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Hadronic Atoms and Effective Interactions

Barry R. Holstein
Institut für Kernphysik
Forschungszentrum Jülich
D-52425 Jülich, Germany
and
Department of Physics and Astronomy
University of Massachusetts
Amherst, MA 01003

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Abstract

We examine the problem of hadronic atom energy shifts using the technique of effective interactions and demonstrate equivalence with the conventional quantum mechanical approach.

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1 Introduction

In a recent paper Kong and Ravndal examined the problem of the decay width of the pionium atom—$\pi^+\pi^-$—using methods of effective field theory.[1] Specifically they employed a quasi-local approximation to the pi-pi interaction and evaluated the width using simple second order perturbation theory to obtain the canonical answer[2]

$$\Gamma^{(0)} = \frac{8\pi}{m_\pi} k_1 |a_{12}|^2 |\Psi(0)|^2 \tag{1}$$

in terms of the $\pi^+\pi^- \rightarrow \pi^0\pi^0$ scattering length a_{12}. Here $k_1 = \sqrt{2m_{\pi^0}(m_{\pi^+} - m_{\pi^0})} \equiv \sqrt{2m_{\pi^0}\Delta m_\pi}$ is the center of mass momentum in the final state—$\pi^0\pi^0$—system and $|\Psi(0)|^2 = m_{\pi^+}^3\alpha^3/8\pi$ is the wavefunction at the origin. In a subsequent paper they examined the problem of pp scattering within a effective local interaction picture and have shown how this matches onto the usual low nuclear physics description of the process.[3] We demonstrate below how these two discussions can be merged—by determining the bound state energy via the presence of a pole in the scattering matrix—and thereby make contact with the traditional quantum mechanical discussion of hadronic atom energy shifts.[4] In the next section we briefly review the usual quantum mechanics approach to the problem, while in section 3 we examine the same problem within the effective field theory procedure and demonstrate equivalence to the quantum mechanical results. In section 4 we discuss applications to pionium and to the $\pi^- p$ atom. Finally, in section 5 we present a short summary.

2 Hadronic Atom Energy Shifts: Quantum Mechanical Approach

The problem of calculating strong interaction energy shifts in hadronic atoms is an old and well-known one, and the methods by which to approach the subject are fairly standard.[4] In cases such as light pionic atoms or pionium the system is essentially nonrelativistic so that simple quantum mechanics can be employed. However, before considering such systems, we first examine the closely related problem of scattering of particles having the *same* charge, as considered by Kong and Ravndal.[3]

Consider then a system consisting of a pair of particles A, B both having charge $+e$ and with reduced mass m_r. Also, suppose that there exists no coupling to any other channel. First neglect the Coulombic interaction and consider only strong scattering. For simplicity, we represent the strong potential between the particles in terms of a simple square well of depth V_0 and radius R—

$$V(r) = \begin{cases} -V_0 & r \leq R \\ 0 & r > R \end{cases} \tag{2}$$

Considering, for simplicity, S-waves the wavefunction in the interior and exterior regions can be written as

$$\psi(r) = \begin{cases} N j_0(Kr) & r \leq R \\ N'(j_0(kr)\cos\delta_0 - n_0(kr)\sin\delta_0) & r > R \end{cases} \tag{3}$$

where j_0, n_0 are spherical harmonics and the interior, exterior wavenumbers are given by $k = \sqrt{2m_r E}$, $K = \sqrt{2m_r(E + V_0)}$ respectively. The connection between the two forms can be made by matching logarithmic derivatives, which yields the result

$$k\cot\delta_0 = -\frac{1}{R}\left[1 + \frac{1}{KRF(KR)}\right] \quad \text{with} \quad F(x) = \cot x - \frac{1}{x} \tag{4}$$

Making an effective range expansion

$$k\cot\delta_0 = -\frac{1}{a_0} + \ldots \tag{5}$$

we find an expression for the scattering length

$$a_0 = R\left[1 - \frac{\tan(K_0 R)}{K_0 R}\right] \quad \text{where} \quad K_0 = \sqrt{2m_r V_0} \tag{6}$$

For later use we note that this can be written in the form

$$a_0 = \frac{m_r}{2\pi}\left(-\frac{4}{3}\pi R^3 V_0\right) + \mathcal{O}(V_0^2) \tag{7}$$

The corresponding scattering amplitude is

$$f(k) = e^{i\delta_0}\frac{\sin\delta_0}{k} = \frac{1}{k\cot\delta_0 - ik} = \frac{1}{-\frac{1}{a_0} - ik} + \ldots \tag{8}$$

Now restore the Coulomb interaction in the exterior region. The analysis of the scattering proceeds as above but with the replacement of the exterior spherical Bessel functions by appropriate Coulomb wavefunctions F_0^+, G_0^+

$$j_0(kr) \to F_0^+(r), \qquad n_0(kr) \to G_0^+(r) \tag{9}$$

whose explicit form can be found in reference [5]. For our purposes we require only the form of these functions in the limit $kr << 1$—

$$\begin{aligned} F_0^+(r) &\stackrel{kr<<1}{\longrightarrow} C(\eta_+(k))(1+\frac{r}{a_B}+\ldots) \\ G_0^+(r) &\stackrel{kr<<1}{\longrightarrow} -\frac{1}{C(\eta_+(k))}\left\{\frac{1}{kr}\right. \\ &+ \left. 2\eta_+(k)\left[h(\eta_+(k))+2\gamma-1+\ln\frac{2r}{a_B}\right]+\ldots\right\} \end{aligned} \tag{10}$$

Here $\gamma = 0.577215..$ is the Euler constant,

$$C^2(x) = \frac{2\pi x}{\exp(2\pi x)-1} \tag{11}$$

is the usual Coulombic enhancement factor, $a_B = 1/m_r\alpha$ is the Bohr radius, $\eta_+(k) = 1/ka_B$, and

$$h(\eta_+(k)) = \mathrm{Re}H(i\eta_+(k)) = \eta_+^2(k)\sum_{n=1}^{\infty}\frac{1}{n(n^2+\eta_+^2(k))} - \ln\eta_+(k) - \gamma \tag{12}$$

where $H(x)$ is the analytic function

$$H(x) = \psi(x) + \frac{1}{2x} - \ln(x) \tag{13}$$

Equating interior and exterior logarithmic derivatives we find

$$\begin{aligned} KF(KR) &= \frac{\cos\delta_0 F_0^{+\prime}(R) - \sin\delta_0 G_0^{+\prime}(R)}{\cos\delta_0 F_0^+(R) - \sin\delta_0 G_0^+(R)} \\ &= \frac{k\cot\delta_0 C^2(\eta_+(k))\frac{1}{a_B} - \frac{1}{R^2}}{k\cot\delta_0 C^2(\eta_+(k)) + \frac{1}{R} + \frac{2}{a_B}\left[h(\eta_+(k)) - \ln\frac{a_B}{2R} + 2\gamma - 1\right]} \end{aligned} \tag{14}$$

Since $R << a_B$ Eq. 14 can be written in the form

$$k\cot\delta_0 C^2(\eta_+(k)) + \frac{2}{a_B}\left[h(\eta_+(k)) - \ln\frac{a_B}{2R} + 2\gamma - 1\right] \simeq -\frac{1}{a_0} \tag{15}$$

The scattering length a_C in the presence of the Coulomb interaction is conventionally defined as[7]

$$k\cot\delta_0 C^2(\eta_+(k)) + \frac{2}{a_B}h(\eta_+(k)) = -\frac{1}{a_C} + \ldots \tag{16}$$

so that we have the relation

$$-\frac{1}{a_0} = -\frac{1}{a_C} - \frac{2}{a_B}(\ln\frac{a_B}{2R} + 1 - 2\gamma) \tag{17}$$

between the experimental scattering length—a_C—and that which would exist in the absence of the Coulomb interaction—a_0.

As an aside we note that a_0 is not itself an observable since the Coulomb interaction *cannot* be turned off. However, in the case of the pp interaction isospin invariance requires $a_0^{pp} = a_0^{nn}$ so that one has the prediction

$$-\frac{1}{a_0^{nn}} = -\frac{1}{a_C^{pp}} - \alpha M_N(\ln\frac{1}{\alpha M_N R} + 1 - 2\gamma) \tag{18}$$

While, strictly speaking, this is a model dependent result, Jackson and Blatt have shown by treating the interior Coulomb interaction perturbatively that a version of this result with $1 \to 0.824$ is approximately valid for a wide range of strong interaction potentials[5] and the correction indicated in Eq. 18 is essential in restoring agreement between the widely discrepant—$a_0^{nn} = -18.8$ fm vs. $a_C^{pp} = -7.82$ fm—values obtained experimentally.

Returning to the problem at hand, the experimental scattering amplitude can then be written as

$$\begin{aligned} f_C^+(k) &= \frac{e^{2i\sigma_0}C^2(\eta_+(k))}{-\frac{1}{a_C} - \frac{2}{a_B}h(\eta_+(k)) - ikC^2(\eta_+(k))} \\ &= \frac{e^{2i\sigma_0}C^2(\eta_+(k))}{-\frac{1}{a_C} - \frac{2}{a_B}H(i\eta_+(k))} \end{aligned} \tag{19}$$

where $\sigma_0 = \arg\Gamma(1 - i\eta_+(k))$ is the Coulomb phase.

Analysis of the situation involving particles of *opposite* charge is similar, except that in this case in the absence of strong interaction effects there will exist, of course, Coulomb bound states at momentum $k_n = i\kappa_n = i/na_B$ and energy $E_n = -\kappa_n^2/2m_r = -m_r\alpha^2/2n^2$ with $n = 1, 2, 3, \ldots$ In the presence of strong interactions between these particles, however, the energies will be shifted. One approach to the calculation of this shift is to examine the corresponding scattering process $A + B \to A + B$. Then the existence of a bound state is indicated by the presence of a pole along the positive imaginary axis—*i.e.* for $\kappa > 0$ under the analytic continuation $k \to i\kappa$. (If we (temporarily) neglect Coulomb effects and look only at strong scattering, this occurs when $\kappa = 1/a_0$ in the case of Eq. 8.)

However, in the case of oppositely charged particles A, B the analysis of the scattering amplitude must be in terms of appropriate Coulomb wavefunctions. If, as before, we include Coulomb effects only in the exterior region, then the appropriate forms of the wavefunctions for $kr << 1$ are given in ref.[6] as

$$\begin{aligned}
F_0^-(r) &\stackrel{kr<<1}{\longrightarrow} C(\eta_+(k))\left(1 - \frac{r}{a_B} + \ldots\right) \\
G_0^-(r) &\stackrel{kr<<1}{\longrightarrow} -\frac{1}{C(\eta_+(k))}\left\{\frac{1}{kr}\right. \\
&- \left. 2\eta_+(k)\left[H(i\eta_+(k)) - i\frac{\pi}{2}\coth(\pi\eta_+(k)) + 2\gamma - 1 + \ln i\frac{2r}{a_B}\right] + \ldots\right\}
\end{aligned} \tag{20}$$

Equating interior and exterior logarithmic deriviatives as before, we find

$$\begin{aligned}
KF(KR) &= \frac{\cos\delta_0 F_0^{-\prime}(R) - \sin\delta_0 G_0^{-\prime}(R)}{\cos\delta_0 F_0^-(R) - \sin\delta_0 G_0^-(R)} \\
&= \frac{-k\cot\delta_0 C^2(-\eta_+(k))\frac{1}{a_B} - \frac{1}{R^2}}{k\cot\delta_0 C^2(-\eta_+(k)) + \frac{1}{R} - \frac{2}{a_B}(h(\eta_+(k)) - \ln\frac{a_B}{2R} + 2\gamma - 1)}
\end{aligned} \tag{21}$$

Thus we have

$$k\cot\delta_0 C^2(-\eta_+(k)) - \frac{2}{a_B}\left[h(\eta_+(k)) - \ln\frac{a_B}{2R} + 2\gamma - 1\right] \simeq -\frac{1}{a_0} \tag{22}$$

In this case the scattering length a_C in the presence of the Coulomb interaction is defined as

$$k\cot\delta_0 C^2(-\eta_+(k)) - \frac{2}{a_B}h(\eta_+(k)) = -\frac{1}{a_C} \tag{23}$$

so that we have the relation

$$-\frac{1}{a_0}=-\frac{1}{a_C}+\frac{2}{a_B}(\ln\frac{a_B}{2R}+1-2\gamma) \tag{24}$$

The corresponding scattering amplitude is then

$$\begin{aligned} f_C^-(k) &= \frac{e^{-2i\sigma_0}C^2(-\eta_+(k))}{-\frac{1}{a_C}+\frac{2}{a_B}h(\eta_+(k))-ikC^2(-\eta_+(k))} \\ &= \frac{e^{-2i\sigma_0}C^2(-\eta_+(k))}{-\frac{1}{a_C}+\frac{2}{a_B}[H(i\eta_+(k))-i\pi\coth\pi\eta_+(k))]} \end{aligned} \tag{25}$$

Under the continuation $k\to i\kappa$ we have

$$H(i\eta_+(k))-i\pi\coth(\pi\eta_+(k))\to H(\xi)+\pi\cot\pi\xi \tag{26}$$

where $\xi=1/\kappa a_B$. The existence of a bound state is then signalled by

$$-\frac{1}{a_C}+\frac{2}{a_B}(H(\xi)+\pi\cot\pi\xi)=0 \tag{27}$$

In the limit of no strong interaction—$a_C\to 0$—we find $\xi_n=1/\kappa_n a_B=n$ and the usual Coulomb bound state energies

$$E_n=-\frac{\kappa_n^2}{2m_r}=-\frac{m_r\alpha^2}{2n^2},\quad n=1,2,3,\ldots \tag{28}$$

while if $a_C\neq 0$ there exists a solution to Eq. 27

$$\xi=\frac{1}{\kappa_n a_B}\approx n+\frac{2a_C}{a_B} \tag{29}$$

and a corresponding energy shift

$$\Delta E_n=-E_n\frac{4a_C}{na_B}+\mathcal{O}(\frac{a_C}{a_B})^2 \tag{30}$$

which is the conventional result.[6],[8]

It is important to note here that Eq. 30 is written in a form that relates one *experimental* quantity—the energy shift ΔE_n—to another—the scattering length a_C. Hence it is *model-independent*, even though, for clarity, we have employed a particular model in its derivation. This feature means that it is an ideal case for an effective interaction approach, as will be shown in the next section. However, we first complete our quantum mechanical discussion.

An alternative approach involves the use of bound state perturbation theory.[1] In this case the problem simplifies because of the feature that the range of the strong interaction—R—is much less than the Bohr radius—a_B. Thus we may write[8]

$$\begin{aligned}\Delta E_n &= <\Psi|V|\Psi> + \sum_{n\neq 0}\frac{<\Psi|V|n><n|V|\Psi>}{E_n - E_0} + \ldots \\ &\simeq |\Psi(0)|^2 \times \lim_{k\rightarrow i\kappa} <\phi_f|V|\psi_i^{(+)}> \end{aligned} \tag{31}$$

Connection with the scattering length may be made via[9]

$$f(k) = -\frac{m_r}{2\pi} <\phi_f|V|\psi_i^{(+)}> \simeq -a \tag{32}$$

so that for weak potentials we have

$$a \approx -\frac{m_r}{2\pi}\int d^3r V(r) = -\frac{m_r}{2\pi}\frac{4}{3}\pi R^3 V_0 \tag{33}$$

in agreement with Eq. 7. Then since for Coulombic wavefunctions

$$|\Psi(0)|^2 = \frac{1}{\pi n^3 a_B^3} \tag{34}$$

we have

$$\Delta E_n = \frac{2\pi a}{m_r}|\Psi(0)|^2 + \ldots = -E_n\frac{4a}{n a_B} + \ldots \tag{35}$$

as found via continuation of the scattering amplitude. (However, in this form it is not completely clear whether the relevant scattering length is the experimental quantity a_C or its Coulomb subtracted analog a_0.)

In the real world, of course, this simple model is no longer valid, since realistic hadronic atoms, such as pionium or $\pi^- p$, are coupled to unbound systems—$\pi^0\pi^0$ or $\pi^0 n, \gamma n$—and must be treated as multi-channel problems. Nevertheless the methods generalize straightforwardly from those given above. Specifically, in the absence of Coulomb interactions the scattering amplitude for the $\pi\pi$ system can be given in the two-channel K-matrix form[1]

$$f^{-1} = -A^{-1} - i\begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} \tag{36}$$

[1]There exists also a coupling of $\pi^+\pi^-$ via the anomaly to the $\pi^0\gamma$ channel, but this is p-wave and can be neglected for s-states such as considered here.

where k_1, k_2 are the center of mass momenta in the 1(neutral),2(charged) channels respectively and

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \tag{37}$$

is a real matrix given in terms of the coupled channel scattering lengths. The inverse of the matrix A is easily determined and the resultant form of the scattering amplitude is

$$\begin{aligned} f &= \frac{1}{1 + ik_1a_{11} + ik_2a_{22} - k_1k_2\mathrm{det}A} \\ &\times \begin{pmatrix} -a_{11}(1 + ik_2a_{22}) + ik_2a_{12}a_{21} & -a_{12} \\ -a_{21} & -a_{22}(1 + ik_1a_{11}) + ik_1a_{12}a_{21} \end{pmatrix} \end{aligned} \tag{38}$$

The existence of a bound state is, as before, indicated by the presence of a pole—*i.e.* by the condition

$$1 + ik_1a_{11} + ik_2a_{22} - k_1k_2\mathrm{det}A = 0 \tag{39}$$

after approporiate analytic continuation. In the case of hadronic atoms, we must, of course, correctly include the Coulomb effects. From our single channel experience above, this is done via the prescription

$$a_{ij} \to a^C_{ij}, \quad -ik_2 \to \frac{2}{a_B}(H(\xi) + \pi \cot \pi\xi) \tag{40}$$

whereby the bound state condition—Eq. 39—reads

$$0 = 1 + \frac{2}{a_B}(H(\xi) + \pi \cot \pi\xi) \left(-a^C_{22} + \frac{ik_1a^C_{12}a^C_{21}}{1 + ik_1a^C_{11}} \right) \tag{41}$$

Thus we find

$$\Delta E_n = -E_n \frac{4}{na_B}(a^C_{22} - \frac{ik_1a^C_{12}a^C_{21}}{1 + ik_1a^C_{11}}) + \mathcal{O}(\frac{a^C_{ij}}{a_B})^2 \tag{42}$$

as the coupled channel generalization of Eq. 30. The real component of the energy shift is, of course, to lowest order identical to the single channel result. What is new is that there has developed an imaginary component, corresponding to a decay width

$$\Gamma_n = -2\mathrm{Im}\Delta E_n = -E_n \frac{8}{na_B} k_1 |a_{12}^C|^2 + \ldots = \frac{4\pi}{m_r^{(2)}} k_1 |a_{12}^C|^2 |\Psi(0)|^2 + \ldots \quad (43)$$

which is precisely what is expected from Fermi's golden rule

$$\begin{aligned} \Gamma_n &= \int \frac{d^3k_1}{(2\pi)^3} 2\pi\delta(\Delta E - \frac{k_1^2}{2m_r^{(1)}}) | < \phi_f | V | \Psi_i > |^2 \\ &= (\frac{2\pi}{\sqrt{m_r^{(1)} m_r^{(2)}}} a_{12}^C)^2 \frac{m_r^{(1)}}{\pi} k_1 |\Psi(0)|^2 \end{aligned} \quad (44)$$

or from second order perturbation theory.

3 Hadronic Atom Energy Shifts: Effective Field Theory

Identical results may be obtained from effective field theory and in many ways the derivation is clearer and more intuitive.[10] First consider the situation that we have two particles A,B interacting only via a local strong interaction, so that the effective Lagrangian can be written as

$$\mathcal{L} = \sum_{i=A}^{B} \Psi_i^\dagger (i\frac{\partial}{\partial t} + \frac{\nabla^2}{2m_i})\Psi_i - C_0 \Psi_A^\dagger \Psi_A \Psi_B^\dagger \Psi_B + \ldots \quad (45)$$

The T-matrix is then given in terms of the multiple scattering series shown in Figure 1

$$T_{fi}(k) = -\frac{2\pi}{m_r} f(k) = C_0 + C_0^2 G_0(k) + C_0^3 G_0^2(k) + \ldots = \frac{C_0}{1 - C_0 G_0(k)} \quad (46)$$

where $G_0(k)$ is the amplitude for particles A, B to travel from zero separation to zero separation—$i.e$ the propagator $D_F(k; \vec{r}' = 0, \vec{r} = 0)$—

$$G_0(k) = \lim_{\vec{r}', \vec{r} \to 0} \int \frac{d^3s}{(2\pi)^3} \frac{e^{i\vec{s}\cdot\vec{r}'} e^{-i\vec{s}\cdot\vec{r}}}{\frac{k^2}{2m_r} - \frac{s^2}{2m_r} + i\epsilon} = \int \frac{d^3s}{(2\pi)^3} \frac{2m_r}{k^2 - s^2 + i\epsilon} \quad (47)$$

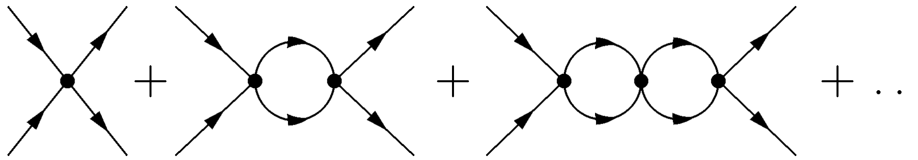
Figure 1: The multiple scattering series.

Equivalently $T_{fi}(k)$ satisfies a Lippman-Schwinger equation

$$T_{fi}(k) = C_0 + C_0 G_0(k) T_{fi}(k). \tag{48}$$

whose solution is given in Eq. 46.

The function $G_0(k)$ is divergent and must be defined via some sort of regularization. There are a number of ways by which to do this. We shall herein use a cutoff regularization with $k_{max} = \mu$ we have

$$G_0(k) = -\frac{m_r}{2\pi}(\frac{2\mu}{\pi} + ik) \tag{49}$$

Equivalently, one could subtract at an unphysical momentum point, as proposed by Gegelia[11]

$$G_0(k) = \int \frac{d^3 s}{(2\pi)^3}(\frac{2m_r}{k^2 - s^2 + i\epsilon} + \frac{2m_r}{\mu^2 + s^2}) = -\frac{m_r}{2\pi}(\mu + ik) \tag{50}$$

which has been shown by Mehen and Stewart[12] to be equivalent to the PDS sceme of Kaplan, Savage and Wise.[10] In any case, the would-be linear divergence is, of course, cancelled by introduction of a counterterm, which renormalizes C_0 to $C_0(\mu)$. The scattering amplitude is then

$$f(k) = -\frac{m_r}{2\pi}\left(\frac{1}{\frac{1}{C_0(\mu)} - G_0(k)}\right) = \frac{1}{-\frac{2\pi}{m_r C_0(\mu)} - \frac{2\mu}{\pi} - ik} \tag{51}$$

Comparing with Eq. 8 we identify the scattering length as

$$-\frac{1}{a_0} = -\frac{2\pi}{m_r C_0(\mu)} - \frac{2\mu}{\pi} \tag{52}$$

More interesting is the case where we restore the Coulomb interaction between the particles. The derivatives in Eq. 45 then become covariant and the bubble sum is evaluated with static photon exchanges between each of

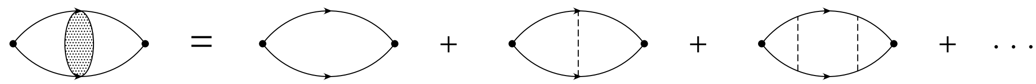

Figure 2: The Coulomb corrected bubble.

the lines—each bubble is replaced by one involving a sum of zero, one, two, etc. Coulomb interactions, as shown in Figure 2.

The net result in the case of same charge scattering is the replacement of the free propagator by its Coulomb analog

$$G_0(k) \to G_C^+(k) = \lim_{\vec{r}',\vec{r}\to 0}\int \frac{d^3s}{(2\pi)^3}\frac{\psi_{\vec{s}}^+(\vec{r}')\psi_{\vec{s}}^{+*}(\vec{r})}{\frac{k^2}{2m_r}-\frac{s^2}{2m_r}+i\epsilon}$$
$$= \int \frac{d^3s}{(2\pi)^3}\frac{2m_r C^2(\eta_+(s))}{k^2-s^2+i\epsilon} \quad (53)$$

where

$$\psi_{\vec{s}}^+(\vec{r}) = C(\eta_+(s))e^{i\sigma_0}e^{i\vec{s}\cdot\vec{r}}{}_1F_1(-i\eta_+(s),1,isr-i\vec{s}\cdot\vec{r}) \quad (54)$$

is the outgoing Coulomb wavefunction for repulsive Coulomb scattering.[13] Also in the initial and final states the influence of static photon exchanges must be included to all orders, which produces the factor $C^2(2\pi\eta_+(k))\exp(2i\sigma_0)$. Thus the repulsive Coulomb scattering amplitude becomes

$$f_C^+(k) = -\frac{m_r}{2\pi}\frac{C_0C^2(\eta_+(k))\exp 2i\sigma_0}{1-C_0G_C^+(k)} \quad (55)$$

The momentum integration in Eq. 53 can be performed as before using cutoff regularization, yielding

$$G_C^+(k) = -\frac{m_r}{2\pi}\left\{\frac{2\mu}{\pi}+\frac{2}{a_B}\left[H(i\eta_+(k))-\ln\frac{\mu a_B}{2\pi}-\zeta\right]\right\} \quad (56)$$

where $\zeta = \ln 2\pi - \gamma$. (Equivalently, in the unphysical momentum subtraction scheme

$$G_C^+(k) = -\frac{m_r}{2\pi}\frac{2}{a_B}(H(i\eta_+(k))-H(\frac{1}{\mu a_B}))$$
$$\simeq -\frac{m_r}{2\pi}(\mu+\frac{2}{a_B}[H(i\eta_+(k))-\ln\mu a_B-\psi(1)]) \quad (57)$$

We have then

$$f_C^+(k) = \frac{C^2(\eta_+(k))e^{2i\sigma_0}}{-\frac{2\pi}{m_r C_0(\mu)} - \frac{2\mu}{\pi} - \frac{2}{a_B}\left[H(i\eta_+(k)) - \ln\frac{\mu a_B}{2\pi} - \zeta\right]}$$
$$= \frac{C^2(\eta_+(k))e^{2i\sigma_0}}{-\frac{1}{a_0} - \frac{2}{a_B}\left[h(\eta_+(k) - \ln\frac{\mu a_B}{2\pi} - \zeta\right] - ikC^2(\eta_+(k))} \quad (58)$$

Comparing with Eq. 19 we identify the Coulomb scattering length as

$$-\frac{1}{a_C} = -\frac{1}{a_0} + \frac{2}{a_B}(\ln\frac{\mu a_B}{2\pi} + \zeta) \quad (59)$$

which matches nicely with Eq. 17 if a reasonable cutoff $\mu \sim m_\pi \sim 1/R$ is employed. The scattering amplitude then has the simple form

$$f_C^+(k) = \frac{C^2(\eta_+(k))e^{2i\sigma_0}}{-\frac{1}{a_C} - \frac{2}{a_B}H(i\eta_+(k))} \quad (60)$$

in agreement with Eq. 19.

Now consider oppositely charged particles. In this case the analysis is parallel to that above, but there exist important new wrinkles in that the intermediate state sum in the Coulomb propagator must now include bound states

$$G_0(k) \to G_C^-(k) = \lim_{\vec{r}',\vec{r}\to 0}\left[\sum_{n\ell m}\frac{\psi_{n\ell m}(\vec{r}')\psi^*_{n\ell m}(\vec{r})}{\frac{k^2}{2m_r} + \frac{m_r\alpha^2}{2n^2}} + \int\frac{d^3s}{(2\pi)^3}\frac{\psi^-_{\vec{s}}(\vec{r}')\psi^{-*}_{\vec{s}}(\vec{r})}{\frac{k^2}{2m_r} - \frac{s^2}{2m_r} + i\epsilon}\right]$$
$$= \frac{2m_r}{\pi a_B}\sum_{n=1}^{\infty}\frac{\eta_+^2(k)}{n(n^2+\eta_+^2(k))} + \int\frac{d^3s}{(2\pi)^3}\frac{2m_r C^2(-\eta(s))}{k^2 - s^2 + i\epsilon} \quad (61)$$

where

$$\psi^-_{\vec{s}}(\vec{r}) = C(-\eta_+(s))e^{-i\sigma_0}e^{i\vec{s}\cdot\vec{r}}{}_1F_1(i\eta_+(s), 1, isr - i\vec{s}\cdot\vec{r}) \quad (62)$$

is the outgoing Coulomb wavefunction for attractive Coulomb scattering and

$$\psi_{n\ell m}(\vec{r}) = \left[(2a_B)^3\frac{(n-\ell-1)!}{2n[(n+\ell)!]^3}\right]^{\frac{1}{2}} e^{-a_B r}(2a_B r)^\ell L^{2\ell+1}_{n-\ell-1}(2a_B r)Y^m_\ell(\theta,\phi) \quad (63)$$

is the bound state wavefunction corresponding to quantum numbers $n\ell m$ with $L^i_j(x)$ being the associated Laguerre polynomial. Using the identity

$$C^2(-\eta_+(k)) = -C^2(\eta_+(k)) + 2\pi\eta_+(k)\coth\pi\eta_+(k) \quad (64)$$

we can write Eq. 61 as

$$
\begin{aligned}
G_C^-(k) &= -G_C^+(k) + 2m_r \int \frac{d^3s}{(2\pi)^3} \frac{2\pi\eta_+(s)\coth\pi\eta_+(s)}{k^2 - s^2 + i\epsilon} \\
&+ \frac{2m_r}{\pi a_B} \sum_{n=1}^{\infty} \frac{\eta_+^2(k)}{n(n^2 + \eta_+^2(k))} \\
&= -\frac{m_r}{2\pi}\left\{\frac{2\mu}{\pi} - \frac{2}{a_B}\left[H(i\eta_+(k)) - i\pi\coth\pi\eta_+(k) - \ln\frac{\mu a_B}{2\pi} - \zeta\right]\right\}
\end{aligned}
\tag{65}
$$

where the integration is done via contour methods and the the contribution from the hyperbolic cotangent poles precisely cancels the bound state term. The resulting attractive Coulomb scattering amplitude is given by

$$
\begin{aligned}
f_C^-(k) &= \frac{C^2(-\eta_+(k))e^{-2i\sigma_0}}{-\frac{2\pi}{m_r C_0(\mu)} - \frac{2\mu}{\pi} + \frac{2}{a_B}\left[H(i\eta_+(k)) - i\pi\coth\pi\eta_+(k) - \ln\frac{\mu a_B}{2\pi} - \zeta\right]} \\
&= \frac{C^2(-\eta_+(k))e^{-2i\sigma_0}}{-\frac{1}{a_0} + \frac{2}{a_B}\left[h(\eta_+(k)) - \ln\frac{\mu a_B}{2\pi} - \zeta\right] - ikC^2(-\eta_+(k))}
\end{aligned}
\tag{66}
$$

Identifying the Coulomb scattering length via

$$
-\frac{1}{a_C} = -\frac{1}{a_0} - \frac{2}{a_B}(\ln\frac{\mu a_B}{2\pi} + \zeta) \tag{67}
$$

this reduces to the simple form

$$
f_C^-(k) = \frac{C^2(-\eta_+(k))e^{-2i\sigma_0}}{-\frac{1}{a_C} + \frac{2}{a_B}[H(i\eta_+(k)) - i\pi\coth\pi\eta_+(k)]} \tag{68}
$$

in agreement with Eq. 25. In order to go to the bound state limit we can utilize the continuation

$$
H(i\eta_+(k)) - i\pi\coth\pi\eta_+(k) \stackrel{k\to i\kappa}{\longrightarrow} H(\xi) + \pi\cot\pi\xi \tag{69}
$$

in which case we find the condition

$$
0 = -\frac{1}{a_C} + \frac{2}{a_B}(H(\xi) + \pi\cot\pi\xi) \tag{70}
$$

in complete agreement with Eq. 27.

In the real world—coupled channel—case the analysis is similar. We must generalize the effective Lagrangian to include off diagonal effects, so that C_0 becomes a matrix $(C_0)_{ij}$ with $i, j = 1, 2$ and the bubble sum must include two forms of intermediate states—charged and neutral. In the absence of Coulomb effects we have then a set of equations

$$T_{ij} = (C_0)_{ij} + \sum_\ell (C_0)_{i\ell} G_0(k_\ell) T_{\ell j} \tag{71}$$

whose solution is

$$T = (1 - C_0 G_0(k))^{-1} C_0 \tag{72}$$

Explicitly, defining

$$D \equiv (1 - (C_0)_{11} G_0(k_1))(1 - (C_0)_{22} G_0(k_2)) - (C_0)_{12}(C_0)_{21} G_0(k_1) G_0(k_2) \tag{73}$$

we find

$$\begin{aligned}
T_{11} &= ((C_0)_{11}(1 - (C_0)_{22} G_0(k_2)) + (C_0)_{12}(C_0)_{21} G_0(k_2))/D \\
T_{21} &= (C_0)_{21}/D \\
T_{12} &= (C_0)_{12}/D \\
T_{22} &= ((C_0)_{22}(1 - (C_0)_{11} G_0(k_1)) + (C_0)_{12}(C_0)_{21} G_0(k_1))/D
\end{aligned} \tag{74}$$

Using the relation between the scattering amplitude and the T-matrix

$$f_{ij} = -\frac{\sqrt{m_r^{(i)} m_r^{(j)}}}{2\pi} T_{ij} \tag{75}$$

the connection between the parameters $(C_0)_{ij}$ and the corresponding scattering lengths a_{ij} is easily found via

$$\begin{aligned}
a_{11} &= \frac{m_r^{(1)}}{2\pi}[(C_0(\mu))_{11} - \mu \frac{m_r^{(2)}}{2\pi} \det C_0(\mu)]/J \\
a_{12} &= \frac{\sqrt{m_r^{(1)} m_r^{(2)}}}{2\pi}(C_0(\mu))_{12}/J \\
a_{21} &= \frac{\sqrt{m_r^{(1)} m_r^{(2)}}}{2\pi}(C_0(\mu))_{21}/J \\
a_{22} &= \frac{m_r^{(2)}}{2\pi}[(C_0(\mu))_{22} - \mu \frac{m_r^{(1)}}{2\pi} \det C_0(\mu)]/J
\end{aligned} \tag{76}$$

with

$$J = 1 - \mu[\frac{m_r^{(1)}}{2\pi}(C_0(\mu))_{11} + \frac{m_r^{(2)}}{2\pi}(C_0(\mu))_{22}] + \mu^2 \frac{m_r^{(1)}}{2\pi}\frac{m_r^{(2)}}{2\pi} \det C_0(\mu) \quad (77)$$

Then Eqs. 76 are seen to be the same as the K-matrix forms Eq. 38. Inclusion of the Coulomb interactions is as before, involving the modifications

$$\begin{aligned} T &\to STS, \quad \text{where} \quad S = \begin{pmatrix} 1 & 0 \\ 0 & C(-\eta_+(k)) \exp -i\sigma_0 \end{pmatrix} \\ a_{ij} &\to a_{ij}^C \quad \text{and} \quad -ik_2 \to \frac{2}{a_B}(H(\xi) + \pi \cot \pi\xi), \end{aligned} \quad (78)$$

and the resulting bound state condition is identical to Eq. 41. Thus the equivalence of the quantum mechanical and effective interaction methods is explicitly demonstrated.

3.1 Effective Range Effects

It is straightforward to go to higher order by inclusion of effective range effects. In the single channel quantum mechanical formulation this is accomplished by the modification

$$a_C \to a_C(1 + \frac{1}{2}a_C r_E k^2 + \ldots) \quad (79)$$

Then Eq. 27 becomes

$$-\frac{1}{a_C} - \frac{r_E}{2\xi^2 a_B^2} + \frac{2}{a_B}(H(\xi) + \pi \cot \pi\xi) \simeq 0 \quad (80)$$

whose solution is

$$\Delta E_n = -E_n \frac{4a_C}{na_B}\left[1 + \frac{a_C}{a_B}(2H(n) - \frac{3}{n}) + \mathcal{O}\left((\frac{a_C}{a_B})^2, \frac{a_C r_E}{a_B^2}\right)\right] \quad (81)$$

so that effects are negligible. Similarly in the two channel case we alter Eq. 37 via

$$a_{ij} \to a_{ij}(1 + \frac{1}{4}a_{ij}(r_E)_{ij}(k_i^2 + k_j^2) + \ldots) \quad (82)$$

and the energy shift and width are modified appropriately

$$\begin{aligned}
\mathrm{Re}\Delta E_n &= -E_n\frac{4a_{22}^C}{na_B}\left[1+\frac{a_{22}^C}{a_B}(2H(n)-\frac{3}{n})+\mathcal{O}\left((\frac{a_{22}^C}{a_B})^2,\frac{a_{22}^C(r_E)_{22}}{sa_B^2}\right)\right] \\
\Gamma_n = 2\mathrm{Im}\Delta E_n &= -E_n\frac{8}{na_B}k_1|a_{12}^C|^2(1+\frac{1}{2}a_{12}^C(r_E)_{12}(k_1^2-\kappa_n^2)-(a_{11}^C)^2k_1^2+\ldots)
\end{aligned} \tag{83}$$

Again for the real component the effective range effect is tiny and can generally be neglected. However, in the case of the decay width there are two types of corrections which should be included—one due to the effective range correction and a second due to multiple rescattering effects in the intermediate state.

Using effective field theory the change is also directly obtained. In the single channel case, one begins by modifying the effective Lagrangian via

$$\mathcal{L}\rightarrow\mathcal{L}+\frac{1}{2}C_2(\Psi_A^\dagger\vec{\nabla}\Psi_A\cdot\Psi_B^\dagger\vec{\nabla}\Psi_B+(\vec{\nabla}\Psi_A^\dagger)\Psi_A\cdot(\vec{\nabla}\Psi_B^\dagger)\Psi_B) \tag{84}$$

The multiple scattering series may summed exactly as before and the solution is found as[14]

$$T_{fi}(k)=(\frac{C_0^R(\mu)+2k^2C_2^R(\mu))C^2(-\eta_+(k))\exp 2i\sigma_0}{1-(C_0^R(\mu)+2k^2C_2^R(\mu))G_C(k)} \tag{85}$$

where $C_0^R(\mu), C_2^R(\mu)$ are the renormalized quantities

$$C_0^R=\frac{C_0+LC_2^2}{(1-C_2J)^2}-2\frac{\alpha m_r\mu}{\pi^2}C_2^R,\qquad C_2^R=\frac{C_2-\frac{1}{2}JC_2^2}{(1-C_2J)^2} \tag{86}$$

and, in cutoff regularization,

$$J=\int\frac{d^3s}{(2\pi)^3}C^2(-\eta_+(s)),\qquad L=\int\frac{d^3s}{(2\pi)^3}s^2C^2(-\eta_+(s)) \tag{87}$$

Thus the scattering amplitude becomes

$$\begin{aligned}
f_C^-(k) &= -\frac{m_r}{2\pi}T_{fi}(k) \\
&= \frac{C^2(-\eta_+(k))e^{-2i\sigma_0}}{-\frac{2\pi}{m_r(C_0^R(\mu)+2k^2C_2^R(\mu))}-\frac{2\mu}{\pi}+\frac{2}{a_B}\left[H(i\eta_+(k))-i\pi\coth\pi\eta_+(k))-\ln\frac{\mu a_B}{2\pi}-\xi\right]} \\
&= \frac{C^2(-\eta_+(k))e^{-2i\sigma_0}}{-\frac{1}{a_C}+\frac{1}{2}r_Ek^2+\frac{2}{a_B}h(\eta_+(k))-ikC^2(-\eta_+(k))}+\ldots
\end{aligned} \tag{88}$$

Thus we identify

$$-\frac{1}{a_C}=-\frac{2\pi}{m_rC_0^R(\mu)}-\frac{2\mu}{\pi}-\frac{2}{a_B}(\ln\frac{\mu a_B}{2\pi}+\zeta),\quad r_E=\frac{8\pi C_2^R(\mu)}{m_r(C_0^R(\mu))^2} \tag{89}$$

and the bound state condition becomes Eq. 80, as expected. As pointed out in ref.[3], the expression for effective range is not affected by Coulombic corrections and consequantly in the case of the NN interaction one expects equality for r_E^{pp} and r_E^{nn}, as found experimentally.

The coupled channel analysis is similar—the momentum dependent effective coupling C_2 becomes a 2×2 matrix and the associated Lippman-Schwinger equation is algebraically somewhat more complex. However, the solution is straightforward and amounts in the end simply to a modification of the effective scattering length parameters a_{ij} defined in Eq. 76 by effective range range effects as given in Eq. 82, where $(r_E)_{ij}$ are given in terms of the matrix elements $C_2(\mu)_{ij}$. Thus the effective interaction bound state condition is identical to Eq. 18, as expected.

4 Applications

While the analysis given above is in some ways old, the applications are not and include two of the most interesting ongoing measurements in contemporary physics. One is the problem of pionium—$\pi^+\pi^-$—whose existence has been claimed by a Russian group[15] and which is now being sought in a major program at CERN. In this case then the first channel represents $\pi^0\pi^0$ while the second is $\pi^+\pi^-$. The reduced mass is

$$m_r^{(1)}=\frac{m_{\pi^0}}{2},\quad m_r^{(2)}=\frac{m_{\pi^+}}{2} \tag{90}$$

and scattering lengths are given in terms of those with total isospin 0,2 via

$$a_{11}=\frac{1}{3}(2a_2+a_0),\quad a_{12}=a_{21}=\frac{\sqrt{2}}{3}(a_2-a_0),\quad a_{22}=\frac{1}{3}(a_2+2a_0) \tag{91}$$

where

$$a_0=-\frac{7m_\pi}{32\pi F_\pi^2},\qquad a_2=\frac{m_\pi}{16\pi F_\pi^2} \tag{92}$$

are the usual Weinberg values with $F_\pi\simeq 92.4$ MeV being the pion decay constant.[16] To lowest order we have a ground state energy shift

$$\Delta E_{gs}^{(0)}(\pi^+\pi^-)=\frac{4\pi}{3m_{\pi^+}}(a_2+2a_0)|\Psi(0)|^2 \tag{93}$$

and a width

$$\Gamma^{(0)}_{gs}(\pi^+\pi^-) = \frac{16\pi}{9m_{\pi^+}}\sqrt{2m_{\pi^0}\Delta m_\pi}(a_2-a_0)^2|\Psi(0)|^2 \qquad (94)$$

which agree with the usual forms.[2] Corrections from higher order effects are found to be

$$\begin{aligned}\frac{\Delta E^{(1)}_{gs}}{\Delta E^{(0)}_{gs}} &= \frac{a_{22}}{a_B}(2H(1)-3)\\ \frac{\Gamma^{(1)}_{gs}}{\Gamma^{(0)}_{gs}} &\simeq (2m_\pi\Delta m_\pi)(\frac{1}{2}a_{12}(r_E)_{12}-a^2_{11})\end{aligned} \qquad (95)$$

Using the lowest order chiral symmetry predictions[16]

$$\frac{1}{2}a_{12}(r_E)_{12} = \frac{4}{3m^2_\pi}, \qquad a^2_{11} = \frac{1}{32\pi^2F^2_\pi} \qquad (96)$$

we find a decay rate correction

$$\frac{\Gamma^{(1)}_{gs}}{\Gamma^{(0)}_{gs}} = 8.3\% \qquad (97)$$

in agreement with the result of Kong and Ravndal.[1] [2] In the case of the energy shift we find a negligible change

$$\frac{\Delta E^{(1)}_{gs}}{\Delta E^{(0)}_{gs}} = -8.2\times 10^{-4} \qquad (98)$$

(Strictly speaking, the charged channel phase shift here—a_{22}—should be replaced by its Coulomb corrected value

$$\frac{1}{a_{22}} \to \frac{1}{a^C_{22}} = \frac{1}{a_{22}} + \frac{2}{a_B}(\ln\frac{a_B}{2R}+1-2\gamma) \qquad (99)$$

However, this is only a small correction numerically.) The future detection of pionium should allow a relatively clean measurement of the $\pi\pi$ scattering lengths.

[2]The dominant effect here is from the effective range, while the rescattering provides only a small correction. The form of our rescattering term differs from that of Kong and Ravndal, as we include only rescattering from physical—$\pi^0\pi^0$—intermediate states while they include also that from (unphysical) charged states.

The other hadronic atom of current experimental interest is π^-p wherein a PSI collaboration has already announced a measurement of the energy shift and decay rate and which plans to further improve these already precise numbers.[17] In this case we are dealing with a three channel system—$\pi^-p, \pi^0n, \gamma n$. However, both the quantum mechanical calculation as well as the effective interaction analysis generalize and the bound state condition—Eq. 41—becomes (after some algebra)

$$\begin{aligned} 0 &= 1+\frac{2}{a_B}(H(\xi)+\pi\cot\pi\xi)\left\{-a_{22}^C\right. \\ &+ \left.\frac{ik_1a_{12}^Ca_{21}^C+ik_3a_{32}^Ca_{23}^C-k_1k_3\left[a_{23}^C(a_{11}^Ca_{32}^C-a_{31}^Ca_{12}^C)+a_{21}^C(a_{12}^Ca_{33}^C-a_{13}^Ca_{32}^C)\right]}{1+ik_1a_{11}^C+ik_3a_{33}^C-k_1k_3(a_{11}^Ca_{33}^C-a_{31}^Ca_{13}^C)}\right\} \end{aligned} \tag{100}$$

To lowest order then we have the results

$$\begin{aligned} \Delta E_{gs}^{(0)} &= -E_1\frac{4a_{22}^C}{a_B} \\ \Gamma_{gs}^{(0)} &= -E_1\frac{8}{a_B}(k_1a_{12}^Ca_{21}^C+k_3a_{13}^Ca_{31}^C) \end{aligned} \tag{101}$$

In the case of the energy shift, the result is as before, but in the case of the decay width we now have two contributions, one from the decay to the π^0n channel and one from decay to γn. However, the πN scattering lengths can then be obtained once the radiative channel is subtracted off via

$$\Gamma_{\pi^0n}=\frac{\Gamma_{tot}}{1+P} \tag{102}$$

where $P=\Gamma_{\gamma n}/\Gamma_{\pi^0n}=1.546\pm0.009$ is the Panofsky ratio.[18]

On the theoretical side the πN scattering lengths can be written in terms of those with total isospin $\frac{1}{2},\frac{3}{2}$ via

$$a_{11}=\frac{1}{3}(a_1+2a_3),\quad a_{12}=a_{21}=\frac{\sqrt{2}}{3}(a_3-a_1),\quad a_{22}=\frac{1}{3}(2a_1+a_3) \tag{103}$$

and the lowest order chiral Lagrangian yields values for these quantities

$$a_1=-2a_3=-\frac{m_\pi}{4\pi F_\pi^2}\frac{1}{1+\frac{m_\pi}{m_N}} \tag{104}$$

and higher order chiral corrections have also been calculated.[19] In the case of the radiative channel, the scattering length is also predicted by chiral symmetry in terms of the Kroll-Ruderman term[20]

$$a_{13} = -\frac{\sqrt{2}eg_A}{8\pi F_\pi}\frac{1}{1+\frac{m_\pi}{M_N}} \tag{105}$$

where $g_A \simeq 1.25$ is the axial coupling in neutron beta decay.

There exist higher order corrections to these lowest order predictions from both rescattering effects—*cf.* Eq. 100—and from inclusion of effective range corrections. However, unlike pionium the latter are not given by chiral symmetry and hence are model dependent. Nevertheless when known corrections are included present experimental results extracted from $\pi^- p$ experiments

$$b_1 = \frac{1}{3}(a_3 - a_1) \simeq 0.096 \pm 0.007 m_\pi^{-1}, \quad b_0 \simeq \frac{1}{3}(a_1 + 2a_3) = 0.0105 \pm 0.007 m_\pi^{-1} \tag{106}$$

are in satisfactory agreement with the chiral symmetry prediction[3]

$$0.096 m_\pi^{-1} \leq b_1 \leq 0.088 m_\pi^{-1} \tag{107}$$

although there is at present a small discrepancy with a parallel measurement of the shift in deuterium.[17] However, our purpose here is not to discuss experimental interpretations, rather only to point out the connection between conventional and effective interaction methods.

5 Conclusions

We have above analyzed the problem of hadronic atom energy shifts due to strong interaction effects both via a traditional quantum mechanical discussion and via a calculation in the effective interaction picture wherein the low energy hadronic interactions are written in terms of an effective local potential and have demonstrated the complete equivalence between the two procedures. Applications have been made to the systems pionium—$\pi^+\pi^-$—and $\pi^- p$ currently being studied experimentally at CERN and PSI respectively.

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[3] The isoscalar prediction is more uncertain, as it has a somewhat senstive dependence on the low energy constants.

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