# Matrix Factorizations and Khovanov Homology

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# MATRIX FACTORIZATIONS AND KHOVANOV HOMOLOGY

A Dissertation Presented

by

ARTHUR WANG

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Department of Mathematics and Statistics

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# ABSTRACT

## MATRIX FACTORIZATIONS AND KHOVANOV HOMOLOGY

SEPTEMBER 2024

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In this thesis we develop a geometric interpretation for Rasmussen's spectral sequences using a construction for Khovanov-Rozansky link homology developed by Oblomkov and Rozansky. In the special case of Khovanov homology, we provide a proof for the geometric construction of Rasmussen's differentials by examining the relationship between matrix factorizations and Soergel bimodules. Finally we leverage the techniques developed in order to provide an alternative method for computing the Khovanov homology of knots and links.

# TABLE OF CONTENTS

CHAPTER

# LIST OF FIGURES

# CHAPTER 1

# INTRODUCTION

## 1.1 Motivation

One of the seminal advances in knot theory occurring when Khovanov defined his categorification of the Jones polynomial [20]. Khovanov homology is a stronger invariant compared to the Jones polynomial and has proven to have useful applications in topology. Most notably Khovanov homology was used by Rasmussen to give an elementary proof of Milnor's conjecture on slice genus. Several years later, Khovanov and Rozansky introduced a new type of link homology theory which categorified the the HOMFLY-PT polynomial [23][21]. The HOMFLY-PT polynomial $\mathcal{P}(L)$ of a link $L$ is a polynomial in two variables $a$ and $q$ which satisfies the following skein relation:

$$a\mathcal{P}(L_-) - a^{-1}\mathcal{P}(L_+) = (q - q^{-1})\mathcal{P}(L_0) \tag{1.1}$$

and has the normalization

$$\mathcal{P}(\bigcirc) = \frac{a - a^{-1}}{q - q^{-1}}. \tag{1.2}$$

It has a particularly interesting specialization when $a = q^N$ for $N > 0$. Let $\mathcal{P}_n(L)(q) = \mathcal{P}(L)(q^N, q)$. The link polynomials $\mathcal{P}_N(L)$ have connections with the representation theory of $U_q(\mathfrak{sl}_N)$. This can be extended to all integers $N$ as follows:

for $N = 0$ corresponds to $U_q(\mathfrak{gl}(1|1))$ while $N < 0$ corresponds to $U_q(\mathfrak{sl}_{-N})$. Note that the equations 1.1 and 1.2 are invariant when exchanging $q$ and $q^{-1}$ so it suffices to consider the case $N \geq 0$. Let us detail some of these polynomials below:

- For $N = 0$, $\mathcal{P}_0(L)$ is the Alexander polynomial.
- For $N = 1$, $\mathcal{P}_1(L)$ is trivial, i.e. for every oriented link $L$, $\mathcal{P}_1(L) = 1$.
- For $N = 2$, $\mathcal{P}_2(L)$ is the Jones polynomial.

In a similar fashion to their construction of Khovanov-Rozansky homology, Khovanov and Rozansky defined a family of bi-graded link homology theories $\mathcal{H}_N(L)$ known as $\mathfrak{sl}_N$ homology which categorify the link polynomials $\mathcal{P}_N(L)$ for $n > 0$ [22].

- For $N = 1$, $\mathcal{H}_1(L)$ is trivial, i.e. for every oriented link $L$, $\mathcal{H}_1(L) \cong \mathbb{Z}$ where $\mathbb{Z}$ sits in degree $(0, 0)$.
- For $N = 2$, $\mathcal{H}_2(L)$ is Khovanov homology.

Knot Floer homology categorifies the Alexander polynomial and was constructed by Ozsváth and Szabó and later independently by Rasmussen. The homology theories $\mathcal{H}_N$ can be defined combinatorially whereas Knot Floer homology cannot. It originates in symplectic geometry and involves counting pseudo-holomorphic curves. Shortly after the publication of Khovanov and Rozansky's work, Dunfeld, Gukov, and Rasmussen put out a series of conjectures about the structure and relationship of these link homology theories [6]. They proposed that Khovanov-Rozansky homology should be a link homology theory which encapsulates all other link homology theories.

Note that when categorifying, the number of gradings increases by one. The HOMFLY-PT polynomial $\mathcal{P}(L)$ is a polynomial in two variables and it categorified by Khovanov-Rozansky homology. Similarly the link polynomials $\mathcal{P}_N(L)$ are polynomials in one variable which are categorified by bi-graded homology theories $\mathcal{H}_N(L)$. Since the specialization $a = q^N$ turns the HOMFLY-PT polynomial into the polynomials $\mathcal{P}_N(L)$, one should hope or expect there should be some analogous procedure at the categorified level. This led to one of their conjectures.

**Conjecture 1.** *Khovanov-Rozansky homology $HHH(L)$ has a family of differentials $\{d_N\}_{N \geq 0}$ such that for $N > 0$, the homology of $(HHH^*(L), d_N)$ is isomorphic to $\mathfrak{sl}_N$ homology. For $N = 0$, the homology of $(HHH^*(L), d_0)$ is isomorphic to Knot Floer homology. In both cases, one must introduce a new grading on the homology groups $HHH^{i,j,k}(L)$ in order for the above statements to make sense.*

Some evidence for the existence of such differentials was given in a follow-up paper [41] by Rasmussen where he proved the following result:

**Theorem 1.1** (Rasmussen)**.** *For $N > 0$ and a link $L \in S^3$, there is a spectral sequence $E_k(N)$ with $E_1(N) \cong HHH(L)$ and $E_\infty(N) \cong \mathcal{H}_N(L)$. For all $k > 0$, the isomorphism type of $E_k(N)$ is a link invariant.*

Rasmussen's result is a weaker version of the conjecture since it is unknown whether the spectral sequence converges after the first differential. Since the publication of Rasmussen's work, progress on the conjecture has stalled; however recent work has revisited Rasmussen's spectral sequences. In [4], the authors constructed a spectral sequence from Khovanov-Rozansky homology to knot Floer homology.

Actually Rasmussen's result was more general. He defined spectral sequences to link homology theories $\mathcal{H}_p$ where $p \in \mathbb{Q}[x]$. The choice of $p(x) = x^{N+1}$ corresponds to the $\mathfrak{sl}_N$-homologies. Motivated by these results, Oblomkov and Rozansky proposed a family of differentials $\{d_{M|K}\}$ which conjecturally solve Rasmussen's original conjecture but also generalize it by defining homology theories corresponding to the super Lie algebra $\mathfrak{gl}(M|K)$. In their work [38], Oblomkov and Rozansky were able to show that these $\mathfrak{gl}(M|K)$ homologies satisfy the Markov moves and computed these homologies for the unknot. When $K = 0$, the $\mathfrak{gl}(M|0)$ invariants were conjectured to agree with the $\mathfrak{sl}_M$ homologies constructed by Khovanov and Rozansky.

## 1.2 Summary of results

Our goal for this thesis is to study Oblomkov and Rozansky's conjecture in depth. We shall prove the statement in the case of $M = 2$ and $K = 0$ that the differential $d_{2|0}$ recovers the Khovanov homology of a link. We prove the following theorem:

**Theorem 1.2.** *Given a braid $\beta$, one can compute the Khovanov homology associated to its braid closure $L(\beta)$ by the formula*

$$\mathrm{Kh}(L(\beta)) = \mathbb{H}(\mathcal{F}_\beta \otimes \mathcal{R}_{2|0}).$$

Here $\mathcal{F}_\beta$ is some object in the derived category of two periodic coherent sheaves on $\mathrm{Hilb}^n(\mathbb{C}^2)$, $D^{\mathrm{per}}(\mathrm{Hilb}^n(\mathbb{C}^2))$ associated to the braid $\beta$. Tensoring with differential defined by $\mathcal{R}_{2|0}$ is understood as the derived tensor product which in this case amounts to taking the usual tensor product with the Koszul complex of $x^2 = 0$.

This theorem provides a geometric interpretation of Rasmussen's spectral sequences which have been conjectured to exist in [38] and [15]. In this geometric setting, Rasmussen's spectral sequences appear more naturally. They arise by taking an appropriate section of the tautological bundle $\mathcal{B}$ on $\mathrm{Hilb}^n(\mathbb{C}^2)$. Furthermore when $\mathcal{F}_\beta$ is a genuine sheaf instead of a complex of sheaves, this leads to a potentially simpler computation compared to Rasmussen's original construction.

## 1.3 Organization of paper

Our paper is organized in the following manner. In Section 2, we recall some background material on knot theory and constructing link invariants. Next in Section 3, we review the constructions of the main link invariants of interest for this paper: the Jones polynomial, Khovanov homology, and Khovanov-Rozansky homology. In section 4, we discuss matrix factorizations, one of the key technical tools used by Oblomkov and Rozansky in their algebro-geometric formulation of Khovanov-Rozansky homology. In Section 5, we explain the connections between the Hilbert scheme of points and link homology. Section 6 contains our main result which proves relationship between Oblomkov and Rozansky's differentials and $\mathfrak{sl}_2$-homology. Finally we end with some computational examples in Section 7.

# CHAPTER 2

# KNOT THEORY AND LINK INVARIANTS

## 2.1 Knot theory

We begin this section with a brief introduction of knot theory. For a more complete overview of the theory, one can refer to the book [19]. We cover only the necessary concepts in order to define and understand what a link invariant is, which is one of the main objects of study in knot theory. Let us start by defining what a knot is.

**Definition 2.1.** *A knot is an embedding of* $S^1$ *into* $S^3$ *or* $\mathbb{R}^3$*.*

**Remark 2.2.** *More generally, one can consider embeddings into other 3-manifolds such as the solid torus which gives rise to annular link invariants, but for the purpose of this paper we will focus on knots in* $S^3$ *or* $\mathbb{R}^3$*. In our case, the compactness of* $S^3$ *is not strictly necessary so the reader may happily consider all knots as being embedded in* $\mathbb{R}^3$ *by assuming that the knot does not pass through the point at infinity.*

A link is a finite collection of nonintersecting or disjoint knots. We consider two links to be equivalent if there is an ambient isotopy between them. The simplest example of a knot is the unknot which is the trivial embedding of $S^1$, while the simplest example of a link is the unlink which is the trivial embedding of two copies of $S^1$. Below we provide some more examples of simple knots and links.

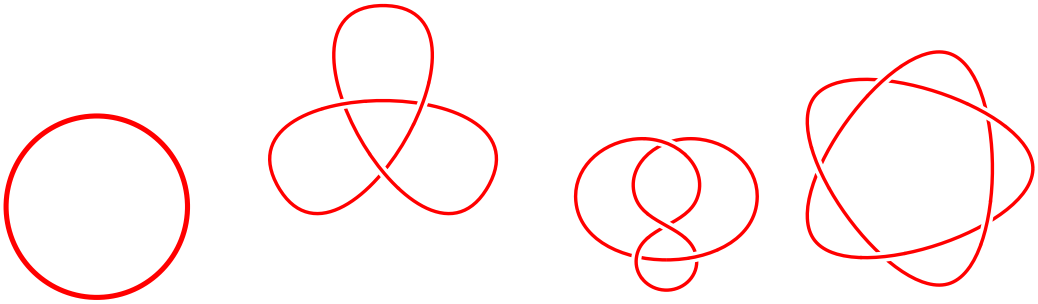


**Figure 1. From left to right: unknot, trefoil knot, figure 8 knot, cinquefoil knot**

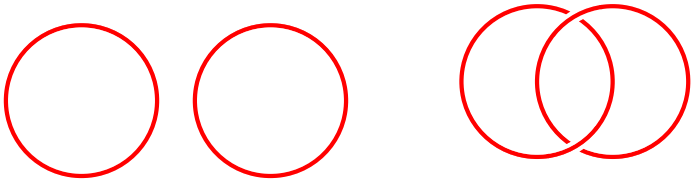


**Figure 2. From left to right: unlink, Hopf link**

Although links are inherently objects that live in three dimensional space, they are often depicted in two dimensions through the use of planar diagrams. More precisely, a planar diagram of a knot is a projection of the link onto a plane. The projection should be a smooth map with no self-tangencies or triple intersections. The preimage of any point in the image of the projection should have size at most two. The points in the planar diagram whose preimage is size two are referred to as crossings. Without loss of generality, such a projection $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ is of the form $(x, y, z) \mapsto (x, y)$. Which part of the link is depicted as lying above the other in planar diagram is determined by whether $z_1 > z_2$ if we have $\pi(x, y, z_1) = \pi(x, y, z_2)$. In the drawings of Figure 1, the unknot has no crossings, the trefoil has 3 crossings, the figure 8 knot has 4 crossings and the cinquefoil knot has 5 crossings.

In practice, one often considers a link along with additional pieces of data. In particular, we will need to consider oriented knots and links. The orientation of a knot is inherited from the orientation of $S^1$. For links, each connected component of the link has an orientation given by the orientation of $S^1$. After adding a choice of orientation to a link, we can further differentiate crossings as being positive crossings or negative crossings. We define ⤱ to be a positive crossing and ⤲ to be a negative crossing.

Although planar diagrams are convenient tools for visualizing links, they are not perfect. There are many different (actually infinitely many) planar diagrams for the same knot or link. Two different planar diagrams can be related by an ambient isotopy in which case they represent the same link. However not only can two seemingly different planar diagrams be associated to the same link, but we can also have two very similar planar diagrams correspond to different links.

Fortunately there is a way to tell whether two planar diagrams represent the same link. There is a collection of modifications one can make to planar diagram graphs known as Reidemeister moves which do not change what type of knot or link the planar diagram represents. We depict the three Reidemeister moves $R1, R2, R3$ in figure 3.

Note that the Reidemeister moves are local procedures in that only a small part of the planar diagram is modified which the rest of the diagram remains unchanged. Reidemeister proved the following theorem which completely solves any ambiguity posed by planar diagrams.

**Theorem 2.3** (Reidemeister's theorem)**.** *Two links can be continuously deformed into one another if and only if their planar diagrams are related by a finite sequence of Reidemeister moves.*

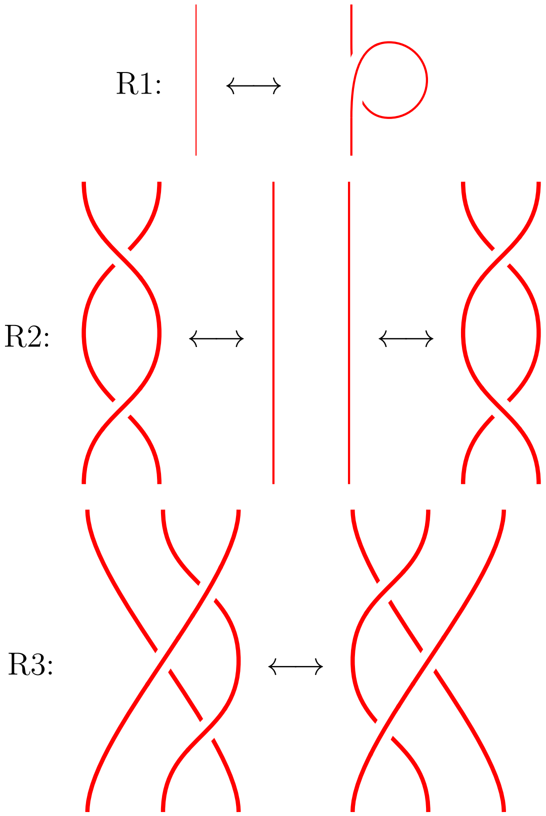


**Figure 3. The three Reidemeister moves**

## 2.2 Constructing link invariants

One of the most basic yet most important questions in knot theory is: when is a knot not the unknot? In the context of the above discussion, one could in principle draw the planar diagram for a knot and exhaustively check sequences of Reidemeister moves. However this is an unsatisfying answer since a priori, one does not know what specific sequence of Reidemeister moves relate two planar diagrams and there can be many such sequences.

An alternative approach to answering such a question is through the use of link invariants. A link invariant is tool that helps to determine whether or not two links are equivalent. These come in various shapes and sizes but two of the most significant link invariants are the Jones polynomial discovered by Vaughan

Jones in 1984 and Khovanov homology developed in the late 1990s by Mikhail Khovanov. Although knots and links are topological objects in nature, one of the modern approaches to constructing link invariants such as the two aforementioned invariants, is through algebra. In order to do so, one must consider braids and the braid group.

**Definition 2.4.** *The braid group on* $n$ *strands* $\mathrm{Br}_n$ *is the group with generators* $\sigma_1, \ldots, \sigma_{n-1}$ *and relations:*

$$\sigma_i\sigma_j = \sigma_j\sigma_i, \quad |i-j| \geq 2$$

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$$

*The second relation is known as the braid relation. Note that this group is closely related to the symmetric group on* $n$ *elements* $S_n$ *which has these relations plus the additional relation of* $\sigma_i^2 = 1$ *for all* $i = 1, \ldots, n-1$.

**Example 1.** *For* $n = 1$*, the braid group* $B_1$ *is the trivial group. For* $n = 2$*,* $B_2$ *is an infinite cyclic group with generator* $\sigma_1$ *and no relations. For* $n \geq 3$*, the braid group is more complicated. In particular, the braid group is nonabelian.*

From the above presentation of the braid group, it is unclear how braids are related to knots and links, but the braid group has another diagrammatic description which will help clarify the relationship. To be more precise, the elements of the braid group are represented by braid diagrams where the diagrams are considered up to ambient isotopy.

The strands of the braid diagram are labeled from 1 to $n$ from bottom to top. Therefore there is a natural embedding of the braid group $\mathrm{Br}_n$ into $\mathrm{Br}_{n+1}$ simply given by adding an extra strand to the top of the braid. By convention, we assign

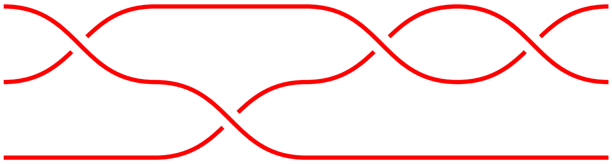


**Figure 4. The diagram above depicts the braid $\beta = \sigma_2\sigma_1\sigma_2^2 \in \mathrm{Br}_3$.**

$\sigma_i$ to positive crossings and $\sigma_i^{-1}$ to negative crossings. Multiplication of elements in the braid group is given by horizontal concatenation of braid diagrams.

Now the relation between braids and links is given by the following two theorems.

**Theorem 2.5** (Alexander's theorem). *Any link is the closure of a braid.*

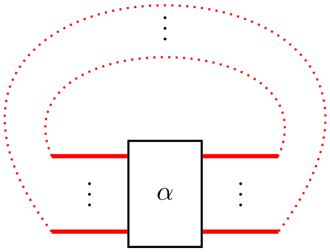


**Figure 5. Taking the closure of the braid $\alpha$ where the corresponding left and right ends of the braid are connected in a non-intersecting fashion.**

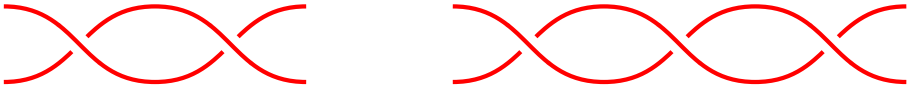


**Figure 6. On the left, the braid $\sigma_1^2 \in \mathrm{Br}_2$ closes up to the Hopf link. On the right, the braid $\sigma_1^3 \in \mathrm{Br}_2$ closes up to the trefoil knot.**

**Theorem 2.6** (Markov's theorem). *The closure of two different braid diagrams represent equivalent links if and only if the braids are related by a finite sequence of moves known as Markov moves.*

M1:

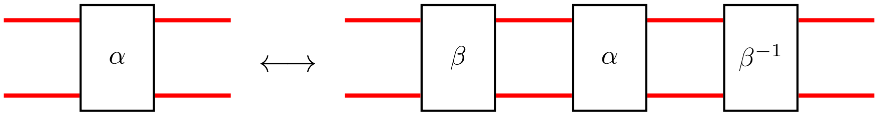


**Figure 7. The first Markov move sends a braid $\alpha \in \mathrm{Br}_n$ to $\beta\alpha\beta^{-1} \in \mathrm{Br}_n$ for any $\beta \in \mathrm{Br}_n$.**

M2:

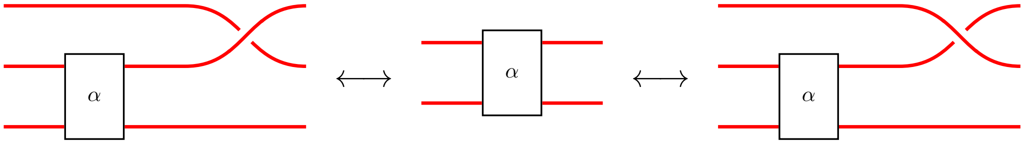


**Figure 8. The second Markov move sends a braid $\alpha \in \mathrm{Br}_n$ to $\alpha\sigma_n^{\pm 1} \in \mathrm{Br}_{n+1}$.**

Note that the first Markov move (M1) means that the closure of a braid $\alpha \in \mathrm{Br}_n$ does not change under conjugation. If one considers the braids $\beta, \beta^{-1}$ and $\alpha$ in the above diagram as beads on a necklace, then taking the braid closure allows one to slide the bead $\beta$ to the other side of $\beta^{-1}$ which allows them to cancel out. The second Markov move (M2) relates a braid $\alpha \in \mathrm{Br}_n$ to another braid $\alpha\sigma_n^{\pm 1} \in \mathrm{Br}_{n+1}$ which can be thought of as extending the topmost strand of the braid $\alpha$ and then twisting it one of two ways back on itself before closing up the braid.

Since Markov's theorem provides necessary and sufficient conditions for two braid closures giving the same link, we now have all the ingredients for constructing a link invariant.

1. Assign objects to the generators $\sigma_i^{\pm 1} \in \mathrm{Br}_n$ for all $i$ and $n$.
2. Check that objects satisfy the relations in the braid group.

3. Construct some operation that mimics taking the closure of a braid.
4. Check that the construction is invariant under the two Markov moves.

**Remark 2.7.** *There are variations on the above procedure. For instance one might only want to consider positive braids in which case the generators* $\sigma_i^{-1}$ *are ignored. It is also common to only impose invariance under the first Markov move (M1) which gives rise to annular link invariants.*

For a more topological approach to constructing link invariants, one can replace the first step in the above procedure by the following two steps.

1. Assign objects to a planar diagram representing a link.
2. Check that objects are invariant under Reidemeister moves.

Although there are many other link invariants, the main invariants we will focus on (Jones polynomial, Khovanov homology and Khovanov-Rozansky homology) all follow the above recipe. This will be detailed in the subsequent section.

# CHAPTER 3

# FROM THE JONES POLYNOMIAL TO LINK HOMOLOGY

In this section we will review the main link invariants of interest: the Jones polynomial, Khovanov homology and triply-graded homology. Although these invariants can be defined more generally with $\mathbb{Z}$-coefficients, we stick to working over $\mathbb{C}$ due to ease of exposition and the fact that our main results cannot be extended from $\mathbb{C}$ to $\mathbb{Z}$. We start by explaining the Jones polynomial.

## 3.1 Jones polynomial

There are two main ways to construct the Jones polynomial: through the Kauffman bracket polynomial or through the Hecke algebra. Both methods are relevant for us, but we will first focus on the latter approach before returning to the first approach as a segue to Khovanov homology. Our discussion summarizes the construction given in [19]. Let us begin by defining the Hecke algebra.

**Definition 3.1.** *The Hecke algebra* $H_n$ *is the unital associative* $\mathbb{C}(Q, Z)$*-algebra* *generated by* $T_1, T_2, \ldots, T_{n-1}$ *with the relations*

$$T_iT_j = T_jT_i \quad \text{for } |i-j| \geq 2 \tag{3.1}$$

$$T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1} \tag{3.2}$$

$$T_i^2 = ZT_i + Q \tag{3.3}$$

The first two relations should be familiar since they are identical to the relations found in the braid group or symmetric while the third relation is known as the quadratic relation.

**Remark 3.2.** *The Hecke algebra is commonly presented using a single parameter* $Q$ *instead of two parameters* $Q$ *and* $Z$ *as above. After possibly extending scalars and setting* $Z = Q - 1$*, these algebras are isomorphic. For the sake of constructing the Ocneanu-Jones trace, we shall stick to the two-parameter version. In the one-parameter version, the quadratic relation is often written as* $T_i^2 = (Q - Q^{-1})T_i + 1$ *or* $T_i^2 = (Q^{1/2} - Q^{-1/2})T_i + 1$ *depending on the convention. Furthermore this single parameter* $Q$ *is often written in lower case, but we use upper case to distinguish it from another* $q$ *we will use later on.*

**Remark 3.3.** *There are two ways to interpret the Hecke algebra. On one hand, the Hecke algebra can be regarded as a quotient of the group algebra of the braid group by the quadratic relation. On the other hand, the Hecke algebra is a deformation of the group algebra of the symmetric group with a formal parameter* $Q$*. After specializing to* $Q = 1$ *and* $Z = 0$*, the group algebra of the symmetric group is recovered. What makes the Hecke algebra a well-suited object for link invariants is precisely that it in some sense interpolates between the braid group and the symmetric group. The Hecke algebra retains the topological properties of braid group while being as well-understood as the symmetric group in terms of representation theory.*

For $w \in S_n$, define $T_w = T_{i_1} \cdots T_{i_r}$ where $s_{i_1} \cdots s_{i_r}$is a reduced expression for $w$. Note that by the above relations, $T_w$ is unique. Then multiplication in the Hecke algebra is defined as

**Lemma 3.4.** *For any $w \in S_n$ and simple transposition $s_i$,*

$$T_i T_w = \begin{cases} T_{s_i w} & \text{if } l(s_i w) > l(w) \\ \\ Q T_{s_i w} + Z T_w & \text{if } l(s_i w) < l(w). \end{cases}$$

We need the following results on the structure of the Hecke algebra $H_n$. We shall omit the proofs since these are standard results.

**Lemma 3.5.** *As a left $\mathbb{C}(Q, Z)$-module $H_n$ is free of rank $n!$ with basis $\{T_w | w \in S_n\}$.*

There is a natural inclusion $\iota : H_n \to H_{n+1}$ given by sending each generator $T_i$ of $H_n$ to the corresponding generator in $H_{n+1}$. This turns $H_{n+1}$ into both a left and right $H_n$-module.

**Lemma 3.6.** *The homomorphism $\iota : H_n \to H_{n+1}$ is injective. As a left $H_n$-module, $H_{n+1}$ is free of rank $n + 1$ with basis*

$$\{1, T_n, T_n T_{n-1}, \ldots, T_n T_{n-1} \cdots T_2 T_1\}$$

**Proposition 3.7.** *For $n \geq 2$, there is an isomorphism of $\mathbb{C}(Q, Z)$-modules*

$$\varphi : H_n \oplus \left(H_n \otimes_{H_{n-1}} H_n\right) \to H_{n+1}$$

*given by*

$$\varphi\left(a + \sum_i b_i \otimes c_i\right) = \iota(a) + \sum_i \iota(b_i) T_n \iota(c_i)$$

*for $a, b_i, c_i \in H_n$ where the summation above is finite.*

Now we construct the Ocneanu-Jones trace $\tau_n : H_n \to \mathbb{C}(Q, Z)$ for $n \geq 1$. These traces are defined inductively. For $n = 1, \tau_1 : H_1 = \mathbb{C}(Q, Z) \to \mathbb{C}(Q, Z)$ is the identity map. For $n = 2, \tau_2 : H_2 \to \mathbb{C}(Q, Z)$ is defined on the basis $\{1, T_1\}$ by

$$\tau_2(1) = \frac{1-Q}{Z} \quad \tau_2(T_1) = 1.$$

Then for $n \geq 2$, we can construct $\tau_{n+1}$ from $\tau_n$ using the isomorphism in Proposition 3.7. For $a, b, c \in H_n$, set

$$\tau_{n+1}(\varphi(a)) = \frac{1-Q}{Z}\tau_n(a) \quad \tau_{n+1}(\varphi(b \otimes c)) = \tau_n(bc)$$

One can check $\tau_n$ is linear and satisfies the following properties.

**Proposition 3.8.** *For all $n \geq 1$ and $a, b \in H_n$, we have*

1. $\tau_n(ab) = \tau_n(ba)$
2. $\tau_{n+1}(T_n a) = \tau_{n+1}(T_n^{-1} a) = \tau_n(a)$

Notice that two properties above are the algebraic analogs of the Markov moves defined earlier. Therefore we get the following result.

**Theorem 3.9.** *Let $\omega_n : \mathrm{Br}_n \to H_n$ be the group homomorphism defined by $\sigma_i \mapsto T_i$ for $i = 1, \ldots, n-1$. Let $L$ be an oriented link in $\mathbb{R}^3$ and $\beta \in \mathrm{Br}_n$ a braid whose braid closure is isotopic to $L$. Then $\tau_n(\omega_n(\beta))$ is a link invariant.*

If we let $Q = t^2$ and $Z = -t^{1/2} + t^{-1/2}$, then the link invariant constructed above is exactly the Jones polynomial of the link $\mathcal{J}(L)$. Let $(L_+, L_-, L_0)$ denote a Conway skein triple. These are three oriented link diagrams that are the same outside of a small ball where the tangles look like the images below.

Then the Jones polynomial satisfies the following skein relation which is useful for computations.

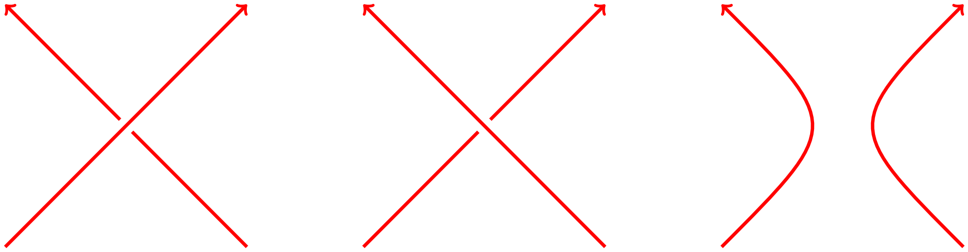


**Figure 9. Diagrams for the Conway triple** $(L_+, L_-, L_0)$**.**

**Corollary 3.10.** *The Jones polynomial of the unknot is 1 and given a Conway skein triple* $(L_+, L_-, L_0)$*, the Jones polynomial satisfies the relation*

$$-t^{-1}\mathcal{J}(L_+) + t\mathcal{J}(L_-) = (t^{1/2} - t^{-1/2})\mathcal{J}(L_0)$$

**Example 2.** *We can use the above corollary along with the diagrams from the previous section to compute the Jones polynomial of the Hopf link and the trefoil.*

$$\mathcal{J}(\text{Hopf link}) = t^{1/2} + t^{5/2}, \quad \mathcal{J}(\text{trefoil}) = t + t^3 - t^4.$$

**Remark 3.11.** *Note that the version of the Jones polynomial constructed above is actually the normalized Jones polynomial since the Jones polynomial of the unknot is trivial. Since the Jones polynomial is a function of* $t^{1/2}$ *and* $t^{-1/2}$*, it is common to introduce a change of variables* $q = t^{1/2}$*. We can then define the unnormalized Jones polynomial* $\overline{\mathcal{J}}(L)$ *(also known as the* $\mathfrak{sl}_2$ *polynomial) which is given by* $\overline{\mathcal{J}}(L) = (q + q^{-1})\mathcal{J}(L)$*. Then*

$$\overline{\mathcal{J}}(\text{Hopf link}) = 1 + q^2 + q^4 + q^6, \quad \overline{\mathcal{J}}(\text{trefoil knot}) = q + q^3 + q^5 - q^9. \tag{3.4}$$

*Our* $q$ *here is less natural than the* $Q$ *parameter used for defining the Hecke algebra, but this* $q$ *is more closely related to the quantum group* $U_q(\mathfrak{sl}_2)$ *and as we shall see later on, this has geometric connections.*

## 3.2 Khovanov homology

In this section, we will review the basics of Khovanov homology. We will paraphrase the presentation given in the papers of Bar-Natan [2][1]. The reader should look through those papers for a more detailed account.

First we need to discuss the Kauffman bracket polynomial. Let $L$ be an oriented link in $\mathbb{R}^3$. Let $\chi$ denote the set of crossings of $L$. Let $n = |\chi|$ and let $n_+$ denote the number of positive (right-handed) crossings ⤱ and let $n_-$ denote the number of negative (left-handed) crossings ⤲.

**Definition 3.12.** *The Kauffman bracket* $\langle L\rangle$ *of a link* $L$ *is a Laurent polynomial in* $\mathbb{Z}[q, q^{-1}]$ *defined by the following axioms*

$$\langle \emptyset \rangle = 1; \qquad \langle \bigcirc \sqcup L\rangle = (q+q^{-1})\langle L\rangle; \qquad \langle ╳ \rangle = \langle ≍ \rangle - q\langle )( \rangle.$$

**Definition 3.13.** *The (unnormalized) Jones polynomial for an oriented link* $L$ *is given by* $\overline{\mathcal{J}}(L) = (-1)^{n_-} q^{n_+ - 2n_-}\langle L\rangle$.

Let us review how compute these polynomials using a cube of resolutions. Let ≍ be the "0"-smoothing of ╳ and )( be the "1"-smoothing. Fix some labeling of the crossing in the link. Then we can form an $n$-dimensional cube $\{0,1\}^{\chi}$ such that each vertex $\alpha \in \{0,1\}^{\chi}$ corresponds to a complete smoothing $S_\alpha$, i.e. all the crossings are resolved by either a "0"-smoothing or "1"-smoothing until the result is a union of a finite number of circles. What this means is that in each link, a crossing is replaced by a "0"-smoothing or "1"-smoothing, until the link no longer possess any crossings at all. At this point, the link has been "resolved" into a finite union of circles.

To compute the Jones polynomial, if $S_\alpha$ consists of $k$ circles, then we associate to $S_\alpha$, the term $(-1)^r q^r (q+q^{-1})^k$ where $r$ is the number of "1"-smoothings used to obtain $S_\alpha$, which one can think of as the height of a smoothing. Then we sum over all terms for all $\alpha$ and multiply by $(-1)^{n_-}q^{n_+-2n_-}$ to obtain the Jones polynomial.

**Example 3.** *Using this procedure for the right-handed trefoil knot* $(n_+, n_-) = (3, 0)$, *one compute its Jones polynomial as* $\overline{\mathcal{J}}($$) = q + q^3 + q^5 - q^9$.

To obtain Khovanov homology, we essentially have to categorify the whole procedure described in the previous section. Khovanov's key insight was to replace polynomials by graded vector spaces with the appropriate grading shifts. This can be thought of as the graded analog of replacing Betti numbers by vector spaces of the corresponding dimension. Before we dive in, let us take a brief digression to set some notation.

**Definition 3.14.** *Let* $W = \bigoplus_m W_m$ *be a graded module with homogeneous components* $\{W_m\}$. *The graded dimension of* $W$ *is given by* $\dim_q W := \sum_m q^m \dim W_m$.

**Definition 3.15.** *Let* $\cdot\{l\}$ *be a degree shift operation on graded modules. If* $W = \bigoplus_m W_m$ *is a graded vector space, then* $W\{l\}_m = W_{m-l}$. *Thus* $\dim_q W\{l\} = q^l \cdot \dim_q W$. *Note that* $\cdot\{l\}$ *extends to chain complexes of graded modules by applying the operation to each degree. One can think of* $\cdot\{l\}$ *as a "vertical shift" on chain complexes.*

**Definition 3.16.** *Let* $\cdot[s]$ *be a degree shift operation on chain complexes. If* $\mathcal{D}^\bullet = \mathcal{C}^\bullet[s]$ *for the chain complex* $\mathcal{C}^\bullet = \cdots \to \mathcal{C}^r \to \mathcal{C}^{r+1} \to \cdots$, *then* $\mathcal{D}^r = \mathcal{C}^{r-s}$. *One can think of* $\cdot[s]$ *as a "horizontal shift" on chain complexes.*

Khovanov's construction replaces the Kauffman bracket $\langle L\rangle$ of a link $L$ by $[[L]]$, a chain complex of graded free $\mathbb{Z}$-modules. This new bracket satisfies the following axioms, which are analogous to the Kauffman bracket axioms:

$$[[\emptyset]] = 0 \to \mathbb{Z} \to 0; \quad [[\bigcirc \sqcup L]] = V \otimes [[L]]; \quad [[\times]] = \mathcal{F}(0 \to [[\asymp]] \xrightarrow{d} [[)(]]\{1\} \to 0).$$

Here $V$ is a graded module of graded dimension $q + q^{-1}$. One can choose to think of $V$ as generated by the basis elements $v_+$ and $v_-$ where $v_+$ has degree $+1$ and $v_-$ has degree $-1$. The operation $\mathcal{F}$ flattens a bicomplex into a single complex by taking direct sums along diagonals. The degree shift operation $\{1\}$ is the analog of multiplying by $q$.

Let us proceed with the actual construction. For every vertex $\alpha \in \{0,1\}^\chi$, we form the graded module $V_\alpha(L) = V^{\otimes k}\{r\}$ where $k$ is the number of circles corresponding to the smoothing for $\alpha$ and $r$ is the height of the smoothing, $r = |\alpha| = \sum\limits_i \alpha_i$ where $\alpha_i$ is the $i^{\text{th}}$ entry in the tuple $\alpha$. We define the $r^{\text{th}}$ chain group $[[L]]^r$ to be the direct sum of all $V_\alpha(L)$ such that $|\alpha| = r$. Then define $\mathcal{C}(L) = [[L]][-n_-]\{n_+ - 2n_-\}$. Note the degree shifts are the analog of multiplying by $(-1)^{n_-}q^{n_+-2n_-}$ when we compute the Jones polynomial. Technically $\mathcal{C}(L)$ is not yet a chain complex since we have not described the differentials between chain groups. Before doing so, the following theorem is already clear based on our construction.

**Definition 3.17.** *Let* $\chi_q(\mathcal{C})$ *denote the graded Euler characteristic of a chain complex* $\mathcal{C}$*, so* $\chi_q(\mathcal{C}) = \sum\limits_r (-1)^r \dim_q \mathcal{C}^r$*.*

**Theorem 3.18.** *The graded Euler characteristic of* $\mathcal{C}(L)$ *is the unnormalized Jones polynomial* $\mathcal{J}(L)$*:*

$$\chi_q(\mathcal{C}(L)) = \mathcal{J}(L).$$

Now let us discuss how to form the differentials. As you travel along an edge of the cube $\{0,1\}^\chi$ from $\mathcal{C}^r$ to $\mathcal{C}^{r+1}$, one of two possibilities will occur: either two circles will merge into one circle or one circle will split into two circles. These correspond to two linear maps

$$\begin{aligned} m : V \otimes V &\to V & \Delta : V &\to V \otimes V \\ v_+ \otimes v_+ &\mapsto v_+ & v_+ &\mapsto v_+ \otimes v_- + v_- \otimes v_+ \\ v_+ \otimes v_- &\mapsto v_- & v_- &\mapsto v_- \otimes v_- \\ v_- \otimes v_+ &\mapsto v_- \\ v_- \otimes v_- &\mapsto 0 \end{aligned}$$

Note that with our choices of $m$ and $\Delta$, all the faces of our cube are commutative squares so we need to decorate the edges of the cube with some signs so our differential actually squares to zero. Signs are added according to the following rule: if we have a directed edge of the cube connecting a vertex $\alpha$ to the vertex $\beta$, there is some $i \in [1, n]$ such that $\alpha_i \neq \beta_i$ and all other entries of $\alpha$ and $\beta$ are equal. Let $\eta = \sum\limits_{j<i} \alpha_j$. Then our edge of the cube is the map $m$ or $\Delta$ and we multiply this by the sign $(-1)^\eta$.

Now that we have equipped $\mathcal{C}(L)$ with the appropriate differentials, we have a genuine chain complex so we can make the following definition.

**Definition 3.19.** *Let $\mathcal{H}^r(L)$ denote the $r^{th}$ homology of $\mathcal{C}(L)$. Then let $Kh(L)$ be the graded Poincaré polynomial of the complex $\mathcal{C}(L)$ in the variable $t$ so*

$$Kh(L) := \sum_r t^r \dim_q \mathcal{H}^r(L)$$

**Theorem 3.20** (Khovanov)**.** *The polynomial $Kh(L)$ is a link invariant which specializes to the unnormalized Jones polynomial at $t = -1$.*

The proof of this theorem is accomplished by checking that the chain complex $\mathcal{C}(L)$ is invariant under Reidemeister moves. This uses standard homological algebra techniques which we shall omit, but the interested reader can check the papers of Bar-Natan [2][1] for further details.

There is an extremely useful technique for simplifying the unwieldy complexes that often occur in link homology. Note that the size of the chain complex in Khovanov homology (and also Khovanov-Rozansky homology) depends exponentially on the number of crossings $n$ in the link. It is often convenient to simplify the complex as much as possible before attempting any computations. The main tool is a homological algebra technique known as Gaussian elimination.

**Lemma 3.21** (Gaussian elimination). *Suppose that $\mathcal{C}$ is an additive category and we have a chain complex of the form*

$$\begin{tikzcd}[ampersand replacement=\&]
\begin{bmatrix} C \end{bmatrix} \arrow[r, "{\begin{pmatrix} \alpha \\ \beta \end{pmatrix}}"] \& \begin{bmatrix} b_1 \\ D \end{bmatrix} \arrow[r, "{\begin{pmatrix} \phi & \delta \\ \gamma & \epsilon \end{pmatrix}}"] \& \begin{bmatrix} b_2 \\ E \end{bmatrix} \arrow[r, "{\begin{pmatrix} \mu & \nu \end{pmatrix}}"] \& \begin{bmatrix} F \end{bmatrix}
\end{tikzcd}$$

*such that $\phi : b_1 \to b_2$ is an isomorphism. Then this complex is isomorphic to the complex*

$$\begin{tikzcd}[ampersand replacement=\&]
\begin{bmatrix} C \end{bmatrix} \arrow[r, "{\begin{pmatrix} 0 \\ \beta \end{pmatrix}}"] \& \begin{bmatrix} b_1 \\ D \end{bmatrix} \arrow[r, "{\begin{pmatrix} \phi & 0 \\ 0 & \epsilon - \gamma\phi^{-1}\delta \end{pmatrix}}"] \& \begin{bmatrix} b_2 \\ E \end{bmatrix} \arrow[r, "{\begin{pmatrix} 0 & \nu \end{pmatrix}}"] \& \begin{bmatrix} F \end{bmatrix}
\end{tikzcd}$$

Both of the above complexes are homotopy equivalent to the complex

$$\begin{tikzcd}[ampersand replacement=\&]
\begin{bmatrix} C \end{bmatrix} \arrow[r, "{\begin{pmatrix} \beta \end{pmatrix}}"] \& \begin{bmatrix} D \end{bmatrix} \arrow[r, "{\begin{pmatrix} \epsilon - \gamma\phi^{-1}\delta \end{pmatrix}}"] \& \begin{bmatrix} E \end{bmatrix} \arrow[r, "{\begin{pmatrix} \nu \end{pmatrix}}"] \& \begin{bmatrix} F \end{bmatrix}
\end{tikzcd}$$

*Proof.* Note that one obtains the second complex from the first complex by a series of suitable invertible column and row operations. The second complex has the contractible subcomplex

$$\begin{tikzcd}
0 \arrow[r] & b_1 \arrow[r, "\phi"] & b_2 \arrow[r] & 0
\end{tikzcd}$$

since by assumption $\phi$ is an isomorphism. Therefore the result follows. $\square$

**Example 4.** *We can compute the Khovanov homology of the Hopf link and trefoil knot using the above technique. We keep track of the homological degree with the variable $t$.*

$$Kh(\text{Hopf link}) = 1 + q^2 + q^4t^2 + q^6t^2 \quad Kh(\text{trefoil knot}) = q + q^3 + q^5t^2 + q^9t^3$$

*We take the Euler characteristic by setting $t = -1$. Compare the results with the computation found in equation 3.4.*

Note that the version of Khovanov homology presented above is a weaker version of the theory since it is not fully functorial. Topologically, one can construct maps between different links through cobordisms which one should hope induces a map on Khovanov homology. To obtain a version of Khovanov homology which is functorial, one must work with a category of cobordisms [1] or more generally with the Clark-Morrison-Walker category of $\mathfrak{sl}_N$-foams [5].

## 3.3 Khovanov-Rozansky homology

We now will discuss Khovanov-Rozansky homology. Our presentation here follows notes from the AIM link homology research community [14]. One can also consult the following sources [12] [26] [21] for a more detailed account.

Let us revisit the (one-parameter) Hecke algebra. Kazhdan and Lusztig famously constructed a basis $\{b_w\}_{w \in S_n}$for the Hecke algebra which has deep connections to combinatorics and representation theory. Let us write $b_i$ instead of $b_{s_i}$. Then the Hecke algebra is generated as an algebra by $b_i, i = 1, \ldots, n-1$ and the usual relations become

$$b_i^2 = (q + q^{-1})b_i \tag{3.5}$$

$$b_i b_j = b_j b_i \quad \text{for } |i - j| \geq 2 \tag{3.6}$$

$$b_i b_{i+1} b_i + b_{i+1} = b_{i+1} b_i b_{i+1} + b_i. \tag{3.7}$$

Soergel introduced the category of Soergel bimodules which categorifies the Hecke algebra [42]. Let us now define Soergel bimodules before explaining his categorification.

Let $R = \mathbb{C}[x_1, \ldots, x_n]$ be the polynomial ring equipped with the action of the symmetric group $S_n$ which permutes the variables $x_i$. We will consider the subrings $R^{s_i}$ which are the rings of invariants of polynomials which are invariant under the transposition $s_i$ which swaps the variables $x_i$ and $x_{i+1}$. We will work with $R - R$ bimodules.

**Remark 3.22.** *Since $R - R$ bimodules are equivalent to $R \otimes R^{\text{op}}$-left modules which are the same as $R \otimes R$-left modules since $R$ is commutative, it is often convenient to think instead of modules over the ring $\mathbb{C}[x_1, \ldots, x_n, x'_1 \ldots, x'_n]$ where the left action of $R$ corresponds to the action of $x_i$ and the right action of $R$ corresponds to the action of $x'_i$. Given two $R - R$ bimodules $M$ and $N$, we can form their tensor product $M \otimes_R N$. The left action of $R$ on $M$ and $N$ will be denoted by $x_i$ and $x'_i$ respectively while the right action of $R$ on $M$ and $N$ will be denoted by $x'_i$ and $x''_i$ respectively.*

The ring $R$ is a graded ring with $\deg(x_i) = 2$ for $i = 1, \ldots, n$. All our bimodules will then be graded $R$-bimodules, so we have a decomposition $M = \bigoplus_{j \in \mathbb{Z}} M_j$ with $x_i M_j, M_j x_i \subseteq M_{j+2}$. We will use $(1)$ to denote a grading shift down by 1, i.e. $M(1)_i = M_{i+1}$.

**Remark 3.23.** *We keep track of the grading with the variable $Q$. With these conventions for grading shifts, if the graded dimension of $M$ is given by* $\mathrm{gdim}(M) = \sum_i Q^i \dim M_i$ *then* $\mathrm{gdim}(M(1)) = Q^{-1}\mathrm{gdim}(M)$.

One of the basic bimodules we will consider is

$$B_i := R \otimes_{R^{s_i}} R(1) = \frac{\mathbb{C}[x_1, \dots, x_n, x_1', \dots, x_n']}{(x_i + x_{i+1} = x_i' + x_{i+1}', x_i x_{i+1} = x_i' x_{i+1}', x_j = x_j' (j \neq i, i+1))}.$$

Note the grading shift in the definition of $B_i$ which means that the degree of $x_j$ in $B_i$ is 1 for all $j$ and the degree of 1 in $B_i$ is $-1$.

**Definition 3.24.** *The Bott-Samelson bimodule corresponding to an expression $w = (s_1, \dots, s_n)$, denoted by $B_w$ is the graded $R - R$ bimodule given by*

$$B_w := B_{s_1} B_{s_2} \cdots B_{s_n}.$$

**Definition 3.25.** *The category* $\mathrm{SBim}_n$ *of Soergel bimodules is the smallest full subcategory of the category of graded $R - R$ bimodules containing $R$ and $B_i$ which is closed under direct sums, grading shifts, tensor products and most importantly direct summands. In other words, a Soergel bimodule is a direct summand of a finite direct sum of grading shifts of Bott-Samelson bimodules.*

**Lemma 3.26.** *We have the $R - R$ bimodule isomorphism*

$$B_i \otimes_R B_i \simeq B_i(1) \oplus B_i(-1)$$

*Proof.* By definition, we have that

$$B_i \otimes_R B_i = (R \otimes_{R^{s_i}} R(1)) \otimes_R (R \otimes_{R^{s_i}} R(1)) = R \otimes_{R^{s_i}} R \otimes_{R^{s_i}} R(2)$$

Note that we can decompose the ring $R$ into its $s_i$-invariant part and its $s_i$-invariant part so $R \simeq R^{s_i} \oplus R^{s_i}(-2)$. Given $f \in R$, the desired isomorphism follows from the map

$$f \mapsto \left( \frac{f + s_i \cdot f}{2}, (x_i - x_{i+1}) \frac{f - s_i \cdot f}{2(x_i - x_{i+1})} \right).$$

Therefore we have

$$\begin{aligned} R \otimes_{R^{s_i}} R \otimes_{R^{s_i}} R(2) &= R \otimes_{R^{s_i}} (R^{s_i} \oplus R^{s_i}(-2)) \otimes_{R^{s_i}} R(2) \\ &= (R \otimes_{R^{s_i}} R(2)) \oplus (R \otimes_{R^{s_i}} R) \\ &= B_i(1) \oplus B_i(-1) \end{aligned}$$

□

**Lemma 3.27.** *We have the $R-R$ bimodule isomorphism*

$$B_i \otimes_R B_j \simeq B_j \otimes_R B_i \quad |i-j| \geq 2$$

*Proof.* This is straightforward. □

**Lemma 3.28.** *We have the $R-R$ bimodule isomorphism*

$$(B_i \otimes_R B_{i+1} \otimes_R B_i) \oplus B_{i+1} \simeq (B_{i+1} \otimes_R B_i \otimes_R B_{i+1}) \oplus B_i$$

*Proof.* Let $R^{s_i,s_{i+1}}$ be the ring of invariants under the actions of $s_i$ and $s_{i+1}$ and define $B_{i,i+1} = R \otimes_{R^{s_i,s_{i+1}}} R(3)$. We need to show the isomorphisms

$$B_i \otimes_R B_{i+1} \otimes_R B_i \simeq B_{i,i+1} \oplus B_i$$

$$B_{i+1} \otimes_R B_i \otimes_R B_{i+1} \simeq B_{i,i+1} \oplus B_{i+1}$$

which imply the desired result. The isomorphisms above follow easily above from a diagrammatic description of the Soergel bimodule category and the corresponding Jones-Wenzl projectors. However showing the isomorphism explicitly can be quite tricky and technical. One can find the explicit details in [12]. □

The lemmas $3.26, 3.27, 3.28$ are parallel to the relations $3.5, 3.6, 3.7$ and help to illustrate Soergel's remarkable result on the categorification of the Hecke algebra.

Let us denote $B_{\underline{w}} = B_{i_1} \otimes_R \cdots \otimes_R B_{i_r}$ for $w = s_{i_1} \cdots s_{i_r}$. Then any $s \in S_n$, we can define the indecomposable Soergel bimodule $B_s$ as the unique indecomposable bimodule which appears as a direct summand of $B_{\underline{w}}$ where $w$ is a reduced expression for $s$ and $B_s$ does not appear as a direct summand for any $B_{\underline{v}}$ where $l(v) < l(w)$.

**Theorem 3.29** (Soergel)**.** *The Hecke algebra $H_n$ is categorified by the category of Soergel bimodules* $\mathrm{SBim}_n$ *in the following sense:*

- *There is a bijection between $S_n$ and the set of all indecomposable objects of* $\mathrm{SBim}_n$ *up to shift and isomorphism:*

$$w \in S_n \leftrightarrow B_w.$$

- *The Hecke algebra is isomorphic to the Grothendieck group $K_0(\mathrm{SBim}_n)$. The isomorphism is given by*

$$b_s \leftrightarrow [B_s].$$

- *Multiplication by $Q$ corresponds to a grading shift: $[M(d)] = Q^d[M]$. Multiplication in the Hecke algebra corresponds to the tensor product of bimodules: $[M] \cdot [N] := [M \otimes_R N]$.*

**Remark 3.30.** *The work done above is enough to classify all indecomposable Soergel bimodules for $S_3$ and verify Soergel's categorification theorem.*

In order to define Khovanov-Rozansky homology, we will need to work in the homotopy category of (bounded) complexes of Soergel bimodules, which we denote as $\mathcal{K}_n := \mathcal{K}^b(\mathrm{SBim}_n)$.

**Remark 3.31.** *One natural question one might ask is why should we work with the homotopy category instead of the derived category. There are a couple of reasons for this. Firstly the category of Soergel bimodules is additive but not abelian. This means that the homotopy category $\mathcal{K}_n$ makes sense, but one cannot naively define* *a derived category of Soergel bimodules. In principle, one can try working with a subcategory of the derived category of all $R-R$ bimodules generated by Soergel bimodules, but this category does not have the desired properties and loses a lot of information. Notably, Rouquier complexes defined in equation (3.8) below become isomorphic up to a grading shift.*

**Proposition 3.32.** *We have the following well-defined morphisms of graded bimodules:*

1. *the map $m_i : B_i(-1) \to R$ which sends 1 to 1.*
2. *the map $\Delta_i : R \to B_i(1)$ which sends $1$ to $x_i - x'_{i+1}$.*

*Proof.* First note that the shifts are added to ensure that we have a genuine map of graded bimodules. Checking that $m$ is a morphism is clear so let us only present the case of $\Delta$.

Let $\frac{1}{2}(x_i - x_{i+1}) = \alpha_i$. Then we have

$$\begin{aligned}\alpha_i \otimes 1 + 1 \otimes \alpha_i &= \frac{1}{2} x_i - x_{i+1} + x'_i - x'_{i+1} \\ &= x_i - \frac{1}{2}(x_i + x_{i+1}) + \frac{1}{2}(x'_i + x'_{i+1}) - x'_{i+1} \\ &= x_i - x'_{i+1}.\end{aligned}$$

We must check that

$$f(\alpha_i \otimes 1 + 1 \otimes \alpha_i) = (\alpha_i \otimes 1 + 1 \otimes \alpha_i)f$$

for any $f \in R$. By the isomorphism $R \simeq R^{s_i} \oplus R^{s_i}(-2)$, we can write $f = g + \alpha_i h$ where $g, h \in R^{s_i}$. Then we compute

$$\begin{aligned}f\alpha_i \otimes 1 + f \otimes \alpha_i &= (g + \alpha_i h)\alpha_i \otimes 1 + (g + \alpha_i h) \otimes \alpha_i \\ &= \alpha_i \otimes g + 1 \otimes \alpha_i h \alpha_i + 1 \otimes \alpha_i g + \alpha_i \otimes h\alpha_i \\ &= \alpha_i \otimes f + 1 \otimes \alpha_i f.\end{aligned}$$

□

Using these maps, we can define Rouquier complexes for braid generators as

$$T_i := [\ B_i(-1) \xrightarrow{m_i} \underline{R}]\ , \quad T_i^{-1} := [\ \underline{R} \xrightarrow{\Delta_i} B_i(1)\ ] \tag{3.8}$$

**Remark 3.33.** *To be more precise, we should consider $B_i(-1)$ and $R$ as one-term complexes and then $T_i$ and $T_i^{-1}$ are the cones of the maps $m_i$ and $\Delta_i$ respectively. The underline denotes the terms living in homological degree zero.*

Rouquier used these complexes to construct a categorification of the braid group. We present a weaker version of his fundamental result.

**Theorem 3.34** (Rouquier)**.** *The complexes $T_i$ and $T_i^{-1}$ satisfy the braid relations up to homotopy.*

Let us check one of the relations that go into this theorem.

**Lemma 3.35.** *The complexes $T_i \otimes_R T_i^{-1}$ and $T_i^{-1} \otimes_R T_i$ are homotopy equivalent to $R$.*

*Proof.* We prove the first homotopy equivalence since the second case is similar. We have

$$\begin{tikzcd}
 & & \underline{R} \arrow[dr] & \\
\left[\ B_i(-1) \xrightarrow{m_i} \underline{R}\ \right] \otimes_R \left[\ \underline{R} \xrightarrow{\Delta_i} B_i(-1)\ \right] \arrow[r, phantom, "="] & B_i(-1) \arrow[ur] \arrow[dr] & & B_i(1) \\
 & & \underline{B_i \otimes B_i} \arrow[ur] &
\end{tikzcd}$$

Now use the isomorphism $B_i \otimes B_i \cong B_i(1) \oplus B_i(-1)$ and Gaussian elimination to simplify the complex. In more detail, there is a subcomplex

$$B_i(-1) \longrightarrow \underline{B_i(1) \oplus B_i(-1)} \longrightarrow B_i(1)$$

which can be contracted. Therefore the complex simplifies to

$$T_i \otimes_R T_i^{-1} \cong [0 \to \underline{R} \to 0] = R. \qquad \square$$

Using Rouquier's result, one can now associate a well-defined complex of Soergel bimodules to any braid $\beta$. Suppose that $\beta = \sigma_{i_1}^{\epsilon_1} \cdots \sigma_{i_k}^{\epsilon_k}$, where $\sigma_i$ are generators in the braid group and $\epsilon = \pm 1$. Then the corresponding Rouquier complex is given by

$$T_\beta := T_{i_1}^{\epsilon_1} \otimes_R \cdots \otimes_R T_{i_k}^{\epsilon_k}$$

**Example 5.** *For $\beta = \sigma_i^2$, we have that*

$$T_\beta := T_i \otimes_R T_i = \left[ \; B_i(-1) \xrightarrow{m_i} \underline{R} \; \right] \otimes_R \left[ \; B_i(-1) \xrightarrow{m_i} \underline{R} \; \right]$$

*Then again using the isomorphism $B_i \otimes B_i \cong B_i(1) \oplus B_i(-1)$ and Gaussian elimination one can simplify the above complex to arrive at*

$$T_\beta \simeq [B_i(-3) \to B_i(-1) \to \underline{R}].$$

*In general, one can use recursion along with the techniques above to show that*

$$T_i^k \cong [\underbrace{B_i(-2k+1) \to B_i(-2k+3) \to \cdots \to B_i(-1)}_{k} \to \underline{R}]$$

The operation that corresponds to taking the closure of a braid is taking the Hochschild cohomology. Note that $R$ itself is an $R{-}R$ bimodule where the bimodule structure is given by left and right multiplication. Then for an $R - R$ bimodule $M$, we define its Hochschild cohomology as

$$HH^i(M) := \operatorname{Ext}^i_{R-\text{bimod}}(R, M)$$

For $R - R$ bimodules $M$ and $N$, we have that $HH^i(M \otimes_R N) = HH^i(N \otimes_R M)$ which is one motivation for why Hochschild cohomology should be though of as the braid closure.

Given a complex $M_\bullet = (M_k, d)$ of Soergel bimodules, we can define the complexes

$$HH^i(M_\bullet) = (HH^i(M_k), d_i),$$

where $d_i$ is the induced differential. More concretely, we apply the functor $HH^i$ to each term in the complex $M_\bullet$ so we end up with a collection of complexes, one for each Hochschild degree.

**Example 6.** *For* $i = 0$*, we have* $HH^i(M) = \mathrm{Hom}(R, M)$ *for a bimodule* $M$*. Then for a complex* $M_\bullet$*,*

$$HH^i(M_\bullet) = \mathrm{Hom}(R, M_\bullet)$$

**Definition 3.36.** *The Khovanov-Rozansky homology of a braid* $\beta$ *is defined as the homology of the Hochschild cohomology of the Rouquier complex* $T_\beta$*:*

$$HHH(\beta) = H(HH(T_\beta))$$

Let us elaborate on the gradings for Khovanov-Rozansky homology. It is a triply-graded homology theory:

- The $Q$-grading is the internal grading of the Soergel bimodules where all $x_i$ have degree 2.
- The $T$-grading is the homological grading of the Rouquier complex.
- The $A$-grading is the Hochschild degree.

**Theorem 3.37** (Khovanov-Rozansky)**.** *The Khovanov-Rozansky homology* $HHH(\beta)$ *is a link invariant.*

**Example 7.** *Let us compute the* $A = 0$ *piece of the Khovanov-Rozansky homology of the trefoil knot. As a braid on two strands, the trefoil knot is the closure of the braid* $\beta = \sigma_1^2$*. From 5, we apply* $HH^0(-) = \mathrm{Hom}(R, -)$ *term-wise to arrive at the complex:*

$$\mathrm{Hom}(R, T_1^3) = [\ R(-6) \xrightarrow{x_1 - x_2} R(-4) \xrightarrow{0} R(-2) \xrightarrow{x_1 - x_2} \underline{R}\ ]$$

*We compute the homology of the complex to get the graded Poincaré polynomial*

$$HHH^{A=0}(T(2,3)) = \frac{1 + Q^4T^{-2}}{1 - Q^2}$$

Computing Khovanov-Rozansky homology is incredibly difficult outside of the case of two strands. The Rouquier complex grows exponentially with the number of crossings and computing Hochschild cohomology is not easy either. Researchers were at an impasse until Elias, Hogancamp and Mellit developed techniques to compute all positive torus links $T(m, n)$ [10][28][18]. Relatively recently, Nakagane and Sano developed a computer program to compute the Khovanov-Rozansky homology of knots up to 11 crossings [30].

# CHAPTER 4

# MATRIX FACTORIZATIONS

## 4.1 Matrix factorizations in homological algebra

We now explain one of the main technical tools used in Oblomkov and Rozansky's geometric construction of link homology: matrix factorizations. Matrix factorizations were first studied by Eisenbud in his work on the homological algebra of complete intersections [8]. In an unpublished manuscript, Buschweitz used matrix factorizations when developing the notion of a stable derived category of a hypersurface ring, a result that was later rediscovered by Orlov [40].

Matrix factorizations have several important applications in mathematics and physics. As shown in Eisenbud's work, they are an important tool for studying hypersurface singularities. In geometry, it was shown by Orlov that the category of matrix factorizations measures the failure of coherent sheaves on a hypersurface to have a finite locally free resolution. Finally matrix factorizations were one of the key ingredients used in Khovanov and Rozansky's original formulation of Khovanov-Rozansky homology [23] and later on used in a more geometric setting by Oblomkov and Rozansky for an equivalent construction of Khovanov-Rozansky homology [37].

We begin by first defining matrix factorizations as they first appeared through Eisenbud's work before redefining matrix factorizations in a slightly more general context and stating some useful results and notations. One can consult [7] for a more complete account of the theory of matrix factorizations.

Let $(R, \mathfrak{m})$ be a regular local ring of finite Krull dimension and fix a non-zero element $W \in \mathfrak{m}$. The category of matrix factorizations $\mathbf{MF}(R, W)$ of $W$ over $R$ is defined to be a differential $\mathbb{Z}/2$-graded category with the following data:

1. the objects of $\mathbf{MF}(R, W)$ are pairs $(M, d)$ where $M = M_0 \oplus M_1$ is a free $\mathbb{Z}/2$-graded $R$-module of finite rank with an $R$-linear map $d$ of degree 1 which satisfies $d^2 = W \cdot \mathrm{id}_M$
2. the morphism complexes $\mathbf{MF}(M, M')$ are given by the $\mathbb{Z}/2$-graded module of $R$-linear maps from $M$ to $M'$ along with the differential given by

$$d(f) = d_{M'} \circ f - (-1)^{|f|} f \circ d_M$$

Note that after choosing bases for $M_0$ and $M_1$, we have a pair of matrices $(d_0, d_1)$

$$\begin{tikzcd} M_0 \arrow[r, shift left, "d_0"] & M_1 \arrow[l, shift left, "d_1"] \end{tikzcd}$$

such that $d_0 \circ d_1 = d_1 \circ d_0 = W \cdot \mathrm{id}$. This immediately implies that $M_0$ and $M_1$ have the same rank so $\phi$ and $\psi$ are square matrices.

**Remark 4.1.** *Matrix factorizations also appear in the literature as $\mathbb{Z}$-graded objects*

$$\begin{tikzcd} \cdots \arrow[r] & \underline{M_0} \arrow[r, "d_0"] & M_1 \arrow[r, "d_1"] & M_0 \arrow[r, "d_0"] & \cdots \end{tikzcd}$$

*where the underlined term denotes homological degree zero. For the most part, whether one chooses to work with $\mathbb{Z}/2$-grading or $\mathbb{Z}$-grading is up to preference but there is some subtlety when defining a homotopy equivalence in the latter case. Matrix factorizations are sometimes called "curved complexes" since they satisfy $d^2 = W$ instead of the usual $d^2 = 0$ for (co)chain complexes.*

**Example 8.** *Let* $R = \mathbb{C}[[x]]$ *and* $W = x^n$*. Then we have a matrix factorization*

$$\begin{tikzcd} R \arrow[r, shift left, "x^k"] & R \arrow[l, shift left, "x^{n-k}"] \end{tikzcd}$$

*For* $k = 1, 2, \ldots, n-1$*, it was shown by Orlov that these matrix factorizations correspond to indecomposable objects in the triangulated category of singularities for* $\mathrm{Spec}(\mathbb{C}[x]/(x^n))$*.*

**Example 9.** *For a more interesting example, consider* $R = \mathbb{C}[[x, y, z]]$ *and* $W = xy + yz + zx$*. Then we have a matrix factorization*

$$\begin{tikzcd} R^2 \arrow[r, shift left, "d_0"] & R^2 \arrow[l, shift left, "d_1"] \end{tikzcd}$$

*where* $d_0 = \begin{bmatrix} z & y \\ x & -x-y \end{bmatrix}$ *and* $d_1 = \begin{bmatrix} x+y & y \\ x & -z \end{bmatrix}$.

## 4.2 Matrix factorizations in algebraic geometry

We now recall some definitions and results from commutative algebra. Let $(R, \mathfrak{m})$ be a regular local ring and let $M$ be a finitely generated $R$-module.

**Definition 4.2.** *We say a sequence of elements* $x_1, \ldots, x_r \in \mathfrak{m}$ *is an* $M$*-regular sequence if each* $x_i$ *is not a zero divisor on* $M/(r_1, \ldots, r_{i-1})$ *for* $i = 1, \ldots, r$*. When* $M = R$*, we simply say that* $(x_1, \ldots, x_r)$ *is a regular sequence.*

Let $S = R/w$ be the ring of a hypersurface singularity where $w$ is singular at the maximal ideal. Let $M$ be a maximal Cohen-Macaulay module over $S$. We will show that such a module $M$ corresponds to a matrix factorization; this was how these objects were originally encountered by Eisenbud.

We can consider $M$ as an $R$ module which is annihilated by $w$. Then by the Auslander-Buchsbaum formula, one can deduce that $M$ has a free resolution of $R$-modules of length 1.

$$0 \longrightarrow X_1 \xrightarrow{\phi} X_0 \longrightarrow M \longrightarrow 0$$

Since multiplication by $w$ annihilates $M$, there exists a homotopy $\psi$ which completes the following commutative diagram.

$$\begin{tikzcd}
X_1 \arrow[r, "\phi"] \arrow[d, "w"] & X_0 \arrow[d, "w"] \arrow[dl, dotted, "\psi"'] \\
X_1 \arrow[r, "\phi"] & X_0
\end{tikzcd}$$

Therefore we have a matrix factorization

$$\begin{tikzcd}
X_1 \arrow[r, shift left, "\phi"] & X_0 \arrow[l, shift left, "\psi"]
\end{tikzcd}$$

and $M$ is isomorphic to $\mathrm{coker}(\phi)$. In Eisenbud's work, he went further by considering free resolutions of $S$-modules from which he proved the following surprising theorem.

**Theorem 4.3.** *Every finitely generated $S$-module admits a free resolution that will eventually become 2-periodic.*

For our purposes, we will use a slightly different definition of matrix factorizations. Let $\mathcal{Z}$ be an affine algebraic variety and $F$ a polynomial in $\mathbb{C}[\mathcal{Z}]$. Then we can define the category of matrix factorizations $\mathbf{MF}(\mathcal{Z}, F)$ similar to the previous definition with $R$ replaced by $\mathbb{C}[\mathcal{Z}]$ and $W$ replaced by $F$. Note that $F$ is often referred to as the potential.

**Definition 4.4.** *The triangulated category $\mathrm{MF}(\mathcal{Z}, F)$ is defined as $H^0(\boldsymbol{MF}(\mathcal{Z}, F))$. In order words, $\mathrm{MF}(\mathcal{Z}, F)$ has the same objects as $\boldsymbol{MF}(\mathcal{Z}, F)$ but the morphism space $\mathrm{MF}(M, M')$ between two matrix factorizations $M, M'$ is given by applying $H^0(-)$ to the morphism complex $\boldsymbol{MF}(M, M')$.*

Let us describe what the shift functor and cone of a map looks like in $\mathrm{MF}(\mathcal{Z}, F)$. We refer the reader to Orlov's paper for further details [40].

The shift of a matrix factorization $(M, d)$ is given by

$$(M,d)[1] := \left( \begin{tikzcd} M_1 \arrow[r, shift left, "-d_1"] & M_0 \arrow[l, shift left, "-d_0"] \end{tikzcd} \right)$$

Given $\phi = (\phi_0, \phi_1) \in \mathrm{MF}((M,d),(M',d'))$, the cone of $\phi$ is defined as

$$C(\phi) = \left( \begin{tikzcd} M_1' \oplus M_0 \arrow[r, shift left, "B_0"] & M_0' \oplus M_1 \arrow[l, shift left, "B_1"] \end{tikzcd} \right)$$

where

$$B_0 = \begin{bmatrix} d_1' & \phi_0 \\ 0 & -d_0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} d_0' & \phi_1 \\ 0 & -d_1 \end{bmatrix}.$$

**Definition 4.5.** *Given potentials $F, F' \in \mathbb{C}[\mathcal{Z}]$, we can define the tensor product bifunctor:*

$$\otimes : \mathrm{MF}(\mathcal{Z}, F) \times \mathrm{MF}(\mathcal{Z}, F') \to \mathrm{MF}(\mathcal{Z}, F + F')$$

*as $(M,d) \otimes (M',d') = (M \otimes_{\mathbb{C}[\mathcal{Z}]} M', d \otimes 1 + (-1)^{|\cdot|} \otimes d')$. The additional minus signs are used to ensure that potential of the tensor product is the sum of the potentials.*

If the potential $F$ can be written as $F = f_1 g_1 + \cdots + f_m g_m$ for $f_i, g_i \in \mathbb{C}[\mathcal{Z}]$, then we can use the tensor product to define a class of matrix factorizations known as Koszul matrix factorizations. These are matrix factorizations of the form

$$\bigotimes_{i=1}^{m} \left( \begin{tikzcd} \mathbb{C}[\mathcal{Z}] \arrow[r, shift left, "f_i"] & \mathbb{C}[\mathcal{Z}] \arrow[l, shift left, "g_i"] \end{tikzcd} \right).$$

This is oftentimes represented simply by the Koszul matrix

$$\begin{bmatrix} f_1 & g_1 \\ f_2 & g_2 \\ \vdots & \vdots \\ f_m & g_m \end{bmatrix}.$$

Koszul matrix factorizations have the following nice properties. Suppose we use a change of basis which replaces the standard basis element $e_i$ of $\mathbb{C}[\mathcal{Z}]^m$ by $e_i + ce_j$. The resulting Koszul matrix factorization is clearly isomorphic to the original one. This operation changes the $i^{\text{th}}$ and $j^{\text{th}}$ rows of the Koszul matrix

$$\begin{bmatrix} f_i & g_i \\ f_j & g_j \end{bmatrix} \mapsto \begin{bmatrix} f_i + cf_j & g_i \\ f_j & g_j - cg_i \end{bmatrix}$$

and leaves all other rows unchanged. Now let us suppose that $\mathcal{Z}$ is now also smooth. We will only ever work with smooth varieties so this additional assumption is never violated.

**Proposition 4.6.** *Suppose that $(f_1, \ldots, f_m)$ forms a regular sequence in $\mathbb{C}[\mathcal{Z}]$. Then if we have $g_1, \ldots, g_m, h_1, \ldots, h_m \in \mathbb{C}[\mathcal{Z}]$ such that*

$$f_1 g_1 + \cdots + f_m g_m = f_1 h_1 + \cdots + f_m h_m = F$$

*then the two Koszul matrix factorizations are isomorphic.*

$$\begin{bmatrix} f_1 & g_1 \\ f_2 & g_2 \\ \vdots & \vdots \\ f_m & g_m \end{bmatrix} \cong \begin{bmatrix} f_1 & h_1 \\ f_2 & h_2 \\ \vdots & \vdots \\ f_m & h_m \end{bmatrix}$$

As a result, we can use the notation $K^F(f_1, \ldots, f_m)$ to represent Koszul matrix factorizations corresponding to regular sequences. These Koszul matrix factorizations have another special property.

**Proposition 4.7.** *Suppose that $(f_1, \ldots, f_m)$ forms a regular sequence in $\mathbb{C}[\mathcal{Z}]$. Given a Koszul matrix factorization $K^F(f_1, \ldots, f_m)$, we have that*

$$H^i(K^F(f_1, \ldots, f_m)) = 0, \quad i \neq 0$$

$$H^0(K^F(f_1, \ldots, f_m)) = \mathbb{C}[\mathcal{Z}]/(f_1, \ldots, f_m).$$

In the work of Oblomkov and Rozansky, they primarily work with matrix factorizations with extra equivariant conditions, but we omit these details since these constructions are much more technical. The interested reader can consult their papers for the details.

# CHAPTER 5

# LINK HOMOLOGY AND HILBERT SCHEME OF POINTS

## 5.1 Algebraic geometry and link homology

We begin by explaining some of the connections between algebraic geometry and knot theory. Let $f(x, y)$ be a polynomial with an isolated singularity at the origin and let $C$ denote the curve $\{f(x, y) = 0\} \subset \mathbb{C}^2$. Consider the intersection $L = C \cap S^3_\epsilon$ of the curve $C$ with a small sphere centered at the origin with radius $\epsilon$. By a well-known result of Milnor [29], for $\epsilon$ small enough $L$ is a smooth link in $S^3$ and the topological type of $L$ does not depend on $\epsilon$. Links that arise in this fashion are known as algebraic links.

**Example 10.** *The curve $\{xy = 0\}$ corresponds to the Hopf link while the curve $\{y^2 = x^3\}$ corresponds to the trefoil knot.*

In [33], Oblomkov, Rasmussen and Shende conjectured the following relation between the Hilbert scheme of $n$ points on a singular curve $C$ and Khovanov-Rozansky homology of the corresponding link $L$.

**Conjecture 2.** *Let $\mathrm{Hilb}^k(C, 0)$ denote the Hilbert scheme of $k$ points on $C$ supported at the origin. The $A = 0$ part of Khovanov-Rozansky homology of $L$ can be* *computed by*

$$HHH^0(L) = \bigoplus_{k=0}^{\infty} H^*(\mathrm{Hilb}^k(C,0))$$

A version of this conjecture was proven by Maulik in [27] where the right-hand side of the conjecture is replaced by the generating function for the Euler characteristics of the Hilbert schemes. Furthermore there is a version of this conjecture that generalizes to higher $A$-degrees of Khovanov-Rozansky homology.

A few years later, Gorsky realized that Haiman's work [16][17] on the Hilbert scheme of points on a plane related to link homology. He conjectured that the $q,t$-Catalan numbers were the same as the $A = 0$ piece of Khovanov-Rozansky homology for the $T(n, n+1)$ torus knot [13]. This led to the idea that in order to study the class of all links instead of just algebraic links, one should work with the Hilbert scheme of points on a plane and other related spaces. The general idea was that to each link, one should be able to associate a sheaf from which one can recover Khovanov-Rozansky homology. These ideas were conjectured in the paper [15] using the flag Hilbert scheme and later proved by Oblomkov-Rozansky in a series of papers [37][36][34][35][39].

## 5.2 Oblomkov-Rozansky theory

Let us now recall some basic facts about the Hilbert scheme of points on a plane. For an in-depth description, one should refer to the book [31].

**Definition 5.1.** *The Hilbert scheme of $n$ points on $\mathbb{C}^2$ is defined as*

$$\mathrm{Hilb}^n(\mathbb{C}^2) := \{I \subset \mathbb{C}[x,y] : \dim \mathbb{C}[x,y]/I = n\}.$$

**Theorem 5.2** (Fogarty)**.** *The Hilbert scheme* $\mathrm{Hilb}^n(\mathbb{C}^2)$ *of* $n$ *points on* $\mathbb{C}^2$ *is a smooth irreducible variety. This variety comes with a Hilbert-Chow morphism*

$$\pi : \mathrm{Hilb}^n(\mathbb{C}^2) \to \mathrm{Sym}^n(\mathbb{C}^2) := (\mathbb{C}^2)^n/S_n$$

*which is a resolution of singularities.*

**Example 11.** *For* $n = 2$*,* $\mathrm{Hilb}^2(\mathbb{C}^2)$ *is the blow-up of* $\mathrm{Sym}^2(\mathbb{C}^2)$ *along the diagonal.*

There are two $\mathbb{C}^*$-actions on $\mathbb{C}^2$ by scaling the coordinates which lift to an action of $(\mathbb{C}^*)^2$ on the Hilbert scheme. We denote the two copies of $\mathbb{C}^*$ in $(\mathbb{C}^*)^2$ by writing $\mathbb{C}^*_q \times \mathbb{C}^*_t$ to emphasize the gradings coming from each factor. There are two distinguished tori in $(\mathbb{C}^*)^2$:

- There is a Hamiltonian torus $H = \{(h, h) : h \in \mathbb{C}^*\}$ which acts via $(x, y) \mapsto (hx, h^{-1}y)$ and preserves the symplectic form.
- There is the scaling torus $S = \{(s, s) : s \in \mathbb{C}^*\}$ which acts via $(x, y) \mapsto (sx, sy)$. This torus scales the symplectic form by $s^2$.

**Remark 5.3.** *With some minor abuse of notation, let* $h$ *and* $s$ *also denote the gradings induced by the actions of tori* $H$ *and* $S$ *respectively. Then these gradings are related to the* $q$*- and* $t$*-gradings by* $h = qt^{-1}$ *and* $s = qt$*.*

There a rank $n$ tautological bundle $\mathcal{B}$ on the Hilbert scheme whose fiber over an ideal $I$ is given by $\mathbb{C}[x, y]/I$. From this, we construct the line bundle

$$\det \mathcal{B} = \wedge^n \mathcal{B} =: \mathcal{O}(1)$$

There are two important subsets of the Hilbert scheme which are worth mentioning.

$$\mathrm{Hilb}^n(\mathbb{C}^2, 0) = \pi^{-1}(\{(0, 0)\})$$

$$\mathrm{Hilb}^n(\mathbb{C}^2, \mathbb{C}) = \pi^{-1}(\{y = 0\})$$

These are subsets of the Hilbert scheme whose ideals are either supported at the origin or on the line $y = 0$ respectively (by support, we mean the set-theoretic support as opposed to the scheme-theoretic support).

**Example 12.** *For $n = 2$, one can easily compute that* $\mathrm{Hilb}^2(\mathbb{C}^2, 0) = \mathbb{P}^1$ *and* $\mathrm{Hilb}^2(\mathbb{C}^2, \mathbb{C}) = (\mathbb{P}^1 \cup \mathbb{C}) \times \mathbb{C}$.

We need the following change of variables:

$$q = Q^2, \quad t = T^2Q^{-2}, \quad a = AQ^{-2} \tag{5.1}$$

**Theorem 5.4** (Oblomkov-Rozansky)**.** *To a braid $\beta$ on $n$ strands we can associate a $\mathbb{C}^* \times \mathbb{C}^*$-equivariant coherent sheaf $\mathcal{F}_\beta$ on* $\mathrm{Hilb}^n(\mathbb{C}^2)$ *with the following properties:*

1. *The Khovanov-Rozansky homology for the braid closure of $\beta$ is given by*

   $$HHH(\beta) \simeq H^*_{\mathbb{C}^* \times \mathbb{C}^*}(\mathrm{Hilb}^n(\mathbb{C}^2, \mathbb{C}), \mathcal{F}_\beta \otimes \wedge^\bullet \mathcal{B}^\vee)$$

   *where the isomorphism above holds as triply-graded vector spaces. The $q$-grading and $t$-grading on the right-hand side come from the $\mathbb{C}^* \times \mathbb{C}^*$-action while the $a$-grading comes from $\wedge^\bullet \mathcal{B}^\vee$. We go from the $(A, Q, T)$-grading on the left-hand side to the $(a, q, t)$-grading on the right-hand side via equations 5.1.*

2. *For a braid $\beta$ which closes up to a knot, the sheaf $\mathcal{F}_\beta$ is supported on* $\mathrm{Hilb}^n(\mathbb{C}^2, 0) \times \mathbb{C} \subset \mathrm{Hilb}^n(\mathbb{C}^2, \mathbb{C})$.

3. *Adding a full twist* $\mathrm{FT} = (\sigma_1 \cdots \sigma_{n-1})^n$ *to the braid $\beta$ corresponds to twisting the sheaf $\mathcal{F}_\beta$ by $\mathcal{O}(1)$. In other words $\mathcal{F}_{\beta \cdot \mathrm{FT}} = \mathcal{F}_\beta \otimes \mathcal{O}(1)$.*

**Example 13.** *The torus braid $\beta = T(n, kn + 1)$ corresponds to the line bundle $\mathcal{O}(k)$ on* $\mathrm{Hilb}^n(\mathbb{C}^2, 0)$. *In particular, the trefoil knot $T(2, 3)$ corresponds to $\mathcal{O}(1)$ on* $\mathrm{Hilb}^2(\mathbb{C}^2, 0) = \mathbb{P}^1$.

For general torus braids $\beta = T(m,n)$, the situation is much more complicated. One must first work with a slightly larger space known as the flag Hilbert scheme [15].

**Definition 5.5.** *The flag Hilbert scheme parameterizes full flags of ideals*

$$\mathrm{FHilb}^n(\mathbb{C}^2) := \{\mathbb{C}[x,y] \supset I_1 \supset \cdots \supset I_n\}$$

*where* $\dim I_i/I_{i+1} = 1$ *for* $i = 1, \ldots, n-1$.

One can similarly define $\mathrm{FHilb}^n(\mathbb{C}^2, 0)$ and $\mathrm{FHilb}^n(\mathbb{C}^2, \mathbb{C})$ as in the case of the usual Hilbert scheme. The flag Hilbert scheme has a projection map

$$p : \mathrm{FHilb}^n(\mathbb{C}^2) \to \mathrm{Hilb}^n(\mathbb{C}^2), \quad (I_1, \ldots, I_n) \mapsto I_n$$

and a collection of line bundles $\mathcal{L}_k = I_{k-1}/I_k$ where $I_0$ is understood to be $\mathbb{C}[x,y]$.

**Theorem 5.6.** *Suppose that gcd*$(m,n) = 1$. *Then the sheaf* $\mathcal{F}_{m,n}$ *corresponding to the braid* $\beta$ *whose closure is* $T(m,n)$ *is given by*

$$\mathcal{F}_{m,n} = p_*(\mathcal{L}_1^{a_1} \cdots \mathcal{L}_n^{a_n})$$

*where* $a_k = \lceil \frac{km}{n} \rceil - \lceil \frac{(k-1)m}{n} \rceil$

Strictly speaking $p$ is the restriction of the projection map to $p : \mathrm{FHilb}^n(\mathbb{C}^2, 0) \to \mathrm{Hilb}^n(\mathbb{C}^2, 0)$ and is the derived pushforward for a morphism of *dg* schemes.

**Remark 5.7.** *More generally one can attempt to consider sheaves* $p_*(\mathcal{L}_1^{a_1} \cdots \mathcal{L}_n^{a_n})$ *for arbitrary exponents* $a_1, \ldots, a_n$. *These sheaves correspond to braids of the form*

$$\beta(a_1, \ldots, a_n) = \delta_1^{a_1} \cdots \delta_n^{a_n} \sigma_1 \cdots \sigma_n$$

*where*

$$\delta_i = \sigma_{i-1} \cdots \sigma_1 \sigma_1 \cdots \sigma_{i-1}$$

*are Jucys-Murphy braids. The class of braids described above are called Coxeter braids and were studied by Oblomkov and Rozansky in their paper on Coxeter links [36]. This is a large class of links that in particular includes torus links* $T(m,n)$.

**Example 14.** *Let us illustrate an example of Oblomkov and Rozansky's construction by computing the Khovanov-Rozansky homology of the trefoil using their techniques. Recall that* $\mathrm{Hilb}^2(\mathbb{C}^2, 0) = \mathbb{P}^1$. *Then the sheaf on* $\mathbb{P}^1 \times \mathbb{C}$ *corresponding to* $T(2,3)$ *is* $\mathcal{O}_{\mathbb{P}^1}(1) \otimes \mathcal{O}_{\mathbb{C}}$ *where* $x$ *is the coordinate on the* $\mathbb{C}$. *The tautological bundle is given by* $(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}) \otimes \mathcal{O}_{\mathbb{C}}$. *We compute the space*

$$\mathcal{F}_\beta \otimes \wedge^\bullet \mathcal{B} = \frac{\boldsymbol{a}}{\boldsymbol{t}} \left( \boldsymbol{a}^2 \mathcal{O} \oplus \boldsymbol{a}^1 \mathcal{O}(1) \oplus \boldsymbol{a}^1 \mathcal{O} \oplus \boldsymbol{a}^0 \mathcal{O}(1) \right) \otimes \mathcal{O}_{\mathbb{C}}$$

*Then the link homology is given*

$$\mathbb{C}[x] \otimes \left( H^*(\mathbb{P}^1, \mathcal{O}(1)) \oplus \boldsymbol{a} H^*(\mathbb{P}^1, \mathcal{O}) \oplus \boldsymbol{a} H^*(\mathbb{P}^1, \mathcal{O}(1)) \oplus \boldsymbol{a}^2 H^*(\mathbb{P}^1, \mathcal{O}) \right),$$

*which is shifted by* $\frac{\boldsymbol{a}}{\boldsymbol{t}}$. *We can recover the* $Q$-,$T$*-gradings from the line bundles by the formulas*

$$\dim_{Q,T} H^0(\mathbb{P}^1, \mathcal{O}(n)) = \sum_{i=0}^{n} Q^{2i} (T/Q)^{2n-2i}$$
$$\dim_{Q,T} H^1(\mathbb{P}^1, \mathcal{O}(n)) = \sum_{i=0}^{-n-2} Q^{2i} (T/Q)^{-2n-2i-4}.$$

*Putting everything together, we get that the Khovanov-Rozansky homology for the trefoil is*

$$\frac{1}{1-Q^2} \cdot \frac{a}{T} \left( Q^2 + \frac{T^2}{Q^2} + a + aQ^2 + a\frac{T^2}{Q^2} + a^2 \right) = \frac{1+a}{1-Q^2} \cdot \frac{a}{T} \left( Q^2 + \frac{T^2}{Q^2} + a \right).$$

*Next we apply the change of variables in equation 5.1 to obtain*

$$\frac{1+a}{1-q} a t^{-1/2} q^{-1/2} (q + t + a).$$

*Notice that the term $q + t + a$ is reduced homology of the trefoil. The term $q + t$ is symmetric in the variables $q$ and $t$ and corresponds to the $q, t$-Catalan number $C_2(q, t)$.*

The theorem 5.4 is proven over a series of papers [37][36][34][35][39]. One can find an overview of their theory and construction in the notes [32]. Let us broadly summarize how each paper fits into proving this remarkable theorem. Their construction utilizes the category of matrix factorizations on an algebraic variety which is closely related to the flag Hilbert scheme $\mathrm{FHilb}^n(\mathbb{C}^2)$. The big picture perspective for motivating the work of Oblomkov and Rozansky is that they wish to categorify the Ocneanu-Jones trace in a suitable way to recover Khovanov-Rozansky homology.

In their first paper [37], they define a homomorphism from the braid group to a particular category of matrix factorizations. They prove that their map satisfies the braid relations along with the Markov moves so they have a genuine link invariant. In a subsequence paper [39], they relate their category of matrix factorizations to the category of Soergel bimodules by constructing an explicit functor between these two categories. This shows that their link invariant agrees with the Khovanov-Rozansky homology originally defined by Khovanov and Rozansky. Finally, in order to associate a sheaf on $\mathrm{Hilb}^n(\mathbb{C}^2)$ to any braid they construct a pair of adjoint functors between their category of matrix factorizations and some version of the derived category on $\mathrm{Hilb}^n(\mathbb{C}^2)$ [35].

# CHAPTER 6

# MAIN THEOREM

## 6.1 Differentials on matrix factorizations and Soergel bimodules

In [38], Oblomkov and Rozansky constructed a family of differentials $\{d_{M|K}\}$ which they used to define $\mathfrak{gl}(M|K)$ link homology. For a braid $\beta$, we denote its $\mathfrak{gl}(M|K)$ link homology as $H_{M|K}(\beta)$.They conjectured that their construction generalizes the original conjectures of Rasmussen in the following sense:

**Conjecture 3.** *For $M \geq 2$, we have for any $\beta \in \mathrm{Br}_n$*

$$H_{M|0}(\beta) = H^{\mathrm{KhR}}_{\mathfrak{gl}(M)}(L(\beta))$$

**Remark 6.1.** *Recall that in the introduction, we defined $\mathfrak{sl}(N)$ link homology, however the right-hand side of conjecture involves $\mathfrak{gl}(M)$ link homology. This is the same as $\mathfrak{sl}(N)$ homology but the change in naming convention is related to reduced vs. unreduced link homology which we elect not to discuss. Although it is somewhat confusing, we choose to stick with the original names to better align with the listed references.*

**Remark 6.2.** *In order to avoid confusion wherever possible, we use capital letters $M, N, K$ to refer to the type of homology ($\mathfrak{sl}_N$-homology, $\mathfrak{gl}(M|K)$-homology) and* *use lower-case letters $n, k$ for other parameters like the number of strands in the braid group* $\mathrm{Br}_n$ *or the number of points on the Hilbert scheme* $\mathrm{Hilb}^n$*.*

We provide a proof of Conjecture 3 in the special case $M = 2$. However before we do so, let us review their construction.

Oblomkov and Rozansky work over the following space:

$$\mathcal{X}_n = \mathfrak{g}_n \times G_n \times \mathfrak{n}_n \times G_n \times \mathfrak{n}_n$$

where $G_n = GL_n$ and $\mathfrak{g}_n = \mathfrak{gl}_n$. Define coordinates on the space $\mathcal{X}_n \times \mathbb{C}^n$ as $(X, g_1, Y_1, g_2, Y_2, v)$. Then we can define a potential on $\mathcal{X}_n$ by

$$W(X, g_1, Y_1, g_2, Y_2) = \mathrm{Tr}(X(\mathrm{Ad}_{g_1} Y_1 - \mathrm{Ad}_{g_2} Y_2))$$

Let $B_n \subset G_n$ denote the Borel subgroup. Then $\mathcal{X}_n$ has an action of $G_n \times B_n^2$ given by

$$(h, b_1, b_2) \cdot (X, g_1, Y_1, g_2, Y_2) = (\mathrm{Ad}_h X, hg_1 b_1^{-1}, \mathrm{Ad}_{b_1} Y_1, hg_2 b_2^{-1}, \mathrm{Ad}_{b_2} Y_2)$$

Observe that the potential $W$ is invariant with respect to this $G_n \times B_n^2$-action, so we can consider $G_n \times B_n^2$-equivariant matrix factorizations on $\mathcal{X}_n$ with potential $W$. We let $\mathrm{MF}_n := \mathrm{MF}_{G_n \times B_n^2}(\mathcal{X}_n, W)$.

Now consider the stable sub-variety of $\mathcal{X}_n \times \mathbb{C}^n$ given by

$$(\mathcal{X}_n \times \mathbb{C}^n)^{\mathrm{st}} = \{(X, g_1, Y_1, g_2, Y_2, v) | \mathbb{C}\langle X, \mathrm{Ad}_{g_i} Y_i \rangle v = \mathbb{C}^n, i = 1, 2\}.$$

Then the main category of interest is given by

$$\mathrm{MF}_n^{\mathrm{st}} := MF_{G_n \times B_n^2}((\mathcal{X}_n \times \mathbb{C}^n)^{\mathrm{st}}, W).$$

Oblomkov and Rozansky define a functor $\mathcal{R}_{M|K}$ which equips an object in $\mathrm{MF}_n^{\mathrm{st}}$ with an extra differential $d_{M|K}$. Given a matrix factorization $\mathcal{F}$, the functor is defined as

$$\mathcal{R}_{M|K}(\mathcal{F}) = \mathcal{F} \otimes (\wedge^{\bullet}\mathbb{C}^n, d_{M|K}), \quad d_{M|K} = \sum_{i=1}^{n} (X^M \mathrm{Ad}_{g_1}(Y_1)^K v)_i \frac{\partial}{\partial \theta_i}$$

where $\theta_i$ for $i = 1, \dots, n$ are the coordinates on $\mathbb{C}^n$.

Let us compare the differentials of Oblomkov and Rozansky with the differentials constructed by Rasmussen. Although there is no direct relation between the two constructions, there are still some similarities and it is hoped that one day the two constructions can be related to each other in a more concrete manner. We need to briefly recall the an alternative way to define Khovanov-Rozansky homology using matrix factorizations. We shall ignore keeping track of gradings in order to simplify the exposition.

In Rasmussen's approach, he associates a complex of matrix factorizations to an oriented planar tangle. Consider an oriented planar tangle where all the edges are labeled distinctly. When encountering a positive or negative crossing, the incoming and outgoing edges should be given different labels as depicted below:

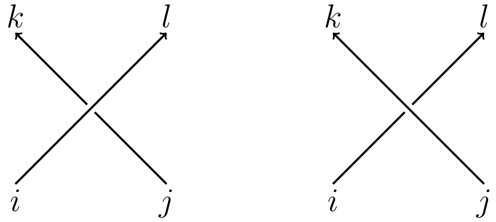


**Figure 10. From left to right: the positive crossing and the negative crossing**

Now consider the ring

$$R = \mathbb{C}[x_i, x_j, x_k, x_l]/(x_k + x_l - x_i - x_j) \cong \mathbb{C}[x_i, x_j, x_k]$$

along with the potential

$$W_p = p(x_k) + p(x_l) - p(x_i) - p(x_j)$$

where $p(x)$ is an arbitrary polynomial (to be specialized later) in $\mathbb{C}[x]$. Note that the potential $W_p$ is divisible by the term $(x_k - x_i)(x_k - x_j)$ (equivalent to $x_k x_l - x_i x_j$ in the ring $R$) since if one substitutes $x_k = x_i$ or $x_k = x_j$ into $W_p$, the result vanishes. Now to the positive and negative crossings, we can associate the complex of matrix factorizations $C_{p,+}$ and $C_{p,-}$ respectively. Then as usual, one can associate a gigantic complex of matrix factorizations to the complete tangle by taking the tensor product of all the various complexes $C_{p,+}$ and $C_{p,-}$.

$$C_{p,+} := \begin{tikzcd} R \arrow[r, shift left, "x_k-x_i"] & R \arrow[l, shift left, "p_i"] \\ R \arrow[u, "x_j-x_k"] \arrow[r, shift left, "-(x_k-x_i)(x_k-x_j)"] & R \arrow[l, shift left, "p_{ij}"] \arrow[u, "1"'] \end{tikzcd}$$

$$C_{p,-} := \begin{tikzcd} R \arrow[r, shift left, "-(x_k-x_i)(x_k-x_j)"] & R \arrow[l, shift left, "p_{ij}"] \\ R \arrow[u, "1"] \arrow[r, shift left, "x_k-x_i"] & R \arrow[l, shift left, "p_i"] \arrow[u, "x_j-x_k"'] \end{tikzcd}$$

where $p_i = W_p/(x_k - x_i)$ and $p_{ij} = W_p/(x_k - x_i)(x_k - x_j)$. At first glance the above construction seems somewhat complicated and unmotivated but in actuality, it is very similar to the construction of Khovanov-Rozansky homology presented in Section 3. For the moment, let us just focus on the complex $C_{p,+}$. Let us just consider the two matrix factorizations defined by the horizontal maps but for the moment ignore the maps $p_i$ and $p_{ij}$. Note that after an appropriate relabeling of variables, we have the isomorphism

$$B_s \cong \left[\ R \xrightarrow{(x_k-x_i)(x_k-x_j)} R\ \right].$$

Similarly the complex $\left[\; R \xrightarrow{x_k - x_i} R \;\right]$ is isomorphic to trivial Soergel bimodule. From this the parallel between these complexes and Rouquier complexes becomes more apparent. The backwards differentials $p_i$ and $p_{ij}$ should be thought of as equipping Soergel bimodules with an "internal" differential.

Denote the differentials going from left to right by $d_+$ and the differentials going from right to left by $d_-$. Now let $p(x) = x^{N+1}$. When we have a closed tangle, the potential $W_p$ vanishes and the total differential $d_+ + d_-$ turns the matrix factorizations into genuine chain complexes. The homology computed with respect to the total differential is isomorphic to the $\mathfrak{gl}_N$-homologies. Furthermore Rasmussen's spectral sequences from Khovanov-Rozansky homology to $\mathfrak{gl}_N$ homology is induced by the differential $d_-$.

The construction of Oblomkov and Rozansky has many similarities with that of Rasmussen's but one key difference is that although matrix factorizations are utilized in both cases, Oblomkov and Rozansky connect matrix factorizations to geometry while Rasmussen uses matrix factorizations purely as a homological algebra tool. In the rest of these section, we develop the tools in order to relate the differentials of Oblomkov and Rozansky to Soergel bimodules in the way we were able to sketch out in Rasmussen's story.

Next we need to better understand how the functor $\mathcal{R}_{M|K}$ behaves in terms of Soergel bimodules. Let $\mathrm{SBim}_n$ denote the category of Soergel bimodules for $S_n$. The category related to Khovanov-Rozansky $\mathfrak{gl}(M)$-invariants is a quotient of $\mathrm{SBim}_n$ by objects of the form

$$B^{(r_1,r_2)} := R \otimes_{R^{S([r_1,r_2])}} R$$

where $R = \mathbb{C}[x_1, \ldots, x_n]$ and $S([r_1, r_2]) \subset S_n$ is the Young subgroup that fixes the elements $1, \ldots, r_1 - 1$ and $r_2 + 2, \ldots, n-1, n$ and $r_2 - r_1 = M$. By quotient, we mean that objects are sent to the zero object and any morphism which factors through the object $B^{(r_1,r_2)}$ becomes zero as well. Let us denote this quotient by $\mathrm{SBim}_n/M$ and let $q_M$ be the quotient map $\mathrm{SBim}_n \to \mathrm{SBim}_n/M$.

**Remark 6.3.** *The objects $B^{(r_1,r_2)}$ can be thought of as living in the 2-category of singular Soergel bimodules* SSBim*, but recall that the category* $\mathrm{Hom}_{\mathrm{SSBim}}(\emptyset, \emptyset)$ *is equal to the category* SBim *when we regard both categories as subcategories of $(R,R)$-bimodules. Therefore we can identify the objects $B^{(r_1,r_2)}$ with ordinary Soergel bimodules.*

Note that the objects $B^{(r_1,r_2)}$ are in fact indecomposable Soergel bimodules by the following useful lemma.

**Lemma 6.4.** *Suppose that $M$ is a graded $R$-bimodule which is generated as an $R$-bimodule by a homogenous element $m \in M$. Then $M$ is indecomposable.*

*Proof.* Suppose that $m$ has degree $d$. Then because $M$ is generated by $m$ we have that $M^d$, the degree $d$ piece of $M$ is given by $\mathbb{C}m$. Now suppose that $M = N \oplus P$. In particular, $M^d = N^d \oplus P^d$. Without loss of generality, assume that $m \in N^d$. Then $M = R \cdot m \cdot R \subset N$ so $P = 0$ and thus $M$ must be indecomposable. $\square$

In the case above, $B^{(r_1,r_2)}$ is generated by the homogeneous element $1 \otimes 1$.

**Remark 6.5.** *Using the previous result, one might hope to give a nicer description for the bimodules $B^{(r_1,r_2)}$. Note that when $r_2 = r_1 + 1$, we have that $B^{(r_1,r_1+1)}$ is isomorphic (up to grading shift) to the indecomposable Soergel bimodule $B_{s_{r_1}}$ where $s_{r_1}$ is the simple reflection that exchanges $r_1$ and $r_1+1$. Similarly when $r_2 = r_1+2$, we have that $B^{(r_1,r_1+2)}$ is isomorphic (up to grading shift) to the indecomposable Soergel bimodule $B_{s_{r_1}s_{(r_1+1)}s_{r_1}}$.*

The author would like to thank Nicolás Libedinsky for informing the author that this is actually a well-known result to experts in Soergel bimodules and can be found in Geordie Williamson's Ph.D. thesis on singular Soergel bimodules [**Wil10**].

**Proposition 6.6.** *The Soergel bimodule $B^{(r_1,r_2)}$ is isomorphic (up to a grading shift) to the indecomposable Soergel bimodule $B_{w([r_1,r_2])}$ where $w([r_1,r_2])$ is the longest word in the Young subgroup $S([r_1,r_2])$.*

In previous work [39] by Oblomkov and Rozansky, they constructed a functor from a subcategory $\mathrm{MF}_n^\flat \subset \mathrm{MF}_n^{\mathrm{st}}$ of matrix factorizations to the category of Soergel bimodules.

$$\mathbb{B} : \mathrm{MF}_n^\flat \to \mathrm{SBim}_n$$

Let us describe what matrix factorizations correspond to the bimodules $B^{(r_1,r_2)}$ under this functor $\mathbb{B}$. Consider the space $\mathcal{X}^{\mathrm{st}} = (\mathfrak{g} \times G \times \mathfrak{b} \times G \times \mathfrak{b} \times \mathbb{C}^n)^{\mathrm{st}}$ with coordinates $(X, g_1, Y_1, g_2, Y_2, v)$. There is a category of matrix factorizations on this space given by

$$\mathrm{MF}_n^{\mathrm{st}} = \mathrm{MF}_{G\times B^2}((\mathfrak{g} \times G \times \mathfrak{b} \times G \times \mathfrak{b} \times V_B)^{\mathrm{st}}, W),$$

where the potential $W$ is given by $\mathrm{Tr}(X(\mathrm{Ad}_{g_1}(Y_1) - \mathrm{Ad}_{g_2}(Y_2)))$ and the stability condition is given by $\mathbb{C}\langle X, \mathrm{Ad}_{g_i}(Y_i)\rangle v = \mathbb{C}^n, i = 1, 2$.

First we consider the special case of $r_1 = 1$ and $r_2 = M$. We claim that the matrix factorizations $\mathcal{C}_\bullet^{(1,M)} \in \mathrm{MF}_n^{\mathrm{st}}$ map to the Soergel bimodules $B^{(1,M)}$ under the functor $\mathbb{B}$. These matrix factorizations are defined as Koszul matrix factorizations. Let $S = \mathbb{C}[(\mathfrak{g} \times G \times \mathfrak{b} \times G \times \mathfrak{b} \times V_B)^{\mathrm{st}}]$ and define $Z_{ij} = (\mathrm{Ad}_{g_1}Y_1 - \mathrm{Ad}_{g_2}Y_2)_{ij}$ for $1 \leq i, j \leq n$. Then $\mathcal{C}_\bullet^{(1,M)}$ is the Koszul matrix factorization given by

$$(S \otimes \wedge^*(\theta_{ij}), D, 0, 0), \quad D = \sum_{i,j=1}^{k} X_{ij}\frac{\partial}{\partial\theta_{ij}} + Z_{ij}\theta_{ij}.$$

The case of $M = 1$ (i.e. $\mathbb{B}(\mathcal{C}^{(1)}) = B^{(1)}$) was proved by Oblomkov and Rozansky through a direct computation [39]. For the sake of completeness, we reproduce their proof below.

**Lemma 6.7.** *For $n = 2$,*

$$\mathbb{B}(\mathcal{C}_{\bullet}^{(1)}) = B^{(1)}.$$

*Proof.* The action of $G_2$ on $\mathcal{X}^{\mathrm{st}}$ is free so we can actually consider matrix factorizations on the $G_2$-quotient space with the following potential:

$$\mathcal{X}^{\circ,\mathrm{st}} \subset \mathfrak{g}_2 \times \mathfrak{b} \times G \times \mathfrak{b} \times V, \quad W^{\circ}(X, Y_1, g_{12}, Y_2, v) = \mathrm{Tr}(X(\mathrm{Ad}_{g_{12}}(Y_1) - Y_2)),$$

and stability conditions $\langle X, Y_2 \rangle v = \mathbb{C}^2$, $\langle X, \mathrm{Ad}_{g_{12}} Y_1 \rangle v = \mathbb{C}^2$ and $B^2$-action given by

$$(b_1, b_2) \cdot v = b_2 v, \quad (b_1, b_2) \cdot Y_i = \mathrm{Ad}_{b_i}(Y_i), \quad (b_1, b_2) \cdot X = \mathrm{Ad}_{b_2}(X), \quad (b_1, b_2) \cdot g_{12} = b_2 g_{12} b_1^{-1}$$

Let us set some notation to help distinguish between the two different $B$-actions so let $B^2 = B^{(1)} \times B^{(2)}$. Note that the stability condition implies that if $v = (v_0, v_1)$, then $v_1 \neq 0$. Therefore we can use the $B^{(2)}$-action to assume that $v$ is of the form

$$v_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Let $G' \subset G$ be an open $B^{(1)}$-equivariant locus inside the group:

$$G' = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a \neq 0 \right\}.$$

We claim that the pull-back along the inclusion map

$$j' : \mathfrak{g}_2 \times \mathfrak{b} \times G' \times \mathfrak{b} \times v_0 \to \mathfrak{g}_2 \times \mathfrak{b} \times G \times \mathfrak{b} \times v_0$$

induces an equivalence of categories of $B^{(1)}$-equivariant matrix factorizations. Suppose that $g \in G \setminus G'$. Then for any $Y_1 \in \mathfrak{b}$, we have that

$$(\mathrm{Ad}_{g_{12}}(Y_1))_{12} = 0.$$

However on the critical locus of $W^\circ$, we have that

$$\mathrm{Ad}_{g_{12}}(Y_1) = Y_2, \quad X = 0,$$

which means that $Y_2$ is diagonal and the regular stability condition $\langle X, Y_2 \rangle = \mathbb{C}^2$ turns into the strong stability condition $\mathbb{C}[Y_2]v = \mathbb{C}^2$. However if $v = v_0$, then it is not possible for $\mathbb{C}[Y_2]v$ to be $\mathbb{C}^2$ which is a contradiction.

Next we can use the $B^{(1)}$ and $B^{(2)}$-actions to move a point in the critical locus to the following standard position:

$$g_{12} = \begin{bmatrix} a_{11} & 0 \\ a_{21} & 1 \end{bmatrix}, \quad Y_i = \begin{bmatrix} y_{11}^{(i)} & 1 \\ 0 & y_{22}^{(i)} \end{bmatrix}, \quad v = v_0.$$

Let us elaborate a bit. As shown above, we can use the $B^{(2)}$-action to turn $v$ into $v_0$. We can simultaneously use the $B^{(2)}$-action to put $Y_2$ into the desired form. Next note that the strong stability condition of $\mathbb{C}[\mathrm{Ad}_{g_{12}}(Y_1)]v = \mathbb{C}^2$ implies that $y_{12}^{(1)} \neq 0$. Then we can use the $B^{(1)}$-action to put $Y_1$ and $g_{12}$ into standard position. We can compute

$$\mathrm{Ad}_{g_{12}}(Y_1) = \begin{bmatrix} y_{11}^{(1)} - a_{21} & a_{11} \\ (y_{11}^{(1)} - y_{22}^{(1)} - a_{21})a_{21}/a_{11} & a_{21} + y_{22}^{(1)} \end{bmatrix}$$

Recall that the potential is given by $W^\circ = \mathrm{Tr}(X(\mathrm{Ad}_{g_{12}}(Y_1) - Y_2))$ so we can use the above computation to obtain

$$W^\circ = x_{11}(y_{11}^{(1)} - y_{11}^{(2)} - a_{21}) + x_{21}(a_{11} - 1) + x_{22}(y_{22}^{(1)} - y_{22}^{(2)} + a_{21}) + x_{12}a_{21}(y_{11}^{(1)} - y_{22}^{(1)} - a_{21})/a_{11}$$

Then we obtain the Koszul matrix factorization

$$\begin{aligned} \mathrm{CE}_{\mathfrak{n}^2}(\mathcal{C}_\bullet)^{T^2} &= [x_{11}, y_{11}^{(1)} - y_{11}^{(2)} - a_{21}] \otimes [x_{21}, a_{11} - 1] \\ &\otimes [x_{22}, y_{22}^{(1)} - y_{22}^{(2)} + a_{21}] \otimes [x_{12}, a_{21}(y_{11}^{(1)} - y_{22}^{(1)} - a_{21})/a_{11}] \end{aligned}$$

Next note that the functor $\mathbb{B}$ sends $X$ to zero and the closed locus has a trivial stabilizer inside $B^2$, so $\mathbb{B}(\mathcal{C}_\bullet)$ is a Koszul complex for the regular sequence

$$a_{21}(y_{11}^{(1)} - y_{22}^{(1)} - a_{21})/a_{11}, \quad y_{11}^{(1)} - y_{11}^{(2)} - a_{21}, \quad a_{11} - 1, \quad y_{22}^{(1)} - y_{22}^{(2)} + a_{21}$$

This sequence is equivalent to the following sequence

$$(y_{22}^{(1)} - y_{22}^{(2)})(y_{11}^{(2)} - y_{22}^{(1)}), \quad y_{11}^{(1)} - y_{11}^{(2)} + y_{22}^{(1)} - y_{22}^{(2)}, \quad a_{11}, \quad y_{22}^{(1)} - y_{22}^{(2)} + a_{21}$$

Then using the first two equations, we see that the variety defined by the regular sequence is the union of twisted diagonals $\{(x, x) : x \in \mathbb{C}^2\} \subset \mathbb{C}^2 \times \mathbb{C}^2$ and $\{(s \cdot x, x) : x \in \mathbb{C}^2\} \subset \mathbb{C}^2 \times \mathbb{C}^2$ where the first copy of $\mathbb{C}^2$ has coordinates given by $y_{11}^{(1)}, y_{22}^{(1)}$ and the second copy of $\mathbb{C}^2$ has coordinates given by $y_{11}^{(2)}, y_{22}^{(2)}$. The action of $s$ is given by $s \cdot y_{11}^{(i)} = y_{22}^{(i)}$ and $s \cdot y_{22}^{(i)} = y_{11}^{(i)}$.

Now recall that for a regular sequence $(f_1, f_2, \ldots, f_m) \subset R$. The associated Koszul complex $K(f_1, \ldots, f_m)$ has the properties

$$H^i(K(f_1, \ldots, f_m)) = 0, \quad i > 0 \quad H^0(K(f_1, \ldots, f_m)) = R/(f_1, \ldots f_m)$$

However this is precisely Soergel's geometric formulation for the Soergel bimodule $B^{(1)}$ and hence the result is proven. $\square$

Next we shall prove the case of $M = 2$ using an alternative geometric approach developed by the Oblomkov and Rozansky in [39]. This method should hopefully generalize though we only need the result for $M = 2$ in order to prove the original conjecture regarding $\mathfrak{sl}_2$-homology. There is a functor

$$i_* : D_G^{b,T_q}(\tilde{\mathfrak{g}}^2) \to MF_n$$

given by composition of the folding functor and the pushforward.

**Proposition 6.8.** *The functor $i_*$ is monoidal:*

$$i_*(\mathcal{B}) \star i_*(\mathcal{B}') = i_*(\mathcal{B} \star \mathcal{B}').$$

Let us define Koszul complexes which map to the generators $\mathcal{C}^{(i)}_\bullet$ under $i_*$:

$$\mathcal{B}^{(i)}_\bullet \in D^{b,T_q}_{G,\mathrm{prop}}(\tilde{\mathfrak{g}}^2), \quad i_*(\mathcal{B}^{(i)}_\bullet) = \mathcal{C}^{(i)}_\bullet.$$

These objects $\mathcal{B}^{(i)}_\bullet$ are Koszul complexes for the subvariety $\mathrm{St}^{(i)} \subset \widetilde{\mathfrak{g}_n} \times \widetilde{\mathfrak{g}_n}$ which is a $B^2$-quotient of the subvariety $\widetilde{\mathrm{St}^{(i)}}$. This variety $\widetilde{\mathrm{St}^{(i)}}$ consists of tuples $(Y_1, g_1, Y_2, g_2)$ satisfying the equations:

$$g_{12} = g_1^{-1} g_2 \in P_i, \quad \mathrm{Ad}_{g_1}(Y_1) = \mathrm{Ad}_{g_2}(Y_2)$$

where we define $P_I \subset G_n$ for $I \subset \{1, \ldots, n-1\}$ to be the parabolic subgroup with the Lie algebra generated by $\mathfrak{b}$ and $E_{i+1,i}$ for $i \in I$.

Similarly we can define $\widetilde{\mathrm{St}^{(i,i+1)}} \subset \tilde{\mathfrak{g}}^2$ as the variety of tuples $(Y_1, g_1, Y_2, g_2)$ satisfying the equations:

$$g_{12} = g_1^{-1} g_2 \in P_{i,i+1}, \quad \mathrm{Ad}_{g_1}(Y_1) = \mathrm{Ad}_{g_2}(Y_2).$$

We have the matrix factorization

$$\mathcal{C}^{(i,i+1)} = i_*(\mathcal{B}^{(i,i+1)}_\bullet).$$

Then we claim that $\mathbb{B}(\mathcal{C}^{(1,2)}) = B^{(1,2)}_\bullet$. However before we can prove this claim, we need to better understand the categories $\mathrm{MF}_n$ and $D^{b,T_q}_G(\tilde{\mathfrak{g}}^2)$. Both of these categories have a duality functor:

$$\mathcal{F} \mapsto \mathcal{F}^\vee := \mathcal{H}om(\mathcal{F}, \mathcal{O}).$$

We have an analog of the subcategory $\mathrm{MF}^\flat_n \subset \mathrm{MF}_n$ for $D^{b,T_q}_G(\tilde{\mathfrak{g}}^2)$. Let $D^\flat \subset D^{b,T_q}_G(\tilde{\mathfrak{g}}^2)$ be the additive subcategory that is spanned by all products of elements

$\mathcal{B}^{(i)}$ and by the products of the direct summands of these products. Our goal is to prove the following proposition.

**Proposition 6.9.** *For any $\mathcal{A}$ in either $\mathrm{MF}_n^\flat$ or $D^\flat$, we have*

$$\mathcal{A}^\vee \cong \mathcal{A}$$

First let us show the following proposition.

**Proposition 6.10.** *For any $\mathcal{A}_1, \mathcal{A}_2$ in $\mathrm{MF}_n^\flat$ or $D^\flat$, we have*

$$(\mathcal{A}_1 \star \mathcal{A}_2)^\vee = \mathcal{A}_1^\vee \star \mathcal{A}_2^\vee$$

*Proof.* Since $\tilde{\mathfrak{g}}_n$ is holomorphic symplectic, it has a trivial canonical class. Therefore the statement follows from Serre duality:

$$\mathcal{A}_1^\vee \star \mathcal{A}_2^\vee = \pi_{13*}\left(\pi_{12}^*(\mathcal{A}_1)^\vee \otimes \pi_{23}^*(\mathcal{A}_2)^\vee\right) = \pi_{13*}\left(\pi_{12}^*(\mathcal{A}_1) \otimes \pi_{23}^*(\mathcal{A}_2)\right)^\vee. \qquad \square$$

Then it suffices to show that the generating objects $\mathcal{B}_\bullet^{(i)}$ and $\mathcal{C}^{(i)}$ are self-dual in order to prove Proposition 6.9. Let us set some notation for the twisting of the $B^2$-action.

Given two characters $\lambda, \mu \in T^\vee$ and a module $M$ with a $B^2$-action

$$(b_1, b_2, m) \mapsto (b_1, b_2) \cdot m, \quad b_1, b_2 \in B, m \in M,$$

we can define the module $M\langle\lambda, \mu\rangle$ as the module $M$ with $B^2$-action twisted by $\lambda$ and $\mu$:

$$(b_1, b_2, m) \mapsto \lambda(b_1)\mu(b_2)(b_1, b_2) \cdot m, \quad b_1, b_2 \in B, m \in M.$$

**Proposition 6.11.** *For any $\mathcal{A}$ in $\mathrm{MF}_n$ or $D^\flat$ and $\lambda, \mu \in T^\vee$, we have*

1. $\mathcal{A} \star \langle\lambda, \mu\rangle = \mathcal{A}\langle 0, \lambda + \mu\rangle$

*2.* $\mathbb{1}\langle\lambda,\mu\rangle \star \mathcal{A} = \mathcal{A}\langle\lambda+\mu,0\rangle$

*3.* $\mathbb{1}\langle\lambda,\mu\rangle = \mathbb{1}\langle\lambda+\mu,0\rangle = \mathbb{1}\langle 0,\lambda+\mu\rangle$

*4.* $\mathbb{1}^\vee = \mathbb{1}$

*Proof.* The first three parts of this proposition were proven in an earlier work [37] of Oblomkov and Rozansky so we only present a proof of the fourth part. Fix coordinates $(g_1, Y_1, g_2, Y_2)$ on $(G \times \mathfrak{b})^2$ and let $\rho$ denote the half sum of the positive roots of $G_n$. Then the element $\mathbb{1}$ is a Koszul complex for the equations:

$$g_2^{-1}g_1 \in B, \quad (\mathrm{Ad}_{g_2^{-1}g_1}(Y_1))_{ij} = (Y_2)_{ij}, \quad i \geq j$$

For the first equation since $(b_1, b_2) \cdot g_2^{-1}g_1 = b_2 g_2^{-1} g_1 b_1^{-1}$, the sum of the $B^2$ weights is $\langle -\rho, -\rho\rangle$. For the second set of equations since $(b_1, b_2) \cdot Y_2 = \mathrm{Ad}_{b_2}(Y_2)$, the weights are $\langle 0, 2\rho\rangle$. Therefore the total sum of the $B^2$ weight is given by $\langle -\rho, \rho\rangle$. To compute the dual of $\mathbb{1}$, we need to invert the Borel weights in the Koszul complex. Then using part 3 of the proposition, we have

$$\mathbb{1}^\vee = \mathbb{1}\langle \rho, -\rho\rangle = \mathbb{1} \qquad \square$$

Now we can finally show that $\mathcal{B}_\bullet^{(i)}$ and $\mathcal{C}^{(i)}$ are self-dual objects. We only show the result for $\mathcal{B}_\bullet^{(i)}$ since the argument for $\mathcal{C}^{(i)}$ is similar.

We begin by computing the dual of $\mathcal{B}_\bullet^{(i)}$ in $D^\flat$. The corresponding variety $\widetilde{\mathrm{St}}^{(i)}$ is described by the two equivalent system of equations. The first system is

$$g_2^{-1}g_1 \in P_i, \quad \left(\mathrm{Ad}_{g_2^{-1}g_1} Y_1\right)_{kj} = (Y_2)_{kj}, \quad k \geq j, \quad \left(\mathrm{Ad}_{g_2^{-1}g_1} Y_1\right)_{i+1,i} = 0.$$

The second system of equations is given by

$$g_1^{-1}g_2 \in P_i, \quad \left(\mathrm{Ad}_{g_1^{-1}g_2} Y_2\right)_{kj} = (Y_1)_{kj}, \quad k \geq j, \quad \left(\mathrm{Ad}_{g_1^{-1}g_2} Y_2\right)_{i+1,i} = 0.$$

The sum of the weights of the first system of equations is $\langle -\rho + \epsilon_{i+1}, \rho - \epsilon_{i+1}\rangle$ and the sum of the weights of the second system of equations is $\langle \rho - \epsilon_{i+1}, -\rho + \epsilon_{i+1}\rangle$. Hence the Koszul complexes of these two systems are dual to each other. Since these two Koszul complexes are two presentations of the element $\mathcal{B}_{\bullet}^{(i)}$, we obtain

$$(\mathcal{B}_{\bullet}^{(i)})^{\vee} = \mathcal{B}_{\bullet}^{(i)}.$$

It follows that all objects in the categories $\mathrm{MF}^{\flat}$ and $D^{\flat}$ are self-dual since these categories are Karoubi envelopes of the categories generated monoidally by $\mathcal{B}_{\bullet}^{(i)}$ and $\mathcal{C}^{(i)}$. It is a general fact that Karoubi envelopes are Krull-Schmidt categories.

Let us denote the unit object in both categories $\mathrm{MF}_n$ and $D_{G_n}(\tilde{\mathfrak{g}}_n^2)$ by $\mathbb{1}$:

$$\mathbb{1} = \mathcal{C}_{\|} \in \mathrm{MF}_n, \quad \mathbb{1} = \mathcal{B}_{\|} \in D_{G_n}(\tilde{\mathfrak{g}}_n^2).$$

**Corollary 6.12.** *For any $\mathcal{A}_1, \mathcal{A}_2$ in either $\mathrm{MF}_n^{\flat}$ or $D^{\flat}$, we have*

$$\mathrm{Ext}^i(\mathcal{A}_1, \mathcal{A}_2) = \mathrm{Tor}_i(\mathcal{A}_1 \star \mathcal{A}_2, \mathbb{1})$$

*Proof.* The proof of this corollary follows from the self-duality of objects in $MF^{\flat}$ and $D^{\flat}$ and the simpler case where $\mathcal{A}_1, \mathcal{A}_2$ are modules over some ring $R$. Then the result follows using the self-duality of objects in the respective categories along with tensor-hom adjunction. Let $P_{\bullet}$ be a projective resolution for $\mathcal{A}_2$.

$$\begin{aligned}
\mathrm{Ext}^i(\mathcal{A}_1, \mathcal{A}_2) &= \mathrm{H}^i(\mathrm{Hom}(\mathcal{A}_1, P_{\bullet})) \\
&= \mathrm{H}^i(\mathrm{Hom}(\mathcal{A}_1, \mathrm{Hom}(P_{\bullet}, R))) \\
&= \mathrm{H}^i(\mathrm{Hom}(\mathcal{A}_1 \otimes P_{\bullet}, R)) \\
&= \mathrm{Tor}_i(\mathcal{A}_1 \otimes \mathcal{A}_2, R) \qquad \square
\end{aligned}$$

Now we can return to the proof of our claim that $\mathbb{B}(\mathcal{C}^{(1,2)}) = B_{\bullet}^{(1,2)}$. Since $\mathbb{B}$ is an additive, monoidal functor, the claim follows from the following proposition.

**Proposition 6.13.** *For* $\mathcal{C}_\bullet^{(1)}, \mathcal{C}_\bullet^{(2)}, \mathcal{C}_\bullet^{(1,2)} \in \mathrm{MF}_3$, *we have that*

$$\mathcal{C}_\bullet^{(1)} \star \mathcal{C}_\bullet^{(2)} \star \mathcal{C}_\bullet^{(1)} = \mathcal{C}_\bullet^{(1)} \oplus \mathcal{C}_\bullet^{(1,2)}.$$

The proof of this proposition relies on the lemma.

**Lemma 6.14.** *There is a short exact sequence of sheaves on* $\tilde{\mathfrak{g}}^2$*:*

$$0 \longrightarrow \mathcal{B}_\bullet^{(1)} \longrightarrow \mathcal{B}_\bullet^{(1)} \star \mathcal{B}_\bullet^{(2)} \star \mathcal{B}_\bullet^{(1)} \longrightarrow \mathcal{B}_\bullet^{(1,2)} \longrightarrow 0$$

If we can show that this sequence splits after applying the pushforward functor, then this implies the desired result. To prove the lemma, we need to give an alternative geometric description of the convolution product of $\mathcal{B}_\bullet^{(1)}$ and $\mathcal{B}_\bullet^{(2)}$.

**Lemma 6.15.** *For* $\mathcal{B}_\bullet^{(1)}, \mathcal{B}_\bullet^{(2)} \in D^{b,T_q}_{G_3,\mathrm{prop}}$, *we have that*

$$\mathcal{B}_\bullet^{(1)} \star \mathcal{B}_\bullet^{(2)} = \mathcal{O}_{Z_{12}},$$

*where* $Z_{12} \subset \tilde{\mathfrak{g}}^2$ *is a complete intersection defined by the equations*

$$(g_1^{-1}g_2)_{3,1} = 0, \quad \mathrm{Ad}_{g_2^{-1}g_1}(Y_1)_{ij} = 0, \quad ij = 32, 21, 31$$
$$(Y_2)_{ij} = \mathrm{Ad}_{g_2^{-1}g_1}(Y_1)_{ij}, \quad ij = 11, 22, 33, 12, 23, 13$$

*Proof.* Note that the projection map

$$\pi_{13} : \pi_{12}^{-1}(\mathrm{St}^{(1)}) \cap \pi_{23}^{-1}(\mathrm{St}^{(2)}) \to Z_{12}$$

is an isomorphism since the defining system of equations on both the left-hand side and the right-hand side are the same. □

Next let us give a geometric description of $Z_{12}$. It is a part of the Steinberg variety $\widetilde{\mathrm{St}}^{(1,2)}$. The Steinberg variety $\mathrm{St}^{(1,2)} \subset \tilde{\mathfrak{g}}_3^2$ has six irreducible components.

$$\mathrm{St}^{(1,2)} = \bigcup_{w \in S_3} \Gamma_w.$$

A generic point of $\Gamma_w$ consists of the triple $(F_\bullet, G_\bullet, X)$ where $X \in \mathfrak{g}_3$ and $F_\bullet, G_\bullet$ are two flags in relative position $w$. The smaller Steinberg varieties have a similar description as a union of irreducible components:

$$\mathrm{St}^{(1)} = \Gamma_1 \cup \Gamma_{s_1}, \quad \mathrm{St}^{(2)} = \Gamma_1 \cup \Gamma_{s_2}$$

Then the previous lemma implies that

$$Z_{12} = \Gamma_1 \cup \Gamma_{s_1} \cup \Gamma_{s_2} \cup \Gamma_{s_1 s_2}.$$

Next we need to compute some convolution products.

**Lemma 6.16.** *For* $\mathcal{B}_\bullet^{(1)}, \mathcal{B}_\bullet^{(1,2)}$, *we have*

$$\mathcal{B}_\bullet^{(1)} \star \mathcal{B}_\bullet^{(1,2)} = \mathcal{B}_\bullet^{(1,2)} \star \mathcal{B}_\bullet^{(1)} = \mathcal{B}_\bullet^{(1,2)} \oplus \mathbf{q}^2 \mathcal{B}_\bullet^{(1,2)}.$$

*Proof.* We shall prove the second inequality since the other case is analogous. The convolution space $\mathcal{X}_{\mathrm{conv}} \subset (G \times \mathfrak{b})^3$ has coordinates $(g_1, Y_1, g_2, Y_2, g_3, Y_3)$ with the constraint $g_2^{-1} g_3 \in P_1$. In these coordinates, the pull-back $\pi_{12}^*(\mathcal{B}_\bullet^{(1,2)})$ is a Koszul complex with generators given by

$$\mathrm{Ad}_{g_1}(Y_1) = \mathrm{Ad}_{g_2}(Y_2)$$

For the Koszul complex $\pi_{23}^*(\mathcal{B}_\bullet^{(1)})$, we can choose a system of generators as:

$$(\mathrm{Ad}_{g_2^{-1} g_3} Y_3)_{21} = 0, \quad (\mathrm{Ad}_{g_2^{-1} g_3}(Y_3))_{ij} = (Y_2)_{ij}, \quad i \le j$$

Then combining the above two systems of equations yields the following:

$$\mathrm{Ad}_{g_2^{-1} g_1}(Y_1) = \mathrm{Ad}_{g_2^{-1} g_3}(Y_3), \quad (\mathrm{Ad}_{g_2^{-1} g_3}(Y_3))_{ij} = (Y_2)_{ij}, \quad ij = 11, 22, 12,$$
$$(\mathrm{Ad}_{g_2^{-1} g_3}(Y_3))_{21} = 0$$

We can use the above equations to eliminate the variable $Y_2$. Furthermore the leftmost equation on the first line can be rewritten as

$$\mathrm{Ad}_{g_1}(Y_1) = \mathrm{Ad}_{g_3}(Y_3).$$

Then the convolution of interest is given by

$$\mathrm{CE}_{\mathfrak{n}}(K[(\mathrm{Ad}_{g_2^{-1}g_3}(Y_3))_{21}])^T \otimes \mathcal{B}_\bullet^{(1,2)}.$$

Now write $g_2^{-1}g_3 = (a_{ij})$. Then we can express the element $\mathrm{Ad}_{g_2^{-1}g_3}(Y_3)_{21}$ as

$$\mathrm{Ad}_{g_2^{-1}g_3}(Y_3)_{21} = a_{21}(a_{22}(y_{22} - y_{11}) + a_{21}y_{12})$$

This element has weight $(2,0)$ with respect to the $T = \mathbb{C}^* \times \mathbb{C}^*$ action since the elements $a_{ij}$ have weight $\epsilon_i$. Now since $P_1/B = \mathbb{P}^1$, it remains to compute the push-forward along the projection of the two-step complex

$$\pi_{\mathbb{P}^1}([\mathcal{O} \to \mathbf{q}^2\mathcal{O}(-2)]), \quad \pi_{\mathbb{P}^1} : \mathbb{P}^1 \times \mathfrak{b} \times G \to \mathfrak{b} \times G.$$

Note that $\pi_{\mathbb{P}^1}(\mathcal{O}) = \mathcal{O}$ and $\pi_{\mathbb{P}^1}(\mathcal{O}(-2)) = \mathcal{O}[1]$ where $[1]$ denotes a homological shift. The desired statement follows from this computation. □

Now let us proceed with the proof of Lemma 6.14.

*Proof.* We need to compute the convolution product

$$\mathcal{O}_{Z_{12}} \star \mathcal{B}_\bullet^{(1)} = \left(\mathcal{B}_\bullet^{(1)} \star \mathcal{B}_\bullet^{(2)}\right) \star \mathcal{B}_\bullet^{(1)}.$$

In geometric terms, the convolution is the push-forward:

$$\pi_{13*}(\mathcal{O}_{Z_{123}}), \quad Z_{123} = \pi_{12}^{-1}(Z_{12}) \cap \pi_{23}^{-1}(\mathrm{St}^{(1)}) \subset \tilde{\mathfrak{g}}_3^3.$$

We can describe the variety $Z_{123}$ as $Z_{123} = Z_{123}^1 \cup Z_{123}^2$ where

$$Z_{123}^1 = \pi_{12}^{-1}(\Gamma_1 \cup \Gamma_{s_1}) \cap \pi_{23}^{-1}(\Gamma_{s_1})$$

$$Z_{123}^2 = \left(\pi_{12}^{-1}(Z_{12}) \cap \pi_{23}^{-1}(\Gamma_1)\right) \bigcup \left(\pi_{12}^{-1}(\Gamma_{s_2} \cup \Gamma_{s_1 s_2}) \cap \pi_{23}^{-1}(\Gamma_{s_1})\right)$$

The intersection of the graph closures $\Gamma_w, \Gamma_{w'}$ is sent by the projection

$$\mu_1 : (G \times \mathfrak{b})^2 \to \mathfrak{b}$$

to the locus defined by the equations:

$$w \cdot \lambda(Y) = w' \cdot \lambda(Y).$$

Hence the intersection $Z^1_{123} \cap Z^2_{123}$ is a divisor inside $Z^1_{123}$ with defining equation given by $(Y^1_{22} - Y^1_{33})(Y^3_{11} - Y^3_{22})$. The vanishing locus of the first factor of the defining equation is the intersection $\pi^{-1}_{12}(\Gamma_1 \cup \Gamma_{s_1}) \cap \pi^{-1}_{12}(\Gamma_{s_2} \cup \Gamma_{s_1 s_2})$ and the second factor defines the intersection $\pi^{-1}_{23}(\Gamma_1) \cap \pi^{-1}_{23}(\Gamma_{s_1})$. Therefore we have the short exact sequence:

$$0 \longrightarrow \mathbf{q}^4 \mathcal{O}_{Z^1_{123}} \longrightarrow \mathcal{O}_{Z_{123}} \longrightarrow \mathcal{O}_{Z^2_{123}} \longrightarrow 0$$

Next we need to compute the push-forward $\pi_{13*}$ of the short exact sequence. Note that

$$\pi_{13*}(\mathcal{O}_{Z^1_{123}}) = \mathcal{B}^{(1)}_\bullet \star \mathcal{O}_{\Gamma_{s_1}}.$$

Then by an analogous argument as the one found in Lemma 6.16, one can compute that

$$\mathcal{B}^{(1)}_\bullet \star \mathcal{O}_{\Gamma_{s_1}} = \mathbf{q}^{-2} \mathcal{B}^{(1)}_\bullet$$

Finally we use the following general fact about the graph closure $\Gamma_w$. Suppose that $w, w' \in S_n$ satisfy $l(ww') = l(w) + l(w')$. Then the projection $\pi_{13} : \tilde{\mathfrak{g}}^3 \to \tilde{\mathfrak{g}}^2$ yields an isomorphism

$$\pi^{-1}_{12}(\Gamma_w) \cap \pi^{-1}_{23}(\Gamma_{w'}) \to \Gamma_{ww'}$$

Applying this to the situation above means that $\pi_{13*}$ restricts to an isomorphism: $Z^2_{123} \to \mathrm{St}$. Thus since we have shown that $\pi_{13*}(\mathcal{O}_{Z^1_{123}}) = \mathcal{B}^{(1)}_\bullet$ and $\pi_{13*}(\mathcal{O}_{Z^2_{123}}) = \mathcal{O}_{\mathrm{St}}$, the desired statement follows. $\square$

Since the pushforward $i_*$ is exact, the short exact sequence in Lemma 6.14 is sent to an exact triangle in $\mathrm{MF}_3$ (not necessarily another short exact sequence). Therefore to complete the proof of the proposition, we need to compute the relevant extension groups.

**Proposition 6.17.** *For $\mathcal{C}_\bullet^{(1)}, \mathcal{C}_\bullet^{(1,2)} \in MF_3$, we have that*

$$\mathrm{Ext}^{>0}(\mathcal{C}_\bullet^{(1,2)}, \mathcal{C}_\bullet^{(1)}) = 0, \quad \mathrm{Ext}^{0}(\mathcal{C}_\bullet^{(1,2)}, \mathcal{C}_\bullet^{(1)}) = \mathbb{C}[y_1, y_2, y_3]^{\oplus 2}, \quad \mathrm{Ext}^{>0}(\mathcal{C}_\bullet^{(1,2)}, \mathcal{C}_{\|}) = 0.$$

*Proof.* Recall that $\widetilde{\mathrm{St}} \subset (G \times \mathfrak{b})^2$ is defined by

$$\mathrm{Ad}_{g_1}(Y_1) = \mathrm{Ad}_{g_2}(Y_2),$$

where $(g_1, Y_1, g_2, Y_2)$ are coordinates on $(G \times \mathfrak{b})^2$. Note that this system of equations is $B^2$-invariant so the corresponding Koszul matrix factorization

$$\mathcal{C}^{(1,2)} = K^W[\mathrm{Ad}_{g_1}(Y_1) - \mathrm{Ad}_{g_2}(Y_2)]$$

is self-dual. It follows from using Corollary 6.12 that

$$\mathrm{Ext}^*(\mathcal{C}_\bullet^{(1,2)}, \mathcal{C}_\bullet^{(1)}) = \mathrm{Tor}_*(\mathcal{C}_\bullet^{(1,2)} * \mathcal{C}_\bullet^{(1)}, \mathcal{C}_{\|}) = \mathrm{Tor}_*(\mathcal{C}_\bullet^{(1,2)}, \mathcal{C}_{\|})^{\oplus 2}$$

where the second equality follows from Lemma 6.16. Therefore it remains to compute the groups on the right-hand side of the above equation. We compute these directly:

$$\begin{aligned}\mathrm{Ext}^*(\mathcal{C}_\bullet^{(1,2)}, \mathcal{C}_{\|}) &= \mathrm{Tor}_*(\mathcal{C}_\bullet^{(1,2)}, \mathcal{C}_{\|}) \\ &= \mathrm{CE}_{\mathfrak{n}^2}(K^W[\mathrm{Ad}_{g_1}(Y_1) - \mathrm{Ad}_{g_2}(Y_2)] \otimes K^{-W}[g_1 g_2^{-1} \in B, Y_1 - Y_2])^{T^2 \times G},\end{aligned}$$

where we used the coordinates $(X, g_1, Y_1, g_2, Y_2)$ for the space $\mathfrak{g} \times (G \times \mathfrak{b})^2$. Oblomkov and Rozansky showed in their previous work that the last group is equal to the pull-back

$$\mathrm{CE}_{\mathfrak{n}}(j_e^*(K^W[\mathrm{Ad}_{g_1}(Y_1) - \mathrm{Ad}_{g_2}(Y_2)]))^T,$$

where $j_e : \mathfrak{g} \times \mathfrak{b} \to \mathfrak{g} \times (G \times \mathfrak{b})^2$ is the $B$-equivariant embedding:

$$j_e(X, Y) = (X, e, Y, e, Y).$$

Since the pull-back $j_e^*$ turns the system of equations $\mathrm{Ad}_{g_1}(Y_1) = \mathrm{Ad}_{g_2}(Y_2)$ into an empty condition, we have that

$$\mathrm{CE}_{\mathfrak{n}}(j_e^*(K^W[\mathrm{Ad}_{g_1}(Y_1) - \mathrm{Ad}_{g_2}(Y_2)]))^T = \mathrm{CE}_{\mathfrak{n}}(K[X])^T = H^*_{\mathrm{Lie}}(\mathfrak{n}, \mathbb{C}[\mathfrak{b}])^T.$$

Since $X$ forms a regular sequence, the Koszul complex $K[X]$ only has homology in degree zero. Then computing the second-to-last term via a spectral sequence, we obtain the last equality. The rightmost term is the homology of the dg-algebra $\mathcal{A} = \mathbb{C}[\mathfrak{b} \otimes \Lambda^*(\theta_{ij})_{i<j}]$ with differential

$$D = \sum_{i<j} \theta_{ij} E_{ij} \cdot,$$

where $E_{ij} \cdot$ is the action of $E_{ij}$ on $\mathcal{A}$. The element $E_{ij}$ acts on $\mathbb{C}[\mathfrak{b}]$ by taking the derivative along the corresponding vector field generated by the adjoint action of $\mathfrak{n}$ on $\mathfrak{b}$ and it acts on $\Lambda^*(\theta_{ij})$ by the Lie bracket. Hence the elements $\theta_{ij}$ have $T$-weight $\epsilon_i - \epsilon_j$ and as a result, all elements in the algebra $\mathcal{A}$ have positive $T$-grading. Therefore the $T$-invariant part of the complex is simply $\mathbb{C}[y_1, y_2, y_3]$ and the desired results follow. □

To recap, the long geometric argument above proves that $\mathbb{B}(\mathcal{C}^{(1,2)}) = B^{(1,2)}$. Next we need to prove the following statement:

**Corollary 6.18.** *For $\mathcal{C}_\bullet^{(r_1,r_2)} \in \mathrm{MF}_n^\flat$ with $r_2 - r_1 = M - 1$:*

$$\mathbb{B} \circ \mathcal{R}_{M|0}(\mathcal{C}_\bullet^{(r_1,r_2)}) = 0$$

*Proof.* Note that using the induction functors constructed in previous work by Oblomkov and Rozansky, one can obtain the matrix factorizations $\mathcal{C}_\bullet^{(r_1,r_2)}$. Since the functor $\mathbb{B}$ commutes with the induction functors, it suffices to consider the case of $\mathcal{C}_\bullet^{(1,n)}$.

Recall that the functor $\mathbb{B}$ is constructed as follows:

$$\mathbb{B}(\mathcal{C}) = \pi_*(j_{x=0}^*(\mathcal{C})^{G\times B^2})$$

where $j_{x=0}$ is the embedding of $((G \times \mathfrak{b})^2 \times V_B)^{\mathrm{st}}$ into $(\mathfrak{g} \times (G \times \mathfrak{b})^2 \times V_B)^{\mathrm{st}}$ and $\pi$ is the projection from $((G \times \mathfrak{b})^2 \times V_B)^{\mathrm{st}}$ to $\mathfrak{h}^2$.

Note that if we pull back $\mathcal{C}_\bullet^{(1,n)}$ along $j_{x=0}$, we obtain a Koszul complex:

$$j_{x=0}^*(\mathcal{C}_\bullet^{(1,n)}) = K[0, Z_{ij}]$$

Next we need to compute the $G \times B^2$ invariant part of

$$i_{z=0}^*((\wedge^* V_B, d_{M|0}))$$

where $i_{z=0} : \{Z_{ij} = 0\} \to ((G \times \mathfrak{b})^2 \times V_B)^{\mathrm{st}}$ is the inclusion map.

**Remark 6.19.** *Recall that the action of* $G \times B^2$ *on* $\mathfrak{g} \times (G \times \mathfrak{b})^2$ *is given by*

$$(g, b_1, b_2) \cdot (X, g_1, Y_1, g_2, Y_2) = (\mathrm{Ad}_g X, g g_1 b_1^{-1}, \mathrm{Ad}_{b_1} Y_1, g g_2 b_2^{-1}, \mathrm{Ad}_{b_2} Y_2)$$

*Then the* $G \times B^2$*-action on the subspace defined by the equations* $Z_{ij} = 0$ *is free.*

A slice to this action has the parameterization:

$$g_1 = g_2 = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}, Y_1 = Y_2 = \begin{bmatrix} y_1 & 1 & 0 & \cdots & 0 \\ 0 & y_2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & y_n \end{bmatrix}, v = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

On this slice, the complex $(\wedge^* V_B, d_{M|0})$ becomes a Koszul complex for the complete intersection variety defined by the equation $Y_1^M v = 0$. Note that $n - M, n$ entry of $Y_1^M$ is 1 so the equation $Y_1^M v = 0$ cuts out the empty set on the slice.

Therefore the contractible complex $i^*_{z=0}((\wedge^* V_B, d_{M|0}))$ is homotopic to $j^*_{x=0}(\mathcal{C}_\bullet^{(1,n)}) \otimes (\wedge^* V_B, d_{M|0})$. It follows that

$$\mathbb{B} \circ \mathcal{R}_{M|0}(\mathcal{C}_\bullet^{(r_1,r_2)}) = 0 \qquad \square$$

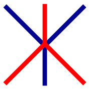
## 6.2 Proof of main theorem

Let us now return to the proof of the original conjecture.

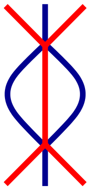
*Proof.* Elias and Khovanov constructed an equivalence of categories between the category of Soergel bimodules $\mathrm{SBim}_n$ and their diagrammatic Soergel category (in type A) $\mathcal{DC}_n$ with $n$ colors in [11]. Let $\mathbb{D} : \mathrm{SBim}_n \to \mathcal{DC}_n$ denote this equivalence (originally $\mathcal{F}$ in their paper). Pedro Vaz constructed a functor $\mathcal{F}_{\mathfrak{sl}_2,n}$ from $\mathcal{DC}_n$ to the category of $\mathfrak{sl}_2$ foams $\mathbf{Foam}_2$ [43]. Let us consider the composition of functors:

$$\mathrm{MF}_n^{\mathrm{st}} \xrightarrow{\mathbb{B}} \mathrm{SBim} \xrightarrow{\mathbb{D}} \mathcal{DC} \xrightarrow{\mathcal{F}_{\mathfrak{sl}_2,n}} \mathbf{Foam}_2$$

The reader should refer to Vaz' original paper for the description of his functor as well as a review of the categories $\mathcal{DC}_n$ and $\mathbf{Foam}_2$, however want to highlight that the key part of the functor $\mathcal{F}_{\mathfrak{sl}_2,n}$ is that the 6-valent vertex is sent to zero.

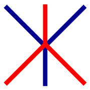
 $\mapsto 0$

The theorem 6.18 above implies that

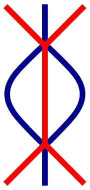


$$\mapsto 0$$

since the diagram on the left-hand side represents projection from the bimodule $B_i \otimes_R B_{i+1} \otimes_R B_i$ onto the indecomposable $B^{(i,i+1)}$. It can be shown that the relations



$$= 0,$$



$$= 0$$

are equivalent to each other by direct computation (they define the same ideal in the category $\mathcal{DC}$). It is clear that the doubled 6-valent vertex in contained in the ideal generated by the 6-valent vertex. The other inclusion follows from the relations in the diagrammatic category and is detailed in Claim 3.1 in [9].

The quotient of the category $\mathcal{DC}$ by either of these relations gives the diagrammatic category $\mathcal{TL}$ which is a categorification of the Temperley-Lieb algebra [9]. Elias showed the functor $\mathcal{F}_{\mathfrak{sl}_2,n}$ descends to a faithful functor between $\mathcal{TL}$ and $\mathbf{Foam}_2$ (see Remark 3.29), i.e. it is injective on hom-sets. We can summarize our construction in the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Br}_n \arrow[r, "\Phi_n"] \arrow[dddrr, bend right, "\mathrm{Kh}"'] & \mathrm{MF}_n^{\mathrm{st}} \arrow[r, "\mathcal{R}_{2|0}"] \arrow[d, "\mathbb{B}"] & \mathrm{MF}_n^{\mathrm{st}} \arrow[d, "\mathbb{B}"] \\
 & \mathrm{SBim}_n \arrow[r, "q_2"] \arrow[d, "\mathbb{D}"] & \mathrm{SBim}_n/2 \arrow[d, "\mathbb{D}"] \\
 & \mathcal{DC}_n \arrow[r, "\mathcal{Q}_2"] & \mathcal{TL}_n \arrow[d, "\mathcal{F}_{\mathfrak{sl}_2,n}"] \\
 & & \mathbf{Foam}_2
\end{tikzcd}$$

The image of $\mathbb{B} \circ \mathcal{R}_{2|0}$ lands in SBim/2 which is equivalent to $\mathcal{TL}_n$ under the functor $\mathbb{D}$. In other words, after complexes in $\mathrm{MF}^{\mathrm{st}}_n$ are equipped with the differential coming for $\mathcal{R}_{2|0}$, the resulting complex can regarded as a complex living in the category $\mathbf{Foam}_2$. Let $\mathcal{C}_2$ denote the composition of functors

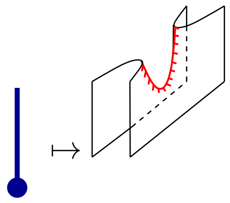
$$\mathcal{C}_2 = \mathcal{F}_{\mathfrak{sl}_2,n} \circ \mathbb{D} \circ \mathbb{B} \circ \mathcal{R}_{2|0}.$$

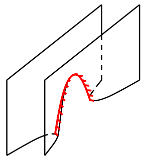
The next step in the proof is to show that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Br}_n \arrow[r, "\Phi_n"] \arrow[dr, "\mathrm{Kh}"] & \mathrm{MF}^{\mathrm{st}}_n \arrow[d, "\mathcal{C}_2"] \\
& \mathbf{Foam}_2
\end{tikzcd}$$

We want to show that the chain complexes obtained through our construction are equivalent to the usual construction used for computing the Khovanov homology of a braid. The map $\Phi_n$ is the homomorphism constructed in [37] which associates a complex of matrix factorizations to any braid. The map Kh is the construction detailed in Section 3 which associates a chain complex to a braid using Khovanov's cube of resolutions. Since the functors defining $\mathcal{C}_2$ are all monoidal, it suffices to check this on generators. Note that in the category $\mathbf{Foam}_2$, the tensor product should be interpreted in the sense of planar algebras.

The generating matrix factorizations are $\mathcal{C}_+^{(i)}$ and $\mathcal{C}_-^{(i)}$ for $i = 1, \ldots, n$. Under the functor $\mathbb{B}$, $\mathcal{C}_+^{(i)}$ and $\mathcal{C}_-^{(i)}$ are sent to the Rouquier complexes $T_i$ and $T_i^{-1}$ respectively. In the language of diagrammatics, these Rouquier complexes are represented by the StartDot and EndDot morphisms. Then it suffices to check where these morphisms go to under the functor $\mathcal{F}_{\mathfrak{sl}_2,n}$. Indeed we have that



 $\mapsto$  .

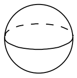
Therefore the diagram above commutes. Finally we need to check that taking the braid closure (i.e. taking the homology) of the complexes in $\mathrm{MF}_n^{\mathrm{st}}$ and $\mathbf{Foam}_2$ yield the desired result.

On the side of $\mathbf{Foam}_2$, the homology of the complex is computed by closing up the strands of the braid and applying a functor $\mathcal{F}$ which lands in the category of graded abelian groups. Then $\mathcal{F}(\bigcirc) = A$ has the structure of a Frobenius algebra.

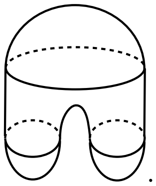
Recall that a Frobenius algebra is a tuple $(A, \epsilon, \mu, \eta, \delta)$ where $A$ is a $\mathbb{C}$-vector space with linear maps $\epsilon, \mu, \eta, \delta$ such that $(A, \epsilon, \mu)$ has the structure of an algebra and $(A, \eta, \delta)$ has the structure of a coalgebra along with a compatibility condition known as Frobenius associativity. The maps $\epsilon, \mu, \eta, \delta$ are known as the unit, multiplication, counit, and comultiplication maps respectively. See Chapter 8 of [12] for more details.

We can identify the functor by direct computation on the $\mathrm{MF}_n^{\mathrm{st}}$ side. One can check that the homology of the unknot is two-dimensional, so it suffices to understand the classification of all two-dimensional Frobenius algebras.

Next recall that the category $\mathbf{Foam}_2$ has the Bar-Natan relation

 $\mapsto 0.$

This imposes restrictions on the type of Frobenius algebra one can have. The maps $\epsilon, \eta, \mu, \delta$ are the images of the cup, cap, pants, and upside-down pants cobodisms under the functor $\mathcal{F}$ respectively. Since we consider coborisms up to ambient isotopy, the sphere is equivalent to the capped pair of pants.



Therefore the relation above implies that

$$\eta \circ \mu \circ (\epsilon(1), \epsilon(1)) = 0.$$

Denote $\sigma = \eta \circ \mu$ as the Frobenius form and let $e = \epsilon(1)$ so the above condition becomes $\sigma(e, e) = 0$. We shall prove the following classification result:

**Proposition 6.20.** *Two-dimensional Frobenius algebras $A$ over $\mathbb{C}$ with $\sigma(e, e) = 0$ are classified as $\mathbb{C}[x]/(x^2 - \alpha)$ for some parameter $\alpha \in \mathbb{C}$.*

*Proof.* First note that since $A$ is two-dimensional, $A$ must be commutative. Let $v$ denote an element of $A$ which lies outside the span of $e$. Then $e$ is a unit for $\mu$ if and only if

$$\mu(e \otimes e) = e, \quad \mu(e \otimes v) = v, \quad \mu(v \otimes e) = v.$$

Then since $e \otimes e, e \otimes v, v \otimes e, v \otimes v$ form a basis for $A \otimes A$, it follows that $A$ is commutative. Furthermore this implies that the Frobenius form $\sigma$ is symmetric. Next we need the following key observation. Since $\sigma$ is nondegenerate, we can find an element $x \in A$ such that $x$ is not in the span of $e$ and $\sigma(x, x) = 0$. Without loss of generality, we can further assume that $\sigma(e, x) = 1$. The elements $e$ and $x$ form a basis for $A$ and we compute:

$$\eta(e) = \eta \circ \mu(e \otimes e) = \sigma(e, e) = 0$$

$$\eta(x) = \eta \circ \mu(e \otimes x) = \sigma(e, x) = 1$$

Now write $\mu(x \otimes x) = \alpha e + \beta x$. We have that

$$\begin{aligned}\sigma(x \otimes x) &= \eta \circ \mu(x \otimes x) \\ &= \alpha\eta(e) + \beta\eta(x) \\ &= \beta.\end{aligned}$$

However by construction, $\sigma(x, x) = 0$ so $\beta = 0$. Note that commutativity implies co-commutativity for Frobenius algebras so we can express $\delta(e)$ and $\delta(x)$ as linear combinations

$$\begin{aligned}\delta(e) &= \alpha_1 e \otimes e + \alpha_2(e \otimes x + x \otimes e) + \alpha_3 x \otimes x \\ \delta(x) &= \beta_1 e \otimes e + \beta_2(e \otimes x + x \otimes e) + \beta_3 x \otimes x.\end{aligned}$$

Then the condition that $\eta$ is a counit for $\delta$ is satisfied if and only if

$$\alpha_2 = 1, \quad \alpha_3 = 0, \quad \beta_2 = 0, \quad \beta_3 = 1.$$

Similarly by checking the Frobenius identity, one concludes that

$$\alpha_1 = 0, \quad \beta_1 = \alpha.$$

Finally one can verify that for any choice of $\alpha$, the maps $\mu$ and $\delta$ are associative and co-associative respectively so $A$ has the structure of a Frobenius algebra where the maps $(\epsilon, \mu, \eta, \delta)$ are defined by

$$\begin{aligned}&\epsilon(1) = e \\ \mu(e \otimes e) = e \quad &\mu(e \otimes x) = \mu(x \otimes e) = x, \quad \mu(x \otimes x) = \alpha e \\ &\eta(e) = 0, \quad \eta(x) = 1 \\ &\delta(e) = e \otimes x + x \otimes e, \quad \delta(x) = \alpha e \otimes e + x \otimes x.\end{aligned}$$

□

When $\alpha = 0$, the homology computed is Khovanov homology while when $\alpha \neq 0$, the homology computed is Lee homology which is a pertubation of Khovanov homology. Recall the following theorem about Lee homology [24]:

**Theorem 6.21** (Lee). *For any link $L$,*

$$H^{\text{Lee}}(L) \cong \oplus_{2^c}\mathbb{C}$$

*where $c$ is the number of connected components of $L$.*

Therefore we can check which homology theory arises from our construction by computing the homology of a nontrivial knot. Indeed we can check (see next section for details) that the homology of the trefoil knot is 4-dimensional. Hence the proof is completed. □

We should note that computing the homology on the matrix factorization side is done using a spectral sequence which converges to Khovanov homology at the $E_2$-page. Therefore although we prove a part of Oblomkov and Rozansky's conjecture, we are unable to answer Rasmussen's original conjecture on the existence of differentials. However our proof above provides additional evidence for the geometric version of the differentials used for the $\mathfrak{sl}_N$ spectral sequences. This geometric formulation has computational benefits due to the action of the full twist braid which we exhibit in the next section.

Furthermore the above constructions transforms Khovanov homology computations from a difficult homological algebra problem into a geometry/commutative algebra problem where one can compute Khovanov homology after understanding the sheaf $S_\beta$ associated to a braid $\beta$. Computing Khovanov homology for general torus knots is still an open problem and above methodology represents a potential avenue for tackling it.

# CHAPTER 7

# EXAMPLES OF COMPUTATION

## 7.1 Two strand torus knots

We will use the results of the main theorem to compute $\mathfrak{sl}_2$-homology for two strand torus knots. Define the space $\mathrm{Hilb}^{\mathrm{free}}_{1,2}$ to be the space

$$\mathrm{Hilb}^{\mathrm{free}}_{1,2} = \{(X, Y, v) \in \mathfrak{b} \times \mathfrak{n} \times \mathbb{C}^2 : \mathbb{C}\langle X, Y \rangle v = \mathbb{C}^2\}/B$$

where $B \subset GL_2$ is the Borel subgroup acting on $X$ and $Y$ by conjugation and $\mathbb{C}^2$ is the fundamental representation. Let us fix notation for an element $(X, Y) \in \mathfrak{b} \times \mathfrak{n}$.

$$X = \begin{bmatrix} x_{11} & x_{12} \\ 0 & x_{22} \end{bmatrix}, Y = \begin{bmatrix} 0 & y_{12} \\ 0 & 0 \end{bmatrix}$$

The stability condition on the Hilbert scheme implies that $x_{12}$ and $y_{12}$ cannot be simultaneously zero. Since they scale the same under a change of basis, $x_{12}$ and $y_{12}$ define a copy of $\mathbb{P}^1$. Because torus knots are Coxeter links, we can work in the locus where $x_{11} = x_{22}$ so denote these by a new variable $x$. Then geometrically we are working a space that looks like $\mathbb{P}^1$ and $\mathbb{C}$ intersecting at a point.

The gradings are given by

$$\mathrm{deg} x_{12} = \mathrm{deg} y_{12} = 1, \mathrm{deg} x = 0$$

$$\mathrm{deg}_{q,t} x_{12} = \mathrm{deg}_{q,t} x = q^2, \mathrm{deg}_{q,t} y_{12} = t^2/q^2$$

**Remark 7.1.** *The $q$ and $t$ used here should actually be $Q$ and $T$ used earlier in the paper, but we stick with $q$ and $t$ since these were originally used by Oblomkov and Rozansky and these variables also match up with the conventions used for Khovanov homology.*

Let $\beta = \sigma$ where $\sigma$ is the positive crossing that generates the braid group on two strands. Recall that in order to compute the triply-graded homology of $\beta$, we need to associate a sheaf (or complex of sheaves) $\mathcal{F}_\beta$ on $\mathrm{Hilb}^{\mathrm{free}}_{1,2}$ to the braid $\beta$. Then the Khovanov-Rozansky homology is obtained by computed the hypercohomology

$$\mathbb{H}^\bullet(\beta) := \mathbb{H}(\mathcal{F}_\beta \otimes \wedge^\bullet \mathcal{B})$$

where $\mathcal{B}$ is the tautological bundle on $\mathrm{Hilb}^{\mathrm{free}}_{1,2}$.

We are focused on the case of torus knots $T_{2,2n+1}$ in which case $\mathcal{F}_\beta = \mathcal{O}_{\mathbb{P}^1}(n) \otimes \mathcal{O}_{\mathbb{C}}$. The tautological bundle $\mathcal{B}$ is given by $(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}) \otimes \mathbb{C}[x]$.

The stability vector is $v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ so the $\mathfrak{sl}_2$ differential is given by contraction with the vector $X^2 v = \begin{pmatrix} 2xx_{12} \\ x^2 \end{pmatrix}$. Let $V = \mathbb{C}^2$ have basis vectors $e_1, e_2$. Then the $\mathfrak{sl}_2$ homology chain complex is given by

$$\begin{tikzcd}
 & \mathbf{a}^2\mathbb{C}e_1 \wedge e_2 \arrow[dl, "2xx_{12}"'] \arrow[dr, "-x^2"] & \\
\mathbf{a}^1\mathbb{C}e_1 \arrow[dr, "x^2"] & & \mathbf{a}^1\mathbb{C}e_2 \arrow[dl, "2xx_{12}"] \\
 & \mathbf{a}^0\mathbb{C} &
\end{tikzcd}$$

Notice the additional shift in $\mathbf{a}$ on each horizontal level of the complex. We shall compute $\mathfrak{sl}_2$ homology for torus links $T_{2,2n+1}$ when $n \geq 0$. Then adding these differentials to the complex $\mathcal{F}_\beta \otimes \wedge^\bullet \mathcal{B}$, we have

$$\begin{tikzcd}
& \mathbf{a}^2 H^0(\mathcal{O}(n-1)) \otimes \mathbb{C}[x] \arrow[dl, "2xx_{12}"'] \arrow[dr, "-x^2"] & \\
\mathbf{a}^1 H^0(\mathcal{O}(n)) \otimes \mathbb{C}[x] \arrow[dr, "x^2"] & & \mathbf{a}^1 H^0(\mathcal{O}(n-1)) \otimes \mathbb{C}[x] \arrow[dl, "2xx_{12}"] \\
& \mathbf{a}^0 H^0(\mathcal{O}(n)) \otimes \mathbb{C}[x] &
\end{tikzcd}$$

shifted by $(\mathbf{a}/\mathbf{t})^n$. Consider the decomposition of the ring $\mathbb{C}[x]$ as $\mathbb{C} \oplus \mathbb{C}x \oplus x^2\mathbb{C}[x]$. We can perform Gaussian elimination to the top right differential in our complex. This leads to the following simplification:

$$\begin{tikzcd}
\mathbf{a}^1 H^0(\mathcal{O}(n)) \otimes \mathbb{C}[x] \arrow[dr, "x^2"] & \mathbf{a}^1 H^0(\mathcal{O}(n-1)) \otimes \mathbb{C} \arrow[d, "2xx_{12}"] & \mathbf{a}^1 H^0(\mathcal{O}(n-1)) \otimes \mathbb{C}x \arrow[dl, "2xx_{12}"] \\
& \mathbf{a}^0 H^0(\mathcal{O}(n)) \otimes \mathbb{C}[x]. &
\end{tikzcd}$$

Now let us explicitly compute $H^0(\mathbb{P}^1, \mathcal{O}(n))$ for $n \geq 0$. Recall that our coordinates on $\mathbb{P}^1$ are given by $x_{12}, y_{12}$. Then $H^0(\mathbb{P}^1, \mathcal{O}(n))$ is the vector space spanned by the the degree $n$ monomials $x_{12}^n, x_{12}^{n-1}y_{12}, \ldots, y_{12}^n$. When $n = 0$, $H^0(\mathbb{P}^1, \mathcal{O}(-1)) = 0$. Then the $\mathfrak{sl}_2$ homology of $T_{2,1}$ or the unknot can be computed from the complex

$$\begin{tikzcd}
H^0(\mathcal{O}(0)) \otimes \mathbb{C}[x] \arrow[d, "x^2"] \\
H^0(\mathcal{O}(0)) \otimes \mathbb{C}[x]
\end{tikzcd}$$

This complex has no kernel and the cokernel is spanned by the elements $1 \otimes 1$ and $1 \otimes x$. Adding in the $q, t$ gradings, we have the $\mathfrak{sl}_2$ homology is given by $1 + q^2$. This is off by a shift of $q^{-1}$ from the expected answer of $q + q^{-1}$.

Next let us compute the case of $n = 1$ or $T_{2,3}$ which is the trefoil knot. We have the complex

$$\begin{tikzcd}
\mathbf{a}^1 H^0(\mathcal{O}(1)) \otimes \mathbb{C}[x] \arrow[dr, "x^2"] & \mathbf{a}^1 H^0(\mathcal{O}(0)) \otimes \mathbb{C} \arrow[d, "2xx_{12}"] & \mathbf{a}^1 H^0(\mathcal{O}(0)) \otimes \mathbb{C}x \arrow[dl, "2xx_{12}"] \\
& \mathbf{a}^0 H^0(\mathcal{O}(1)) \otimes \mathbb{C}[x] &
\end{tikzcd}$$

with a shift by $\mathbf{a}/\mathbf{t}$. The kernel is spanned by the element $\mathbf{a}(1 \otimes x - 2x_{12} \otimes 1)$ and the cokernel is spanned by the elements $x_{12} \otimes 1, y_{12} \otimes 1, y_{12} \otimes x$. Adding in the $q, t$-grading and setting $\mathbf{a} = q^4/t$, we have that the kernel is given by $q^{10}t^{-3}$ and the cokernel is given by $q^6t^{-2} + q^2 + q^4$. After shifting again by $q^{-1}$, we see that this computation agrees with the Khovanov homology for the trefoil $(q+q^3+q^5t^2+q^9t^3)$ when $t$ is exchanged with $t^{-1}$.

We summarize the results of our computation method in the following table for $n = 0, 1, 2$:

| Khovanov homology for $T_{2,2n+1}$ | |
|---|---|
| $T_{2,1}$ | $q + q^{-1}$ |
| $T_{2,3}$ | $q^9t^{-3} + q^5t^{-2} + q^3 + q$ |
| $T_{2,5}$ | $q^{15}t^{-5} + q^{11}t^{-4} + q^{11}t^{-3} + q^7t^{-2} + q^5 + q^3$ |

By performing a similar computation as above one can recover exactly the $\mathfrak{sl}_N$-homology $(N > 2)$ for the torus knots $T_{2,2n+1}$. We produce the results for $N = 3, 4, 5$ and $n = 0, 1, 2$ in the following tables:

| $\mathfrak{sl}_3$-homology for $T_{2,2n+1}$ | |
|---|---|
| $T_{2,1}$ | $q^2 + 1 + q^{-2}$ |
| $T_{2,3}$ | $q^{14}t^{-3} + q^{12}t^{-3} + q^8t^{-2} + q^6t^{-2} + q^6 + q^4 + q^2$ |
| $T_{2,5}$ | $q^{22}t^{-5} + q^{20}t^{-5} + q^{16}t^{-4} + q^{14}t^{-4} + q^{18}t^{-3} +$<br>$q^{16}t^{-3} + q^{12}t^{-2} + q^{10}t^{-2} + q^{10}q^8 + q^6$ |

| $\mathfrak{sl}_4$-homology for $T_{2,2n+1}$ | |
|---|---|
| $T_{2,1}$ | $q^3+q+q^{-1}+q^{-3}$ |
| $T_{2,3}$ | $q^{19}t^{-3}+q^{17}t^{-3}+q^{15}t^{-3}+q^{11}t^{-1}+q^9t^{-2}+q^7t^{-2}+q^9+q^7+q^5+q^3$ |
| $T_{2,5}$ | $q^{29}t^{-5}+q^{27}t^{-5}+q^{25}t^{-5}+q^{21}t^{-4}+q^{19}t^{-4}+q^{17}t^{-4}+q^{25}t^{-3}+q^{23}t^{-3}+q^{17}t^{-2}+q^{15}t^{-2}+q^{13}t^{-2}+q^{15}+q^{13}+q^{11}+q^9$ |

| $\mathfrak{sl}_5$-homology for $T_{2,2n+1}$ | |
|---|---|
| $T_{2,1}$ | $q^4+q^2+1+q^{-2}+q^{-4}$ |
| $T_{2,3}$ | $q^{24}t^{-3}+q^{22}t^{-3}+q^{20}t^{-3}+q^{18}t^{-3}+q^{14}t^{-2}+q^{12}t^{-2}+q^{10}t^{-2}+q^8t^{-2}+q^{12}+q^{10}+q^8+q^6+q^4$ |
| $T_{2,5}$ | $q^{36}t^{-4}+q^{34}t^{-5}+q^{32}t^{-5}+q^{30}t^{-5}+q^{26}t^{-4}+q^{24}t^{-4}+q^{22}t^{-4}+q^{20}t^{-4}+q^{30}t^{-3}+q^{28}t^{-3}+q^{26}t^{-3}+q^{22}t^{-2}+q^{20}t^{-2}+q^{18}t^{-2}+q^{16}t^{-2}+q^{20}+q^{18}+q^{16}+q^{14}+q^{12}$ |

In this case, one can check that our computations above agree with Lewark's computation for $\mathfrak{sl}_3$-homology using foams [25]. One can check that for all $N \geq 3$, the two different methods of computation match up. However unlike the case of $N = 2$, a formal proof for why this geometric construction should compute $\mathfrak{sl}_N$-homology for arbitrary knots and links is still out of reach.

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