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Item Type	article;article
Authors	Burstall, F;Donaldson, N;Pedit, F;Pinkall, U
Download date	2024-08-08 19:41:26
Link to Item	<a href="https://hdl.handle.net/20.500.14394/34213">https://hdl.handle.net/20.500.14394/34213</a>

# ISOTHERMIC SUBMANIFOLDS OF SYMMETRIC $R$ -SPACES

FRANCIS E. BURSTALL, NEIL M. DONALDSON, FRANZ PEDIT, AND ULRICH PINKALL

ABSTRACT. We extend the classical theory of isothermic surfaces in conformal 3-space, due to Bour, Christoffel, Darboux, Bianchi and others, to the more general context of submanifolds of symmetric  $R$ -spaces with essentially no loss of integrable structure.

## INTRODUCTION

**Background.** A surface in  $\mathbb{R}^3$  is *isothermic* if, away from umbilics, it admits coordinates which are simultaneously conformal and curvature line or, more invariantly, if it admits a holomorphic quadratic differential  $q$  which commutes with the (trace-free) second fundamental form (then the coordinates are  $z = x + iy$  for which  $q = dz^2$ ). Examples include surfaces of revolution, cones and cylinders; quadrics and constant mean curvature surfaces (where  $q$  is the Hopf differential).

Starting with the work of Bour [9, §54], isothermic surfaces were the focus of intensive study by the geometers of the late 19th and early 20th centuries with contributions from Christoffel, Cayley, Darboux, Demoulin, Bianchi, Calapso, Tzitzéica and many others. There has been a recent revival of interest in the topic thanks to Cieřliński–Goldstein–Sym [26] who pointed out the links with soliton theory: indeed, isothermic surfaces constitute an integrable system with a particularly beautiful and intricate transformation theory, some aspects of which we list below:

*Conformal invariance.* Since the trace-free second fundamental form is invariant (up to scale) under conformal diffeomorphisms of  $\mathbb{R}^3$ , such diffeomorphisms preserve the class of isothermic surfaces. Thus isothermic surfaces are more properly to be viewed as surfaces in the conformal 3-sphere.

*Deformations.* At least locally, isothermic surfaces admit a 1-parameter deformation preserving the conformal structure and trace-free second fundamental form: this is the  $T$ -transformation of Bianchi [3] and Calapso [19]. In fact, isothermic surfaces are characterised by the existence of such a deformation [23] (see [47, 17, 14] for modern treatments).

*Darboux transformations.* According to Darboux [27], given an isothermic surface, one may locally construct a 4-parameter family of new isothermic surfaces. Analytically, this is accomplished by solving an integrable system of linear differential equations; geometrically, the two surfaces envelop a conformal Ribaucour sphere congruence<sup>1</sup>. These transformations, the *Darboux transformations*, are analogous to the Bäcklund transformations of constant curvature surfaces (indeed, specialise to the latter in certain circumstances [40, 43]) and there is a Bianchi permutability theorem [2] relating iterated Darboux transforms. There are additional permutability theorems relating  $T$ -transforms and Darboux transforms [38] and also relating Darboux transforms with the Euclidean Christoffel transform [25] (see also [9]) via  $T$ -transforms [3].

*Curved flats.* Curved flats were introduced by Ferus–Pedit [30] and are an integrable system defined on submanifolds of a symmetric space  $G/H$  which, in non-degenerate cases, coincides with the  $G/H$ -system of Terng [54]. In particular, the space  $S^3 \times S^3 \setminus \Delta$  of pairs of distinct points in  $S^3$  is a (pseudo-Riemannian) symmetric space for the diagonal action of the conformal diffeomorphism group and curved flats in this space are the same as pairs of isothermic surfaces related by a Darboux transformation [16].

*Discrete theory.* Bobenko–Pinkall [6] show that the combinatorics of iterated Darboux transforms give rise to an integrable system on quad-graphs that is a convincing discretisation of isothermic surfaces and shares with them all the classical transformation properties listed above [39, 37, 8].

2000 *Mathematics Subject Classification.* 53C42;37K35,53C30.

<sup>1</sup>That is, the two surfaces admit a common parametrisation under which both conformal structures and curvature lines coincide and, through each pair of corresponding points, there is a 2-sphere tangent to both surfaces.

For more information on isothermic surfaces, we refer the Reader to Hertrich-Jeromin's encyclopedic monograph [38].

**Manifesto.** In this paper, we will show that the preceding theory continues to hold, in almost every detail, when we replace the conformal 3-sphere by an arbitrary *symmetric R-space*. There are a number of approaches to symmetric *R-spaces* [53]: they comprise Hermitian symmetric spaces and their real forms; they are the compact Riemannian symmetric spaces which admit a Lie group of diffeomorphisms strictly larger than their isometry group [48] and, basic for us, they are the conjugacy classes of parabolic subalgebras of a real semisimple Lie algebra with abelian nilradicals. Examples include projective spaces and Grassmannians (real, complex and quaternionic) and quadrics (thus conformal spheres of arbitrary signature). The symmetric *R-spaces*  $G/P$ , for  $G$  simple, are enumerated in Table 1 on page 33.

To see how an isothermic submanifold could be defined in this setting, let us begin with the following manifestly conformally invariant reformulation of the isothermic condition. View the holomorphic quadratic differential  $q$  of an isothermic surface  $f : \Sigma \rightarrow S^3$  as a  $T^*\Sigma$ -valued 1-form and thus, via  $df$ , as a  $f^{-1}T^*S^3$ -valued 1-form. Now contemplate the group  $G$  of conformal diffeomorphisms of  $S^3$ , an open subgroup of  $O(4, 1)$  and so semisimple, and its Lie algebra  $\mathfrak{g}$ . Since  $G$  acts transitively on  $S^3$ , we have, for each  $x \in S^3$ , an isomorphism  $T_x S^3 \cong \mathfrak{g}/\mathfrak{p}_x$ , where  $\mathfrak{p}_x$  is the infinitesimal stabiliser of  $x$ , and so, via the Killing form, an isomorphism  $T_x^* S^3 \cong \mathfrak{p}_x^\perp \subset \mathfrak{g}$ . Thus  $q$  may be identified with a certain  $\mathfrak{g}$ -valued 1-form  $\eta$ . The key observation now is that  $q$  is a holomorphic quadratic differential commuting with the second fundamental form of  $f$  if and only if  $\eta$  is a *closed* 1-form [14].

The decisive property enjoyed by the conformal sphere is that each  $\mathfrak{p}_x^\perp$  is an abelian subalgebra of  $\mathfrak{g}$  from which it quickly follows that each of the connections  $d + t\eta$ ,  $t \in \mathbb{R}$ , is flat. This provides a zero-curvature formulation of isothermic surfaces from which the whole theory may ultimately be deduced.

However, according to Grothendieck [35], the condition that an infinitesimal stabiliser  $\mathfrak{p}_x$  has abelian Killing polar is precisely that  $\mathfrak{p}_x$  is a parabolic subalgebra with abelian nilradical  $\mathfrak{p}_x^\perp$ . The map  $x \mapsto \mathfrak{p}_x$  then identifies  $S^3$  with a conjugacy class of such parabolic subalgebras.

It is now clear that this formulation of the isothermic condition makes sense for maps into any symmetric *R-space*. For a semisimple Lie algebra  $\mathfrak{g}$  and conjugacy class  $N$  of parabolic subalgebras  $\mathfrak{p} < \mathfrak{g}$  with abelian nilradicals, we may view  $T_{\mathfrak{p}}^* N$  as an abelian subalgebra  $\mathfrak{p}^\perp < \mathfrak{g}$  and say that a map  $f : \Sigma \rightarrow N$  is isothermic if there is an  $f^{-1}T^*N$ -valued 1-form  $\eta$  which is closed as an element of  $\Omega_\Sigma^1 \otimes \mathfrak{g}$ . We immediately get a pencil of flat connections  $d + t\eta$  and, as we will show, the theory of isothermic maps can be developed in complete analogy with the conformal theory sketched above.

**Road-map.** To orient the Reader, we briefly sketch the contents of the paper.

The first two sections are preparatory in nature. We recall in §1 the basic definitions and facts about parabolic subalgebras  $\mathfrak{p}$  of a real semisimple Lie algebra  $\mathfrak{g}$ . Particular emphasis is given to the notion of parabolic subalgebras *complementary* to a given parabolic  $\mathfrak{p}$ . Then, in §2, we rehearse what we need of the theory of *R-spaces* and symmetric *R-spaces* viewed as conjugacy classes of parabolics. In particular, we note that such spaces have convenient charts generalising those given by stereoprojection on spheres. Complementary parabolic subalgebras arise in two different ways. Firstly, the set of parabolic subalgebras complementary to some element of a conjugacy class  $M$  is itself a conjugacy class  $M^*$  which we call the *dual R-space*. It may happen that  $M = M^*$  (conformal spheres are an example) and then we say that  $M$  is *self-dual*. Another, non-self-dual, example of this duality is given by the dual projective spaces of lines and hyperplanes in a vector space. Secondly, the set  $Z \subset M \times M^*$  of pair-wise complementary parabolic subalgebras is shown to be a pseudo-Riemannian symmetric space (in fact, a *para-Hermitian* symmetric space as studied by Kaneyuki [42, 41]). This symmetric space plays the role of  $S^3 \times S^3 \setminus \Delta$  in the conformal theory.

With these preliminaries out of the way, in §3 we define isothermic maps into symmetric *R-spaces* and explore their transformation properties. Just as in the conformal case, we can define *T-transforms*, *Darboux transforms* and *Christoffel transforms* and prove some permutability theorems. We also show how Christoffel transforms are limits of Darboux transforms: a result which may have some novelty even in the classical context.

Analogues of the classical Bianchi Permutability Theorem for Darboux transforms are explored in §4. Here we must restrict attention to self-dual targets but, in that setting, find that the classical results carry through in every detail. An important role is played in this analysis by a  $G$ -invariant family of circles in a self-dual symmetric  $R$ -space which share many properties of circles in a conformal sphere: they are determined by three generic points and carry an invariant projective structure (and thus a cross-ratio). We then find that four isothermic maps in the configuration of the permutability theorem have corresponding points concircular with prescribed constant cross-ratio (a theorem of Demoulin [29] in the classical case). Our methods here seem to be more efficient than those known in the conformal case and yield a version of Bianchi's cube theorem with almost no extra work. A side-benefit of our approach is an integrable theory of *discrete* isothermic nets in self-dual symmetric  $R$ -spaces which we sketch in §4.5. The cube theorem in this context amounts to 3D consistency of our discrete integrable system in the sense of Bobenko–Suris [5].

Curved flats are considered in §5: a point-wise complementary pair of isothermic maps define a map  $\phi = (f, \hat{f}) : \Sigma \rightarrow Z \subset M \times M^*$  and, in strict analogy with the conformal case [16],  $\phi$  is a curved flat if and only if  $f$  and  $\hat{f}$  are Darboux transforms of each other. In addition, we consider the dressing transforms [11, 13] of curved flats and prove that, in the self-dual case, the four isothermic maps comprising a curved flat and its dressing transform are a Bianchi quadrilateral of isothermic maps related by Darboux transforms. This last appears to be a new result even in the conformal case.

Finally, in §6, we show that the 1-form  $\eta$  defines a quadratic differential on  $\Sigma$  and consider the case where this is non-degenerate (in particular, such isothermic maps immerse). We show that there is a sharp Lie-theoretic upper bound on the dimension of these non-degenerate isothermic submanifolds and so, in particular, settle the existence question for isothermic maps.

Isothermic submanifolds of maximal dimension in  $\mathbb{R}P^n$  turn out to be curves and we contemplate the simplest case of isothermic curves in  $\mathbb{R}P^1$ . These are the same as parametrised curves but, perhaps surprisingly, our theory still has something to offer: at the level of curvatures, the relation between a curved flat and its constituent isothermic maps amounts to the Miura transform and we find an integrable dynamics of isothermic curves, commuting with  $T$  and Darboux transforms. Passing to curvatures, the dynamics of isothermic curves gives the KdV equation, that of curved flats the mKdV equation while the Darboux transform extends to yield the Wahlquist–Estabrook Bäcklund transform [59] of the KdV equation.

**Acknowledgements.** During the preparation of this work, we have derived great benefit from conversations with David Calderbank; Udo Hertrich-Jeromin; Nigel Hitchin; Soji Kaneyuki; Katrin Leschke and Chuu-Lian Terng. Special thanks are due from the first author to Yoshihiro Ohnita who first taught him about symmetric  $R$ -spaces over green tea ice cream in 1998.

## 1. ALGEBRAIC PRELIMINARIES

An  $R$ -space is a conjugacy class of parabolic subalgebras in a non-compact semisimple Lie algebra. We begin by collecting some elements of the underlying algebra.

Here, and throughout this paper,  $\mathfrak{g}$  will be a real non-compact semisimple Lie algebra with adjoint group  $G$ . We denote the adjoint action of  $G$  on  $\mathfrak{g}$  as well as the induced action on subalgebras of  $\mathfrak{g}$  by juxtaposition: thus, for  $g \in G$ ,  $\zeta \in \mathfrak{g}$ , we write  $g\zeta$  for  $\text{Ad}(g)\zeta$ .

**1.1. Parabolic subalgebras.** A subalgebra  $\mathfrak{p} \leq \mathfrak{g}$  is parabolic if its complexification  $\mathfrak{p}^{\mathbb{C}} \leq \mathfrak{g}^{\mathbb{C}}$  is parabolic in the sense that it contains a Borel (thus maximal solvable) subalgebra of  $\mathfrak{g}^{\mathbb{C}}$ . However, it will be useful for us to adopt the following equivalent definition [18, Proposition 4.2] (see also [35, Lemma 4.2] and [20, Definition 2.1]):

**Definition 1.1.** A subalgebra  $\mathfrak{p}$  of a non-compact semisimple Lie algebra  $\mathfrak{g}$  is *parabolic* if and only if the polar  $\mathfrak{p}^{\perp}$  with respect to the Killing form is a nilpotent subalgebra.

In this case,  $\mathfrak{p}^{\perp}$  is the nilradical of  $\mathfrak{p}$  (in particular,  $\mathfrak{p}^{\perp} < \mathfrak{p}$ ) and we say that  $\mathfrak{p}$  has *height*  $n$  if  $\mathfrak{p}^{\perp}$  is  $n$ -step nilpotent.

In particular,  $\mathfrak{p}$  has height one if and only if  $\mathfrak{p}^{\perp}$  is abelian.

There are a finite number of conjugacy classes of parabolic subalgebras of  $\mathfrak{g}$ . If  $\mathfrak{g}$  is complex<sup>2</sup>, they are in bijective correspondence with subsets of nodes (the *crossed nodes*) of the Dynkin diagram of  $\mathfrak{g}$ . If, in addition,  $\mathfrak{g}$  is simple, one computes the height of a parabolic subalgebra by summing the coefficients in the highest root of the simple roots corresponding to crossed nodes (see, for example, [1]). If  $\mathfrak{g}$  is semisimple, any parabolic subalgebra is a direct sum of parabolic subalgebras (possibly not proper<sup>3</sup>) of simple ideals and the height is the maximum of the heights of these. If  $\mathfrak{g}$  is not complex, the same analysis is available if Dynkin diagrams are replaced by Satake diagrams with the caveat that the crossed nodes must all be white and any node joined by an arrow to a crossed node must also be crossed [46, Théorème 3.1].

In particular, height one parabolic subalgebras correspond to simple roots with coefficient one in the highest root and if  $\mathfrak{g}$  is not complex, the corresponding node on the Satake diagram must be white with no arrows. It is now an easy matter to arrive at the classification of conjugacy classes given in Table 1 on page 33. For example, there are no height one parabolic subalgebras in any  $\mathfrak{g}_2$ ,  $\mathfrak{f}_4$  or  $\mathfrak{e}_8$  (no coefficient one simple roots) and none in  $\mathfrak{su}(p, q)$  for  $p \neq q$  (no white nodes without arrows).

**1.2. Complementary subalgebras.** A parabolic subalgebra  $\mathfrak{p}$  of height  $n$  makes  $\mathfrak{g}$  into a filtered algebra: define inductively

$$\mathfrak{p}^{(0)} = \mathfrak{p}, \quad \mathfrak{p}^{(-1)} = \mathfrak{p}^\perp, \quad \mathfrak{p}^{(j)} = \begin{cases} [\mathfrak{p}^\perp, \mathfrak{p}^{(j+1)}] & j \leq -2, \\ \mathfrak{p}^{(-1-j)\perp} & j \geq 1. \end{cases}$$

and observe that

$$\mathfrak{g} = \mathfrak{p}^{(n)} \supseteq \cdots \supseteq \mathfrak{p}^{(1)} \supseteq \mathfrak{p} \supseteq \mathfrak{p}^{(-1)} \supseteq \cdots \supseteq \mathfrak{p}^{(-n)} \supseteq \mathfrak{p}^{(-n-1)} = \{0\}$$

while  $[\mathfrak{p}^{(j)}, \mathfrak{p}^{(k)}] \subset \mathfrak{p}^{(j+k)}$ .

**Definition 1.2.** Two parabolic subalgebras  $\mathfrak{p}, \mathfrak{q}$  of height  $n$  are *complementary* if

$$\mathfrak{g} = \mathfrak{p}^{(j)} \oplus \mathfrak{q}^{(-1-j)},$$

for all  $j = -n, \dots, n$ .

Any non-trivial parabolic subalgebra admits complementary subalgebras:

**Lemma 1.3** ([20, Lemma 2.2]). *For any Cartan involution  $\theta$  of  $\mathfrak{g}$  and any parabolic  $\mathfrak{p} < \mathfrak{g}$ ,  $\mathfrak{p}$  and  $\theta\mathfrak{p}$  are complementary.*

Moreover, the complementary subalgebras to a fixed  $\mathfrak{p}$  are all conjugate:

**Lemma 1.4** ([20, Lemma 2.5]).  *$\exp \mathfrak{p}^\perp < G$  acts simply transitively on the set of parabolic subalgebras complementary to  $\mathfrak{p}$ .*

Since  $\mathfrak{p}^\perp$  is nilpotent,  $\exp|_{\mathfrak{p}^\perp} : \mathfrak{p}^\perp \rightarrow \exp \mathfrak{p}^\perp$  is a diffeomorphism so that Lemma 1.4 tells us that this set of complementary parabolics is an affine space modelled on  $\mathfrak{p}^\perp$ .

A complementary pair  $(\mathfrak{p}, \mathfrak{q})$  of height  $n$  parabolic subalgebras splits the filtered structure of  $\mathfrak{g}$  into a graded one: set  $\mathfrak{g}_j := \mathfrak{p}^{(j)} \cap \mathfrak{q}^{(-j)}$  and then  $\mathfrak{g} = \bigoplus_{j=-n}^n \mathfrak{g}_j$  while  $[\mathfrak{g}_j, \mathfrak{g}_k] \subset \mathfrak{g}_{j+k}$ . Thus the map given by  $\zeta \mapsto j\zeta$ , for  $\zeta \in \mathfrak{g}_j$  is a derivation and so, since  $\mathfrak{g}$  is semisimple, given by the adjoint action of a unique element  $\xi$  in the centre of  $\mathfrak{g}_0$ . We call  $\xi = \xi_{\mathfrak{p}}^{\mathfrak{q}}$  the *grading* or *canonical element* of the pair  $(\mathfrak{p}, \mathfrak{q})$ .

Of course, we can recover the complementary pair from the grading element:

$$(1.1) \quad \mathfrak{p} = \bigoplus_{j=-n}^0 \mathfrak{g}_j \quad \mathfrak{q} = \bigoplus_{j=0}^n \mathfrak{g}_j.$$

*Remark 1.5.* When  $\mathfrak{p}$  has height 1, the theory simplifies considerably. In particular,  $\mathfrak{q}$  is complementary to  $\mathfrak{p}$  if and only if  $\mathfrak{p}^\perp \cap \mathfrak{q}^\perp = \{0\}$  if and only if  $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{q}^\perp$ .

<sup>2</sup>Real parabolic subalgebras of a complex  $\mathfrak{g}$  are in fact complex: indeed, (1.1) gives a decomposition into complex subspaces.

<sup>3</sup>Thus height zero.

As a simple application of the existence of grading elements, we see that parabolic subalgebras are self-normalising:

**Lemma 1.6.** *For  $\zeta \in \mathfrak{g}$ ,  $[\zeta, \mathfrak{p}] \subset \mathfrak{p}$  if and only if  $\zeta \in \mathfrak{p}$ .*

*Proof.* We know that  $\text{ad } \xi$  is invertible on  $\mathfrak{p}^\perp$  so that  $[\xi, \mathfrak{p}^\perp] = \mathfrak{p}^\perp$  giving  $[\mathfrak{p}, \mathfrak{p}^\perp] = \mathfrak{p}^\perp$ . Now, if  $[\zeta, \mathfrak{p}] \subset \mathfrak{p}$  then  $[\zeta, \mathfrak{p}]$  is Killing orthogonal to  $\mathfrak{p}^\perp$  or, equivalently,  $\zeta \in [\mathfrak{p}, \mathfrak{p}^\perp]^\perp = \mathfrak{p}$ .  $\square$

## 2. $R$ -SPACES AND SYMMETRIC $R$ -SPACES

### 2.1. $R$ -spaces.

**Definition 2.1** ([57]). An  $R$ -space of height  $n$  is a conjugacy class of height  $n$  parabolic subalgebras.

Let  $M$  be an  $R$ -space. Thus  $M$  is a homogeneous space of  $G$  with stabilisers which are, by definition, *parabolic subgroups* of  $G$ . By Lemma 1.6, the stabiliser  $P$  of  $\mathfrak{p} \in M$  has Lie algebra  $\mathfrak{p}$ .

Now let  $K$  be a maximal compact subgroup of  $G$ . By virtue of the Iwasawa decomposition of  $G$ ,  $K$  acts transitively on  $M$  [53, Theorem 7] and so, in particular,  $M$  is compact.

Let  $\mathfrak{p}_0 \in M$  and fix a complementary subalgebra  $\mathfrak{p}_\infty$ . Set

$$\Omega_{\mathfrak{p}_\infty} = \{\mathfrak{p} \in M : \mathfrak{p} \text{ is complementary to } \mathfrak{p}_\infty\} \subset M.$$

Then  $\Omega_{\mathfrak{p}_\infty}$  is an open, dense neighbourhood of  $\mathfrak{p}_0$  in  $M$ : it is the “big cell” in the cell decomposition of  $M$  induced by the Bruhat decomposition of  $G$  [53, Theorem 8]. Moreover, from Lemma 1.4, we see that the map

$$(2.1) \quad \zeta \mapsto \exp \zeta \mathfrak{p}_0 : \mathfrak{p}_\infty^\perp \rightarrow \Omega_{\mathfrak{p}_\infty}$$

is a diffeomorphism. We call this map *inverse-stereoprojection* with respect to  $(\mathfrak{p}_0, \mathfrak{p}_\infty)$ .

If  $\mathfrak{g}$  is a complex Lie algebra, the  $R$ -spaces are the well-known generalised flag manifolds: these are homogeneous complex (in fact, projective, algebraic) manifolds with a  $K$ -invariant Kähler structure. The remaining  $R$ -spaces are real forms of generalised flag manifolds. Indeed, let  $\mathfrak{g}$  be non-complex and  $M$  an  $R$ -space for  $G$ . For  $\mathfrak{p} \in M$ ,  $\mathfrak{p}^\mathbb{C}$  is a parabolic subalgebra of  $\mathfrak{g}^\mathbb{C}$  of the same height as  $\mathfrak{p}$  and we denote by  $M^\mathbb{C}$  the  $G^\mathbb{C}$ -conjugacy class of the  $\mathfrak{p}^\mathbb{C}$ . The map  $\mathfrak{p} \mapsto \mathfrak{p}^\mathbb{C}$  embeds  $M$  in  $M^\mathbb{C}$  as the fixed set of an anti-holomorphic involution of  $M^\mathbb{C}$ : indeed, the conjugation  $\sigma$  of  $\mathfrak{g}^\mathbb{C}$  across  $\mathfrak{g}$  induces such an involution  $\sigma : \mathfrak{p} \mapsto \sigma \mathfrak{p}$  on  $M^\mathbb{C}$  with  $M \subset \text{Fix}(\sigma)$  and equality in this last inclusion follows from Matsumoto’s classification result [46, Théorème 3.1] since there is at most one real conjugacy class of parabolic subalgebras in any complex one (see [53, Theorem 10] for an alternative argument).

**2.2. Symmetric  $R$ -spaces: definition and examples.** The following definition is due to Takeuchi [53].

**Definition 2.2.** A *symmetric  $R$ -space* is an  $R$ -space of height 1.

Symmetric  $R$ -spaces are distinguished by the fact that they are Riemannian symmetric spaces for the maximal compact subgroup  $K$ . Indeed, with  $\theta$  the corresponding Cartan involution of  $\mathfrak{g}$ , the involution at  $\mathfrak{p} \in M$  is given by  $\tau_{\mathfrak{p}} = \exp(i\pi\xi) \in \text{Aut}(\mathfrak{g})$  where  $\xi$  is the grading element for the complementary pair  $(\mathfrak{p}, \theta\mathfrak{p})$ .

Thus symmetric  $R$ -spaces are compact Riemannian symmetric spaces admitting a Lie group of diffeomorphisms  $G$  properly containing the isometry group. It is a celebrated theorem of Nagano [48, Theorem 3.1] that, up to covers, these are essentially the only compact Riemannian symmetric spaces admitting such a group.

However, in what follows, we will be exclusively interested in the  $G$ -invariant geometry of symmetric  $R$ -spaces and shall relegate their Riemannian structure (which requires the additional choice of a maximal compact subgroup  $K$ ) firmly to the background (see, however, §6.2.1 and §6.3.2).

For  $\mathfrak{g}$  complex, the symmetric  $R$ -spaces  $M$  are precisely the Hermitian symmetric spaces of compact type. Here  $G$  is the group of biholomorphisms. As we saw in §2.1, any symmetric  $R$ -space for non-complex  $\mathfrak{g}$  is a real form of such a Hermitian symmetric space.

Let us now illustrate some of the above theory with some concrete examples:

2.2.1. *Example: the conformal  $n$ -sphere.* Contemplate the Lorentzian vector space  $\mathbb{R}^{n+1,1}$ ; an  $(n+2)$ -dimensional vector space with an inner product  $(\cdot, \cdot)$  of signature  $(n+1, 1)$ <sup>4</sup>. We distinguish the light-cone  $\mathcal{L} = \{v \in \mathbb{R}_\times^{n+1,1} : (v, v) = 0\}$  and consider its projectivisation  $\mathbb{P}(\mathcal{L}) \subset \mathbb{P}(\mathbb{R}^{n+1,1})$  on which  $G = \text{Ad SO}(n+1, 1) \cong \text{SO}_0(n+1, 1)$  acts transitively.

There is a  $G$ -invariant conformal structure on  $\mathbb{P}(\mathcal{L})$ : view a section  $s$  of the principal bundle  $\mathcal{L} \rightarrow \mathbb{P}(\mathcal{L})$  as a map into  $\mathbb{R}^{n+1,1}$  and set  $g_s = s^*(\cdot, \cdot)$ . Thus

$$g_s(X, Y) = (d_X s, d_Y s).$$

It is not difficult to see that  $g_s$  is positive definite and that, for any  $u : \mathbb{P}(\mathcal{L}) \rightarrow \mathbb{R}$ ,  $g_{e^u s} = e^{2u} g_s$  so that we have a well-defined conformal structure.

Let us identify the infinitesimal stabiliser  $\text{stab}(\Lambda) \leq \mathfrak{g} = \mathfrak{so}(n+1, 1)$  of a null line  $\Lambda \in \mathbb{P}(\mathcal{L})$ . First  $\Lambda$  defines a flag  $\Lambda \subset \Lambda^\perp \subset \mathbb{R}^{n+1,1}$  and then, under the canonical isomorphism

$$(2.2) \quad \wedge^2 \mathbb{R}^{n+1,1} \cong \mathfrak{so}(n+1, 1) : u \wedge v \mapsto (u, \cdot)v - (v, \cdot)u,$$

we readily see that

$$\text{stab}(\Lambda) = \Lambda \wedge \mathbb{R}^{n+1,1} + \wedge^2 \Lambda^\perp$$

The Killing polar of  $\text{stab}(\Lambda)$  is then the abelian subalgebra

$$\text{stab}(\Lambda)^\perp = \Lambda \wedge \Lambda^\perp.$$

Thus  $\text{stab}(\Lambda)$  is a parabolic subalgebra of  $\mathfrak{g}$  and the map  $\Lambda \mapsto \text{stab}(\Lambda)$  is a  $G$ -isomorphism from  $\mathbb{P}(\mathcal{L})$  to the conjugacy class of  $\text{stab}(\Lambda)$  which is a symmetric  $R$ -space.

The general theory now tells us that  $\mathbb{P}(\mathcal{L})$  is a Riemannian symmetric  $K$ -space for  $K \cong \text{SO}(n+1)$  a maximal compact subgroup of  $G$ . In fact,  $\mathbb{P}(\mathcal{L}) \cong S^n$  as  $K$ -spaces. Indeed, a choice of maximal compact amounts to a choice of unit time-like vector  $t_0 \in \mathbb{R}^{n+1,1}$ . Then  $K$  is the identity component of  $\text{Stab}(t_0) \leq \text{SO}(n+1, 1)$  and we identify  $\mathbb{P}(\mathcal{L})$  with the unit sphere  $S^n \subset \langle t_0 \rangle^\perp$  via:

$$x \mapsto \langle x + t_0 \rangle : S^n \rightarrow \mathbb{P}(\mathcal{L}).$$

For  $s$  the unique section of  $\mathcal{L}$  with  $(s, t_0) \equiv -1$ , this map is an isometry  $(S^n, g_{\text{can}}) \rightarrow (\mathbb{P}(\mathcal{L}), g_s)$  so that we conclude that  $\mathbb{P}(\mathcal{L})$  is conformally equivalent to  $S^n$ . Moreover, this equivalence identifies  $G$  with the group of orientation-preserving conformal diffeomorphisms of  $S^n$ .

This model of the conformal  $n$ -sphere is due to Darboux [28, Chapitre VI].

For  $\Lambda \in \mathbb{P}(\mathcal{L})$ , the parabolic subalgebras of  $\mathfrak{g}$  complementary to  $\text{stab}(\Lambda)$  are precisely the stabilisers  $\text{stab}(\hat{\Lambda})$  for  $\hat{\Lambda} \in \mathbb{P}(\mathcal{L}) \setminus \{\Lambda\}$ . Indeed, given such a  $\hat{\Lambda}$ , set  $W = (\Lambda \oplus \hat{\Lambda})^\perp$  to get a decomposition

$$\mathbb{R}^{n+1,1} = \Lambda \oplus W \oplus \hat{\Lambda}$$

with  $\Lambda \oplus W = \Lambda^\perp$ ,  $\hat{\Lambda}^\perp = W \oplus \hat{\Lambda}$ . Thus  $\text{stab}(\Lambda)^\perp = \Lambda \wedge W$  and  $\text{stab} \hat{\Lambda}^\perp = \hat{\Lambda} \wedge W$  and these have zero intersection whence  $\text{stab}(\Lambda)$  and  $\text{stab}(\hat{\Lambda})$  are complementary (see Remark 1.5). The corresponding grading element has eigenvalues  $-1, 0, 1$  on  $\Lambda, W, \hat{\Lambda}$  respectively. Conversely, if  $\xi$  is the grading element of a complementary pair  $(\text{stab}(\Lambda), \mathfrak{q})$ , then  $\mathfrak{q} = \text{stab}(\hat{\Lambda})$  where  $\hat{\Lambda}$  is the  $+1$ -eigenspace for the action of  $\xi$  on  $\mathbb{R}^{n+1,1}$ .

In particular, we note that the parabolic subalgebras complementary to some  $\text{stab}(\Lambda)$  lie in a single conjugacy class which coincides with that of  $\text{stab}(\Lambda)$ .

The inverse-stereoprojection introduced in §2.1 coincides with the classical notion: choose  $\Lambda_0 \neq \Lambda_\infty \in \mathbb{P}(\mathcal{L})$  with infinitesimal stabilisers  $\mathfrak{p}_0, \mathfrak{p}_\infty$  and set  $\mathbb{R}^n = (\Lambda_0 \oplus \Lambda_\infty)^\perp$ . Then  $\Omega_{\mathfrak{p}_\infty} = \mathbb{P}(\mathcal{L}) \setminus \{\Lambda_\infty\}$  and (2.1) reads:

$$\zeta \mapsto \exp(\zeta)\Lambda_0 : \Lambda_\infty \wedge \mathbb{R}^n \rightarrow \mathbb{P}(\mathcal{L}) \setminus \{\Lambda_\infty\}.$$

If we now choose  $v_0 \in \Lambda_0$ ,  $v_\infty \in \Lambda_\infty$  with  $(v_0, v_\infty) = -1$ , we can identify  $\Lambda_\infty \wedge \mathbb{R}^n$  with  $\mathbb{R}^n$  via  $x \wedge v_\infty \mapsto x = (x \wedge v_\infty)v_0$  and then the diffeomorphism is

$$x \mapsto \langle \exp(x \wedge v_\infty)v_0 \rangle = \langle v_0 + x + \frac{1}{2}(x, x)v_\infty \rangle$$

which last is the expression for classical inverse-stereoprojection that can be found in [12].

One can perform a similar analysis of the projective lightcone  $\mathbb{P}(\mathcal{L}^{p+1, q+1})$  of  $\mathbb{R}^{p+1, q+1}$  to get a symmetric  $R$ -space diffeomorphic to  $(S^q \times S^q)/\mathbb{Z}_2$  which is the conformal compactification via (2.1) of  $\mathbb{R}^{p, q}$ .

<sup>4</sup>Thus  $n+1$  positive directions and 1 negative direction.

2.2.2. *Example: Grassmannians.* Contemplate the Grassmannian  $G_k(\mathbb{C}^n)$  of  $k$ -dimensional linear subspaces of  $\mathbb{C}^n$  on which  $G = \mathrm{PSL}(n, \mathbb{C})$  acts transitively (thus, for  $k = 1$ , we are studying the projective geometry of  $\mathbb{C}\mathbb{P}^n$ ). The infinitesimal stabiliser of  $W \in G_k(\mathbb{C}^n)$  has Killing polar  $\mathrm{stab}(W)^\perp = \mathrm{hom}(\mathbb{C}^n/W, W)$  which is an abelian subalgebra of  $\mathfrak{sl}(n, \mathbb{C})$  so that the conjugacy class  $M$  of these stabilisers is a symmetric  $R$ -space. The map  $W \mapsto \mathrm{stab}(W) : G_k(\mathbb{C}^n) \rightarrow M$  is now an isomorphism of  $G$ -spaces.

For  $W \in G_k(\mathbb{C}^n)$ , the parabolic subalgebras complementary to  $\mathrm{stab}(W)$  are all of the form  $\mathrm{stab}(\hat{W})$  for  $\hat{W} \in G_{n-k}(\mathbb{C}^n)$  with  $W \oplus \hat{W} = \mathbb{C}^n$ . The grading element of the pair  $(W, \hat{W})$  has eigenvalues  $(k - n)/n, k/n$  on  $W, \hat{W}$  respectively. Again we see that the set of parabolic subalgebras complementary to some  $\mathrm{stab}(W)$ ,  $W \in G_k(\mathbb{C}^n)$ , lies in a single conjugacy class but, in contrast to the case of spheres, this conjugacy class does not coincide with that of the  $\mathrm{stab}(W)$  unless  $2k = n$ .

For Grassmannians, stereoprojection coincides with the usual affine chart: given complementary  $W_0 \in G_k(\mathbb{C}^n)$  and  $W_\infty \in G_{n-k}(\mathbb{C}^n)$  with infinitesimal stabilisers  $\mathfrak{p}_0, \mathfrak{p}_\infty$ , we identify  $\Omega_{\mathfrak{p}_\infty}$  with  $\{W \in G_k(\mathbb{C}^n) : W \cap W_\infty = \{0\}\}$ ,  $\mathbb{C}^n/W_\infty$  with  $W_0$  and thus  $\mathfrak{p}_\infty^\perp$  with  $\mathrm{hom}(W_0, W_\infty)$ . Now (2.1) reads

$$T \mapsto (1 + T)W_0 : \mathrm{hom}(W_0, W_\infty) \cong \Omega_{\mathfrak{p}_\infty}.$$

A similar analysis is available for the real Grassmannians  $G_k(\mathbb{R}^n)$  with  $G = \mathrm{PSL}(n, \mathbb{R})$  and the quaternionic Grassmannians  $G_k(\mathbb{H}^n)$  with  $G = \mathrm{PSU}^*(2n)$ . These symmetric  $R$ -spaces are real forms of the complex Grassmannians: indeed, they may be viewed as the fixed set of the anti-holomorphic involution  $W \mapsto jW$  on  $G_k(\mathbb{C}^n)$  induced by a real or quaternionic structure  $j$  on  $\mathbb{C}^n$  (thus  $j$  is an antilinear endomorphism of  $\mathbb{C}^n$  with  $j^2 = \pm 1$ ).

There is one other real form of  $G_n(\mathbb{C}^{2n})$ : this is the set  $G_n^\perp(\mathbb{C}^{n,n})$  of maximal isotropic subspaces for a Hermitian inner product of signature  $(n, n)$  and so is the fixed set of the anti-holomorphic involution  $W \mapsto W^\perp$ .

2.2.3. *Example: isotropic Grassmannians.* Equip  $\mathbb{C}^{2n}$  with a nondegenerate complex bilinear inner product  $q$  and contemplate the set of maximal isotropic subspaces of  $\mathbb{C}^{2n}$ . The group  $\mathrm{Ad} \mathrm{SO}(2n, \mathbb{C})$  acts on these and there are two orbits: for  $W \subset \mathbb{C}^{2n}$  maximal isotropic,  $\bigwedge^n W$  lies in an eigenspace of the Hodge  $*$ -operator and so we have two orbits  $J^+(\mathbb{C}^{2n})$  and  $J^-(\mathbb{C}^{2n})$  according to the sign of the eigenvalue.

Under the isomorphism (2.2), the infinitesimal stabiliser of  $W \in J^\pm(\mathbb{C}^{2n})$  is  $\mathbb{C}^{2n} \wedge W$  with Killing polar  $\bigwedge^2 W$ , an abelian subalgebra of  $\mathfrak{so}(2n, \mathbb{C})$ . Thus the conjugacy class  $M^\pm$  of  $\mathrm{stab}(W)$  is a symmetric  $R$ -space and  $W \mapsto \mathrm{stab}(W) : J^\pm(\mathbb{C}^{2n}) \rightarrow M^\pm$  is an isomorphism of  $G$ -spaces.

The parabolic subalgebras complementary to  $\mathrm{stab}(W)$  are those of the form  $\mathrm{stab}(\hat{W})$  where  $\hat{W} \in J^\pm(\mathbb{C}^{2n})$  and  $\mathbb{C}^{2n} = W \oplus \hat{W}$ . Now, for  $W^\pm \in J^\pm(\mathbb{C}^{2n})$ , one uses the characterisation of the orbits via the  $*$ -operator to see that  $\dim W^+ \cap W^-$  has the opposite parity to  $n$ . Thus complementary  $W, \hat{W}$  lie in the same orbit if and only if  $n$  is even.

The real forms of  $J^\pm(\mathbb{C}^{2n})$  arise from real or quaternionic structures  $j$  on  $\mathbb{C}^{2n}$  compatible with  $q$ :  $j^*q = \bar{q}$ . This leads to two orbits  $J^\pm(\mathbb{R}^{n,n})$  of real maximal isotropic subspaces of  $\mathbb{R}^{n,n}$  and one<sup>5</sup> quaternionic orbit  $J(\mathbb{H}^{2m})$  when  $n = 2m$ .

The choice of a maximal compact subgroup of  $\mathrm{SO}(n, n)$  amounts to a decomposition  $\mathbb{R}^{n,n} = \mathbb{R}^{n,0} \oplus \mathbb{R}^{0,n}$  into orthogonal definite spaces. Now any  $W \in J^\pm(\mathbb{R}^{n,n})$  is precisely the graph of an anti-isometry  $\mathbb{R}^{n,0} \rightarrow \mathbb{R}^{0,n}$  and  $\mathrm{SO}(n) = \mathrm{SO}(n, 0)$  acts freely and transitively on these by precomposition. In this way, both  $J^\pm(\mathbb{R}^{n,n})$  are identified as  $K$ -spaces with the group manifold  $\mathrm{SO}(n)$ . It is an amusing exercise to see that, in this setting, inverse-stereoprojection amounts to the Cayley parametrisation of  $\mathrm{SO}(n)$ .

A similar analysis can also be made for  $\mathbb{C}^{2n}$  equipped with a complex symplectic form and leads to the complex Lagrangian Grassmannian  $\mathrm{Lag}(\mathbb{C}^{2n})$  with real forms  $\mathrm{Lag}(\mathbb{R}^{2n})$  and  $\mathrm{Lag}(\mathbb{H}^{2m})$ , the latter being isomorphic as a  $K$ -space to the group manifold  $\mathrm{Sp}(m)$ .

<sup>5</sup>Only one of  $J^\pm(\mathbb{C}^{4m})$  can contain quaternionic elements since the intersection of any two such has even dimension.



**2.3. Duality of  $R$ -spaces.** Let  $M$  be an  $R$ -space. Let  $M^*$  be the set of parabolic subalgebras  $\mathfrak{q}$  of  $\mathfrak{g}$  for which  $\mathfrak{q}$  is complementary to some  $\mathfrak{p} \in M$ . In the examples we have just inspected,  $M^*$  is a single conjugacy class which may or may not coincide with  $M$ . This is an entirely general phenomenon and leads to a notion of duality among  $R$ -spaces which will be important to us.

**Proposition 2.3.**  $M^*$  comprises a single conjugacy class and so is an  $R$ -space of the same height as  $M$ .

*Proof.* Let  $\mathfrak{q}, \mathfrak{q}' \in M^*$  with  $\mathfrak{q}$  complementary to  $\mathfrak{p} \in M$  and  $\mathfrak{q}'$  complementary to  $g\mathfrak{p}$ . We must show that  $\mathfrak{q}, \mathfrak{q}'$  are conjugate. Now  $g^{-1}\mathfrak{q}'$  is complementary to  $\mathfrak{p}$  so by Lemma 1.4 there is a unique  $n \in \exp \mathfrak{p}^\perp$  with  $ng^{-1}\mathfrak{q}' = \mathfrak{q}$  and we are done.

$M$  and  $M^*$  have the same height since the heights of complementary parabolic subalgebras coincide.  $\square$

Clearly  $(M^*)^* = M$  so we make the following:

**Definition 2.4.**  $M^*$  is the *dual* of  $M$ .

It can happen that  $M = M^*$  in which case we say that  $M$  is *self-dual*. Otherwise,  $M$  is said to be *non-self-dual*. For example, we have seen that the conformal  $n$ -sphere is self-dual but the Grassmannians  $G_k(\mathbb{R}^n)$ , for  $n \neq 2k$ , are non-self-dual.

Even in the non-self-dual case,  $M$  and  $M^*$  are non-canonically diffeomorphic: for  $\theta$  the Cartan involution corresponding to a maximal compact  $K < G$ , the map  $\mathfrak{p} \mapsto \theta\mathfrak{p} : M \rightarrow M^*$  is a diffeomorphism, in fact an isomorphism of  $K$ -spaces.

In §1.1, we described how  $R$ -spaces are parametrised by certain subsets of nodes of a Dynkin or Satake diagram. One can detect the duality relation from a certain automorphism of the diagram. For this, first note that  $\mathfrak{p}, \mathfrak{q} \leq \mathfrak{g}$  are complementary if and only if  $\mathfrak{p}^\mathbb{C}, \mathfrak{q}^\mathbb{C} \leq \mathfrak{g}^\mathbb{C}$  are, so that  $M$  is self-dual if and only if its complexification  $M^\mathbb{C} = \{g\mathfrak{p}^\mathbb{C} : \mathfrak{p} \in M, g \in G^\mathbb{C}\}$  is self-dual (recall  $\mathfrak{p}$  is  $G$ -conjugate to  $\mathfrak{q}$  if and only if  $\mathfrak{p}^\mathbb{C}$  is  $G^\mathbb{C}$ -conjugate to  $\mathfrak{q}^\mathbb{C}$ ). This reduces the question to the case when  $\mathfrak{g}$  is complex. For  $\mathfrak{p} \leq \mathfrak{g}$  parabolic, fix a Borel subalgebra  $\mathfrak{b} \leq \mathfrak{p}$  and a Cartan subalgebra  $\mathfrak{h} < \mathfrak{b}$ . The Borel subalgebra determines a Weyl chamber  $C$  and there is a unique element  $w_0$  of the Weyl group such that  $w_0C = -C$  [10, V, 1.6]. Define  $\sigma = -w_0$  so that  $\sigma$  preserves  $C$  and therefore permutes the corresponding simple roots giving an automorphism of the Dynkin diagram. If  $\mathfrak{p} \in M$  corresponds to a set of simple roots  $I$ , then the parabolic subalgebra corresponding to  $\sigma I$  lies in  $M^*$ : it is conjugate via  $w_0$  to the opposite parabolic subalgebra to  $\mathfrak{p}$  which comprises  $\mathfrak{h}$  and the root spaces  $\mathfrak{g}_{-\alpha}$  for  $\mathfrak{g}_\alpha \subset \mathfrak{p}$  and this last is clearly complementary to  $\mathfrak{p}$ .

Thus our duality relation is implemented on subsets of the Dynkin diagram by  $\sigma$ . For  $\mathfrak{g}$  simple,  $\sigma$  is readily determined: it is the identity if and only if  $-1$  lies in the Weyl group (since the Weyl group acts simply transitively on Weyl chambers) and this is the case unless  $\mathfrak{g}$  is one of  $\mathfrak{sl}(n)$ ,  $n > 1$ ;  $\mathfrak{so}(2n)$ ,  $n$  odd; or  $\mathfrak{e}_6$  [10, VI, 4.5–13]. In these cases,  $\sigma$  is the usual involution of the Dynkin diagram.

It is now a simple matter to determine which  $R$ -spaces are self-dual: we simply ask that  $\sigma$  preserve the corresponding set of crossed nodes. The results for symmetric  $R$ -spaces are tabulated in Table 1 on page 33.

**2.4. The space of complementary pairs.** Let  $M$  be a symmetric  $R$ -space. While the Riemannian symmetric structure of  $M$  will not concern us, there is a pseudo-Riemannian symmetric  $G$ -space associated to  $M$  which will play an important role in what follows. This is the *space of complementary pairs*  $Z \subset M \times M^*$ :

$$Z = \{(\mathfrak{p}, \mathfrak{q}) \in M \times M^* : \mathfrak{p} \text{ and } \mathfrak{q} \text{ are complementary}\}.$$

We have:

**Proposition 2.5.**  $G$  acts transitively on  $Z$  so that  $Z$  is a pseudo-Riemannian symmetric  $G$ -space.

*Proof.* Let  $(\mathfrak{p}, \mathfrak{q}), (\mathfrak{p}', \mathfrak{q}') \in Z$ . There is  $h \in G$  with  $h\mathfrak{p} = \mathfrak{p}'$  and then  $h^{-1}\mathfrak{q}'$  is complementary to  $\mathfrak{p}$ . Lemma 1.4 now supplies  $n \in \exp \mathfrak{p}^\perp \leq \text{Stab}(\mathfrak{p})$  with  $n\mathfrak{q} = h^{-1}\mathfrak{q}'$  and then with  $g = hn$  we immediately get  $g\mathfrak{p} = \mathfrak{p}'$  and  $g\mathfrak{q} = \mathfrak{q}'$ . Thus  $G$  acts transitively on  $Z$ .

The stabiliser of  $(\mathfrak{p}, \mathfrak{q}) \in Z$  is clearly  $\text{Stab}(\mathfrak{p}) \cap \text{Stab}(\mathfrak{q})$  with Lie algebra  $\mathfrak{p} \cap \mathfrak{q}$ . This last is the centraliser of the grading element  $\xi_{\mathfrak{p}}^{\mathfrak{q}}$  of the pair so that  $\text{Stab}(\mathfrak{p}) \cap \text{Stab}(\mathfrak{q})$  is open in the fixed set of the involution  $\tau_{\mathfrak{p}, \mathfrak{q}} = \exp(i\pi\xi) \in \text{Aut}(\mathfrak{g})$ . Thus  $Z$  is a symmetric  $G$ -space with invariant pseudo-Riemannian metric induced by the Killing form of  $\mathfrak{g}$ .  $\square$

Kaneyuki [41, Theorem 3.1] shows that  $Z$  is a dense open subset of  $M \times M^*$ . Moreover  $Z$  is a *parahermitian symmetric space* in the sense of Kaneyuki–Kozai [42] who show that essentially all such spaces for semisimple  $G$  arise this way.

**2.5. Homogeneous geometry and the solder form.** The geometry of a homogeneous  $G$ -space is tied to the Lie theory of  $G$  via the *solder form* which is defined as follows.

Let  $N$  be a homogeneous  $G$ -space. Let  $H_x \leq G$  be the stabiliser of  $x \in N$  with Lie algebra  $\mathfrak{h}_x$ . Since  $G$  acts transitively, we have a surjection  $\mathfrak{g} \rightarrow T_x N$  given by

$$\zeta \mapsto \left. \frac{d}{dt} \right|_{t=0} \exp(t\zeta) x$$

with kernel  $\mathfrak{h}_x$  and so an isomorphism  $\mathfrak{g}/\mathfrak{h}_x \cong T_x N$  whose inverse is the solder form  $\beta_x^N : T_x N \rightarrow \mathfrak{g}/\mathfrak{h}_x$ . Globally, the  $\mathfrak{h}_x$  are the fibres of a vector subbundle  $\mathfrak{h}$  of the trivial bundle<sup>6</sup>  $\underline{\mathfrak{g}}_N = N \times \mathfrak{g}$  and the solder form is then a bundle isomorphism  $\beta^N : TN \cong \underline{\mathfrak{g}}_N/\mathfrak{h}$ . Dually, the Killing form now gives an isomorphism  $T^*N \cong \mathfrak{h}^\perp \leq \underline{\mathfrak{g}}$ .

In particular, applying this analysis to an  $R$ -space  $M$ , the bundle of infinitesimal stabilisers is just the tautological bundle of parabolic subalgebras

$$\mathfrak{h}_{\mathfrak{p}} = \mathfrak{p}$$

and we have a canonical identification of  $T_{\mathfrak{p}}^*N$  with  $\mathfrak{p}^\perp$ , for each  $\mathfrak{p} \in M$ . It follows that the fibration of the symmetric space  $Z \subset M \times M^*$  over  $M$  is non-canonically isomorphic to the cotangent bundle  $T^*M \rightarrow M$  [42, Theorem 4.3]: indeed, the fibre  $Z_{\mathfrak{p}}$  at  $\mathfrak{p}$  is an affine space modelled on  $\mathfrak{p}^\perp$  (see Lemma 1.4 and the remarks following it) so that any section of  $Z \rightarrow M$  allows us to identify  $Z_{\mathfrak{p}}$  with  $\mathfrak{p}^\perp$  and so with  $T_{\mathfrak{p}}^*M$ . In particular, a Cartan involution  $\theta$  of  $\mathfrak{g}$  with fixed set  $K$  gives such a section  $\mathfrak{p} \mapsto (\mathfrak{p}, \theta\mathfrak{p})$  and  $Z \cong T^*M$  as  $K$ -spaces. Moreover, this section gives a totally geodesic embedding of  $M$ , *qua* Riemannian symmetric  $K$ -space, in  $Z$ . We shall return to this last point in §6.3.2.

The following lemma provides a helpful characterisation of the solder form:

**Lemma 2.6.** *Let  $s \in \Gamma\mathfrak{h}$  then*

$$(2.3) \quad ds \equiv [\beta^N, s] \pmod{\mathfrak{h}}.$$

*Moreover, if the fibres of  $\mathfrak{h}$  are self-normalising,  $\beta^N$  is the unique  $\underline{\mathfrak{g}}_N/\mathfrak{h}$ -valued 1-form with this property.*

*Proof.* Let  $X \in T_x N$ ,  $s \in \Gamma\mathfrak{h}$  and  $\zeta \in \mathfrak{g}$  a representative of  $\beta_X^N \in \mathfrak{g}/\mathfrak{h}_x$ . For any  $g \in G$ ,  $\mathfrak{h}_{g \cdot x} = \text{Ad}(g)\mathfrak{h}_x$  so that  $s(\exp(t\zeta)x) = \text{Ad}\exp(t\zeta)\sigma(t)$  for some curve  $\sigma$  in  $\mathfrak{h}_x$  with  $\sigma(0) = s(x)$ . Differentiating this at  $t = 0$  yields

$$d_X s = [\zeta, s(x)] + \sigma'(0) \equiv [\beta_X^N, s(x)] \pmod{\mathfrak{h}}.$$

The uniqueness assertion is clear.  $\square$

If  $N$  is reductive<sup>7</sup> so that each  $\mathfrak{h}_x$  has an  $\text{Ad } H_x$ -invariant complement  $\mathfrak{m}_x$ , we have a decomposition into  $H$ -bundles

$$\underline{\mathfrak{g}} = \mathfrak{h} \oplus \mathfrak{m}$$

which allows us to identify  $\underline{\mathfrak{g}}/\mathfrak{h}$  with  $\mathfrak{m}$  and so view  $\beta^N$  as an  $\mathfrak{m}$ -valued 1-form. With this understood, the corresponding reduction of  $d$  to an  $H$ -connection now reads

$$d = \mathcal{D} + \beta^N$$

(see, for example, [18, Chapter 1]).

<sup>6</sup>We shall consistently denote the trivial bundle over a manifold  $N$  with fibre  $V$  by  $\underline{V}_N$  or even just  $\underline{V}$  if the context is clear.

<sup>7</sup> $R$ -spaces are emphatically *not* reductive but the symmetric spaces  $Z$  of complementary pairs are.

## 3. ISOTHERMIC MAPS

We now come to the main object of our discussion: the isothermic maps to and submanifolds of a symmetric  $R$ -space and their transformation theory.

Henceforth, we fix a symmetric  $R$ -space  $M$ , a conjugacy class of parabolic subalgebras of  $\mathfrak{g}$ , with dual space  $M^*$  and space of complementary pairs  $Z \subset M \times M^*$ .

**3.1. Definition and  $G$ -invariance.** Let  $f : \Sigma \rightarrow M$  be a map of a manifold  $\Sigma$ . We identify  $f$  with a subbundle, also called  $f$ , of parabolic subalgebras of the trivial bundle  $\underline{\mathfrak{g}} = \Sigma \times \mathfrak{g}$  via

$$f_x = f(x),$$

for  $x \in \Sigma$ . Any such subbundle over connected  $\Sigma$  arises in this way from a map to an  $R$ -space since the conjugacy classes are the components of the set of all parabolic subalgebras inside the Grassmannian of  $\mathfrak{g}$ .

**Definition 3.1.** A map  $f : \Sigma \rightarrow M$  is *isothermic* if there is a non-zero closed 1-form  $\eta \in \Omega_\Sigma^1(f^\perp)$ .

If  $f$  immerses,  $(f, \eta)$  is called an *isothermic submanifold* of  $M$ .

Thus  $\eta$  is a 1-form taking values in the bundle  $f^\perp$  of nilradicals which is closed when viewed as a  $\mathfrak{g}$ -valued form. Alternatively, the solder isomorphism identifies  $f^\perp$  with  $f^{-1}T^*M$  and we may therefore view  $\eta$  as an  $f^{-1}T^*M$ -valued 1-form: we shall exploit this in §6.1.

*Remark 3.2.* For some symmetric  $R$ -spaces, the 1-form  $\eta$  is uniquely determined up to (constant) scale by the isothermic map  $f$  when  $f$  immerses. However, for other symmetric  $R$ -spaces, this is far from the case as we shall see in §6.3.1.

We note that the isothermic property is manifestly  $G$ -invariant: for isothermic  $(f, \eta)$  and  $g \in G$ ,  $(gf, g\eta)$  is also isothermic.

**3.2. Stereoprojection and the Christoffel transform.** We saw in §2.1 that a fixed complementary pair  $(\mathfrak{p}_0, \mathfrak{p}_\infty)$  gives us distinguished charts  $\mathfrak{p}_0 \in \Omega_{\mathfrak{p}_\infty} \subset M$ ,  $\mathfrak{p}_\infty \in \Omega_{\mathfrak{p}_0} \subset M^*$  and stereoprojections  $\Omega_{\mathfrak{p}_\infty} \cong \mathfrak{p}_\infty^\perp$ ,  $\Omega_{\mathfrak{p}_0} \cong \mathfrak{p}_0^\perp$ .

We now show that this data also induces a duality between isothermic maps. Indeed, suppose that  $f : \Sigma \rightarrow M$  has image in  $\Omega_{\mathfrak{p}_\infty}$  and let  $F : \Sigma \rightarrow \mathfrak{p}_\infty^\perp$  be its stereoprojection. Thus

$$f = \exp(F) \mathfrak{p}_0.$$

If  $\eta \in \Omega_\Sigma^1(f^\perp)$ , we have a 1-form  $\omega \in \mathfrak{p}_0^\perp$  such that  $\eta = \exp(F) \omega$ . Since  $F$  takes values in the fixed abelian subalgebra  $\mathfrak{p}_\infty^\perp$ , we readily compute the exterior derivative of  $\eta$ :

$$d\eta = \exp(F) (d\omega + [dF \wedge \omega]).$$

The two summands on the right lie in  $\mathfrak{p}_0^\perp$  and  $\mathfrak{p}_0 \cap \mathfrak{p}_\infty$  respectively and so must vanish separately if and only if  $d\eta = 0$ . We therefore conclude:

**Proposition 3.3.** *A map  $f : \Sigma \rightarrow M$  with stereoprojection  $F : \Sigma \rightarrow \mathfrak{p}_\infty^\perp$  with respect to  $(\mathfrak{p}_0, \mathfrak{p}_\infty)$  is isothermic if and only if there is a 1-form  $\omega \in \Omega_\Sigma^1(\mathfrak{p}_0^\perp)$  such that*

- (1)  $\omega$  is closed;
- (2)  $[dF \wedge \omega] = 0$ .

In this situation, we can (locally) integrate to find  $F^c : \Sigma \rightarrow \mathfrak{p}_0^\perp$  with  $dF^c = \omega$  and thus  $[dF \wedge dF^c] = 0$ . Everything is now symmetric in  $F$  and  $F^c$  and we conclude that  $f^c = \exp(F^c) \mathfrak{p}_\infty : \Sigma \rightarrow M^*$  is isothermic with 1-form  $\eta^c = \exp(F^c) dF$ . By strict analogy with the classical situation (see §3.2.1 below), we call  $f^c$  (or  $F^c$ ) the *Christoffel transform* of  $f$  (or  $F$ ).

Of course, if we change our choice of initial complementary pair  $(\mathfrak{p}_0, \mathfrak{p}_\infty)$  then we will obtain a quite different Christoffel transform. However, these (or rather their stereoprojections) are all obtained in a very simple way from a primitive of  $\eta$ . For this, observe that

$$\eta = \exp(F) \omega = \omega + [F, \omega] + \frac{1}{2}[F, [F, \omega]]$$

with the latter three summands taking values in  $\mathfrak{p}_0^\perp$ ,  $\mathfrak{p}_0 \cap \mathfrak{p}_\infty$  and  $\mathfrak{p}_\infty^\perp$  respectively. Thus  $\omega = \pi_{\mathfrak{p}_0^\perp} \eta$  for  $\pi_{\mathfrak{p}_0^\perp}$  the projection onto  $\mathfrak{p}_0^\perp$  along  $\mathfrak{p}_\infty$ . We therefore conclude:

**Proposition 3.4.** *Let  $(f, \eta)$  be isothermic and  $\Phi : \Sigma \rightarrow \mathfrak{g}$  a primitive of  $\eta$ :  $d\Phi = \eta$ . Then the Christoffel transform of  $(f, \eta)$  with respect to  $(\mathfrak{p}_0, \mathfrak{p}_\infty)$  has stereoprojection  $F^c = \pi_{\mathfrak{p}_0^\perp} \Phi$ .*

Thus  $\Phi$  is a universal Christoffel transform for  $f$ .

3.2.1. *Example: isothermic surfaces in  $S^n$ .* Let us take  $M$  to be the conformal  $n$ -sphere as in §2.2.1 and show that, in this case, our theory recovers the classical notion of an isothermic surface.

A map  $f : \Sigma \rightarrow M = \mathbb{P}(\mathcal{L})$  is the same as a null line-subbundle  $\Lambda \subset \underline{\mathbb{R}}^{n+1,1}$  and the bundle of nilradicals is  $f^\perp = \Lambda \wedge \Lambda^\perp$ . Now  $f$  is isothermic if there is a non-zero, closed  $\mathfrak{so}(n+1,1)$ -valued 1-form  $\eta$  taking values in  $\Lambda \wedge \Lambda^\perp$ .

We stereoproject to compare with the classical definition. Thus, choose  $\Lambda_0 = \langle v_0 \rangle, \Lambda_\infty = \langle v_\infty \rangle \in \mathbb{P}(\mathcal{L})$  with  $\langle v_0, v_\infty \rangle = -1$  and, as in §2.2.1, identify both  $\text{stab}(\Lambda_0)$  and  $\text{stab}(\Lambda_\infty)$  with  $\mathbb{R}^n = \langle v_0, v_\infty \rangle^\perp$ . With both  $F, F^c$  viewed as maps  $\Sigma \rightarrow \mathbb{R}^n$ , the bracket  $[dF \wedge dF^c]$  takes values in  $\mathfrak{so}(n) \oplus \mathfrak{so}(1,1)$  and the vanishing of these two components amounts to

$$\begin{aligned} (dF_X, dF_Y^c) &= (dF_Y, dF_X^c) \\ dF_X \wedge dF_Y^c &= dF_Y \wedge dF_X^c, \end{aligned}$$

for vector fields  $X, Y$  on  $\Sigma$ . These, in turn, can be conveniently packaged as

$$(3.1) \quad dF \wedge_{Cl_n} dF^c = 0$$

where here we use Clifford multiplication to multiply coefficients so that the left hand side of (3.1) is a 2-form with values in the Clifford algebra of  $\mathbb{R}^n$ . To summarise:

**Proposition 3.5.**  *$F : \Sigma \rightarrow \mathbb{R}^n$  is the stereoprojection of an isothermic map  $\Sigma \rightarrow \mathbb{P}(\mathcal{L})$  if and only if there is (locally) a map  $F^c : \Sigma \rightarrow \mathbb{R}^n$  such that (3.1) holds.*

According to [13, §2.1], this coincides with the classical formulation of the isothermic property due to Christoffel [25], for  $n = 3$ , and Palmer [50] for  $n > 3$ .

One can show [13, Lemma 1.6] that an isothermic  $F : \Sigma \rightarrow \mathbb{R}^n$  has rank at most 2 so that isothermic submanifolds are necessarily surfaces. As we shall see in §6.2.1, similar restrictions hold for any symmetric  $R$ -space.

**3.3. Zero curvature representation and spectral deformation.** Isothermic maps comprise an integrable system with a transformation theory that exactly mirrors the classical theory of isothermic surfaces in  $S^3$ . This is all a consequence of the following simple observation: for  $f : \Sigma \rightarrow M$  and  $\eta \in \Omega_\Sigma^1(f^\perp)$ , view  $\eta$  as a gauge potential and contemplate the pencil of  $G$ -connections on  $\underline{\mathfrak{g}}$  given by

$$(3.2) \quad \nabla^t = d + t\eta,$$

for  $t \in \mathbb{R}$ . Of course,  $\nabla^0 = d$ .

We have:

**Proposition 3.6.**  *$(f, \eta)$  is isothermic if and only if  $\nabla^t$  is a flat connection for each  $t \in \mathbb{R}$ .*

*Proof.* We compute:

$$R^{\nabla^t} = R^d + t d\eta + \frac{1}{2} t^2 [\eta \wedge \eta] = t d\eta$$

since  $d$  is flat and  $\eta$  takes values in the bundle of abelian<sup>8</sup> subalgebras  $f^\perp$ . Thus  $d\eta = 0$  if and only if  $R^{\nabla^t} = 0$  for all  $t \in \mathbb{R}$ .  $\square$

As a first application, we show that isothermic maps possess a *spectral deformation*: that is, they come locally in 1-parameter families. There are two ways to see this corresponding to the active and passive viewpoints on gauge transformations but the key observation is that

$$d^{\nabla^t} \eta = d\eta + t[\eta \wedge \eta] = 0.$$

Thus  $(f, \eta)$  remains isothermic when the flat connection  $d$  on  $\underline{\mathfrak{g}}$  is replaced by the flat connection  $\nabla^t$ .

<sup>8</sup>It is here that  $M$  being a *symmetric*  $R$ -space comes to the fore.

Equivalently, we can trivialise  $\nabla^t$ : locally, we can find gauge transformations  $\Phi_t$ , unique up to left multiplication by a constant element of  $G$ , such that  $\Phi_t \cdot \nabla^t = d$  and then  $(\Phi_t f, \Phi_t \eta)$  is isothermic in the usual sense. Here, and below,  $\Phi_t \cdot \nabla^t = \Phi_t \circ \nabla^t \circ \Phi_t^{-1}$  is the usual left action of gauge transformations on connections.

We write  $\mathcal{T}_t f$  for  $\Phi_t f$  and, following Bianchi [3], say that  $\mathcal{T}_t f$  is a  $T$ -transform<sup>9</sup> of  $f$ .

The gauge transformation  $\Phi_t$  does more than intertwine  $\nabla^t$  and  $d$ : we clearly have  $\Phi_t \cdot (\nabla^t + s\eta) = d + s\Phi_t \eta$ , for all  $s \in \mathbb{R}$ . Now, if  $\Phi_s^t$  implements the transform  $\mathcal{T}_s$  of  $\mathcal{T}_t f$  so that  $\Phi_s^t \cdot (d + s\Phi_t \eta) = d$ , we have  $(\Phi_s^t \Phi_t) \cdot \nabla^{s+t} = d$  so that  $\Phi_s^t \Phi_t = \Phi_{s+t}$ . We therefore obtain an identity on  $T$ -transforms due to Hertrich-Jeromin–Musso–Nicolodi [36] for the case where  $M$  is the conformal 3-sphere:

$$(3.3) \quad \mathcal{T}_{s+t} = \mathcal{T}_s \circ \mathcal{T}_t.$$

**3.4. Darboux transforms.** Flat connections have many parallel sections locally and, in our situation, we can use the parallel sections of  $\nabla^t$  to construct new isothermic surfaces in the dual symmetric  $R$ -space.

**Definition 3.7.** Let  $(f, \eta)$  be isothermic and  $m \in \mathbb{R}^\times$ . A map  $\hat{f} : \Sigma \rightarrow M^*$  into the dual  $R$ -space is a *Darboux transform of  $f$  with parameter  $m$*  if

- (1)  $\hat{f} \subset \underline{\mathfrak{g}}$  is  $\nabla^m$ -parallel;
- (2)  $f$  and  $\hat{f}$  are pointwise complementary parabolic subalgebras.

In this case, we write  $\hat{f} = \mathcal{D}_m f$ .

*Remark 3.8.*

- (1) The condition that  $\hat{f}$  be  $\nabla^m$ -parallel means that any section  $\zeta$  of  $\hat{f}$  has  $\nabla^m \zeta \in \Omega_\Sigma^1(\hat{f})$ . Since  $\nabla^m$  is flat, such  $\hat{f}$  are determined by their value at a fixed point  $p_0 \in \Sigma$  and any complement to  $f(p_0)$  extends locally to a parallel  $\hat{f}$  (which may eventually fail to be pointwise complementary to  $f$ ).
- (2) That  $(f, \hat{f})$  be pointwise complementary amounts to demanding that  $(f, \hat{f})$  takes values in the symmetric space  $Z$  of complementary pairs. As we shall see in §5,  $\hat{f}$  is a Darboux transform if and only if  $(f, \hat{f})$  is a *curved flat* in the sense of Ferus–Pedit [30].

We are going to show that  $\hat{f}$  is isothermic and that the relationship between  $f$  and  $\hat{f}$  is reciprocal:  $f = \mathcal{D}_m \hat{f}$ . In fact, we shall show more and exhibit an explicit gauge transformation between the two pencils of flat connections. All of this will take a little preparation.

So let  $(f, \eta)$  be isothermic,  $\hat{f} = \mathcal{D}_m f$  and contemplate the bundle decomposition

$$(3.4) \quad \underline{\mathfrak{g}} = f^\perp \oplus (f \cap \hat{f}) \oplus \hat{f}^\perp.$$

This is the eigenbundle decomposition of  $\text{ad } \xi$  where  $\xi(p) = \xi_{f(p)}^{\hat{f}(p)}$  is the grading element of the complementary pair at  $p \in \Sigma$ . There is a corresponding decomposition of the flat connection  $d$ : since  $\xi$  takes values in a single conjugacy class,  $d\xi$  takes values in  $\text{Im } \text{ad } \xi = f^\perp + \hat{f}^\perp$  on which  $\text{ad } \xi$  is invertible and we may write  $d\xi = [\xi, \beta + \hat{\beta}]$  where  $\beta \in \Omega_\Sigma^1(\hat{f}^\perp)$  and  $\hat{\beta} \in \Omega_\Sigma^1(f^\perp)$ . Now define a new  $G$ -connection  $\mathcal{D}$  by

$$(3.5) \quad d = \mathcal{D} - \beta - \hat{\beta}$$

and note that  $\mathcal{D}\xi = 0$  so that each summand in (3.4) is  $\mathcal{D}$ -parallel.

We have that  $\hat{f}$  is  $\nabla^m$ -parallel. Since  $\hat{f}$  is also  $\mathcal{D}$ -parallel and preserved by  $\text{ad } \beta$ , we conclude that it is also preserved by  $\text{ad}(m\eta - \hat{\beta})$ . Since  $\hat{f}$  is self-normalising, this gives that  $m\eta - \hat{\beta}$  takes values in  $\hat{f} \cap f^\perp = \{0\}$  so that  $\hat{\beta} = m\eta$ .

The symmetry of the situation now suggests that if we define  $\hat{\eta} = \frac{1}{m}\beta \in \Omega_\Sigma^1(\hat{f}^\perp)$  then  $(\hat{f}, \hat{\eta})$  should be isothermic. This will be the case if and only if the connections  $\hat{\nabla}^t = d + t\hat{\eta}$  are flat for all  $t \in \mathbb{R}$  and we will prove this by writing down an explicit gauge transformation intertwining  $\nabla^t$  and  $\hat{\nabla}^t$ .

<sup>9</sup>In the case of isothermic surfaces in  $S^3$ , it will follow from the results of §5.2 that our notion of  $T$ -transform coincides with the classical one due to Calapso [19] and Bianchi [3] since both are given by the curved flat spectral deformation of Darboux pairs [16].

For this, we introduce a class of elements of  $\text{Aut}(\mathfrak{g})$  that will appear frequently in what follows. For  $(\mathfrak{p}, \mathfrak{q}) \in M \times M^*$  complementary and  $s \in \mathbb{R}^\times$ , define  $\Gamma_{\mathfrak{p}}^{\mathfrak{q}}(s)$  by

$$(3.6) \quad \Gamma_{\mathfrak{p}}^{\mathfrak{q}}(s) = \begin{cases} s & \text{on } \mathfrak{q}^\perp \\ 1 & \text{on } \mathfrak{p} \cap \mathfrak{q} \\ s^{-1} & \text{on } \mathfrak{p}^\perp \end{cases}$$

We note that  $s \mapsto \Gamma_{\mathfrak{p}}^{\mathfrak{q}}(s)$  is a homomorphism  $\mathbb{R}^\times \rightarrow \text{Aut}(\mathfrak{g})$ , that  $\Gamma_{\mathfrak{p}}^{\mathfrak{q}}(s) = \exp((\ln s) \xi_{\mathfrak{p}}^{\mathfrak{q}}) \in G$ , for  $s > 0$ , and that  $\Gamma(-1)$  is the involution at  $(\mathfrak{p}, \mathfrak{q})$  defining the symmetric space structure of  $Z$  in Proposition 2.5.

*Remark 3.9.* These  $\Gamma_{\mathfrak{p}}^{\mathfrak{q}}$  are the analogues, for real semisimple  $G$ , of the *simple factors* considered by Terng–Uhlenbeck [55]. We shall return to this point in §5.1.

Now consider the action of the gauge transformation  $\Gamma_f^{\hat{f}}(s)$  on the connection  $\nabla^t$ :

$$(3.7) \quad \begin{aligned} \Gamma_f^{\hat{f}}(s) \cdot \nabla^t &= \Gamma_f^{\hat{f}}(s) \cdot (\mathcal{D} - \beta - \frac{m-t}{m} \hat{\beta}) \\ &= \mathcal{D} - s\beta - s^{-1} \frac{m-t}{m} \hat{\beta}, \end{aligned}$$

since  $\Gamma_f^{\hat{f}}(s)$  is  $\mathcal{D}$ -parallel, having constant eigenvalues and  $\mathcal{D}$ -parallel eigenspaces, and both  $\beta, \hat{\beta}$  take values in eigenspaces of  $\Gamma_f^{\hat{f}}$ .

In particular,

$$\Gamma_f^{\hat{f}}(1 - \frac{t}{m}) \cdot \nabla^t = \mathcal{D} - \frac{m-t}{m} \beta - \hat{\beta} = \hat{\nabla}^t$$

whence each  $\hat{\nabla}^t$  is flat (for  $t \neq m$  and then for all  $t$  by continuity) since  $\nabla^t$  is flat. Moreover,  $\hat{\nabla}^m = \mathcal{D} - \hat{\beta}$  so that  $f$  is clearly  $\hat{\nabla}^m$ -parallel, that is  $f = \mathcal{D}_m \hat{f}$ . To summarise:

**Theorem 3.10.** *Let  $(f, \eta)$  be isothermic and  $\hat{f} = \mathcal{D}_m f$  a Darboux transform of  $f$  with  $\hat{\eta}$  defined as above. Then:*

- (1)  $(\hat{f}, \hat{\eta})$  is isothermic;
- (2)  $\Gamma_f^{\hat{f}}(1 - \frac{t}{m}) \cdot (d + t\eta) = d + t\hat{\eta}$  for all  $t \neq m$ ;
- (3)  $f = \mathcal{D}_m \hat{f}$ .

There is a converse to Theorem 3.10 which detects when two isothermic surfaces are Darboux transforms of each other:

**Proposition 3.11.** *Let  $(f, \eta), (\hat{f}, \hat{\eta})$  be pointwise complementary isothermic surfaces such that  $\Gamma_f^{\hat{f}}(1 - \frac{t}{m}) \cdot (d + t\eta) = d + t\hat{\eta}$ , for all  $t \neq m$ .*

*Then  $\hat{f} = \mathcal{D}_m f$  (whence  $f = \mathcal{D}_m \hat{f}$ ).*

*Proof.* As above, write  $d = \mathcal{D} - \beta - \hat{\beta}$  so that we have

$$\mathcal{D} - \frac{m-t}{m} \beta - \frac{m-t}{m-t} (\hat{\beta} - t\eta) = \mathcal{D} - \beta - \hat{\beta} + t\hat{\eta}$$

and compare components in  $f^\perp, \hat{f}^\perp$  to get

$$m\beta = \hat{\eta}, \quad m\hat{\beta} = \eta.$$

In particular,  $d + m\eta = \mathcal{D} - \beta$  so that  $\hat{f}$  is  $(d + m\eta)$ -parallel and thus a Darboux transform of  $f$  with parameter  $m$ .  $\square$

Part (2) of Theorem 3.10 enables us to prove a permutability theorem relating Darboux and  $T$ -transforms: indeed, with  $\hat{f} = \mathcal{D}_s f$  and  $\hat{\nabla}^t = d + t\hat{\eta}$ , the  $T$ -transform  $\mathcal{T}_f \hat{f}$  is given by  $\hat{\Phi}_t \hat{f}$  for  $\hat{\Phi}_t$  a gauge transformation with  $\hat{\Phi}_t \cdot \hat{\nabla}^t = d$ . On the other hand, the  $T$ -transform of  $f$  is implemented by  $\Phi_t$  with  $\Phi_t \cdot \nabla^t = d$  so that we conclude that

$$\hat{\Phi}_t \Gamma_f^{\hat{f}}(1 - \frac{t}{s}) = \Phi_t,$$

up to left multiplication by constants in  $G$ . Apply this to  $\hat{f}$  to conclude that  $\hat{\Phi}_t \hat{f} = \Phi_t \hat{f}$ . Now,  $\hat{f}$  is  $\nabla^s = \nabla^t + (s-t)\eta$ -parallel, whence  $\Phi_t \hat{f}$  is  $d + (s-t)\Phi_t \eta$ -parallel. Thus  $\Phi_t \hat{f}$  is a Darboux

transform of  $\Phi_t f$  with parameter  $s - t$ . To summarise, we have proved a result which can be found in [38, Theorem 5.6.15] for the classical case:

**Theorem 3.12.** *For  $t, s \in \mathbb{R}$  with  $0 \neq s \neq t$ ,  $\mathcal{J}_t \mathcal{D}_s = \mathcal{D}_{s-t} \mathcal{J}_t$ .*

There is a similar result, due to Bianchi [3, §3] in the classical setting, that uses the  $T$ -transform to intertwine Christoffel and Darboux transformations. Again, the key for us is to find the gauge transformation relating the pencils of flat connections corresponding to an isothermic map and its Christoffel transform. For this, let  $(f, \eta)$  be isothermic and fix a complementary pair  $(\mathfrak{p}_0, \mathfrak{p}_\infty) \in M \times M^*$ . Then we have maps  $F : \Sigma \rightarrow \mathfrak{p}_\infty^\perp, F^c : \Sigma \rightarrow \mathfrak{p}_0^\perp$  such that  $f = \exp(F) \mathfrak{p}_0, \eta = \exp(F) dF^c$  while the corresponding Christoffel transform is  $(f^c, \eta^c)$  with  $f^c = \exp(F^c) \mathfrak{p}_\infty$  and  $\eta^c = \exp(F^c) dF$ . Since  $F$  takes values in a fixed abelian subalgebra, we have

$$d = \exp(F) \cdot (d + dF)$$

so that

$$d + t\eta = \exp(F) \cdot (d + dF + tdF^c)$$

and, similarly,

$$d + t\eta^c = \exp(F^c) \cdot (d + tdF + dF^c).$$

Thus define  $\Gamma^c(t) = \exp(F^c) \Gamma_{\mathfrak{p}_0^\infty}^{\mathfrak{p}_0^\infty}(t) \exp(-F)$  and conclude:

**Lemma 3.13.** *With  $f, f^c$  a Christoffel pair of isothermic surfaces and  $\Gamma^c(t)$  defined as above,  $\Gamma^c(t) \cdot (d + t\eta) = d + t\eta^c$ , for all  $t \neq 0$ .*

Note that  $\Gamma^c(t)f \equiv \mathfrak{p}_0$  while  $\Gamma^c(t)\mathfrak{p}_\infty \equiv f^c$ .

We can now argue as for Theorem 3.12 above: the  $T$ -transform of  $f^c$  is  $\Phi_t^c f^c$  where  $\Phi_t^c \cdot (d + t\eta^c) = d$ . Thus Lemma 3.13 tells us that

$$\Phi_t^c \Gamma^c(t) = \Phi_t$$

so that  $\Phi_t^c f^c = \Phi_t \Gamma^c(t)^{-1} f^c = \Phi_t \mathfrak{p}_\infty$ . Now the constant section  $\mathfrak{p}_\infty$  is  $d = \nabla^t - t\eta$ -parallel so that  $\Phi_t \mathfrak{p}_\infty$  is  $d - t\Phi_t \eta$ -parallel, that is,  $\Phi_t \mathfrak{p}_\infty$  is a Darboux transform of  $\Phi_t f$  with parameter  $-t$ . To summarise:

**Theorem 3.14.** *For  $t \in \mathbb{R}^\times$ ,  $\mathcal{J}_t f^c = \mathcal{D}_{-t} \mathcal{J}_t f$ .*

**3.5. Christoffel transform as a blow-up of Darboux transforms.** If we let  $t \rightarrow 0$  in Theorem 3.14, it appears that the Christoffel transform  $f^c$  is some kind of limit of Darboux transforms. This is indeed the case as we now show.

Fix  $(f, \eta)$  isothermic,  $f : \Sigma \rightarrow M$ , and contemplate the gauge transformations  $\Phi_s : \Sigma \rightarrow G$  that implement the  $T$ -transforms of  $f$ . Thus  $\Phi_s \cdot \nabla^s = d$  and each  $\Phi_s$  is determined uniquely up to left multiplication by a constant in  $\text{Aut}(\mathfrak{g})$ . In particular, we may assume that  $\Phi_s$  depends smoothly on  $s$  and that  $\Phi_0 = 1$ . Now define  $\dot{\Phi} : \Sigma \rightarrow \mathfrak{g}$  by

$$\dot{\Phi} = \left. \frac{\partial \Phi_s}{\partial s} \right|_{s=0}.$$

Then:

**Proposition 3.15.**  $d\dot{\Phi} = \eta$ .

*Proof.* The defining property of the  $\Phi_s$  amounts to

$$\Phi_s^{-1} d\Phi_s = s\eta$$

and we differentiate this with respect to  $s$  at  $s = 0$  to draw the conclusion.  $\square$

Thus  $\dot{\Phi}$  is a primitive for  $\eta$  and so, by Proposition 3.4, is a universal Christoffel transform for  $f$ . That is, for  $(\mathfrak{p}_0, \mathfrak{p}_\infty) \in M \times M^*$  a complementary pair with  $\mathfrak{p}_\infty$  pointwise complementary to  $f$ ,  $f^c := \exp(\pi_{\mathfrak{p}_0^\perp} \dot{\Phi}) \mathfrak{p}_\infty : \Sigma \rightarrow M^*$  is a Christoffel transform of  $(f, \eta)$  with respect to  $(\mathfrak{p}_0, \mathfrak{p}_\infty)$ .

We use this to give two alternative characterisations of  $f^c$ . First we see that  $f^c$  is a blow-up of Darboux transforms of  $f$ . Indeed let  $\hat{f}_s : \Sigma \rightarrow M^*$  be a smooth variation of maps through  $\hat{f}_0 \equiv \mathfrak{p}_\infty$  with each  $\hat{f}_s$   $\nabla^s$ -parallel and pointwise complementary to  $f$ . Thus, for  $s \neq 0$ ,  $\hat{f}_s$  is a Darboux transform of  $(f, \eta)$  with parameter  $s$ :  $\hat{f}_s = \mathcal{D}_s f$ . Then  $\partial/\partial s|_{s=0} \hat{f}_s : \Sigma \rightarrow T_{\mathfrak{p}_\infty} M^*$ . Now the

derivative at 1 of  $g \mapsto g \mathfrak{p}_\infty$  induces the soldering isomorphism  $\mathfrak{g}/\mathfrak{p}_\infty \cong T_{\mathfrak{p}_\infty} M^*$  and thus we get an identification  $\mathfrak{p}_0^\perp \cong T_{\mathfrak{p}_\infty} M^*$ . With this in mind, observe that we may choose  $\hat{f}_s$  so that  $\hat{f}_s = \Phi_s^{-1} \mathfrak{p}_\infty$  whence, since  $\partial/\partial s|_{s=0} \Phi_s^{-1} = -\dot{\Phi}$ , we conclude:

**Corollary 3.16.** *Under the identification  $T_{\mathfrak{p}_\infty} M^* \cong \mathfrak{p}_0^\perp$ ,  $\partial/\partial s|_{s=0} \hat{f}_s = -\pi_{\mathfrak{p}_\infty} \dot{\Phi}$  so that*

$$f^c = \exp(-\partial/\partial s|_{s=0} \hat{f}_s) \mathfrak{p}_\infty.$$

This leads us to another realisation of both  $f^c$  and its accompanying 1-form  $\eta^c$ , this time as a limit of conjugates of  $(\hat{f}_s, \hat{\eta}_s)$  which will be useful below.

**Theorem 3.17.** *Let  $(\hat{f}_s, \hat{\eta}_s)$  be a smooth family of Darboux transforms of  $(f, \eta)$  with parameter  $s$  such that  $\lim_{s \rightarrow 0} \hat{f}_s = \mathfrak{p}_\infty$ . Then a Christoffel transform  $(f^c, \eta^c)$  of  $(f, \eta)$  with respect to  $(\mathfrak{p}_0, \mathfrak{p}_\infty)$  is given by*

$$\begin{aligned} f^c &= \lim_{s \rightarrow 0} \Gamma_{\mathfrak{p}_0}^{\mathfrak{p}_\infty}(-s) \hat{f}_s, \\ \eta^c &= \lim_{s \rightarrow 0} \Gamma_{\mathfrak{p}_0}^{\mathfrak{p}_\infty}(-s) \hat{\eta}_s. \end{aligned}$$

*In particular,  $d + t\eta^c = \lim_{s \rightarrow 0} \Gamma_{\mathfrak{p}_0}^{\mathfrak{p}_\infty}(-s) \cdot (d + t\hat{\eta}_s)$ , for all  $t \in \mathbb{R}$ .*

*Proof.* Let  $\hat{F}_s : \Sigma \rightarrow \mathfrak{p}_0^\perp$  be the stereoprojection of  $\hat{f}_s$  from  $\mathfrak{p}_0$  so that  $\hat{f}_s = \exp(\hat{F}_s) \mathfrak{p}_\infty$ . The derivative of stereoprojection at  $\mathfrak{p}_\infty$  is precisely our identification  $T_{\mathfrak{p}_\infty} M^* \cong \mathfrak{p}_0^\perp$  under which  $\partial/\partial s|_{s=0} \hat{f}_s = \partial/\partial s|_{s=0} \hat{F}_s$  so that Corollary 3.16 reads

$$f^c = \exp(-\partial/\partial s|_{s=0} \hat{F}_s) \mathfrak{p}_\infty.$$

However,  $\lim_{s \rightarrow 0} \hat{F}_s = 0$  so that

$$f^c = \lim_{s \rightarrow 0} \exp(-\hat{F}_s/s) \mathfrak{p}_\infty = \lim_{s \rightarrow 0} \Gamma_{\mathfrak{p}_0}^{\mathfrak{p}_\infty}(-s) \exp(\hat{F}_s) \mathfrak{p}_\infty = \lim_{s \rightarrow 0} \Gamma_{\mathfrak{p}_0}^{\mathfrak{p}_\infty}(-s) \hat{f}_s$$

as required.

As for the 1-forms, recall that  $\hat{\eta}_s = \beta_s/s$  where  $\beta_s \in \Omega_\Sigma^1(\hat{f}_s^\perp)$  and  $\hat{\beta}_s \in \Omega_\Sigma^1(f^\perp)$  are given by  $d\xi_f^{\hat{f}_s} = [\xi_f^{\hat{f}_s}, \beta_s + \hat{\beta}_s]$ . Meanwhile,  $\eta^c = \exp(F^c) dF$  where  $F : \Sigma \rightarrow \mathfrak{p}_\infty^\perp$  is the stereoprojection of  $F$  and, as we have just seen,  $F^c = -\partial/\partial s|_{s=0} \hat{F}_s$  is the stereoprojection of  $f^c$ . Write  $\hat{\eta}_s = \exp(\hat{F}_s) b_s/s$  with  $b_s \in \Omega_\Sigma^1(\mathfrak{p}_\infty^\perp)$ . Then

$$\lim_{s \rightarrow 0} \Gamma_{\mathfrak{p}_0}^{\mathfrak{p}_\infty}(-s) \hat{\eta}_s = \lim_{s \rightarrow 0} \Gamma_{\mathfrak{p}_0}^{\mathfrak{p}_\infty}(-s) \exp(\hat{F}_s) b_s/s = - \lim_{s \rightarrow 0} \exp(-\hat{F}_s/s) b_s = - \exp(-\partial/\partial s|_{s=0} \hat{F}_s) b_0.$$

Now  $b_0 = \beta_0$  and it remains to identify this with  $-dF$ . However,  $\xi_f^{\mathfrak{p}_\infty} = \exp(F) \xi_{\mathfrak{p}_0}^{\mathfrak{p}_\infty}$  and we differentiate to get  $d\xi_f^{\mathfrak{p}_\infty} = [dF, \xi_f^{\mathfrak{p}_\infty}]$  yielding  $\beta_0 = -dF$  as required (and  $\hat{\beta}_0 = 0$ ).  $\square$

#### 4. BIANCHI PERMUTABILITY IN SELF-DUAL SPACES

Let us recall Bianchi's celebrated Permutability Theorem [2, §5–§9] for Darboux transforms of isothermic surfaces in  $\mathbb{R}^3$ : given such an isothermic surface  $f$  and two distinct Darboux transforms  $f_1 = \mathcal{D}_{m_1} f$ ,  $f_2 = \mathcal{D}_{m_2} f$  with  $m_1 \neq m_2$ , then there is a fourth isothermic surface  $\hat{f}$  which is a simultaneous Darboux transform of  $f_1$  and  $f_2$ :  $\hat{f} = \mathcal{D}_{m_2} f_1 = \mathcal{D}_{m_1} f_2$ . Moreover,  $\hat{f}$  can be constructed algebraically from the first three surfaces: in fact, according to Demoulin [29, p. 157], corresponding points on the four surfaces are concircular with constant cross-ratio  $m_2/m_1$ .

We now show that these results still hold *in every detail* for isothermic maps into any *self-dual* symmetric  $R$ -space. For this, we begin by describing a distinguished family of circles in such a space.

**4.1. Circles in self-dual symmetric  $R$ -spaces.** Three distinct points in  $S^n$  determine a unique circle on which they lie. This fact generalises to self-dual symmetric  $R$ -spaces  $M$ : three pairwise complementary points in  $M$  determine a submanifold of  $M$ , conformally diffeomorphic to a circle, on which they lie.

Let  $\mathfrak{p}_0, \mathfrak{p}_\infty \in M$  be complementary points in a self-dual symmetric  $R$ -space. The set of parabolic subalgebras in  $M$  that are complementary to both  $\mathfrak{p}_0$  and  $\mathfrak{p}_\infty$  is the dense open set  $\Omega_{\mathfrak{p}_0} \cap \Omega_{\mathfrak{p}_\infty}$ . We begin with a simple criterion for membership of this set.



**Lemma 4.1.** *Let  $x_\infty \in \mathfrak{p}_\infty^\perp$ . Then  $\exp(x_\infty) \mathfrak{p}_0 \in \Omega_{\mathfrak{p}_0} \cap \Omega_{\mathfrak{p}_\infty}$  if and only if  $\ker(\text{ad } x_\infty)^2 = \mathfrak{p}_\infty$ .*

*Proof.*  $\exp(x_\infty) \mathfrak{p}_0 \in \Omega_{\mathfrak{p}_0} \cap \Omega_{\mathfrak{p}_\infty}$  if and only if  $\exp(x_\infty) \mathfrak{p}_0 \in \Omega_{\mathfrak{p}_0}$ , that is,  $\exp(x_\infty) \mathfrak{p}_0$  and  $\mathfrak{p}_0$  are complementary. This means that  $\exp(x_\infty) \mathfrak{p}_0^\perp \cap \mathfrak{p}_0 = \{0\}$ . However, for  $x \in \mathfrak{p}_0^\perp$ , we have

$$\exp(x_\infty) x = x + [x_\infty, x] + \frac{1}{2}(\text{ad } x_\infty)^2 x$$

with the first two summands in  $\mathfrak{p}_0$  and the last in  $\mathfrak{p}_\infty^\perp$ . Now  $\mathfrak{g} = \mathfrak{p}_0 \oplus \mathfrak{p}_\infty^\perp$  so  $\exp(x_\infty) x \in \exp(x_\infty) \mathfrak{p}_0^\perp \cap \mathfrak{p}_0$  if and only if  $(\text{ad } x_\infty)^2 x = 0$ . Thus  $\exp(x_\infty) \mathfrak{p}_0 \in \Omega_{\mathfrak{p}_0} \cap \Omega_{\mathfrak{p}_\infty}$  if and only if  $(\text{ad } x_\infty)^2$  injects on  $\mathfrak{p}_0^\perp$ . Since  $\mathfrak{g} = \mathfrak{p}_0^\perp \oplus \mathfrak{p}_\infty$  and  $\mathfrak{p}_\infty \subset \ker(\text{ad } x_\infty)^2$ , for any  $x_\infty \in \mathfrak{p}_\infty^\perp$ , the lemma follows.  $\square$

Note that the condition on  $x_\infty$  is independent of the choice of complementary  $\mathfrak{p}_0$ .

Suppose now that we have three mutually complementary points  $\mathfrak{p}_0, \mathfrak{p}_1, \mathfrak{p}_\infty \in M$  and write  $\mathfrak{p}_1 = \exp(x_\infty) \mathfrak{p}_0$  for a unique  $x_\infty \in \mathfrak{p}_\infty^\perp$ . For  $t \in \mathbb{R}^\times$ ,  $(\text{ad } tx_\infty)^2 = t^2(\text{ad } x_\infty)^2$  and so has kernel  $\mathfrak{p}_\infty$  also whence, by Lemma 4.1,  $\mathfrak{p}_t := \exp(tx_\infty) \mathfrak{p}_0$  is again complementary to  $\mathfrak{p}_0$  and  $\mathfrak{p}_\infty$ .

**Definition 4.2.** The *circle through*  $\mathfrak{p}_0, \mathfrak{p}_1, \mathfrak{p}_\infty$  is the subset  $C \subset M$  given by

$$C = \{\mathfrak{p}_t : t \in \mathbb{R} \cup \{\infty\}\}.$$

Note that, for  $t \neq s$ ,  $\mathfrak{p}_t, \mathfrak{p}_s$  are the images under  $\exp sx_\infty$  of the complementary pair  $(\mathfrak{p}_{t-s}, \mathfrak{p}_0)$  and so are also complementary.

We now show that our construction is independent of choices, that is, any three points of  $C$  determine the same circle. For this we give an alternative approach to  $C$  which also shows that the projective structure on  $C$  given by the coordinate  $t$  is also independent of choices.

For  $i \neq j \in \{0, 1, \infty\}$ , let  $\xi_j^i$  be the grading element of the pair  $(\mathfrak{p}_j, \mathfrak{p}_i)$ . Thus

$$\text{ad } \xi_j^i = \begin{cases} 1 & \text{on } \mathfrak{p}_i^\perp \\ 0 & \text{on } \mathfrak{p}_i \cap \mathfrak{p}_j \\ -1 & \text{on } \mathfrak{p}_j^\perp \end{cases}$$

and  $\xi_i^j = -\xi_j^i$ .

**Proposition 4.3.** *The span  $\mathfrak{s} = \langle \xi_0^1, \xi_1^\infty, \xi_\infty^0 \rangle \subset \mathfrak{g}$  is a subalgebra of  $\mathfrak{g}$  isomorphic to  $\mathfrak{sl}(2, \mathbb{R})$ . Indeed, with  $\mathfrak{p}_1 = \exp(x_\infty) \mathfrak{p}_0 = \exp(x_0) \mathfrak{p}_\infty$ , for  $x_i \in \mathfrak{p}_i^\perp$ , we have  $x_0, x_\infty \in \mathfrak{s}$  and*

$$[\xi_0^\infty, x_\infty] = x_\infty \quad [x_\infty, x_0] = 2\xi_0^\infty \quad [\xi_0^\infty, x_0] = -x_0.$$

*Proof.* For distinct  $i, j, k \in \{0, 1, \infty\}$ , there is, by Lemma 1.4,  $x_{jk} \in \mathfrak{p}_i^\perp$  such that  $\exp(x_{jk}) \xi_i^j = \xi_i^k$ . That is,

$$(4.1) \quad \xi_i^k = \xi_i^j + [x_{jk}, \xi_i^j] = \xi_i^j + x_{jk}.$$

Thus  $\xi_i^k - \xi_i^j = x_{jk} \in \mathfrak{p}_i^\perp$  so that

$$(4.2) \quad [\xi_i^j, \xi_i^k] = [\xi_i^j, \xi_i^k - \xi_i^j] = \xi_i^j - \xi_i^k$$

which shows that  $\mathfrak{s}$  is a subalgebra.

Moreover, we have  $\exp(x_\infty) \xi_\infty^0 = \xi_\infty^1$  so (4.1) gives  $x_\infty = \xi_\infty^1 - \xi_\infty^0 \in \mathfrak{s}$  and, similarly,  $x_0 = \xi_0^1 - \xi_0^\infty \in \mathfrak{s}$ . Now use (4.2) and  $\xi_i^j = -\xi_j^i$  to compute:

$$\begin{aligned} [x_\infty, x_0] &= [\xi_\infty^1 - \xi_\infty^0, \xi_0^1 - \xi_0^\infty] \\ &= [\xi_\infty^1, \xi_0^1] + [\xi_\infty^1, \xi_0^\infty] + [\xi_\infty^0, \xi_0^1] \\ &= 2\xi_0^\infty. \end{aligned}$$

$\square$

The Killing form  $(\cdot, \cdot)_\mathfrak{s}$  of  $\mathfrak{s}$  has signature  $(2, 1)$  (indeed,  $x_0, x_\infty$  are null and  $(\xi_0^\infty, \xi_0^\infty)_\mathfrak{s} = 2$ ). The light-cone  $\mathcal{L}_\mathfrak{s} \subset \mathfrak{s}$  for this inner product is precisely the set of nilpotent elements of  $\mathfrak{s}$ . The projective light-cone  $\mathbb{P}(\mathcal{L}_\mathfrak{s})$  is then, on the one hand, a conformal diffeomorph of  $S^1$  and, on the other, the set of nilradicals of the single conjugacy class of parabolic subalgebras of  $\mathfrak{s}$ .

Let  $S \leq G$  be the analytic subgroup of  $G$  with Lie algebra  $\mathfrak{s}$ . Then  $S \cong \mathrm{PSL}(2, \mathbb{R})$  and acts transitively on  $\mathbb{P}(\mathcal{L}_{\mathfrak{s}})$ . Thus any  $\langle x \rangle \in \mathbb{P}(\mathcal{L}_{\mathfrak{s}})$  is of the form  $h \langle x_{\infty} \rangle$ , for some  $h \in S$ , so that  $\ker(\mathrm{ad} x)^2 = h \ker(\mathrm{ad} x_{\infty})^2 = h \mathfrak{p}_{\infty} \in M$ . We have therefore defined an  $S$ -equivariant map  $\Psi : \mathbb{P}(\mathcal{L}) \rightarrow M$  by

$$\Psi(\langle x \rangle) = \ker(\mathrm{ad} x)^2$$

which injects since  $\mathfrak{s} \cap \Psi(\langle x \rangle)^{\perp} = \langle x \rangle$ . It is easy to see that the circle  $C$  constructed earlier coincides with the image of  $\Psi$ : indeed, since  $\mathfrak{p}_0 = \Psi(\langle x_0 \rangle)$ , the  $S$ -equivariance of  $\Psi$  gives

$$(4.3) \quad \mathfrak{p}_t = \exp(tx_{\infty}) \mathfrak{p}_0 = \exp(tx_{\infty}) \Psi(\langle x_0 \rangle) = \Psi(\exp(tx_{\infty}) \langle x_0 \rangle),$$

for  $t \in \mathbb{R}$ . Moreover, since  $S$  acts (simply) transitively on triples of distinct points of  $\mathbb{P}(\mathcal{L}_{\mathfrak{s}})$ , we see that any three distinct (and so complementary)  $\mathfrak{p}'_0, \mathfrak{p}'_1, \mathfrak{p}'_{\infty} \in C$  define the same  $\mathfrak{s}$  (the grading elements  $\xi_i^j$  lie in  $\mathfrak{s}$  and therefore span it) and thus the same  $S$  and  $C$ .

We have now equipped our circle  $C$  with a conformal structure, or, what is the same thing in dimension one, a projective structure. Indeed, fix a double cover  $\mathrm{SL}(2, \mathbb{R}) \rightarrow S$  and then we have an equivariant isomorphism  $\mathbb{R}\mathbb{P}^1 \cong \mathbb{P}(\mathcal{L}_{\mathfrak{s}})$  given by  $\ell \mapsto \mathrm{stab}(\ell)^{\perp}$ . From (4.3), we see that the coordinate  $\mathfrak{p}_t \mapsto t$  on  $C$  is the pull-back by  $\Psi^{-1}$  of the coordinate on  $S^1$  given by stereoprojection. According to §2.2.2, this last is an affine coordinate on  $\mathbb{R}\mathbb{P}^1$ .

Here are some consequences of this circle of ideas. First, four distinct points on  $C$  have, via the identification with  $\mathbb{R}\mathbb{P}^1$ , an  $S$ -invariant cross-ratio and, in particular, the cross-ratio of  $\mathfrak{p}_1, \mathfrak{p}_{\infty}, \mathfrak{p}_t, \mathfrak{p}_0$  is exactly  $t$ :

$$(4.4) \quad \mathrm{cr}(\mathfrak{p}_1, \mathfrak{p}_{\infty}, \mathfrak{p}_t, \mathfrak{p}_0) = t.$$

For the second, let us first relate the parametrisation  $t \mapsto \mathfrak{p}_t$  to the gauge transformations  $\Gamma_{\mathfrak{p}}^{\mathfrak{q}}$  of §3.4: for  $t \in \mathbb{R}^{\times}$ ,

$$\begin{aligned} \Gamma_{\mathfrak{p}_0}^{\mathfrak{p}_1}(t) \mathfrak{p}_1 &= \Gamma_{\mathfrak{p}_0}^{\mathfrak{p}_1}(t) \exp(x_{\infty}) \mathfrak{p}_0 \\ &= \exp(\Gamma_{\mathfrak{p}_0}^{\mathfrak{p}_1}(t) x_{\infty}) (\Gamma_{\mathfrak{p}_0}^{\mathfrak{p}_1}(t) \mathfrak{p}_0) \\ &= \exp(tx_{\infty}) \mathfrak{p}_0 = \mathfrak{p}_t, \end{aligned}$$

since  $\Gamma_{\mathfrak{p}_0}^{\mathfrak{p}_1}(t)$  preserves  $\mathfrak{p}_0$  and has  $\mathfrak{p}_1^{\perp}$  as eigenspace with eigenvalue  $t$ . Thus we may set  $\Gamma_{\mathfrak{p}_0}^{\mathfrak{p}_1}(0) \mathfrak{p}_1 = \mathfrak{p}_0$ ,  $\Gamma_{\mathfrak{p}_0}^{\mathfrak{p}_1}(\infty) \mathfrak{p}_1 = \mathfrak{p}_{\infty}$  and see that  $t \mapsto \Gamma_{\mathfrak{p}_0}^{\mathfrak{p}_1}(t) \mathfrak{p}_1$  is the parametrisation of  $C$  inverting the affine coordinate  $\mathfrak{p}_t \mapsto t$ . Now any two affine coordinates on  $\mathbb{R}\mathbb{P}^1$  are related by a unique linear fractional transformation so the same is true of the corresponding parametrisations. We summarise the situation in the following proposition which will enable us to avoid several tedious computations below.

**Proposition 4.4.** *Let  $(\mathfrak{p}_0, \mathfrak{p}_1, \mathfrak{p}_{\infty})$ ,  $(\mathfrak{p}'_0, \mathfrak{p}'_1, \mathfrak{p}'_{\infty})$  be two triples of mutually complementary points determining the same circle  $C \subset M$ .*

(1) *There is a unique linear fractional transformation  $s(t)$  such that*

$$\Gamma_{\mathfrak{p}_0}^{\mathfrak{p}_1}(t) \mathfrak{p}_1 = \Gamma_{\mathfrak{p}'_0}^{\mathfrak{p}'_1}(s(t)) \mathfrak{p}'_1,$$

*for all  $t \in \mathbb{R} \cup \{\infty\}$ . In particular,  $\Gamma_{\mathfrak{p}_0}^{\mathfrak{p}_1} \mathfrak{p}_1$  is determined by its values at any three distinct points of  $\mathbb{R} \cup \{\infty\}$ .*

(2)  *$\mathrm{cr}(\mathfrak{p}_1, \mathfrak{p}_{\infty}, \Gamma_{\mathfrak{p}_0}^{\mathfrak{p}_1}(t) \mathfrak{p}_1, \mathfrak{p}_0) = t$ , for all  $t \in \mathbb{R} \cup \{\infty\}$ .*

*Remark 4.5.* If  $\mathfrak{g}$  is a complex Lie algebra so that  $M$  is a conjugacy class of complex parabolic subalgebras, the entire discussion goes through unchanged with  $t \in \mathbb{C}$ . Thus, in this case, pairwise complementary  $\mathfrak{p}_0, \mathfrak{p}_1, \mathfrak{p}_{\infty}$  determine a rational curve  $C$  in  $M$  with  $C \cong \mathbb{C}\mathbb{P}^1$  carrying a complex projective structure and well-defined complex cross-ratio. Moreover, Proposition 4.4 is still valid so long as  $t$  is taken to be complex. We will persist in calling  $C$  the circle through  $\mathfrak{p}_0, \mathfrak{p}_1, \mathfrak{p}_{\infty}$ .

With this in hand, we conclude this section with a lemma whose unpromising statement is tailored for an application in §5.2.2.

**Lemma 4.6.** *Let  $\mathfrak{g}$  be complex and  $M$  a self-dual symmetric R-space for  $\mathfrak{g}$ . Let  $\mathfrak{p}, \mathfrak{p}_1, \mathfrak{p}_2 \in M$  with  $(\mathfrak{p}, \mathfrak{p}_1)$  complementary and set  $\tau = \Gamma_{\mathfrak{p}}^{\mathfrak{p}_1}(-1) \in \mathrm{Aut}(\mathfrak{g})$ . Finally, let  $w \in \mathbb{C}^{\times}$  and put  $\mathfrak{r} = \Gamma_{\mathfrak{p}}^{\mathfrak{p}_2}(w) \mathfrak{p}_2$ .*

*Then  $\mathfrak{p}, \mathfrak{p}_1, \mathfrak{p}_2$  are pairwise complementary if and only if  $\mathfrak{r}, \tau\mathfrak{r}$  are complementary. In this case, all five parabolic subalgebras are concircular.*

*Proof.* First suppose that  $\mathfrak{p}, \mathfrak{p}_1, \mathfrak{p}_2$  are pairwise complementary then

$$\tau\mathfrak{r} = \Gamma_{\mathfrak{p}}^{\mathfrak{p}_1}(-1)\Gamma_{\mathfrak{p}}^{\mathfrak{p}_1}(w)\mathfrak{p}_2 = \Gamma_{\mathfrak{p}}^{\mathfrak{p}_1}(-w)\mathfrak{p}_2.$$

Thus, since  $w \neq -w$ ,  $\mathfrak{r}$  and  $\tau\mathfrak{r}$  are distinct points on the circle through  $\mathfrak{p}, \mathfrak{p}_1, \mathfrak{p}_2$ .

For the converse, we must work a little harder. So suppose that  $\mathfrak{r}, \tau\mathfrak{r}$  are complementary and let  $\xi = \xi_{\mathfrak{r}}^{\tau\mathfrak{r}}$  be the corresponding grading element. Clearly we have  $\tau\xi = -\xi$  and we use this to see that  $\mathfrak{r}$  (and so  $\tau\mathfrak{r}$ ) is complementary to both  $\mathfrak{p}$  and  $\mathfrak{p}_1$ . For this, let  $x \in (\mathfrak{p}^{\perp} \oplus \mathfrak{p}_1^{\perp}) \cap \mathfrak{r}^{\perp}$  so that, on the one hand,  $\tau x = -x$  and, on the other hand,  $[\xi, x] = -x$ . Then

$$x = -\tau x = \tau[\xi, x] = [-\xi, -x] = -x$$

so that  $x = 0$ . Thus  $\mathfrak{p}^{\perp} \cap \mathfrak{r}^{\perp} = \{0\} = \mathfrak{p}_1^{\perp} \cap \mathfrak{r}^{\perp}$  so that both  $(\mathfrak{p}, \mathfrak{r})$  and  $(\mathfrak{p}_1, \mathfrak{r})$  are complementary pairs. We now have  $\mathfrak{p}_2 = \Gamma_{\mathfrak{p}}^{\mathfrak{p}_1}(1/w)\mathfrak{r}$  so that  $\mathfrak{p}_2$  lies on the circle through  $\mathfrak{p}, \mathfrak{p}_1, \mathfrak{r}$  and, since  $1/w \neq 0, \infty$ , is distinct from and so complementary to  $\mathfrak{p}$  and  $\mathfrak{p}_1$ .  $\square$

**4.2. Bianchi permutability for Darboux transforms.** Let  $(f, \eta) : \Sigma \rightarrow M$  be an isothermic map to a self-dual symmetric  $R$ -space  $M$  with Darboux transforms  $(f_1, \eta_1), (f_2, \eta_2)$  with parameters  $m_1 \neq m_2$  respectively:  $f_1 = \mathcal{D}_{m_1}f$ ,  $f_2 = \mathcal{D}_{m_2}f$ . Suppose, in addition, that the  $f_i$  are pointwise complementary. We seek a fourth isothermic map  $\hat{f}$  so that  $\hat{f} = \mathcal{D}_{m_2}f_1 = \mathcal{D}_{m_1}f_2$ .

Recall from Theorem 3.10 that, for  $i = 1, 2$  we have  $\Gamma_f^{f_i}(1 - \frac{t}{m_i}) \cdot (d + t\eta) = d + t\eta_i$ . In particular, since  $f_2$  is  $(d + m_2\eta)$ -parallel,  $f_{12} = \Gamma_f^{f_1}(1 - \frac{m_2}{m_1})f_2$  is  $(d + m_2\eta_1)$ -parallel. Moreover  $(f_1, f_{12})$  is the image by  $\Gamma_f^{f_1}(1 - \frac{m_2}{m_1})$  of the complementary pair  $(f_1, f_2)$  and so is a complementary pair also. Thus  $(f_{12}, \eta_{12})$  is a Darboux transform of  $(f_1, \eta_1)$  with parameter  $m_2$  and  $\eta_{12}$  is determined by the requirement that  $\Gamma_{f_1}^{f_{12}}(1 - \frac{t}{m_2}) \cdot (d + t\eta_1) = d + t\eta_{12}$ .

Similarly  $(f_{21}, \eta_{21})$  is a Darboux transform of  $(f_2, \eta_2)$  with parameter  $m_1$  where  $f_{21} = \Gamma_f^{f_2}(1 - \frac{m_1}{m_2})f_1$  and  $\eta_{21}$  is determined by the requirement that  $\Gamma_{f_2}^{f_{21}}(1 - \frac{t}{m_1}) \cdot (d + t\eta_2) = d + t\eta_{21}$ . It therefore suffices to prove that  $f_{12} = f_{21}$  and  $\eta_{12} = \eta_{21}$ .

For the first of these identities, note that  $\Gamma_f^{f_1}(t)f_2$  and  $\Gamma_f^{f_2}(t)f_1$  parametrise the circles through corresponding points of  $f, f_1, f_2$  and so differ by a linear fractional transformation of  $t$  by Proposition 4.4. This linear fractional transformation must fix 0 and swap 1 and  $\infty$  so that, for all  $t$ ,

$$\Gamma_f^{f_1}(t)f_2 = \Gamma_f^{f_2}\left(\frac{t}{t-1}\right)f_1$$

and evaluating this at  $t = 1 - m_2/m_1$  yields  $f_{12} = f_{21}$ .

The second identity is equivalent to  $d + t\eta_{12} = d + t\eta_{21}$  and since these connections are gauges of  $d + t\eta$  by  $\Gamma_{f_1}^{\hat{f}}(1 - \frac{t}{m_2})\Gamma_f^{f_1}(1 - \frac{t}{m_1})$  and  $\Gamma_{f_2}^{\hat{f}}(1 - \frac{t}{m_1})\Gamma_f^{f_2}(1 - \frac{t}{m_2})$ , it suffices to prove:

**Lemma 4.7.**  $\Gamma_{f_1}^{\hat{f}}(1 - \frac{t}{m_2})\Gamma_f^{f_1}(1 - \frac{t}{m_1}) = \Gamma_{f_2}^{\hat{f}}(1 - \frac{t}{m_1})\Gamma_f^{f_2}(1 - \frac{t}{m_2}) = \Gamma_{f_2}^{f_1}\left(\frac{1-t/m_1}{1-t/m_2}\right)$ , for all  $t$ .

*Proof.* It suffices to prove that  $\Gamma_{f_1}^{\hat{f}}(1 - \frac{t}{m_2})\Gamma_f^{f_1}(1 - \frac{t}{m_1}) = \Gamma_{f_2}^{f_1}\left(\frac{1-t/m_1}{1-t/m_2}\right)$  for then the other equality follows by swapping the roles of  $f_1$  and  $f_2$ . With  $\Gamma(t)$  denoting  $\Gamma_{f_1}^{\hat{f}}(1 - \frac{t}{m_2})\Gamma_f^{f_1}(1 - \frac{t}{m_1})$ , we observe that, by definition,  $\Gamma(t)$  acts as  $\frac{1-t/m_1}{1-t/m_2}$  on  $f_1^{\perp}$  and as 1 on  $f_1/f_1^{\perp}$ . Moreover,  $\Gamma(t)$  preserves  $f_2$ : indeed, this amounts to demanding that  $\Gamma_{f_1}^{\hat{f}}(1 - \frac{t}{m_2})f_2 = \Gamma_f^{f_1}(1 - \frac{t}{m_1})f_2$ . We readily check that these agree at  $t = 0, m_1, \infty$  and so everywhere by Proposition 4.4.

Now, since  $\Gamma(t)$  is orthogonal with respect to the Killing form, we see that it must preserve the decomposition  $\mathfrak{g} = f_1^{\perp} \oplus (f_1 \cap f_2) \oplus f_2^{\perp}$  and so acts as 1 on  $f_1 \cap f_2$ . Moreover, since  $f_1^{\perp}$  and  $f_2^{\perp}$  are dual with respect to the Killing form,  $\Gamma(t)$  acts as  $\frac{1-t/m_2}{1-t/m_1}$  on  $f_2^{\perp}$  and the lemma follows.  $\square$

Finally, apply the identity of Lemma 4.7 to  $f$  to get

$$\Gamma_{f_2}^{f_1}\left(\frac{1-t/m_1}{1-t/m_2}\right)f = \Gamma_{f_1}^{\hat{f}}\left(1 - \frac{t}{m_2}\right)\Gamma_f^{f_1}\left(1 - \frac{t}{m_1}\right)f = \Gamma_{f_1}^{\hat{f}}\left(1 - \frac{t}{m_2}\right)f$$

and evaluate at  $t = \infty$  to conclude:

$$\hat{f} = \Gamma_{f_2}^{f_1}(m_2/m_1)f$$

whence  $\text{cr}(f, f_1, \hat{f}, f_2) = m_2/m_1$ . To summarise:

**Theorem 4.8.** *Let  $f_1, f_2 : \Sigma \rightarrow M$  be complementary Darboux transforms of an isothermic map  $f : \Sigma \rightarrow M$  with parameters  $m_1 \neq m_2$  respectively. Then there is a common Darboux transform  $\hat{f} = \mathcal{D}_{m_1} f_2 = \mathcal{D}_{m_2} f_1$  of  $f_1$  and  $f_2$  which is pointwise concircular with  $f, f_1, f_2$  and has constant cross-ratio  $m_2/m_1$  with these.*

We call such a configuration of four isothermic maps a *Bianchi quadrilateral* (see Figure 1).

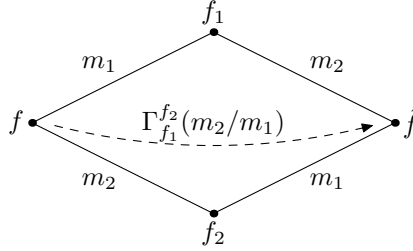


FIGURE 1. A Bianchi quadrilateral

**4.3. The Cube Theorem.** With almost no extra effort, we can compute the effect of a third Darboux transform on a Bianchi quadrilateral and so prove the analogue of Bianchi's Cube Theorem [2, §11] in our context.

So start with an isothermic map  $f$ , distinct  $m_1, m_2, m_3 \in \mathbb{R}^\times$  and pairwise complementary Darboux transforms  $f_1, f_2, f_3$  of  $f$ :  $f_i = \mathcal{D}_{m_i} f$ . Then Theorem 4.8 provides three simultaneous Darboux transforms  $f_{ij} = f_{ji} = \mathcal{D}_{m_i} f_j = \mathcal{D}_{m_j} f_i$ , for  $i, j \in \{1, 2, 3\}$  distinct. Suppose now that the  $f_{ij}$  are also pairwise complementary and appeal once more to Theorem 4.8 to construct three more Bianchi quadrilaterals  $(f_i, f_{ij}, f_{i(jk)}, f_{ik})$  so that  $f_{i(jk)} = \mathcal{D}_{m_k} f_{ij} = \mathcal{D}_{m_j} f_{ik}$ . The claim is that these three new maps  $f_{i(jk)}$  coincide (as do the corresponding  $\eta_{i(jk)}$ ) so that we have eight isothermic maps arranged into six Bianchi quadrilaterals with the combinatorics of a cube (see Figure 2).

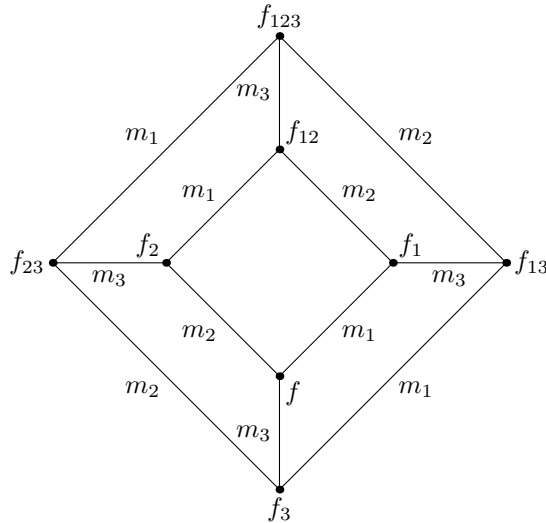


FIGURE 2. A Bianchi cube

To prove this, it is enough to show that  $f_{1(23)} = f_{2(31)}$  and that  $\eta_{1(23)} = \eta_{2(31)}$  for the rest follows by permuting indices. Now

$$\begin{aligned} f_{1(23)} &= \Gamma_{f_1}^{f_{12}} \left(1 - \frac{m_3}{m_2}\right) f_{13} \\ &= \Gamma_{f_1}^{f_{12}} \left(1 - \frac{m_3}{m_2}\right) \Gamma_f^{f_1} \left(1 - \frac{m_3}{m_1}\right) f_3 \end{aligned}$$

and similarly

$$f_{2(31)} = \Gamma_{f_2}^{f_{12}} \left(1 - \frac{m_3}{m_1}\right) \Gamma_f^{f_2} \left(1 - \frac{m_3}{m_2}\right) f_3.$$

The identity of Lemma 4.7 evaluated at  $t = m_3$  now yields

$$f_{1(23)} = f_{2(31)} = \Gamma_{f_2}^{f_1 \left( \frac{1-m_3/m_1}{1-m_3/m_2} \right)} f_3$$

and we learn a little more: the eighth surface  $f_{123}$  is concircular with  $f_1, f_2, f_3$  with constant cross-ratio:

$$\text{cr}(f_3, f_1, f_{123}, f_2) = \frac{1-m_3/m_1}{1-m_3/m_2}.$$

For the classical case of isothermic surfaces in  $S^n$ , this last statement is due to Bobenko–Suris [5].

Finally, we have  $d + t\eta_{1(23)} = \Gamma_{f_{12}}^{f_{123}} \left( 1 - \frac{t}{m_3} \right) \cdot (d + t\eta_{12}) = d + t\eta_{2(31)}$  so that  $\eta_{1(23)} = \eta_{2(31)}$ .

To summarise:

**Theorem 4.9.** *Let  $f : \Sigma \rightarrow M$  be an isothermic map and  $m_1, m_2, m_3 \in \mathbb{R}^\times$  distinct parameters. Let  $f_1, f_2, f_3$  be pairwise complementary Darboux transforms of  $f$ ,  $f_i = \mathcal{D}_{m_i} f$ , and  $f_{12}, f_{23}, f_{13}$  pairwise complementary simultaneous Darboux transforms of the  $f_i$  as in Theorem 4.8:  $f_{ij} = \mathcal{D}_{m_i} f_j = \mathcal{D}_{m_j} f_i$ . Then there is an isothermic map  $(f_{123}, \eta_{123})$  which is a simultaneous Darboux transform of each  $f_{ij}$ :  $f_{123} = \mathcal{D}_{m_k} f_{ij}$ , for  $i, j, k$  distinct.*

Moreover,  $f_{123}, f_1, f_2, f_3$  are concircular with cross-ratio  $\frac{1/m_3 - 1/m_1}{1/m_3 - 1/m_2}$ .

*Remark 4.10.* The cube configuration is highly symmetrical and, in particular, starting the analysis at  $f_{123}$  rather than  $f$ , one readily deduces that the other tetrahedron in the cube has concircular vertices with the same cross-ratio:

$$\text{cr}(f_{12}, f_{23}, f, f_{13}) = \frac{1-m_3/m_2}{1-m_3/m_1}.$$

**4.4. Permutability of Christoffel and Darboux transforms.** Theorem 3.17 realises the Christoffel transform of  $f$  as a limit of conjugated Darboux transforms. Taken together with the preceding permutability theorems for Darboux transforms, this allows us to prove the analogue of two more permutability results of Bianchi.

First we prove that Christoffel and Darboux transforms commute (c.f. Bianchi [2, §3]):

**Theorem 4.11.** *Let  $f, f^c, \hat{f} : \Sigma \rightarrow M$  be isothermic maps into a self-dual symmetric R-space with  $(f^c, \eta^c)$  a Christoffel transform of  $f$  and  $(\hat{f}, \hat{\eta})$  a Darboux transform of  $f$  with parameter  $m$ .*

*Then there is a fourth isothermic map  $\hat{f}^c : \Sigma \rightarrow M$  which is simultaneously a Christoffel transform of  $\hat{f}$  and a Darboux transform of  $f^c$  with parameter  $m$ :  $\hat{f}^c = (\hat{f})^c = \mathcal{D}_m f^c$ .*

*Proof.* Let  $f^c$  be a Christoffel transform with respect to  $(\mathfrak{p}_0, \mathfrak{p}_\infty)$  and find, via Theorem 3.17, a family  $(f_s, \eta_s)$  of Darboux transforms of  $f$  with parameter  $s$  such that

$$\lim_{s \rightarrow 0} f_s = \mathfrak{p}_\infty, \quad \lim_{s \rightarrow 0} \Gamma_{\mathfrak{p}_0}^{\mathfrak{p}_\infty}(-s) f_s = f^c, \quad \lim_{s \rightarrow 0} \Gamma_{\mathfrak{p}_0}^{\mathfrak{p}_\infty}(-s) \eta_s = \eta^c.$$

Now apply Theorem 4.8 to  $f_s, \hat{f}$  to get  $(\hat{f}_s, \hat{\eta}_s)$  with  $\hat{f}_s = \mathcal{D}_m f_s = \mathcal{D}_s \hat{f}$ . We have  $\hat{f}_s = \Gamma_{\hat{f}}^{\hat{f}}(1-s/m) f_s$  so that

$$\lim_{s \rightarrow 0} \hat{f}_s = \Gamma_{\hat{f}}^{\hat{f}}(1) \mathfrak{p}_\infty = \mathfrak{p}_\infty$$

so that Theorem 3.17 applies to the  $\hat{f}_s$  and we may define a Christoffel transform  $(\hat{f}^c, \hat{\eta}^c)$  of  $\hat{f}$  by

$$\hat{f}^c = \lim_{s \rightarrow 0} \Gamma_{\mathfrak{p}_0}^{\mathfrak{p}_\infty}(-s) \hat{f}_s, \quad \hat{\eta}^c = \lim_{s \rightarrow 0} \Gamma_{\mathfrak{p}_0}^{\mathfrak{p}_\infty}(-s) \hat{\eta}_s.$$

It remains to show that  $(\hat{f}^c, \hat{\eta}^c)$  is a Darboux transform with parameter  $m$  of  $f^c$ . By Proposition 3.11, this is the case if and only if  $\Gamma_{\hat{f}^c}^{\hat{f}^c} \left( 1 - \frac{t}{m} \right) \cdot (d + t\eta^c) = d + t\hat{\eta}^c$  for all  $t \neq m$ . However, we already know that, for each  $s$ ,  $\Gamma_{\hat{f}_s}^{\hat{f}_s} \left( 1 - \frac{t}{m} \right) \cdot (d + t\eta_s) = d + t\hat{\eta}_s$  and applying  $\Gamma_{\mathfrak{p}_0}^{\mathfrak{p}_\infty}(-s)$  to both sides yields

$$\Gamma_{\mathfrak{p}_0}^{\mathfrak{p}_\infty}(-s) \Gamma_{\hat{f}_s}^{\hat{f}_s} \left( 1 - \frac{t}{m} \right) \Gamma_{\mathfrak{p}_0}^{\mathfrak{p}_\infty}(-s) \cdot (d + t\eta_s) = \Gamma_{\mathfrak{p}_0}^{\mathfrak{p}_\infty}(-s) \cdot (d + t\hat{\eta}_s).$$

Now let  $s \rightarrow 0$  and use the last assertion of Theorem 3.17 to get  $\Gamma_{\hat{f}^c}^{\hat{f}^c} \left( 1 - \frac{t}{m} \right) \cdot (d + t\eta^c) = d + t\hat{\eta}^c$  as required.  $\square$

Similarly, we can take  $m_3 = s$  and let  $s \rightarrow 0$  in the Cube Theorem 4.9 to obtain, without further argument, the following result due to Bianchi [2, §9] in the classical case:

**Theorem 4.12.** *Let  $f, f_1, f_2, f_{12} : \Sigma \rightarrow M$  be isothermic maps in a self-dual symmetric  $R$ -space forming a Bianchi quadrilateral. Let  $f^c$  be a Christoffel transform of  $f$ .*

*Then there are Christoffel transforms  $f_1^c, f_2^c, f_{12}^c$  of  $f_1, f_2, f_{12}$  respectively so that  $f^c, f_1^c, f_2^c, f_{12}^c$  also form a Bianchi quadrilateral.*

**4.5. Discrete isothermic surfaces in self-dual symmetric  $R$ -spaces.** It is an experimental fact [7, 6] (with some theoretical underpinning [4]) that the combinatorics of the Bäcklund transforms of an integrable system provide an integrable discretisation of that system. In particular, such an analysis is available for isothermic surfaces in the conformal sphere [6, 37].

We now indicate how the same ideas may be applied to give a satisfying theory of discrete isothermic nets in self-dual symmetric  $R$ -spaces which replicates essentially all features of the smooth theory we have been developing.

For this we begin with a subset  $\Omega \subset \mathbb{Z}^2$  of a 2-dimensional lattice. If  $i, j, k, l \in \Omega$  are the vertices of an elementary quadrilateral as in Figure 3, we denote the oriented edge from  $i$  to  $j$  by  $(j, i)$  and the quadrilateral by  $(i, j, k, l)$ .

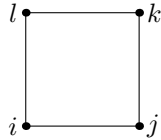


FIGURE 3. An elementary quadrilateral

Thanks to the discussion in §4.1, Hertrich-Jeromin's definition of an isothermic net in the conformal sphere [37] carries straight over into our setting.

**Definition 4.13.** Let  $f : \Omega \rightarrow M$  be a map into a self-dual symmetric  $R$ -space.

We say that  $f$  is an *isothermic net* if, for each elementary quadrilateral  $(i, j, k, l)$ , the values  $f(i), f(j), f(k), f(l)$  are concircular and pairwise complementary with

$$\text{cr}(f(i), f(j), f(k), f(l)) = m(i, l)/m(i, j),$$

for some *factorising function*  $m$  which is a real valued function on the unoriented edges of  $\Omega$  with equal values on opposite edges<sup>10</sup>.

Just as in the smooth case, isothermic nets have a zero-curvature representation: let  $(f, m)$  be such a net and let  $V = \Omega \times \mathfrak{g}$  be the trivial  $\mathfrak{g}$ -bundle. For each oriented edge  $(j, i)$  and  $t \in \mathbb{R}$ , define  $\Gamma^t(j, i) : V_i \rightarrow V_j$  by

$$\Gamma^t(j, i) = \Gamma_{f(i)}^{f(j)} \left( 1 - \frac{t}{m(i, j)} \right).$$

It is immediate that  $\Gamma^0(j, i) = 1$  while  $\Gamma^t(j, i)\Gamma^t(i, j) = 1_{V(j)}$ , so that each  $\Gamma^t$  is (the holonomy of) a discrete connection on the bundle  $V$ . A mild reorganisation of the discussion leading to the proof of Theorem 4.8 now gives:

**Theorem 4.14.**  *$(f, m)$  is an isothermic net if and only if each  $\Gamma^t$  is a flat connection: that is, on each elementary quadrilateral  $(i, j, k, l)$ , we have*

$$\Gamma^t(k, j)\Gamma^t(j, i) = \Gamma^t(k, l)\Gamma^t(l, i).$$

(See [5, 15] for this gauge-theoretic approach to isothermic nets in the conformal case.)

With this in hand, the transformation theory of isothermic nets can be developed along the same lines as the smooth case. Firstly,  $T$ -transforms arise by trivialising the family of connections  $\Gamma^t$ : locally we may find  $\Phi_t : \Omega \rightarrow \text{Aut}(\mathfrak{g})$ , unique up to left multiplication by constants in  $\text{Aut}(\mathfrak{g})$ , such that

$$\Gamma^t(j, i) = \Phi_t(j)^{-1}\Phi_t(i)$$

and then define the  $T$ -transforms  $\mathcal{T}_s f$  of  $f$  by  $\mathcal{T}_s f = \Phi_s f$ . One can show:

<sup>10</sup>Thus  $m(i, j) = m(j, i)$  for all edges and  $m(i, j) = m(l, k)$ ,  $m(i, l) = m(j, k)$  on elementary quadrilaterals.

**Theorem 4.15.** *Let  $(f, m)$  be an isothermic net. Then each  $T$ -transform  $\mathcal{T}_s f$  is an isothermic net with factorising function  $m - s$ .*

Indeed, an easy calculation shows that the connection  ${}^s\Gamma^t$  associated to  $(\mathcal{T}_s f, m - s)$  is the gauge transform<sup>11</sup> by  $\Phi_s$  of  $\Gamma^{t+s}$  and so is flat.

Again, Darboux transforms amount to parallel bundles of parabolic subalgebras: a map  $\hat{f} : \Omega \rightarrow M$  is a *Darboux transform of  $f$  with parameter  $\hat{m} \in \mathbb{R}$*  if  $\hat{f}$  is  $\Gamma^{\hat{m}}$ -parallel. Thus, for all edges  $(i, j)$ ,

$$\Gamma^{\hat{m}}(j, i)\hat{f}(i) = \hat{f}(j).$$

It follows from the definition that, for each edge  $(j, i)$ , the points  $f(i), f(j), \hat{f}(j), \hat{f}(i)$  are concircular with cross-ratio  $\hat{m}/m(i, j)$  and then Lemma 4.7 applies to show that the connections  $\hat{\Gamma}^t$  of  $(\hat{f}, m)$  are the gauges of  $\Gamma^t$  by  $i \mapsto \Gamma_{f(i)}^{\hat{f}(i)}(1 - \frac{t}{\hat{m}})$  so that:

**Theorem 4.16.**  *$\hat{f}$  is isothermic with the same factorising function  $m$ .*

Moreover, the argument of Theorem 4.9 now gives

$$\hat{f}(k) = \Gamma_{f(l)}^{f(j)} \left( \frac{1 - \hat{m}/m(i, j)}{1 - \hat{m}/m(i, l)} \right) \hat{f}(i)$$

so that  $\hat{f}(i), f(j), \hat{f}(k), f(l)$  are concircular with

$$\text{cr}(\hat{f}(i), f(j), \hat{f}(k), f(l)) = \frac{1 - \hat{m}/m(i, j)}{1 - \hat{m}/m(i, l)}.$$

In particular,  $\hat{f}(k)$  depends only on  $\hat{f}(i), f(j)$  and  $f(l)$  which is the *tetrahedron property* of Bobenko–Suris [5].

One can go further and construct Christoffel transforms via blow-ups of Darboux transforms or (with a little more work) via stereoprojections and then see that all our permutability theorems relating these transforms hold in the discrete setting (with essentially the same proofs). However, all this would take us too far afield for the present. We may return to these topics elsewhere.

## 5. CURVED FLATS AND DARBOUX PAIRS

The main result of [16] asserts that Darboux pairs of isothermic surfaces in  $S^3$  are precisely the curved flats in the symmetric space of point-pairs  $S^3 \times S^3 \setminus \Delta$ . This was extended in [13] to isothermic surfaces in  $S^n$  where the dressing transformation of curved flats was related to Darboux transforms of Christoffel pairs.

We now show that this circle of ideas carries through in our general context of isothermic maps to symmetric  $R$ -spaces. We begin by rehearsing the relevant details of the theory of curved flats.

**5.1. Curved flats and their transformations.** Let  $N$  be a symmetric  $G$ -space<sup>12</sup>: thus each  $x \in N$  has stabiliser  $H_x$  open in the fixed set of an involution  $\tau_x \in \text{Aut}(G)$ . The derivative of  $\tau_x$ , also called  $\tau_x \in \text{Aut}(\mathfrak{g})$ , has eigenvalues  $\pm 1$  with eigenspaces  $\mathfrak{h}_x, \mathfrak{m}_x$  where  $\mathfrak{h}_x$  is the Lie algebra of  $H_x$  and  $T_x N \cong \mathfrak{m}_x$  via the solder form  $\beta_x^N$ .

**Definition 5.1** ([30]). A *curved flat* in a symmetric space  $N$  is a map  $\phi : \Sigma \rightarrow N$  such that each  $\phi^* \beta^N(T_p \Sigma)$  is an abelian subalgebra of  $\mathfrak{m}_{\phi(p)}$ ,  $p \in \Sigma$ .

Curved flats have a gauge-theoretic formulation that will be basic for us. Let  $\phi : \Sigma \rightarrow N$  be a map and contemplate the section  $\tau$  of  $\text{Aut}(\mathfrak{g})$  given by  $\tau(p) = \tau_{\phi(p)}$ . There is a corresponding eigenbundle decomposition  $\mathfrak{g} = \phi^{-1}\mathfrak{h} \oplus \phi^{-1}\mathfrak{m}$  with  $\phi^{-1}\mathfrak{m} \cong \phi^{-1}TN$  and thus a decomposition of the flat connection  $d$ :

$$(5.1) \quad d = \mathcal{D} + \mathcal{N}$$

where  $\mathcal{D}\tau = 0$  while  $\mathcal{N} \in \Omega_{\Sigma}^1(\phi^{-1}\mathfrak{m})$  (explicitly,  $\mathcal{N} = -\frac{1}{2}\tau d\tau$ ). From §2.5, we have  $\mathcal{N} = \phi^* \beta^N$  so the curved flat condition is precisely

$$[\mathcal{N} \wedge \mathcal{N}] = 0.$$

<sup>11</sup>Thus  ${}^s\Gamma^t(j, i) = \Phi_s(j)\Gamma^{t+s}(j, i)\Phi_s(i)^{-1}$ , for all edges  $(j, i)$ .

<sup>12</sup>We retain our standing assumption that  $G$  is the adjoint group of  $\mathfrak{g}$  but much of the theory of curved flats carries through without it.

On the other hand, the flatness of  $d$  yields

$$0 = R^d = R^{\mathcal{D}} + d^{\mathcal{D}}\mathcal{N} + \frac{1}{2}[\mathcal{N} \wedge \mathcal{N}]$$

the  $\phi^{-1}\mathfrak{h}$ - and  $\phi^{-1}\mathfrak{m}$ -components of which are, respectively, the Gauss and Codazzi equations of the situation:

$$(5.2a) \quad 0 = R^{\mathcal{D}} + \frac{1}{2}[\mathcal{N} \wedge \mathcal{N}]$$

$$(5.2b) \quad 0 = d^{\mathcal{D}}\mathcal{N}.$$

We therefore conclude:

**Proposition 5.2.** *For a map  $\phi : \Sigma \rightarrow N$ , the following are equivalent:*

- (1)  $\phi$  is a curved flat;
- (2)  $\mathcal{D}$  is a flat connection;
- (3)  $[\mathcal{N} \wedge \mathcal{N}] = 0$ .

5.1.1. *Zero curvature representation and spectral deformation.* Curved flats comprise an integrable system which, in the case where each  $\phi^*\beta^N(T_p\Sigma)$  is maximal abelian semisimple, is gauge-equivalent to the “ $G/H$ -system” introduced by Terng [54, 11]. At the root of all this is the observation that, just like isothermic maps, curved flats are characterised by the flatness of a pencil of connections. Indeed, for  $u \in \mathbb{R}$ , define  $G$ -connections  $d^u$  on  $\underline{\mathfrak{g}}$  by

$$(5.3) \quad d^u = \mathcal{D} + u\mathcal{N}$$

so that, in particular,  $d^0 = \mathcal{D}$  and  $d^1 = d$ .

**Proposition 5.3** (c.f. [30]).  *$\phi : \Sigma \rightarrow N$  is a curved flat if and only if  $d^u$  is flat for all  $u \in \mathbb{R}$ .*

*Proof.* The coefficients of  $u$  in  $R^{d^u}$  are  $R^{\mathcal{D}}$ ,  $d^{\mathcal{D}}\mathcal{N}$  and  $\frac{1}{2}[\mathcal{N} \wedge \mathcal{N}]$  of which  $d^{\mathcal{D}}\mathcal{N}$  vanishes automatically from the Codazzi equation (5.2b) while, by Proposition 5.2, the others vanish exactly if  $\phi$  is a curved flat.  $\square$

We may now argue as in §3.3 to see that curved flats come in 1-parameter families: for  $u \in \mathbb{R}$ , we can trivialise  $d^u$ : locally we can find gauge transformations  $\Psi_u$ , unique up to left multiplication by a constant element of  $G$ , with  $\Psi_u \cdot d^u = d$ . We have:

**Proposition 5.4.**  *$\phi_u := \Psi_u \phi : \text{dom}(\Psi_u) \subset \Sigma \rightarrow N$  is a curved flat.*

*Moreover, for  $u, v \in \mathbb{R}$ ,  $(\phi_u)_v = \phi_{uv}$  modulo the action of  $G$ .*

*Proof.* We have  $\Psi_u(\phi^{-1}\mathfrak{h}) = \phi_u^{-1}\mathfrak{h}$ ,  $\Psi_u(\phi^{-1}\mathfrak{m}) = \phi_u^{-1}\mathfrak{m}$  so that in the decomposition of  $d$  with respect to  $\underline{\mathfrak{g}} = \phi_u^{-1}\mathfrak{h} \oplus \phi_u^{-1}\mathfrak{m}$ :

$$d = \mathcal{D}^u + \mathcal{N}^u$$

we have  $\mathcal{D}^u = \Psi_u \cdot \mathcal{D}$ ,  $\mathcal{N}^u = \Psi_u \mathcal{N}$  and thus  $\Psi_u \cdot (\mathcal{D} + uv\mathcal{N}) = \mathcal{D}^u + v\mathcal{N}^u$ . In particular  $\mathcal{D}^u + v\mathcal{N}^u$  is flat for all  $v \in \mathbb{R}$  so that  $\phi_u$  is a curved flat by Proposition 5.3.

Further, with  $\Psi_v \cdot (\mathcal{D}^u + v\mathcal{N}^u) = d$ , we have  $\Psi_{uv} = \Psi_v \Psi_u$  whence  $\phi_{uv} = (\phi_u)_v$ , up to the action of  $G$ .  $\square$

We call  $\{\phi_u : u \in \mathbb{R}\}$  the *associated family* or *spectral deformation* of  $\phi$ . We note that  $\phi_0$  is constant.

5.1.2. *Dressing transformations.* There is another class of transformations of curved flats which shares some of the structure of the Darboux transformations discussed above in that the initial data is a parallel bundle of height one parabolic subalgebras with respect to a certain connection. However, for curved flats, the construction of the new curved flat from this data is somewhat more elaborate.

We begin with a curved flat  $\phi : \Sigma \rightarrow N$  with associated field of involutions  $\tau \in \Gamma \text{Aut}(\mathfrak{g})$  and pencil of flat connections  $d^u$ ,  $u \in \mathbb{R}$ . We extend the definition of  $d^u$  by taking  $u \in \mathbb{C}$  to get a holomorphic pencil of  $G^{\mathbb{C}}$ -connections on  $\underline{\mathfrak{g}}^{\mathbb{C}}$ . We observe that this pencil is uniquely characterised by the following properties:

1.  $u \rightarrow d^u$  is holomorphic on  $\mathbb{C}$  with a simple pole at  $\infty$ ;



2.  $\tau \circ d^u \circ \tau^{-1} = d^{-u}$ , for  $u \in \mathbb{C}$ ;
3.  $\overline{d^u} = d^{\bar{u}}$ , for  $u \in \mathbb{C}$ ;
4.  $d^1 = d$ .

Now let  $w \in \mathbb{C}^\times \setminus \{\pm 1\}$  with  $w^2 \in \mathbb{R}$  and suppose we are given a bundle of height one parabolic subalgebras  $\mathfrak{r} \leq \underline{\mathfrak{g}}^{\mathbb{C}}$  with the following properties:

- (a)  $\mathfrak{r}$  is  $d^w$ -parallel;
- (b)  $\mathfrak{r}$  and  $\tau\mathfrak{r}$  are complementary;
- (c)  $\mathfrak{r} = \bar{\mathfrak{r}}$  if  $w \in \mathbb{R}$  and  $\mathfrak{r} = \tau\bar{\mathfrak{r}}$  if  $w \in i\mathbb{R}$ .

*Remark 5.5.* The local existence of such  $\mathfrak{r}$  is a point-wise affair: given a height one parabolic subalgebra  $\mathfrak{r}_0 \leq \mathfrak{g}$  and  $p \in \Sigma$  with  $\mathfrak{r}_0, \tau(p)\mathfrak{r}_0$  complementary and satisfying condition (c), one can define  $\mathfrak{r}$  by parallel transport since  $d^u$  is flat and preserves condition (c). However, the existence of such a  $\mathfrak{r}_0$  imposes strong restrictions on the symmetric space  $N$ . We have already noted that the existence of any height one parabolic subalgebra  $\mathfrak{r}_0 \leq \mathfrak{g}^{\mathbb{C}}$  excludes  $\mathfrak{g} = \mathfrak{g}_2, \mathfrak{f}_4$  or  $\mathfrak{e}_8$  and conditions (b) and (c) further restrict the possibilities for  $\tau(p)$ . Thus, for example, for  $N = G_k(\mathbb{K}^n)$  a Grassmannian for  $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ , viewed as a Riemannian symmetric space, such  $\mathfrak{r}_0$  are only available when  $n = 2k$ .

Given such a bundle  $\mathfrak{r}$ , we define the family of gauge transformations  $\Gamma(u) := \Gamma_{\tau\mathfrak{r}}^{\mathfrak{r}}\left(\frac{u-w}{u+w}\right) \in \Gamma \text{Aut}(\underline{\mathfrak{g}}^{\mathbb{C}})$  and observe that  $\Gamma(u)$  enjoys similar properties to  $d^u$ :

- i.  $u \mapsto \Gamma(u)$  is holomorphic on  $\mathbb{C}P^1$  except for simple poles at  $\pm w$ ;
- ii.  $\tau\Gamma(u)\tau^{-1} = \Gamma(-u)$ , for  $u \in \mathbb{C}$ ;
- iii.  $\overline{\Gamma(u)} = \Gamma(\bar{u})$ .

Now contemplate the family of connections

$$\hat{d}^u = \Gamma(u) \cdot d^u.$$

We readily check that they share properties 2 and 3 with  $d^u$  and that  $u \mapsto \hat{d}^u$  is holomorphic on  $\mathbb{C}$  except possibly at  $\pm w$ . In fact, the singularities there are removable:

**Lemma 5.6.**  $u \mapsto \hat{d}^u$  is holomorphic near  $\pm w$ .

*Proof.* We show that  $u \mapsto \hat{d}^u$  is holomorphic near  $w$  and then appeal to the symmetry  $u \mapsto -u$ . Contemplate the eigenbundle decomposition of  $\text{ad}\xi_{\tau\mathfrak{r}}^{\mathfrak{r}}$ :

$$\underline{\mathfrak{g}} = \mathfrak{r}^\perp \oplus (\mathfrak{r} \cap \tau\mathfrak{r}) \oplus \tau\mathfrak{r}^\perp$$

and the corresponding decomposition of  $d^w$ :

$$d^w = D - b - \hat{b}$$

with  $D\xi_{\tau\mathfrak{r}}^{\mathfrak{r}} = 0$ ,  $b \in \Omega^1(\mathfrak{r}^\perp)$  and  $\hat{b} \in \Omega^1(\tau\mathfrak{r}^\perp)$ . Since  $\mathfrak{r}$  is  $d^w$ -parallel and preserved by both  $D$  and  $b$ , we conclude that  $\hat{b}$  takes values in  $\mathfrak{r} \cap \tau\mathfrak{r}^\perp$  and so vanishes. Thus

$$d^u = D - b + (u - w)\mathcal{N}$$

whence

$$\begin{aligned} \hat{d}^u &= \Gamma(u) \cdot D - \Gamma(u)b + (u - w)\Gamma(u)\mathcal{N} \\ &= D - \frac{u-w}{u+w}b + (u - w)\Gamma(u)\mathcal{N}, \end{aligned}$$

which last is clearly holomorphic near  $w$ . □

Thus  $u \mapsto \hat{d}^u$  is holomorphic on  $\mathbb{C}$  and has a simple pole at  $\infty$ , since  $\Gamma$  is holomorphic and  $\text{Aut}(\underline{\mathfrak{g}})$ -valued there.

Finally, set  $\hat{\phi} := \Gamma(1)^{-1}\phi : \Sigma \rightarrow N$  with corresponding field of involutions  $\hat{\tau} := \Gamma(1)^{-1}\tau\Gamma(1)$  (we assume<sup>13</sup>, if necessary, that  $G^{\mathbb{C}} \cap \text{Aut}(\underline{\mathfrak{g}})$  acts on  $N$ ). Now  $u \mapsto \Gamma(1)^{-1} \cdot \hat{d}^u$  has properties 1–4 of  $d^u$  with  $\hat{\tau}$  replacing  $\tau$  so we conclude that this is the holomorphic pencil of connections associated with  $\hat{\phi}$ . Since these connections are flat, we deduce that  $\hat{\phi}$  is a curved flat. To summarise:

<sup>13</sup>This assumption is justified for our applications in §5.2.

**Proposition 5.7.** *Let  $\phi : \Sigma \rightarrow N$  be a curved flat and  $\mathfrak{r} \leq \underline{\mathfrak{g}}$  a bundle of height one parabolic subalgebras with properties (a)–(c) above for some  $w \in \mathbb{C}^\times \setminus \{\pm 1\}$  with  $w^2 \in \mathbb{R}$ . Then  $\hat{\phi} : \Sigma \rightarrow N$  defined by*

$$\hat{\phi} = \Gamma_{\tau\mathfrak{r}}^{\mathfrak{r}}\left(\frac{1+w}{1-w}\right)\phi$$

*is also a curved flat. We call  $\hat{\phi}$  a dressing transform of  $\phi$ .*

5.1.3. *Relationship with loop group formalism.* This construction is essentially an invariant<sup>14</sup> reformulation of the Terng–Uhlenbeck construction [55] of *dressing by simple factors*. We digress to give a brief account of this.

In the loop group formalism, one works with frames of the underlying maps into homogeneous spaces. In the case at hand, this means we fix a base-point  $o \in N$  and, for  $\phi : \Sigma \rightarrow N$ , contemplate maps  $F : \Sigma \rightarrow G$  with  $\phi = F o$ . Given such a frame  $F$ , let  $\alpha = F^{-1}dF \in \Omega_\Sigma^1(\mathfrak{g})$  and write  $\alpha = \alpha_{\mathfrak{h}_o} + \alpha_{\mathfrak{m}_o}$  for the decomposition of  $\alpha$  according to the decomposition of  $\mathfrak{g}$  into eigenspaces of  $\tau_o$ :

$$\mathfrak{g} = \mathfrak{h}_o \oplus \mathfrak{m}_o.$$

Viewing  $F$  as a gauge transformation, we see that  $F \cdot (d + \alpha) = d$  and that  $F$  intertwines the decompositions  $\underline{\mathfrak{g}} = \underline{\mathfrak{h}}_o \oplus \underline{\mathfrak{m}}_o$  and  $\underline{\mathfrak{g}} = (\phi^{-1}\mathfrak{h}) \oplus (\phi^{-1}\mathfrak{m})$ . It follows at once that  $F \cdot (d + \alpha_{\mathfrak{h}_o}) = \mathcal{D}$  and  $F \alpha_{\mathfrak{m}_o} = \mathcal{N}$  so that  $\phi$  is a curved flat if and only if  $[\alpha_{\mathfrak{m}_o} \wedge \alpha_{\mathfrak{m}_o}] = 0$ .

In this case, for  $u \in \mathbb{C}$ , set  $\alpha_u = \alpha_{\mathfrak{h}_o} + u\alpha_{\mathfrak{m}_o}$  and note that  $d + \alpha_u = F^{-1} \cdot d^u$  and so is flat. We may therefore integrate, at least locally, to find  $F_u : \Sigma \rightarrow G^\mathbb{C}$  with  $F_u^{-1}dF_u = \alpha_u$  or, equivalently,  $F_u \cdot (d + \alpha_u) = d$ . The family  $\{F_u : u \in \mathbb{C}\}$  constitute an *extended frame* of  $\phi$ .

Observe that  $(F_u F^{-1}) \cdot d^u = d$  so that  $\Psi_u = F_u F^{-1}$  giving

$$\phi_u = \Psi_u \phi = F_u o$$

so that  $F_u$  is a frame of the spectral deformation  $\phi_u$ .

The philosophy of the loop group formalism is to view the maps  $F_u$  as a single map into a loop group on which there is a (local) action of a second loop group via Birkhoff factorisation. In the case at hand, let  $\mathcal{G}_+$  denote the group (under pointwise multiplication) of holomorphic maps  $g : \mathbb{C} \rightarrow G^\mathbb{C}$  which are *real* in that  $\bar{g}(u) = g(\bar{u})$  and *twisted* in that  $\tau_o g(u) = g(-u)$ . If the constants of integration are chosen correctly, each  $u \mapsto F_u(p)$ ,  $p \in \Sigma$ , is an element of  $\mathcal{G}_+$ . Now let  $\mathcal{G}_-$  be the group of real, twisted rational maps  $g$  on  $\mathbb{C}P^1$  with values in  $G^\mathbb{C}$  which are holomorphic near  $\infty$  with  $g(\infty) = 1$ . It follows from the Birkhoff decomposition theorem [52] that a generic real, twisted  $G^\mathbb{C}$ -valued holomorphic map  $g$  on  $\mathbb{C}P^1 \setminus D$ ,  $D$  a divisor, has a unique factorisation

$$g = g_+ g_-$$

with  $g_\pm \in \mathcal{G}_\pm$ . This leads to a local action of  $\mathcal{G}_-$  on  $\mathcal{G}_+$  by

$$g_- \# g_+ = (g_- g_+)_+$$

which can be shown to preserve the set of extended frames.

In general, the Birkhoff factorisation is non-local and so difficult to compute but the key observation of Terng and Uhlenbeck [55, 58] is that, for certain basic elements of  $\mathcal{G}_-$ , the *simple factors*, the factorisation can be carried out explicitly (see [13, §4.3] for a conceptual discussion of this). For our setting, we define the simple factors as follows: let  $\mathfrak{r}_o \leq \mathfrak{g}^\mathbb{C}$  be a parabolic subalgebra of height one and  $w \in \mathbb{C}^\times \setminus \{\pm 1\}$  such that  $(\mathfrak{r}_o, \tau_o \mathfrak{r}_o)$  are a complementary pair and  $\mathfrak{r} = \bar{\mathfrak{r}}_o$ , for  $w \in \mathbb{R}$  and  $\mathfrak{r} = \tau_o \bar{\mathfrak{r}}_o$  otherwise. Our simple factor is now defined to be  $u \mapsto \Gamma_o(u) := \Gamma_{\tau_o \mathfrak{r}_o}^{\mathfrak{r}_o} \left( \frac{u-w}{u+w} \right)$ . One can show (see [13, Proposition 4.11] for the case where  $\mathfrak{g} = \mathfrak{so}(n+1, 1)$  and  $\mathfrak{r}_o$  is the stabiliser of a null line in  $(\mathbb{R}^{n+1, 1})^\mathbb{C}$ ) that here the Birkhoff factorisation is given by

$$\Gamma_o g_+ = \hat{g}_+ \hat{\Gamma}$$

<sup>14</sup>Thus without the use of frames.

with  $\hat{\Gamma}(u) = \Gamma_{\tau_o \hat{\tau}}^{\hat{\tau}}(\frac{u-w}{u+w})$  and  $\hat{\tau} = g_+(w)^{-1} \tau_o$ . We apply this to the extended frame  $u \mapsto F_u$  to get a new extended frame  $\hat{F}_u$  which we evaluate at  $u = 1$  to get a new curved flat  $\hat{\phi}$ . Thus

$$\begin{aligned} \hat{\phi} &= \Gamma_{\tau_o \tau_o}^{\tau_o}(\frac{1-w}{1+w}) F(\Gamma_{\tau_o \hat{\tau}}^{\hat{\tau}}(\frac{1-w}{1+w}))^{-1} o \\ &= \Gamma_{\tau_o \tau_o}^{\tau_o}(\frac{1-w}{1+w}) F(\Gamma_{\tau_o \hat{\tau}}^{\hat{\tau}}(\frac{1-w}{1+w}))^{-1} F^{-1} \phi \\ &= \Gamma_{\tau_o \tau_o}^{\tau_o}(\frac{1-w}{1+w}) \Gamma_{F \tau_o \hat{\tau}}^{F \hat{\tau}}(\frac{1+w}{1-w}) \phi. \end{aligned}$$

Now set  $\tau = F \hat{\tau} = F F_w^{-1} \tau_o = \Psi_w^{-1} \tau_o$  which is  $d^u$ -parallel since  $\tau_o$  is constant so  $d$ -parallel. Then  $F \tau_o \hat{\tau} = \tau \tau$  since  $\tau = F \tau_o F^{-1}$  and we conclude that  $\hat{\phi}$  yielded by the dressing action coincides with that of Proposition 5.7 up to the (irrelevant) constant factor  $\Gamma_{\tau_o \tau_o}^{\tau_o}(\frac{1-w}{1+w})$ .

**5.2. Curved flats in  $Z$  are Darboux pairs.** We now apply this theory to the case where the symmetric space is the space  $Z \subset M \times M^*$  of complementary pairs of parabolic subalgebras in dual symmetric  $R$ -spaces  $M$  and  $M^*$ .

A map  $\phi = (f, \hat{f}) : \Sigma \rightarrow Z$  can be viewed as a pair of maps  $f : \Sigma \rightarrow M$ ,  $\hat{f} : \Sigma \rightarrow M^*$  which are pointwise complementary. Our first result is that  $\phi$  is a curved flat if and only if  $f$  and  $\hat{f}$  are a Darboux pair of isothermic maps. For this, first note that the field of involutions  $\tau$  along  $\phi$  is given by  $\tau = \Gamma_f^{\hat{f}}(-1)$  so that

$$\phi^{-1} \mathfrak{h} = f \cap \hat{f}, \quad \phi^{-1} \mathfrak{m} = f^\perp \oplus \hat{f}^\perp.$$

Thus the decomposition (3.5) of  $d$  coincides with the decomposition (5.1) yielding  $\mathcal{N} = -\beta - \hat{\beta}$ . With this in mind, we revisit the argument that established Theorem 3.10: we have

$$(5.4) \quad \Gamma_f^{\hat{f}}(1/u) \cdot d^u = \mathcal{D} - \beta - u^2 \hat{\beta} = d + m(1 - u^2) \hat{\beta}/m = \nabla^{m(1-u^2)}$$

and, similarly,

$$(5.5) \quad \Gamma_{\hat{f}}^f(1/u) \cdot d^u = d + m(1 - u^2) \beta/m = \hat{\nabla}^{m(1-u^2)}.$$

Thus, if  $\phi = (f, \hat{f})$  is a curved flat, each  $d^u$  is flat whence each  $\nabla^{m(1-u^2)}, \hat{\nabla}^{m(1-u^2)}$  is flat so that both  $(f, \hat{\beta}/m)$  and  $(\hat{f}, \beta/m)$  are isothermic. Clearly  $\hat{f}$  is  $\nabla^m$ -parallel while  $f$  is  $\hat{\nabla}^m$ -parallel and we conclude that they are  $m$ -Darboux transforms of each other.<sup>15</sup>

Conversely, if  $(f, \eta)$  is isothermic and  $\hat{f} = \mathcal{D}_m f$ , we know that  $\eta = \hat{\beta}/m$  so that flatness of  $\nabla^t$ , for all  $t$ , forces flatness of  $d^u$ , for all  $u \neq 0$  and so, by continuity, flatness of  $d^0$  also whence  $\phi = (f, \hat{f})$  is a curved flat. We have therefore proved:

**Theorem 5.8.** *A map  $\phi = (f, \hat{f}) : \Sigma \rightarrow Z$  is a curved flat if and only if  $f, \hat{f}$  are a Darboux pair of isothermic maps.*

**5.2.1. Spectral deformation is  $T$ -transform.** It is now straightforward to relate the spectral deformation of curved flats in  $Z$  to  $T$ -transforms of the constituent isothermic maps. Indeed, the spectral deformation of  $\phi = (f, \hat{f})$  is given by  $\phi_u = \Psi_u \phi$  where  $\Psi \cdot d^u = d$  while  $\mathcal{T}_t f = \Phi_t f$ ,  $\mathcal{T}_t \hat{f} = \hat{\Phi}_t \hat{f}$  where  $\Phi_t \cdot \nabla^t = d$  and  $\hat{\Phi}_t \cdot \hat{\nabla}^t = d$ . It follows at once from (5.4) and (5.5) that we may take

$$\Psi_u = \Phi_{m(1-u^2)} \circ \Gamma_f^{\hat{f}}(1/u) = \hat{\Phi}_{m(1-u^2)} \circ \Gamma_{\hat{f}}^f(u).$$

Since  $\Gamma_f^{\hat{f}}$  preserves both  $f$  and  $\hat{f}$  we conclude:

**Theorem 5.9.** *Let  $f, \hat{f} : \Sigma \rightarrow M$  be isothermic with  $\hat{f} = \mathcal{D}_m f$  and let  $\phi_u = (f_u, \hat{f}_u) : \Sigma \rightarrow Z$  be a spectral deformation of the curved flat  $\phi = (f, \hat{f})$ . Then, for  $u \neq 0$ ,*

$$f_u = \mathcal{T}_{m(1-u^2)} f \quad \hat{f}_u = \mathcal{T}_{m(1-u^2)} \hat{f}.$$

<sup>15</sup>Note that  $m \neq 0$  may be chosen arbitrarily.

5.2.2. *Dressing pairs of curved flats are Bianchi quadrilaterals.* Suppose now that  $M$  is self-dual and recall the discussion of Bianchi permutability in §4.2: given  $(f, \eta) : \Sigma \rightarrow M$  isothermic with pointwise complementary Darboux transforms  $f_i = \mathcal{D}_{m_i} f$ ,  $m_1 \neq m_2$ , there is a fourth isothermic map  $\hat{f}$  given by

$$\hat{f} = \Gamma_{f_2}^{f_1}(m_2/m_1)f = \Gamma_f^{f_1}(1 - \frac{m_2}{m_1})f_2 = \Gamma_f^{f_2}(1 - \frac{m_1}{m_2})f_1$$

such that  $\hat{f} = \mathcal{D}_{m_1} f_2 = \mathcal{D}_{m_2} f_1$ . We claim that the curved flat  $(f_2, \hat{f})$  is a dressing transform of  $(f, f_1)$  and that all dressing pairs of curved flats  $\Sigma \rightarrow Z \subset M \times M$  arise this way.

For this, let  $\phi = (f, f_1) : \Sigma \rightarrow Z$  with  $\tau = \Gamma_f^{f_1}(-1)$  the field of involutions along  $\phi$  and  $d^u$  the pencil of flat connections. Choose  $w \in \mathbb{C}^\times \setminus \{\pm 1\}$  such that  $m_2 = m_1(1 - w^2)$  and note that  $\tau := \Gamma_f^{f_1}(w)f_2^{\mathbb{C}}$  is  $d^w$ -parallel if and only if  $f_2$  is  $\nabla^{m_2}$ -parallel since  $\Gamma_f^{f_1}(w) \cdot \nabla^{m_2} = d^w$  by virtue of (5.4). Moreover, Lemma 4.6 tells us that  $f, f_1, f_2$  are pairwise complementary at each point if and only if  $\tau, \tau\tau$  are pointwise complementary. Finally,

$$\bar{\tau} = \Gamma_f^{f_1}(\overline{w})\overline{f_2^{\mathbb{C}}}$$

so that  $f_2^{\mathbb{C}} = \overline{f_2^{\mathbb{C}}}$  if and only if  $\bar{\tau} = \tau$  or  $\tau\tau$  according to the sign of  $w^2$  since  $\Gamma_f^{f_1}(-w) = \tau\Gamma_f^{f_1}(w)$ . Thus  $f, f_1, f_2$  define a Bianchi quadrilateral of isothermic maps if and only if  $\tau$  satisfies the conditions to define a dressing transformation. We now inspect the effect of that dressing transformation: from Proposition 4.4, we have, for all  $\lambda \in \mathbb{C} \cup \{\infty\}$ ,

$$(5.6) \quad \Gamma_{\tau\tau}^{\tau}(\frac{w+\lambda}{w-\lambda})f = \Gamma_f^{f_1}(\lambda)f_2$$

$$(5.7) \quad \Gamma_{\tau\tau}^{\tau}(\frac{\lambda+w}{\lambda-w})f_1 = \Gamma_f^{f_1}(\lambda)f_2$$

since both sides of (5.6) agree at  $\lambda = 0, \pm w$  while both sides of (5.7) agree at  $\lambda = \infty, \pm w$ . Now evaluate (5.6) at  $\lambda = w^2 = 1 - m_2/m_1$  and (5.7) at  $\lambda = 1$  to conclude that

$$\Gamma_{\tau\tau}^{\tau}(\frac{1+w}{1-w})(f, f_1) = (\hat{f}, f_2)$$

as required.

In summary:

**Theorem 5.10.** *Let  $M$  be a self-dual symmetric  $R$ -space and  $Z \subset M \times M$  the space of complementary pairs. Let  $\phi = (f, f_1) : \Sigma \rightarrow Z$  and  $\hat{\phi} = (\hat{f}, f_2) : \Sigma \rightarrow Z$  be Darboux pairs of isothermic maps with  $f_1 = \mathcal{D}_{m_1} f$ .*

*Then  $\hat{\phi}$  is a dressing transform of  $\phi$  with data  $\tau : \Sigma \rightarrow M^{\mathbb{C}}$  and  $w \in \mathbb{C}$  if and only if  $f, f_1, f_2, \hat{f}$  form a Bianchi quadrilateral. Moreover, in this case,  $\tau = \Gamma_f^{f_1}(w)f_2^{\mathbb{C}}$  and  $f_2 = \mathcal{D}_{m_2} f$  where  $m_2 = m_1(1 - w^2)$ .*

*Remark 5.11.* What can be said when  $M$  is not self-dual? Darboux pairs of isothermic maps are still curved flats and there may still be dressing transformations available although the  $\tau$  must necessarily take values in a self-dual complex  $R$ -space as  $\tau$  and  $\tau\tau$  are  $G^{\mathbb{C}}$ -conjugate (via  $\tau$ ). An example of this situation is provided by Darboux pairs of isothermic maps into Grassmannians  $f : \Sigma \rightarrow G_k(\mathbb{R}^{2n})$ ,  $f_1 : \Sigma \rightarrow G_{n-k}(\mathbb{R}^{2n})$ ,  $k < n$ , with  $\tau : \Sigma \rightarrow G_n(\mathbb{C}^{2n})$ . In such a case, the dressing transformation will provide a new Darboux pair but its relation with the original pair requires further investigation. We may return to this elsewhere.

## 6. NONDEGENERATE ISOTHERMIC SUBMANIFOLDS

6.1. **A quadratic form.** Let  $(f, \eta) : \Sigma \rightarrow M$  be an isothermic map to a symmetric  $R$ -space. Recall from §3.1 that we may view  $\eta$  as an  $f^{-1}T^*M$ -valued 1-form and so define a 2-tensor  $q_f$  on  $\Sigma$  by contracting  $\eta$  with  $df$ :  $q_f(X, Y) = \eta_X(df_Y)$ . We are about to see that  $q_f$  is symmetric and so we call it the *quadratic form associated to  $(f, \eta)$* .

This quadratic form is invariant under all the transformations of isothermic maps we have discussed:

**Proposition 6.1.** *Let  $(f, \eta) : \Sigma \rightarrow M$  be an isothermic map with associated quadratic form  $q$ . Then*

- (1)  $q$  is symmetric.
- (2) Any Christoffel, Darboux or  $T$ -transform of  $(f, \eta)$  also has associated quadratic form  $q$ .

(3) Let  $\phi = (f, \hat{f}) : \Sigma \rightarrow Z \subset M \times M^*$  be a Darboux pair with  $\hat{f} = \mathcal{D}_m f$ . Then

$$q = \frac{1}{2m} \phi^* g_Z,$$

where  $g_Z$  is the (neutral signature) metric on  $Z$  induced by the Killing form.

*Proof.* To get a convenient formulation of  $q$ , we pull back the soldering isomorphism of §2.5 to view  $df$  as a 1-form with values in  $\underline{\mathfrak{g}}/f$  so that  $q = (\eta, df)$ .

Now equation (2.3) pulls back to give

$$ds \equiv [df, s] \pmod{f},$$

for all  $s \in \Gamma f$ , and, since the fibres of  $f$  are self-normalising,  $df$  is uniquely characterised by this property.

With this in hand, let  $\hat{f} = \mathcal{D}_m f$  be a Darboux transform of  $f$  and recall the decomposition  $\underline{\mathfrak{g}} = f^\perp \oplus (f \cap \hat{f}) \oplus \hat{f}^\perp$  into eigenbundles of  $\xi_f^{\hat{f}}$  with the accompanying decomposition of connections (3.5):

$$d = \mathcal{D} - \beta - \hat{\beta}.$$

We see immediately that

$$m\hat{\eta} = \beta \equiv df \pmod{f} \qquad m\eta = \hat{\beta} \equiv d\hat{f} \pmod{\hat{f}}$$

so that  $q(X, Y) = \frac{1}{m}(\hat{\beta}_X, \beta_Y) = q_{\hat{f}}(Y, X)$ .

On the other hand, the  $f \cap \hat{f}$ -component of  $d\eta$  is  $-\beta \wedge \eta$  so that  $[\beta \wedge \hat{\beta}]$  vanishes and we have

$$\begin{aligned} q(X, Y) &= -([\xi_f^{\hat{f}}, \hat{\beta}_X], \beta_Y) = -(\xi_f^{\hat{f}}, [\hat{\beta}_X, \beta_Y]) \\ &= -(\xi_f^{\hat{f}}, [\hat{\beta}_Y, \beta_X]) = q(Y, X). \end{aligned}$$

This settles the symmetry of  $q$  and yields  $q = q_f$  also. Moreover, with  $\phi = (f, \hat{f}) : \Sigma \rightarrow Z$ , we have

$$\phi^* g_Z = (\mathcal{N}, \mathcal{N})$$

where  $\mathcal{N} = -\beta - \hat{\beta}$  is the pull-back by  $\phi$  of the solder form of  $Z$  from which assertion 3 follows.

It remains to treat Christoffel and  $T$ -transforms of  $f$ . For the first of these, let  $(f^c, \eta^c)$  be a Christoffel transform of  $(f, \eta)$  with respect to  $(\mathfrak{p}_0, \mathfrak{p}_\infty)$  with corresponding stereoprojections  $F, F^c$ . Thus  $f = \exp F \mathfrak{p}_0$  and  $\eta = \exp F dF^c$ . By writing  $s \in \Gamma f$  as  $s = \exp F \sigma$  for  $\sigma : \Sigma \rightarrow \mathfrak{p}_0$  and differentiating, one readily checks that

$$dF = \exp F dF \equiv df \pmod{f}$$

so that  $q(X, Y) = (dF_X^c, dF_Y) = q_{f^c}(Y, X)$ . We therefore deduce from the symmetry of  $q$  that  $q = q_{f^c}$ .

Finally let  $(f_t, \eta_t) = (\Phi_t f, \Phi_t \eta)$  be a  $T$ -transform of  $(f, \eta)$  where  $\Phi_t \cdot (d + t\eta) = d$ . We observe that, for  $s \in \Gamma f$ ,

$$(d + t\eta)s \equiv ds \pmod{f}$$

from which it follows that  $df_t = \Phi_t df$  and thus, since  $\Phi_t$  is isometric for the Killing form,  $q = q_{f_t}$ .  $\square$

## 6.2. Nondegenerate isothermic submanifolds.

**Definition 6.2.** We say that an isothermic map  $(f, \eta)$  is *nondegenerate* if its associated quadratic form  $q_f$  is nondegenerate.

Clearly, in this case  $f$  immerses so that  $(f, \eta)$  is an isothermic submanifold and we note from Proposition 6.1(2) that all transforms of  $(f, \eta)$  are nondegenerate also.

*Remark 6.3.* For the classical case of isothermic surfaces in  $S^n$ ,  $q$  is, in fact, a holomorphic quadratic differential (see, for example, [13, Lemma 2.1]) and so is nondegenerate off a divisor.

6.2.1. *Dimension bounds.* The dimension of a nondegenerate isothermic submanifold of a symmetric  $R$ -space  $M$  is bounded by the rank of the associated symmetric space  $Z \subset M \times M^*$  of complementary pairs.

To recall what is involved in this, we begin with a (not necessarily Riemannian) symmetric  $G$ -space  $N$  and the associated symmetric decomposition

$$\mathfrak{g} = \mathfrak{h}_x \oplus \mathfrak{m}_x,$$

for some  $x \in N$ . A *Cartan subspace* of  $\mathfrak{m}_x$  is a maximal abelian subspace  $\mathfrak{a}$  of  $\mathfrak{m}_x$  all of whose elements are semisimple. It is known that there are a finite number of  $\text{Ad } H_x$  conjugacy classes of these [49, Theorem 3] and they all have the same dimension [31, page 14]: this is the *rank* of  $N$ .

With this understood, we have:

**Theorem 6.4.** *Let  $(f, \eta) : \Sigma \rightarrow M$  be a nondegenerate isothermic submanifold and  $Z \subset M \times M^*$  the symmetric space of complementary pairs. Then*

$$(6.1) \quad \dim \Sigma \leq \text{rank } Z.$$

To prove this, let  $\hat{f} = \mathcal{D}_m f$  be a Darboux transform of  $f$  and contemplate the curved flat  $\phi = (f, \hat{f}) : \Sigma \rightarrow Z$ . From Proposition 6.1(3),  $q_f$  coincides up to scale with the metric induced by  $\phi$  so that each  $\phi^* \beta^Z(T_p \Sigma)$  is an abelian subspace of  $\mathfrak{m}_{\phi(p)}$  on which the Killing form of  $\mathfrak{g}$  is nondegenerate. Our result therefore follows from:

**Proposition 6.5.** *Let  $U \subset \mathfrak{m}_x$ ,  $x \in Z$  be an abelian subspace on which the Killing form of  $\mathfrak{g}$  is nondegenerate. Then  $\dim U \leq \text{rank } Z$  and equality holds if and only if  $U$  is a Cartan subspace of  $\mathfrak{m}_x$ .*

*Proof.* Let  $W \subset \mathfrak{m}_x$  be a maximal abelian subspace of  $\mathfrak{m}_x$  containing  $U$ . We argue as in Carleson–Toledo [22, Lemma 4.2]: recall that any  $X \in \mathfrak{g}$  has a unique Jordan decomposition  $X = X_s + X_n$  with  $[X_s, X_n] = 0$ ,  $\text{ad } X_s$  semisimple and  $\text{ad } X_n$  nilpotent. Moreover both  $\text{ad } X_s, \text{ad } X_n$  are polynomials without constant term in  $\text{ad } X$ . The uniqueness of the decomposition yields  $X_s, X_n \in \mathfrak{m}_x$  whenever  $X \in \mathfrak{m}_x$  while it follows from the maximal abelian property of  $W$ , that, for  $X \in W$ , we have  $X_s, X_n \in W$  also. Thus we may write  $W = W_s \oplus W_n$  where  $W_s, W_n$  consist respectively of the elements of  $W$  which are semisimple, respectively, nilpotent (these are linear subspaces of  $W$  since the sum of commuting semisimples is semisimple and similarly for nilpotents). Moreover, for  $X \in W_n$  and  $Y \in W$ ,  $\text{ad } X \circ \text{ad } Y$  is nilpotent so that the Killing inner product  $(X, Y) = \text{trace}(\text{ad } X \circ \text{ad } Y) = 0$  and we have  $W_n \subset W \cap W^\perp$ . Thus, since the Killing form is nondegenerate on  $U$ , we must have  $U \cap W_n = \{0\}$  so that  $\dim U \leq \dim W_s$ .

Now choose a maximal toral subspace  $\mathfrak{a}$  of  $\mathfrak{m}_x$  with  $W_s \subset \mathfrak{a}$ . Thus  $\mathfrak{a}$  is a (necessarily abelian) subspace in  $\mathfrak{m}_x$  all of whose elements are semisimple and maximal for this last property. Lepowsky–McCollum [45, Corollary to Theorem 5.2] prove that  $\mathfrak{a}$  is then maximal abelian also and so a Cartan subspace of  $\mathfrak{m}_x$ . We therefore have

$$\dim U \leq \dim W_s \leq \dim \mathfrak{a} = \text{rank } Z.$$

Moreover, in the case of equality,  $W_s = \mathfrak{a}$  is maximal abelian so that  $W_n = \{0\}$  and then  $U = \mathfrak{a}$  also.  $\square$

We can draw a geometric corollary of this development:

**Corollary 6.6.** *Let  $(f, \eta) : \Sigma \rightarrow M$  be a nondegenerate isothermic submanifold of maximal dimension:  $\dim \Sigma = \text{rank } Z$ . Then the associated quadratic form  $q_f$  is a flat pseudo-Riemannian metric on  $\Sigma$ .*

*Proof.* Let  $\hat{f}$  be a Darboux transform of  $f$  and again contemplate  $\phi = (f, \hat{f}) : \Sigma \rightarrow Z$ . We know that  $q_f$  is, up to scale, the metric induced on  $\Sigma$  by  $\phi$  while Proposition 6.5 tells us that the soldering image of each  $d\phi(T_p M)$  is a Cartan subspace. That the metric induced by  $\phi$  is flat is now a result of Ferus–Pedit [30, Theorem 2] (see also Remark 1 of that paper).  $\square$

*Remark 6.7.* Corollary 6.6 provides distinguished coordinates on nondegenerate isothermic submanifolds. For isothermic surfaces in  $S^n$ , these include the conformal curvature line coordinates that

provided the original definition of an isothermic surface [9, 24]. It seems likely that similar coordinates may be available, at least for self-dual  $M$ , which are related to the *generalised conformal structure* defined on  $M$  by Gindikin–Kaneyuki [33]. We may return to this elsewhere.

For  $\mathfrak{g}$  simple, we can readily compute  $\text{rank } Z$  in terms of more familiar invariants. To do this, we define the rank of a symmetric  $R$ -space  $M$  to be the rank of  $M$  when viewed as a compact Riemannian symmetric space with isometry group a maximal compact subgroup  $K$  of  $G$ . In more detail, the symmetric decomposition of  $\mathfrak{k}$  at  $\mathfrak{p} \in M$  is given by

$$\mathfrak{k} = (\mathfrak{p} \cap \theta\mathfrak{p} \cap \mathfrak{k}) \oplus (\mathfrak{p}^\perp \oplus \theta\mathfrak{p}^\perp) \cap \mathfrak{k},$$

where  $\theta$  is the Cartan involution fixing  $\mathfrak{k}$ , and the rank of  $M$  is the dimension of a maximal abelian subspace of the second summand.

We now have:

**Proposition 6.8.** *Let  $\mathfrak{g}$  be simple and  $M$  be a symmetric  $R$ -space for  $G$  with  $Z \subset M \times M^*$  the symmetric space of complementary pairs.*

- (1) *If  $\mathfrak{g}$  is complex so that  $M$  is a Hermitian symmetric  $K$ -space, then  $\text{rank } Z = 2 \text{rank } M$ .*
- (2) *Otherwise  $\text{rank } Z = \text{rank } M^{\mathbb{C}}$ .*

*Proof.* We begin with  $\mathfrak{g}$  complex so that  $\mathfrak{g} = \mathfrak{k}^{\mathbb{C}}$  with  $\mathfrak{k}$  a compact simple Lie algebra and the corresponding Cartan involution is just complex conjugation across  $\mathfrak{k}$ . At  $x = (\mathfrak{p}, \bar{\mathfrak{p}}) \in Z$ , we have  $\mathfrak{m}_x = \mathfrak{p}^\perp \oplus \bar{\mathfrak{p}}^\perp = (\mathfrak{m}_x \cap \mathfrak{k})^{\mathbb{C}}$ . Let  $\mathfrak{a}$  be a Cartan subspace of  $\mathfrak{m}_x$ . Since the Lie bracket on  $\mathfrak{g}$  is complex linear,  $\mathfrak{a}$  is necessarily a complex subspace. Now, let  $\mathfrak{c}$  be a Cartan subspace of  $\mathfrak{m}_x \cap \mathfrak{k}$ . Then  $\mathfrak{c}^{\mathbb{C}}$  is another Cartan subspace of  $\mathfrak{m}_x$  and, by a theorem of Kostant–Rallis [44, Theorem 1], all such Cartan subspaces are  $G^x$ -conjugate and so have the same dimension. Thus

$$\text{rank } Z = \dim_{\mathbb{R}} \mathfrak{a} = 2 \dim_{\mathbb{C}} \mathfrak{a} = 2 \dim_{\mathbb{C}} \mathfrak{c}^{\mathbb{C}} = 2 \dim_{\mathbb{R}} \mathfrak{c} = 2 \text{rank } M.$$

Now consider the case where  $\mathfrak{g}$  is not complex. Let  $(\mathfrak{p}, \mathfrak{q}) \in Z$  and  $\mathfrak{a} \subset \mathfrak{p}^\perp \oplus \mathfrak{q}^\perp = \mathfrak{m}_{(\mathfrak{p}, \mathfrak{q})}$  a Cartan subspace so that  $\text{rank } Z = \dim \mathfrak{a}$ . Then it is easy to see that  $\mathfrak{a}^{\mathbb{C}} \subset \mathfrak{m}_{(\mathfrak{p}, \mathfrak{q})}^{\mathbb{C}} = \mathfrak{m}_{(\mathfrak{p}^{\mathbb{C}}, \mathfrak{q}^{\mathbb{C}})}$  is a Cartan subspace at  $(\mathfrak{p}^{\mathbb{C}}, \mathfrak{q}^{\mathbb{C}}) \in Z^{\mathbb{C}} \subset M^{\mathbb{C}} \times (M^{\mathbb{C}})^*$ . Thus, the first part of the proposition applied to  $M^{\mathbb{C}}$  yields

$$2 \text{rank } Z = \dim_{\mathbb{R}} \mathfrak{a}^{\mathbb{C}} = \text{rank } Z^{\mathbb{C}} = 2 \text{rank } M^{\mathbb{C}}$$

whence the result.  $\square$

The outcome of this analysis is shown in Table 1 on page 33.

6.2.2. *Existence.* We now show that the dimension bounds of the last section are sharp: for any symmetric  $R$ -space  $M$ , we can find a nondegenerate isothermic submanifold of maximal dimension and so, by applying the transformation theory of §3, infinitely many such.

For this, let  $(\mathfrak{p}_0, \mathfrak{p}_\infty) \in M \times M^*$  be a complementary pair and choose a Cartan subspace  $\mathfrak{a} \subset \mathfrak{p}_0^\perp \oplus \mathfrak{p}_\infty^\perp$ . Define  $F : \mathfrak{a} \rightarrow \mathfrak{p}_\infty^\perp$  and  $F^c : \mathfrak{a} \rightarrow \mathfrak{p}_0^\perp$  to be the restrictions to  $\mathfrak{a}$  of the projections onto  $\mathfrak{p}_\infty^\perp, \mathfrak{p}_0^\perp$  along  $\mathfrak{p}_0^\perp, \mathfrak{p}_\infty^\perp$ , respectively. Then both  $F, F^c$  are injective immersions ( $\mathfrak{a}$  has no intersection with either  $\mathfrak{p}_0^\perp$  or  $\mathfrak{p}_\infty^\perp$  since all its elements are semisimple). For any  $X \in T_H \mathfrak{a}$ , we have  $dF_H(X) + dF_H^c(X) = F(X) + F^c(X) = X$  so that, since  $\mathfrak{a}, \mathfrak{p}_0^\perp, \mathfrak{p}_\infty^\perp$  are all abelian, we have

$$[dF \wedge dF^c] = 0.$$

Thus  $(F, F^c)$  is the stereoprojection of a Christoffel pair of isothermic submanifolds. Otherwise said, define  $f : \mathfrak{a} \rightarrow M$  and  $\eta \in \Omega_{\mathfrak{a}}^1(f/f^\perp)$  by

$$f = \exp(F) \mathfrak{p}_0, \quad \eta = \exp(F) dF^c$$

and conclude that  $(f, \eta)$  is an isothermic submanifold. Moreover, in the proof of Proposition 6.1, we saw that  $q_f(X, Y) = (dF_X, dF_Y^c) = (dF_Y, dF_X^c)$  but this last is just  $\frac{1}{2}(X, Y)$  so that  $q_f$  is nondegenerate.

We can also write down an explicit gauge transformation relating the flat connections  $\nabla^t = d + t\eta$  to the trivial connection so that the computation of iterated Darboux and  $T$ -transforms of  $(f, \eta)$  is

a purely algebraic matter. Indeed,  $\nabla^t = \exp(F) \cdot (d + dF + tdF^c)$  and we observe that  $F + tF^c$  takes values in a fixed abelian (in fact, Cartan, for  $t \neq 0$ ) subspace so that

$$d = \exp(F + tF^c) \cdot (d + dF + tdF^c) = \exp(F + tF^c) \exp(-F) \cdot \nabla^t$$

To summarise:

**Theorem 6.9.** *Let  $M$  be a symmetric  $R$ -space,  $(\mathfrak{p}_0, \mathfrak{p}_\infty) \in M \times M^*$  be a complementary pair and  $\mathfrak{a} \subset \mathfrak{p}_0^\perp \oplus \mathfrak{p}_\infty^\perp$  a Cartan subspace.*

*Then  $(f, \eta) : \mathfrak{a} \rightarrow M$  defined as above is a nondegenerate isothermic submanifold of maximal dimension with*

$$\exp(F + tF^c) \exp(-F) \cdot (d + t\eta) = d.$$

**6.3. Examples.** The main motivating example for our theory is the case when  $M$  is the projectivisation of some real quadric where we recover the rich theory of isothermic surfaces in conformal spheres of arbitrary signature. We conclude our account by briefly contemplating what is known for some other classical symmetric  $R$ -spaces.

**6.3.1. Projective spaces and higher flows.** It follows from the results of §6.2.1 that a nondegenerate isothermic submanifold of  $\mathbb{R}P^n$  is necessarily a curve. Moreover any curve  $f$  in  $\mathbb{R}P^n$  is isothermic with respect to any  $\eta \in \Omega^1(f^{-1}T^*\mathbb{R}P^n)$ . In particular, in sharp contrast to the conformal case,  $\eta$  is not uniquely determined by  $f$ .

Similarly, a non-degenerate isothermic submanifold of  $\mathbb{C}P^n$  of maximal dimension is a surface and, in fact, it is not difficult to see that it must be a holomorphic curve. Again, any holomorphic curve  $f$  in  $\mathbb{C}P^n$  is isothermic with respect to any holomorphic  $\eta \in \Omega^1(f^{-1}T^*\mathbb{C}P^n)$ .

However, even the case  $n = 1$  is not completely banal if we consider the *dynamics* of isothermic curves. For this, contemplate a non-degenerate isothermic curve  $(f, \eta) : \Sigma \rightarrow \mathbb{R}P^1$ , viewed as a line subbundle of a trivial  $\mathbb{R}^2$ -bundle over a 1-manifold  $\Sigma$ . The 1-form  $\eta$  determines a coordinate  $x$  on  $\Sigma$  (unique up to sign and translations) for which  $q_f = dx^2$ : thus, for any  $\psi \in \Gamma f$ ,

$$\eta(\partial/\partial x)\psi_x = \psi.$$

We normalise  $\psi$  so that  $\psi \wedge \psi_x$  is a constant section of  $\wedge^2 \underline{\mathbb{R}}^2$ . It follows that  $\psi \wedge \psi_{xx} = 0$  so that

$$(6.2) \quad \psi_{xx} = p\psi$$

where  $p$ , so defined, is the *projective curvature* of  $f$ .

Following Pinkall [51] (see also [17, 21, 34]) we suppose now that  $f$  evolves so that

$$(6.3) \quad f_t = pf_x.$$

Thus we have a map  $f : \Sigma \times I \rightarrow \mathbb{R}P^1$  and we again contemplate the normalised lift  $\psi \in \underline{\mathbb{R}}^2_{\Sigma \times I}$  with  $\psi \wedge \psi_x$  constant. This last, together with (6.3), yields

$$\psi_t = -\frac{p_x}{2}\psi + p\psi_x$$

and then  $\psi_{xxt} = \psi_{xtx}$  yields the KdV equation:

$$(6.4) \quad p_t = -\frac{p_{xxx}}{2} + 3pp_x.$$

Moreover, the converse is true and any solution of (6.4) gives rise, at least locally, to  $f : \Sigma \times I \rightarrow \mathbb{R}P^1$ , unique up to the action of  $\text{PSL}(2, \mathbb{R})$ , solving (6.3).

The key point now is that this flow of isothermic curves commutes with the transformation theory of §§3–4 and so provides symmetries of the KdV equation. For this, we extend the connections  $\nabla^m = d + m\eta$  on  $\underline{\mathbb{R}}^2_\Sigma$  in the  $t$ -direction to get connections  $\nabla^m$  on  $\underline{\mathbb{R}}^2_{\Sigma \times I}$  by

$$\begin{aligned} \nabla_{\partial/\partial t}^m \psi &= -\frac{p_x}{2}\psi + (p - 2m)\psi_x \\ \nabla_{\partial/\partial t}^m \psi_x &= \left(-\frac{p_{xx}}{2} + (p - 2m)(p + m)\right)\psi + \frac{p_x}{2}\psi_x. \end{aligned}$$

One readily checks that the connections  $\nabla^m$  are flat for all  $m \in \mathbb{R}$  exactly when (6.4) holds. In fact, these connections give the AKNS zero-curvature formulation of KdV (see, for example [32]), albeit in a less familiar gauge.



Now fix  $\hat{m} \in \mathbb{R}^\times$  and let  $\hat{f}$  be a  $\nabla^{\hat{m}}$ -parallel complement to  $f$ :

$$\underline{\mathbb{R}}_{\Sigma \times I}^2 = f \oplus \hat{f}.$$

In particular, for each  $t \in I$ ,  $\hat{f}|_{\Sigma \times \{t\}} = \mathcal{D}_{\hat{m}f}|_{\Sigma \times \{t\}}$ . To analyse  $\hat{f}$ , we choose  $\hat{\psi} \in \Gamma \hat{f}$  with  $\psi \wedge \hat{\psi} = \psi \wedge \psi_x$  so that

$$(6.5) \quad \hat{\psi} = a\psi + \psi_x.$$

One computes that  $\hat{f}$  is  $\nabla^{\hat{m}}$ -parallel if and only if

$$(6.6a) \quad a^2 - a_x - \hat{m} = p$$

$$(6.6b) \quad a_t - \frac{p_{xx}}{2} + p^2 - ap_x - a^2p = -\hat{m}(2a^2 - 2\hat{m} - p)$$

We recognise (6.6a) as the Miura transform and from it deduce first that  $\hat{\psi}$  is a normalised section of  $\hat{f}$ :  $\hat{\psi} \wedge \hat{\psi}_x = \hat{m}\psi \wedge \psi_x$  is constant; then that the projective curvature  $\hat{p}$  of  $\hat{f}$  is given by

$$\hat{p} = a^2 + a_x - \hat{m} = p + 2a_x.$$

Moreover, it is not difficult to check that

$$\begin{aligned} \hat{\psi}_x &\equiv -\hat{m}\psi \pmod{\hat{f}} \\ \hat{\psi}_t &\equiv -\hat{m}\hat{p}\psi \pmod{\hat{f}} \end{aligned}$$

so that  $\hat{f}_t = \hat{p}\hat{f}_x$  whence  $\hat{p}$  is a new solution of the KdV equation (6.4). Thus  $\hat{f}$  also gives rise to a family  $\hat{\nabla}^{\hat{m}}$  of flat connections on  $\underline{\mathbb{R}}_{\Sigma \times I}^2$  and one can verify that the identity of Theorem 3.10 holds in this extended context:

$$\Gamma_f^{\hat{f}}(1 - m/\hat{m}) \cdot \nabla^m = \nabla^{\hat{m}}$$

and then argue exactly as in §4.2 to establish Bianchi permutability of our extended Darboux transformations.

Of course, this transformation of KdV solutions is not new: it is the Bäcklund transformation discovered by Wahlquist–Estabrook [59] who used precisely the system (6.6). Following [59], we can eliminate  $p$  from (6.6) and arrive at an mKdV equation for  $a$ :

$$(6.7) \quad a_t = -\frac{a_{xxx}}{2} + 3(a^2 - \hat{m})a_x.$$

Conversely, any solution  $a$  of (6.7) gives rise to a Bäcklund pair  $(p, \hat{p})$  of KdV solutions. All this also admits a geometric interpretation: consider the map  $\phi = (f, \hat{f}) : \Sigma \times I \rightarrow Z = \mathbb{R}P^1 \times \mathbb{R}P^1 \setminus \Delta$  into the symmetric space of complementary pairs—a space-form with indefinite metric. For fixed  $t$ ,  $\phi$  is a curve of constant velocity,  $(\phi_x, \phi_x) = -4\hat{m}$  and curvature  $\kappa$  given by

$$\kappa = a/\sqrt{|\hat{m}|}$$

so that the mKdV equation is also an equation on a curvature.

Taken as a whole,  $\phi$  evolves by

$$(6.8) \quad \phi_t = (a^2 - \hat{m})\phi_x - 2\sqrt{|\hat{m}|}a_x n,$$

for  $n$  a positively oriented unit vector field orthogonal to  $\phi_x$ . This may be viewed as a higher flow of the curved flat system and any solution has  $a$  solving the mKdV equation (6.7). Conversely, a solution of (6.7) locally determines  $\phi$  solving (6.8) and then solutions  $(f, \hat{f})$  of (6.3).

To summarise: the KdV and mKdV flows correspond, via appropriate curvatures, to flows on isothermic curves and curved flats respectively and then the Miura transform relating the curvatures amounts to projecting the curved flats onto the isothermic curves.

A similar analysis is available in higher dimensions starting from the observation that the Davey–Stewartson equations amount to a flow on conformal immersions of a surface in  $S^4$  which preserves the class of isothermic immersions [17]. We shall return to this elsewhere.

6.3.2. *Curved flats in symmetric  $R$ -spaces.* Recall that if  $\theta \in \text{Aut}(\mathfrak{g})$  is the Cartan involution of the compact real form  $K$ , then we have a  $K$ -equivariant inclusion  $\iota_\theta : M \rightarrow Z$  into the associated symmetric space of complementary pairs given by  $\mathfrak{p} \mapsto (\mathfrak{p}, \theta\mathfrak{p})$ . The solder forms of  $M$  and  $Z$  are related by  $\beta^M = \iota_\theta^* \beta^Z$  so that if  $f : \Sigma \rightarrow M$  is a curved flat in  $M$ , where the latter is viewed as a Riemannian symmetric  $K$ -space, then  $\iota_\theta \circ f = (f, \theta f) : \Sigma \rightarrow Z$  is a curved flat. Now Theorem 5.8 applies and we conclude that  $f : \Sigma \rightarrow M$  is isothermic and  $\theta f$  is a Darboux transform of  $f$ .

Moreover, it is often the case that the rank of  $M$ , *qua* Riemannian symmetric  $K$ -space, coincides<sup>16</sup> with that of  $Z$ :  $\text{rank } M = \text{rank } Z$ . In this situation, curved flats of maximal rank in  $M$  are non-degenerate isothermic submanifolds.

Thus we obtain, for example, isothermic submanifolds of the Grassmannian  $G_{k+1}(\mathbb{R}^{n+1})$  from the Gauss maps of isometric immersions of certain  $k$ -dimensional space-forms into  $S^n$  [30] and isothermic submanifolds of the Lagrangian Grassmannian  $\text{Lag}(\mathbb{R}^{2n})$  from Egoroff nets and the Gauss maps of flat Lagrangian submanifolds in  $\mathbb{C}^n$  and  $\mathbb{C}\mathbb{P}^{n-1}$  [56].

In fact, it is not even necessary to require that  $\theta$  is a Cartan involution: all that is needed is to restrict attention to the open subset  $\Omega_\theta = \{\mathfrak{p} \in M : (\mathfrak{p}, \theta\mathfrak{p}) \in Z\}$ , for an arbitrary involution  $\theta \in \text{Aut}(\mathfrak{g})$ . For example, the Grassmannians  $G_{k,l}(\mathbb{R}^{p,q}) \subset G_{k+l}(\mathbb{R}^{p+q})$  of signature  $(k, l)$  subspaces of  $\mathbb{R}^{p,q}$  are of this kind and the many examples of curved flats in these Grassmannians [11, 13, 14, 30] are all isothermic submanifolds of the real Grassmannian  $G_{k+l}(\mathbb{R}^{p+q})$ .

#### APPENDIX A. SUMMARY OF SIMPLE SYMMETRIC $R$ -SPACES

Table 1 lists the symmetric  $R$ -spaces for simple  $G$ . For each such space  $M$ , we give the group  $G$ ; the realisation of  $M$  as a Riemannian symmetric  $K$ -space (recall that  $K$  is a maximal compact subgroup of  $G$ ); the dimension of  $M$ ; whether  $M$  is self-dual and rank  $Z$ , the maximal dimension of a nondegenerate isothermic submanifold of  $M$ .

$M$	$G$	$K$ -space	$\dim M$	$M = M^*$	rank $Z$
$G_p(\mathbb{C}^{p+q})$	$\text{SL}(p+q, \mathbb{C})$	$\frac{\text{SU}(p+q)}{\text{SU}(p) \times \text{SU}(q) \times S^1}$	$2pq$	$p = q$	$2 \min(p, q)$
$G_p(\mathbb{R}^{p+q})$	$\text{SL}(p+q)$	$\frac{\text{SO}(p+q)}{\text{SO}(p) \times \text{SO}(q) \times \mathbb{Z}_2}$	$pq$	$p = q$	$\min(p, q)$
$G_p(\mathbb{H}^{p+q})$	$\text{SU}^*(2p+2q)$	$\frac{\text{Sp}(p+q)}{\text{Sp}(p) \times \text{Sp}(q)}$	$4pq$	$p = q$	$2 \min(p, q)$
$G_n^\perp(\mathbb{C}^{n,n})$	$\text{SU}(n, n)$	$\frac{\text{SU}(n) \times \text{SU}(n) \times S^1}{\text{SU}(n)} = \text{U}(n)$	$n^2$	Yes	$n$
$\mathbb{P}(\mathcal{L}^\mathbb{C})$	$\text{SO}(n+2, \mathbb{C})$	$\frac{\text{SO}(n+2)}{\text{SO}(n) \times \text{SO}(2) \times \mathbb{Z}_2}$	$2n$	Yes	4
$\mathbb{P}(\mathcal{L}^{p+1, q+1})$	$\text{SO}(p+1, q+1)$	$\frac{\text{SO}(p+1) \times \text{SO}(q+1)}{\text{SO}(p) \times \text{SO}(q) \times \mathbb{Z}_2}$	$p+q$	Yes	2
$J^\pm(\mathbb{C}^{2n})$	$\text{SO}(2n, \mathbb{C})$	$\text{SO}(2n)/\text{U}(n)$	$n(n-1)$	$n$ even	$2 \lfloor n/2 \rfloor$
$J^\pm(\mathbb{R}^{n,n})$	$\text{SO}(n, n)$	$\frac{\text{SO}(n) \times \text{SO}(n)}{\text{SO}(n)} = \text{SO}(n)$	$\frac{1}{2}n(n-1)$	$n$ even	$\lfloor n/2 \rfloor$
$J(\mathbb{H}^{2n})$	$\text{SO}^*(4n)$	$\text{U}(2n)/\text{Sp}(n)$	$n(2n-1)$	Yes	$n$
$\text{Lag}(\mathbb{C}^{2n})$	$\text{Sp}(n, \mathbb{C})$	$\text{Sp}(n)/\text{U}(n)$	$n(n+1)$	Yes	$2n$
$\text{Lag}(\mathbb{R}^{2n})$	$\text{Sp}(n, \mathbb{R})$	$\text{U}(n)/\text{SO}(n)$	$\frac{1}{2}n(n+1)$	Yes	$n$
$\text{Lag}(\mathbb{H}^{2n})$	$\text{Sp}(n, n)$	$\frac{\text{Sp}(n) \times \text{Sp}(n)}{\text{Sp}(n)} = \text{Sp}(n)$	$n(2n+1)$	Yes	$2n$
$\mathbb{P}^2(\mathbb{O})$	$E_6^\mathbb{C}$	$E_6/\text{Spin}(10) \times S^1$	32	No	4
	$E_6I$	$\text{Sp}(4)/\text{Sp}(2) \times \text{Sp}(2)$	16	No	2
	$E_6IV$	$F_4/\text{Spin}(9)$	16	No	2
	$E_7^\mathbb{C}$	$E_7/E_6 \times S^1$	54	Yes	6
	$E_7V$	$\text{SU}(8)/\text{Sp}(4)$	27	Yes	3
	$E_7VII$	$E_6 \times S^1/F_4$	27	Yes	3

TABLE 1. The symmetric  $R$ -spaces of simple type

<sup>16</sup>Indeed, among the simple symmetric  $R$ -spaces,  $\text{rank } M = \text{rank } Z$  for all the non-complex examples except the conformal sphere  $\mathbb{P}(\mathcal{L}^{n+1,1})$ , the Cayley plane, the quaternionic Grassmannians and the quaternionic Lagrangian Grassmannian.

## REFERENCES

- [1] Robert J. Baston and Michael G. Eastwood, *The Penrose transform*, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 1989, Its interaction with representation theory, Oxford Science Publications. MR1038279 (92j:32112)
- [2] ———, *Ricerche sulle superficie isoterme e sulla deformazione delle quadriche*, Ann. di Mat. **11** (1905), 93–157.
- [3] L. Bianchi, *Complimenti alle ricerche sulle superficie isoterme*, Ann. di Mat. **12** (1905), 19–54.
- [4] A. I. Bobenko, D. Matthes, and Yu. B. Suris, *Nonlinear hyperbolic equations in surface theory: integrable discretizations and approximation results*, Algebra i Analiz **17** (2005), no. 1, 53–83. MR2140674 (2006b:53006)
- [5] A. I. Bobenko and Yu. B. Suris, *Integrable noncommutative equations on quad-graphs. The consistency approach*, Lett. Math. Phys. **61** (2002), no. 3, 241–254. MR1942362 (2003j:37106)
- [6] Alexander Bobenko and Ulrich Pinkall, *Discrete isothermic surfaces*, J. Reine Angew. Math. **475** (1996), 187–208. MR1396732 (97f:53004)
- [7] ———, *Discrete surfaces with constant negative Gaussian curvature and the Hirota equation*, J. Differential Geom. **43** (1996), no. 3, 527–611. MR1412677 (97m:53008)
- [8] Alexander I. Bobenko and Yuri B. Suris, *Isothermic surfaces in sphere geometries as Moutard nets*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. **463** (2007), no. 2088, 3171–3193. MR2386657 (2009e:53020)
- [9] Edmond Bour, *Théorie de la déformation des surfaces*, J. L'École Impériale Polytechnique (1862), 1–148.
- [10] N. Bourbaki, *Éléments de mathématique. Fasc. XXXIV. Groupes et algèbres de Lie. Chapitre IV: Groupes de Coxeter et systèmes de Tits. Chapitre V: Groupes engendrés par des réflexions. Chapitre VI: systèmes de racines*, Actualités Scientifiques et Industrielles, No. 1337, Hermann, Paris, 1968. MR0240238 (39 #1590)
- [11] Martina Brück, Xi Du, Joonsang Park, and Chuu-Lian Terng, *The submanifold geometries associated to Grassmannian systems*, Mem. Amer. Math. Soc. **155** (2002), no. 735, viii+95. MR1875645 (2002k:53103)
- [12] Robert L. Bryant, *A duality theorem for Willmore surfaces*, J. Differential Geom. **20** (1984), no. 1, 23–53. MR772125 (86j:58029)
- [13] F. E. Burstall, *Isothermic surfaces: conformal geometry, Clifford algebras and integrable systems*, Integrable systems, geometry, and topology, AMS/IP Stud. Adv. Math., vol. 36, Amer. Math. Soc., Providence, RI, 2006, pp. 1–82. MR2222512 (2008b:53006)
- [14] F. E. Burstall and D. M. J. Calderbank, *Conformal submanifold geometry*, in preparation.
- [15] F.E. Burstall, U. Hertrich-Jeromin, W. Rossman, and S. Santos, *Discrete surfaces of constant mean curvature*, arXiv:0804.2707 [math.DG], 2008.
- [16] F. Burstall, U. Hertrich-Jeromin, F. Pedit, and U. Pinkall, *Curved flats and isothermic surfaces*, Math. Z. **225** (1997), no. 2, 199–209. MR1464926 (98j:53004)
- [17] Francis Burstall, Franz Pedit, and Ulrich Pinkall, *Schwarzian derivatives and flows of surfaces*, Differential geometry and integrable systems (Tokyo, 2000), Contemp. Math., vol. 308, Amer. Math. Soc., Providence, RI, 2002, pp. 39–61. MR1955628 (2004f:53010)
- [18] Francis E. Burstall and John H. Rawnsley, *Twistor theory for Riemannian symmetric spaces*, Lecture Notes in Mathematics, vol. 1424, Springer-Verlag, Berlin, 1990. MR1059054 (91m:58039)
- [19] P. Calapso, *Sulle superficie a linee di curvatura isoterme*, Rendiconti Circolo Matematico di Palermo **17** (1903), 275–286.
- [20] David M. J. Calderbank, Tammo Diemer, and Vladimír Souček, *Ricci-corrected derivatives and invariant differential operators*, Differential Geom. Appl. **23** (2005), no. 2, 149–175. MR2158042 (2006j:53073)
- [21] Annalisa Calini, Thomas Ivey, and Gloria Mari Beffa, *Remarks on KdV-type flows on Star-Shaped curves*, arXiv:0808.3593 [nlin.SI], 2008.
- [22] J.A. Carlson and D. Toledo, *Harmonic mappings of Kähler manifolds to locally symmetric spaces*, Publ. Math. I.H.E.S. **69** (1989), 173–201. MR1019964 (91c:58032)
- [23] E. Cartan, *Les espaces à connexion conforme.*, Ann. de la soc. polon. de math. **2** (1923), 171–221 (French).
- [24] A. Cayley, *On the surfaces divisible into Squares by their Curves of Curvature*, Proc. London Math. Soc. **4** (1872), 8–9, 120–121.
- [25] E. Christoffel, *Ueber einige allgemeine Eigenschaften der Minimumsflächen*, Crelle's J. **67** (1867), 218–228.
- [26] Jan Cieřliński, Piotr Goldstein, and Antoni Sym, *Isothermic surfaces in  $E^3$  as soliton surfaces*, Phys. Lett. A **205** (1995), no. 1, 37–43. MR1352426 (96g:53005)
- [27] Gaston Darboux, *Sur les surfaces isothermiques*, Ann. Sci. École Norm. Sup. (3) **16** (1899), 491–508. MR1508975
- [28] ———, *Leçons sur la théorie générale des surfaces et les applications géométriques du calcul infinitésimal. Première partie*, Chelsea Publishing Co., Bronx, N. Y., 1972, Généralités. Coordonnées curvilignes. Surfaces minima, Réimpression de la deuxième édition de 1914. MR0396211 (53 #79)
- [29] A. Demoulin, *Sur les systèmes et les congruences K*, C. R. Acad. Sci. Paris Sér. I Math. **150** (1910), 150, 156–159, 310–312.
- [30] D. Ferus and F. Pedit, *Curved flats in symmetric spaces*, Manuscripta Math. **91** (1996), no. 4, 445–454. MR1421284 (97k:53074)
- [31] Mogens Flensted-Jensen, *Analysis on non-Riemannian symmetric spaces*, CBMS Regional Conference Series in Mathematics, vol. 61, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1986. MR837420 (87h:43013)
- [32] A. P. Fordy, *A historical introduction to solitons and Bäcklund transformations*, Harmonic maps and integrable systems, Aspects Math., E23, Vieweg, Braunschweig, 1994, pp. 7–28. MR1264180
- [33] S. Gindikin and S. Kaneyuki, *On the automorphism group of the generalized conformal structure of a symmetric R-space*, Diff. Geom. Appl. **8** (1998), 21–33. MR1601542 (99a:53068)
- [34] Raymond E. Goldstein and Dean M. Petrich, *The Korteweg-de Vries hierarchy as dynamics of closed curves in the plane*, Phys. Rev. Lett. **67** (1991), no. 23, 3203–3206. MR1135964 (92g:58050)

- [35] A. Grothendieck, *Sur la classification des fibrés holomorphes sur la sphère de Riemann*, Amer. J. Math. **79** (1957), 121–138. MR0087176 (19,315b)
- [36] U. Hertrich-Jeromin, E. Musso, and L. Nicolodi, *Möbius geometry of surfaces of constant mean curvature 1 in hyperbolic space*, Ann. Global Anal. Geom. **19** (2001), no. 2, 185–205. MR1826401 (2002a:53079)
- [37] Udo Hertrich-Jeromin, *Transformations of discrete isothermic nets and discrete cmc-1 surfaces in hyperbolic space*, Manuscripta Math. **102** (2000), no. 4, 465–486. MR1785326 (2001g:53013)
- [38] ———, *Introduction to Möbius differential geometry*, London Mathematical Society Lecture Note Series, vol. 300, Cambridge University Press, Cambridge, 2003. MR2004958 (2004g:53001)
- [39] Udo Hertrich-Jeromin, Tim Hoffmann, and Ulrich Pinkall, *A discrete version of the Darboux transform for isothermic surfaces*, Discrete integrable geometry and physics (Vienna, 1996), Oxford Lecture Ser. Math. Appl., vol. 16, Oxford Univ. Press, New York, 1999, pp. 59–81. MR1676683 (2000j:53010)
- [40] Udo Hertrich-Jeromin and Franz Pedit, *Remarks on the Darboux transform of isothermic surfaces*, Doc. Math. **2** (1997), 313–333 (electronic). MR1487467 (99k:53006)
- [41] Soji Kaneyuki, *On orbit structure of compactifications of para-Hermitian symmetric spaces*, Japan. J. Math. (N.S.) **13** (1987), no. 2, 333–370. MR921587 (88m:53094)
- [42] Soji Kaneyuki and Masato Kozai, *Paracomplex structures and affine symmetric spaces*, Tokyo J. Math. **8** (1985), no. 1, 81–98. MR800077 (87c:53078)
- [43] Shimpei Kobayashi and Jun-Ichi Inoguchi, *Characterizations of Bianchi-Bäcklund transformations of constant mean curvature surfaces*, Internat. J. Math. **16** (2005), no. 2, 101–110. MR2121843 (2006d:53006)
- [44] B. Kostant and S. Rallis, *Orbits and representations associated with symmetric spaces*, Amer. J. Math. **93** (1971), 753–809. MR0311837 (47 #399)
- [45] J. Lepowsky and G. W. McCollum, *Cartan subspaces of symmetric Lie algebras*, Trans. Amer. Math. Soc. **216** (1976), 217–228. MR0404361 (53 #8163)
- [46] Hideya Matsumoto, *Quelques remarques sur les groupes de Lie algébriques réels*, J. Math. Soc. Japan **16** (1964), 419–446. MR0183816 (32 #1292)
- [47] Emilio Musso, *Deformation of surfaces in Möbius space*, Rend. Istit. Mat. Univ. Trieste **27** (1995), no. 1-2, 25–45 (1996). MR1421046 (97i:53013)
- [48] Tadashi Nagano, *Transformation groups on compact symmetric spaces*, Trans. Amer. Math. Soc. **118** (1965), 428–453. MR0182937 (32 #419)
- [49] Toshio Ōshima and Toshihiko Matsuki, *Orbits on affine symmetric spaces under the action of the isotropy subgroups*, J. Math. Soc. Japan **32** (1980), no. 2, 399–414. MR567427 (81f:53043)
- [50] Bennett Palmer, *Isothermic surfaces and the Gauss map*, Proc. Amer. Math. Soc. **104** (1988), no. 3, 876–884. MR964868 (90a:53077)
- [51] Ulrich Pinkall, *Hamiltonian flows on the space of star-shaped curves*, Results Math. **27** (1995), no. 3-4, 328–332. MR1331105 (96i:58078)
- [52] Andrew Pressley and Graeme Segal, *Loop groups*, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 1986, Oxford Science Publications. MR900587 (88i:22049)
- [53] Masaru Takeuchi, *Cell decompositions and Morse equalities on certain symmetric spaces*, J. Fac. Sci. Univ. Tokyo Sect. I **12** (1965), 81–192. MR0216517 (35 #7350)
- [54] C.-L. Terng, *Soliton equations and differential geometry*, J. Diff. Geom. **45** (1997), 407–445. MR1449979 (98e:58102)
- [55] Chuu-Lian Terng and Karen Uhlenbeck, *Bäcklund transformations and loop group actions*, Comm. Pure Appl. Math. **53** (2000), no. 1, 1–75. MR1715533 (2000k:37116)
- [56] Chuu-Lian Terng and Erxiao Wang, *Transformations of flat Lagrangian immersions and Egoroff nets*, Asian J. Math. **12** (2008), no. 1, 99–119. MR2415015
- [57] Jacques Tits, *Sur les  $R$ -espaces*, C. R. Acad. Sci. Paris **239** (1954), 850–852. MR0066395 (16,568b)
- [58] Karen Uhlenbeck, *Harmonic maps into Lie groups: classical solutions of the chiral model*, J. Differential Geom. **30** (1989), no. 1, 1–50. MR1001271 (90g:58028)
- [59] Hugo D. Wahlquist and Frank B. Estabrook, *Bäcklund transformation for solutions of the Korteweg-de Vries equation*, Phys. Rev. Lett. **31** (1973), 1386–1390. MR0415100 (54 #3191)

DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF BATH, BATH BA2 7AY, UK

*E-mail address:* feb@maths.bath.ac.uk

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA AT IRVINE, IRVINE, CA 92697-3875, USA

*E-mail address:* ndonalds@math.uci.edu

MATHEMATISCHES INSTITUT DER UNIVERSITÄT TÜBINGEN, AUF DER MORGENSTELLE 10, 72076 TÜBINGEN, GERMANY, AND DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MASSACHUSETTS, AMHERST, MA 01003, USA

*E-mail address:* pedit@mathematik.uni-tuebingen.de

FACHBEREICH MATHEMATIK, MA 3-2, TECHNISCHE UNIVERSITÄT BERLIN, STRASSE DES 17. JUNI 136, 10623 BERLIN, GERMANY

*E-mail address:* pinkall@math.tu-berlin.de