EQUIVARIANT (K-)HOMOLOGY OF AFFINE GRASSMANNIAN AND TODA LATTICE

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1. Introduction

1.1. Let G be an almost simple complex algebraic group, and let Gr_G be its affine Grassmannian. Recall that if we set $\mathbf{O} = \mathbb{C}[[\mathsf{t}]]$, $\mathbf{F} = \mathbb{C}((\mathsf{t}))$, then $\mathrm{Gr}_G = G(\mathbf{F})/G(\mathbf{O})$.

It is well-known that the subgroup ΩK of polynomial loops into a maximal compact subgroup $K \subset G$ projects isomorphically to Gr_G; thus Gr_G acquires the structure of a topological group. An algebro-geometric counterpart of this structure is provided by *the convolution diagram* $G(\mathbf{F}) \times_{G(\mathbf{O})} \mathrm{Gr}_G \to \mathrm{Gr}_G$.

It allows one to define the *convolution* of two $G(\mathbf{O})$ equivariant geometric objects (such as sheaves, or constrictible functions) on Gr_G. A famous example of such a structure is the category of $G(\mathbf{O})$ equivariant perverse sheaves on Gr ("Satake category" in the terminology of Beilinson and Drinfeld); this is a semi-simple abelian category, and convolution provides it with a symmetric monoidal structure. By results of [10], [19], [2] this category is identified with the category of (algebraic) representations of the Langlands dual group.

The starting point for the present work was the observation that a similar definition works in another setting, yielding a monoidal structure on the category of $G(\mathbf{O})$ equivariant *perverse coherent sheaves* on Gr (the "coherent Satake category"). The latter is a non-semisimple artinian abelian category, the heart of the middle perversity t-structure on the derived category of $G(\mathbf{O})$ equivariant coherent sheaves on Gr_G; existence of this t-structure is due to the fact that dimensions of all $G(\mathbf{O})$-orbits inside a given component of Gr_G are of the same parity, cf. [3]. The resulting monoidal category turns out to be non-symmetric, though its Grothendieck ring $K^{G(\mathbf{O})}(\mathrm{Gr}_G)$ is commutative. One of the results of this paper is a computation of this ring. Along with $K^{G(\mathbf{O})}(\mathrm{Gr}_G)$ we compute its "graded version", the ring $H^{G(\mathbf{O})}(\mathrm{Gr})$ of equivariant homology of Gr, where the algebra structure is again provided by convolution.[1] (The ring $H_\bullet^{G(\mathbf{O})}(\mathrm{Gr}_G)$ was essentially computed by Dale Peterson [20], cf. also [15].)

To describe the answer, let $\check{G}$ be the Langlands dual group to G, and let $\check{\mathfrak{g}}$ be its Lie algebra. Consider the *universal centralizers* $\mathfrak{Z}^{\check{G}}_{\check{\mathfrak{g}}}$ and $\mathfrak{Z}^{\check{G}}_{\check{G}}$: if we denote by $C_{\check{G},\check{\mathfrak{g}}} \subset \check{G} \times \check{\mathfrak{g}}$ (resp. $C_{\check{G},\check{G}} \subset \check{G} \times \check{G}$) the locally closed subvariety formed by all the pairs (g, x) such that $Ad_g(x) = x$ and x is regular (resp. all the pairs (g_1, g_2) such that $Ad_{g_1} g_2 = g_2$ and

[1]The two rings are related via the Chern character homomorphism from $K^{G(\mathbf{O})}(\mathrm{Gr})$ to the completion of $H^{G(\mathbf{O})}(\mathrm{Gr})$.

g_2 is regular), then $\mathfrak{Z}^{\check{G}}_{\check{\mathfrak{g}}}$ (resp. $\mathfrak{Z}^{\check{G}}_{\check{G}}$) is the categorical quotient $C_{\check{G},\check{\mathfrak{g}}}//\check{G}$ (resp. $C_{\check{G},\check{G}}//\check{G}$) with respect to the diagonal adjoint action of $\check{G}$.

We identify Spec $\left(H^{G(\mathbf{O})}_{\bullet}(\mathrm{Gr}_G)\right)$ with $\mathfrak{Z}^{\check{G}}_{\check{\mathfrak{g}}}$. Also, we identify Spec $\left(K^{G(\mathbf{O})}(\mathrm{Gr}_G)\right)$ with *a variant* of $\mathfrak{Z}^{\check{G}}_{\check{G}}$ (the isomorphism Spec $\left(K^{G(\mathbf{O})}(\mathrm{Gr}_G)\right) \simeq \mathfrak{Z}^{\check{G}}_{\check{G}}$ holds true iff G is of type E_8).

Notice that $\mathfrak{Z}^{\check{G}}_{\check{\mathfrak{g}}}$ inherits a canonical symplectic structure as a hamiltonian reduction of the cotangent bundle $\mathsf{T}^*\check{G}$. Also, $\mathfrak{Z}^{\check{G}}_{\check{G}}$ inherits a canonical Poisson structure as a q-Hamiltonian reduction of the q-Hamiltonian $\check{G}$-space *internal fusion double* $\mathbf{D}(\check{G})$ (see [1]); this Poisson structure is in fact symplectic iff $\check{G}$ is simply connected (that is, G is adjoint).

The corresponding Poisson structures on $K^{G(\mathbf{O})}(\mathrm{Gr}_G)$, $H^{G(\mathbf{O})}(\mathrm{Gr}_G)$ come from a deformation of these commutative algebras to non-commutative algebras $H^{G(\mathbf{O})\ltimes\mathbb{G}_m}_{\bullet}(\mathrm{Gr}_G)$ (resp. $K^{G(\mathbf{O})\ltimes\mathbb{G}_m}(\mathrm{Gr}_G)$); here $\mathbb{G}_m$ acts on Gr_G by loop rotation. We conjecture that the non-commutative algebra $H^{G(\mathbf{O})\ltimes\mathbb{G}_m}_{\bullet}(\mathrm{Gr}_G)$ can also be obtained from the ring of differential operators on $\check{G}$ by quantum Hamiltonian reduction.

The space $\mathfrak{Z}^{\check{G}}_{\check{\mathfrak{g}}}$ contains an open piece $\mathfrak{Z}(\check{G})$ which for $\check{G}$ adjoint (that is, for G simply connected) is a complexification of the Kostant's phase space of the classical Toda lattice ([14], Theorem 2.6). We remark in passing that Toda lattice also appears in the (apparently related) computations by Givental, Kim and others of quantum cohomology of flag varietites (see e.g. [13]).

Our computation should be compared with (and is to a large extent inspired by) [10] where equivariant cohomology $H_{G(\mathbf{O})}(\mathrm{Gr}_G)$ were computed[2] in terms of the $\check{G}$. (The precise relation between the two computations is spelled out in Remark 2.13).

The second main object considered in the paper is another derived category of coherent sheaves with a convolution monoidal structure, namely the derived category $D^bCoh^{G(\mathbf{O})}_{\Lambda_G}(T^*\,\mathrm{Gr})$ of $G(\mathbf{O})$-equivariant coherent sheaves on the cotangent bundle of Gr_G supported on the union Λ_G of conormal bundles to the $G(\mathbf{O})$-orbits (the definition of involved objects requires extra work since Gr_G is infinite dimensional). (In this case we do not find a t-structure compatible with convolution, so all we get is a monoidal triangulated category). Notice that the singular support of a $G(\mathbf{O})$-equivariant D-module on Gr_G is an object of $Coh^{G(\mathbf{O})}_{\Lambda_G}(T^*\,\mathrm{Gr})$, thus this category can be considered a "classical limit" of the (derived) Satake category. We compute the Grothendieck ring of $D^bCoh^{G(\mathbf{O})}_{\Lambda_G}(T^*\,\mathrm{Gr})$ identifying its spectrum with $(T\times\check{T})/W$, where $T\subset G$, and $\check{T}\subset\check{G}$ are Cartan subgroups. This is a singular variety birationally equivalent to Spec $\left(K^{G(\mathbf{O})}(\mathrm{Gr}_G)\right)$. Unlike the latter, the former remains unchanged if we replace G by $\check{G}$. This motivates a conjecture that the corresponding triangulated monoidal categories for G and $\check{G}$ are equivalent. The conjecture is compatible with a "classical

[2]Another description for $H_{G(\mathbf{O})}(\mathrm{Gr}_G)$ is provided by a general result of [16]; in fact, its extension from [17] gives also an answer for $K^{G(\mathbf{O})}(\mathrm{Gr}_G)$, and a similar technique can be applied to compute $H^{G(\mathbf{O})}(\mathrm{Gr}_G)$. However, this form of the answer does not make the relation to the (dual) group geometry explicit.

limit" of the geometric Langlands conjecture of Beilinson and Drinfeld (see 7.9 below for a more precise statement of the conjecture).

Finally, we remark that the convolution of $G(\mathbf{O})$-equivariant perverse coherent sheaves is closely related to the *fusion product* of $G(\mathbf{O})$-modules introduced by B. Feigin[3] [6] (see Section 8). In fact, our desire to understand the category $\mathcal{P}^{G(\mathbf{O})}(\mathrm{Gr}_G)$, and the work [6] of B. Feigin and S. Loktev, was one of the motivations for the present work.

1.2. **Acknowledgments.** We are obliged to D. Gaitsgory, V. Ginzburg, D. Kazhdan and S. Loktev for help with the references, and especially to P. Etingof for the explanations about integrable systems. R.B. is grateful to the Independent Moscow University and the American Embassy in Moscow for granting him an opportunity to complete the work on this paper; he is partially supported by NSF grant DMS-0071967. M.F. is grateful to the University of Massachusetts at Amherst and the University of Chicago for the hospitality and support. His research was conducted for the Clay Mathematical Institute and partially supported by the CRDF award RM1-2545-MO-03.

2. Notations and statements of the results

2.1. **Kostant slices.** G is an almost simple algebraic group with the Lie algebra $\mathfrak{g}$. We choose a principal $\mathfrak{sl}_2$ triple (e,h,f) in $\mathfrak{g}$. Let $\phi:\ \mathfrak{sl}_2\to\mathfrak{g}$ (resp. $\Phi:\ SL_2\to G$) be the corresponding homomorphism. We denote by e_G (resp. f_G) the image $\Phi\begin{pmatrix}1&1\\0&1\end{pmatrix}$ (resp. $\Phi\begin{pmatrix}1&0\\1&1\end{pmatrix}$). We denote by $\mathfrak{z}(e)$ the centralizer of e in $\mathfrak{g}$, and by $Z(e)$ (resp. $Z^0(e)$) the centralizer of e (equivalently, of e_G) in G (resp. its neutral connected component). We denote by $\Sigma_{\mathfrak{g}}\subset\mathfrak{g}$ (resp. $\Sigma_G\subset G$) the *Kostant slice* $\mathfrak{z}(e)+f$ (resp. $Z^0(e)\cdot f_G$). It is known that $\Sigma_{\mathfrak{g}}\subset\mathfrak{g}^{reg}$ (resp. $\Sigma_G\subset G^{reg}$), and the projection to the categorical quotient $\Sigma_{\mathfrak{g}}\hookrightarrow\mathfrak{g}\twoheadrightarrow\mathfrak{g}//Ad_G=\mathfrak{t}/W$ induces an isomorphism $\Sigma_{\mathfrak{g}}\simeq\mathfrak{t}/W$. Similarly, if G is simply connected, the projection to the categorical quotient $\Sigma_G\hookrightarrow G\twoheadrightarrow G//Ad_G=T/W$ induces an isomorphism $\Sigma_G\simeq T/W$.

2.2. **The universal centralizers.** We consider the locally closed subvariety $C_{\mathfrak{g},\mathfrak{g}}\subset\mathfrak{g}\times\mathfrak{g}$ (resp. $C_{\mathfrak{g},G}\subset\mathfrak{g}\times G$, $C_{G,\mathfrak{g}}\subset G\times\mathfrak{g}$, $C_{G,G}\subset G\times G$) formed by all the pairs (x_1,x_2) such that $[x_1,x_2]=0$ and x_2 is regular (resp. all the pairs (x,g) such that $Ad_g(x)=x$ and g is regular; all the pairs (g,x) such that $Ad_g(x)=x$ and x is regular; all the pairs (g_1,g_2) such that $Ad_{g_1}(g_2)=g_2$ and g_2 is regular). The categorical quotients with respect to the diagonal adjoint action of G are denoted respectively $C_{\mathfrak{g},\mathfrak{g}}//G=\mathfrak{Z}^{\mathfrak{g}}_{\mathfrak{g}}$, $C_{\mathfrak{g},G}//G=\mathfrak{Z}^{\mathfrak{g}}_{G}$, $C_{G,\mathfrak{g}}//G=\mathfrak{Z}^{G}_{\mathfrak{g}}$, $C_{G,G}//G=\mathfrak{Z}^{G}_{G}$. The projections to the second (regular) factor are denoted by $\varpi:\ \mathfrak{Z}^{\mathfrak{g}}_{\mathfrak{g}}\to\mathfrak{g}^{reg}//G=\mathfrak{t}/W$, $\varpi:\ \mathfrak{Z}^{\mathfrak{g}}_{G}\to G^{reg}//G=T/W$, $\varpi:\ \mathfrak{Z}^{G}_{\mathfrak{g}}\to\mathfrak{g}^{reg}//G=\mathfrak{t}/W$, $\varpi:\ \mathfrak{Z}^{G}_{G}\to G^{reg}//G=T/W$. In all the four cases ϖ is flat.

We consider the restrictions of our centralizer varieties to the Kostant slices: $C^{\Sigma}_{\mathfrak{g},\mathfrak{g}}=C_{\mathfrak{g},\mathfrak{g}}\cap(\mathfrak{g}\times\Sigma_{\mathfrak{g}})$, $C^{\Sigma}_{\mathfrak{g},G}=C_{\mathfrak{g},G}\cap(\mathfrak{g}\times\Sigma_G)$, $C^{\Sigma}_{G,\mathfrak{g}}=C_{G,\mathfrak{g}}\cap(G\times\Sigma_{\mathfrak{g}})$, $C^{\Sigma}_{G,G}=C_{G,G}\cap(G\times\Sigma_G)$.

[3]The relation between convolution and fusion was known to B. Feigin since 1997.

Then the locally closed embedding $C^{\Sigma}_{\mathfrak{g},\mathfrak{g}} \hookrightarrow C_{\mathfrak{g},\mathfrak{g}} \twoheadrightarrow \mathfrak{Z}^{\mathfrak{g}}_{\mathfrak{g}}$ induces an isomorphism $C^{\Sigma}_{\mathfrak{g},\mathfrak{g}} \simeq \mathfrak{Z}^{\mathfrak{g}}_{\mathfrak{g}}$. Similarly, we have isomorphisms $C^{\Sigma}_{G,\mathfrak{g}} \simeq \mathfrak{Z}^{G}_{\mathfrak{g}}$ and (for simply connected G) $C^{\Sigma}_{\mathfrak{g},G} \simeq \mathfrak{Z}^{\mathfrak{g}}_{G}$, $C^{\Sigma}_{G,G} \simeq \mathfrak{Z}^{G}_{G}$.

Thus both $\mathfrak{Z}^{\mathfrak{g}}_{\mathfrak{g}} \to \mathfrak{t}/W$ and $\mathfrak{Z}^{\mathfrak{g}}_{G} \to T/W$ (for simply connected G) are the sheaves of abelian Lie algebras, while both $\mathfrak{Z}^{G}_{\mathfrak{g}} \to \mathfrak{t}/W$ and $\mathfrak{Z}^{G}_{G} \to T/W$ (for simply connected G) are the sheaves of abelian Lie groups.

2.3. **Isogenies.** The center $Z(G)$ acts naturally on $\mathfrak{Z}^{\mathfrak{g}}_{G}$ (resp. $\mathfrak{Z}^{G}_{\mathfrak{g}}$) by $z(x,g) = (x, zg)$ (resp. $z(g,x) = (zg, x)$). The center $Z(G)$ acts on $\mathfrak{Z}^{G}_{G}$ on both sides: $z_1(g_1,g_2)z_2 = (z_1g_1, z_2g_2)$. Let $\tilde{G}$ denote the universal cover of G. Then the fundamental group $\pi_1(G)$ is embedded into $Z(\tilde{G})$, and we have $\mathfrak{Z}^{\mathfrak{g}}_{G} = \pi_1(G)\backslash\mathfrak{Z}^{\mathfrak{g}}_{\tilde{G}}$, $\mathfrak{Z}^{G}_{\mathfrak{g}} = \pi_1(G)\backslash\mathfrak{Z}^{\tilde{G}}_{\mathfrak{g}}$, $\mathfrak{Z}^{G}_{G} = \pi_1(G)\backslash\mathfrak{Z}^{\tilde{G}}_{\tilde{G}}/\pi_1(G)$.

2.4. **Symplectic structures.** We fix an invariant identification $\mathfrak{g} \simeq \mathfrak{g}^*$, hence $\mathfrak{t} \simeq \mathfrak{t}^*$. Then $\mathfrak{g} \times \mathfrak{g}$ gets identified with $\mathfrak{g} \times \mathfrak{g}^* = \mathsf{T}^*\mathfrak{g}$ (the cotangent bundle), and $G \times \mathfrak{g}$ gets identified with $G \times \mathfrak{g}^* = \mathsf{T}^*G$. After this $\mathfrak{Z}^{\mathfrak{g}}_{\mathfrak{g}}$ (resp. $\mathfrak{Z}^{G}_{\mathfrak{g}}$) can be viewed as a hamiltonian reduction of $\mathsf{T}^*\mathfrak{g}$ (resp. T^*G); thus it inherits a canonical symplectic structure.

Identifying $\mathfrak{g} \times G$ with $\mathfrak{g}^* \times G = \mathsf{T}^*G$ we can view $\mathfrak{Z}^{\mathfrak{g}}_{G}$ as a hamiltonian reduction of T^*G as well; thus it inherits a canonical Poisson structure. Note that $\mathfrak{Z}^{\mathfrak{g}}_{G}$ is smooth and symplectic iff G is simply connected. We have symplectic isomorphisms $\mathfrak{Z}^{\mathfrak{g}}_{\mathfrak{g}} \simeq \mathsf{T}^*(\mathfrak{t}/W)$, and (in case G is simply connected) $\mathfrak{Z}^{\mathfrak{g}}_{G} \simeq \mathsf{T}^*(T/W)$.

Note that $\mathfrak{Z}^{\mathfrak{g}}_{G}$ and $\mathfrak{Z}^{G}_{\mathfrak{g}}$ share a common open piece $\mathsf{Z}(G)$ formed by the classes of pairs (g,x) where both g and x are regular. The canonical symplectic structures agree on $\mathfrak{Z}^{\mathfrak{g}}_{G} \supset \mathsf{Z}(G) \subset \mathfrak{Z}^{G}_{\mathfrak{g}}$. Note also that for adjoint G the space $\mathsf{Z}(G)$ contains (a complexification of) the Kostant's phase space $\mathfrak{Z}(G)$ of the classical Toda lattice [14], and the embedding $\mathfrak{Z}(G) \hookrightarrow \mathfrak{Z}^{G}_{\mathfrak{g}}$ is given by the Theorem 2.6 of *loc. cit.*

A. Alexeev, A. Malkin and E. Meinrenken introduced in [1] Example 6.1 the q-Hamiltonian G-space *internal fusion double* $\mathbf{D}(G)$. Its q-Hamiltonian reduction is $\mathfrak{Z}^{G}_{G}$, so it inherits a canonical Poisson structure. For a simply connected G the space $\mathfrak{Z}^{G}_{G}$ is smooth and symplectic.

2.5. **Affine blow-ups.** The set of roots of G (resp. $\check{G}$) is denoted by R (resp. $\check{R}$). We will view $\alpha \in R$ (resp. $\check{\alpha} \in \check{R}$) as a homomorphism $\mathfrak{t} \to \mathbb{C}$ (resp. $\check{\mathfrak{t}} \to \mathbb{C}$) or as a homomorphism $T \to \mathbb{C}^*$ (resp. $\check{T} \to \mathbb{C}^*$) depending on a context. Also, for a root $\alpha \in R$ we denote by ${}^1\alpha$ (resp. ${}^2\alpha$) the linear function on $\mathfrak{t}\times\mathfrak{t}$ obtained as a composition of α with the projection on the first (resp. second) factor.

We consider the following affine blow-up of $\mathfrak{t} \times \mathfrak{t}$ at the diagonal walls: $\overset{\bullet}{\mathfrak{B}}{}^{\mathfrak{g}}_{\mathfrak{g}} = \operatorname{Spec}(\mathbb{C}[\mathfrak{t} \times \mathfrak{t}, \frac{{}^1\alpha}{{}^2\alpha}, \alpha \in R])$. We also set $\overset{\bullet}{\mathfrak{B}}{}^{G}_{\mathfrak{g}} = \operatorname{Spec}(\mathbb{C}[\mathfrak{t} \times T, \frac{{}^1\alpha}{{}^2\alpha-1}, \alpha \in R])$; $\overset{\bullet}{\mathfrak{B}}{}^{G}_{G} = \operatorname{Spec}(\mathbb{C}[T \times T, \frac{{}^1\alpha-1}{{}^2\alpha-1}, \alpha \in R])$, $\overset{\bullet}{\mathfrak{B}}{}^{\check{G}}_{G} = \operatorname{Spec}(\mathbb{C}[\check{T} \times T, \frac{{}^1\check{\alpha}-1}{{}^2\alpha-1}, \alpha \in R])$; and let $\mathfrak{B}^{\mathfrak{g}}_{\mathfrak{g}} = \overset{\bullet}{\mathfrak{B}}{}^{\mathfrak{g}}_{\mathfrak{g}}/W$, $\mathfrak{B}^{G}_{G} = \overset{\bullet}{\mathfrak{B}}{}^{G}_{G}/W$, $\mathfrak{B}^{\check{G}}_{G} = \overset{\bullet}{\mathfrak{B}}{}^{\check{G}}_{G}/W$ (thus $\mathfrak{B}^{\mathfrak{g}}_{\mathfrak{g}} = \operatorname{Spec}(\mathbb{C}[\mathfrak{t} \times \mathfrak{t}, \frac{{}^1\alpha}{{}^2\alpha}, \alpha \in R]^W)$, etc.). We denote by ϖ the projection of $\mathfrak{B}$ to the second factor; thus we have $\varpi:\ \mathfrak{B}^{\mathfrak{g}}_{\mathfrak{g}} \to \mathfrak{t}/W$, $\mathfrak{B}^{\mathfrak{g}}_{G} \to T/W$, $\mathfrak{B}^{G}_{\mathfrak{g}} \to \mathfrak{t}/W$, $\mathfrak{B}^{G}_{G} \to T/W$, $\mathfrak{B}^{\check{G}}_{G} \to T/W$.

2.6. **Poisson structures.** We have the canonical trivializations of the tangent bundles $\mathsf{T}(\mathfrak{t}\times\mathfrak{t}) = (\mathfrak{t}\times\mathfrak{t})\times(\mathfrak{t}\times\mathfrak{t})$, $\mathsf{T}(\mathfrak{t}\times T) = (\mathfrak{t}\times T)\times(\mathfrak{t}\times\mathfrak{t})$, $\mathsf{T}(T\times\mathfrak{t}) = (T\times\mathfrak{t})\times(\mathfrak{t}\times\mathfrak{t})$, $\mathsf{T}(T\times T) = (T\times T)\times(\mathfrak{t}\times\mathfrak{t})$, $\mathsf{T}(T\times\check{T}) = (T\times\check{T})\times(\mathfrak{t}\times\check{\mathfrak{t}})$. Making use of the identification $\check{\mathfrak{t}} = \mathfrak{t}^* \simeq \mathfrak{t}$ we obtain the W-invariant symplectic structures on the above varieties. Thus the above affine blow-ups carry the rational Poisson structures (regular off the discriminants $\mathbf{D}\subset\mathfrak{B}$).

Proposition 2.7. *The Poisson structure on* $\mathfrak{B}^{\mathfrak{g}}_{\mathfrak{g}} - \mathbf{D}$ *(resp.* $\mathfrak{B}^{\mathfrak{g}}_{G} - \mathbf{D}$, $\mathfrak{B}^{G}_{\mathfrak{g}} - \mathbf{D}$, $\mathfrak{B}^{G}_{G} - \mathbf{D}$, $\mathfrak{B}^{\check{G}}_{G} - \mathbf{D}$*) extends to the global Poisson structure; it is a symplectic structure if the corresponding variety is smooth.*

Proposition 2.8. *We are in the setup of 2.5.*

a) ϖ *is flat if* G *is simply connected;*

b) There are natural identifications $\mathfrak{B}^{\mathfrak{g}}_{\mathfrak{g}} \simeq \mathfrak{Z}^{\mathfrak{g}}_{\mathfrak{g}}$, $\mathfrak{B}^{\mathfrak{g}}_{G} \simeq \mathfrak{Z}^{\mathfrak{g}}_{G}$, $\mathfrak{B}^{G}_{\mathfrak{g}} \simeq \mathfrak{Z}^{G}_{\mathfrak{g}}$, $\mathfrak{B}^{G}_{G} \simeq \mathfrak{Z}^{G}_{G}$ *commuting with* ϖ.

c) If G *is simply laced and adjoint, we have an identification* $\mathfrak{B}^{G}_{\check{G}} \simeq Z(\check{G})\backslash\mathfrak{Z}^{\check{G}}_{\check{G}}$ *commuting with* ϖ*;*

d) If G *is simply laced and simply connected, we have an identification* $\mathfrak{B}^{G}_{\check{G}} \simeq \mathfrak{Z}^{G}_{G}/Z(G)$ *commuting with* ϖ*;*

e) The above identifications respect the Poisson structures.

2.9. **Flat group sheaves.** We consider the functor $\mathfrak{F}^{\mathfrak{g}}_{\mathfrak{g}}$ on the category $\mathrm{Flat}_{\mathfrak{t}/W}$ of schemes flat over $\mathfrak{t}/W$ to the category of sets, sending a test scheme S to the set of W-invariant morphisms $\left(\mathrm{Mor}(S\times_{\mathfrak{t}/W}\mathfrak{t},\mathfrak{t})\right)^W$. Similarly, we consider the functor $\mathfrak{F}^{\mathfrak{g}}_{G}$ on the category $\mathrm{Flat}_{T/W}$ sending a test scheme S to the set of W-invariant morphisms $\left(\mathrm{Mor}(S\times_{T/W}T,\mathfrak{t})\right)^W$. Also, we consider the functor $\mathfrak{F}^{G}_{\mathfrak{g}}$ on the category $\mathrm{Flat}_{\mathfrak{t}/W}$ sending a test scheme S to the set of W-invariant morphisms $\left(\mathrm{Mor}(S\times_{\mathfrak{t}/W}\mathfrak{t},T)\right)^W_0 \subset \left(\mathrm{Mor}(S\times_{\mathfrak{t}/W}\mathfrak{t},T)\right)^W$ subject to the condition (cf. [5] 4.2)

$$\alpha\left(f(\alpha^{-1}(0))\right) = 1 \ \forall\ \alpha\in R. \tag{1}$$

(note that the W-invariance condition automatically implies $\alpha\left(f(\alpha^{-1}(0))\right) = \pm1\ \forall\ \alpha\in R$.)

Furthermore, we consider the functor $\mathfrak{F}^{G}_{G}$ on the category $\mathrm{Flat}_{T/W}$ sending a test scheme S to the set of W-invariant morphisms $\left(\mathrm{Mor}(S\times_{T/W}T,T)\right)^W_0 \subset \left(\mathrm{Mor}(S\times_{T/W}T,T)\right)^W$ subject to the condition

$$\alpha\left(f(\alpha^{-1}(1))\right) = 1 \ \forall\ \alpha\in R. \tag{2}$$

(note that the W-invariance condition automatically implies $\alpha\left(f(\alpha^{-1}(1))\right) = \pm1\ \forall\ \alpha\in R$.)

Finally, we consider the functor $\mathfrak{F}^{\check{G}}_{G}$ on the category $\mathrm{Flat}_{T/W}$ sending a test scheme S to the set of W-invariant morphisms $\left(\mathrm{Mor}(S\times_{T/W}T,\check{T})\right)^W_0 \subset \left(\mathrm{Mor}(S\times_{T/W}T,\check{T})\right)^W$ subject to the condition

$$\check{\alpha}\left(f(\alpha^{-1}(1))\right) = 1 \ \forall\ \alpha\in R. \tag{3}$$

(note that the W-invariance condition automatically implies $\check{\alpha}\left(f(\alpha^{-1}(1))\right) = \pm 1\ \forall\ \alpha \in R$.)

The following Proposition is a generalization of [5] 11.6.

Proposition 2.10. *Assume that G is simply connected. The functor $\mathfrak{F}_{\mathfrak{g}}^{\mathfrak{g}}$ (resp. $\mathfrak{F}_G^{\mathfrak{g}}$, $\mathfrak{F}_{\mathfrak{g}}^G$, $\mathfrak{F}_G^G$, $\mathfrak{F}_G^{\check{G}}$) is representable by the scheme $\mathfrak{B}_{\mathfrak{g}}^{\mathfrak{g}}$ (resp. $\mathfrak{B}_G^{\mathfrak{g}}$, $\mathfrak{B}_{\mathfrak{g}}^G$, $\mathfrak{B}_G^G$, $\mathfrak{B}_G^{\check{G}}$).*

2.11. **Equivariant Borel-Moore Homology.** For the definition of convolution in equivariant Borel-Moore Homology we refer the reader to [4] 2.7, 8.3 or [18] Chapter 2.

We have $H_\bullet^{G(\mathbf{O})}(pt) = H^\bullet_{G(\mathbf{O})}(pt) = \mathbb{C}[\mathfrak{t}/W]$, and $H_\bullet^{G(\mathbf{O})\ltimes\mathbb{G}_m}(pt) = H^\bullet_{G(\mathbf{O})\ltimes\mathbb{G}_m}(pt) = \mathbb{C}[\mathfrak{t}/W][\hbar]$ where $\hbar$ is the generator of $H^2_{\mathbb{G}_m}(pt)$. We will consider the $\mathbb{C}[\mathfrak{t}/W]$-algebra (resp. $\mathbb{C}[\mathfrak{t}/W][\hbar]$-algebra) (with respect to convolution) $H_\bullet^{G(\mathbf{O})}(\mathrm{Gr}_G)$ (resp. $H_\bullet^{G(\mathbf{O})\ltimes\mathbb{G}_m}(\mathrm{Gr}_G)$). Note that setting $\hbar = 0$ in $H_\bullet^{G(\mathbf{O})\ltimes\mathbb{G}_m}(\mathrm{Gr}_G)$ we obtain $H_\bullet^{G(\mathbf{O})}(\mathrm{Gr}_G)$; indeed for any group H, a space X with an $H \times \mathbb{G}_m$ action, and an $H \times \mathbb{G}_m$-equivariant complex $\mathcal{F}$ on X we have a long exact sequence $\cdots \to H^{i-2}_{H\times\mathbb{G}_m}(X,\mathcal{F}) \xrightarrow{\hbar} H^i_{H\times\mathbb{G}_m}(X,\mathcal{F}) \to H^i_H(X,\mathcal{F}) \to H^{i-1}_{H\times\mathbb{G}_m}(X,\mathcal{F}) \to \dots$ coming from the principal $\mathbb{G}_m$-bundle $E(H\times\mathbb{G}_m)\times_H X \to E(H\times\mathbb{G}_m)\times_{H\times\mathbb{G}_m} X$; if the space of $H \times \mathbb{G}_m$-equivariant cohomology is $\hbar$-torsion free, then we get $H^\bullet_H(X,\mathcal{F}) = H^\bullet(X,\mathcal{F})|_{\hbar=0}$.

Theorem 2.12. *a) The algebra $H_\bullet^{G(\mathbf{O})}(\mathrm{Gr}_G)$ is commutative;*

b) Its spectrum together with the projection onto $\mathfrak{t}/W = \check{\mathfrak{t}}/W$ is naturally isomorphic to $\mathfrak{Z}_{\check{\mathfrak{g}}}^{\check{G}} \xrightarrow{\varpi} \check{\mathfrak{t}}/W$;

c) The Poisson structure on $H_\bullet^{G(\mathbf{O})}(\mathrm{Gr}_G)$ arising from the $\hbar$-deformation $H_\bullet^{G(\mathbf{O})\ltimes\mathbb{G}_m}(\mathrm{Gr}_G)$, corresponds under the above identification to the Poisson structure of 2.4 on $\mathfrak{Z}_{\check{\mathfrak{g}}}^{\check{G}}$.

Remark 2.13. The equivariant cohomology ring $H^\bullet_{G(\mathbf{O})}(\mathrm{Gr}_G,\mathbb{C}) = H^\bullet_{G(\mathbf{O})}(\mathrm{Gr}_G)$ was computed by V. Ginzburg [10]. More precisely, the projection to the second (regular) factor $\mathfrak{Z}_{\check{\mathfrak{g}}}^{\check{\mathfrak{g}}} \to \check{\mathfrak{g}}^{reg}//\check{G} = \check{\mathfrak{t}}/W$ makes $\mathfrak{Z}_{\check{\mathfrak{g}}}^{\check{\mathfrak{g}}}$ a sheaf of abelian Lie algebras. V. Ginzburg identifies $H^\bullet_{G(\mathbf{O})}(\mathrm{Gr}_G)$ with the global sections of the relative universal enveloping algebra $U_{\check{\mathfrak{t}}/W}\left(\mathfrak{Z}_{\check{\mathfrak{g}}}^{\check{\mathfrak{g}}}\right)$. One can easily check that this result is compatible with our Theorem 2.12(b) as follows. For a group scheme A over a base S one has a natural pairing $U(\mathfrak{a}) \times \mathcal{O}(A) \to \mathcal{O}(S)$ where $U(\mathfrak{a})$ is the enveloping (over $\mathcal{O}(S)$) of the Lie algebra of A; the pairing sends (ξ, f) to $\xi(f)$ restricted to the identity of A. On the other hand, for a compact (or ind-compact) H-space X we have a pairing $H^\bullet_H(X) \times H_\bullet^H(X) \to H^\bullet_H(pt)$ induced by the action of cohomology on homology, and the push-forward map in Borel-Moore homology $H_\bullet^H(X) \to H^\bullet_H(pt)$. The isomorphisms of [10] and of Theorem 2.12 take the first pairing into the second one.

2.14. **Equivariant K-theory.** For the definition of convolution in equivariant K-theory we refer the reader to Chapter 5 of [4].

We have $K^{G(\mathbf{O})}(pt) = \mathbb{C}[T/W]$, and $K^{G(\mathbf{O})\ltimes\mathbb{G}_m}(pt) = \mathbb{C}[T/W][q^{\pm 1}]$. We will consider the $\mathbb{C}[T/W]$-algebra (resp. $\mathbb{C}[T/W][q^{\pm 1}]$-algebra) (with respect to convolution)

$K^{G(\mathbf{O})}(\mathrm{Gr}_G)$ (resp. $K^{G(\mathbf{O})\ltimes\mathbb{G}_m}(\mathrm{Gr}_G)$). Note that setting $q=1$ in $K^{G(\mathbf{O})\ltimes\mathbb{G}_m}(\mathrm{Gr}_G)$ we obtain $K^{G(\mathbf{O})}(\mathrm{Gr}_G)$.

Theorem 2.15. *a) The algebra $K^{G(\mathbf{O})}(\mathrm{Gr}_G)$ is commutative;*

b) Its spectrum together with the projection onto T/W is naturally isomorphic to $\mathfrak{B}_G^{\check{G}} \stackrel{\varpi}{\to} T/W$;

c) The Poisson structure on $K^{G(\mathbf{O})}(\mathrm{Gr}_G)$ arising from the q-deformation $K^{G(\mathbf{O})\ltimes\mathbb{G}_m}(\mathrm{Gr}_G)$, corresponds under the above identification to the Poisson structure of 2.7 on $\mathfrak{B}_G^{\check{G}}$ in case the latter variety is smooth, i.e. G is simply connected.

3. Calculations in rank 1

In this section $G = SL_2$, and $\check{G} = PGL_2$. The Weyl group $W = \mathbb{Z}/2\mathbb{Z}$, the Cartan torus $T = \mathbb{G}_m = \mathbb{C}^*$ with a coordinate z, and the only simple root $\alpha(z) = z^2$. The dual torus $\check{T} = \mathbb{G}_m = \mathbb{C}^*$ with a coordinate t, and $\check{\alpha}(t) = t$. The Cartan Lie algebra $\mathfrak{t} = \mathbb{C}$ with a coordinate $x = \alpha(x)$. We fix a $\sqrt{-1}$.

3.1. **$\mathfrak{Z}_G^G$ and $\mathfrak{B}_G^G$.** We choose the standard $\mathfrak{sl}_2$-triple $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Then the Kostant slice $\Sigma_G = \left\{ \begin{pmatrix} a-1 & a-2 \\ 1 & 1 \end{pmatrix},\ a \in \mathbb{C} \right\}$. One checks that a matrix $\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$ commutes with $\begin{pmatrix} a-1 & a-2 \\ 1 & 1 \end{pmatrix}$ iff $\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \sqrt{-1} \begin{pmatrix} (1-a)c+b & (2-a)c \\ -c & b-c \end{pmatrix}$ for $b, c \in \mathbb{C}$. Then the condition $\det = 1$ reads as

$$1 = abc - b^2 - c^2. \tag{4}$$

Thus, $\mathfrak{Z}_G^G$ is identified with a hypersurface $\mathcal{S}$ in $\mathbb{A}^3$ given by the equation (4). The left (resp. right) multiplication by $-1 \in Z(SL_2)$ is an involution $\imath$ (resp. $\jmath$) on $\mathcal{S}$ given by $\imath(a,b,c) = (a,-b,-c)$ (resp. $\jmath(a,b,c) = (-a,b,-c)$). Hence, $\mathfrak{Z}_{\check{G}}^{\check{G}} = \imath\backslash\mathcal{S}/\jmath$.

Generically, we can diagonalize two commuting matrices simultaneously, that is, there is $g \in SL_2$ such that $g\sqrt{-1}\begin{pmatrix} (1-a)c+b & (2-a)c \\ -c & b-c \end{pmatrix} g^{-1} = \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix}$ and $g\begin{pmatrix} a-1 & a-2 \\ 1 & 1 \end{pmatrix} g^{-1} = \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}$ for some $y, z \in \mathbb{G}_m = \mathbb{C}^* = T$ defined up to simultaneous inversion. Then we have

$$a = z + z^{-1},\ b = \frac{-\sqrt{-1}}{2}\left(y + y^{-1} + \frac{(y-y^{-1})(z+z^{-1})}{z - z^{-1}}\right),\ c = -\sqrt{-1}\frac{y-y^{-1}}{z-z^{-1}}. \tag{5}$$

We conclude that $\mathbb{C}[\mathcal{S}] = \mathbb{C}[y^{\pm1}, z^{\pm1}, \frac{y-y^{-1}}{z-z^{-1}}]^W$ where the nontrivial element $w \in W$ acts by $w(y,z) = (y^{-1}, z^{-1})$. We can rewrite $\mathbb{C}[y^{\pm1}, z^{\pm1}, \frac{y-y^{-1}}{z-z^{-1}}]^W$ as $\mathbb{C}[y^{\pm1}, z^{\pm1}, \frac{y^2-1}{z^2-1}]^W$ to manifest its coincidence with $\mathbb{C}[\mathfrak{B}_G^G]$. All in all, we have $\mathfrak{B}_G^G \simeq \mathcal{S} \simeq \mathfrak{Z}_G^G$. Since we can identify $\check{T}$ with $T/Z(G)$, the identifications $\mathfrak{B}^G_{\check{G}} \simeq \mathcal{S}/\jmath$, $\mathfrak{B}^{\check{G}}_G \simeq \imath\backslash\mathcal{S}$, $\mathfrak{B}^{\check{G}}_{\check{G}} \simeq \imath\backslash\mathcal{S}/\jmath \simeq \mathfrak{Z}^{\check{G}}_{\check{G}}$ follow immediately.

3.2. $\mathfrak{Z}^G_{\mathfrak{g}}$ **and** $\mathfrak{B}^G_{\mathfrak{g}}$**.** The Kostant slice $\Sigma_{\mathfrak{g}} = \left\{ \begin{pmatrix} 0 & \delta \\ 1 & 0 \end{pmatrix},\ \delta \in \mathbb{C} \right\}$. One checks that a matrix $\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$ commutes with $\begin{pmatrix} 0 & \delta \\ 1 & 0 \end{pmatrix}$ iff $\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} \xi & \delta\eta \\ \eta & \xi \end{pmatrix}$ for $\xi, \eta \in \mathbb{C}$. Then the condition det $= 1$ reads as

$$1 = \xi^2 - \delta\eta^2. \tag{6}$$

Thus, $\mathfrak{Z}^G_{\mathfrak{g}}$ is identified with a hypersurface $\mathcal{S}'$ in $\mathbb{A}^3$ given by the equation (6). The action of $-1 \in Z(SL_2)$ is an involution $\imath$ on $\mathcal{S}'$ given by $\imath(\delta, \xi, \eta) = (\delta, -\xi, -\eta)$. Hence, $\mathfrak{Z}^{\check{G}}_{\check{\mathfrak{g}}} = \imath\backslash\mathcal{S}'$.

Generically, we can diagonalize two commuting matrices simultaneously, that is, there is $g \in SL_2$ such that $g \begin{pmatrix} \xi & \delta\eta \\ \eta & \xi \end{pmatrix} g^{-1} = \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix}$ and $g \begin{pmatrix} 0 & \delta \\ 1 & 0 \end{pmatrix} g^{-1} = \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix}$ for some $y \in \mathbb{G}_m = \mathbb{C}^* = T,\ x \in \mathbb{C} = \mathfrak{t}$, defined up to $(y, x) \mapsto (y^{-1}, -x)$. Then we have

$$\delta = x^2,\ \xi = \frac{y + y^{-1}}{2},\ \eta = \frac{y - y^{-1}}{2x}.$$

We conclude that $\mathbb{C}[\mathcal{S}'] = \mathbb{C}[y^{\pm 1}, x, \frac{y-y^{-1}}{x}]^W$ where the nontrivial element $w \in W$ acts by $w(y, x) = (y^{-1}, -x)$. We can rewrite $\mathbb{C}[y^{\pm 1}, x, \frac{y-y^{-1}}{x}]^W$ as $\mathbb{C}[y^{\pm 1}, x, \frac{y^2-1}{x}]^W$ to manifest its coincidence with $\mathbb{C}[\mathfrak{B}^G_{\mathfrak{g}}]$. All in all, we have $\mathfrak{B}^G_{\mathfrak{g}} \simeq \mathcal{S}' \simeq \mathfrak{Z}^G_{\mathfrak{g}}$. Since we can identify $\check{T}$ with $T/Z(G)$, the identification $\mathfrak{B}^{\check{G}}_{\check{\mathfrak{g}}} \simeq \imath\backslash\mathcal{S}' \simeq \mathfrak{Z}^{\check{G}}_{\check{\mathfrak{g}}}$ follows immediately.

3.3. $\mathfrak{Z}^{\mathfrak{g}}_G$ **and** $\mathfrak{B}^{\mathfrak{g}}_G$**.** Recall the Kostant slice $\Sigma_G = \left\{ \begin{pmatrix} a-1 & a-2 \\ 1 & 1 \end{pmatrix},\ a \in \mathbb{C} \right\}$. One checks that a traceless matrix $\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & -x_{11} \end{pmatrix}$ commutes with $\begin{pmatrix} a-1 & a-2 \\ 1 & 1 \end{pmatrix}$ iff $\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & -x_{11} \end{pmatrix} = \zeta \begin{pmatrix} 2-a & 4-2a \\ -2 & a-2 \end{pmatrix}$ for $\zeta \in \mathbb{C}$.

Thus, $\mathfrak{Z}^{\mathfrak{g}}_G$ is identified with $\mathbb{A}^2$ with coordinates a, ζ. The action of $-1 \in Z(SL_2)$ is an involution $\jmath$ on $\mathbb{A}^2$ given by $\jmath(a, \zeta) = (-a, -\zeta)$. Hence, $\mathfrak{Z}^{\check{\mathfrak{g}}}_{\check{G}} = \mathbb{A}^2/\jmath$.

Generically, we can diagonalize two commuting matrices simultaneously, that is, there is $g \in SL_2$ such that $g\zeta \begin{pmatrix} 2-a & 4-2a \\ -2 & a-2 \end{pmatrix} g^{-1} = \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix}$ and $g \begin{pmatrix} a-1 & a-2 \\ 1 & 1 \end{pmatrix} g^{-1} = \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}$ for some $x \in \mathbb{C} = \mathfrak{t},\ z \in \mathbb{G}_m = \mathbb{C}^* = T$ defined up to $(x, z) \mapsto (-x, z^{-1})$. Then we have

$$a = z + z^{-1},\ \zeta = \frac{x}{z - z^{-1}}.$$

We conclude that $\mathbb{C}[\mathbb{A}^2]=\mathbb{C}[x,z^{\pm1},\frac{x}{z-z^{-1}}]^W$ where the nontrivial element $w\in W$ acts by $w(x,z)=(-x,z^{-1})$. We can rewrite $\mathbb{C}[x,z^{\pm1},\frac{x}{z-z^{-1}}]^W$ as $\mathbb{C}[x,z^{\pm1},\frac{x}{z^2-1}]^W$ to manifest its coincidence with $\mathbb{C}[\mathfrak{B}^{\mathfrak{g}}_G]$. All in all, we have $\mathfrak{B}^{\mathfrak{g}}_G\simeq\mathbb{A}^2\simeq\mathfrak{Z}^{\mathfrak{g}}_G$. Since we can identify $\check{T}$ with $T/Z(G)$, the identification $\mathfrak{B}^{\check{\mathfrak{g}}}_{\check{G}}\simeq\mathbb{A}^2/\jmath\simeq\mathfrak{Z}^{\check{\mathfrak{g}}}_{\check{G}}$ follows immediately.

3.4. $\mathfrak{Z}^{\mathfrak{g}}_{\mathfrak{g}}$ **and** $\mathfrak{B}^{\mathfrak{g}}_{\mathfrak{g}}$**.** Recall the Kostant slice $\Sigma_{\mathfrak{g}}=\left\{\begin{pmatrix}0&\delta\\1&0\end{pmatrix},\ \delta\in\mathbb{C}\right\}$. One checks that a traceless matrix $\begin{pmatrix}x_{11}&x_{12}\\x_{21}&-x_{11}\end{pmatrix}$ commutes with $\begin{pmatrix}0&\delta\\1&0\end{pmatrix}$ iff $\begin{pmatrix}x_{11}&x_{12}\\x_{21}&-x_{11}\end{pmatrix}=\begin{pmatrix}0&\delta\theta\\\theta&0\end{pmatrix}$ for $\theta\in\mathbb{C}$. Thus, $\mathfrak{Z}^{\mathfrak{g}}_{\mathfrak{g}}$ is identified with $\mathbb{A}^2$ with coordinates δ,θ.

Generically, we can diagonalize two commuting matrices simultaneously, that is, there is $g\in SL_2$ such that $g\begin{pmatrix}0&\delta\theta\\\theta&0\end{pmatrix}g^{-1}=\begin{pmatrix}u&0\\0&-u\end{pmatrix}$ and $g\begin{pmatrix}0&\delta\\1&0\end{pmatrix}g^{-1}=\begin{pmatrix}x&0\\0&-x\end{pmatrix}$ for some $u,x\in\mathbb{C}=\mathfrak{t}$, defined up to $(u,x)\mapsto(-u,-x)$. Then we have

$$\delta=x^2,\ \theta=\frac{u}{x}.$$

We conclude that $\mathbb{C}[\mathbb{A}^2]=\mathbb{C}[u,x,\frac{u}{x}]^W$ where the nontrivial element $w\in W$ acts by $w(u,x)=(-u,-x)$. Hence we get an identification $\mathfrak{B}^{\mathfrak{g}}_{\mathfrak{g}}\simeq\mathbb{A}^2\simeq\mathfrak{Z}^{\mathfrak{g}}_{\mathfrak{g}}$.

3.5. $\mathfrak{B}^G_{\mathfrak{g}}$ **and** $\mathfrak{F}^G_{\mathfrak{g}}$**.** Recall the setup of Proposition 2.10. We will prove that the functor $\mathfrak{F}^G_{\mathfrak{g}}$ is representable by the scheme $\mathfrak{B}^G_{\mathfrak{g}}$; the other parts of the Proposition are proved absolutely similarly, as well as the Proposition for G replaced by $\check{G}$. For a scheme S flat over $\mathfrak{t}/W$ we will denote by $S_{\mathfrak{t}}$ the cartesian product $S\times_{\mathfrak{t}/W}\mathfrak{t}$. Our usual coordinate x on $\mathfrak{t}$ gives rise to the same named function on $S_{\mathfrak{t}}$. The nontrivial element $w\in W$ acts by the involution of $S_{\mathfrak{t}}$. Finally, we denote by $(\mathfrak{B}^G_{\mathfrak{g}})_{S_{\mathfrak{t}}}$ the affine blow-up of $S_{\mathfrak{t}}\times T$, that is $S_{\mathfrak{t}}\times_{\mathfrak{t}}\mathfrak{B}^G_{\mathfrak{g}}$. Clearly, w acts as an involution of $(\mathfrak{B}^G_{\mathfrak{g}})_{S_{\mathfrak{t}}}$.

Note that the condition (1) is void in the case under consideration. Given a w-equivariant morphism $f:\ S_{\mathfrak{t}}\to T=\mathbb{G}_m$ we see that f^2-1 is divisible by x, hence f lifts uniquely to a section $\hat{f}$ of $(\mathfrak{B}^G_{\mathfrak{g}})_{S_{\mathfrak{t}}}$ over $S_{\mathfrak{t}}$. Evidently, $\hat{f}$ is w-invariant. If we consider $\hat{f}$ as a closed subscheme of $(\mathfrak{B}^G_{\mathfrak{g}})_{S_{\mathfrak{t}}}$, then $\hat{f}/W$ is a closed subscheme of $(\mathfrak{B}^G_{\mathfrak{g}})_{S_{\mathfrak{t}}}/W=S\times_{\mathfrak{t}/W}\mathfrak{B}^G_{\mathfrak{g}}$ which is the graph of a morphism $\tilde{f}:\ S\to\mathfrak{B}^G_{\mathfrak{g}}$.

Conversely, given a morphism $\tilde{f}:\ S\to\mathfrak{B}^G_{\mathfrak{g}}$ we consider its graph $\Gamma_{\tilde{f}}$ as a closed subscheme of $S\times_{\mathfrak{t}/W}\mathfrak{B}^G_{\mathfrak{g}}$, and then the cartesian product $\Gamma_{\tilde{f}}\times_{S\times_{\mathfrak{t}/W}\mathfrak{B}^G_{\mathfrak{g}}}(\mathfrak{B}^G_{\mathfrak{g}})_{S_{\mathfrak{t}}}$ is a section $\hat{f}$ of $(\mathfrak{B}^G_{\mathfrak{g}})_{S_{\mathfrak{t}}}$ over $S_{\mathfrak{t}}$. Evidently, $\hat{f}$ gives rise to a w-equivariant function $f:\ S_{\mathfrak{t}}\to T$.

3.6. **A basis in equivariant K-theory.** We recall a few standard facts about the affine Grassmannians Gr_G and $\mathrm{Gr}_{\check{G}}$. The $G(\mathbf{O})$-orbits (equivalently, $\check{G}(\mathbf{O})$-orbits) on $\mathrm{Gr}_{\check{G}}$ are numbered by nonnegative integers and denoted by $\mathrm{Gr}_{\check{G},n},\ n\in\mathbb{N}$. The orbits $\mathrm{Gr}_{\check{G},2n},\ n\in\mathbb{N}$, form a connected component of $\mathrm{Gr}_{\check{G}}$ equal to Gr_G. The open embedding of an orbit into its closure will be denoted by $j_n:\ \mathrm{Gr}_{\check{G},n}\hookrightarrow\overline{\mathrm{Gr}}_{\check{G},n}$ or simply by j if no confusion is likely. We have $\dim\mathrm{Gr}_{\check{G},n}=n$; in particular, $\mathrm{Gr}_{\check{G},0}$ is a point.

We have $K^{G(\mathbf{O})}(\mathrm{Gr}_{\check{G},0}) = \mathrm{Rep}(G)$ with a basis $\mathbf{v}(n),\ n \in \mathbb{N}$, formed by the classes of irreducible G-modules $\mathcal{V}(n)$. Also, $K^{\check{G}(\mathbf{O})}(\mathrm{Gr}_{\check{G},0}) = \mathrm{Rep}(\check{G}) \subset \mathrm{Rep}(G)$ has a basis $\mathbf{v}(2n),\ n \in \mathbb{N}$.

For $m > 0$ the $G(\mathbf{O})$-equivariant line bundles in $\mathrm{Gr}_{\check{G},m}$ are numbered by integers and denoted by $\mathcal{L}(n)_m$. Among them, the $\check{G}(\mathbf{O})$-equivariant line bundles are exactly $\mathcal{L}(2n)_m,\ n \in \mathbb{Z}$. We define $\mathcal{V}(n)_m$ as $j_*\mathcal{L}(n)_m[\frac{m}{2}]$, that is, the (nonderived) direct image to the orbit closure placed in the homological degree $-\frac{m}{2}$. Note that since the complement $\overline{\mathrm{Gr}}_{\check{G},m} - \mathrm{Gr}_{\check{G},m}$ has codimension 2, the above direct image is a coherent sheaf. The degree shift will become clear later. The class $[\mathcal{L}(n)_m]$ in $K^{G(\mathbf{O})}(\mathrm{Gr}_{\check{G}})$ will be denoted by $\mathbf{v}(n)_m$. Thus, it is natural to denote $\mathbf{v}(n)$ above by $\mathbf{v}(n)_0$; we will keep both names.

The collection $\{\mathbf{v}(n)_m :\ n \in \mathbb{N} \text{ if } m = 0;\ n \in \mathbb{Z} \text{ if } m \in \mathbb{N} - 0\}$ forms a basis in $K^{G(\mathbf{O})}(\mathrm{Gr}_{\check{G}})$. Among this collection, all the $\mathbf{v}(n)_m$ with n even (resp. m even) form a basis in $K^{\check{G}(\mathbf{O})}(\mathrm{Gr}_{\check{G}})$ (resp. $K^{G(\mathbf{O})}(\mathrm{Gr}_G)$).

3.7. **Convolution: commutativity.** In this subsection G is an arbitrary semisimple group. We prove 2.15 (a). We refer the reader to [7] for the basics of Beilinson-Drinfeld Grassmannian. Recall that $\mathrm{Gr}_G^{BD} \xrightarrow{\pi} \mathbb{A}^1$ is a flat ind-scheme such that $\pi^{-1}(\mathbb{A}^1 - 0) = (\mathbb{A}^1 - 0) \times \mathrm{Gr}_G \times \mathrm{Gr}_G$, while $\pi^{-1}(0) = \mathrm{Gr}_G$. We also have the deformed convolution diagram $\mathrm{Gr}_G^{BD,conv} \xrightarrow{\Pi} \mathrm{Gr}_G^{BD}$ such that Π is an isomorphism over $\mathbb{A}^1 - 0$, while over $0 \in \mathbb{A}^1$ our Π is the usual convolution diagram $G(\mathbf{F}) \times_{G(\mathbf{O})} \mathrm{Gr}_G \xrightarrow{\Pi_0} \mathrm{Gr}_G$.

Given two $G(\mathbf{O})$-equivariant complexes of coherent sheaves $\mathcal{A}, \mathcal{B}$ on Gr_G, we can form their "deformed convolution" complex $\mathcal{A}\widetilde{\star}\mathcal{B}$ on $\mathrm{Gr}_G^{BD,conv}$ such that over $\mathbb{A}^1 - 0$ it is isomorphic to $\mathcal{O}_{\mathbb{A}^1-0} \boxtimes \mathcal{A} \boxtimes \mathcal{B}$, while over $0 \in \mathbb{A}^1$ it is isomorphic to the usual twisted product $\mathcal{A} \ltimes \mathcal{B}$ on the convolution diagram $G(\mathbf{F}) \times_{G(\mathbf{O})} \mathrm{Gr}_G$. In addition, if $\mathcal{A}, \mathcal{B}$ are coherent sheaves, then $\mathcal{A}\widetilde{\star}\mathcal{B}$ is flat over $\mathbb{A}^1$. It implies that in the K-group the class $[\mathcal{A} \ltimes \mathcal{B}]$ is the *specialization* (see [4] 5.3) of the class $[\mathcal{O}_{\mathbb{A}^1-0} \boxtimes \mathcal{A} \boxtimes \mathcal{B}]$ in the family $\mathrm{Gr}_G^{BD,conv} \xrightarrow{\pi\circ\Pi} \mathbb{A}^1$, and also the class $[\mathcal{A} \star \mathcal{B}] = [\Pi_{0*}(\mathcal{A} \ltimes \mathcal{B})]$ is the specialization of the class $[\mathcal{O}_{\mathbb{A}^1-0} \boxtimes \mathcal{A} \boxtimes \mathcal{B}]$ in the family $\mathrm{Gr}_G^{BD} \xrightarrow{\pi} \mathbb{A}^1$. Hence the desired commutativity.

3.8. **Convolution: relations.** We return to the setup of 3.6. Note that $\mathrm{Gr}_{\check{G},1} \simeq \mathbb{P}^1$, and $\mathcal{V}(n)_1$ is the line bundle $\mathcal{O}(n)$ on $\mathbb{P}^1$. The twisted product $\mathcal{V}(n)_1 \ltimes \mathcal{V}(l)_1$ is the line bundle $\mathcal{O}(n,l)$ on the 2-dimensional subvariety $\mathcal{H}_2 \subset \check{G}(\mathbf{F}) \times_{\check{G}(\mathbf{O})} \mathrm{Gr}_{\check{G}}$ isomorphic to the Hirzebruch surface $\mathbb{P}(\mathcal{O}(2) \oplus \mathcal{O})$ over $\mathbb{P}^1$. The projection $\Pi_0 :\ \mathcal{H}_2 \to \mathrm{Gr}_{\check{G},2}$ is the contraction of the -2-section $\mathbb{P}^1 \hookrightarrow \mathcal{H}_2$.

Now it is easy to compute $\mathbf{v}(n)_1 \star \mathbf{v}(n)_1 = \mathbf{v}(2n)_2,\ \mathbf{v}(1)_1 \star \mathbf{v}(-1)_1 = \mathbf{v}(0)_2 + 1$. Taking into account the evident relation $\mathbf{v}(1)_0 \star \mathbf{v}(0)_1 = \mathbf{v}(1)_1 + \mathbf{v}(-1)_1$ we arrive at

$$\mathbf{v}(1)_0 \star \mathbf{v}(0)_1 \star \mathbf{v}(1)_1 = \mathbf{v}(1)_1 \star \mathbf{v}(1)_1 + \mathbf{v}(0)_1 \star \mathbf{v}(0)_1 + 1. \tag{7}$$

A moment of reflection shows that $K^{G(\mathbf{O})}(\mathrm{Gr}_G)$ is generated as algebra by $\mathbf{v}(1)_0,\ \mathbf{v}(0)_2 = \mathbf{v}(0)_1 \star \mathbf{v}(0)_1,\ \mathbf{v}(2)_2 = \mathbf{v}(1)_1 \star \mathbf{v}(1)_1,\ \mathbf{v}(1)_2 = \mathbf{v}(1)_1 \star \mathbf{v}(0)_1$ (one has to use that $\mathbf{v}(k)_{2l} \star \mathbf{v}(n)_{2m} = \mathbf{v}(k+n)_{2l+2m}$ plus the terms supported on the smaller orbits). Similarly, $K^{\check{G}(\mathbf{O})}(\mathrm{Gr}_{\check{G}})$ is generated as algebra by $\mathbf{v}(2)_0 = \mathbf{v}(1)_0 \star \mathbf{v}(1)_0 - 1,\ \mathbf{v}(0)_1,\ \mathbf{v}(2)_2 = \mathbf{v}(1)_1 \star \mathbf{v}(1)_1,\ \mathbf{v}(2)_1 = \mathbf{v}(1)_1 \star \mathbf{v}(1)_0 - \mathbf{v}(0)_1$.

Note that both algebras $K^{G(\mathbf{O})}(\mathrm{Gr}_G)$ and $K^{\check{G}(\mathbf{O})}(\mathrm{Gr}_{\check{G}})$ lie in the vector space $K^{G(\mathbf{O})}(\mathrm{Gr}_{\check{G}})$, and their intersection is the common subalgebra $K^{\check{G}(\mathbf{O})}(\mathrm{Gr}_G)$. The tensor product algebra $K^{G(\mathbf{O})}(\mathrm{Gr}_G) \otimes_{K^{\check{G}(\mathbf{O})}(\mathrm{Gr}_G)} K^{\check{G}(\mathbf{O})}(\mathrm{Gr}_{\check{G}})$ can be identified as a vector space with $K^{G(\mathbf{O})}(\mathrm{Gr}_{\check{G}})$, and then it is generated by the three basic elements $\mathbf{v}(1)_0, \mathbf{v}(0)_1, \mathbf{v}(1)_1$ subject to the only relation (7).

The comparison of equations (7) and (4) shows that the assignment $a \mapsto \mathbf{v}(1)_0,\ b \mapsto \mathbf{v}(0)_1,\ c \mapsto \mathbf{v}(1)_1$ establishes an isomorphism $\mathbb{C}[\mathcal{S}] \simeq K^{G(\mathbf{O})}(\mathrm{Gr}_{\check{G}})$. It identifies the spectrum of $K^{G(\mathbf{O})}(\mathrm{Gr}_G)$ with $\imath\backslash\mathcal{S} \simeq \mathfrak{B}_G^{\check{G}}$, and the spectrum of $K^{\check{G}(\mathbf{O})}(\mathrm{Gr}_{\check{G}})$ with $\mathcal{S}/\jmath \simeq \mathfrak{B}_{\check{G}}^G$.

3.9. **Iwahori-equivariant K-theory.** Let $I \subset G(\mathbf{O})$ be the Iwahori subgroup. The space $K^I(\mathrm{Gr}_G) = K^T(\mathrm{Gr}_G) = K^{T(\mathbf{O})}(\mathrm{Gr}_G) = K(T(\mathbf{O})\backslash G(\mathbf{F})/G(\mathbf{O}))$ is equipped with the two commuting actions: $K(T(\mathbf{O})\backslash T(\mathbf{F})/T(\mathbf{O}))$ acts by convolutions on the left, and $K^G(\mathrm{Gr}_G) = K^{G(\mathbf{O})}(\mathrm{Gr}_G) = K(G(\mathbf{O})\backslash G(\mathbf{F})/G(\mathbf{O}))$ acts by convolutions on the right. Also, W acts on $K^T(\mathrm{Gr}_G)$ commuting with the right action of $K^G(\mathrm{Gr}_G)$. Clearly, the algebra $K(T(\mathbf{O})\backslash T(\mathbf{F})/T(\mathbf{O}))$ is isomorphic to $\mathbb{C}[\check{T}\times T]$. The action of W on $K^T(\mathrm{Gr}_G)$ normalizes the action of $K(T(\mathbf{O})\backslash T(\mathbf{F})/T(\mathbf{O}))$ and induces the natural (diagonal) action of W on $\mathbb{C}[\check{T}\times T]$.

Our aim in this subsection is to identify the $K(T(\mathbf{O})\backslash T(\mathbf{F})/T(\mathbf{O})) \ltimes W - K^G(\mathrm{Gr}_G)$-bimodule $K^T(\mathrm{Gr}_G)$ with the $\mathbb{C}[\check{T}\times T] \ltimes W - \mathbb{C}[\mathfrak{B}_G^{\check{G}}]$-bimodule $\mathbb{C}[\dot{\mathfrak{B}}_G^{\check{G}}]$ (and similarly for G replaced by $\check{G}$). As in 3.8, it suffices to identify the $K(T(\mathbf{O})\backslash\check{T}(\mathbf{F})/\check{T}(\mathbf{O})) \ltimes W - K^G(\mathrm{Gr}_{\check{G}})$-bimodule $K^T(\mathrm{Gr}_{\check{G}})$ with the $\mathbb{C}[T\times T] \ltimes W - \mathbb{C}[\mathfrak{B}_G^G]$-bimodule $\mathbb{C}[\dot{\mathfrak{B}}_G^G]$.

Note that $K^G(\mathrm{Gr}_{\check{G}}) \subset K^T(\mathrm{Gr}_{\check{G}})$, and the $K^G(\mathrm{Gr}_{\check{G}})$-module $K^T(\mathrm{Gr}_{\check{G}})$ is free of rank 2 with the generators $1, z$ where z is the generator of $K^T(pt) = \mathbb{C}[T]$ (so that, e.g. $\mathbf{v}(1)_0 = z + z^{-1}$). Furthermore, $\mathbb{C}[y^{\pm1}, z^{\pm1}] = \mathbb{C}[T\times T] = K(T(\mathbf{O})\backslash\check{T}(\mathbf{F})/\check{T}(\mathbf{O})) \subset K^T(\mathrm{Gr}_{\check{G}})$, and one can check

$$y^{-1} = \sqrt{-1}(\mathbf{u}_0 - \mathbf{u}_2),\ y = \sqrt{-1}(\mathbf{v}(0)_1 - \mathbf{v}(2)_1 + \mathbf{u}_2 - \mathbf{u}_0) \tag{8}$$

where $\mathbf{u}_0 \in K^T(\mathrm{Gr}_{\check{G}})$ (resp. $\mathbf{u}_2$) is the class of the irreducible skyscraper sheaf supported at the one-point Iwahori orbit in $\mathrm{Gr}_{\check{G},1} = \mathbb{P}^1$ with the trivial action of T (resp. with the action of T given by z^2), and placed in the homological degree $-\frac{1}{2}$. Hence

$$y + y^{-1} = \sqrt{-1}(2\mathbf{v}(0)_1 - \mathbf{v}(1)_0 \star \mathbf{v}(1)_1),\ y - y^{-1} = \sqrt{-1}(z - z^{-1})\mathbf{v}(1)_1. \tag{9}$$

Comparing (9) with (5) we get the desired identification of the $K(T(\mathbf{O})\backslash\check{T}(\mathbf{F})/\check{T}(\mathbf{O})) \ltimes W - K^G(\mathrm{Gr}_{\check{G}})$-bimodule $K^T(\mathrm{Gr}_{\check{G}})$ with the $\mathbb{C}[y^{\pm1}, z^{\pm1}] \ltimes W - \mathbb{C}[y^{\pm1}, z^{\pm1}, \frac{y-y^{-1}}{z-z^{-1}}]^W$-bimodule $\mathbb{C}[y^{\pm1}, z^{\pm1}, \frac{y-y^{-1}}{z-z^{-1}}]$.

3.10. **Borel-Moore Homology.** For an arbitrary semisimple G one proves the commutativity of $H_\bullet^{G(\mathbf{O})}(\mathrm{Gr}_G)$ (Theorem 2.12 a) exactly as in 3.7 using the Beilinson-Drinfeld Grassmannian and the *specialization* in Borel-Moore Homology (see [4] 2.6.30).

For $\check{G} = PGL_2$, let us denote by $\delta \in H^4_{\check{G}(\mathbf{O})}(pt, \mathbb{Z}) = H_4^{\check{G}(\mathbf{O})}(pt, \mathbb{Z})$ the generator of the equivariant (co)homology. Furthermore, we denote by η (resp. ξ) the generator of $H_{-2}^{\check{G}(\mathbf{O})}(\mathrm{Gr}_{\check{G},1}, \mathbb{Z})$ (resp. the generator of $H_0^{\check{G}(\mathbf{O})}(\mathrm{Gr}_{\check{G},1}, \mathbb{Z})$). Then it is easy to see that δ, ξ, η generate $H_\bullet^{\check{G}(\mathbf{O})}(\mathrm{Gr}_{\check{G}})$ (while $\delta, \xi^2, \eta^2, \xi\eta$ generate the subalgebra $H_\bullet^{G(\mathbf{O})}(\mathrm{Gr}_G)$), and we claim that

$$1 = \xi^2 - \delta\eta^2. \tag{10}$$

In effect, this is an equality in $H_0^{\check{G}(\mathbf{O})}(\mathrm{Gr}_{\check{G},2})$. Since $\mathrm{Gr}_{\check{G},2}$ is rationally smooth, $H_0^{\check{G}(\mathbf{O})}(\mathrm{Gr}_{\check{G},2}) = H^4_{\check{G}(\mathbf{O})}(\mathrm{Gr}_{\check{G},2})$. Let us denote by $\mathbf{B}\,\mathrm{Gr}_{\check{G},2} \xrightarrow{p} \mathbf{B}\check{G}(\mathbf{O})$ the associated fibre bundle over the classifying space of $\check{G}(\mathbf{O})$ with the fiber $\mathrm{Gr}_{\check{G},2}$. Then $1 \in H^4_{\check{G}(\mathbf{O})}(\mathrm{Gr}_{\check{G},2}) = H^4(\mathbf{B}\,\mathrm{Gr}_{\check{G},2})$ is the Poincaré dual class of the codimension 2 cycle $\mathbf{B}\check{G}(\mathbf{O}) = \mathbf{B}\,\mathrm{Gr}_{\check{G},0} \hookrightarrow \mathbf{B}\,\mathrm{Gr}_{\check{G},2}$, and $\delta\eta^2 = p^*\delta$.

Recall the convolution morphism $\Pi_0 : \mathcal{H}_2 \to \mathrm{Gr}_{\check{G},2}$ of 3.8. This is a morphism of $\check{G}(\mathbf{O})$-varieties, and we denote by $\Pi_0 : \mathbf{B}\mathcal{H}_2 \to \mathbf{B}\,\mathrm{Gr}_{\check{G},2}$ the corresponding morphism of associated fibre bundles. Note that (additively) $H^\bullet(\mathbf{B}\mathcal{H}_2) = H^\bullet(\mathbf{B}\,\mathrm{Gr}_{\check{G},1}) \otimes_{H^\bullet(\mathbf{B}\check{G}(\mathbf{O}))} H^\bullet(\mathbf{B}\,\mathrm{Gr}_{\check{G},1})$. Recall that ξ is the generator of $H_0^{\check{G}(\mathbf{O})}(\mathrm{Gr}_{\check{G},1}) = H^2_{\check{G}(\mathbf{O})}(\mathrm{Gr}_{\check{G},1}) = H^2(\mathbf{B}\,\mathrm{Gr}_{\check{G},1})$. Finally, we have $\xi^2 = \Pi_{0*}(\xi \otimes \xi)$. Now (10) follows easily.

Comparing the sizes of $H_\bullet^{\check{G}(\mathbf{O})}(\mathrm{Gr}_{\check{G}})$ and $\mathbb{C}[\delta, \xi, \eta]/(\xi^2 - \delta\eta^2 - 1)$ we conclude that they are isomorphic. The comparison with the equation (6) establishes an isomorphism $H_\bullet^{\check{G}(\mathbf{O})}(\mathrm{Gr}_{\check{G}}) \simeq \mathbb{C}[\mathcal{S}']$, and identifies the spectrum of $H_\bullet^{\check{G}(\mathbf{O})}(\mathrm{Gr}_{\check{G}})$ with $\mathcal{S}' \simeq \mathfrak{Z}_{\mathfrak{g}}^G$, and the spectrum of $H_\bullet^{G(\mathbf{O})}(\mathrm{Gr}_G)$ with $\imath\backslash\mathcal{S}' \simeq \mathfrak{Z}_{\check{\mathfrak{g}}}^{\check{G}}$.

4. Centralizers and blow-ups

The aim of this section is a proof of Proposition 2.8. We will consider $\mathfrak{B}_G^G$ and $\mathfrak{Z}_G^G$, the other cases being similar. Till the further notice G is assumed simply connected.

Lemma 4.1. *$\varpi : \mathfrak{B}_G^G \to T/W$ is flat.*

Proof It suffices to prove that the first projection of $\overset{\bullet}{\mathfrak{B}}{}_G^G$ to T is smooth (recall that $\overset{\bullet}{\mathfrak{B}}{}_G^G$ is defined as $\mathrm{Spec}(\mathbb{C}[T \times T, \frac{^1\alpha - 1}{^2\alpha - 1}, \alpha \in R]))$. In effect, then $\mathbb{C}[T \times T, \frac{^1\alpha - 1}{^2\alpha - 1}, \alpha \in R]$ is a flat $\mathbb{C}[T]$-module; hence it is a flat $\mathbb{C}[T]^W$-module (since $\mathbb{C}[T]$ is free over $\mathbb{C}[T]^W$, see [21]). Finally, $\mathbb{C}[T \times T, \frac{^1\alpha - 1}{^2\alpha - 1}, \alpha \in R]^W$ is a direct summand of a flat $\mathbb{C}[T]^W$-module $\mathbb{C}[T \times T, \frac{^1\alpha - 1}{^2\alpha - 1}, \alpha \in R]$; hence it is flat.

The affine blow-up $\overset{\bullet}{\mathfrak{B}}{}_G^G$ is the result of the following successive blow up of $T \times T$. We choose an ordering $\alpha_1, \ldots, \alpha_\nu$ of the set of positive roots R^+. We define $\mathfrak{B}_1$ as the blow up of $T\times T$ at the diagonal wall ${}^1\alpha_1 = {}^2\alpha_1 = 1$ with the proper preimage of the divisor ${}^1\alpha_1 = 1$ removed. We define $\mathfrak{B}_2$ as the blow up of $\mathfrak{B}_1$ at the proper transform of the diagonal wall ${}^1\alpha_2 = {}^2\alpha_2 = 1$ with the proper preimage of the divisor ${}^1\alpha_2 = 1$ removed. Going on like this we construct $\mathfrak{B}_\nu$; evidently, it coincides with $\dot{\mathfrak{B}}^G_G$.

Note that at each step the center of the blow-up is smooth over the corresponding wall ${}^2\alpha_i = 1$ in T by the following Claim. Thus the desired flatness assertion follows inductively from the

Claim. Let $p:\ X\to Y$ be a smooth morphism of smooth varieties; let $X'\subset X$ be a subvariety such that $Y'=f(X')\subset Y$ is a smooth hypersurface, and $p:\ X'\to Y'$ is also smooth. Then the blow-up $\mathrm{Bl}_{X'}X$ with the proper preimage of the divisor $p^{-1}(p(X'))$ removed is smooth over Y.

The smoothness is checked in the formal neighbourhoods of points by direct calculation in coordinates. This completes the proof of the lemma.

4.2. **The simultaneous resolution.** Recall that $\{(g,B):\ g\in B\}=\dot{G}\xrightarrow{p}G$ is the Grothendieck simultaneous resolution; here B is a Borel subgroup, and $p(g,B)=g$. We also have the projection $\varrho:\ \dot{G}\to T$ to the abstract Cartan, which we identify with T; namely, $\varrho(g,B)=g \pmod{\mathrm{rad}(B)}$. The preimage $p^{-1}(\Sigma_G)\subset\dot{G}$ is identified with T by ϱ. We denote by $\dot{\mathfrak{Z}}^G_G\subset G\times\dot{G}$ the subset of triples (g_1,g_2,B) such that $Ad_{g_1}=g_2$ and $(g_2,B)\in p^{-1}(\Sigma_G)$. Note that necessarily $g_1\in B$ (as well as $g_2\in B$); hence we have the projections $\varrho_1,\varrho_2:\ \dot{\mathfrak{Z}}^G_G\to T$; namely, $\varrho_i(g_1,g_2,B)=g_i \pmod{\mathrm{rad}(B)}$.

The natural projection $\dot{\mathfrak{Z}}^G_G\to\mathfrak{Z}^G_G$ (forgetting B) is a Galois W-covering. Finally, $\varrho_2:\ \dot{\mathfrak{Z}}^G_G\to T$ is flat.

4.3. **The proof of Proposition 2.8.** In order to identify $\mathfrak{Z}^G_G$ and $\mathfrak{B}^G_G$ it suffices to identify their Galois W-coverings $\dot{\mathfrak{Z}}^G_G\to T$ and $\dot{\mathfrak{B}}^G_G\to T$ in an equivariant way. Let $\mathbf{D}\subset T$ denote the discriminant, so that $T-\mathbf{D}=T^{reg}$. Let $\Delta\in\mathbb{C}[T]^W$ denote the product $\prod_{\alpha\in R}(\alpha-1)$, so that $\mathbf{D}$ is the divisor cut out by Δ.

Evidently, both $\dot{\mathfrak{Z}}^G_G|_{T^{reg}}$ and $\dot{\mathfrak{B}}^G_G|_{T^{reg}}$ are isomorphic to $T\times T^{reg}$. Hence both $\mathbb{C}[\dot{\mathfrak{Z}}^G_G]$ and $\mathbb{C}[\dot{\mathfrak{B}}^G_G]$ are the flat $\mathbb{C}[T]$-modules embedded into $\mathbb{C}[T\times T](\Delta^{-1})$. We must prove that the identification of $\dot{\mathfrak{Z}}^G_G|_{T^{reg}}$ and $\dot{\mathfrak{B}}^G_G|_{T^{reg}}$ extends to the identification over the whole T. To this end it suffices to check that the identification extends over the codimension 1 points of T (indeed, for a flat quasi-coherent sheaf $\mathcal{F}$ on a normal irreducible scheme we have $\mathcal{F}\xrightarrow{\sim}j_*j^*\mathcal{F}$ if j is an open imbedding with complement of codimension 2). Let $g\in T$ be a regular point of $\mathbf{D}$; that is, g is a semisimple element of G such that the centralizer $Z(g)$ has semisimple rank 1.

We must construct an isomorphism between localizations $(\dot{\mathfrak{Z}}^G_G)_g$ and $(\dot{\mathfrak{B}}^G_G)_g$ which is compatible with the above isomorphism at the generic point. To this end note that the embedding of reductive groups $Z(g)\hookrightarrow G$ (note that $Z(g)$ is connected since G is simply connected) induces the morphisms $\overset{\bullet}{\mathfrak{Z}}{}^{Z(g)}_{Z(g)} \to \overset{\bullet}{\mathfrak{Z}}{}^{G}_{G}$ and $\overset{\bullet}{\mathfrak{B}}{}^{Z(g)}_{Z(g)} \to \overset{\bullet}{\mathfrak{B}}{}^{G}_{G}$ which become isomorphisms after localizations: $\left(\overset{\bullet}{\mathfrak{Z}}{}^{Z(g)}_{Z(g)}\right)_g \simeq \left(\overset{\bullet}{\mathfrak{Z}}{}^{G}_{G}\right)_g$ and $\left(\overset{\bullet}{\mathfrak{B}}{}^{Z(g)}_{Z(g)}\right)_g \simeq \left(\overset{\bullet}{\mathfrak{Z}}{}^{G}_{G}\right)_g$. Now the desired identification $\left(\overset{\bullet}{\mathfrak{Z}}{}^{Z(g)}_{Z(g)}\right)_g \simeq \left(\overset{\bullet}{\mathfrak{B}}{}^{Z(g)}_{Z(g)}\right)_g$ follows from the calculations in 3.1.

This completes the identification $\mathfrak{Z}^G_G \simeq \mathfrak{B}^G_G$ for a simply connected G. Evidently, this identification respects the left and right actions of the center $Z(G)$, so the isomorphism for an arbitrary G follows from the one for its universal cover. The other isomorphisms in 2.8 (b) are proved in a similar way.

To prove 2.8 (c), (d) it suffices to notice that the minimal level (viewed as a W-equivariant homomorphism $T \twoheadrightarrow \check{T}$) for a simply laced simply connected G identifies $\check{T}$ with $T/Z(G)$; also, $\check{G} = G/Z(G)$.

5. W-invariant sections and blow-ups

The aim of this section is a proof of Proposition 2.10. We concentrate on the last statement, the other being completely similar.

Let $T^{reg} \subset T$, $T^{reg}_\alpha \subset T$ be the open subschemes defined by $T^{reg} = \{t \mid \alpha(t) \neq 1$ for all roots $\alpha\}$; $T^{reg}_\alpha = \{t \mid \beta(t) \neq 1$ for all roots $\beta \neq \alpha\}$; and $\overset{\circ}{T} = \bigcup_\alpha T^{reg}_\alpha$ (thus $T - \overset{\circ}{T}$ has codimension 2 in T (where the empty subscheme in a curve is considered to be of codimension 2)). Notice that since G is simply connected the action of W on T^{reg} is free.

We start with a

Lemma 5.1. *The map $\overset{\bullet}{\mathfrak{B}}{}^{\check{G}}_{G} \times_T \overset{\circ}{T} \to \mathfrak{B}^{\check{G}}_{G}/W \times_{T/W} \overset{\circ}{T}$ is an isomorphism.*

Proof Let $X \to Y$ be a flat morphism of semi-separated (which means that the diagonal embedding is affine) schemes of finite type over a characteristic zero field, and let a finite group W act on X, Y so that the map is W-equivariant. Assume that Y is flat over Y/W. We then claim that the map $X \to X/W \times_{Y/W} Y$ is an isomorphism provided that for every Zariski point $y \in Y$ the action of $\mathrm{Stab}_W(y)$ on the scheme-theoretic fiber X_y is trivial (here X/W, Y/W stand for categorical quotients). To check this claim we can assume X is affine: by semi-separatedness every W-invariant subset in X has a W-invariant affine neighborhood. Let us first assume also that Y/W is a point; then (by replacing Y by its connected component, and W by the stabilizer of that component) we can assume that Y is nilpotent. Then $\mathcal{O}_X$ is free over $\mathcal{O}_Y$, and the generators of $\mathcal{O}_X$ as an $\mathcal{O}_Y$ module can be chosen to be W-invariant (by semi-simplicity of the W action on $\mathcal{O}_X$, and triviality of the W-action on $\mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathsf{k}$); since $\mathcal{O}_Y^W = \mathsf{k}$ (where k is the base field) we see that $\mathcal{O}_X^W \otimes \mathcal{O}_Y \widetilde{\longrightarrow} \mathcal{O}_X$ as claimed. Now for a general Y we see that the morphism in question is a morphism of flat schemes of finite type over Y/W, which induces an isomorphism on every fiber; and such a morphism is necessarily an isomorphism.

Now it remains to check that the above conditions hold for $X = \overset{\bullet}{\mathfrak{B}}{}^{\check{G}}_{G} \times_T \overset{\circ}{T}$, $Y = T$. For $y \in T^{reg}$ the stabilizer of y is trivial, so there is nothing to check. Consider now

$y \in T_\alpha^{reg}$, $y \notin T^{reg}$. Then the stabilizer of y is $\{1, s_\alpha\}$. The ring of functions on $\dot{\mathfrak{B}}_G^{\check{G}}$ is generated by ${}^1\check{\lambda}$, ${}^2\mu$, t_α where $\check{\lambda}$, μ run over weights of $\check{T}$, T respectively, $\alpha \in R^+$, and $t_\alpha({}^2\alpha - 1) = {}^1\check{\alpha} - 1$. We have $s_\alpha^*({}^1\check{\lambda}) = {}^1\check{\lambda} \cdot ({}^1\check{\alpha})^{\langle -\alpha, \check{\lambda}\rangle}$, $s_\alpha^*({}^2\mu) = {}^2\mu \cdot ({}^2\alpha)^{\langle -\mu, \check{\alpha}\rangle}$, and $s_\alpha^*(t_\alpha) = t_\alpha \cdot \frac{{}^2\alpha}{{}^1\check{\alpha}}$. On the fiber we have ${}^2\alpha = 1$, hence ${}^1\check{\alpha} = 1$, so the action of s_α on the fiber is trivial. $\square$

Proposition 2.10 clearly follows from the $(ii) \iff (iv)$ part of the next

Proposition 5.2. *Let $S \to T/W$ be a flat morphism, and set $\phi : S \times_{T/W} T^{reg}/W \to (\check{T} \times T)/W$ be a T^{reg}/W-morphism. Then the following are equivalent:*

(i) ϕ extends to a morphism $S \times_{T/W} \overset{\circ}{T}/W \to \mathfrak{B}_G^{\check{G}} \times_{T/W} \overset{\circ}{T}$.

(ii) ϕ extends to a morphism $S \to \mathfrak{B}_G^{\check{G}}$.

(iii) For every $\alpha \in R$ the morphism $\phi \times id_{T^{reg}} : S \times_{T/W} T^{reg} \to \check{T} \times T^{reg}$ extends to a morphism $S \times_{T/W} T_\alpha^{reg} \to \check{T} \times T_\alpha^{reg}$ such that (3) holds.

(iv) $\phi \times id_{T^{reg}} : S \times_{T/W} T^{reg} \to \check{T} \times T^{reg}$ extends to a morphism $S \times_{T/W} T \to \check{T} \times T$, such that (3) holds for every $\alpha \in R$.

Proof It is enough to assume that S is affine. Indeed, a morphism from S extends iff its restriction to every affine open in S does, because compatibility on intersections follows from uniqueness of such an extension; this uniqueness follows from flatness: if S is flat affine, then tensoring the injection $\mathcal{O} \to j_*\mathcal{O}$ with $\mathcal{O}_S$ we get an imbedding $\mathcal{O}_S \hookrightarrow j_*j^*\mathcal{O}_S$, where j stands for the imbedding $T^{reg}/W \to T/W$, or $T^{reg} \to T$. So we will assume S affine from now on.

(iv) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (i) are obvious.

To check that (iii) $\Rightarrow$ (iv) we tensor (over $\mathcal{O}_{T/W}$) the exact sequence of $\mathcal{O}_T$-modules

$$0 \to \mathcal{O} \to \mathcal{O}_{T^{reg}} \to \bigoplus_\alpha (\mathcal{O}_{T_\alpha^{reg}}/\mathcal{O}_T) \tag{11}$$

with $\mathcal{O}_S$. The resulting exact sequence shows that a regular function on $S \times_{T/W} T^{reg}$ extends to a regular function on $S \times_{T/W} T$ iff it extends to $S \times_T T_\alpha^{reg}$ for all α. Applying this observation to $(\phi \times \mathrm{id})^*(f|_{\check{T} \times T^{reg}})$ for each regular function f on $\check{T} \times T$ we see that (iii) implies extendability of $\phi \times \mathrm{id}$ to $S \times_{T/W} T$. It is also clear that (3) holds if it holds on $\overset{\circ}{T}$.

Verification of (i) $\Rightarrow$ (ii) is similar (with (11) replaced by the W-invariant part of (11)).

It remains to check (i) $\iff$ (iii). If (i) holds, i.e. ϕ extends to a map $S \times_{T/W} \overset{\circ}{T}/W \to \mathfrak{B}_G^{\check{G}} \times_{T/W} \overset{\circ}{T}$ then we can take the fiber product of this map with $\mathrm{id}_{\overset{\circ}{T}}$ over T/W. By Lemma 5.1 it yields a map $S \times_{T/W} \overset{\circ}{T} \to \dot{\mathfrak{B}}_G^{\check{G}} \times_T \overset{\circ}{T}$, which can be composed with the projection $\dot{\mathfrak{B}}_G^{\check{G}} \to \check{T} \times T$ to produce a map $S \times_{T/W} \overset{\circ}{T} \to \check{T} \times \overset{\circ}{T}$. It is clear that this map satisfies (3), because the image of the map $\dot{\mathfrak{B}}_G^{\check{G}} \to \check{T} \times T$ intersected with $\check{T} \times \mathrm{Ker}({}^2\alpha)$ is contained in $\mathrm{Ker}({}^1\check{\alpha}) \times T$.

Conversely, if (iii) holds then restricting the given map $S \times_{T/W} \overset{\circ}{T} \to \check{T} \times \overset{\circ}{T}$ to $S \times_{T/W} (\mathrm{Ker}(\alpha) \cap \overset{\circ}{T})$ we get a map into $\mathrm{Ker}(\check{\alpha}) \times T$ (this is immediate from (3)). This means that the map lifts to a map into $\overset{\bullet}{\mathfrak{B}}{}^{\check{G}}_{G}$. Replacing both the source and the target by their quotients by W we get the map required in (i). □

6. K-theory and blow-ups

The aim of this section is a proof of Proposition 2.15. Recall that 2.15 (a) was already proved in 3.7. G is assumed simply connected till the further notice.

6.1. **Reminder on the affine Grassmannians.** Let $X = X_G$ be the lattice of characters of T, and let $Y = Y_G$ be the lattice of cocharacters of G. Note that $X_G = Y_{\check{G}}$, $Y_G = X_{\check{G}}$. Let $X^+ \subset X$ (resp. $Y^+ \subset Y$) be the cone of dominant weights (resp. dominant coweights). It is well known that the $G(\mathbf{O})$-orbits in Gr_G are numbered by the dominant coweights: $\mathrm{Gr}_G = \bigsqcup_{\check{\lambda} \in Y^+} \mathrm{Gr}_{G,\check{\lambda}}$. The adjacency relation of orbits corresponds to the standard partial order on coweights: $\overline{\mathrm{Gr}}_{G,\check{\lambda}} = \bigsqcup_{\check{\mu} \leq \check{\lambda}} \mathrm{Gr}_{G,\check{\mu}}$. The open embedding $\mathrm{Gr}_{G,\check{\lambda}} \hookrightarrow \overline{\mathrm{Gr}}_{G,\check{\lambda}}$ will be denoted by $j_{\check{\lambda}}$ or simply by j if no confusion is likely. The dimension $\dim(\mathrm{Gr}_{G,\check{\lambda}}) = \langle 2\rho, \check{\lambda} \rangle$ where $2\rho = \sum_{\alpha \in R^+} \alpha$, and $\langle , \rangle : \ X \times Y \to \mathbb{Z}$ is the canonical perfect pairing.

Recall that the T-fixed points in Gr_G are naturally numbered by Y; a point $\check{\mu}$ lies in an orbit $\mathrm{Gr}_{G,\check{\lambda}}$ iff $\check{\mu}$ lies in the W-orbit of $\check{\lambda}$. Each $G(\mathbf{O})$-orbit $\mathrm{Gr}_{G,\check{\lambda}}$ is partitioned into Iwahori orbits isomorphic to affine spaces and numbered by $\check{\mu} \in W\check{\lambda}$. Hence the basics of [4] Chapter 5 are applicable in our situation.

In particular, $K^T(\mathrm{Gr}_{G,\check{\lambda}})$ is a free $K^T(pt)$-module, and $K^{G(\mathbf{O})}(\mathrm{Gr}_{G,\check{\lambda}}) = K^G(\mathrm{Gr}_{G,\check{\lambda}})$ is a free $K^G(pt)$-module (recall that $K^T(pt) = \mathbb{C}[T]$, and $K^G(pt) = \mathbb{C}[T/W]$). Moreover, the natural map $K^T(pt) \otimes_{K^G(pt)} K^G(\mathrm{Gr}_{G,\check{\lambda}}) \to K^T(\mathrm{Gr}_{G,\check{\lambda}})$ is an isomorphism, and $K^G(\mathrm{Gr}_{G,\check{\lambda}}) = K^T(\mathrm{Gr}_{G,\check{\lambda}})^W$, cf. [4] 6.1.22.

Since $K^{T(\mathbf{O})}(\mathrm{Gr}_G) = K^T(\mathrm{Gr}_G)$ (resp. $K^{G(\mathbf{O})}(\mathrm{Gr}_G) = K^G(\mathrm{Gr}_G)$) is filtered by the support in $G(\mathbf{O})$-orbit closures, with the associated graded $\bigoplus_{\check{\lambda} \in Y^+} K^T(\mathrm{Gr}_{G,\check{\lambda}})$ (resp. $\bigoplus_{\check{\lambda} \in Y^+} K^G(\mathrm{Gr}_{G,\check{\lambda}})$), we arrive at the following

Lemma 6.2. *$K^{T(\mathbf{O})}(\mathrm{Gr}_G) = K^T(\mathrm{Gr}_G)$ is a flat $K^T(pt)$-module, and $K^{G(\mathbf{O})}(\mathrm{Gr}_G) = K^G(\mathrm{Gr}_G)$ is a flat $K^G(pt)$-module. Moreover, the natural map $K^T(pt) \otimes_{K^G(pt)} K^G(\mathrm{Gr}_G) \to K^T(\mathrm{Gr}_G)$ is an isomorphism, and $K^G(\mathrm{Gr}_G) = \left(K^T(\mathrm{Gr}_G)\right)^W$.*

6.3. **Localization.** The space $K^T(\mathrm{Gr}_G) = K^{T(\mathbf{O})}(\mathrm{Gr}_G) = K(T(\mathbf{O}) \backslash G(\mathbf{F})/G(\mathbf{O}))$ is equipped with the two commuting actions: $K(T(\mathbf{O}) \backslash T(\mathbf{F})/T(\mathbf{O}))$ acts by convolutions on the left, and $K^G(\mathrm{Gr}_G) = K^{G(\mathbf{O})}(\mathrm{Gr}_G) = K(G(\mathbf{O}) \backslash G(\mathbf{F})/G(\mathbf{O}))$ acts by convolutions on the right. Also, W acts on $K^T(\mathrm{Gr}_G)$ commuting with the right action of $K^G(\mathrm{Gr}_G)$. Clearly, the algebra $K(T(\mathbf{O}) \backslash T(\mathbf{F})/T(\mathbf{O}))$ is isomorphic to $\mathbb{C}[\check{T} \times T]$. The action of W on $K^T(\mathrm{Gr}_G)$ normalizes the action of $K(T(\mathbf{O}) \backslash T(\mathbf{F})/T(\mathbf{O}))$ and induces the natural (diagonal) action of W on $\mathbb{C}[\check{T} \times T]$.

Let g be a general (regular) element of T. Then the fixed point set $(\mathrm{Gr}_G)^g = (\mathrm{Gr}_G)^T = Y$ coincides with the image of the embedding $\mathrm{Gr}_T \hookrightarrow \mathrm{Gr}_G$. According to Thomason Localization Theorem (see e.g. [4] 5.10), after localization, $\left(K^T(\mathrm{Gr}_G)\right)_g$ becomes a free rank one $\left(K(T(\mathbf{O})\backslash T(\mathbf{F})/T(\mathbf{O}))\right)_g$-module. This means that after restriction to $T^{reg} \subset T = \mathrm{Spec}(K^T(pt))$ we have an isomorphism $K^T(\mathrm{Gr}_G)|_{T^{reg}} \simeq \mathbb{C}[\check{T} \times T]|_{T^{reg}}$ compatible with the natural W-actions. The localized algebra $K^{G(\mathbf{O})}(\mathrm{Gr}_G)|_{T^{reg}/W}$ is embedded into $\left(\mathrm{End}_{K(T(\mathbf{O})\backslash T(\mathbf{F})/T(\mathbf{O}))|_{T^{reg}}}(K^T(\mathrm{Gr}_G)|_{T^{reg}})\right)^W$. According to Lemma 6.2, $K^G(\mathrm{Gr}_G) = \left(K^T(\mathrm{Gr}_G)\right)^W$; hence this embedding is an isomorphism, and we have $K^{G(\mathbf{O})}(\mathrm{Gr}_G)|_{T^{reg}/W} \simeq \mathbb{C}[\check{T} \times T]^W|_{T^{reg}/W}$.

Hence both $\mathbb{C}[\mathfrak{B}_G^{\check{G}}]$ and $K^{G(\mathbf{O})}(\mathrm{Gr}_G)$ are the flat $\mathbb{C}[T]^W$-modules embedded into $\mathbb{C}[\check{T} \times T](\Delta^{-1})$ (see 4.3). We must prove that the identification of $\mathbb{C}[\mathfrak{B}_G^{\check{G}}]|_{T^{reg}/W}$ and $K^{G(\mathbf{O})}(\mathrm{Gr}_G)|_{T^{reg}/W}$ extends to the identification over the whole T/W. To this end it suffices to check that the identification extends over the codimension 1 points of T/W. Let $g \in T/W$ be a regular point of $\mathbf{D}$; that is, g is represented by a semisimple element of G such that the centralizer $Z(g)$ has semisimple rank 1.

We must prove that the localizations $\mathbb{C}[\mathfrak{B}_G^{\check{G}}]_g$ and $\left(K^{G(\mathbf{O})}(\mathrm{Gr}_G)\right)_g$ are isomorphic. To this end it suffices to identify $\mathbb{C}\left[\check{T} \times T,\ \frac{1}{2}\frac{\check{\alpha}-1}{\alpha-1},\ \alpha \in R\right]_g$ (which we denote by $\mathbb{C}[\dot{\mathfrak{B}}_G^{\check{G}}]_g$ for short) and $\left(K^T(\mathrm{Gr}_G)\right)_g$. Note that the embedding of reductive groups $Z(g) \hookrightarrow G$ (the neutral connected component) induces the isomorphism $\mathrm{Gr}_{Z(g)} = (\mathrm{Gr}_G)^g \hookrightarrow \mathrm{Gr}_G$. According to Thomason Localization Theorem, we have an isomorphism of localizations $\left(K^T(\mathrm{Gr}_{Z(g)})\right)_g \simeq \left(K^T(\mathrm{Gr}_G)\right)_g$. Finally, the isomorphism $K^T\left(\mathrm{Gr}_{Z(g)}\right) \simeq \mathbb{C}\left[\dot{\mathfrak{B}}_{Z(g)}^{\check{Z}(g)}\right]$ follows from the calculations in 3.8, 3.9, and together with the evident isomorphism of localizations $\mathbb{C}\left[\dot{\mathfrak{B}}_{Z(g)}^{\check{Z}(g)}\right]_g \simeq \mathbb{C}\left[\dot{\mathfrak{B}}_G^{\check{G}}\right]_g$ establishes the desired isomorphism $\left(K^T(\mathrm{Gr}_G)\right)_g \simeq \mathbb{C}\left[\dot{\mathfrak{B}}_G^{\check{G}}\right]_g$.

This completes the proof of 2.15 (b).

6.4. **Comparison of Poisson structures.** In order to compare the Poisson structures on $K^{G(\mathbf{O})}(\mathrm{Gr}_G)$ and $\mathbb{C}[\mathfrak{B}_G^{\check{G}}]$ it suffices to identify them on the open subset $K^{G(\mathbf{O})}(\mathrm{Gr}_G)|_{T^{reg}/W} = \mathbb{C}[\mathfrak{B}_G^{\check{G}}]|_{T^{reg}/W} = \mathbb{C}[\check{T} \times T^{reg}]^W$. The space

$$K^{T\times\mathbb{G}_m}(\mathrm{Gr}_G) = K^{T(\mathbf{O})\ltimes\mathbb{G}_m}(\mathrm{Gr}_G) = K\left(T(\mathbf{O}) \ltimes \mathbb{G}_m \backslash G(\mathbf{F}) \ltimes \mathbb{G}_m / G(\mathbf{O}) \ltimes \mathbb{G}_m\right)$$

is equipped with the two commuting actions: $K\left(T(\mathbf{O}) \ltimes \mathbb{G}_m \backslash T(\mathbf{F}) \ltimes \mathbb{G}_m / T(\mathbf{O}) \ltimes \mathbb{G}_m\right)$ acts by convolutions on the left, and

$$K^{G\times\mathbb{G}_m}(\mathrm{Gr}_G) = K^{G(\mathbf{O})\ltimes\mathbb{G}_m}(\mathrm{Gr}_G) = K\left(G(\mathbf{O}) \ltimes \mathbb{G}_m \backslash G(\mathbf{F}) \ltimes \mathbb{G}_m / G(\mathbf{O}) \ltimes \mathbb{G}_m\right)$$

acts by convolutions on the right. Also, W acts on $K^{T(\mathbf{O})\ltimes\mathbb{G}_m}(\mathrm{Gr}_G)$ commuting with the right action of $K^{G(\mathbf{O})\ltimes\mathbb{G}_m}(\mathrm{Gr}_G)$. Clearly, the algebra

$K\left(T(\mathbf{O})\ltimes\mathbb{G}_m\backslash T(\mathbf{F})\ltimes\mathbb{G}_m/T(\mathbf{O})\ltimes\mathbb{G}_m\right)$ is isomorphic to the group algebra $\mathbb{C}[\Gamma]$ of the following Heisenberg group Γ.

It is a $\mathbb{Z}$-central extension of $Y\times X$ with the multiplication (written multiplicatively)

$$(q^{n_1},e^{\check\lambda_1},e^{\mu_1})\cdot(q^{n_2},e^{\check\lambda_2},e^{\mu_2})=(q^{n_1+n_2+\langle\mu_1,\check\lambda_2\rangle},e^{\check\lambda_1+\check\lambda_2},e^{\mu_1+\mu_2})$$

where $\langle,\rangle:\ X\times Y\to\mathbb{Z}$ is the canonical perfect pairing.

Finally, the action of the Weyl group W on $K^{T(\mathbf{O})\ltimes\mathbb{G}_m}(\mathrm{Gr}_G)$ normalizes the action of $K\left(T(\mathbf{O})\ltimes\mathbb{G}_m\backslash T(\mathbf{F})\ltimes\mathbb{G}_m/T(\mathbf{O})\ltimes\mathbb{G}_m\right)$ and induces the natural (diagonal) action of W on $\mathbb{C}[\Gamma]$. From this we deduce, exactly as in 6.3, that $K^{G(\mathbf{O})\ltimes\mathbb{G}_m}(\mathrm{Gr}_G)|_{T^{reg}/W}\simeq\mathbb{C}[\Gamma]|_{T^{reg}/W}$. It follows that the Poisson structure on $K^{G(\mathbf{O})}(\mathrm{Gr}_G)|_{T^{reg}/W}$ coincides with the standard Poisson structure on $\mathbb{C}[\check T\times T^{reg}]^W$.

This completes the proof of 2.15 (c).

6.5. **The case of non simply connected G.** For general G let $\tilde G$ denote its universal cover, and let $\tilde T$ stand for the Cartan of $\tilde G$. Note that the dual torus is $\check{\tilde T}/\pi_1(G)$. As in 6.3, we have $K^G(\mathrm{Gr}_G)=\left(\mathrm{End}_{K(T(\mathbf{O})\backslash T(\mathbf{F})/T(\mathbf{O}))}(K^T(\mathrm{Gr}_G))\right)^W$, so it suffices to identify the $K(T(\mathbf{O})\backslash T(\mathbf{F})/T(\mathbf{O}))\ltimes W=\mathbb{C}[\check T\times T]\ltimes W$-module $K^T(\mathrm{Gr}_G)$ with $\mathbb{C}\left[\check T\times T,\ \frac{1}{2}\frac{\check\alpha-1}{\alpha-1},\ \alpha\in R\right]=\operatorname{Spec}\mathbb{C}[\dot{\mathfrak{B}}_{\check G}^{\check G}]$. We do this by reduction to the known case of $\tilde G$.

Evidently, the $K(T(\mathbf{O})\backslash T(\mathbf{F})/T(\mathbf{O}))\ltimes W=\mathbb{C}[\check T\times T]\ltimes W$-module $K^T(\mathrm{Gr}_G)$ equals $\mathbb{C}[\check T\times T]\ltimes W\otimes_{\mathbb{C}[(\check T/\pi_1(G))\times T]\ltimes W}K^T(\mathrm{Gr}_{\tilde G})$. On the other hand, it follows from 6.3 that the $K(T(\mathbf{O})\backslash\tilde T(\mathbf{F})/\tilde T(\mathbf{O}))\ltimes W=\mathbb{C}[(\check T/\pi_1(G))\times T]\ltimes W$-module $K^T(\mathrm{Gr}_{\tilde G})$ equals the invariants of $\pi_1(G)$ in $K^{\tilde T}(\mathrm{Gr}_{\tilde G})$, that is $\mathbb{C}\left[(\check T/\pi_1(G))\times\tilde T,\ \frac{1}{2}\frac{\check\alpha-1}{\alpha-1},\ \alpha\in R\right]^{\pi_1(G)}=\mathbb{C}\left[(\check T/\pi_1(G))\times T,\ \frac{1}{2}\frac{\check\alpha-1}{\alpha-1},\ \alpha\in R\right]$.

This completes the proof of 2.15 for general G.

6.6. **Borel-Moore Homology and blow-ups.** Theorem 2.12 is proved absolutely parallelly to the proof of Theorem 2.15.

7. Computation of $K_{G(\mathbf{O})}(\Lambda)$.

7.1. **The affine Grassmannian Steinberg variety.** We denote by $\mathfrak{u}\subset\mathfrak{g}(\mathbf{O})$ (resp. $U\subset G(\mathbf{O})$) the nilpotent (resp. unipotent) radical. It has a filtration $\mathfrak{u}=\mathfrak{u}^{(0)}\supset\mathfrak{u}^{(1)}\supset\ldots$ by congruence subalgebras. The trivial (Tate) vector bundle $\underline{\mathfrak{g}(\mathbf{F})}$ with the fiber $\mathfrak{g}(\mathbf{F})$ over Gr_G has a structure of an ind-scheme. It contains a profinite dimensional vector subbundle $\underline{\mathfrak{u}}$ whose fiber over a point $g\in\mathrm{Gr}_G$ represented by a compact subalgebra in $\mathfrak{g}(\mathbf{F})$ is the pronilpotent radical of this subalgebra. The trivial vector bundle $\underline{\mathfrak{g}(\mathbf{F})}=\mathfrak{g}(\mathbf{F})\times\mathrm{Gr}_G$ also contains a trivial vector subbundle $\mathfrak{u}\times\mathrm{Gr}_G$.

We will call $\underline{\mathfrak{u}}$ *the cotangent bundle* of Gr_G, and we will call the intersection $\Lambda:=\underline{\mathfrak{u}}\cap(\mathfrak{u}\times\mathrm{Gr}_G)$ *the affine Grassmannian Steinberg variety.* It has a structure of an ind-scheme of ind-infinite type. Namely, if p stands for the natural projection $\Lambda\to\mathrm{Gr}_G$, then $\Lambda_{\le\check\lambda}:=p^{-1}(\overline{\mathrm{Gr}}_{G,\check\lambda})$ is a scheme of infinite type, and $\Lambda=\bigcup\Lambda_{\le\check\lambda}$.

Note that for a fixed $\check\lambda$ and $l \gg 0$ the intersection of fibers of $\underline{\mathfrak{u}}$ over all points of $\overline{\mathrm{Gr}}_{G,\check\lambda}$ (as vector subspaces of $\mathfrak{g}(\mathbf{F})$) contains $\mathfrak{u}^{(l)}$. Thus $\mathfrak{u}^{(l)}$ acts freely (by fiberwise translations) on $\Lambda_{\leq\check\lambda}$, and the quotient is a scheme of finite type, to be denoted by $\Lambda^l_{\leq\check\lambda}$. For $k > l$ we have evident affine fibrations $p^k_l:\ \Lambda^k_{\leq\check\lambda} \to \Lambda^l_{\leq\check\lambda}$, and $\Lambda_{\leq\check\lambda}$ coincides with the inverse limit of this system.

Similarly, the total space of the vector bundle $\underline{\mathfrak{u}}$ (to be denoted by the same symbol) is a union of infinite type schemes $\underline{\mathfrak{u}}_{\leq\check\lambda}$, and for fixed $\check\lambda$ and $l \gg 0$, the scheme $\underline{\mathfrak{u}}_{\leq\check\lambda}$ is the inverse limit of affine fibrations $p^k_l:\ \underline{\mathfrak{u}}^k_{\leq\check\lambda} \to \underline{\mathfrak{u}}^l_{\leq\check\lambda}$ $(k > l)$. Note that the proalgebraic group $G(\mathbf{O})$ acts on all the above schemes, and the fibrations p^k_l are $G(\mathbf{O})$-equivariant.

A *$G(\mathbf{O})$-equivariant coherent sheaf $\mathcal{F}$ on $\underline{\mathfrak{u}}$* is by definition supported on some $\underline{\mathfrak{u}}_{\leq\check\lambda}$. There, it is defined as a collection of $G(\mathbf{O})$-equivariant sheaves $\mathcal{F}^l$ on $\underline{\mathfrak{u}}^l_{\leq\check\lambda}$ for $l \gg 0$ together with isomorphisms $(p^k_l)^*\mathcal{F}^l \simeq \mathcal{F}^k$. We will consider the $G(\mathbf{O})$-equivariant coherent sheaves on $\underline{\mathfrak{u}}$ *supported on* Λ, and $D^bCoh^{G(\mathbf{O})}_\Lambda(\underline{\mathfrak{u}})$ stands for the derived category of such sheaves, and $K^{G(\mathbf{O})}(\Lambda)$ stands for the K-group of such sheaves.

7.2. **Convolution in $D^bCoh^{G(\mathbf{O})}_\Lambda(\underline{\mathfrak{u}})$.** We have a principal $G(\mathbf{O})$-bundle $G(\mathbf{F}) \to \mathrm{Gr}_G$. Given a $G(\mathbf{O})$-(ind)-scheme A we can form an associated bundle $\widetilde{A} = G(\mathbf{F}) \times_{G(\mathbf{O})} A \to \mathrm{Gr}_G$. Given a coherent $G(\mathbf{O})$-equivariant sheaf $\mathcal{F}$ on A we can form an associated sheaf $\widetilde{\mathcal{F}}$ on $\widetilde{A}$ as $G(\mathbf{O})$-invariants in the direct image of $\mathcal{O}_{G(\mathbf{F})} \boxtimes \mathcal{F}$ from $G(\mathbf{F}) \times A$ to $G(\mathbf{F}) \times_{G(\mathbf{O})} A$. If $A = \mathrm{Gr}_G$, apart from the natural projection $p_1:\ \widetilde{A} \to \mathrm{Gr}_G$, we have a multiplication map $G(\mathbf{F}) \times_{G(\mathbf{O})} \mathrm{Gr}_G \to \mathrm{Gr}_G$, to be denoted p_2. Then (p_1, p_2) identifies $\widetilde{\mathrm{Gr}_G}$ with $\mathrm{Gr}_G \times \mathrm{Gr}_G$. Furthermore, $\widetilde{\underline{\mathfrak{u}}}$ is a vector bundle over $\widetilde{\mathrm{Gr}_G} = \mathrm{Gr}_G \times \mathrm{Gr}_G$ which is naturally identified with $p_2^*\underline{\mathfrak{u}}$. Thus we have an ind-proper morphism $p_2:\ \widetilde{\underline{\mathfrak{u}}} \to \underline{\mathfrak{u}}$.

Note that both $\widetilde{\underline{\mathfrak{u}}} = p_2^*\underline{\mathfrak{u}}$ and $p_1^*\underline{\mathfrak{u}}$ are subbundles in the trivial (Tate) vector bundle $\underline{\mathfrak{g}(\mathbf{F})}$ over $\mathrm{Gr}_G \times \mathrm{Gr}_G$ with the fiber $\mathfrak{g}(\mathbf{F})$. Their intersection is naturally identified with $\widetilde\Lambda$. In particular, we have an embedding $\widetilde\Lambda \subset p_1^*\underline{\mathfrak{u}} \oplus p_2^*\underline{\mathfrak{u}}$, and an ind-proper morphism $p_2:\ \widetilde\Lambda \to \underline{\mathfrak{u}}$.

Hence given $G(\mathbf{O})$-equivariant coherent sheaves $\mathcal{F}, \mathcal{G}$ on Λ we can consider the $G(\mathbf{O})$-equivariant complex $\mathcal{F} \star \mathcal{G} := (p_2)_*(p_1^*\mathcal{F} \overset{L}{\otimes} \widetilde{\mathcal{G}})$ (tensor product over the structure sheaf of the profinite dimensional vector bundle $p_1^*\underline{\mathfrak{u}} \oplus p_2^*\underline{\mathfrak{u}}$). Clearly, $\mathcal{F} \star \mathcal{G}$ is supported on Λ. Hence we get a convolution operation on $D^bCoh^{G(\mathbf{O})}_\Lambda(\underline{\mathfrak{u}})$ and on $K^{G(\mathbf{O})}(\Lambda)$ once we check that $p_1^*\mathcal{F} \overset{L}{\otimes} \widetilde{\mathcal{G}}$ is bounded.

To this end, note that $\widetilde{\mathcal{G}}$ is flat over the first copy of Gr_G, and for some $\check\lambda$ the sheaf $\mathcal{F}$ is supported on $\Lambda_{\leq\check\lambda}$, so the tensor product $p_1^*\mathcal{F} \overset{L}{\otimes} \widetilde{\mathcal{G}}$ can actually be computed over the structure sheaf of $(p_1^*\underline{\mathfrak{u}} \oplus p_2^*\underline{\mathfrak{u}})|_{\overline{\mathrm{Gr}}_{G,\check\lambda} \times \mathrm{Gr}_G} = \underline{\mathfrak{u}}_{\leq\check\lambda} \times \underline{\mathfrak{u}} \subset \underline{\mathfrak{u}} \times \underline{\mathfrak{u}} = p_1^*\underline{\mathfrak{u}} \oplus p_2^*\underline{\mathfrak{u}}$. That is, $p_1^*\mathcal{F} \overset{L}{\otimes} \widetilde{\mathcal{G}}$ is the direct image of $p_1^*\mathcal{F}|_{\underline{\mathfrak{u}}_{\leq\check\lambda} \times \underline{\mathfrak{u}}} \overset{L}{\otimes}_{\mathcal{O}_{\underline{\mathfrak{u}}_{\check\lambda} \times \underline{\mathfrak{u}}}} \widetilde{\mathcal{G}}|_{\underline{\mathfrak{u}}_{\leq\check\lambda} \times \underline{\mathfrak{u}}}$ under the closed embedding $\underline{\mathfrak{u}}_{\leq\check\lambda} \times \underline{\mathfrak{u}} \hookrightarrow \underline{\mathfrak{u}} \times \underline{\mathfrak{u}}$. On the other hand, $p_1^*\mathcal{F}$ is flat over the second copy of Gr_G, while the support of $\widetilde{\mathcal{G}}$ intersected with $\underline{\mathfrak{u}}_{\leq\check\lambda} \times \underline{\mathfrak{u}}$ is contained in $\underline{\mathfrak{u}}_{\leq\check\lambda} \times \underline{\mathfrak{u}}_{\leq\check\mu}$ for some $\check\mu$. Hence the tensor product $p_1^*\mathcal{F} \overset{L}{\otimes} \widetilde{\mathcal{G}}$ can actually be computed over the structure sheaf of $\underline{\mathfrak{u}}_{\leq\check{\lambda}} \times \underline{\mathfrak{u}}_{\leq\check{\mu}}$. There exists $l \gg 0$ such that the diagonal fiberwise action of $\mathfrak{u}^{(l)}$ on $\underline{\mathfrak{u}}_{\leq\check{\lambda}} \times \underline{\mathfrak{u}}_{\leq\check{\mu}}$ is free, and both $p_1^*\mathcal{F}$ and $\widetilde{\mathcal{G}}$ restricted to $\underline{\mathfrak{u}}_{\leq\check{\lambda}} \times \underline{\mathfrak{u}}_{\leq\check{\mu}}$ are $\mathfrak{u}^{(l)}$-equivariant, that is, they are lifted from the sheaves on $(\underline{\mathfrak{u}}_{\leq\check{\lambda}} \times \underline{\mathfrak{u}}_{\leq\check{\mu}})/\mathfrak{u}^{(l)} =: V$; we abuse notation by keeping the same names for these sheaves. So the tensor product $p_1^*\mathcal{F} \overset{L}{\otimes} \widetilde{\mathcal{G}}$ can actually be computed as the tensor product of coherent sheaves over the structure sheaf of the profinite dimensional vector bundle V over the finite dimensional scheme $\overline{\mathrm{Gr}}_{G,\check{\lambda}} \times \overline{\mathrm{Gr}}_{G,\check{\mu}}$.

Now there exists a vector subbundle $V' \subset V$ such that the quotient $\overline{V} := V/V'$ is a finite dimensional vector bundle, $p_1^*\mathcal{F}$ is lifted from $\overline{V}$, and the support of $\widetilde{\mathcal{G}}$ in V projects isomorphically onto its image in $\overline{V}$. Moreover, recall that $p_1^*\mathcal{F}$ is flat over $\overline{\mathrm{Gr}}_{G,\check{\mu}}$, while $\widetilde{\mathcal{G}}$ is flat over $\overline{\mathrm{Gr}}_{G,\check{\lambda}}$. Clearly, in this situation $p_1^*\mathcal{F} \overset{L}{\otimes} \widetilde{\mathcal{G}} \in D^b(V)$. This explains why for $G(\mathbf{O})$-equivariant coherent sheaves $\mathcal{F}, \mathcal{G}$ on Λ the tensor product $p_1^*\mathcal{F} \overset{L}{\otimes} \widetilde{\mathcal{G}}$ is a bounded complex of coherent sheaves on $p_1^*\underline{\mathfrak{u}} \oplus p_2^*\underline{\mathfrak{u}}$ supported on $\widetilde{\Lambda}$. Hence the same is true for the bounded complexes of $G(\mathbf{O})$-equivariant coherent sheaves $\mathcal{F}, \mathcal{G}$ on $\underline{\mathfrak{u}}$ supported on Λ. Thus, $D^bCoh_\Lambda^{G(\mathbf{O})}(\underline{\mathfrak{u}})$ is closed with respect to convolution.

Theorem 7.3. *$K^{G(\mathbf{O})}(\Lambda)$ is a commutative algebra isomorphic to $\mathbb{C}[\check{T} \times T]^W$.*

Remark 7.4. Since Λ_G is an affine Grassmannian analogue of the classical Steinberg variety, this result agrees well with the geometric realization of the Cherednik double affine Hecke algebra in [8], [23]. In effect, $K^{G(\mathbf{O})}(\Lambda_G)$ is the spherical subalgebra of the Cherednik algebra with both parameters trivial: $q = t = 1$.

7.5. **Bialynicki-Birula stratifications.** The proof of Theorem 7.3 uses the following lemma on K-theory of cellular spaces. Let M be a normal quasiprojective variety equipped with a torus H-action with finitely many fixed points. We assume that M is equipped with an H-invariant stratification $M = \bigsqcup_{\mu \in M^H} M_\mu$ such that each stratum M_μ contains exactly one H-fixed point μ, and M_μ is isomorphic to an affine space. For $\mu \in M^H$ we denote by $j_\mu : M_\mu \hookrightarrow M$ the locally closed embedding of the corresponding stratum. We denote by $i_\mu : \mu \hookrightarrow M_\mu$ the closed embedding of an H-fixed point in the corresponding stratum, or in the whole of M when no confusion is likely. We denote by $\mu \leq \nu$ the closure relation of strata. We denote by $M_{\leq\mu} \subset M$ the union $\bigcup_{\nu\leq\mu} M_\nu$.

Given an H-equivariant closed embedding of M into a smooth H-variety M' (for the existence see [22]) we denote by T^*M the restriction of the cotangent bundle T^*M' to $M \subset M'$. We denote by $\imath : M \hookrightarrow T^*M$ the embedding of the zero section. We also denote by i_μ the closed embedding of the conormal bundle $T^*_\mu M' \hookrightarrow T^*M$ when no confusion is likely. Finally, we denote by $\mathcal{L}'$ the union of conormal bundles $\bigcup_\mu T^*_{M_\mu} M'$, and $\jmath$ stands for the closed embedding $\mathcal{L}' \hookrightarrow T^*M$. We denote by $\mathcal{L}'_{\leq\mu} \subset \mathcal{L}'$ the union $\bigcup_{\nu\leq\mu} T^*_{\mathcal{M}_\nu} M'$; it is a closed subvariety of $\mathcal{L}'$. It has a closed subvariety $\mathcal{L}'_{<\mu} := \bigcup_{\nu<\mu} T^*_{M_\nu} M'$.

For $\mu \in M^H$ we have an embedding $i_{\mu *}: K^H(\mu) \hookrightarrow K^H(M)$. We have an embedding $\jmath_*: K^H(\mathcal{L}') \hookrightarrow K^H(T^*M) \overset{\imath^*}{\simeq} K^H(M)$. Indeed, the exact sequences (see [4] Chapter 5)

$$0 \to K^H(\mathcal{L}'_{<\mu}) \to K^H(\mathcal{L}'_{\leq\mu}) \to K^H(T^*_{M_\mu}M') \to 0,$$

$$0 \to K^H(T^*M'|_{M_{<\mu}}) \to K^H(T^*M'|_{M_{\leq\mu}}) \to K^H(T^*M'|_{M_\mu})$$

give rise to the support filtrations on $K^H(\mathcal{L}')$ and $K^H(T^*M)$ with associated graded $\bigoplus_{\mu\in M^H} K^H(T^*_{M_\mu}M')$ and $\bigoplus_{\mu\in M^H} K^H(T^*M'|_{M_\mu})$. Now $\jmath_*$ is strictly compatible with the support filtrations and clearly injective on the associated graded.

Note that the image $\jmath_*(K^H(\mathcal{L}')) \subset K^H(M)$ is independent of the choice of the closed embedding $M \hookrightarrow M'$. In effect, given another embedding $M \hookrightarrow \widetilde{M}$, we can consider the diagonal embedding $M \hookrightarrow M'' := M' \times \widetilde{M}$. Clearly, we have a projection $p: T^*M''|_M \to T^*M'|_M$ which realizes $T^*M''|_M$ as a vector bundle over $T^*M'|_M$. Moreover, if we denote by $\mathcal{L}''$ the union of conormal bundles $\bigcup_\mu T^*_{M_\mu}M'' \subset T^*M''|_M$ then $\mathcal{L}'' = p^{-1}\mathcal{L}'$. This shows that the images of $K^H(\mathcal{L}')$ and $K^H(\mathcal{L}'')$ in $K^H(M)$ coincide, and thus $\jmath_*(K^H(\mathcal{L}')) \subset K^H(M)$ is well-defined.

Lemma 7.6. *In $K^H(M)$ we have an equality $\jmath_*(K^H(\mathcal{L}')) = \oplus_\mu i_{\mu*}(K^H(\mu))$.*

Proof Let $K^H(D_M)$ stand for the K-group of weakly H-equivariant D-modules on M' supported on $M \subset M'$. Given such a D-module and passing to associated graded with respect to a good filtration, we obtain an H-equivariant coherent sheaf on T^*M, and this way one obtains a homomorphism $SS: K^H(D_M) \to K^H(T^*M) \overset{\imath^*}{\simeq} K^H(M)$ (see e.g. [11]). Let δ_μ stand for a δ-function D-module at the point $\mu \in M^H$ with its obvious H-equivariance. Then, evidently, $SS(\delta_\mu)$ generates $i_{\mu*}(K^H(\mu))$ as a module over $K^H(pt)$. Moreover, $\{SS(j_{\mu!}\mathcal{O}_{M_\mu}),\ \mu \in M^H\}$ forms a basis of $\jmath_*(K^H(\mathcal{L}'))$.

In effect, the closed embedding $\mathcal{L}'_{<\mu} \hookrightarrow \mathcal{L}'_{\leq\mu}$ gives rise to the exact sequence

$$0 \to K^H(\mathcal{L}'_{<\mu}) \to K^H(\mathcal{L}'_{\leq\mu}) \to K^H(T^*_{M_\mu}M') \to 0$$

(see [4] Chapter 5), and the image of $SS(j_{\mu!}\mathcal{O}_{M_\mu})$ in $K^H(T^*_{M_\mu}M')$ clearly generates it.

So it is enough to check the equality in $K^H(T^*M)$:

$$SS(\delta_\mu) = SS(j_{\mu!}\mathcal{O}_{M_\mu}) \cdot (-1)^{\dim M_\mu} \det(T_\mu M_\mu) \tag{12}$$

where $\det(T_\mu M_\mu)$ is the character of H (thus an invertible element of $K^H(pt) = \mathbb{C}[H]$) acting in the determinant of the tangent bundle of M_μ at μ.

To this end note that restriction to the H-fixed points gives rise to an embedding $\oplus_\nu i^*_\nu \imath^*: K^H(T^*M) \hookrightarrow \oplus_\nu K^H(\nu)$. This is checked by induction in ν using the exact sequences

$$0 \to K^H(T^*M'|_{M_{<\nu}}) \to K^H(T^*M'|_{M_{\leq\nu}}) \to K^H(T^*M'|_{M_\nu}) \to 0.$$

It is clear that for $\nu = \mu$ the restrictions $i^*_\mu \imath^*$ of the LHS and RHS of (12) coincide. We are going to check that for $\nu \neq \mu$ the restrictions $i^*_\nu \imath^*$ of the LHS and RHS of (12) both vanish. Evidently, $i^*_\nu \imath^* SS(\delta_\mu) = 0$.

Recall that i_ν also stands for the closed embedding $T^*_\nu M' \hookrightarrow T^*M$, so we just have to check that $i^*_\nu SS(j_{\mu !}\mathcal{O}_{M_\mu}) = 0 \in K^H(T^*_\nu M')$. Note that the functor of global sections of H-equivariant coherent sheaves on the vector space $T^*_\nu M'$ gives rise to an embedding $\Gamma :\ K^H(T^*_\nu M') \hookrightarrow \mathbb{Z}^{X^*(H)}$ where $X^*(H)$ stands for the lattice of characters of H. Now for a D-module $\mathcal{F}$ we have $\Gamma(i^*_\nu SS\mathcal{F}) = \mathbf{i}^*_\nu\mathcal{F}$ where $\mathbf{i}^*_\nu\mathcal{F}$ stands for the fiber at $\nu \in M$ of the H-equivariant quasicoherent $\mathcal{O}_{M'}$-module $\mathcal{F}$. Finally, for $\mathcal{F} = j_{\mu !}\mathcal{O}_{M_\mu}$ and $\nu \neq \mu$ we have $\mathbf{i}^*_\nu j_{\mu !}\mathcal{O}_{M_\mu} = 0$. This completes the proof of the lemma.

7.7. **Bialynicki-Birula stratification of** Gr_G**.** We consider the stratification of Gr_G by the Iwahori orbits $\mathrm{Gr}_G = \bigsqcup_{\check\mu\in Y}\mathrm{Gr}_G^{\check\mu}$. This is a refinement of the stratification by the $G(\mathbf{O})$-orbits: $\mathrm{Gr}_{G,\check\lambda} = \bigsqcup_{\check\mu\in W\check\lambda}\mathrm{Gr}_G^{\check\mu}$. Let us denote by $\mathfrak{n}\supset\mathfrak{u}$ the nilpotent radical of the Iwahori subalgebra in $\mathfrak{g}(\mathbf{F})$. The union of conormal bundles to the Iwahori orbits is the following subvariety Λ_I of the cotangent bundle $\underline{\mathfrak{u}}$: by definition, $\Lambda_I := \underline{\mathfrak{u}} \cap (\mathfrak{n}\times\mathrm{Gr}_G)$. We have a closed embedding $\Lambda\subset\Lambda_I$.

Lemma 7.6 allows us to compute $K^T(\Lambda_I) = \oplus_{\check\mu\in Y}K^T(\check\mu) \subset K^T(\mathrm{Gr}_G)$, i.e. $K^T(\Lambda_I)\simeq \mathbb{C}[\check T\times T]$ (note that the natural W-action on $K^T(\mathrm{Gr}_G)$ induces the diagonal W-action on $\mathbb{C}[\check T\times T]\simeq K^T(\Lambda_I)\subset K^T(\mathrm{Gr}_G)$). Although Lemma 7.6 was formulated for finite dimensional varieties M, its proof goes through for Gr_G without changes: we only need to have the singular support map $SS : K^T(D_{\mathrm{Gr}_G}) \to K^T(\underline{\mathfrak{u}}) \simeq K^T(\mathrm{Gr}_G)$. For this see [12], [2] (Chapter 15), [8].

The embedding $\Lambda\hookrightarrow\Lambda_I$ gives rise to the embedding $K^T(\Lambda)\hookrightarrow K^T(\Lambda_I)\hookrightarrow K^T(\underline{\mathfrak{u}}) = K^T(\mathrm{Gr}_G)$. Note that W acts naturally on both $K^T(\Lambda)$ and $K^T(\mathrm{Gr}_G)$, and the embedding $K^T(\Lambda)\hookrightarrow K^T(\mathrm{Gr}_G)$ is W-equivariant. Also, $(K^T(\Lambda))^W = K^G(\Lambda) = K^{G(\mathbf{O})}(\Lambda)$. Hence, the image of the embedding $K^{G(\mathbf{O})}(\Lambda)\hookrightarrow K^T(\Lambda_I)\simeq\mathbb{C}[\check T\times T]\subset K^T(\mathrm{Gr}_G)$ lies in the invariants of the diagonal W-action on $\mathbb{C}[\check T\times T]$. Thus to prove Theorem 7.3 we must check that the image of this embedding contains $\mathbb{C}[\check T\times T]^W$.

We have projections $\pi :\ \Lambda\to\mathrm{Gr}_G$, and $\pi_I :\ \Lambda_I\to\mathrm{Gr}_G$. For $\check\lambda\in Y^+$ we denote by $\Lambda_{\check\lambda}$ (resp. $\Lambda_{\le\check\lambda}$, $\Lambda_{<\check\lambda}$) the preimage $\pi^{-1}(\mathrm{Gr}_{G,\check\lambda})$ (resp. $\pi^{-1}(\overline{\mathrm{Gr}}_{G,\check\lambda})$, $\pi^{-1}(\overline{\mathrm{Gr}}_{G,\check\lambda}-\mathrm{Gr}_{G,\check\lambda})$). For $\check\lambda\in Y^+$ we denote by $\Lambda_{I,\check\lambda}$ (resp. $\Lambda_{I,\le\check\lambda}$, $\Lambda_{I,<\check\lambda}$) the preimage $\pi_I^{-1}(\mathrm{Gr}_{G,\check\lambda})$ (resp. $\pi_I^{-1}(\overline{\mathrm{Gr}}_{G,\check\lambda})$, $\pi_I^{-1}(\overline{\mathrm{Gr}}_{G,\check\lambda}-\mathrm{Gr}_{G,\check\lambda})$). Clearly, $\Lambda_{<\check\lambda}$ (resp. $\Lambda_{I,<\check\lambda}$) is closed in $\Lambda_{\le\check\lambda}$ (resp. $\Lambda_{I,\le\check\lambda}$), with the open complement $\Lambda_{\check\lambda}$ (resp. $\Lambda_{I,\check\lambda}$). In K-groups we have exact sequences (see [4] Chapter 5)

$$0\to K^T(\Lambda_{<\check\lambda})\to K^T(\Lambda_{\le\check\lambda})\to K^T(\Lambda_{\check\lambda})\to 0,$$

$$0\to K^T(\Lambda_{I,<\check\lambda})\to K^T(\Lambda_{I,\le\check\lambda})\to K^T(\Lambda_{I,\check\lambda})\to 0.$$

Thus we obtain a support filtration on $K^T(\Lambda_I)$ (resp. $K^T(\Lambda)$) with associated graded $\bigoplus_{\check\lambda\in Y^+}K^T(\Lambda_{I,\check\lambda})$ (resp. $\bigoplus_{\check\lambda\in Y^+}K^T(\Lambda_{\check\lambda})$).

We have the embeddings $K^T(\Lambda_{\check\lambda})\hookrightarrow K^T(\Lambda_{I,\check\lambda})\hookrightarrow K^T(\underline{\mathfrak{u}}|_{\mathrm{Gr}_{\check\lambda}})\simeq K^T(\mathrm{Gr}_{\check\lambda})$. The Weyl group W acts naturally both on $K^T(\Lambda_{\check\lambda})$ and $K^T(\mathrm{Gr}_{\check\lambda})$, and to prove Theorem 7.3 it suffices to check that the image of $(K^T(\Lambda_{\check\lambda}))^W$ in $K^T(\Lambda_{I,\check\lambda})$ contains (equivalently, coincides with) the intersection $K^T(\Lambda_{I,\check\lambda})\cap(K^T(\mathrm{Gr}_{\check\lambda}))^W$.

To this end recall that $\mathrm{Gr}_{G,\check\lambda}$ can be G-equivariantly identified with the total space $\widetilde{\mathcal{B}}$ of a vector bundle over a certain partial flag variety $\mathcal{B}$ of the group G (the quotient $G/P_{\check\lambda}$ by a parabolic subgroup depending on $\check\lambda$). The Borel subgroup $B \subset G$ acts on $\mathcal{B}$ with finitely many orbits numbered by the cosets of parabolic Weyl subgroup $W^{\check\lambda} = W/W_{\check\lambda}$: we have $\mathcal{B} = \bigsqcup_{w\in W^{\check\lambda}} \mathcal{B}_w$. Let us denote by $\mathcal{L} \subset T^*\mathcal{B}$ the union of conormal bundles $\mathcal{L} = \bigsqcup_{w\in W^{\check\lambda}} T^*_{\mathcal{B}_w}\mathcal{B}$. Let us also denote by $\widetilde{\mathcal{B}}_w$ the preimage of $\mathcal{B}_w$ in $\widetilde{\mathcal{B}}$ (it coincides with a certain Iwahori orbit $\mathrm{Gr}_G^{\check\mu} \subset \mathrm{Gr}_{G,\check\lambda} = \widetilde{\mathcal{B}}$). We define $\widetilde{\mathcal{L}} := \bigsqcup_{w\in W^{\check\lambda}} T^*_{\widetilde{\mathcal{B}}_w}\widetilde{\mathcal{B}} \subset T^*\widetilde{\mathcal{B}}$. Then there exists a G-equivariant profinite dimensional vector bundle $\mathcal{V} \xrightarrow{p} T^*\widetilde{\mathcal{B}}$ such that $\mathcal{V} \simeq \underline{\mathfrak{u}}|_{\mathrm{Gr}_{\check\lambda}}$, and under this isomorphism we have $\mathcal{V}|_{\widetilde{\mathcal{L}}} \simeq \Lambda_{I,\check\lambda}$, $\mathcal{V}|_{\widetilde{\mathcal{B}}\hookrightarrow T^*\widetilde{\mathcal{B}}} \simeq \Lambda_{\check\lambda}$. Thus to prove Theorem 7.3 it is enough to check that the image of $(K^T(\widetilde{\mathcal{B}}))^W$ in $K^T(T^*\widetilde{\mathcal{B}})$ contains the intersection $K^T(\widetilde{\mathcal{L}}) \cap (K^T(T^*\widetilde{\mathcal{B}}))^W$. Equivalently, we have to check that the image of $(K^T(\mathcal{B}))^W$ in $K^T(T^*\mathcal{B})$ contains the intersection $K^T(\mathcal{L}) \cap (K^T(T^*\mathcal{B}))^W$. This is the subject of the following lemma.

Lemma 7.8. *Let $\imath:\ \mathcal{B} \hookrightarrow T^*\mathcal{B}$ denote the embedding of the zero section, and let $\jmath:\ \mathcal{L} \hookrightarrow T^*\mathcal{B}$ denote the natural closed embedding. Then $\imath_*(K^T(\mathcal{B}))^W$ coincides with* $\mathrm{Im}\left(\jmath_*:\ K^T(\mathcal{L}) \hookrightarrow K^T(T^*\mathcal{B})\right) \bigcap \left(K^T(T^*\mathcal{B})\right)^W$.

Proof For $w \in W^{\check\lambda}$ we denote by $w \in \mathcal{B}_w \subset \mathcal{B}$ the corresponding T-fixed point. We denote by i_w the closed embedding $T^*_w\mathcal{B} \hookrightarrow T^*\mathcal{B}$ (and also the closed embedding $w \hookrightarrow \mathcal{B}$, when the confusion is unlikely), and we denote by $\mathfrak{i}_w$ the closed embedding $w \hookrightarrow T^*\mathcal{B}$. According to Lemma 7.6, the image of $\jmath_*:\ K^T(\mathcal{L}) \hookrightarrow K^T(T^*\mathcal{B})$ coincides with the image of $\oplus_{w\in W^{\check\lambda}} i_{w*}:\ \oplus_{w\in W^{\check\lambda}} K^T(T^*_w\mathcal{B}) \to K^T(T^*\mathcal{B})$. We have an embedding $\oplus_{w\in W^{\check\lambda}} \mathfrak{i}^*_w:\ K^T(T^*\mathcal{B}) \hookrightarrow \oplus_{w\in W^{\check\lambda}} K^T(w)$, and similarly an embedding $\oplus_{w\in W^{\check\lambda}} i^*_w:\ K^T(\mathcal{B}) \hookrightarrow \oplus_{w\in W^{\check\lambda}} K^T(w)$.

Clearly, the W-invariants project injectively into any direct summand: $K^G(\mathcal{B}) = (K^T(\mathcal{B}))^W \overset{i^*_w}{\hookrightarrow} K^T(w)$ (resp. $K^G(T^*\mathcal{B}) = (K^T(T^*\mathcal{B}))^W \overset{\mathfrak{i}^*_w}{\hookrightarrow} K^T(w)$) for any $w \in W^{\check\lambda}$. Thus it suffices to check that for any $w \in W^{\check\lambda}$ we have a coincidence $\mathrm{Im}(\mathfrak{i}^*_w i_{w*}:\ K^T(T^*_w\mathcal{B})^W \to K^T(w)) = \mathrm{Im}(\mathfrak{i}^*_w \imath_* \mathrm{Res}^G_T:\ K^G(\mathcal{B}) \to K^T(w))$. Note that if $w = e$ (the identity coset of $W_{\check\lambda}$ in W), then the image $i^*_e(K^T(\mathcal{B}))^W \subset K^T(e)$ (resp. $\mathfrak{i}^*_e(K^T(T^*\mathcal{B}))^W \subset K^T(e)$) coincides with $(K^T(e))^{W_{\check\lambda}} = \mathbb{C}[T]^{W_{\check\lambda}}$. Moreover, under identification $K^T(T^*_e\mathcal{B}) = K^T(e) = \mathbb{C}[T]$, we have $K^T(T^*_e\mathcal{B}) \cap (K^T(T^*\mathcal{B}))^W = \mathbb{C}[T]^{W_{\check\lambda}}$.

Identifying both $K^T(T^*_e\mathcal{B})$ and $K^T(e)$ with $\mathbb{C}[T]$, the map $\mathfrak{i}^*_e i_{e*}$ is a multiplication by the product $\Delta_1 = \prod_{k=1}^{\dim\mathcal{B}}(1-\chi_k)$ where χ_k run through the characters of T in the tangent space $T_e(T^*_e\mathcal{B}) = T^*_e\mathcal{B}$. Furthermore, identifying $K^G(\mathcal{B})$ with $\mathbb{C}[T]^{W_{\check\lambda}}$, and $K^T(e)$ with $\mathbb{C}[T]$, the map $\mathfrak{i}^*_e \imath_* \mathrm{Res}^G_T$ is a multiplication by the product $\Delta_2 = \prod_{k=1}^{\dim\mathcal{B}}(1-\chi'_k)$ where χ'_k run through the characters of T in the tangent space $T_e\mathcal{B}$. We can arrange the characters χ'_k so that we have $\chi'_k = \chi_k^{-1}$. Then we see that $\Delta_1 = \Delta_2 \cdot \prod_{k=1}^{\dim\mathcal{B}}(-\chi_k)$, so they differ by an invertible function, hence the corresponding images coincide: $\Delta_1 \cdot \mathbb{C}[T]^{W_{\check\lambda}} = \Delta_2 \cdot \mathbb{C}[T]^{W_{\check\lambda}}$.

This completes the proof of the lemma along with Theorem 7.3.

7.9. In this subsection we describe (without striving for high precision) a conjectural picture motivating Theorem 7.3.

We hope that the isomorphism $K^{G(\mathbf{O})}(\Lambda_G) = \mathbb{C}[\check{T}\times T]^W = \mathbb{C}[T\times\check{T}]^W = K^{\check{G}(\mathbf{O})}(\Lambda_{\check{G}})$ lifts to an equivalence of monoidal categories $F:\ D^bCoh^{G(\mathbf{O})}_{\Lambda_G}(\underline{\mathfrak{u}}_G) \simeq D^bCoh^{\check{G}(\mathbf{O})}_{\Lambda_{\check{G}}}(\underline{\mathfrak{u}}_{\check{G}})$. The conjectural equivalence F is related to the Langlands correspondence in the following way.

Recall that the conjectural (for $G = GL(n)$ mostly proven in [9]) geometric Langlands correspondence is an equivalence of triangulated categories between the derived category of D-modules on the stack Bun_G of G-bundles on a given smooth projective curve C, and the derived category of coherent sheaves on the stack of $\check{G}$ local systems on the same curve. One might expect its "classical limit" to be an equivalence between the derived categories of coherent sheaves $L:\ D(T^*\mathrm{Bun}_G) \simeq D(T^*\mathrm{Bun}_{\check{G}})$ where $T^*\mathrm{Bun}_G$ is the cotangent bundle to the moduli stack of G-bundles on C. Given a point $c\in C$, and identifying $\mathbf{O}$ with the algebra of functions on the formal neighbourhood of c, one gets an action of $D^bCoh^{G(\mathbf{O})}_{\Lambda_G}(\underline{\mathfrak{u}}_G)$ on $D(T^*\mathrm{Bun}_G)$. The "classical limit" of the Hecke eigen-property of geometric Langlands correspondence (see [2]) should be stated in terms of this action; it should say that the global equivalence L is compatible with our local equivalence F.

8. Perverse sheaves and fusion

We refer the reader to [3] for the definition of perverse equivariant coherent sheaves and related objects.

8.1. Recall the setup of 6.1. Note that all the $G(\mathbf{O})$-orbits in a connected component of Gr_G have dimensions of the same parity. Thus it makes sense to consider the middle perversity function $p(\mathrm{Gr}_{G,\check{\lambda}}) = -\frac{1}{2}\dim(\mathrm{Gr}_{G,\check{\lambda}}) = -\langle\rho,\check{\lambda}\rangle$. It is obviously strictly monotone and comonotone, but at some connected components of Gr_G it takes values in half-integers. This means that we consider equivariant complexes formally placed in half-integer homological degrees. The theory of [3] defines the artinian abelian category $\mathcal{P}^{G(\mathbf{O})}(\mathrm{Gr}_G)$ of perverse $G(\mathbf{O})$-equivariant coherent sheaves (with respect to the above middle perversity). Let $D^{b,G(\mathbf{O})}(\mathrm{Gr}_G)$ denote the bounded derived category of $G(\mathbf{O})$-equivariant coherent sheaves on Gr_G (with the same convention that the complexes at "odd" connected components are placed in half-integer homological degrees).

Given two complexes $\mathcal{F},\mathcal{G} \in D^{b,G(\mathbf{O})}(\mathrm{Gr}_G)$ we have their convolution $\mathcal{F}\star\mathcal{G}\in D^{b,G(\mathbf{O})}(\mathrm{Gr}_G)$. Recall that $\mathcal{F}\star\mathcal{G} = \Pi_{0*}(\mathcal{F}\ltimes\mathcal{G})$ where $\Pi_0:\ G(\mathbf{F})\times_{G(\mathbf{O})}\mathrm{Gr}_G\to\mathrm{Gr}_G$ is the convolution diagram, and $\mathcal{F}\ltimes\mathcal{G}$ is the twisted product of $\mathcal{F}$ and $\mathcal{G}$ on $G(\mathbf{F})\times_{G(\mathbf{O})}\mathrm{Gr}_G$.

Proposition 8.2. *The convolution preserves perverse sheaves: for* $\mathcal{F},\mathcal{G}\in\mathcal{P}^{G(\mathbf{O})}(\mathrm{Gr}_G)$ *we have* $\mathcal{F}\star\mathcal{G}\in\mathcal{P}^{G(\mathbf{O})}(\mathrm{Gr}_G)$.

Proof Denote the projection $G(\mathbf{F})\to G(\mathbf{F})/G(\mathbf{O}) = \mathrm{Gr}_G$ by p, and consider a stratification $G(\mathbf{F})\times_{G(\mathbf{O})}\mathrm{Gr}_G = \bigsqcup_{\check{\lambda},\check{\mu}\in Y^+} p^{-1}(\mathrm{Gr}_{G,\check{\lambda}})\times_{G(\mathbf{O})}\mathrm{Gr}_{G,\check{\mu}}$. Clearly, $\mathcal{F}\ltimes\mathcal{G}$ is smooth (locally free) along this stratification, and perverse (with respect to the middle perversity). According to [19] 2.7, the map Π_0 is stratified semismall with respect to the above stratification. Now the perversity of $\Pi_{0*}(\mathcal{F} \ltimes \mathcal{G})$ follows in the same manner as in the constructible case, cf. *loc. cit.*

8.3. **The absence of commutativity constraint.** According to Proposition 8.2, $\mathcal{P}^{G(\mathbf{O})}(\mathrm{Gr}_G)$ acquires the structure of abelian artinian monoidal category. Moreover, according to 2.15 (a), its K-ring is commutative. Nevertheless, $\mathcal{P}^{G(\mathbf{O})}(\mathrm{Gr}_G)$ admits no commutativity constraint, as can be seen in the following example.

We recall the setup of 3.6, and consider Gr_{PGL_2}. One can check that there are the nonsplit exact sequences in $\mathcal{P}^{PGL_2(\mathbf{O})}(\mathrm{Gr}_{PGL_2})$:

$$0 \to \mathcal{V}(0)_0 \to \mathcal{V}(0)_1 \star \mathcal{V}(-2)_1 \to \mathcal{V}(-2)_2 \to 0$$
$$0 \to \mathcal{V}(-2)_2 \to \mathcal{V}(-2)_1 \star \mathcal{V}(0)_1 \to \mathcal{V}(0)_0 \to 0$$

Thus $\mathcal{V}(0)_1 \star \mathcal{V}(-2)_1$ and $\mathcal{V}(-2)_1 \star \mathcal{V}(0)_1$ are nonisomorphic.

8.4. $G(\mathbf{O}) \ltimes \mathbb{G}_m$**-equivariant sheaves and fusion.** The orbits of $G(\mathbf{O}) \ltimes \mathbb{G}_m$ on Gr_G coincide with the $G(\mathbf{O})$-orbits, so one can consider the abelian artinian monoidal category $\mathcal{P}^{G(\mathbf{O})\ltimes\mathbb{G}_m}(\mathrm{Gr}_G)$ of $G(\mathbf{O}) \ltimes \mathbb{G}_m$-equivariant coherent perverse sheaves on Gr_G. For $\mathcal{F} \in \mathcal{P}^{G(\mathbf{O})\ltimes\mathbb{G}_m}(\mathrm{Gr}_G)$ we have $R\Gamma(\mathrm{Gr}_G, \mathcal{F}) \in D^b(G(\mathbf{O}) \ltimes \mathbb{G}_m - mod)$.

B. Feigin and S. Loktev define (under certain restrictions) in [6] the *fusion product* $V_1 \star \ldots \star V_k \in G(\mathbf{O}) \ltimes \mathbb{G}_m - mod$ of $G(\mathbf{O}) \ltimes \mathbb{G}_m$-modules $V_1, \ldots, V_k$. We recall some of their results in case $G = PGL_2$.

Let $V(n)$ be the $n+1$-dimensional $G(\mathbf{O}) \ltimes \mathbb{G}_m$-module factoring through $G(\mathbf{O}) \ltimes \mathbb{G}_m \twoheadrightarrow G \times \mathbb{G}_m \twoheadrightarrow G$. Recall the irreducible $PGL_2(\mathbf{O})$-equivariant perverse sheaf $\mathcal{V}(n)_m$ introduced in 3.6. It can be lifted to the same named $PGL_2(\mathbf{O}) \ltimes \mathbb{G}_m$-equivariant perverse sheaf, where the action of $\mathbb{G}_m$ in the fiber over a $\mathbb{G}_m$-fixed point in the orbit $\mathrm{Gr}_{PGL_2,m}$ is set *trivial*. In particular, $R\Gamma(\mathrm{Gr}_{PGL_2}, \mathcal{V}(n)_1) = V(n)[\frac{1}{2}]$ for $n \geq 0$.

Now we can reformulate Theorem 2.5 of [6] as follows.

Proposition 8.5. *Let $n_1 \geq n_2 \geq \ldots \geq n_k$. Then*
(a) $R\Gamma(\mathrm{Gr}_{PGL_2}, \mathcal{V}(n_1)_1 \star \ldots \star \mathcal{V}(n_k)_1)$ is concentrated in degree $-\frac{k}{2}$;
(b) $R\Gamma(\mathrm{Gr}_{PGL_2}, \mathcal{V}(n_1)_1 \star \ldots \star \mathcal{V}(n_k)_1)[-\frac{k}{2}] \simeq V(n_k) \star \ldots \star V(n_1)$.

8.6. **Multiplication table.** According to Proposition 8.5, the calculation of fusion product in $K(G(\mathbf{O}) \ltimes \mathbb{G}_m - mod)$ is closely related to the ring structure of $K^{G(\mathbf{O})\ltimes\mathbb{G}_m}(\mathrm{Gr}_G)$. Let us formulate the recurrence relations in $K^{G(\mathbf{O})\ltimes\mathbb{G}_m}(\mathrm{Gr}_G)$, compare [6], end of section 2.1. So $\mathbf{v}(n)_m$ is the class of $\mathcal{V}(n)_m$ in $K^{G(\mathbf{O})\ltimes\mathbb{G}_m}(\mathrm{Gr}_G)$. We assume that $n \geq 0$.

$$q^{-l}\mathbf{v}(l+n)_1 0 \star \mathbf{v}(l)_1 = q^{-2l}\mathbf{v}(2l+n)_2 + q^2\mathbf{v}(n-2)_0 + q^4\mathbf{v}(n-4)_0 + \ldots \tag{13}$$

(the last summand being $q^n\mathbf{v}(0)_0$ if n is even, and $q^{n-1}\mathbf{v}(1)_0$ if n is odd.)

$$q^{-l-2}\mathbf{v}(l-n)_1 0 \star \mathbf{v}(l)_1 = q^{-2l-2}\mathbf{v}(2l-n)_2 + q^{-2}\mathbf{v}(n-2)_0 + q^{-4}\mathbf{v}(n-4)_0 + \ldots \tag{14}$$

(the last summand being $q^{-n}\mathbf{v}(0)_0$ if n is even, and $q^{-n+1}\mathbf{v}(1)_0$ if n is odd.)

$$\mathbf{v}(l+1)_1^{\star a} \star \mathbf{v}(l)_1^{\star b} = q^{\frac{1}{2}(a(1-a)+l(a+b)(1-a-b))}\mathbf{v}(a+l(a+b))_{a+b} \tag{15}$$

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