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The bullet problem with discrete speeds

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Abstract

Bullets are fired from the origin of the positive real line, one per second, with independent speeds sampled uniformly from a discrete set. Collisions result in mutual annihilation. We show that a bullet with the second largest speed survives with positive probability, while a bullet with the smallest speed does not. This also holds for exponential spacings between firing times. Our results imply that the middle-velocity particle survives with positive probability in a two-sided version of the bullet process with three speeds known to physicists as ballistic annihilation.

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1 Introduction

The bullet process is a deceptively simple interacting particle system. Each second, a bullet is fired from the origin along the positive real line with a speed uniformly sampled from $(0, 1)$. When a faster bullet collides with a slower one, they mutually annihilate. The *bullet problem* is to show there exists $s_c > 0$ such that if the first bullet has speed faster than s_c it survives with positive probability, and if it has speed slower than s_c it is almost surely annihilated. In this work, we prove an analogous transition occurs when speeds are instead sampled uniformly from a discrete set.

Let $(S_i)_{i\geq 1}$ be i.i.d. speeds sampled according to a probability measure μ supported on $(0,\infty)$. These define bullet trajectories $b_i(t) = S_i(t - i)$ defined for $t \geq i$. We identify bullets with the trajectory and refer to $b_i(t)$ as the bullet b_i. A deterministic delay between firings is convenient for our argument, but not needed. All of the results here hold for exponentially distributed firing times (see Remark 2.4). When two or more bullets collide, all of them are annihilated.

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The classical bullet problem is formulated with μ the Lebesgue measure on $(0, 1)$. In this work, we consider the discrete analogue with μ a sum of equally weighted point-masses on a set of $n \geq 3$ distinct speeds: $0 < s_n < \cdots < s_2 < s_1 < \infty$. We will refer to this as a *discrete bullet process*.

Let $\{b_j \mapsto b_i\}$ denote the event of bullets b_j and b_i colliding with b_j faster, thus resulting in their mutual annihilation. We say that b_j *catches* b_i. Note that this can only happen if $i < j$, $S_i < S_j$, and all bullets fired at times $k \in (i, j)$ annihilate before the time at which $b_j(t_k) = b_k(t_k)$. Define $\tilde{\tau}$ to be the minimum index with $b_{\tilde{\tau}} \mapsto b_1$. The minimum is to account for the possibility of a simultaneous collision of several bullets. If b_1 is never caught by another bullet, set $\tilde{\tau} = \infty$. When $\tilde{\tau} = \infty$, we say that b_1 *survives*. When $\tilde{\tau} < \infty$, we say that b_1 *perishes*. Our main result is that, when the bullet speeds are uniformly sampled from a finite set, a second fastest bullet survives with positive probability, while the slowest bullet does not.

Theorem 1.1. *In the discrete bullet process it holds that*

(i) $\mathbf{P}[b_1$ *survives* $\mid S_1 = s_2] > 0$*, and*

(ii) $\mathbf{P}[b_1$ *survives* $\mid S_1 = s_n] = 0$.

That b_1 survives when it has maximal speed is obvious because no bullet can catch it. This is not the case with the second fastest bullet. There will a.s. be infinitely many faster bullets trailing it. So, its survival hinges on interference of slower bullets.

Theorem 1.1 solves the discrete analogue of the bullet problem. The coupling between two bullet processes with bullet speeds (S_i) and (S'_i) in which $S_1 > S'_1$ and $S_i = S'_i$ for $i \geq 2$ has b_1 surviving for every realization in which b'_1 survives. This guarantees that, when μ is fixed, the probability the first bullet survives is non-decreasing with respect to its speed. This monotonicity combined with Theorem 1.1 implies that there is a speed at which an initial bullet with that speed will perish, while one with faster speed will survive with positive probability. An interesting further question, that relates back to the original bullet problem, is to locate where the phase transition occurs when μ is uniform on the set $\{1/n, 2/n, \ldots, 1\}$. Currently it is open to to prove that when b_1 has the third fastest speed it survives with positive probability for some choice of n.

1.1 Application to ballistic annihilation

Ballistic annihilation is a physics model that was introduced to isolate intriguing features observed in more complicated systems, such as diffusions in random media and irreversible aggregation [6, 1]. Particles are placed on the real line according to a unit intensity Poisson point process. Each particle is assigned a speed from a measure ν on $\mathbb{R}$. Particles move at their assigned speed and mutually annihilate upon colliding. This model received considerable attention from physicists in the 1990s (see [4, 12, 11] for a start).

Although it appears to have arisen independently, it was observed in [13] that the bullet problem is equivalent to one-sided ballistic annihilation on $[0, \infty)$. If one considers the graphical representation of bullet locations, it is easy to see that inverting time and space coordinates makes the process into ballistic annihilation with inverted speeds (see Figure 1). To make the bullet process two-sided, we extend the definition $b_i(t) = S_i(t - i)$ to all integers i.

Systems with three velocities (and not necessarily the uniform measure) are canonical in ballistic annihilation [5]. A corollary of Theorem 1.1 (i) is survival of the second fastest particle for asymmetric three-element sets with the uniform measure.

Corollary 1.2. *Let* $-\infty < r_3 < r_2 < r_1 < \infty$ *and* ν *be the uniform measure on* $\{r_3, r_2, r_1\}$. *For ballistic annihilation with either unit or exponential spacings, a particle with speed*

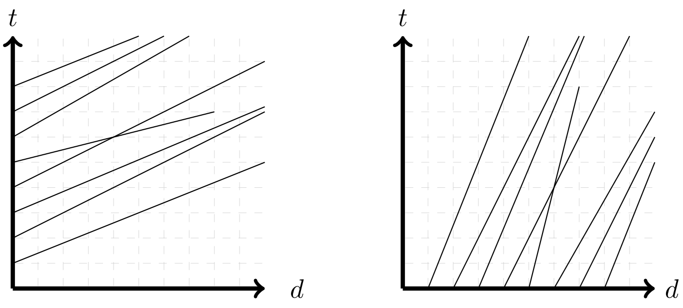

Figure 1: The bullet process is equivalent to one-sided ballistic annihilation.

r_2 *will survive with positive probability.*

A similar observation was made in [13] for the case of symmetric speeds $\{-1, 0, 1\}$. Our result is especially relevant given the recent developments for the three-velocity case that occurred while this article was under review [3, 7, 10]. These concern the existence of a phase transition for the survival probability of the middle-velocity particle.

1.2 History

The IBM problem of the month from May in 2014 credits a version of the problem to an engineer named David Wilson. The question there is to fire exactly $2m$ bullets with independent uniform $(0, 1)$ speeds and compute the probability of the event $E_m = \{\text{no bullets survive}\}$. There is an unpublished result of Fedor Nazarov that

$$\mathbf{P}[E_m] = \prod_{i=1}^{m}\Big(1 - \frac{1}{2i}\Big) = O(m^{-1/2}). \tag{1.1}$$

Note this is a special case of the more general formula (1.2) described below.

Letting $E_{m,s}$ be the event E_m conditioned on $S_1 = s$, it is conjectured that

$$\mathbf{P}[E_{m,s}] = O(m^{-c_s}) \text{ with } c_s \to \infty \text{ as } s \to 1.$$

It is surprising that changing one bullet speed out of the $2n$ total bullets affects the exponent. One would naively expect it only changes $\mathbf{P}[E_m]$ by a constant factor. This conjecture along with a guess that $s_c \approx 0.9$, come from simulations performed by Kostya Makarychev.

Understanding $\mathbf{P}[E_{m,s}]$ would be an important step towards solving the bullet problem. Suppose one could prove that $c_s > 1$ for some value of s, and consider a sequence of finite bullet processes with $2m$ total bullets that are coupled to have the same speeds in the overlap. The Borel-Cantelli lemma would imply that, for m greater than or equal to some almost surely finite M, there are always at least two surviving bullets. The two earliest fired surviving bullets in the process with $2M$ bullets must survive for all larger bullet processes, otherwise there would be a process with more than $2M$ bullets with no surviving bullets. Since the first fired bullet is one of the surviving bullets in the process with $2M$ bullets with positive probability, we have the first bullet survives with positive probability.

The bullet process with n bullets fired was recently studied by Nicolas Broutin and Jean-François Marckert [2]. They consider arbitrary non-atomic speed distributions on $[0, \infty)$ and find that the distribution $\mathbf{q}_n$ for the number of surviving bullets is invariant for several different spacings and acceleration functions for the bullets. The distribution shows up in other contexts such as random permutations and random matrices. It is characterized by the following recurrence relation:

$$q_0(0) = 1, \quad q_1(1) = 1, \quad q_1(0) = 0,$$

and for $n \geq 2$ and any $0 \leq n$,

$$q_n(k) = \frac{1}{n} q_{n-1}(k-1) + \Big(1 - \frac{1}{n}\Big) q_{n-2}(k) \tag{1.2}$$

with $q_n(-1) = q_n(k) = 0$ for $k > n$.

This formula generalizes (1.1), which describes $q_{2m}(0)$. The equation for $\mathbf{q}_n$ can be analyzed to prove a central limit theorem that says $\approx \log n$ bullets survive (see [2, Proposition 2]). Unfortunately, this does not imply survival with infinitely many bullets. Although the number of surviving bullets is growing like $\log n$, we cannot rule out the possibility that the number of bullets alive at time n in the process is 0 infinitely often. Indeed, there are instances of $\mathbf{q}_n$ for which this happens and others where it does not. These results suggest that it is equally challenging to analyze variants of the bullet problem.

1.3 Overview of proofs

Let τ be the index of the bullet that destroys b_1 in a discrete bullet process with $S_1 = s_2$. The idea in Proposition 2.3 is to condition on S_2 and derive a recursive distributional inequality for τ. If $S_2 = s_1$, then b_1 is caught no matter what. However, if $S_2 = s_2$, then b_1 survives "twice" as long as it would have otherwise. If the second bullet is slower than s_2, then it acts as a shield for b_1—thus increasing the survival time of b_1. These arguments hinge on the renewal properties described in Lemma 2.1 and Lemma 2.2, and a fortuitous dependence that makes fast bullets less likely to appear behind the bullet that catches b_2 when $S_2 < s_2$.

We deduce Theorem 1.1 (i) in Section 2.2 by proving that τ is stochastically larger than the number of leftward steps required for a rightward-biased random walk to reach 0 when starting at 1. As this is infinite with positive probability, so is τ. The recursive technique we employ is partly inspired by the work of Christopher Hoffman, Tobias Johnson, and Matthew Junge on the frog model on trees [8, 9].

We use Theorem 1.1 (i) to prove Theorem 1.1 (ii) via contradiction. If the slowest bullet survives with positive probability, then monotonicity implies that the second slowest bullet also survives with positive probability. When we extend the bullet process to be two-sided, the two slowest speeds become the two fastest speeds from the perspective of bullets fired before them. Theorem 1.1 then implies that both speeds survive with positive probability in the two-sided process. Because the two-sided process is ergodic, the Birkhoff ergodic theorem gives a positive density of both speeds that survive. This is a contradiction since these surviving bullets with different speeds must eventually meet, and thus cannot survive.

1.4 Organization

The proofs of Theorem 1.1 (i), (ii), and Corollary 1.2 are in Sections 2, 3, and 4, respectively.

2 Survival of a second fastest bullet

We assume that μ is the uniform measure on a set of n discrete speeds $0 < s_n < \cdots < s_2 < s_1$. As defined just above, let τ be the minimum index with $b_\tau \mapsto b_1$ in the process with $S_1 = s_2$. The goal of this section is to prove that $\mathbf{P}[\tau = \infty] > 0$. To this end, we will throughout assume that $\mathbf{P}[\tau = \infty] = 0$ and derive a contradiction.

2.1 Obtaining a recursive inequality

We start with two lemmas describing a renewal property in bullet processes with discrete speeds. The first states that the bullet speeds behind a maximal speed bullet are independent of any event involving this bullet.

Lemma 2.1. *Let $1 \le i < j$ be fixed indices. Conditional on $\{b_j \mapsto b_i, S_j = s_1)\}$, the random variables $(S_k)_{k>j}$ are independent and have distribution μ.*

Proof. The bullet b_j has the fastest speed, so the bullets behind it do not interfere. Thus the event $\{b_h \mapsto b_i, S_j = s_1\}$ depends only on the bullet speeds $S_1, S_2, \ldots, S_{j-1}$. □

A longer range renewal property holds for other annihilation events where, outside of a particular window, the bullet speeds become independent.

Lemma 2.2. *Set $i < j$, fix $x < y$ from the set $\{s_n, \ldots, s_1\}$, and let*

$$E = E(S, x, y, i, j) = \{b_j \mapsto b_i, S_i = x, S_j = y\}.$$

There exists a positive integer $a = a(x, y, i, j)$ such that, conditional on E, the bullet speeds $S_{j+a}, S_{j+a+1}, \ldots$ are independent and have distribution μ. Moreover, no bullet fired after time $j + a$ can reach b_j before it reaches b_i.

Proof. Given i, j, x, and y, let a be the smallest integer such that a maximal speed bullet fired at time $j + a$ cannot reach b_j before the collision time of b_j and b_i. The event $b_j \mapsto b_i$ is thus unaffected by the bullet speeds $S_{j+a}, S_{j+a+1}, \ldots$. The independence claim follows.

Because we will need it later, we write down an explicit formula for a. A collision between b_j and b_i would occur at time t_0 and location z_0 given by

$$t_0 = \frac{jy - ix}{y - x}, \qquad z_0 = \frac{j - i}{y - x} xy.$$

We then define k_0 to be the smallest firing time at which a fastest bullet could not interfere with $b_j \mapsto b_i$:

$$k_0 := \min\{k > j : s_1(t_0 - k) > z_0\}. \tag{2.1}$$

We conclude by setting $a = k_0 - j$. □

We will occasionally refer to the interval $[j, j + a - 1]$ as the *window of dependence of* E. This is because, as described more precisely above in Lemma 2.2, the bullet speeds in this interval are influenced by E, while those beyond it are again i.i.d. Note, however, that bullets behind the window of dependence may still interfere with bullets inside the window, just not in a way that prevents b_j from catching b_i.

We will write $X \sim Y$ if X and Y are random variables with the same distribution. Recall that one of the several equivalent forms of stochastic dominance $X \succeq Y$ is that there is a coupling with marginals $X' \sim X$ and $Y' \sim Y$ such that $X' \ge Y'$ almost surely. We let $\mathbf{1}_{\{\cdot\}}$ denote an indicator function.

Proposition 2.3. *Suppose that τ is almost surely finite. Let $\tau_1, \ldots, \tau_5$ be independent copies of τ that are also independent of S_2. There exists an event $F \subseteq \{S_2 < s_2\}$ independent of the τ_i with $\mathbf{P}[F] = \epsilon > 0$ so that*

$$\tau \succeq \mathbf{1}_{\{S_2 = s_1\}} \tag{2.2}$$

$$+ \mathbf{1}_{\{S_2 = s_2\}}(\tau_1 + \tau_2) \tag{2.3}$$

$$+ \mathbf{1}_{\{S_2 < s_2\}}(\mathbf{1}_F(\tau_3 + \tau_4) + \mathbf{1}_{F^c}\tau_5). \tag{2.4}$$

The value of ϵ may depend on the underlying set of speeds.

Proof. We will establish each line of the above by conditioning on the value of S_2. When $S_2 = s_1$ as in (2.2), we have $b_2 \mapsto b_1$ deterministically. Although $\tau = 2$ on this event, it will simplify our calculations later to use the indicator function as a lower bound.

When $S_2 = s_2$ as in (2.3), suppose that b_σ destroys b_2. We have translated the original setup by one index, so $\sigma \sim \tau_1 + 1$. Only a bullet with the fastest speed can catch b_2, thus $S_\sigma = s_1$. Lemma 2.1 ensures that $S_{\sigma+1}, S_{\sigma+2}, \dots$ are independent of σ. Suppose that $b_{\sigma'} \mapsto b_1$. Once again this is the first unobstructed speed-s_1 bullet fired after b_σ. Thus $\sigma' - \sigma \sim \tau_2 - 1$, and this difference is independent of σ. Summing $(\sigma' - \sigma) + \sigma$ we obtain the term $\tau_1 + \tau_2$ in (2.3).

The pivotal case is (2.4), when $S_2 < s_2$. The idea is that b_2 acts as a shield, and causes an ϵ-bias for the bullets close behind it to have speed s_2. The reasoning in (2.3) then ensures that b_1 will survive twice as long on this ϵ-likely event. To see this rigorously, suppose that b_γ is the earliest bullet catching b_2.

Since $\gamma \preceq \tau$ (by the coupling mentioned in the introduction), we have γ is almost surely finite by our hypothesis $\mathbf{P}[\tau = \infty] = 0$. We will start by describing the ϵ-likely event F for which we obtain an extra copy of τ. When b_2 is caught, there is a finite window of dependence behind the catching bullet (see Lemma 2.2). With positive probability this window contains only bullets with speed s_2.

A minor nuisance is showing that there is enough room in the window behind b_γ for a speed-s_2 bullet. We start by restricting to the event that $S_2 = s_n$ and show that $\mathbf{P}[\gamma > M] > 0$ for all $M > 0$. Let $m \geq 2$. With positive probability, there are alternating fastest and slowest bullets from index 3 up to $2m$, and then a speed-s_2 bullet. Call this event

$$A = \{S_2 = s_n, S_3 = s_n, S_4 = s_1, \dots, S_{2m-1} = s_n, S_{2m} = s_1, S_{2m+1} = s_2\}.$$

On the event A, we have $\gamma = 2m + 1$ and $S_\gamma = s_2$ so long as nothing catches b_γ before it reaches b_2. We track the size of the window of dependence behind b_γ with the function

$$h(m) = a(s_2, s_n, 2m + 1, 2), \qquad m \geq 2.$$

Here $a(s_2, s_n, 2m + 1, 2) \geq 1$ is as in Lemma 2.2; it is the index distance behind $2m + 1$ at which bullets resume being i.i.d. conditioned on the event $\{b_{2m+1} \mapsto b_2, S_{2m+1} = s_2, S_2 = s_n\}$. We remark that, because we are fixing the indices and speeds in a, the function h is deterministic.

Plugging our conditions into the explicit formula at (2.1), we have $t_0 \to \infty$ as $m \to \infty$, and also $s_2/s_1 < 1$. Thus, $h(m)$ is non-decreasing with $\lim_{m\to\infty} h(m) = \infty$. Let $m_0 = \min\{m \geq 2 \colon h(m) > 1\}$. As bullet speeds are between s_n and s_1, we must have $m_0 < \infty$ and thus $1 < h(m_0) < \infty$. Let B be the event that all of the bullets in this window have speed s_2. Formally,

$$B = \{S_{2m_0+1+i} = s_2 \text{ for all } i = 1, \dots, h(m_0) - 1\}.$$

Let $F = A \cap B$. This event specifies the speeds of $2m_0 + h(m_0) - 1$ bullets, and by independence we have

$$\mathbf{P}[F] = p_1^{m_0-1} p_n^{m_0} p_2^{h(m_0)} > 0, \tag{2.5}$$

where $p_i = \mu(s_i)$.

Conditioned on F, all of $b_2, \dots, b_{2m_0+1}$ mutually annihilate. Moreover, $S_{2m_0+1+i} = s_2$ for $i = 1, \dots, h(m_0) - 1$. The trailing bullets speeds $(S_{2m_0+1+\ell})_{\ell \geq h(m_0)}$ are i.i.d. μ-distributed. The reasoning that yields the additional copy of τ in (2.3) then gives $h(m_0) - 1 \geq 1$ additional copies of τ when F occurs. We take only one of them and set $\epsilon = \mathbf{P}[F]$ as in (2.5). This accounts for the term $\mathbf{1}_F(\tau_3 + \tau_4)$ in (2.4).

Now that we have constructed the ϵ-likely event to have b_1 survive for at least two copies of τ, it remains to show that b_1 survives for at least a τ-distributed amount of time on the event $\{S_2 < s_2\} \cap F^c$. This will give the term $\mathbf{1}_{F^c}\tau_5$ in (2.4).

We borrow an idea from [3] to recursively construct a sequence of windows of dependence induced by the event $\{b_\gamma \mapsto b_2\}$. Let $\eta_1 = \gamma + a(S_\gamma, S_2, \gamma, 2)$. By Lemma 2.2, this is the index at which bullet speeds are once again independent and μ-distributed, conditional on $\{b_\gamma \mapsto b_2, S_2 < s_2\} \cap F^c$. On the event $\{b_\gamma \mapsto b_2\}$, the bullet process restricted to bullets fired in the interval $[2, \eta_1)$ may contain surviving particles with speeds $\leq s_2$, but cannot contain surviving speed-s_1 particles. For $t \geq 1$, we recursively define η_{t+1} to be the largest value in the union of all windows of dependence induced by collision events in the process restricted to $[2, \eta_t)$. The quantity η_{t+1} is finite since τ is almost surely finite and thus all of the bullets fired at times in $[2, \eta_t)$ will eventually be caught.

Let t_0 be the first time that $\eta_{t_0+1} = \eta_{t_0}$. Since τ is almost surely finite, we must have $t_0 < \infty$. By the definition of a window of dependence, there are no surviving particles in the process restricted to $[2, \eta_{t_0})$ and the bullets fired at times $\geq \eta_{t_0}$ are independent and μ-distributed. Moreover, the bullets fired after time η_{t_0} will not interfere with any collision events in the process restricted to $[2, \eta_{t_0}]$. Thus, on the event $\{S_2 < s_2\} \cap F^c$, we have $\tau - \eta_{t_0} + 2 \sim \tau_5$ with τ_5 an independent copy of τ. □

Remark 2.4. The same recursive inequality as in Proposition 2.3 holds for exponential spacings. Let (ζ_i) be i.i.d. unit exponential random variables and consider a discrete bullet process where we fire b_1 at time $t_1 = \zeta_1$, and b_i at time $t_i = t_{i-1} + \zeta_i$ for $i \geq 2$. As before, let τ be the random index of the first bullet to catch b_1 conditional on $S_1 = 2$. We claim that τ still satisfies Proposition 2.3, but with a different event $F \subseteq \{S_2 < s_2\}$.

As before, if $S_2 = s_1$, then $\tau = 2$. So, (2.2) still holds. Next, if $S_2 = s_2$, then b_1 survives twice as long in the same sense as (2.3). This is because a bullet with speed s_1 must catch b_2, and the bullets trailing it have independent speeds and firing times that keep the exponential spacings just as in Lemma 2.1.

Lastly, if $S_2 < s_2$, then we let γ be the index $b_\gamma \mapsto b_2$. The construction is simpler than before. Just as in Lemma 2.2 the event $b_\gamma \mapsto b_2$ induces a finite window of dependence $t_\gamma + a$. Let N be the number of bullets fired in the window of dependence. We take

$$F = \{N = 1, S_2 = s_n, S_3 = s_2, S_4 = s_2\} \tag{2.6}$$

to be the event that b_2 is caught by b_3 when it has speed s_2. The conditions $N = 1, S_4 = s_2$ ensure that there is one speed-s_2 bullet in the window of dependence and no others. It is important that the spacings have the memoryless property, otherwise the times bullets are fired after $t_\gamma + a$ would not have the same distribution as at the start of the process.

We will see in the next section that satisfying the recursive distributional inequality in Proposition 2.3 is sufficient to deduce a nonnegative random variable places some mass at ∞. So, our results extend to exponential spacings.

2.2 Analyzing the recursive inequality

Our goal now is to show that any random variable satisfying the recursive distributional inequality in Proposition 2.3 must be infinite with positive probability. With ϵ as in Proposition 2.3, we introduce an operator $\mathcal{A} = \mathcal{A}(\mu)$ that acts on probability measures supported on the positive integers. It will be more convenient to represent such a measures by the random variable T with law μ. To define $\mathcal{A}$, we let S be sampled according to μ, and U be an independent uniform $(0, 1)$ random variable. Take $T_1, \dots, T_5$ to be i.i.d. copies of T that are also independent of U and S. Let ϵ be as in Proposition 2.3.

We obtain a new distribution

$$\mathcal{A}T \overset{d}{=} \mathbf{1}_{\{S=s_1\}} + \mathbf{1}_{\{S=s_2\}}(T_1+T_2) + \mathbf{1}_{\{S<s_2\}}(\mathbf{1}_{\{U\le\epsilon\}}(T_3+T_4) + \mathbf{1}_{\{U>\epsilon\}}T_5).$$

By Proposition 2.3, assuming $\mathbf{P}[\tau=\infty]=0$, it holds that

$$\tau \succeq \mathcal{A}\tau. \tag{2.7}$$

We first observe that $\mathcal{A}$ is monotone.

Lemma 2.5. *If $T \succeq T'$, then $\mathcal{A}T \succeq \mathcal{A}T'$.*

Proof. This follows from the canonical coupling which sets each $T_i \ge T_i'$. □

Let $\mathcal{A}^m$ denote m iterations of $\mathcal{A}$. We next prove that $\mathcal{A}^m\tau$ converges to a fixed point of $\mathcal{A}$. We write $T(m) \implies T$ for random variables $(T(m))$ supported on the positive integers, if $\mathbf{P}[T(m)\le k] \to \mathbf{P}[T\le k]$ for all $k\ge 1$.

Lemma 2.6. *Let τ be as in Proposition 2.3 and assume $\mathbf{P}[\tau=\infty]=0$. It holds that $\mathcal{A}^m\tau \implies \tau^*$ with $\tau^* \overset{d}{=} \mathcal{A}\tau^*$. Moreover, τ^* is the unique fixed point of $\mathcal{A}$.*

Proof. Let $F_m(k) = \mathbf{P}[\mathcal{A}^m\tau \le k]$ be the cumulative distribution function of $\mathcal{A}^m\tau$. By the previous lemma and (2.7), we have $\mathcal{A}^m\tau \succeq \mathcal{A}^{m+1}\tau$ for all $m\ge 0$. Let $F(k) = \lim_{m\to\infty} F_m(k)$. The limit exists because the definition of stochastic dominance implies that $\{F_m(k)\}_{m=0}^\infty$ is an increasing bounded sequence. The function F is non-decreasing and belongs to $[0,1]$. Moreover, since $\tau \preceq \mathcal{A}^m\tau$ by (2.7) and Lemma 2.5, we have $F(k) \ge \mathbf{P}[\tau\le k]$ for all k. In particular, $\lim_{k\to\infty}F(k)=1$ by our assumption $\mathbf{P}[\tau<\infty]=1$. Thus, $F(k)$ is the cumulative distribution function of some random variable τ^*.

To see that $\tau^* = \mathcal{A}\tau^*$ observe that

$$\tau^* = \lim_{m\to\infty}\mathcal{A}^{m+1}\tau = \lim_{m\to\infty}\mathcal{A}(\mathcal{A}^m\tau) = \mathcal{A}(\lim_{m\to\infty}\mathcal{A}^m\tau) = \mathcal{A}\tau^*.$$

The limit commutes with $\mathcal{A}$ in the third step, because $\mathcal{A}T$ decomposes into a sum of indicator random variables and i.i.d. copies of T. Thus, if a collection of random variables $T(m) \implies T$, it is easy to deduce that $\mathcal{A}T(m) \implies \mathcal{A}T$.

As for uniqueness, one way to see this is to assume that $T=\mathcal{A}T$ is a fixed point and explicitly compute the generating function $f(x) := \mathbf{E}x^T = \mathbf{E}x^{\mathcal{A}T}$. This gives a quadratic equation in $f(x)$ that can be solved for explicitly. Choosing the proper branch is straightforward since $f(0)=0$. Since the probability generating function uniquely specifies the distribution of a random variable, this proves that all fixed points of $\mathcal{A}$ have the same distribution. □

Next we observe that τ^* couples to the return time to zero of a lazy biased random walk on the integers.

Proposition 2.7. *Let τ^* be as in Lemma 2.6. It holds that $\mathbf{P}[\tau^*=\infty]>0$.*

Proof. Since $\{S=s_2\}\cap\{S<s_2\}=\varnothing$, an equivalent definition of $\mathcal{A}\tau^*$, that reuses copies of τ^* for disjoint events, is

$$\mathcal{A}\tau^* = \mathbf{1}_{\{S=s_1\}} + (\mathbf{1}_{\{S=s_2\}} + \mathbf{1}_{\{S<s_2,U\le\epsilon\}})(\tau_1^*+\tau_2^*) + \mathbf{1}_{\{S<s_2,U>\epsilon\}}\tau_2^*.$$

Let T be the number of leftward steps to reach 0 for a discrete-time lazy random walk on $\mathbb{Z}$ started at 1. The walk moves left with probability $p_\ell = \mathbf{P}[S=s_1]=1/n$, moves right with probability

$$p_r = \mathbf{P}[\{S=s_2\}\cup\{S<s_2,U\le\epsilon] = \frac{1}{n} + \frac{n-2}{n}\epsilon,$$

and stays put with probability $1-(p_\ell+p_r)$. Because $\epsilon>0$, we have $p_\ell<p_r$ and such a biased random walk does not return to 0 with probability $(1+(n-2)\epsilon)^{-1}>0$. Hence $\mathbf{P}[T=\infty]>0$. Using the Markov property and translation invariance of simple random walk, it is easy to see that $T=\mathcal{A}T$. By Lemma 2.6, we have τ^* is the unique fixed point of $\mathcal{A}$. Thus, $T\overset{d}{=}\tau^*$ and $\mathbf{P}[\tau^*=\infty]>0$. ☐

We are now ready to establish survival of a bullet with the second largest speed.

Proof of Theorem 1.1 *(i).* Assume that $\mathbf{P}[\tau=\infty]=0$. The relation at (2.7) along with Proposition 2.7 immediately give the contradiction $\tau\succeq\tau^*$ with $\mathbf{P}[\tau^*=\infty]>0$. ☐

3 The slowest bullet does not survive

In this section we continue to assume that μ is the uniform measure on a discrete set of positive speeds with at least three elements. In the usual bullet process the bullet b_i has position $S_i(t-i)$. We can extend this definition to all integers $i\in\mathbb{Z}$ to make the *two-sided bullet process*. In this process bullets are removed the first time their position coincides with another. Now bullets can be destroyed from both sides.

We say that b_i *survives*$^+$ if the position of b_i never coincides with the position of any other b_j for $j>i$. Alternatively, we say that b_j *survives*$^-$ if its position never coincides with the position of a b_j for $j<i$. If both occur, we say that b_j *survives*$^{+,-}$.

Survival$^+$ only depends on bullets fired after a given bullet, so it describes whether a bullet catches the survivor. So, survival$^+$ favors faster bullets. On the other hand, survival$^-$ favors slower bullets since it describes whether a bullet catches one fired before it. As bullet speeds are independent, we can describe survival$^{+,-}$ as a product of the probabilities of one-sided survival.

Lemma 3.1. *For all $i\in\mathbb{Z}$ it holds that* $\mathbf{P}[b_i$ *survives*$^{+,-}\ ]=\mathbf{P}[b_i$ *survives*$^+]\mathbf{P}[b_i$ *survives*$^-]$.

The advantage of the two-sided process is that it is ergodic, and so there cannot be two different bullet speeds that survive with positive probability.

Proposition 3.2. *Only one bullet speed can survive*$^{+,-}$ *with positive probability in the two-sided discrete bullet process.*

Proof. Notice that the two-sided process is translation invariant with i.i.d. speeds and thus ergodic. If two or more different speeds survived$^{+,-}$ with positive probability, then by the Birkhoff ergodic theorem, we would have a positive fraction of surviving$^{+,-}$ bullets of each speed. Suppose that b_i is one of these surviving bullets. For some $j,k>0$ there almost surely are surviving$^{+,-}$ bullets b_{i+j} and b_{i-k} with the same speed as one another, but different speed than b_i. With different speeds, one of these must collide with b_i, or perhaps some other surviving$^{+,-}$ bullet. In either case, this contradicts that these bullets survive$^{+,-}$. ☐

Proof of Theorem 1.1 *(ii).* If b_1 survives$^+$ then b_1 survives in the discrete bullet process. So it suffices to prove that $\mathbf{P}[b_1$ survives$^+\mid S_1=s_n]=0$. To show a contradiction suppose this probability is equal to $q>0$. A bullet with speed s_n is the easiest to catch for bullets fired at times after it, but it is uncatchable by bullets fired before it. Thus, $\mathbf{P}[b_1$ survives$^-\mid S_1=s_n]=1$.

Let s_2' be the second slowest speed in the support of μ. The monotonicity for survival of bullets discussed in the introduction following the statement of Theorem 1.1 ensures that $\mathbf{P}[b_1$ survives$^+\mid S_1=s_2']\geq q$. Moreover, a bullet with speed s_2' is the second fastest bullet from the perspective of bullets fired before it. Since μ is uniform, we can apply Theorem 1.1 (i) and deduce $\mathbf{P}[b_1$ survives$^-\mid S_1=s_2']=p>0$.

The one-sided survival probabilities above are all positive. By Lemma 3.1, a bullet with speed s_n or s_2' survives$^{+,-}$ with positive probability. This contradicts Proposition 3.2. □

4 Application to ballistic annihilation

Corollary 1.2 follows from Theorem 1.1 (i) and Lemma 3.1.

Proof of Corollary 1.2. Start with ballistic annihilation with the uniform measure on three speeds: $r_3 < r_2 < r_1$. If $r_1 > 0$, then this is equivalent to a two-sided discrete bullet process with speeds $s_i = 1/r_i$. If $r_1 \leq 0$ we can use the fact that the manner in which collisions happen in ballistic annihilation is translation invariant (this is referred to as the *linear speed-change invariance property* in [13, Section 2]). Namely, the same particle collisions will occur (although at different times) in ballistic annihilation with shifted-speeds $r_i' = r_i - r_1 + 1$. The r_i' are positive and, so this process is equivalent to a two-sided discrete bullet process with speeds $s_i = 1/r_i'$. In both cases we have $s_n < s_2 < s_1$ and μ the uniform measure.

In the two-sided discrete bullet process from the previous section, a bullet with speed s_2 is the second fastest from the perspective of bullets fired before and after it. So, Theorem 1.1 (i) guarantees that both

$$\mathbf{P}[b_1 \text{ survives}^+ \mid S_1 = s_2], \mathbf{P}[b_1 \text{ survives}^- \mid S_1 = s_2] > 0,$$

Note that these probabilities are positive, but may not be equal. Combine this with Lemma 3.1 and we have

$$\mathbf{P}[b_1 \text{ survives}^{+,-} \mid S_1 = s_2] > 0.$$

We conclude by noting that equivalence of the two processes ensures that a speed-s_2 bullet surviving with positive probability is the same as a speed-r_2 particle surviving in ballistic annihilation. □

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