δ.

Evidently, both $\dot{\mathfrak{Z}}^G_G|_{T^{reg}}$ and $\dot{\mathfrak{B}}^G_G|_{T^{reg}}$ are isomorphic to $T\times T^{reg}$. Hence both $\mathbb{C}[\dot{\mathfrak{Z}}^G_G]$ and $\mathbb{C}[\dot{\mathfrak{B}}^G_G]$ are the flat $\mathbb{C}[T]$-modules embedded into $\mathbb{C}[T\times T](\Delta^{-1})$. We must prove that the identification of $\dot{\mathfrak{Z}}^G_G|_{T^{reg}}$ and $\dot{\mathfrak{B}}^G_G|_{T^{reg}}$ extends to the identification over the whole T. To this end it suffices to check that the identification extends over the codimension 1 points of T (indeed, for a flat quasi-coherent sheaf $\mathcal{F}$ on a normal irreducible scheme we have $\mathcal{F}\xrightarrow{\sim} j_*j^*\mathcal{F}$ if j is an open imbedding with complement of codimension 2). Let $g\in T$ be a regular point of $\mathbf{D}$; that is, g is a semisimple element of G such that the centralizer $Z(g)$ has semisimple rank 1.

We must construct an isomorphism between localizations $(\dot{\mathfrak{Z}}^G_G)_g$ and $(\dot{\mathfrak{B}}^G_G)_g$ which is compatible with the above isomorphism at the generic point. To this end note that the embedding of reductive groups $Z(g)\hookrightarrow G$ (note that $Z(g)$ is connected since G is simply connected) induces the morphisms $\overset{\bullet}{\mathfrak{Z}}{}^{Z(g)}_{Z(g)} \to \overset{\bullet}{\mathfrak{Z}}{}^{G}_{G}$ and $\overset{\bullet}{\mathfrak{B}}{}^{Z(g)}_{Z(g)} \to \overset{\bullet}{\mathfrak{B}}{}^{G}_{G}$ which become isomorphisms after localizations: $\left(\overset{\bullet}{\mathfrak{Z}}{}^{Z(g)}_{Z(g)}\right)_g \simeq \left(\overset{\bullet}{\mathfrak{Z}}{}^{G}_{G}\right)_g$ and $\left(\overset{\bullet}{\mathfrak{B}}{}^{Z(g)}_{Z(g)}\right)_g \simeq \left(\overset{\bullet}{\mathfrak{Z}}{}^{G}_{G}\right)_g$. Now the desired identification $\left(\overset{\bullet}{\mathfrak{Z}}{}^{Z(g)}_{Z(g)}\right)_g \simeq \left(\overset{\bullet}{\mathfrak{B}}{}^{Z(g)}_{Z(g)}\right)_g$ follows from the calculations in 3.1.

This completes the identification $\mathfrak{Z}^G_G \simeq \mathfrak{B}^G_G$ for a simply connected G. Evidently, this identification respects the left and right actions of the center $Z(G)$, so the isomorphism for an arbitrary G follows from the one for its universal cover. The other isomorphisms in 2.8 (b) are proved in a similar way.

To prove 2.8 (c), (d) it suffices to notice that the minimal level (viewed as a W-equivariant homomorphism $T \twoheadrightarrow \check{T}$) for a simply laced simply connected G identifies $\check{T}$ with $T/Z(G)$; also, $\check{G} = G/Z(G)$.

5. W-invariant sections and blow-ups

The aim of this section is a proof of Proposition 2.10. We concentrate on the last statement, the other being completely similar.

Let $T^{reg} \subset T$, $T^{reg}_\alpha \subset T$ be the open subschemes defined by $T^{reg} = \{t \mid \alpha(t) \neq 1$ for all roots $\alpha\}$; $T^{reg}_\alpha = \{t \mid \beta(t) \neq 1$ for all roots $\beta \neq \alpha\}$; and $\overset{\circ}{T} = \bigcup_\alpha T^{reg}_\alpha$ (thus $T - \overset{\circ}{T}$ has codimension 2 in T (where the empty subscheme in a curve is considered to be of codimension 2)). Notice that since G is simply connected the action of W on T^{reg} is free.

We start with a

Lemma 5.1. *The map* $\overset{\bullet}{\mathfrak{B}}{}^{\check{G}}_{G} \times_T \overset{\circ}{T} \to \mathfrak{B}^{\check{G}}_G/W \times_{T/W} \overset{\circ}{T}$ *is an isomorphism.*

Proof Let $X \to Y$ be a flat morphism of semi-separated (which means that the diagonal embedding is affine) schemes of finite type over a characteristic zero field, and let a finite group W act on X, Y so that the map is W-equivariant. Assume that Y is flat over Y/W. We then claim that the map $X \to X/W \times_{Y/W} Y$ is an isomorphism provided that for every Zariski point $y \in Y$ the action of $\mathrm{Stab}_W(y)$ on the scheme-theoretic fiber X_y is trivial (here X/W, Y/W stand for categorical quotients). To check this claim we can assume X is affine: by semi-separatedness every W-invariant subset in X has a W-invariant affine neighborhood. Let us first assume also that Y/W is a point; then (by replacing Y by its connected component, and W by the stabilizer of that component) we can assume that Y is nilpotent. Then $\mathcal{O}_X$ is free over $\mathcal{O}_Y$, and the generators of $\mathcal{O}_X$ as an $\mathcal{O}_Y$ module can be chosen to be W-invariant (by semi-simplicity of the W action on $\mathcal{O}_X$, and triviality of the W-action on $\mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathsf{k}$); since $\mathcal{O}_Y^W = \mathsf{k}$ (where k is the base field) we see that $\mathcal{O}_X^W \otimes \mathcal{O}_Y \xrightarrow{\sim} \mathcal{O}_X$ as claimed. Now for a general Y we see that the morphism in question is a morphism of flat schemes of finite type over Y/W, which induces an isomorphism on every fiber; and such a morphism is necessarily an isomorphism.

Now it remains to check that the above conditions hold for $X = \overset{\bullet}{\mathfrak{B}}{}^{\check{G}}_{G} \times_T \overset{\circ}{T}$, $Y = T$. For $y \in T^{reg}$ the stabilizer of y is trivial, so there is nothing to check. Consider now

$y \in T_\alpha^{reg}$, $y \notin T^{reg}$. Then the stabilizer of y is $\{1, s_\alpha\}$. The ring of functions on $\overset{\bullet}{\mathfrak{B}}{}_G^{\check{G}}$ is generated by ${}^1\check{\lambda}$, ${}^2\mu$, t_α where $\check{\lambda}$, μ run over weights of $\check{T}$, T respectively, $\alpha \in R^+$, and $t_\alpha({}^2\alpha - 1) = {}^1\check{\alpha} - 1$. We have $s_\alpha^*({}^1\check{\lambda}) = {}^1\check{\lambda} \cdot ({}^1\check{\alpha})^{\langle -\alpha, \check{\lambda}\rangle}$, $s_\alpha^*({}^2\mu) = {}^2\mu \cdot ({}^2\alpha)^{\langle -\mu, \check{\alpha}\rangle}$, and $s_\alpha^*(t_\alpha) = t_\alpha \cdot \frac{{}^2\alpha}{{}^1\check{\alpha}}$. On the fiber we have ${}^2\alpha = 1$, hence ${}^1\check{\alpha} = 1$, so the action of s_α on the fiber is trivial. □

Proposition 2.10 clearly follows from the $(ii) \iff (iv)$ part of the next

Proposition 5.2. *Let $S \to T/W$ be a flat morphism, and set $\phi : S \times_{T/W} T^{reg}/W \to (\check{T} \times T)/W$ be a T^{reg}/W-morphism. Then the following are equivalent:*

(i) ϕ extends to a morphism $S \times_{T/W} \overset{\circ}{T}/W \to \mathfrak{B}_G^{\check{G}} \times_{T/W} \overset{\circ}{T}$.

(ii) ϕ extends to a morphism $S \to \mathfrak{B}_G^{\check{G}}$.

(iii) For every $\alpha \in R$ the morphism $\phi \times id_{T^{reg}} : S \times_{T/W} T^{reg} \to \check{T} \times T^{reg}$ extends to a morphism $S \times_{T/W} T_\alpha^{reg} \to \check{T} \times T_\alpha^{reg}$ such that (3) *holds.*

(iv) $\phi \times id_{T^{reg}} : S \times_{T/W} T^{reg} \to \check{T} \times T^{reg}$ extends to a morphism $S \times_{T/W} T \to \check{T} \times T$, such that (3) *holds for every $\alpha \in R$.*

Proof It is enough to assume that S is affine. Indeed, a morphism from S extends iff its restriction to every affine open in S does, because compatibility on intersections follows from uniqueness of such an extension; this uniqueness follows from flatness: if S is flat affine, then tensoring the injection $\mathcal{O} \to j_*\mathcal{O}$ with $\mathcal{O}_S$ we get an imbedding $\mathcal{O}_S \hookrightarrow j_*j^*\mathcal{O}_S$, where j stands for the imbedding $T^{reg}/W \to T/W$, or $T^{reg} \to T$. So we will assume S affine from now on.

(iv) ⇒ (iii) and (ii) ⇒ (i) are obvious.

To check that (iii) ⇒ (iv) we tensor (over $\mathcal{O}_{T/W}$) the exact sequence of $\mathcal{O}_T$-modules

$$0 \to \mathcal{O} \to \mathcal{O}_{T^{reg}} \to \bigoplus_\alpha (\mathcal{O}_{T_\alpha^{reg}}/\mathcal{O}_T) \tag{11}$$

with $\mathcal{O}_S$. The resulting exact sequence shows that a regular function on $S \times_{T/W} T^{reg}$ extends to a regular function on $S \times_{T/W} T$ iff it extends to $S \times_T T_\alpha^{reg}$ for all α. Applying this observation to $(\phi \times \mathrm{id})^*(f|_{\check{T} \times T^{reg}})$ for each regular function f on $\check{T} \times T$ we see that (iii) implies extendability of $\phi \times \mathrm{id}$ to $S \times_{T/W} T$. It is also clear that (3) holds if it holds on $\overset{\circ}{T}$.

Verification of (i) ⇒ (ii) is similar (with (11) replaced by the W-invariant part of (11)).

It remains to check (i) $\iff$ (iii). If (i) holds, i.e. ϕ extends to a map $S \times_{T/W} \overset{\circ}{T}/W \to \mathfrak{B}_G^{\check{G}} \times_{T/W} \overset{\circ}{T}$ then we can take the fiber product of this map with $\mathrm{id}_{\overset{\circ}{T}}$ over T/W. By Lemma 5.1 it yields a map $S \times_{T/W} \overset{\circ}{T} \to \overset{\bullet}{\mathfrak{B}}{}_G^{\check{G}} \times_T \overset{\circ}{T}$, which can be composed with the projection $\overset{\bullet}{\mathfrak{B}}{}_G^{\check{G}} \to \check{T} \times T$ to produce a map $S \times_{T/W} \overset{\circ}{T} \to \check{T} \times \overset{\circ}{T}$. It is clear that this map satisfies (3), because the image of the map $\overset{\bullet}{\mathfrak{B}}{}_G^{\check{G}} \to \check{T} \times T$ intersected with $\check{T} \times \mathrm{Ker}({}^2\alpha)$ is contained in $\mathrm{Ker}({}^1\check{\alpha}) \times T$.

Conversely, if (iii) holds then restricting the given map $S \times_{T/W} \overset{\circ}{T} \to \check{T} \times \overset{\circ}{T}$ to $S \times_{T/W} (\mathrm{Ker}(\alpha) \cap \overset{\circ}{T})$ we get a map into $\mathrm{Ker}(\check{\alpha}) \times T$ (this is immediate from (3)). This means that the map lifts to a map into $\overset{\bullet}{\mathfrak{B}}{}^{\check{G}}_{G}$. Replacing both the source and the target by their quotients by W we get the map required in (i). □

6. K-theory and blow-ups

The aim of this section is a proof of Proposition 2.15. Recall that 2.15 (a) was already proved in 3.7. G is assumed simply connected till the further notice.

6.1. **Reminder on the affine Grassmannians.** Let $X = X_G$ be the lattice of characters of T, and let $Y = Y_G$ be the lattice of cocharacters of G. Note that $X_G = Y_{\check{G}}$, $Y_G = X_{\check{G}}$. Let $X^+ \subset X$ (resp. $Y^+ \subset Y$) be the cone of dominant weights (resp. dominant coweights). It is well known that the $G(\mathbf{O})$-orbits in Gr_G are numbered by the dominant coweights: $\mathrm{Gr}_G = \bigsqcup_{\check{\lambda} \in Y^+} \mathrm{Gr}_{G,\check{\lambda}}$. The adjacency relation of orbits corresponds to the standard partial order on coweights: $\overline{\mathrm{Gr}}_{G,\check{\lambda}} = \bigsqcup_{\check{\mu} \leq \check{\lambda}} \mathrm{Gr}_{G,\check{\mu}}$. The open embedding $\mathrm{Gr}_{G,\check{\lambda}} \hookrightarrow \overline{\mathrm{Gr}}_{G,\check{\lambda}}$ will be denoted by $j_{\check{\lambda}}$ or simply by j if no confusion is likely. The dimension $\dim(\mathrm{Gr}_{G,\check{\lambda}}) = \langle 2\rho, \check{\lambda} \rangle$ where $2\rho = \sum_{\alpha \in R^+} \alpha$, and $\langle , \rangle : \ X \times Y \to \mathbb{Z}$ is the canonical perfect pairing.

Recall that the T-fixed points in Gr_G are naturally numbered by Y; a point $\check{\mu}$ lies in an orbit $\mathrm{Gr}_{G,\check{\lambda}}$ iff $\check{\mu}$ lies in the W-orbit of $\check{\lambda}$. Each $G(\mathbf{O})$-orbit $\mathrm{Gr}_{G,\check{\lambda}}$ is partitioned into Iwahori orbits isomorphic to affine spaces and numbered by $\check{\mu} \in W\check{\lambda}$. Hence the basics of [4] Chapter 5 are applicable in our situation.

In particular, $K^T(\mathrm{Gr}_{G,\check{\lambda}})$ is a free $K^T(pt)$-module, and $K^{G(\mathbf{O})}(\mathrm{Gr}_{G,\check{\lambda}}) = K^G(\mathrm{Gr}_{G,\check{\lambda}})$ is a free $K^G(pt)$-module (recall that $K^T(pt) = \mathbb{C}[T]$, and $K^G(pt) = \mathbb{C}[T/W]$). Moreover, the natural map $K^T(pt) \otimes_{K^G(pt)} K^G(\mathrm{Gr}_{G,\check{\lambda}}) \to K^T(\mathrm{Gr}_{G,\check{\lambda}})$ is an isomorphism, and $K^G(\mathrm{Gr}_{G,\check{\lambda}}) = K^T(\mathrm{Gr}_{G,\check{\lambda}})^W$, cf. [4] 6.1.22.

Since $K^{T(\mathbf{O})}(\mathrm{Gr}_G) = K^T(\mathrm{Gr}_G)$ (resp. $K^{G(\mathbf{O})}(\mathrm{Gr}_G) = K^G(\mathrm{Gr}_G)$) is filtered by the support in $G(\mathbf{O})$-orbit closures, with the associated graded $\bigoplus_{\check{\lambda} \in Y^+} K^T(\mathrm{Gr}_{G,\check{\lambda}})$ (resp. $\bigoplus_{\check{\lambda} \in Y^+} K^G(\mathrm{Gr}_{G,\check{\lambda}})$), we arrive at the following

Lemma 6.2. *$K^{T(\mathbf{O})}(\mathrm{Gr}_G) = K^T(\mathrm{Gr}_G)$ is a flat $K^T(pt)$-module, and $K^{G(\mathbf{O})}(\mathrm{Gr}_G) = K^G(\mathrm{Gr}_G)$ is a flat $K^G(pt)$-module. Moreover, the natural map $K^T(pt) \otimes_{K^G(pt)} K^G(\mathrm{Gr}_G) \to K^T(\mathrm{Gr}_G)$ is an isomorphism, and $K^G(\mathrm{Gr}_G) = \big(K^T(\mathrm{Gr}_G)\big)^W$.*

6.3. **Localization.** The space $K^T(\mathrm{Gr}_G) = K^{T(\mathbf{O})}(\mathrm{Gr}_G) = K(T(\mathbf{O})\backslash G(\mathbf{F})/G(\mathbf{O}))$ is equipped with the two commuting actions: $K(T(\mathbf{O})\backslash T(\mathbf{F})/T(\mathbf{O}))$ acts by convolutions on the left, and $K^G(\mathrm{Gr}_G) = K^{G(\mathbf{O})}(\mathrm{Gr}_G) = K(G(\mathbf{O})\backslash G(\mathbf{F})/G(\mathbf{O}))$ acts by convolutions on the right. Also, W acts on $K^T(\mathrm{Gr}_G)$ commuting with the right action of $K^G(\mathrm{Gr}_G)$. Clearly, the algebra $K(T(\mathbf{O})\backslash T(\mathbf{F})/T(\mathbf{O}))$ is isomorphic to $\mathbb{C}[\check{T} \times T]$. The action of W on $K^T(\mathrm{Gr}_G)$ normalizes the action of $K(T(\mathbf{O})\backslash T(\mathbf{F})/T(\mathbf{O}))$ and induces the natural (diagonal) action of W on $\mathbb{C}[\check{T} \times T]$.

Let g be a general (regular) element of T. Then the fixed point set $(\mathrm{Gr}_G)^g = (\mathrm{Gr}_G)^T = Y$ coincides with the image of the embedding $\mathrm{Gr}_T \hookrightarrow \mathrm{Gr}_G$. According to Thomason Localization Theorem (see e.g. [4] 5.10), after localization, $\left(K^T(\mathrm{Gr}_G)\right)_g$ becomes a free rank one $(K(T(\mathbf{O})\backslash T(\mathbf{F})/T(\mathbf{O})))_g$-module. This means that after restriction to $T^{reg} \subset T = \mathrm{Spec}(K^T(pt))$ we have an isomorphism $K^T(\mathrm{Gr}_G)|_{T^{reg}} \simeq \mathbb{C}[\check{T} \times T]|_{T^{reg}}$ compatible with the natural W-actions. The localized algebra $K^{G(\mathbf{O})}(\mathrm{Gr}_G)|_{T^{reg}/W}$ is embedded into $\left(\mathrm{End}_{K(T(\mathbf{O})\backslash T(\mathbf{F})/T(\mathbf{O}))|_{T^{reg}}}(K^T(\mathrm{Gr}_G)|_{T^{reg}})\right)^W$. According to Lemma 6.2, $K^G(\mathrm{Gr}_G) = \left(K^T(\mathrm{Gr}_G)\right)^W$; hence this embedding is an isomorphism, and we have $K^{G(\mathbf{O})}(\mathrm{Gr}_G)|_{T^{reg}/W} \simeq \mathbb{C}[\check{T} \times T]^W|_{T^{reg}/W}$.

Hence both $\mathbb{C}[\mathfrak{B}_G^{\check{G}}]$ and $K^{G(\mathbf{O})}(\mathrm{Gr}_G)$ are the flat $\mathbb{C}[T]^W$-modules embedded into $\mathbb{C}[\check{T} \times T](\Delta^{-1})$ (see 4.3). We must prove that the identification of $\mathbb{C}[\mathfrak{B}_G^{\check{G}}]|_{T^{reg}/W}$ and $K^{G(\mathbf{O})}(\mathrm{Gr}_G)|_{T^{reg}/W}$ extends to the identification over the whole T/W. To this end it suffices to check that the identification extends over the codimension 1 points of T/W. Let $g \in T/W$ be a regular point of $\mathbf{D}$; that is, g is represented by a semisimple element of G such that the centralizer $Z(g)$ has semisimple rank 1.

We must prove that the localizations $\mathbb{C}[\mathfrak{B}_G^{\check{G}}]_g$ and $\left(K^{G(\mathbf{O})}(\mathrm{Gr}_G)\right)_g$ are isomorphic. To this end it suffices to identify $\mathbb{C}\left[\check{T} \times T,\ \frac{1}{2}\frac{\check{\alpha}-1}{\alpha-1},\ \alpha \in R\right]_g$ (which we denote by $\mathbb{C}[\dot{\mathfrak{B}}_G^{\check{G}}]_g$ for short) and $\left(K^T(\mathrm{Gr}_G)\right)_g$. Note that the embedding of reductive groups $Z(g) \hookrightarrow G$ (the neutral connected component) induces the isomorphism $\mathrm{Gr}_{Z(g)} = (\mathrm{Gr}_G)^g \hookrightarrow \mathrm{Gr}_G$. According to Thomason Localization Theorem, we have an isomorphism of localizations $\left(K^T(\mathrm{Gr}_{Z(g)})\right)_g \simeq \left(K^T(\mathrm{Gr}_G)\right)_g$. Finally, the isomorphism $K^T\left(\mathrm{Gr}_{Z(g)}\right) \simeq \mathbb{C}\left[\dot{\mathfrak{B}}_{Z(g)}^{\check{Z}(g)}\right]$ follows from the calculations in 3.8, 3.9, and together with the evident isomorphism of localizations $\mathbb{C}\left[\dot{\mathfrak{B}}_{Z(g)}^{\check{Z}(g)}\right]_g \simeq \mathbb{C}\left[\dot{\mathfrak{B}}_G^{\check{G}}\right]_g$ establishes the desired isomorphism $\left(K^T(\mathrm{Gr}_G)\right)_g \simeq \mathbb{C}\left[\dot{\mathfrak{B}}_G^{\check{G}}\right]_g$.

This completes the proof of 2.15 (b).

6.4. **Comparison of Poisson structures.** In order to compare the Poisson structures on $K^{G(\mathbf{O})}(\mathrm{Gr}_G)$ and $\mathbb{C}[\mathfrak{B}_G^{\check{G}}]$ it suffices to identify them on the open subset $K^{G(\mathbf{O})}(\mathrm{Gr}_G)|_{T^{reg}/W} = \mathbb{C}[\mathfrak{B}_G^{\check{G}}]|_{T^{reg}/W} = \mathbb{C}[\check{T} \times T^{reg}]^W$. The space

$$K^{T\times\mathbb{G}_m}(\mathrm{Gr}_G) = K^{T(\mathbf{O})\ltimes\mathbb{G}_m}(\mathrm{Gr}_G) = K\left(T(\mathbf{O}) \ltimes \mathbb{G}_m \backslash G(\mathbf{F}) \ltimes \mathbb{G}_m / G(\mathbf{O}) \ltimes \mathbb{G}_m\right)$$

is equipped with the two commuting actions: $K\left(T(\mathbf{O}) \ltimes \mathbb{G}_m \backslash T(\mathbf{F}) \ltimes \mathbb{G}_m / T(\mathbf{O}) \ltimes \mathbb{G}_m\right)$ acts by convolutions on the left, and

$$K^{G\times\mathbb{G}_m}(\mathrm{Gr}_G) = K^{G(\mathbf{O})\ltimes\mathbb{G}_m}(\mathrm{Gr}_G) = K\left(G(\mathbf{O}) \ltimes \mathbb{G}_m \backslash G(\mathbf{F}) \ltimes \mathbb{G}_m / G(\mathbf{O}) \ltimes \mathbb{G}_m\right)$$

acts by convolutions on the right. Also, W acts on $K^{T(\mathbf{O})\ltimes\mathbb{G}_m}(\mathrm{Gr}_G)$ commuting with the right action of $K^{G(\mathbf{O})\ltimes\mathbb{G}_m}(\mathrm{Gr}_G)$. Clearly, the algebra

$K\left(T(\mathbf{O})\ltimes\mathbb{G}_m\backslash T(\mathbf{F})\ltimes\mathbb{G}_m/T(\mathbf{O})\ltimes\mathbb{G}_m\right)$ is isomorphic to the group algebra $\mathbb{C}[\Gamma]$ of the following Heisenberg group Γ.

It is a $\mathbb{Z}$-central extension of $Y\times X$ with the multiplication (written multiplicatively)

$$(q^{n_1},e^{\check\lambda_1},e^{\mu_1})\cdot(q^{n_2},e^{\check\lambda_2},e^{\mu_2})=(q^{n_1+n_2+\langle\mu_1,\check\lambda_2\rangle},e^{\check\lambda_1+\check\lambda_2},e^{\mu_1+\mu_2})$$

where $\langle,\rangle:\ X\times Y\to\mathbb{Z}$ is the canonical perfect pairing.

Finally, the action of the Weyl group W on $K^{T(\mathbf{O})\ltimes\mathbb{G}_m}(\mathrm{Gr}_G)$ normalizes the action of $K\left(T(\mathbf{O})\ltimes\mathbb{G}_m\backslash T(\mathbf{F})\ltimes\mathbb{G}_m/T(\mathbf{O})\ltimes\mathbb{G}_m\right)$ and induces the natural (diagonal) action of W on $\mathbb{C}[\Gamma]$. From this we deduce, exactly as in 6.3, that $K^{G(\mathbf{O})\ltimes\mathbb{G}_m}(\mathrm{Gr}_G)|_{T^{reg}/W}\simeq\mathbb{C}[\Gamma]|_{T^{reg}/W}$. It follows that the Poisson structure on $K^{G(\mathbf{O})}(\mathrm{Gr}_G)|_{T^{reg}/W}$ coincides with the standard Poisson structure on $\mathbb{C}[\check T\times T^{reg}]^W$.

This completes the proof of 2.15 (c).

6.5. **The case of non simply connected G.** For general G let $\tilde G$ denote its universal cover, and let $\tilde T$ stand for the Cartan of $\tilde G$. Note that the dual torus is $\check{\tilde T}/\pi_1(G)$. As in 6.3, we have $K^G(\mathrm{Gr}_G)=\left(\mathrm{End}_{K(T(\mathbf{O})\backslash T(\mathbf{F})/T(\mathbf{O}))}(K^T(\mathrm{Gr}_G))\right)^W$, so it suffices to identify the $K(T(\mathbf{O})\backslash T(\mathbf{F})/T(\mathbf{O}))\ltimes W=\mathbb{C}[\check T\times T]\ltimes W$-module $K^T(\mathrm{Gr}_G)$ with $\mathbb{C}\left[\check T\times T,\ \frac{1}{2}\frac{\check\alpha-1}{\alpha-1},\ \alpha\in R\right]=\mathrm{Spec}\,\mathbb{C}[\dot{\mathfrak{B}}^{\check G}_{G}]$. We do this by reduction to the known case of $\tilde G$.

Evidently, the $K(T(\mathbf{O})\backslash T(\mathbf{F})/T(\mathbf{O}))\ltimes W=\mathbb{C}[\check T\times T]\ltimes W$-module $K^T(\mathrm{Gr}_G)$ equals $\mathbb{C}[\check T\times T]\ltimes W\otimes_{\mathbb{C}[(\check T/\pi_1(G))\times T]\ltimes W}K^T(\mathrm{Gr}_{\tilde G})$. On the other hand, it follows from 6.3 that the $K(T(\mathbf{O})\backslash\tilde T(\mathbf{F})/\tilde T(\mathbf{O}))\ltimes W=\mathbb{C}[(\check T/\pi_1(G))\times T]\ltimes W$-module $K^T(\mathrm{Gr}_{\tilde G})$ equals the invariants of $\pi_1(G)$ in $K^{\tilde T}(\mathrm{Gr}_{\tilde G})$, that is $\mathbb{C}\left[(\check T/\pi_1(G))\times\tilde T,\ \frac{1}{2}\frac{\check\alpha-1}{\alpha-1},\ \alpha\in R\right]^{\pi_1(G)}=\mathbb{C}\left[(\check T/\pi_1(G))\times T,\ \frac{1}{2}\frac{\check\alpha-1}{\alpha-1},\ \alpha\in R\right]$.

This completes the proof of 2.15 for general G.

6.6. **Borel-Moore Homology and blow-ups.** Theorem 2.12 is proved absolutely parallelly to the proof of Theorem 2.15.

7. Computation of $K_{G(\mathbf{O})}(\Lambda)$.

7.1. **The affine Grassmannian Steinberg variety.** We denote by $\mathfrak{u}\subset\mathfrak{g}(\mathbf{O})$ (resp. $U\subset G(\mathbf{O})$) the nilpotent (resp. unipotent) radical. It has a filtration $\mathfrak{u}=\mathfrak{u}^{(0)}\supset\mathfrak{u}^{(1)}\supset\dots$ by congruence subalgebras. The trivial (Tate) vector bundle $\underline{\mathfrak{g}(\mathbf{F})}$ with the fiber $\mathfrak{g}(\mathbf{F})$ over Gr_G has a structure of an ind-scheme. It contains a profinite dimensional vector subbundle $\underline{\mathfrak{u}}$ whose fiber over a point $g\in\mathrm{Gr}_G$ represented by a compact subalgebra in $\mathfrak{g}(\mathbf{F})$ is the pronilpotent radical of this subalgebra. The trivial vector bundle $\underline{\mathfrak{g}(\mathbf{F})}=\mathfrak{g}(\mathbf{F})\times\mathrm{Gr}_G$ also contains a trivial vector subbundle $\mathfrak{u}\times\mathrm{Gr}_G$.

We will call $\underline{\mathfrak{u}}$ *the cotangent bundle* of Gr_G, and we will call the intersection $\Lambda:=\underline{\mathfrak{u}}\cap(\mathfrak{u}\times\mathrm{Gr}_G)$ *the affine Grassmannian Steinberg variety.* It has a structure of an ind-scheme of ind-infinite type. Namely, if p stands for the natural projection $\Lambda\to\mathrm{Gr}_G$, then $\Lambda_{\le\check\lambda}:=p^{-1}(\overline{\mathrm{Gr}}_{G,\check\lambda})$ is a scheme of infinite type, and $\Lambda=\bigcup\Lambda_{\le\check\lambda}$.

Note that for a fixed $\check\lambda$ and $l \gg 0$ the intersection of fibers of $\underline{\mathfrak{u}}$ over all points of $\overline{\mathrm{Gr}}_{G,\check\lambda}$ (as vector subspaces of $\mathfrak{g}(\mathbf{F})$) contains $\mathfrak{u}^{(l)}$. Thus $\mathfrak{u}^{(l)}$ acts freely (by fiberwise translations) on $\Lambda_{\leq\check\lambda}$, and the quotient is a scheme of finite type, to be denoted by $\Lambda^l_{\leq\check\lambda}$. For $k > l$ we have evident affine fibrations $p^k_l:\ \Lambda^k_{\leq\check\lambda} \to \Lambda^l_{\leq\check\lambda}$, and $\Lambda_{\leq\check\lambda}$ coincides with the inverse limit of this system.

Similarly, the total space of the vector bundle $\underline{\mathfrak{u}}$ (to be denoted by the same symbol) is a union of infinite type schemes $\underline{\mathfrak{u}}_{\leq\check\lambda}$, and for fixed $\check\lambda$ and $l \gg 0$, the scheme $\underline{\mathfrak{u}}_{\leq\check\lambda}$ is the inverse limit of affine fibrations $p^k_l:\ \underline{\mathfrak{u}}^k_{\leq\check\lambda} \to \underline{\mathfrak{u}}^l_{\leq\check\lambda}$ $(k > l)$. Note that the proalgebraic group $G(\mathbf{O})$ acts on all the above schemes, and the fibrations p^k_l are $G(\mathbf{O})$-equivariant.

A $G(\mathbf{O})$-*equivariant coherent sheaf* $\mathcal{F}$ *on* $\underline{\mathfrak{u}}$ is by definition supported on some $\underline{\mathfrak{u}}_{\leq\check\lambda}$. There, it is defined as a collection of $G(\mathbf{O})$-equivariant sheaves $\mathcal{F}^l$ on $\underline{\mathfrak{u}}^l_{\leq\check\lambda}$ for $l \gg 0$ together with isomorphisms $(p^k_l)^*\mathcal{F}^l \simeq \mathcal{F}^k$. We will consider the $G(\mathbf{O})$-equivariant coherent sheaves on $\underline{\mathfrak{u}}$ *supported on* Λ, and $D^bCoh^{G(\mathbf{O})}_\Lambda(\underline{\mathfrak{u}})$ stands for the derived category of such sheaves, and $K^{G(\mathbf{O})}(\Lambda)$ stands for the K-group of such sheaves.

7.2. **Convolution in $D^bCoh^{G(\mathbf{O})}_\Lambda(\underline{\mathfrak{u}})$.** We have a principal $G(\mathbf{O})$-bundle $G(\mathbf{F}) \to \mathrm{Gr}_G$. Given a $G(\mathbf{O})$-(ind)-scheme A we can form an associated bundle $\widetilde{A} = G(\mathbf{F}) \times_{G(\mathbf{O})} A \to \mathrm{Gr}_G$. Given a coherent $G(\mathbf{O})$-equivariant sheaf $\mathcal{F}$ on A we can form an associated sheaf $\widetilde{\mathcal{F}}$ on $\widetilde{A}$ as $G(\mathbf{O})$-invariants in the direct image of $\mathcal{O}_{G(\mathbf{F})} \boxtimes \mathcal{F}$ from $G(\mathbf{F}) \times A$ to $G(\mathbf{F}) \times_{G(\mathbf{O})} A$. If $A = \mathrm{Gr}_G$, apart from the natural projection $p_1:\ \widetilde{A} \to \mathrm{Gr}_G$, we have a multiplication map $G(\mathbf{F}) \times_{G(\mathbf{O})} \mathrm{Gr}_G \to \mathrm{Gr}_G$, to be denoted p_2. Then (p_1, p_2) identifies $\widetilde{\mathrm{Gr}_G}$ with $\mathrm{Gr}_G \times \mathrm{Gr}_G$. Furthermore, $\widetilde{\underline{\mathfrak{u}}}$ is a vector bundle over $\widetilde{\mathrm{Gr}_G} = \mathrm{Gr}_G \times \mathrm{Gr}_G$ which is naturally identified with $p_2^*\underline{\mathfrak{u}}$. Thus we have an ind-proper morphism $p_2:\ \widetilde{\underline{\mathfrak{u}}} \to \underline{\mathfrak{u}}$.

Note that both $\widetilde{\underline{\mathfrak{u}}} = p_2^*\underline{\mathfrak{u}}$ and $p_1^*\underline{\mathfrak{u}}$ are subbundles in the trivial (Tate) vector bundle $\underline{\mathfrak{g}(\mathbf{F})}$ over $\mathrm{Gr}_G \times \mathrm{Gr}_G$ with the fiber $\mathfrak{g}(\mathbf{F})$. Their intersection is naturally identified with $\widetilde\Lambda$. In particular, we have an embedding $\widetilde\Lambda \subset p_1^*\underline{\mathfrak{u}} \oplus p_2^*\underline{\mathfrak{u}}$, and an ind-proper morphism $p_2:\ \widetilde\Lambda \to \underline{\mathfrak{u}}$.

Hence given $G(\mathbf{O})$-equivariant coherent sheaves $\mathcal{F}, \mathcal{G}$ on Λ we can consider the $G(\mathbf{O})$-equivariant complex $\mathcal{F} \star \mathcal{G} := (p_2)_*(p_1^*\mathcal{F} \overset{L}{\otimes} \widetilde{\mathcal{G}})$ (tensor product over the structure sheaf of the profinite dimensional vector bundle $p_1^*\underline{\mathfrak{u}} \oplus p_2^*\underline{\mathfrak{u}}$). Clearly, $\mathcal{F} \star \mathcal{G}$ is supported on Λ. Hence we get a convolution operation on $D^bCoh^{G(\mathbf{O})}_\Lambda(\underline{\mathfrak{u}})$ and on $K^{G(\mathbf{O})}(\Lambda)$ once we check that $p_1^*\mathcal{F} \overset{L}{\otimes} \widetilde{\mathcal{G}}$ is bounded.

To this end, note that $\widetilde{\mathcal{G}}$ is flat over the first copy of Gr_G, and for some $\check\lambda$ the sheaf $\mathcal{F}$ is supported on $\Lambda_{\leq\check\lambda}$, so the tensor product $p_1^*\mathcal{F} \overset{L}{\otimes} \widetilde{\mathcal{G}}$ can actually be computed over the structure sheaf of $(p_1^*\underline{\mathfrak{u}} \oplus p_2^*\underline{\mathfrak{u}})|_{\overline{\mathrm{Gr}}_{G,\check\lambda} \times \mathrm{Gr}_G} = \underline{\mathfrak{u}}_{\leq\check\lambda} \times \underline{\mathfrak{u}} \subset \underline{\mathfrak{u}} \times \underline{\mathfrak{u}} = p_1^*\underline{\mathfrak{u}} \oplus p_2^*\underline{\mathfrak{u}}$. That is, $p_1^*\mathcal{F} \overset{L}{\otimes} \widetilde{\mathcal{G}}$ is the direct image of $p_1^*\mathcal{F}|_{\underline{\mathfrak{u}}_{\leq\check\lambda} \times \underline{\mathfrak{u}}} \overset{L}{\otimes}_{\mathcal{O}_{\underline{\mathfrak{u}}_{\check\lambda} \times \underline{\mathfrak{u}}}} \widetilde{\mathcal{G}}|_{\underline{\mathfrak{u}}_{\leq\check\lambda} \times \underline{\mathfrak{u}}}$ under the closed embedding $\underline{\mathfrak{u}}_{\leq\check\lambda} \times \underline{\mathfrak{u}} \hookrightarrow \underline{\mathfrak{u}} \times \underline{\mathfrak{u}}$. On the other hand, $p_1^*\mathcal{F}$ is flat over the second copy of Gr_G, while the support of $\widetilde{\mathcal{G}}$ intersected with $\underline{\mathfrak{u}}_{\leq\check\lambda} \times \underline{\mathfrak{u}}$ is contained in $\underline{\mathfrak{u}}_{\leq\check\lambda} \times \underline{\mathfrak{u}}_{\leq\check\mu}$ for some $\check\mu$. Hence the tensor product $p_1^*\mathcal{F} \overset{L}{\otimes} \widetilde{\mathcal{G}}$ can actually be computed over the structure sheaf of $\underline{\mathfrak{u}}_{\leq\check{\lambda}} \times \underline{\mathfrak{u}}_{\leq\check{\mu}}$. There exists $l \gg 0$ such that the diagonal fiberwise action of $\mathfrak{u}^{(l)}$ on $\underline{\mathfrak{u}}_{\leq\check{\lambda}} \times \underline{\mathfrak{u}}_{\leq\check{\mu}}$ is free, and both $p_1^*\mathcal{F}$ and $\widetilde{\mathcal{G}}$ restricted to $\underline{\mathfrak{u}}_{\leq\check{\lambda}} \times \underline{\mathfrak{u}}_{\leq\check{\mu}}$ are $\mathfrak{u}^{(l)}$-equivariant, that is, they are lifted from the sheaves on $(\underline{\mathfrak{u}}_{\leq\check{\lambda}} \times \underline{\mathfrak{u}}_{\leq\check{\mu}})/\mathfrak{u}^{(l)} =: V$; we abuse notation by keeping the same names for these sheaves. So the tensor product $p_1^*\mathcal{F} \overset{L}{\otimes} \widetilde{\mathcal{G}}$ can actually be computed as the tensor product of coherent sheaves over the structure sheaf of the profinite dimensional vector bundle V over the finite dimensional scheme $\overline{\mathrm{Gr}}_{G,\check{\lambda}} \times \overline{\mathrm{Gr}}_{G,\check{\mu}}$.

Now there exists a vector subbundle $V' \subset V$ such that the quotient $\overline{V} := V/V'$ is a finite dimensional vector bundle, $p_1^*\mathcal{F}$ is lifted from $\overline{V}$, and the support of $\widetilde{\mathcal{G}}$ in V projects isomorphically onto its image in $\overline{V}$. Moreover, recall that $p_1^*\mathcal{F}$ is flat over $\overline{\mathrm{Gr}}_{G,\check{\mu}}$, while $\widetilde{\mathcal{G}}$ is flat over $\overline{\mathrm{Gr}}_{G,\check{\lambda}}$. Clearly, in this situation $p_1^*\mathcal{F} \overset{L}{\otimes} \widetilde{\mathcal{G}} \in D^b(V)$. This explains why for $G(\mathbf{O})$-equivariant coherent sheaves $\mathcal{F}, \mathcal{G}$ on Λ the tensor product $p_1^*\mathcal{F} \overset{L}{\otimes} \widetilde{\mathcal{G}}$ is a bounded complex of coherent sheaves on $p_1^*\underline{\mathfrak{u}} \oplus p_2^*\underline{\mathfrak{u}}$ supported on $\widetilde{\Lambda}$. Hence the same is true for the bounded complexes of $G(\mathbf{O})$-equivariant coherent sheaves $\mathcal{F}, \mathcal{G}$ on $\underline{\mathfrak{u}}$ supported on Λ. Thus, $D^bCoh_\Lambda^{G(\mathbf{O})}(\underline{\mathfrak{u}})$ is closed with respect to convolution.

Theorem 7.3. *$K^{G(\mathbf{O})}(\Lambda)$ is a commutative algebra isomorphic to $\mathbb{C}[\check{T} \times T]^W$.*

Remark 7.4. Since Λ_G is an affine Grassmannian analogue of the classical Steinberg variety, this result agrees well with the geometric realization of the Cherednik double affine Hecke algebra in [8], [23]. In effect, $K^{G(\mathbf{O})}(\Lambda_G)$ is the spherical subalgebra of the Cherednik algebra with both parameters trivial: $q = t = 1$.

7.5. **Bialynicki-Birula stratifications.** The proof of Theorem 7.3 uses the following lemma on K-theory of cellular spaces. Let M be a normal quasiprojective variety equipped with a torus H-action with finitely many fixed points. We assume that M is equipped with an H-invariant stratification $M = \bigsqcup_{\mu\in M^H} M_\mu$ such that each stratum M_μ contains exactly one H-fixed point μ, and M_μ is isomorphic to an affine space. For $\mu \in M^H$ we denote by $j_\mu : M_\mu \hookrightarrow M$ the locally closed embedding of the corresponding stratum. We denote by $i_\mu : \mu \hookrightarrow M_\mu$ the closed embedding of an H-fixed point in the corresponding stratum, or in the whole of M when no confusion is likely. We denote by $\mu \leq \nu$ the closure relation of strata. We denote by $M_{\leq\mu} \subset M$ the union $\bigcup_{\nu\leq\mu} M_\nu$.

Given an H-equivariant closed embedding of M into a smooth H-variety M' (for the existence see [22]) we denote by T^*M the restriction of the cotangent bundle T^*M' to $M \subset M'$. We denote by $\imath : M \hookrightarrow T^*M$ the embedding of the zero section. We also denote by i_μ the closed embedding of the conormal bundle $T^*_\mu M' \hookrightarrow T^*M$ when no confusion is likely. Finally, we denote by $\mathcal{L}'$ the union of conormal bundles $\bigcup_\mu T^*_{M_\mu} M'$, and $\jmath$ stands for the closed embedding $\mathcal{L}' \hookrightarrow T^*M$. We denote by $\mathcal{L}'_{\leq\mu} \subset \mathcal{L}'$ the union $\bigcup_{\nu\leq\mu} T^*_{\mathcal{M}_\nu} M'$; it is a closed subvariety of $\mathcal{L}'$. It has a closed subvariety $\mathcal{L}'_{<\mu} := \bigcup_{\nu<\mu} T^*_{M_\nu} M'$.

For $\mu \in M^H$ we have an embedding $i_{\mu *}: K^H(\mu) \hookrightarrow K^H(M)$. We have an embedding $\jmath_*: K^H(\mathcal{L}') \hookrightarrow K^H(T^*M) \overset{\imath^*}{\simeq} K^H(M)$. Indeed, the exact sequences (see [4] Chapter 5)

$$0 \to K^H(\mathcal{L}'_{<\mu}) \to K^H(\mathcal{L}'_{\leq\mu}) \to K^H(T^*_{M_\mu}M') \to 0,$$

$$0 \to K^H(T^*M'|_{M_{<\mu}}) \to K^H(T^*M'|_{M_{\leq\mu}}) \to K^H(T^*M'|_{M_\mu})$$

give rise to the support filtrations on $K^H(\mathcal{L}')$ and $K^H(T^*M)$ with associated graded $\bigoplus_{\mu\in M^H} K^H(T^*_{M_\mu}M')$ and $\bigoplus_{\mu\in M^H} K^H(T^*M'|_{M_\mu})$. Now $\jmath_*$ is strictly compatible with the support filtrations and clearly injective on the associated graded.

Note that the image $\jmath_*(K^H(\mathcal{L}')) \subset K^H(M)$ is independent of the choice of the closed embedding $M \hookrightarrow M'$. In effect, given another embedding $M \hookrightarrow \widetilde{M}$, we can consider the diagonal embedding $M \hookrightarrow M'' := M' \times \widetilde{M}$. Clearly, we have a projection $p: T^*M''|_M \to T^*M'|_M$ which realizes $T^*M''|_M$ as a vector bundle over $T^*M'|_M$. Moreover, if we denote by $\mathcal{L}''$ the union of conormal bundles $\bigcup_\mu T^*_{M_\mu}M'' \subset T^*M''|_M$ then $\mathcal{L}'' = p^{-1}\mathcal{L}'$. This shows that the images of $K^H(\mathcal{L}')$ and $K^H(\mathcal{L}'')$ in $K^H(M)$ coincide, and thus $\jmath_*(K^H(\mathcal{L}')) \subset K^H(M)$ is well-defined.

Lemma 7.6. *In $K^H(M)$ we have an equality $\jmath_*(K^H(\mathcal{L}')) = \oplus_\mu i_{\mu*}(K^H(\mu))$.*

Proof Let $K^H(D_M)$ stand for the K-group of weakly H-equivariant D-modules on M' supported on $M \subset M'$. Given such a D-module and passing to associated graded with respect to a good filtration, we obtain an H-equivariant coherent sheaf on T^*M, and this way one obtains a homomorphism $SS: K^H(D_M) \to K^H(T^*M) \overset{\imath^*}{\simeq} K^H(M)$ (see e.g. [11]). Let δ_μ stand for a δ-function D-module at the point $\mu \in M^H$ with its obvious H-equivariance. Then, evidently, $SS(\delta_\mu)$ generates $i_{\mu*}(K^H(\mu))$ as a module over $K^H(pt)$. Moreover, $\{SS(j_{\mu!}\mathcal{O}_{M_\mu}),\ \mu \in M^H\}$ forms a basis of $\jmath_*(K^H(\mathcal{L}'))$.

In effect, the closed embedding $\mathcal{L}'_{<\mu} \hookrightarrow \mathcal{L}'_{\leq\mu}$ gives rise to the exact sequence

$$0 \to K^H(\mathcal{L}'_{<\mu}) \to K^H(\mathcal{L}'_{\leq\mu}) \to K^H(T^*_{M_\mu}M') \to 0$$

(see [4] Chapter 5), and the image of $SS(j_{\mu!}\mathcal{O}_{M_\mu})$ in $K^H(T^*_{M_\mu}M')$ clearly generates it.

So it is enough to check the equality in $K^H(T^*M)$:

$$SS(\delta_\mu) = SS(j_{\mu!}\mathcal{O}_{M_\mu}) \cdot (-1)^{\dim M_\mu} \det(T_\mu M_\mu) \tag{12}$$

where $\det(T_\mu M_\mu)$ is the character of H (thus an invertible element of $K^H(pt) = \mathbb{C}[H]$) acting in the determinant of the tangent bundle of M_μ at μ.

To this end note that restriction to the H-fixed points gives rise to an embedding $\oplus_\nu i^*_\nu \imath^*: K^H(T^*M) \hookrightarrow \oplus_\nu K^H(\nu)$. This is checked by induction in ν using the exact sequences

$$0 \to K^H(T^*M'|_{M_{<\nu}}) \to K^H(T^*M'|_{M_{\leq\nu}}) \to K^H(T^*M'|_{M_\nu}) \to 0.$$

It is clear that for $\nu = \mu$ the restrictions $i^*_\mu \imath^*$ of the LHS and RHS of (12) coincide. We are going to check that for $\nu \neq \mu$ the restrictions $i^*_\nu \imath^*$ of the LHS and RHS of (12) both vanish. Evidently, $i^*_\nu \imath^* SS(\delta_\mu) = 0$.

Recall that i_ν also stands for the closed embedding $T^*_\nu M' \hookrightarrow T^*M$, so we just have to check that $i^*_\nu SS(j_{\mu!}\mathcal{O}_{M_\mu}) = 0 \in K^H(T^*_\nu M')$. Note that the functor of global sections of H-equivariant coherent sheaves on the vector space $T^*_\nu M'$ gives rise to an embedding $\Gamma:\ K^H(T^*_\nu M') \hookrightarrow \mathbb{Z}^{X^*(H)}$ where $X^*(H)$ stands for the lattice of characters of H. Now for a D-module $\mathcal{F}$ we have $\Gamma(i^*_\nu SS\mathcal{F}) = \mathbf{i}^*_\nu\mathcal{F}$ where $\mathbf{i}^*_\nu\mathcal{F}$ stands for the fiber at $\nu \in M$ of the H-equivariant quasicoherent $\mathcal{O}_{M'}$-module $\mathcal{F}$. Finally, for $\mathcal{F} = j_{\mu!}\mathcal{O}_{M_\mu}$ and $\nu \neq \mu$ we have $\mathbf{i}^*_\nu j_{\mu!}\mathcal{O}_{M_\mu} = 0$. This completes the proof of the lemma.

7.7. **Bialynicki-Birula stratification of** Gr_G**.** We consider the stratification of Gr_G by the Iwahori orbits $\mathrm{Gr}_G = \bigsqcup_{\check\mu\in Y}\mathrm{Gr}^{\check\mu}_G$. This is a refinement of the stratification by the $G(\mathbf{O})$-orbits: $\mathrm{Gr}_{G,\check\lambda} = \bigsqcup_{\check\mu\in W\check\lambda}\mathrm{Gr}^{\check\mu}_G$. Let us denote by $\mathfrak{n} \supset \mathfrak{u}$ the nilpotent radical of the Iwahori subalgebra in $\mathfrak{g}(\mathbf{F})$. The union of conormal bundles to the Iwahori orbits is the following subvariety Λ_I of the cotangent bundle $\underline{\mathfrak{u}}$: by definition, $\Lambda_I := \underline{\mathfrak{u}} \cap (\mathfrak{n}\times\mathrm{Gr}_G)$. We have a closed embedding $\Lambda \subset \Lambda_I$.

Lemma 7.6 allows us to compute $K^T(\Lambda_I) = \oplus_{\check\mu\in Y}K^T(\check\mu) \subset K^T(\mathrm{Gr}_G)$, i.e. $K^T(\Lambda_I) \simeq \mathbb{C}[\check T\times T]$ (note that the natural W-action on $K^T(\mathrm{Gr}_G)$ induces the diagonal W-action on $\mathbb{C}[\check T\times T] \simeq K^T(\Lambda_I) \subset K^T(\mathrm{Gr}_G)$). Although Lemma 7.6 was formulated for finite dimensional varieties M, its proof goes through for Gr_G without changes: we only need to have the singular support map $SS:\ K^T(D_{\mathrm{Gr}_G}) \to K^T(\underline{\mathfrak{u}}) \simeq K^T(\mathrm{Gr}_G)$. For this see [12], [2] (Chapter 15), [8].

The embedding $\Lambda \hookrightarrow \Lambda_I$ gives rise to the embedding $K^T(\Lambda) \hookrightarrow K^T(\Lambda_I) \hookrightarrow K^T(\underline{\mathfrak{u}}) = K^T(\mathrm{Gr}_G)$. Note that W acts naturally on both $K^T(\Lambda)$ and $K^T(\mathrm{Gr}_G)$, and the embedding $K^T(\Lambda) \hookrightarrow K^T(\mathrm{Gr}_G)$ is W-equivariant. Also, $(K^T(\Lambda))^W = K^G(\Lambda) = K^{G(\mathbf{O})}(\Lambda)$. Hence, the image of the embedding $K^{G(\mathbf{O})}(\Lambda) \hookrightarrow K^T(\Lambda_I) \simeq \mathbb{C}[\check T\times T] \subset K^T(\mathrm{Gr}_G)$ lies in the invariants of the diagonal W-action on $\mathbb{C}[\check T\times T]$. Thus to prove Theorem 7.3 we must check that the image of this embedding contains $\mathbb{C}[\check T\times T]^W$.

We have projections $\pi:\ \Lambda \to \mathrm{Gr}_G$, and $\pi_I:\ \Lambda_I \to \mathrm{Gr}_G$. For $\check\lambda \in Y^+$ we denote by $\Lambda_{\check\lambda}$ (resp. $\Lambda_{\leq\check\lambda}$, $\Lambda_{<\check\lambda}$) the preimage $\pi^{-1}(\mathrm{Gr}_{G,\check\lambda})$ (resp. $\pi^{-1}(\overline{\mathrm{Gr}}_{G,\check\lambda})$, $\pi^{-1}(\overline{\mathrm{Gr}}_{G,\check\lambda} - \mathrm{Gr}_{G,\check\lambda})$). For $\check\lambda \in Y^+$ we denote by $\Lambda_{I,\check\lambda}$ (resp. $\Lambda_{I,\leq\check\lambda}$, $\Lambda_{I,<\check\lambda}$) the preimage $\pi_I^{-1}(\mathrm{Gr}_{G,\check\lambda})$ (resp. $\pi_I^{-1}(\overline{\mathrm{Gr}}_{G,\check\lambda})$, $\pi_I^{-1}(\overline{\mathrm{Gr}}_{G,\check\lambda} - \mathrm{Gr}_{G,\check\lambda})$). Clearly, $\Lambda_{<\check\lambda}$ (resp. $\Lambda_{I,<\check\lambda}$) is closed in $\Lambda_{\leq\check\lambda}$ (resp. $\Lambda_{I,\leq\check\lambda}$), with the open complement $\Lambda_{\check\lambda}$ (resp. $\Lambda_{I,\check\lambda}$). In K-groups we have exact sequences (see [4] Chapter 5)

$$0 \to K^T(\Lambda_{<\check\lambda}) \to K^T(\Lambda_{\leq\check\lambda}) \to K^T(\Lambda_{\check\lambda}) \to 0,$$

$$0 \to K^T(\Lambda_{I,<\check\lambda}) \to K^T(\Lambda_{I,\leq\check\lambda}) \to K^T(\Lambda_{I,\check\lambda}) \to 0.$$

Thus we obtain a support filtration on $K^T(\Lambda_I)$ (resp. $K^T(\Lambda)$) with associated graded $\bigoplus_{\check\lambda\in Y^+}K^T(\Lambda_{I,\check\lambda})$ (resp. $\bigoplus_{\check\lambda\in Y^+}K^T(\Lambda_{\check\lambda})$).

We have the embeddings $K^T(\Lambda_{\check\lambda}) \hookrightarrow K^T(\Lambda_{I,\check\lambda}) \hookrightarrow K^T(\underline{\mathfrak{u}}|_{\mathrm{Gr}_{\check\lambda}}) \simeq K^T(\mathrm{Gr}_{\check\lambda})$. The Weyl group W acts naturally both on $K^T(\Lambda_{\check\lambda})$ and $K^T(\mathrm{Gr}_{\check\lambda})$, and to prove Theorem 7.3 it suffices to check that the image of $(K^T(\Lambda_{\check\lambda}))^W$ in $K^T(\Lambda_{I,\check\lambda})$ contains (equivalently, coincides with) the intersection $K^T(\Lambda_{I,\check\lambda}) \cap (K^T(\mathrm{Gr}_{\check\lambda}))^W$.

To this end recall that $\mathrm{Gr}_{G,\check\lambda}$ can be G-equivariantly identified with the total space $\widetilde{\mathcal{B}}$ of a vector bundle over a certain partial flag variety $\mathcal{B}$ of the group G (the quotient $G/P_{\check\lambda}$ by a parabolic subgroup depending on $\check\lambda$). The Borel subgroup $B\subset G$ acts on $\mathcal{B}$ with finitely many orbits numbered by the cosets of parabolic Weyl subgroup $W^{\check\lambda}=W/W_{\check\lambda}$: we have $\mathcal{B}=\bigsqcup_{w\in W^{\check\lambda}}\mathcal{B}_w$. Let us denote by $\mathcal{L}\subset T^*\mathcal{B}$ the union of conormal bundles $\mathcal{L}=\bigsqcup_{w\in W^{\check\lambda}}T^*_{\mathcal{B}_w}\mathcal{B}$. Let us also denote by $\widetilde{\mathcal{B}}_w$ the preimage of $\mathcal{B}_w$ in $\widetilde{\mathcal{B}}$ (it coincides with a certain Iwahori orbit $\mathrm{Gr}_G^{\check\mu}\subset\mathrm{Gr}_{G,\check\lambda}=\widetilde{\mathcal{B}}$). We define $\widetilde{\mathcal{L}}:=\bigsqcup_{w\in W^{\check\lambda}}T^*_{\widetilde{\mathcal{B}}_w}\widetilde{\mathcal{B}}\subset T^*\widetilde{\mathcal{B}}$. Then there exists a G-equivariant profinite dimensional vector bundle $\mathcal{V}\xrightarrow{p}T^*\widetilde{\mathcal{B}}$ such that $\mathcal{V}\simeq\underline{\mathfrak{u}}|_{\mathrm{Gr}_{\check\lambda}}$, and under this isomorphism we have $\mathcal{V}|_{\widetilde{\mathcal{L}}}\simeq\Lambda_{I,\check\lambda}$, $\mathcal{V}|_{\widetilde{\mathcal{B}}\hookrightarrow T^*\widetilde{\mathcal{B}}}\simeq\Lambda_{\check\lambda}$. Thus to prove Theorem 7.3 it is enough to check that the image of $(K^T(\widetilde{\mathcal{B}}))^W$ in $K^T(T^*\widetilde{\mathcal{B}})$ contains the intersection $K^T(\widetilde{\mathcal{L}})\cap(K^T(T^*\widetilde{\mathcal{B}}))^W$. Equivalently, we have to check that the image of $(K^T(\mathcal{B}))^W$ in $K^T(T^*\mathcal{B})$ contains the intersection $K^T(\mathcal{L})\cap(K^T(T^*\mathcal{B}))^W$. This is the subject of the following lemma.

Lemma 7.8. *Let $\imath:\ \mathcal{B}\hookrightarrow T^*\mathcal{B}$ denote the embedding of the zero section, and let $\jmath:\ \mathcal{L}\hookrightarrow T^*\mathcal{B}$ denote the natural closed embedding. Then $\imath_*(K^T(\mathcal{B}))^W$ coincides with* $\mathrm{Im}\left(\jmath_*:\ K^T(\mathcal{L})\hookrightarrow K^T(T^*\mathcal{B})\right)\bigcap\left(K^T(T^*\mathcal{B})\right)^W$.

Proof For $w\in W^{\check\lambda}$ we denote by $w\in\mathcal{B}_w\subset\mathcal{B}$ the corresponding T-fixed point. We denote by i_w the closed embedding $T^*_w\mathcal{B}\hookrightarrow T^*\mathcal{B}$ (and also the closed embedding $w\hookrightarrow\mathcal{B}$, when the confusion is unlikely), and we denote by $\mathfrak{i}_w$ the closed embedding $w\hookrightarrow T^*\mathcal{B}$. According to Lemma 7.6, the image of $\jmath_*:\ K^T(\mathcal{L})\hookrightarrow K^T(T^*\mathcal{B})$ coincides with the image of $\oplus_{w\in W^{\check\lambda}}i_{w*}:\ \oplus_{w\in W^{\check\lambda}}K^T(T^*_w\mathcal{B})\to K^T(T^*\mathcal{B})$. We have an embedding $\oplus_{w\in W^{\check\lambda}}\mathfrak{i}^*_w:\ K^T(T^*\mathcal{B})\hookrightarrow\oplus_{w\in W^{\check\lambda}}K^T(w)$, and similarly an embedding $\oplus_{w\in W^{\check\lambda}}i^*_w:\ K^T(\mathcal{B})\hookrightarrow\oplus_{w\in W^{\check\lambda}}K^T(w)$.

Clearly, the W-invariants project injectively into any direct summand: $K^G(\mathcal{B})=(K^T(\mathcal{B}))^W\overset{i^*_w}{\hookrightarrow}K^T(w)$ (resp. $K^G(T^*\mathcal{B})=(K^T(T^*\mathcal{B}))^W\overset{\mathfrak{i}^*_w}{\hookrightarrow}K^T(w)$) for any $w\in W^{\check\lambda}$. Thus it suffices to check that for any $w\in W^{\check\lambda}$ we have a coincidence $\mathrm{Im}(\mathfrak{i}^*_w i_{w*}:\ K^T(T^*_w\mathcal{B})^W\to K^T(w))=\mathrm{Im}(\mathfrak{i}^*_w\imath_*\mathrm{Res}^G_T:\ K^G(\mathcal{B})\to K^T(w))$. Note that if $w=e$ (the identity coset of $W_{\check\lambda}$ in W), then the image $i^*_e(K^T(\mathcal{B}))^W\subset K^T(e)$ (resp. $\mathfrak{i}^*_e(K^T(T^*\mathcal{B}))^W\subset K^T(e)$) coincides with $(K^T(e))^{W_{\check\lambda}}=\mathbb{C}[T]^{W_{\check\lambda}}$. Moreover, under identification $K^T(T^*_e\mathcal{B})=K^T(e)=\mathbb{C}[T]$, we have $K^T(T^*_e\mathcal{B})\cap(K^T(T^*\mathcal{B}))^W=\mathbb{C}[T]^{W_{\check\lambda}}$.

Identifying both $K^T(T^*_e\mathcal{B})$ and $K^T(e)$ with $\mathbb{C}[T]$, the map $\mathfrak{i}^*_e i_{e*}$ is a multiplication by the product $\Delta_1=\prod_{k=1}^{\dim\mathcal{B}}(1-\chi_k)$ where χ_k run through the characters of T in the tangent space $T_e(T^*_e\mathcal{B})=T^*_e\mathcal{B}$. Furthermore, identifying $K^G(\mathcal{B})$ with $\mathbb{C}[T]^{W_{\check\lambda}}$, and $K^T(e)$ with $\mathbb{C}[T]$, the map $\mathfrak{i}^*_e\imath_*\mathrm{Res}^G_T$ is a multiplication by the product $\Delta_2=\prod_{k=1}^{\dim\mathcal{B}}(1-\chi'_k)$ where χ'_k run through the characters of T in the tangent space $T_e\mathcal{B}$. We can arrange the characters χ'_k so that we have $\chi'_k=\chi_k^{-1}$. Then we see that $\Delta_1=\Delta_2\cdot\prod_{k=1}^{\dim\mathcal{B}}(-\chi_k)$, so they differ by an invertible function, hence the corresponding images coincide: $\Delta_1\cdot\mathbb{C}[T]^{W_{\check\lambda}}=\Delta_2\cdot\mathbb{C}[T]^{W_{\check\lambda}}$.

This completes the proof of the lemma along with Theorem 7.3.

7.9. In this subsection we describe (without striving for high precision) a conjectural picture motivating Theorem 7.3.

We hope that the isomorphism $K^{G(\mathbf{O})}(\Lambda_G) = \mathbb{C}[\check{T}\times T]^W = \mathbb{C}[T\times\check{T}]^W = K^{\check{G}(\mathbf{O})}(\Lambda_{\check{G}})$ lifts to an equivalence of monoidal categories $F: \ D^bCoh^{G(\mathbf{O})}_{\Lambda_G}(\underline{\mathfrak{u}}_G) \simeq D^bCoh^{\check{G}(\mathbf{O})}_{\Lambda_{\check{G}}}(\underline{\mathfrak{u}}_{\check{G}})$. The conjectural equivalence F is related to the Langlands correspondence in the following way.

Recall that the conjectural (for $G = GL(n)$ mostly proven in [9]) geometric Langlands correspondence is an equivalence of triangulated categories between the derived category of D-modules on the stack Bun_G of G-bundles on a given smooth projective curve C, and the derived category of coherent sheaves on the stack of $\check{G}$ local systems on the same curve. One might expect its "classical limit" to be an equivalence between the derived categories of coherent sheaves $L: \ D(T^*\mathrm{Bun}_G) \simeq D(T^*\mathrm{Bun}_{\check{G}})$ where $T^*\mathrm{Bun}_G$ is the cotangent bundle to the moduli stack of G-bundles on C. Given a point $c \in C$, and identifying $\mathbf{O}$ with the algebra of functions on the formal neighbourhood of c, one gets an action of $D^bCoh^{G(\mathbf{O})}_{\Lambda_G}(\underline{\mathfrak{u}}_G)$ on $D(T^*\mathrm{Bun}_G)$. The "classical limit" of the Hecke eigen-property of geometric Langlands correspondence (see [2]) should be stated in terms of this action; it should say that the global equivalence L is compatible with our local equivalence F.

8. Perverse sheaves and fusion

We refer the reader to [3] for the definition of perverse equivariant coherent sheaves and related objects.

8.1. Recall the setup of 6.1. Note that all the $G(\mathbf{O})$-orbits in a connected component of Gr_G have dimensions of the same parity. Thus it makes sense to consider the middle perversity function $p(\mathrm{Gr}_{G,\check{\lambda}}) = -\frac{1}{2}\dim(\mathrm{Gr}_{G,\check{\lambda}}) = -\langle\rho,\check{\lambda}\rangle$. It is obviously strictly monotone and comonotone, but at some connected components of Gr_G it takes values in half-integers. This means that we consider equivariant complexes formally placed in half-integer homological degrees. The theory of [3] defines the artinian abelian category $\mathcal{P}^{G(\mathbf{O})}(\mathrm{Gr}_G)$ of perverse $G(\mathbf{O})$-equivariant coherent sheaves (with respect to the above middle perversity). Let $D^{b,G(\mathbf{O})}(\mathrm{Gr}_G)$ denote the bounded derived category of $G(\mathbf{O})$-equivariant coherent sheaves on Gr_G (with the same convention that the complexes at "odd" connected components are placed in half-integer homological degrees).

Given two complexes $\mathcal{F},\mathcal{G} \in D^{b,G(\mathbf{O})}(\mathrm{Gr}_G)$ we have their convolution $\mathcal{F}\star\mathcal{G} \in D^{b,G(\mathbf{O})}(\mathrm{Gr}_G)$. Recall that $\mathcal{F}\star\mathcal{G} = \Pi_{0*}(\mathcal{F}\ltimes\mathcal{G})$ where $\Pi_0: \ G(\mathbf{F})\times_{G(\mathbf{O})}\mathrm{Gr}_G \to \mathrm{Gr}_G$ is the convolution diagram, and $\mathcal{F}\ltimes\mathcal{G}$ is the twisted product of $\mathcal{F}$ and $\mathcal{G}$ on $G(\mathbf{F})\times_{G(\mathbf{O})}\mathrm{Gr}_G$.

Proposition 8.2. *The convolution preserves perverse sheaves: for $\mathcal{F},\mathcal{G} \in \mathcal{P}^{G(\mathbf{O})}(\mathrm{Gr}_G)$ we have $\mathcal{F}\star\mathcal{G} \in \mathcal{P}^{G(\mathbf{O})}(\mathrm{Gr}_G)$.*

Proof Denote the projection $G(\mathbf{F}) \to G(\mathbf{F})/G(\mathbf{O}) = \mathrm{Gr}_G$ by p, and consider a stratification $G(\mathbf{F})\times_{G(\mathbf{O})}\mathrm{Gr}_G = \bigsqcup_{\check{\lambda},\check{\mu}\in Y^+} p^{-1}(\mathrm{Gr}_{G,\check{\lambda}})\times_{G(\mathbf{O})}\mathrm{Gr}_{G,\check{\mu}}$. Clearly, $\mathcal{F}\ltimes\mathcal{G}$ is smooth (locally free) along this stratification, and perverse (with respect to the middle perversity). According to [19] 2.7, the map Π_0 is stratified semismall with respect to the above stratification. Now the perversity of $\Pi_{0*}(\mathcal{F} \ltimes \mathcal{G})$ follows in the same manner as in the constructible case, cf. *loc. cit.*

8.3. **The absence of commutativity constraint.** According to Proposition 8.2, $\mathcal{P}^{G(\mathbf{O})}(\mathrm{Gr}_G)$ acquires the structure of abelian artinian monoidal category. Moreover, according to 2.15 (a), its K-ring is commutative. Nevertheless, $\mathcal{P}^{G(\mathbf{O})}(\mathrm{Gr}_G)$ admits no commutativity constraint, as can be seen in the following example.

We recall the setup of 3.6, and consider Gr_{PGL_2}. One can check that there are the nonsplit exact sequences in $\mathcal{P}^{PGL_2(\mathbf{O})}(\mathrm{Gr}_{PGL_2})$:

$$0 \to \mathcal{V}(0)_0 \to \mathcal{V}(0)_1 \star \mathcal{V}(-2)_1 \to \mathcal{V}(-2)_2 \to 0$$
$$0 \to \mathcal{V}(-2)_2 \to \mathcal{V}(-2)_1 \star \mathcal{V}(0)_1 \to \mathcal{V}(0)_0 \to 0$$

Thus $\mathcal{V}(0)_1 \star \mathcal{V}(-2)_1$ and $\mathcal{V}(-2)_1 \star \mathcal{V}(0)_1$ are nonisomorphic.

8.4. $G(\mathbf{O}) \ltimes \mathbb{G}_m$**-equivariant sheaves and fusion.** The orbits of $G(\mathbf{O}) \ltimes \mathbb{G}_m$ on Gr_G coincide with the $G(\mathbf{O})$-orbits, so one can consider the abelian artinian monoidal category $\mathcal{P}^{G(\mathbf{O})\ltimes\mathbb{G}_m}(\mathrm{Gr}_G)$ of $G(\mathbf{O}) \ltimes \mathbb{G}_m$-equivariant coherent perverse sheaves on Gr_G. For $\mathcal{F} \in \mathcal{P}^{G(\mathbf{O})\ltimes\mathbb{G}_m}(\mathrm{Gr}_G)$ we have $R\Gamma(\mathrm{Gr}_G, \mathcal{F}) \in D^b(G(\mathbf{O}) \ltimes \mathbb{G}_m - mod)$.

B. Feigin and S. Loktev define (under certain restrictions) in [6] the *fusion product* $V_1 \star \ldots \star V_k \in G(\mathbf{O}) \ltimes \mathbb{G}_m - mod$ of $G(\mathbf{O}) \ltimes \mathbb{G}_m$-modules $V_1, \ldots, V_k$. We recall some of their results in case $G = PGL_2$.

Let $V(n)$ be the $n+1$-dimensional $G(\mathbf{O}) \ltimes \mathbb{G}_m$-module factoring through $G(\mathbf{O}) \ltimes \mathbb{G}_m \twoheadrightarrow G \times \mathbb{G}_m \twoheadrightarrow G$. Recall the irreducible $PGL_2(\mathbf{O})$-equivariant perverse sheaf $\mathcal{V}(n)_m$ introduced in 3.6. It can be lifted to the same named $PGL_2(\mathbf{O}) \ltimes \mathbb{G}_m$-equivariant perverse sheaf, where the action of $\mathbb{G}_m$ in the fiber over a $\mathbb{G}_m$-fixed point in the orbit $\mathrm{Gr}_{PGL_2,m}$ is set *trivial*. In particular, $R\Gamma(\mathrm{Gr}_{PGL_2}, \mathcal{V}(n)_1) = V(n)[\frac{1}{2}]$ for $n \geq 0$.

Now we can reformulate Theorem 2.5 of [6] as follows.

Proposition 8.5. *Let $n_1 \geq n_2 \geq \ldots \geq n_k$. Then*
(a) $R\Gamma(\mathrm{Gr}_{PGL_2}, \mathcal{V}(n_1)_1 \star \ldots \star \mathcal{V}(n_k)_1)$ is concentrated in degree $-\frac{k}{2}$;
(b) $R\Gamma(\mathrm{Gr}_{PGL_2}, \mathcal{V}(n_1)_1 \star \ldots \star \mathcal{V}(n_k)_1)[-\frac{k}{2}] \simeq V(n_k) \star \ldots \star V(n_1)$.

8.6. **Multiplication table.** According to Proposition 8.5, the calculation of fusion product in $K(G(\mathbf{O}) \ltimes \mathbb{G}_m - mod)$ is closely related to the ring structure of $K^{G(\mathbf{O})\ltimes\mathbb{G}_m}(\mathrm{Gr}_G)$. Let us formulate the recurrence relations in $K^{G(\mathbf{O})\ltimes\mathbb{G}_m}(\mathrm{Gr}_G)$, compare [6], end of section 2.1. So $\mathbf{v}(n)_m$ is the class of $\mathcal{V}(n)_m$ in $K^{G(\mathbf{O})\ltimes\mathbb{G}_m}(\mathrm{Gr}_G)$. We assume that $n \geq 0$.

$$q^{-l}\mathbf{v}(l+n)_1 0 \star \mathbf{v}(l)_1 = q^{-2l}\mathbf{v}(2l+n)_2 + q^2\mathbf{v}(n-2)_0 + q^4\mathbf{v}(n-4)_0 + \ldots \tag{13}$$

(the last summand being $q^n\mathbf{v}(0)_0$ if n is even, and $q^{n-1}\mathbf{v}(1)_0$ if n is odd.)

$$q^{-l-2}\mathbf{v}(l-n)_1 0 \star \mathbf{v}(l)_1 = q^{-2l-2}\mathbf{v}(2l-n)_2 + q^{-2}\mathbf{v}(n-2)_0 + q^{-4}\mathbf{v}(n-4)_0 + \ldots \tag{14}$$

(the last summand being $q^{-n}\mathbf{v}(0)_0$ if n is even, and $q^{-n+1}\mathbf{v}(1)_0$ if n is odd.)

$$\mathbf{v}(l+1)_1^{\star a} \star \mathbf{v}(l)_1^{\star b} = q^{\frac{1}{2}(a(1-a)+l(a+b)(1-a-b))}\mathbf{v}(a+l(a+b))_{a+b} \tag{15}$$

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