The Stability of Hot Curved Space*

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The propagator is evaluated for the situation of a non-interacting scalar field at non-zero temperature in curved space-time. The coincidence limit is taken and the corresponding effective Lagrangian is constructed. Fluctuations are determined using a background metric which satisfies the Einstein equations. The resulting energy-momentum tensor is found to be that for an imperfect fluid. Metric fluctuations are found to be unstable (for most values of the cosmological constant) with an exponential growth characterized by a Jeans mass squared which is twice the classical value. Comparisons with previous calculations are also made.

I. INTRODUCTION

In this paper we examine the Green's function and the one-loop effective Lagrangian for a quantized scalar field in a curved space-time at finite temperature. These objects should be of interest in situations involving both large curvatures and high temperatures, e.g., the early universe [3]. While techniques have previously been developed for finding effective Lagrangians and propagators in a wide class of curved space-times [2, 10], progress in evaluating these quantities at non-zero temperature has, for the most part, been restricted to static space-times where the well-known relation between Euclidean field theory and finite temperature field theory may be used [1, 2, 11, 16, 18, 24]. Drummond's work [13] is an exception, but his attention was restricted to the Robertson–Walker metric. This has led to the present unsatisfactory situation wherein curvature contributions are often calculated in an unspecified background at zero temperature while the finite temperature contributions are determined in flat space [3], which is certainly inconsistent with conditions in the early universe. Some of the more recent work has attempted to rectify this problem by developing techniques specifically meant for dealing with non-equilibrium systems [5–8, 14, 19, 20].

A specific application of these techniques is the study of the stability of space-time at non-zero temperature. Gross *et al.* [17] were the first to examine this question by using the methods of finite temperature field theory. These authors

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argued that an instability of hot, *flat* space would be indicated by the generation of a tachyonic mass in a specific component of the graviton propagator. Classically this phenomenon is known as the Jeans instability [23, 28] and is characterized by the "Jeans mass," $m_J^2 = -4\pi G\rho$, with ρ being the energy density. While the propagator calculated by these authors did indeed develop a tachyonic mass, its value did not agree with classical value of the Jeans mass. Since that time others have attempted to derive the Jeans mass via finite temperature field theory [21, 24], also without success. We examine the Jeans instability problem in some detail in Section V and its relation to the Jeans mass.

In examining the finite temperature Green's function we shall utilize the real-time formulation of finite temperature field theory [12, 22, 25]. The alternative to this technique is the imaginary-time formalism which involves analytic continuation of the time-like coordinate to complex values and the imposition of periodic boundary conditions [12]. However, the latter procedure requires a global choice of time, which restricts the space-times that can be considered. In contrast, the real-time formalism requires only a local choice of time and seems particularly well-suited to the adiabatic expansion of the Green's function that we will develop. It has the additional advantage of allowing the use of momentum space techniques familiar from zero temperature field theory.

An interesting property of the finite temperature Green's function that is explicitly displayed in the real-time formalism is the presence of terms that explicitly break the local Lorentz invariance present at zero temperature. Such terms arise because the presence of a background heat bath implies the existence of a preferred frame of reference (the rest frame of the heat bath). Thus, the introduction of temperature is more than a minor addition to the study of quantum fields in curved space.

The paper is organized as follows. In Section II we derive the real-time Feynman Green's function $G(x, x')$, correct to second order in the curvature in a manner similar to that employed by Bunch and Parker [4] for the zero temperature Green's function. In Section III, the coincidence limit $G(x, x)$ and the effective action are calculated while the energy-momentum tensor is evaluated in Section IV. We discuss the Jeans instability in the Newtonian limit and in a curved background metric in Section V. Finally, Section VI is reserved for concluding remarks. We use the conventions of [2] throughout.

II. Derivation of the Green's Function

We consider a massive scalar field $\phi(x)$ in the presence of a classical gravitational field. The action for such a field is given by

$$S = -\tfrac{1}{2} \int d^n x \sqrt{-g(x)}\, \phi(x)[\Box + m^2 + \zeta R(x)]\, \phi(x), \tag{2.1}$$

where $R(x)$ is the scalar curvature and $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$ is the D'Alembertian. The scalar field $\phi(x)$ satisfies the resulting wave equation

$$[\Box + m^2 + \zeta R(x)]\, \phi(x) = 0. \tag{2.2}$$

Construction of the effective action requires the Feynman Green's function $G(x, x')$, which is the solution to the differential equation

$$[\Box + m^2 + \zeta R(x)]\, G(x, x') = \frac{-1}{\sqrt{-g(x)}}\, \delta^n(x - x') \tag{2.3}$$

with appropriate boundary conditions, to be specified later.

Equation (2.3) can be solved exactly only in a few, highly symmetric and specialized space-times and often only for the case of a massless scalar field. Since we do not wish to impose such restrictions, we shall be forced to seek an approximate solution. To this end we follow the method of Bunch and Parker [4], who describe an asymptotic expansion for $G(x, x')$ which is valid locally so long as $\delta x = x - x' \ll \sqrt{1/R_{\mu\nu}}$, where $R_{\mu\nu}$ is a typical component of the Ricci curvature. The primary ingredient of this method is the use of Riemann normal co-ordinates, characterized by $\Gamma^\lambda_{\mu\nu}(x') = 0$ where x' specifies the origin. Riemann normal co-ordinate expansions are valid so long as the background metric changes slowly compared to the characteristic scales of the situation under consideration.

Introducing Riemann normal co-ordinates $y = x - x'$, with origin at x', we have

$$\begin{aligned}
g_{\mu\nu}(x) &= \eta_{\mu\nu} + \tfrac{1}{3}R_{\mu\alpha\nu\beta}\, y^\alpha y^\beta + \tfrac{1}{6}R_{\mu\alpha\nu\beta;\lambda}\, y^\alpha y^\beta y^\lambda \\
&\quad + (\tfrac{1}{20}R_{\mu\alpha\nu\beta;\sigma\lambda} + \tfrac{2}{45}R_{\alpha\mu\beta\gamma}\, R^\gamma{}_{\sigma\nu\lambda})\, y^\alpha y^\beta y^\sigma y^\lambda + \cdots \\
-g(x) &= 1 + \tfrac{1}{3}R_{\alpha\beta}\, y^\alpha y^\beta + \tfrac{1}{6}R_{\alpha\beta;\lambda}\, y^\alpha y^\beta y^\lambda \\
&\quad + (\tfrac{1}{18}R_{\sigma\lambda}\, R_{\alpha\beta} - \tfrac{1}{90}R^\mu{}_\alpha{}^\nu{}_\beta\, R_{\mu\sigma\nu\lambda} + \tfrac{1}{20}R_{\alpha\beta;\sigma\lambda})\, y^\alpha y^\beta y^\sigma y^\lambda + \cdots.
\end{aligned} \tag{2.4}$$

All coefficients in the expansions are understood to be evaluated at $y^\alpha = 0$. It is useful to define a modified Green's function $\bar{G}(x, x')$ by

$$\begin{aligned}
G(x, x') &= [-g(x)]^{-1/4}\, \bar{G}(x, x')[-g(x')]^{-1/4} \\
&= [-g(x)]^{-1/4}\, \bar{G}(x, x').
\end{aligned} \tag{2.5}$$

Then using the expansions for $g_{\mu\nu}(x)$ and $g(x)$ given in Eq. (2.4) we determine that $\bar{G}(x, x')$ satisfies

$$\begin{aligned}
&\eta^{\mu\nu}\partial_\mu\partial_\nu\bar{G} + [m^2(\zeta - \tfrac{1}{6})\, R]\, \bar{G} - \tfrac{1}{3}[R^\mu{}_\alpha{}^\nu{}_\beta\, y^\alpha y^\beta\partial_\mu\partial_\nu - R_\alpha{}^\nu y^\alpha\partial_\nu]\, \bar{G} \\
&\quad + (\zeta - \tfrac{1}{6})\, R_{;\alpha}\, y^\alpha\bar{G} + \tfrac{1}{3}[(R_\alpha{}^\nu{}_{;\beta} - \tfrac{1}{2}R_{\alpha\beta;}{}^\nu)\, y^\alpha y^\beta\partial_\nu - \tfrac{1}{2}R^\mu{}_\alpha{}^\nu{}_{\beta;\lambda}\, y^\alpha y^\beta y^\lambda\partial_\mu\partial_\nu]\, \bar{G} \\
&\quad + a_{\alpha\beta}\, y^\alpha y^\beta\bar{G} + [-\tfrac{1}{20}R^\mu{}_\alpha{}^\nu{}_{\beta;\sigma\lambda} + \tfrac{1}{15}R^\mu{}_{\alpha\varepsilon\beta}\, R^\varepsilon{}_\sigma{}^\nu{}_\lambda]\, y^\alpha y^\beta y^\sigma y^\lambda\partial_\mu\partial_\nu\bar{G} \\
&\quad + [\tfrac{3}{20}R^\nu{}_{\alpha;\beta\lambda} - \tfrac{1}{10}R_{\alpha\beta;}{}^\nu{}_\lambda - \tfrac{1}{60}R^\mu{}_\alpha{}^\nu{}_\beta\, R_{\mu\lambda} + \tfrac{1}{15}R^\sigma{}_{\alpha\mu\beta}\, R_\sigma{}^\nu{}_\lambda{}^\mu]\, y^\alpha y^\beta y^\lambda\partial_\nu\bar{G} = -\delta^n(y),
\end{aligned} \tag{2.6}$$

where $\partial_\mu = \partial/\partial y^\mu$ and

$$a_{\alpha\beta} = \tfrac{1}{2}(\zeta - \tfrac{1}{6})\, R_{;\alpha\beta} + \tfrac{1}{120} R_{;\alpha\beta} - \tfrac{1}{40} \Box R_{\alpha\beta} - \tfrac{1}{30} R^\lambda_\alpha R_{\lambda\beta}$$
$$+ \tfrac{1}{60} R^{\sigma}{}_{\alpha}{}^{\lambda}{}_{\beta} R_{\sigma\lambda} + \tfrac{1}{60} R^{\lambda\mu\nu}{}_{\alpha} R_{\lambda\mu\nu\beta}. \tag{2.7}$$

In deriving Eq. (2.6) we have retained only terms involving up to four derivatives of the metric. Note that in normal co-ordinates $\bar{G}(x, x')$ is a function of both y *and* the origin x'; $\bar{G}(x, x') = \bar{G}(y; x')$.

We can construct $\bar{G}(x, x')$ iteratively by defining

$$\bar{G}(x, x') = \sum_{l=0} \bar{G}_l(x, x'), \tag{2.8}$$

where $\bar{G}_l(x, x')$ contains l derivatives of the metric. Substituting this expansion for $\bar{G}(x, x')$ into Eq. (2.6), we observe that the $\bar{G}_l(x, x')$ solve

$$(\eta^{\mu\nu}\partial_\mu\partial_\nu + m^2)\, \bar{G}_0(y) = -\delta^n(y) \tag{2.9a}$$

$$\bar{G}_1(y) = 0 \tag{2.9b}$$

$$(\eta^{\mu\nu}\partial_\mu\partial_\nu + m^2)\, \bar{G}_2(y) + (\zeta - \tfrac{1}{6})\, R\bar{G}_0(y)$$
$$- \tfrac{1}{3}[R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta}\, y^\alpha y^\beta \partial_\mu\partial_\nu - R^\nu_\alpha\, y^\alpha \partial_\nu]\, \bar{G}_0(y) = 0 \tag{2.9c}$$

$$(\eta^{\mu\nu}\partial_\mu\partial_\nu + m^2)\, \bar{G}_3(y) + (\zeta - \tfrac{1}{6})\, R_{;\alpha}\, y^\alpha \bar{G}_0(y)$$
$$+ \tfrac{1}{3}[(R^\nu_{\alpha;\beta} - \tfrac{1}{2} R_{\alpha\beta;}{}^{\nu})\, y^\alpha y^\beta \partial_\nu - \tfrac{1}{2} R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta;\lambda}\, y^\alpha y^\beta y^\lambda \partial_{\mu\nu}]\, \bar{G}_0(y) = 0 \tag{2.9d}$$

$$(\eta^{\mu\nu}\partial_\mu\partial_\nu + m^2)\, \bar{G}_4(y)$$
$$+ (\zeta - \tfrac{1}{6})\, R\bar{G}_2(y) - \tfrac{1}{3}[R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta}\, y^\alpha y^\beta \partial_\mu\partial_\nu - R^\nu_\alpha\, y^\alpha \partial_\nu]\, \bar{G}_2(y)$$
$$+ a_{\alpha\beta}\, y^\alpha y^\beta \bar{G}_0(y) + [-\tfrac{1}{20} R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta;\sigma\lambda} + \tfrac{1}{15} R^{\mu}{}_{\alpha\varepsilon\beta} R^{\varepsilon}{}_{\sigma}{}^{\nu}{}_{\lambda}]\, y^\alpha y^\beta y^\sigma y^\lambda \partial_\mu \partial_\nu \bar{G}_0(y)$$
$$+ [\tfrac{3}{20} R^\nu_{\alpha;\beta\lambda} - \tfrac{1}{10} R_{\alpha\beta;}{}^{\nu}{}_{\lambda} - \tfrac{1}{60} R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta} R_{\mu\lambda} + \tfrac{1}{15} R^{\sigma}{}_{\alpha\mu\beta} R_{\sigma}{}^{\nu}{}_{\lambda}{}^{\mu}]$$
$$\times y^\alpha y^\beta y^\lambda \partial_\nu \bar{G}_0(y) = 0. \tag{2.9e}$$

It is evident then that each of the $\bar{G}_l(y)$ can be expressed in terms of the lowest order piece $\bar{G}_0(y)$, so that specifying the form of $\bar{G}_0(y)$ is sufficient to determine the entire propagator. We take the boundary conditions on $\bar{G}_0(y)$ to be those appropriate for finite temperature in flat space, i.e.,

$$\bar{G}_0(x, x') = -i\langle \beta |\, T\phi(x)\, \phi(x') | \beta\rangle, \tag{2.10}$$

where $|\beta\rangle$ is the thermal vacuum.

In order to have a convenient form for this Green's function, we introduce a momentum space representation associated with the point x', by employing the Fourier transform of $\bar{G}(x, x')$,

$$\bar{G}(x, x') = \int \frac{d^n p}{(2\pi)^n}\, e^{-ip\cdot y} \bar{G}(p), \tag{2.11}$$

where $p \cdot y = \eta^{\mu\nu} p_\mu y_\nu$. Strictly speaking, $\bar{G}(p)$ is also a function of x', $\bar{G}(p; x')$, and should be defined by the inverse of Eq. (2.11). Of course, explicitly evaluating the inverse requires specification of $\bar{G}(x, x')$ for *all* y and requires knowledge that we do not in general possess. Instead we shall take $\bar{G}(p; x')$ to be the Fourier transform of a function which agrees with the solution to Eq. (2.6) in an open set containing x' and has compact support in a neighborhood of x'. Despite the much better convergence of the temperature-dependent integrals compared to their zero temperature counterparts this limitation applies equally well to them because concepts such as temperature and thermal equilibrium can, in general, be defined only locally in the presence of a gravitational field.

Introducing a derivative expansion for $\bar{G}(p)$ like that for $\bar{G}(x, x')$ and Fourier transforming the $\bar{G}_l(x, x')$ yield

$$\bar{G}_l(x, x') = \int \frac{d^n p}{(2\pi)^n} e^{-ip \cdot y} \bar{G}_l(p). \tag{2.12}$$

It is the $\bar{G}_l(p)$ that we will actually evaluate. Thus, casting Eqs. (2.9) into momentum space form gives

$$(p^2 - m^2)\, \bar{G}_0(p) = 1 \tag{2.13a}$$

$$\begin{aligned}(p^2 - m^2)\, \bar{G}_2(p) = {} & \left(\zeta - \frac{1}{6}\right) R\bar{G}_0(p) \\ & - \frac{1}{3} \left[R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta}\, p_\mu p_\nu \partial^\alpha \partial^\beta - R^{\nu}_{\alpha}\, p_\nu \partial^\alpha\right] \bar{G}_0(p)\end{aligned} \tag{2.13b}$$

$$\begin{aligned}(p^2 - m^2)\, \bar{G}_3(p) = {} & -i \left(\zeta - \frac{1}{6}\right) R_{;\alpha} \partial^\alpha \bar{G}_0(p) \\ & + \frac{i}{3} \left[\frac{1}{2} R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta;\lambda}\, p_\mu p_\nu \partial^\alpha \partial^\beta \partial^\lambda + \left(R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta;\mu} - \frac{1}{2} R_{\alpha\beta;}{}^{\nu}\right) p_\nu \partial^\alpha \partial^\beta \right. \\ & \left. - \left(R^{\mu}{}_{\alpha;\mu} + \frac{1}{2} R_{;\alpha}\right) \partial^\alpha \right] \bar{G}_0(p)\end{aligned} \tag{2.13c}$$

$$\begin{aligned}(p^2 - m^2)\, \bar{G}_4(p) = {} & \left(\zeta - \frac{1}{6}\right) R\bar{G}_2(p) - \frac{1}{3} \left[R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta}\, p_\mu p_\nu \partial^\alpha \partial^\beta - R_{\alpha}{}^{\nu} p_\nu \partial^\alpha\right] \bar{G}_2(p) \\ & - a_{\alpha\beta} \partial^\alpha \partial^\beta \bar{G}_0(p) \\ & + \left[\left(\frac{1}{20} R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta;\sigma\lambda} - \frac{1}{15} R^{\mu}{}_{\alpha\varepsilon\beta} R^{\varepsilon}{}_{\sigma}{}^{\nu}{}_{\lambda}\right) \partial^\alpha \partial^\beta \partial^\sigma \partial^\lambda p_\mu p_\nu \right. \\ & + \left(\frac{3}{20} R_{\alpha}{}^{\nu}{}_{;\beta\lambda} - \frac{1}{10} R_{\alpha\beta;}{}^{\nu}{}_{\lambda} - \frac{1}{60} R^{\varepsilon}{}_{\alpha}{}^{\nu}{}_{\beta} R_{\varepsilon\lambda} \right. \\ & \left.\left. + \frac{1}{15} R^{\sigma}{}_{\alpha\varepsilon\beta} R_{\sigma}{}^{\nu}{}_{\lambda}{}^{\varepsilon}\right) \partial^\alpha \partial^\beta \partial^\lambda p_\nu \right] \bar{G}_0(p),\end{aligned} \tag{2.13d}$$

where we use $\partial^\alpha \equiv \partial/\partial p_\alpha$ henceforth. Note that the terms in brackets on the right hand side of Eqs. (2.13) vanish when operating on Lorentz-invariant functions. Since $\bar{G}_0(p)$ is *not* explicitly Lorentz-invariant when $T \neq 0$ such terms *will* in general contribute to the propagator. These terms were apparently missed in the work of Fujimoto [15].

$\bar{G}_0(p)$ is obviously the flat space finite temperature propagator, given in the real-time formalism by

$$\bar{G}_0(p) = \begin{pmatrix} 1/(p^2 - m^2 + i\varepsilon) & 0 \\ 0 & -1/(p^2 - m^2 - i\varepsilon) \end{pmatrix} + \delta(p^2 - m^2)\, F(p). \tag{2.14}$$

$F(p)$ is a temperature-dependent two-by-two matrix. For situations at or very near equilibrium $F(p)$ is given by

$$F(p) = -2\pi i \begin{pmatrix} n_{\mathrm{B}}(|u \cdot p|) & \frac{1}{2}\operatorname{sech}(\beta/2)|u \cdot p| \\ \frac{1}{2}\operatorname{sech}(\beta/2)|u \cdot p| & n_{\mathrm{B}}(|u \cdot p|) \end{pmatrix}, \tag{2.15}$$

where $n_{\mathrm{B}}(|u \cdot p|)$ is the Bose–Einstein distribution and u^μ is a time-like unit vector [25]. When considering situations not near equilibrium $F(p)$ is modified. So as to be able to deal with such cases we will not specify the form of $F(p)$ at present.

By using the real-time formalism instead of imaginary-time we are able to leave the geometry undetermined until one looks for solutions of the field equations. This is in contrast to the imaginary-time method where the geometry must be specified from the beginning. Real-time has the added advantages that it is better suited to dynamical situations and one can make use of all of the familiar zero temperature momentum space techniques.

We turn now to finding forms for the higher order ($l \geqslant 2$) components of the propagator, beginning with $\bar{G}_2(p)$. One cannot simply divide both sides of Eq. (2.13b) by $(p^2 - m^2)$ as is possible at $T = 0$, since this would result in a singular term of the form $(p^2 - m^2)^{-1}\, \delta(p^2 - m^2)$. Instead, we will use the relevant identities for derivatives of $\delta(p^2 - m^2)$ given in Appendix 1 in order to introduce a factor of $(p^2 - m^2)$. This gives

$$\bar{G}_2(p) = \left(\left(\zeta - \frac{1}{6}\right) R - \frac{1}{3}\left[R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta}\, p_\mu p_\nu \partial^\alpha \partial^\beta - R_\alpha{}^\nu p_\nu \partial^\alpha\right]\right) \frac{\partial}{\partial m^2}\, \bar{G}_0(p). \tag{2.16}$$

The higher order pieces $\bar{G}_3(p)$ and $\bar{G}_4(p)$ are found in exactly the same manner as $\bar{G}_2(p)$. The results are

$$\begin{aligned} \bar{G}_3(p) = \frac{i}{2}\Bigg[&\left(\frac{1}{6} - \zeta\right) R_{;\alpha} \partial^\alpha + \frac{1}{3}\big(R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta;\lambda}\, \partial^\alpha \partial^\beta \partial^\lambda p_\mu p_\nu - R_{\alpha\beta;}{}^{\nu} \partial^\alpha \partial^\beta p_\nu \\ &+ 2R_\alpha{}^\nu{}_{;\beta}\, \partial^\alpha \partial^\beta p_\nu\big)\Bigg] \frac{\partial}{\partial m^2}\, \bar{G}_0(p) + \frac{i}{2}\frac{\partial}{\partial m^2}\Bigg[\left(\frac{1}{6} - \zeta\right) R_{;\alpha}\, \delta(p^2 - m^2)\, \partial^\alpha \\ &+ \frac{1}{6}\{\partial^\alpha \delta(p^2 - m^2)\}\big(R_{\alpha\beta;}{}^{\nu} p_\nu \partial^\beta - R^{\mu}{}_{\lambda}{}^{\nu}{}_{\beta;\alpha}\, p_\mu p_\nu \partial^\beta \partial^\lambda\big)\Bigg] F(p) \end{aligned} \tag{2.17}$$

$$
\begin{aligned}
\bar{G}_4(p)=&\left[\frac{1}{2}\left(\frac{1}{6}-\zeta\right)^2 R^2\left(\frac{\partial}{\partial m^2}\right)^2+\frac{1}{3}a^{\lambda}{}_{\lambda}\left(\frac{\partial}{\partial m^2}\right)^2\right.\\
&\left.-\frac{1}{3}a_{\alpha\beta}\partial^{\alpha}\partial^{\beta}\left(\frac{\partial}{\partial m^2}\right)\right]\begin{pmatrix}1/(p^2-m^2+i\varepsilon) & 0\\ 0 & -1/(p^2-m^2-i\varepsilon)\end{pmatrix}\\
&+\left\{\frac{1}{2}\left(\frac{1}{6}-\zeta\right)^2 R^2+\frac{1}{3}a^{\lambda}{}_{\lambda}-\frac{1}{6}\left(\zeta-\frac{1}{6}\right)R[R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta}\partial^{\alpha}\partial^{\beta}p_{\mu}p_{\nu}+R_{\alpha}{}^{\nu}\partial^{\alpha}p_{\nu}]\right.\\
&\left.+\frac{1}{18}[R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta}\partial^{\alpha}\partial^{\beta}p_{\mu}p_{\nu}+R_{\alpha}{}^{\nu}\partial^{\alpha}p_{\nu}]^2\right\}\left(\frac{\partial}{\partial m^2}\right)^2\delta(p^2-m^2)\,F(p)\\
&+\left\{-\frac{1}{2}\left(\zeta-\frac{3}{20}\right)R_{;\alpha\beta}-\frac{3}{40}\Box R_{\alpha\beta}-\frac{1}{10}R_{\alpha}{}^{\nu}{}_{;\nu\beta}+\frac{1}{10}R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta;\mu\nu}-\frac{1}{15}R_{\alpha}{}^{\nu}R_{\nu\beta}\right.\\
&\left.+\frac{1}{12}R_{\mu\nu}R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta}-\frac{1}{60}R_{\lambda\mu\nu\alpha}R^{\lambda\mu\nu}{}_{\beta}\right\}\frac{\partial.}{\partial m^2}\left[\partial^{\alpha}\partial^{\beta}\{\delta(p^2-m^2)\,F(p)\}\right.\\
&\left.-\partial^{(\alpha}F(p)\,\partial^{\beta)}\,\delta(p^2-m^2)-\frac{2}{3}F(p)\,\partial^{\alpha}\partial^{\beta}\delta(p^2-m^2)\right]\\
&+\left\{\frac{1}{20}R_{\alpha}{}^{\nu}{}_{;\beta\lambda}-\frac{1}{10}R_{\alpha\beta;}{}^{\nu}{}_{\lambda}+\frac{1}{10}R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta;\mu\lambda}+\frac{1}{10}R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta;\lambda\mu}+\frac{7}{60}R_{\mu\lambda}R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta}\right.\\
&\left.-\frac{1}{5}R^{\nu}{}_{\alpha\sigma\mu}R^{\mu}{}_{\beta}{}^{\sigma}{}_{\lambda}+\frac{1}{15}R^{\nu}{}_{\mu\alpha\sigma}R^{\mu}{}_{\beta}{}^{\sigma}{}_{\lambda}\right\}p_{\nu}\frac{\partial}{\partial m^2}\left[\partial^{\alpha}\partial^{\beta}\partial^{\lambda}\{\delta(p^2-m^2)\,F(p)\}\right.\\
&-\frac{2}{3}\{\partial^{\alpha}F(p)\,\partial^{\beta}\partial^{\lambda}+\partial^{\beta}F(p)\,\partial^{\alpha}\partial^{\lambda}+\partial^{\lambda}F(p)\,\partial^{\alpha}\partial^{\beta}\}\,\delta(p^2-m^2)\\
&-\frac{3}{4}F(p)\,\partial^{\alpha}\partial^{\beta}\partial^{\lambda}\delta(p^2-m^2)\\
&\left.-\frac{1}{2}\{\partial^{\alpha}\partial^{\beta}F(p)\,\partial^{\lambda}+\partial^{\alpha}\partial^{\lambda}F(p)\,\partial^{\beta}+\partial^{\beta}\partial^{\lambda}F(p)\,\partial^{\alpha}\}\,\delta(p^2-m^2)\right]\\
&-\left[\frac{-1}{20}R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta;\sigma\lambda}+\frac{1}{15}R^{\mu}{}_{\alpha\varepsilon\beta}R^{\nu}{}_{\lambda}{}^{\varepsilon}{}_{\sigma}\right]p_{\mu}p_{\nu}\frac{\partial}{\partial m^2}\left[\frac{1}{5}F(p)\,\partial^{\alpha}\partial^{\beta}\partial^{\sigma}\partial^{\lambda}\delta(p^2-m^2)\right.\\
&+\frac{1}{2}\{\partial^{(\alpha}F(p)\,\partial^{\beta)}\partial^{\sigma}\partial^{\lambda}+\partial^{(\sigma}F(p)\,\partial^{\lambda)}\partial^{\alpha}\partial^{\beta}\}\,\delta(p^2-m^2)\\
&+\frac{2}{3}\{\partial^{\alpha}\partial^{(\beta}F(p)\,\partial^{\sigma)}\partial^{\lambda}+\partial^{\beta}\partial^{(\sigma}F(p)\,\partial^{\lambda)}\partial^{\alpha}+\partial^{\lambda}\partial^{(\sigma}F(p)\,\partial^{\alpha)}\partial^{\beta}\}\,\delta(p^2-m^2)\\
&+\{\partial^{\alpha}\partial^{\beta}\partial^{(\sigma}F(p)\,\partial^{\lambda)}+\partial^{\sigma}\partial^{\lambda}\partial^{(\alpha}F(p)\,\partial^{\beta)}\}\,\delta(p^2-m)^2\\
&\left.+\delta(p^2-m^2)\,\partial^{\alpha}\partial^{\beta}\partial^{\sigma}\partial^{\lambda}F(p)\right]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{3}\Bigg[\frac{-1}{10}R_{;}{}^{\nu}{}_{\alpha}+\frac{1}{20}\Box R_{\alpha}{}^{\nu}+\frac{3}{20}R^{\mu\nu}{}_{;\mu\alpha}+\frac{3}{20}R^{\mu\nu}{}_{;\alpha\mu}\\
&-\frac{1}{5}R_{\mu\alpha;}{}^{\nu\mu}+\frac{1}{10}R^{\nu\sigma\lambda}{}_{\alpha;\sigma\lambda}+\frac{1}{10}R^{\nu\sigma\lambda}{}_{\alpha;\lambda\sigma}+\frac{7}{60}R^{\mu\nu}R_{\mu\alpha}\\
&+\frac{1}{20}R_{\sigma\lambda}R^{\nu\sigma\lambda}{}_{\alpha}+\frac{1}{15}R^{\nu}{}_{\mu\sigma\lambda}R^{\mu\sigma\lambda}{}_{\alpha}-\frac{1}{15}R^{\nu}{}_{\mu\sigma\lambda}R^{\sigma\lambda\mu}{}_{\alpha}\Bigg]\\
&\times p_{\nu}\left(\frac{\partial}{\partial m^2}\right)^2\left[\frac{1}{2}F(p)\,\partial^{\alpha}\delta(p^2-m^2)-\partial^{\alpha}\{F(p)\,\delta(p^2-m^2)\}\right]\\
&+\Bigg[\frac{-1}{20}(R^{\mu\nu}{}_{;\alpha\beta}+\Box R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta}+2R^{\mu}{}_{\alpha}{}^{\nu}{}_{\lambda;\beta}{}^{\lambda}+2R^{\mu}{}_{\alpha}{}^{\nu}{}_{\lambda;}{}^{\lambda}{}_{\beta})+\frac{1}{15}(2R^{\mu}{}_{\lambda}R^{\nu}{}_{\alpha}{}^{\lambda}{}_{\beta}\\
&+2R^{\mu}{}_{\alpha\sigma\lambda}R^{\nu\lambda\sigma}{}_{\beta}+R^{\mu}{}_{\alpha\sigma\lambda}R^{\nu}{}_{\beta}{}^{\sigma\lambda}+R^{\mu}{}_{\sigma\lambda\alpha}R^{\nu\sigma\lambda}{}_{\beta})\Bigg]p_{\mu}p_{\nu}\left(\frac{\partial}{\partial m^2}\right)^2\\
&\times\Bigg[\frac{1}{20}F(p)\,\partial^{\alpha}\partial^{\beta}\delta(p^2-m^2)+\frac{1}{3}\partial^{(\alpha}F(p)\,\partial^{\beta)}\delta(p^2-m^2)\\
&+\frac{1}{3}\delta(p^2-m^2)\,\partial^{\alpha}\partial^{\beta}F(p)\Bigg]-\frac{1}{15}\Bigg[\frac{1}{15}(R^{\mu}{}_{\lambda}R^{\lambda\nu}+R^{\mu}{}_{\alpha\lambda\beta}R^{\nu\alpha\lambda\beta}+R^{\mu}{}_{\alpha\,\lambda\beta}R^{\nu\beta\lambda\alpha})\\
&-\frac{1}{20}(\Box R^{\mu\nu}+2R^{\mu}{}_{\nu}{}^{\nu}{}_{\beta;}{}^{\alpha\beta})\Bigg]p_{\mu}p_{\nu}F(p)\left(\frac{\partial}{\partial m^2}\right)^3\delta(p^2-m^2).
\end{aligned}
\tag{2.18}
$$

Similarly, other higher order contributions to the finite temperature, curved space propagator can be found. However, they become increasingly complex in form.

We emphasize again that these results do not agree with the previous calculation of Fujimoto [15], due to his failure to take into account the presence of terms which are non-vanishing when $G_0(p)$ is not explicitly Lorentz-invariant.

III. Coincidence Limit of $G(x, x')$ and the Effective Lagrangian

In order to determine the effective Lagrangian, we will use the relation between the action and the Green's function

$$S_{\text{eff}}=\int dm^2\,\frac{i}{4}\int d^4x\,\operatorname{Tr}G(x,x). \tag{3.1}$$

Thus we shall require the coincidence limit $G(x, x)$. Due to the complexity of the $\bar{G}_i(p)$ this is a somewhat non-trivial task. However, there exist two features that will simplify matters. First, since $g_{\mu\nu}(x')=\eta_{\mu\nu}$ and $G(x, x')=(-g(x))^{-1/4}\,\bar{G}(x, x')$ we observe that $G(x, x) = \bar{G}(x, x)$—so determining the coincidence limit of $\bar{G}(x, x')$ is sufficient. Second, we may drop any terms in the $\bar{G}_l(p)$ which are total derivatives since the Fourier transform of $\partial^\alpha f(p)$ is equivalent to that of $i\, y^\alpha f(p)$.

We begin with Eqs. (2.14) and (2.16), from which we determine the coincident limits for $G_0(x, x)$ and $G_2(x, x)$,

$$G_0(x, x) = \int \frac{d^n p}{(2\pi)^n} \bar{G}_0(p) \tag{3.2}$$

$$G_2(x, x) = \int \frac{d^n p}{(2\pi)^n} \left(\frac{1}{6} - \zeta\right) R \left(-\frac{\partial}{\partial m^2}\right) \bar{G}_0(p). \tag{3.3}$$

Similarly, the expression for $\bar{G}_3(p)$, Eq. (2.17), gives

$$\begin{aligned} G_3(x, x) = \frac{i}{2} \int \frac{d^n p}{(2\pi)^n} \frac{\partial}{\partial m^2} \bigg[&\left(\frac{1}{6} - \zeta\right) R_{;\alpha} \delta(p^2 - m^2)\, \partial^\alpha \\ &+ \frac{1}{6} (\partial^\alpha (p^2 - m^2))(R_{\alpha\beta;}{}^{\nu} p_\nu \partial^\beta - R^{\mu}{}_{\lambda}{}^{\nu}{}_{\beta;\alpha}\, p_\mu p_\nu \partial^\beta \partial^\lambda)\bigg] F(p). \end{aligned} \tag{3.4}$$

Integrating the second term by parts, this becomes

$$\begin{aligned} G_3(x, x) = \frac{i}{2} \int \frac{d^n p}{(2\pi)^n} \frac{\partial}{\partial m^2} \delta(p^2 - m^2) \bigg[&\left(\frac{1}{6} - \zeta\right) R_{;\alpha} \partial^\alpha \\ &- \frac{1}{6} (R_{\nu\alpha;}{}^{\nu} \partial^\alpha + (R_{\alpha\beta;}{}^{\nu} - 2R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta;\mu})\, p_\nu \partial^\alpha \partial^\beta \\ &- R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta;\lambda}\, p_\mu p_\nu \partial^\alpha \partial^\beta \partial^\lambda)\bigg] F(p). \end{aligned} \tag{3.5}$$

There are two features of $G_3(x, x)$ which are worthy of comment. (Actually these features are shared by any of the $G_l(x, x)$ with l odd.) First, unlike $G_0(x, x)$ and $G_2(x, x)$, $G_3(x, x)$ does not contain a zero temperature term. Second, $G_3(x, x)$ vanishes if the elements of the matrix $F(p)$ are even functions of p_μ as for the case of the equilibrium form, Eq. (2.15), but Eq. (3.5) need not vanish in general.

The term $G_4(x, x)$ can be evaluated in like fashion, with terms involving derivatives of delta functions with respect to momentum being integrated by parts so that any derivatives act on $F(p)$. The result of this procedure after some simplification is

$$\begin{aligned} G_4(x, x) = &\int \frac{d^n p}{(2\pi)^n} \left[\frac{1}{2}\left(\frac{1}{6} - \zeta\right)^2 R^2 + \frac{1}{3} a^{\lambda}{}_{\lambda}\right] \left(\frac{\partial}{\partial m^2}\right)^2 \bar{G}_0(p) \\ &+ \int \frac{d^n p}{(2\pi)^n} \frac{\partial}{\partial m^2} \delta(p^2 - m^2) \\ &\times [b_{\alpha\beta}(x)\, \partial^\alpha \partial^\beta + b^{\nu}_{\alpha\beta\lambda}(x)\, p_\nu \partial^\alpha \partial^\beta \partial^\lambda + b^{\mu\nu}_{\alpha\sigma\lambda}(x)\, p_\mu p_\nu \partial^\alpha \partial^\beta \partial^\sigma \partial^\lambda]\, F(p) \end{aligned}$$

$$
\begin{aligned}
&+\int \frac{d^np}{(2\pi)^n}\left(\frac{\partial}{\partial m^2}\right)^2 \delta(p^2-m^2) \\
&\times [c(x)+c^{\nu}_{\alpha}(x)\,p_\nu \partial^\alpha + c^{\mu\nu}_{\alpha\beta}(x)\,p_\mu p_\nu \partial^\alpha\partial^\beta]\,F(p) \\
&+\int \frac{d^np}{(2\pi)^n}\left(\frac{\partial}{\partial m^2}\right)^3 \delta(p^2-m^2)\,d^{\mu\nu}(x)\,p_\mu p_\nu F(p),
\end{aligned} \tag{3.6}
$$

with the geometric factors in $G_4(x, x)$ being given by

$$
\begin{aligned}
b_{\alpha\beta}(x) = &-\tfrac{1}{6}(\zeta-\tfrac{1}{2})\,R_{;\alpha\beta} - \tfrac{1}{6}\Box R_{\alpha\beta} - \tfrac{1}{120}R_{\alpha}{}^{\nu}{}_{;\nu\beta} + \tfrac{1}{48}R_{\alpha}{}^{\nu}{}_{;\beta\nu} \\
&- \tfrac{49}{720}R_{\alpha}{}^{\nu}R_{\nu\beta} + \tfrac{37}{720}R_{\mu\nu}R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta} - \tfrac{17}{360}R_{\lambda\mu\nu\alpha}R^{\lambda\mu\nu}{}_{\beta}
\end{aligned} \tag{3.7a}
$$

$$
b^{\nu}_{\alpha\beta\lambda}(x) = -\tfrac{3}{80}R_{\alpha}{}^{\nu}{}_{;\beta\lambda} - \tfrac{1}{40}R_{\alpha\beta;}{}^{\nu}{}_{\lambda} - \tfrac{1}{240}R_{\mu\lambda}R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta} + \tfrac{1}{60}R\mu_{\alpha}{}^{\sigma}{}_{\beta}R^{\nu}{}_{\mu\sigma\lambda} \tag{3.7b}
$$

$$
b^{\mu\nu}_{\alpha\beta\sigma\lambda}(x) = \tfrac{1}{100}R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta;\lambda} - \tfrac{1}{75}R^{\mu}{}_{\alpha\varepsilon\beta}R^{\nu}{}_{\lambda}{}^{\varepsilon}{}_{\sigma} \tag{3.7c}
$$

$$
c(x) = -\tfrac{7}{1200}\Box R - \tfrac{7}{600}R^{\mu\nu}{}_{;\mu\nu} + \tfrac{7}{900}R^{\mu\nu}R_{\mu\nu} + \tfrac{7}{600}R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta} \tag{3.7d}
$$

$$
\begin{aligned}
c^{\nu}_{\alpha}(x) = &\tfrac{1}{60}R_{;\alpha}{}^{\nu} - \tfrac{1}{50}\Box R_{\alpha}{}^{\nu} - \tfrac{1}{40}R^{\mu\nu}{}_{;\mu\alpha} - \tfrac{1}{40}R^{\mu\nu}{}_{;\alpha\mu} \\
&+ \tfrac{1}{30}R^{\mu}{}_{\alpha;}{}^{\nu}{}_{\mu} - \tfrac{1}{200}R^{\nu\sigma\lambda}{}_{\alpha;\sigma\lambda} + \tfrac{1}{200}R^{\nu\sigma\lambda}{}_{\alpha;\lambda\sigma} \\
&- \tfrac{7}{1800}R^{\mu\nu}R_{\mu\alpha} - \tfrac{1}{120}R^{\sigma\lambda}R^{\nu}{}_{\sigma\lambda\alpha} - \tfrac{17}{600}R^{\sigma\lambda\nu}R_{\sigma\lambda\mu\alpha}
\end{aligned} \tag{3.7e}
$$

$$
\begin{aligned}
c^{\mu\nu}_{\alpha\beta}(x) = &\tfrac{1}{20}[-\tfrac{1}{20}R^{\mu\nu}{}_{;\alpha\beta} - \tfrac{1}{20}\Box R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta} - \tfrac{1}{10}R^{\mu}{}_{\alpha}{}^{\nu}{}_{\lambda;}{}^{\lambda}{}_{\beta} \\
&- \tfrac{1}{10}R^{\mu}{}_{\alpha}{}^{\nu}{}_{\lambda;\beta}{}^{\lambda} + \tfrac{2}{15}R^{\mu}{}_{\lambda}R^{\nu}{}_{\alpha}{}^{\lambda}{}_{\beta} - \tfrac{2}{15}R^{\mu}{}_{\alpha}{}^{\lambda}{}_{\sigma}R^{\nu}{}_{\lambda}{}^{\sigma}{}_{\beta} \\
&- \tfrac{1}{15}R^{\mu}{}_{\alpha}{}^{\lambda}{}_{\sigma}R^{\nu}{}_{\beta}{}^{\sigma}{}_{\lambda} + \tfrac{1}{15}R^{\mu\sigma\lambda}{}_{\alpha}R^{\nu}{}_{\sigma\lambda\beta}]
\end{aligned} \tag{3.7f}
$$

$$
\begin{aligned}
d^{\mu\nu}(x) = &-\tfrac{1}{15}[-\tfrac{1}{20}\Box R^{\mu\nu} - \tfrac{1}{10}R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta;}{}^{\alpha\beta} \\
&+ \tfrac{1}{15}R^{\mu}{}_{\lambda}R^{\nu\lambda} + \tfrac{1}{10}R^{\mu}{}_{\sigma\lambda\alpha}R^{\nu\sigma\lambda\alpha}].
\end{aligned} \tag{3.7g}
$$

The finite temperature effective Lagrangian L_β may then be written in terms of $G(x, x)$ as (cf. Eq. (3.1))

$$
L_\beta = \frac{i}{4}\int_{m^2}^{\infty} dm^2\,\mathrm{Tr}\,G(x, x). \tag{3.8}
$$

It is convenient to separate L_β into temperature-independent and temperature-dependent pieces, $L(0)$ and $L(\beta)$, respectively, and indeed the use of the real-time formalism allows this decomposition to be performed straightforwardly. The temperature-independent integrals can be evaluated by dimensional regularization and yield the standard result [2]

$$
L(0) = \frac{1}{2}(4\pi)^{-n/2}\left(\frac{m}{\mu}\right)^{n-4} \sum_{l=0} a_l(x)\,\Gamma\left(l-\frac{n}{2}\right) m^{4-2l}, \tag{3.9}
$$

where

$$a_0 = 1, \qquad a_1 = (\tfrac{1}{6} - \zeta)\, R$$
$$a_2 = \tfrac{1}{2}(\tfrac{1}{6} - \zeta)^2\, R^2 + \tfrac{1}{3} a^{\lambda}{}_{\lambda}, \qquad \text{etc.} \tag{3.10}$$

Here μ is an arbitrary mass scale introduced in order to keep the dimensionality of $L(0)$ fixed.

In order to give concrete forms for the temperature-dependent integrals we need to specify $\mathrm{Tr}\, F(p)$. For simplicity, we shall take $F(p)$ to be given by Eq. (2.15). With this choice

$$\begin{aligned} L(\beta) = {} & \sum_{l=0}^{2} a_l(x) \left(-\frac{\partial}{\partial m^2} \right)^l I(\beta) + \left[c(x) + \frac{1}{2} d^{\mu}_{\mu}(x) \right] \left(-\frac{\partial}{\partial m^2} \right)^2 I(\beta) \\ & - [b_{\alpha\beta} u^{\alpha} u^{\beta} B_1(\beta) + b^{\nu}_{\alpha\beta\lambda} u^{\alpha} u^{\beta} u^{\lambda} u_{\nu} B_2(\beta) \\ & + b^{\mu\nu}_{\alpha\beta\sigma\lambda}\, g_{\mu\nu} u^{\alpha} u^{\beta} u^{\sigma} u^{\lambda} B_2(\beta)] \\ & - \frac{\partial}{\partial m^2} [c^{\nu}_{\alpha} u_{\nu} u^{\alpha} C_1(\beta) + c^{\mu\nu}_{\alpha\beta} u^{\alpha} u^{\beta} (g_{\mu\nu} C_1(\beta) + u_{\mu} u_{\nu} C_2(\beta))] \\ & - d^{\mu\nu} u_{\mu} u_{\nu} \left(-\frac{\partial}{\partial m^2} \right)^2 D_1(\beta), \end{aligned} \tag{3.11}$$

where the temperature-dependent functions in $L(\beta)$, etc., are collected in Appendix 2. Since we chose $F(p)$ to be the matrix appropriate for equilibrium in flat space, and the action—Eq. (2.1)—contains no self-interactions for $\phi(x)$, any non-equilibrium effects that arise will be due to the background gravitational field. We observe this explicitly in the next section when the stress tensor is evaluated. Thus to lowest order we find

$$L(\beta) = \left(1 - \left(\frac{1}{6} - \zeta \right) R \frac{\partial}{\partial m^2} \right) I(\beta) + \cdots \tag{3.12}$$

The high and low temperature expansions of $I(\beta)$ are found in the usual way, yielding

$$\begin{aligned} T \gg m, \qquad & I(\beta) = \frac{\pi^2}{90} \beta^{-4} - \frac{1}{24} \frac{m^2}{\beta^2} + \frac{1}{12\pi} \frac{m^3}{\beta} + \cdots \\ T \ll m, \qquad & I(\beta) = \frac{1}{2\pi^2} \frac{m^2}{\beta^2} K_1(m\beta) + \cdots. \end{aligned} \tag{3.13}$$

Then, for example, the high temperature limit of L_{β} becomes

$$L_{\beta} \approx \frac{\pi^2}{90} \beta^{-4} + \frac{1}{24} \left[\left(\frac{1}{6} - \zeta \right) R - m^2 \right] \beta^{-2} + \cdots, \qquad T \gg m. \tag{3.14}$$

To this order our result agrees with that of Nakazawa and Fukuyama [24], except for the apparently spurious non-covariant term involving

$$\ln(-g_{00})_{,i}{}^{i}$$

which appears in their formula.

We are finally ready to list the effective action S_β, which is given by

$$S_\beta = \int d^n x \sqrt{-g}\, L_\beta \tag{3.15}$$

with

$$\begin{aligned} L_\beta = {} & \frac{1}{2}(4\pi)^{-n/2}\left(\frac{m}{\mu}\right)^{n-4} \sum_{l=0} a_l(x)\, \Gamma\left(l-\frac{n}{2}\right) m^{4-2l} \\ & + \sum_{l=0}^{2} \tilde{a}_l(x)\left(-\frac{\partial}{\partial m^2}\right)^l I(\beta) \\ & - [b_{\alpha\beta} u^\alpha u^\beta B_1(\beta) + (b^{\nu}_{\alpha\beta\lambda} u^\alpha u^\beta u^\lambda u_\nu + b^{\mu\nu}_{\alpha\beta\sigma\lambda}\, g_{\mu\nu} u^\alpha u^\beta u^\sigma u^\lambda)\, B_2(\beta)] \\ & - \frac{\partial}{\partial m^2}[(c^{\nu}_{\alpha} u^\alpha u_\nu + c^{\mu\nu}_{\alpha\beta}\, g_{\mu\nu} u^\alpha u^\beta)\, C_1(\beta) + c^{\mu\nu}_{\alpha\beta} u^\alpha u^\beta u_\mu u_\nu C_2(\beta)] \\ & - d^{\mu\nu} u_\mu u_\nu \left(-\frac{\partial}{\partial m^2}\right)^2 D_1(\beta) + \cdots, \end{aligned} \tag{3.16}$$

where we have defined the $\tilde{a}_l(x)$ by

$$\begin{aligned} &\tilde{a}_0 = a_0, \qquad \tilde{a}_1 = a_1 \\ &\tilde{a}_2 = a_2 + c + \tfrac{1}{2} d^{\mu}_{\mu}. \end{aligned} \tag{3.17}$$

The total action is obtained by adding S_β to the gravitational action

$$\begin{aligned} S_G = \int d^4x \sqrt{-g}\, \bigg[& \frac{1}{16\pi G}(R - 2\Lambda) + \varepsilon_1 R^2 + \varepsilon_2 R_{\mu\nu} R^{\mu\nu} \\ & + \varepsilon_3 R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} + \varepsilon_4 \Box R \bigg] \end{aligned} \tag{3.18}$$

and renormalizing S_G as we take the limit $n \to 4$ in S_β. The higher derivative terms in S_G are needed for renormalization purposes.

IV. Stress-Energy Tensor

The energy-momentum tensor $T^{\mu\nu}$ is defined by

$$\delta S_\beta = -\tfrac{1}{2} \int d^n x \sqrt{-g}\, T^{\mu\nu} h_{\mu\nu}, \tag{4.1}$$

where $h_{\mu\nu} = \delta g_{\mu\nu}$ is a small metric fluctuation. In order to construct $T^{\mu\nu}$ we must take into account that the temperature-dependent integrals in L_β *implicitly* depend on the metric. We will demonstrate how this is accomplished by considering first the ordinary perfect fluid, for which the energy-momentum tensor can be written in the form

$$T_{\mu\nu} = \rho u_\mu u_\nu + p P_{\mu\nu}, \tag{4.2}$$

where ρ and p are the energy density and internal pressure of the fluid, respectively, and

$$P_{\mu\nu} = u_\mu u_\nu - g_{\mu\nu} \tag{4.3}$$

is a space-like projection operator.

The Lagrangian for the perfect fluid, as with all other thermodynamic systems, is

$$L = -f \tag{4.4}$$

with f being the free energy density [26]. For the perfect fluid, f depends solely on the temperature. (We are assuming here the use of a comoving co-ordinate system.) The variation of the action corresponding to Eq. (4.4) is given by

$$\delta S = -\int d^n x \sqrt{-g}\, [\tfrac{1}{2} g^{\mu\nu} h_{\mu\nu} f + \delta f], \tag{4.5}$$

where we have used

$$\delta(\sqrt{-g}) = \tfrac{1}{2}\sqrt{-g}\, g^{\mu\nu} h_{\mu\nu}. \tag{4.6}$$

In order to find δf we employ the conditions of thermodynamic equilibrium,

$$\rho = sT + f \tag{4.7a}$$

$$\rho = \frac{\partial}{\partial\beta}(\beta f) = f - T\frac{\partial f}{\partial T}, \tag{4.7b}$$

which imply that

$$s = -\frac{\partial f}{\partial T}, \tag{4.8}$$

with s being the entropy density of the fluid. We see then that δf can be written in the form

$$\delta f = -s\, \delta T. \tag{4.9}$$

Since the co-ordinate system is co-moving, δT is given in terms of the gravitational red-shift

$$\delta T = -\tfrac{1}{2} T u^\mu u^\nu h_{\mu\nu}. \tag{4.10}$$

Finally, substituting Eqs. (4.9) and (4.10) into Eq. (4.5), we determine the first order variation

$$\delta S = \tfrac{1}{2} \int d^n x \sqrt{-g}\, [f g^{\mu\nu} + sTu^\mu u^\nu]\, h_{\mu\nu}. \tag{4.11}$$

As a check, we note that comparing with the perfect fluid stress tensor, Eq. (4.2), we can identify the pressure and energy density as

$$\begin{aligned} p &= -f \\ \rho &= sT + f \end{aligned} \tag{4.12}$$

as required.

We now proceed to the variation of the finite temperature effective action derived in the previous section, in order to yield the energy-momentum tensor for the self-gravitating fluid. All of the variations of the temperature-dependent functions are performed in exactly the same manner as in the perfect fluid case. First, however, we set our formalism. It is convenient to write $T^{\mu\nu}$ in the form

$$T^{\mu\nu} \equiv T_0^{\mu\nu} + \Pi^{\mu\nu}. \tag{4.13}$$

Here $T_0^{\mu\nu}$ is given by the simple perfect fluid form

$$T_0^{\mu\nu} = \rho_0 u^\mu u^\nu + p_0 P^{\mu\nu} \tag{4.14a}$$

$$\rho_0 = \frac{\partial}{\partial \beta} (\beta f), \tag{4.14b}$$

$$p_0 = -f, \tag{4.14c}$$

$$f = -L_\beta. \tag{4.14d}$$

$\Pi^{\mu\nu}$ is the portion of full energy-momentum tensor $T^{\mu\nu}$ obtained by the variation of the explicitly metric-dependent quantities in L_β,

$$\int d^n x \sqrt{-g}\, \Pi^{\mu\nu} h_{\mu\nu} = -2 \int d^n x \sqrt{-g}\, \frac{\partial L_\beta}{\partial g_{\mu\nu}} h_{\mu\nu}. \tag{4.15}$$

Clearly, in general, ρ_0 is *not* the energy density and p_0 is *not* the internal pressure. Indeed the full energy density and pressure are determined by

$$\begin{aligned} \rho &= u_\mu u_\nu T^{\mu\nu} = \rho_0 + u_\mu u_\nu \Pi^{\mu\nu} \\ p &= \tfrac{1}{3} P_{\mu\nu} T^{\mu\nu} = p_0 + \tfrac{1}{3} P_{\mu\nu} \Pi^{\mu\nu}. \end{aligned} \tag{4.16}$$

Rather, ρ_0 and p_0 are the "static" values of these quantities which one would obtain if the metric were held fixed, while the additional contributions indicated above arise from the metric variation.

Of course, in many cases the perfect fluid form of the energy-momentum tensor is not expected. Instead, the most general form of $T_{\mu\nu}$ can be written as

$$T^{\mu\nu} = \rho u^\mu u^\nu + pP^{\mu\nu} + 2q^{(\mu}u^{\nu)} + \tilde{\Pi}^{\mu\nu}, \tag{4.17}$$

where q^μ is the heat-flow vector and $\tilde{\Pi}^{\mu\nu}$ is the viscosity tensor [23]. $\tilde{\Pi}^{\mu\nu}$ and q^μ must satisfy

$$u_\mu q^\mu = 0 \tag{4.18a}$$

$$u_\mu \tilde{\Pi}^{\mu\nu} = 0. \tag{4.18b}$$

Solving for q^μ and $\tilde{\Pi}^{\mu\nu}$ gives

$$q^\mu = u_\nu \Pi^{\nu\mu} - (u_\alpha u_\beta \Pi^{\alpha\beta})\, u^\mu \tag{4.19a}$$

$$\tilde{\Pi}^{\mu\nu} = \Pi^{\mu\nu} - \tfrac{1}{3}(P_{\alpha\beta}\Pi^{\alpha\beta})\, P^{\mu\nu} + (u_\alpha u_\beta \Pi^{\alpha\beta})\, u^\mu u^\nu - 2u_\alpha \Pi^{\alpha\,(\mu}u^{\nu)}. \tag{4.19b}$$

Non-vanishing q^μ and $\tilde{\Pi}^{\mu\nu}$ indicate non-equilibrium for the fluid.

We turn now to finding an explicit form for the stress tensor $T^{\mu\nu}$. For simplicity, we will restrict our attention to the terms in the effective action only up to order R. We need not include the zero temperature contributions since, to this order, such terms merely go into renormalizing Λ and G in Eq. (3.18) and, therefore, do not contribute to $T^{\mu\nu}$. To this order then, S_β is given by

$$S_\beta = \int d^4x \sqrt{-g} \left[1 - \left(\frac{1}{6} - \zeta\right) R \frac{\partial}{\partial m^2}\right] I(\beta) + \cdots. \tag{4.20}$$

According to Eqs. (4.14b) and (4.14c) the static energy density and pressure terms are given by

$$\begin{aligned}
\rho_0 &= \left[1 - \left(\frac{1}{6} - \zeta\right) R \frac{\partial}{\partial m^2}\right] \frac{\partial}{\partial \beta}(-\beta I(\beta)) + \cdots \\
&= \frac{\pi^2}{30\beta^4} - \frac{m^2}{24\beta^2} + \left(\frac{1}{6} - \zeta\right) \frac{R}{24\beta^2} + \cdots, \qquad T \gg m \\
p_0 &= \left[1 - \left(\frac{1}{6} - \zeta\right) R \frac{\partial}{\partial m^2}\right] I(\beta) + \cdots \\
&= \frac{\pi^2}{90\beta^4} - \frac{m^2}{24\beta^2} + \frac{m^3}{12\pi\beta} \\
&\quad + \left(\frac{1}{6} - \zeta\right) R \left(\frac{1}{24\beta^2} - \frac{m}{8\pi\beta}\right) + \cdots, \qquad T \gg m.
\end{aligned} \tag{4.21}$$

We observe that in the high temperature limit the energy density and pressure have the expected Stefan–Boltzmann form. In order to determine the metric contribution $\Pi^{\mu\nu}$ we require the variation of the scalar curvature R, which is found to be

$$\delta R = -R^{\mu\nu}h_{\mu\nu} + [g^{\mu\nu}\Box - \nabla^{(\mu}\nabla^{\nu)}]\, h_{\mu\nu}. \tag{4.22}$$

Hence, using Eq. (4.15) we find

$$\begin{aligned}
&\int d^nx\,\sqrt{-g}\,\Pi^{\mu\nu}h_{\mu\nu} \\
&\quad = 2\int d^nx\,\sqrt{-g}\left(\frac{1}{6}-\zeta\right)[-R^{\mu\nu}h_{\mu\nu} + (g^{\mu\nu}\Box - \nabla^{(\mu}\nabla^{\nu)})\,h_{\mu\nu}]\,\frac{\partial I(\beta)}{\partial m^2} + \cdots \\
&\quad = -2\left(\frac{1}{6}-\zeta\right)\int d^nx\,\sqrt{-g}\,h_{\mu\nu}[R^{\mu\nu} - g^{\mu\nu}\Box + \nabla^{(\mu}\nabla^{\nu)}]\,\frac{\partial I(\beta)}{\partial m^2} + \cdots,
\end{aligned} \tag{4.23}$$

yielding

$$\begin{aligned}
\Pi^{\mu\nu} &= -2\left(\frac{1}{6}-\zeta\right)[R^{\mu\nu} - g^{\mu\nu}\Box + \nabla^{(\mu}\nabla^{\nu)}]\,\frac{\partial I(\beta)}{\partial m^2} + \cdots \\
&\cong -2\left(\frac{1}{6}-\zeta\right)[R^{\mu\nu} - g^{\mu\nu}\Box + \nabla^{(\mu}\nabla^{\nu)}] \\
&\quad \times\left(-\frac{1}{24\beta^2} + \frac{1}{8\pi}\frac{m}{\beta} + \cdots\right), \qquad T \gg m.
\end{aligned} \tag{4.24}$$

In comparing with previous work, we note that in [24], the derivative terms in $\Pi^{\mu\nu}$ were not included. This omission is not justified, however. While at $T=0$ these terms can be dropped, in general at finite temperature such pieces must be retained, as they contribute through the space-time dependence of the temperature. Thus, since

$$\beta \cong \sqrt{g_{00}}\,\beta_0 \tag{4.25}$$

in the weak field limit, where β_0 is the inverse temperature at infinite distance, we see that $\nabla_\mu \beta \neq 0$ in general because of the space-time dependence of the metric.

Finally, we can combine the results in Eqs. (4.21) and (4.22) to find the physically relevant quantities, yielding

$$\begin{aligned}
\rho &= \left[1 - \left(\frac{1}{6}-\zeta\right) R\,\frac{\partial}{\partial m^2}\right]\frac{\partial}{\partial \beta}(-\beta I) - 2\left(\frac{1}{6}-\zeta\right) \\
&\quad \times [u_\mu u_\nu R^{\mu\nu} - \Box + u_\mu u_\nu \nabla^\mu\nabla^\nu]\,\frac{\partial I}{\partial m^2} \\
&\cong \rho_0 - 2\left(\frac{1}{6}-\zeta\right)[u_\mu u_\nu R^{\mu\nu} - \Box + u_\mu u_\nu \nabla^\mu\nabla^\nu] \\
&\quad \times\left(-\frac{1}{24\beta^2} + \frac{m}{8\pi\beta} + \cdots\right)
\end{aligned} \tag{4.26a}$$

$$p=\left[1-\frac{2}{3}\left(\frac{1}{6}-\zeta\right)\left(u^\mu u^\nu R_{\mu\nu}+\frac{1}{2}R+2\Box+u_\mu u_\nu\nabla^\mu\nabla^\nu\right)\frac{\partial}{\partial m^2}\right]I$$

$$=p_0-\frac{2}{3}\left(\frac{1}{6}-\zeta\right)\left(u^\mu u^\nu R_{\mu\nu}+\frac{1}{2}R+2\Box+u_\mu u_\nu\nabla^\mu\nabla^\nu\right)$$

$$\times\left(-\frac{1}{24\beta^2}+\frac{m}{8\pi\beta}+\cdots\right) \tag{4.26b}$$

$$q^\mu=-2\left(\frac{1}{6}-\zeta\right)[u_\nu R^{\nu\mu}-(u_\alpha u_\beta R^{\alpha\beta})\,u^\mu+u_\nu\nabla^{(\mu}\nabla^{\nu)}$$

$$-u^\mu(u_\alpha u_\beta\nabla^\alpha\nabla^\beta)]\frac{\partial I}{\partial m^2}$$

$$=-2\left(\frac{1}{6}-\zeta\right)[u_\nu R^{\nu\mu}-(u_\alpha u_\beta R^{\alpha\beta})\,u^\mu+u_\nu\nabla^{(\mu}\nabla^{\nu)}$$

$$-u^\mu(u_\alpha u_\beta\nabla^\alpha\nabla^\beta)]\left(-\frac{1}{24\beta^2}+\frac{m}{8\pi\beta}+\cdots\right) \tag{4.26c}$$

$$\tilde{\Pi}^{\mu\nu}=-2\left(\frac{1}{6}-\zeta\right)\left[R^{\mu\nu}+g^{\mu\nu}P_{\alpha\beta}R^{\alpha\beta}+\frac{1}{3}g^{\mu\nu}P_{\alpha\beta}\nabla^\alpha\nabla^\beta+\nabla^{(\mu}\nabla^{\nu)}\right]\frac{\partial I}{\partial m^2}$$

$$-2\left(\frac{1}{6}-\zeta\right)u^\mu u^\nu[2u_\alpha u_\beta R^{\alpha\beta}+\Box+2u_\alpha u_\beta\nabla^\alpha\nabla^\beta]\frac{\partial I}{\partial m^2}$$

$$+2\left(\frac{1}{6}-\zeta\right)u_\alpha[2R^{\alpha(\mu}u^{\nu)}+u^\mu\nabla^{(\nu}\nabla^{\alpha)}+u^\nu\nabla^{(\mu}\nabla^{\alpha)}]\frac{\partial I}{\partial m^2}+\cdots$$

$$=-2\left(\frac{1}{6}-\zeta\right)\Big[R^{\mu\nu}+\frac{1}{3}g^{\mu\nu}P_{\alpha\beta}R^{\alpha\beta}+\frac{1}{3}g^{\mu\nu}P_{\alpha\beta}\nabla^\alpha\nabla^\beta+\nabla^{(\mu}\nabla^{\nu)}$$

$$+u^\mu u^\nu(2u_\alpha u_\beta R^{\alpha\beta}+\Box+2u_\alpha u_\beta\nabla^\alpha\nabla^\beta)$$

$$-(2R^{\alpha(\mu}u^{\nu)}+u^\mu\nabla^{(\nu}\nabla^{\alpha)}+u^\nu\nabla^{(\mu}\nabla^{\alpha)})\,u_\alpha\Big]$$

$$\times\left(-\frac{1}{24\beta^2}+\frac{m}{8\pi\beta}+\cdots\right). \tag{4.26d}$$

These are the forms which we have been seeking and in the next section we apply them to the Jeans mass problem. Before proceeding its useful to look at the Robertson–Walker metric as a consistency check. In a Robertson–Walker space $T^{\mu\nu}$ is restricted to the perfect fluid form by the geometry [27] and the temperature is a function of time only. Thus, we can verify that

$$u_\nu R^{\nu\mu}-u_\alpha u_\beta R^{\alpha\beta}u^\mu=R^{00}\frac{1}{\sqrt{g_{00}}}\delta_0^\mu-R^{00}u^\mu=0$$

and

$$[u_\nu \nabla^{(\mu}\nabla^{\nu)} - u^\mu(u_\alpha u_\beta \nabla^\alpha \nabla^\beta)] \frac{\partial}{\partial m^2} I(\beta) = 0 \tag{4.27}$$

so that $q^\mu = 0$. Similarly we find that

$$\left(\nabla^{(\mu}\nabla^{\nu)} + \frac{1}{3} g^{\mu\nu} P_{\alpha\beta} \nabla^\alpha \nabla^\beta + u^\mu u^\nu (\Box + 2u_\alpha u_\beta \nabla^\alpha \nabla^\beta) - (u^\mu \nabla^{(\nu}\nabla^{\alpha)} + u^\nu \nabla^{(\mu}\nabla^{\alpha)} u_\alpha\right)$$
$$\times \frac{\partial I(\beta)}{\partial m^2} = 0 \tag{4.28a}$$

and

$$R^{\mu\nu} + \frac{1}{3} g^{\mu\nu} P_{\alpha\beta} R^{\alpha\beta} + 2u^\mu u^\nu (u_\alpha u_\beta R^{\alpha\beta}) - 2R^{\alpha(\mu} u^{\nu)} u_\alpha = 0 \tag{4.28b}$$

so that

$$\tilde{\Pi}^{\mu\nu} = 0. \tag{4.29}$$

Thus, for a Roberton–Walker metric, the non-equilibrium effects vanish and we are left with a perfect fluid stress tensor as expected with ρ and p given by Eqs. (4.26a) and (4.26b).

V. Jeans Mass

As an application of the formalism developed above, we now examine the Jeans mass and its relation to the stability of hot, curved space. First recall that, restricting ourselves to terms with up to two derivatives of the metric, the total action after renormalization can be written as

$$S[g] = \int d^4x \sqrt{-g} \left[\frac{1}{2\kappa} (R - 2\Lambda) + L(\beta) \right], \tag{5.1}$$

where $\kappa \equiv 8\pi G$. The variation of the action with respect to $g_{\mu\nu}$ yields the Einstein equations

$$0 = \delta S[g] = \int d^4x \sqrt{-g} \left[\frac{1}{2\kappa} \left(-R^{\mu\nu} + \frac{1}{2} R g^{\mu\nu} - \Lambda g^{\mu\nu} \right) - \frac{1}{2} T^{\mu\nu} \right] h_{\mu\nu}, \tag{5.2}$$

where we have defined $h_{\mu\nu} \equiv \delta g_{\mu\nu}$. But the stability question requires us to go further and examine the equations of motion which govern the behavior of the metric fluctuation $h_{\mu\nu}$. Thus, let $g_{\mu\nu}+h_{\mu\nu}$ be two neighboring solutions of Eq. (5.2), i.e.,

$$\delta S[g_{\mu\nu}]=0, \tag{5.3a}$$

$$\delta S[g_{\mu\nu}+h_{\mu\nu}]=0. \tag{5.3b}$$

Expanding Eq. (5.3b) about $h_{\mu\nu}=0$, we find

$$\delta S[g_{\mu\nu}+h_{\mu\nu}]=\delta S[g_{\mu\nu}]+\tfrac{1}{2}\delta^2 S[g_{\mu\nu}]+\cdots=0. \tag{5.4}$$

Therefore, to lowest order, the equation of motion for the metric disturbance $h_{\mu\nu}$ is given by [9]

$$\delta^2 S[g_{\mu\nu}]=0. \tag{5.5}$$

(Equation (5.5) is often called the Jacobi equation for $h_{\mu\nu}$.) Performing the expansion of $\delta S[g_{\mu\nu}]$, we find that

$$\begin{aligned}0=\int d^4x\,\sqrt{-g}\,h_{\mu\nu}\Bigg[&-\frac{1}{4}\left(\frac{\delta T^{\mu\nu}}{\delta g_{\alpha\beta}}+\frac{\delta T^{\alpha\beta}}{\delta g_{\mu\nu}}\right)\\&+\frac{1}{2\kappa}\Big\{g^{\mu(\alpha}R^{\beta)\nu}-\frac{1}{4}(g^{\mu\nu}R^{\alpha\beta}+g^{\alpha\beta}R^{\mu\nu})-\frac{1}{2}g^{\mu(\alpha}g^{\beta)\nu}R+g^{\nu(\alpha}R^{\beta)\mu}\\&+\Lambda g^{\mu(\alpha}g^{\beta)\nu}+\frac{1}{2}(g^{\mu\nu}g^{\alpha\beta}-g^{\mu(\alpha}g^{\beta)\nu})\,\Box-\frac{1}{2}(g^{\mu\nu}\nabla^{(\alpha}\nabla^{\beta)}+g^{\alpha\beta}\nabla^{(\mu}\nabla^{\nu)})\\&+\frac{1}{2}(g^{\alpha(\mu}\nabla^{\nu)}\nabla^{\beta}+g^{\beta(\mu}\nabla^{\nu)}\nabla^{\alpha})\Big\}\Bigg]h_{\alpha\beta}.\end{aligned} \tag{5.6}$$

The Einstein equations can now be used to eliminate the curvature terms, yielding the simplified form

$$\begin{aligned}0=\frac{1}{2}\int d^4x\,\sqrt{-g}\,h_{\mu\nu}\Bigg[&\frac{1}{2\kappa}K^{\mu\nu\alpha\beta}+\frac{1}{4}(g^{\mu\nu}T^{\alpha\beta}+g^{\alpha\beta}T^{\mu\nu})\\&-g^{\mu(\alpha}\,T^{\beta)\nu}-g^{\nu(\alpha}\,T^{\beta)\nu}\\&+\frac{1}{2}T^{\lambda}_{\lambda}\left(g^{\mu(\alpha}\,g^{\beta)\nu}-\frac{1}{2}g^{\mu\nu}g^{\alpha\beta}\right)-\frac{1}{2}\left(\frac{\delta T^{\mu\nu}}{\delta g^{\alpha\beta}}+\frac{\delta T^{\alpha\beta}}{\delta g^{\mu\nu}}\right)\Bigg]h_{\alpha\beta}.\end{aligned} \tag{5.7}$$

Here $K^{\mu\nu\alpha\beta}$ is the massless spin-two inverse propagator

$$\begin{aligned}K^{\mu\nu\alpha\beta}=2\Lambda(g^{\mu(\alpha}g^{\beta)\nu}-\tfrac{1}{2}g^{\mu\nu}g^{\alpha\beta})+(g^{\mu\nu}g^{\alpha\beta}-g^{\mu(\alpha}g^{\beta)\nu})\,\Box\\-g^{\mu\nu}\nabla^{(\alpha}\nabla^{\beta)}-g^{\alpha\beta}\nabla^{(\mu}\nabla^{\nu)}+g^{\alpha(\mu}\nabla^{\nu)}\nabla^{\beta}+g^{\beta(\mu}\nabla^{\nu)}\nabla^{\alpha}.\end{aligned} \tag{5.8}$$

It is convenient at this point to define an effective mass tensor $\Pi^{\mu\nu\alpha\beta}$ via

$$\Pi^{\mu\nu\alpha\beta} = 2\kappa \left[\frac{1}{4} (g^{\mu\nu} T^{\alpha\beta} + g^{\alpha\beta} T^{\mu\nu}) - g^{\mu(\alpha} T^{\beta)\nu} - g^{\nu(\alpha} T^{\beta)\mu} + \frac{1}{2} T^{\lambda}_{\lambda} \left(g^{\mu(\alpha} g^{\beta)\nu} - \frac{1}{2} g^{\mu\nu} g^{\alpha\beta} \right) - \frac{1}{2} \left(\frac{\delta T^{\mu\nu}}{\delta g_{\alpha\beta}} + \frac{\delta T^{\alpha\beta}}{\delta g_{\mu\nu}} \right) \right] \tag{5.9}$$

allowing the Jacobi equation for $h_{\alpha\beta}$ to be written as

$$0 = \int d^4x \sqrt{-g}\, h_{\mu\nu} (K^{\mu\nu\alpha\beta} + \Pi^{\mu\nu\alpha\beta})\, h_{\alpha\beta}. \tag{5.10}$$

Finally, $\Pi^{\mu\nu\alpha\beta}$ can be evaluated by use of the stress tensor found in the previous section. Of course, when performing the variation $\delta T^{\mu\nu}/\delta g_{\alpha\beta}$ we must preserve the normalization of u^μ, $u^2 = 1$. The result of the variation is then found to be

$$\frac{\delta T^{\mu\nu}}{\delta g_{\alpha\beta}} = p_0 g^{\mu(\alpha} g^{\beta)\nu} - \frac{1}{2} (\rho_0 + p_0) [g^{\mu(\alpha} u^{\beta)} u^\nu + g^{\nu(\alpha} u^{\beta)} u^\mu] + (u^\mu u^\nu - g^{\mu\nu}) \frac{\delta p_0}{\delta g_{\alpha\beta}} + u^\mu u^\nu \frac{\delta \rho_0}{\delta g_{\alpha\beta}} + \frac{\delta \Pi^{\mu\nu}}{\delta g_{\alpha\beta}}. \tag{5.11}$$

The quantities $\delta\rho_0/\delta g_{\alpha\beta}$, $\delta p_0/\delta g_{\alpha\beta}$, and $\delta\Pi^{\mu\nu}/\delta g_{\alpha\beta}$ are given by

$$\begin{aligned} \frac{\delta\rho_0}{\delta g_{\alpha\beta}} &= \frac{1}{2} \left(1 - a_1(x) \frac{\partial}{\partial m^2} \right) \beta \frac{\partial^2}{\partial \beta^2} (-\beta I)\, u^\alpha u^\beta + \frac{1}{2} \frac{\partial}{\partial \beta} (\beta \Pi^{\alpha\beta}) \\ &\cong \left(-\frac{\pi^2}{15} \beta^{-4} + \frac{1}{24} \beta^{-2} \left(m^2 - \left(\frac{1}{6} - \zeta \right) R \right) \right) u^\alpha u^\beta \\ &\quad + \frac{1}{24} \left(\frac{1}{6} - \zeta \right) (R^{\alpha\beta} - g^{\alpha\beta} \Box + \nabla^{(\alpha} \nabla^{\beta)})\, \beta^{-2} + \cdots \end{aligned} \tag{5.12a}$$

$$\begin{aligned} \frac{\delta p_0}{\delta g_{\alpha\beta}} &= \frac{1}{2} \left(1 - a_1(x) \frac{\partial}{\partial m^2} \right) \beta \frac{\partial I}{\partial \beta} u^\alpha u^\beta - \frac{1}{2} \Pi^{\alpha\beta} \\ &\cong \left(-\frac{\pi^2}{45} \beta^{-4} + \frac{1}{24} \beta^{-2} \left(m^2 - \left(\frac{1}{6} - \zeta \right) R \right) \right) u^\alpha u^\beta \\ &\quad - \frac{1}{24} \left(\frac{1}{6} - \zeta \right) [R^{\alpha\beta} - g^{\alpha\beta} \Box + \nabla^{(\alpha} \nabla^{\beta)}]\, \beta^{-2} + \cdots \end{aligned} \tag{5.12b}$$

$$\frac{\delta \Pi^{\mu\nu}}{\delta g_{\alpha\beta}} = -\left(\frac{1}{6}-\zeta\right) u^\alpha u^\beta [R^{\mu\nu} - g^{\mu\nu}\Box + \nabla^\mu \nabla^\nu]\, \beta \frac{\partial}{\partial \beta}\frac{\partial I}{\partial m^2}$$

$$+2\left(\frac{1}{6}-\zeta\right)\left[g^{\mu(\alpha}R^{\beta)\nu} + g^{\nu(\alpha}R^{\beta)\mu} + \frac{3}{2}\,(g^{\mu(\alpha}\nabla^{\beta)}\nabla^\nu + g^{\nu(\alpha}\nabla^{\beta)}\nabla^\mu)\right.$$

$$\left. -\frac{1}{2}\,(g^{\alpha(\mu}\nabla^{\nu)}\nabla^\beta + g^{\beta(\mu}\nabla^{\nu)}\nabla^\alpha + g^{\alpha\beta}\nabla^{(\mu}\nabla^{\nu)} + g^{\alpha\beta}g^{\mu\nu}\Box)\right]\frac{\partial I}{\partial m^2}$$

$$\cong -\frac{1}{12}\left(\frac{1}{6}-\zeta\right)\left[u^\alpha u^\beta (R^{\mu\nu} - g^{\mu\nu}\Box + \nabla^{(\mu}\nabla^{\nu)})\right.$$

$$+ g^{\mu(\alpha}R^{\beta)\nu} + g^{\nu(\alpha}R^{\beta)\mu} + \frac{3}{2}\,(g^{\mu(\alpha}\nabla^{\beta)}\nabla^\nu + g^{\nu(\alpha}\nabla^{\beta)}\nabla^\mu)$$

$$\left. -\frac{1}{2}\,(g^{\alpha(\mu}\nabla^{\nu)}\nabla^\beta + g^{\beta(\mu}\nabla^{\nu)}\nabla^\alpha + g^{\alpha\beta}\nabla^{(\nu}\nabla^{\mu)} + g^{\alpha\beta}g^{\mu\nu}\Box)\right]\beta^{-2} + \cdots. \tag{5.12c}$$

In order to obtain Eqs. (5.12) we have utilized the identity $\delta F(\beta) = \frac{1}{2}\beta(\partial F/\partial\beta)\, u^\mu u^\nu h_{\mu\nu}$ arising from the graviational red-shift. Also in evaluating Eq. (5.12c) we used the result

$$\delta R^{\mu\nu} = -2R^{\lambda(\mu}h^{\nu)}{}_\lambda + \tfrac{1}{2}[\nabla^\mu\nabla^\nu h^\lambda_\lambda + \Box h^{\mu\nu} - 2\nabla^\lambda\nabla^{(\mu}h^{\nu)}{}_\lambda]. \tag{5.13}$$

Finally, combining these results, we find that the mass tensor $\Pi^{\mu\nu\alpha\beta}$ is given by

$$\Pi^{\mu\nu\alpha\beta} = 2\kappa\left[p_0\left(\frac{1}{2}g^{\mu\nu}g^{\alpha\beta} - g^{\mu(\alpha}g^{\beta)\nu}\right)\right.$$

$$+(\rho_0+p_0)\left\{\frac{1}{4}\,(g^{\mu\nu}u^\alpha u^\beta + g^{\alpha\beta}u^\mu u^\nu) - \frac{1}{2}\,(g^{\mu(\alpha}u^{\beta)}u^\nu + g^{\nu(\alpha}u^{\beta)}u^\mu)\right.$$

$$\left. +\frac{1}{2}g^{\mu(\alpha}g^{\beta)\nu} - \frac{1}{4}g^{\mu\nu}g^{\alpha\beta}\right\}$$

$$+\frac{1}{4}\,(g^{\mu\nu}\Pi^{\alpha\beta} + g^{\alpha\beta}\Pi^{\mu\nu}) - g^{\mu(\alpha}\Pi^{\beta)\nu} - g^{\nu(\alpha}\Pi^{\beta)\mu}$$

$$+\frac{1}{2}\Pi^\lambda{}_\lambda\left(g^{\mu(\alpha}g^{\beta)\nu} - \frac{1}{2}g^{\mu\nu}g^{\alpha\beta}\right) - \frac{1}{2}\,(u^\mu u^\nu - g^{\mu\nu})\frac{\delta p_0}{\delta g_{\alpha\beta}}$$

$$-\frac{1}{2}\,(u^\alpha u^\beta - g^{\alpha\beta})\frac{\delta p_0}{\delta g_{\mu\nu}} - \frac{1}{2}u^\mu u^\nu\frac{\delta\rho_0}{\delta g_{\alpha\beta}} - \frac{1}{2}u^\alpha u^\beta\frac{\delta\rho_0}{\delta g_{\mu\nu}}$$

$$\left. -\frac{1}{2}\frac{\delta\Pi^{\mu\nu}}{\delta g_{\alpha\beta}} - \frac{1}{2}\frac{\delta\Pi^{\alpha\beta}}{\delta g_{\mu\nu}}\right]. \tag{5.14}$$

Before making the connection with the Jeans mass, it is instructive to review the classical Newtonian calculation [17, 23, 28], which was originally proposed by Jeans as a model for galaxy formation. Consider the universe to be filled with a non-gravitating, non-relativistic fluid with density ρ, pressure p, and velocity $\mathbf{v}$. The fluid is governed by both the continuity stricture

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \tag{5.15}$$

and the Euler equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\, \mathbf{v} = -\frac{1}{\rho} \nabla p. \tag{5.16}$$

The unperturbed solution of Eqs. (5.15) and (5.16) is taken to be a static, uniform fluid at rest: $\rho = \text{constant}$, $p = \text{constant}$, $\mathbf{v} = \mathbf{0}$. Now add a small gravitational fluctuation $\delta\mathbf{g}$ and examine the linearized forms of the fluid equations. We find that the perturbations $\delta\rho$ and $\delta\mathbf{v}$ obey

$$\frac{\partial}{\partial t}(\delta\rho) + \rho \nabla \cdot \delta\mathbf{v} = 0 \tag{5.17}$$

$$\frac{\partial}{\partial t}(\delta\mathbf{v}) = -\frac{1}{\rho} v_s^2 \nabla(\delta\rho) + \delta\mathbf{g}, \tag{5.18}$$

where $v_s = (\delta p/\delta\rho)^{1/2}$ is the speed of sound in the fluid. When supplemented by the equations of Newtonian gravity

$$\nabla \times \delta\mathbf{g} = 0, \qquad \nabla \cdot \delta\mathbf{g} = -\frac{\kappa}{2} \delta\rho, \tag{5.19}$$

one can obtain an equation for $\delta\rho$ alone,

$$\left(\frac{\partial^2}{\partial t^2} - v_s^2 \nabla^2 - \frac{\kappa}{2} \rho\right) \delta\rho = 0, \tag{5.20}$$

which determines the evolution of density fluctuations. Looking for a solution of the form

$$\delta\rho \sim \exp(i\mathbf{k} \cdot \mathbf{x} - i\omega t) \tag{5.21}$$

we find

$$\omega^2 = (v_s^2 \mathbf{k}^2 + m_J^2), \tag{5.22}$$

where

$$m_J^2 = -\frac{\kappa}{2}\rho \tag{5.23}$$

defines the Jeans mass. Obviously long-wavelength perturbations, i.e.,

$$k < \frac{|m_J|}{v_s}, \tag{5.24}$$

are unstable in that ω is imaginary, allowing $\delta\rho$ to grow (or decay) exponentially with time. Unfortunately, as a theory of galaxy formation Jeans'idea did not pass muster. The problem is that the *maximum* rate of growth from Eq. (5.22) is given by

$$|\omega_{\max}| = \left(\frac{\kappa}{2}\rho\right)^{1/2} \tag{5.25}$$

and is the same order as the expansion rate of the universe

$$\frac{\dot{R}}{R} \sim \left(\frac{1}{3}\kappa\rho\right)^{1/2}. \tag{5.26}$$

Thus his proposition of constant background density and pressure is clearly incorrect, and it becomes necessary to examine the question from a more realistic point of view, as in the effective Lagrangian formalism developed above.

One must exercise some caution in this process, however, as will become clear. Previous investigators have attempted to address this problem by beginning with the flat space-time limit of Eq. (5.6) and evaluating the effective graviton propagator in order to isolate the Jeans mass. Before doing so, however, it is important to realize that the metric fluctuation $h_{\alpha\beta}$ is subject to certain constraints associated with being a spin-two field. Thus consider the example of a massive spin-two field $\phi_{\mu\nu}$ in flat space-time. The appropriate Lagrangian is then given by [9]

$$L[\phi_{\mu\nu}] = \tfrac{1}{2}\phi_{\mu\nu}(K^{\mu\nu\alpha\beta} + m^2(\eta^{\mu\nu}\eta^{\alpha\beta} - \eta^{\mu(\alpha}\eta^{\beta)\nu}))\,\phi_{\alpha\beta}, \tag{5.27}$$

where $K^{\mu\nu\alpha\beta}$ is defined in Eq. (5.8), with $\Lambda = 0$ and the covariant derivative ∇_μ replaced by the ordinary partial derivative ∂_μ. The Euler–Lagrange equation for $\phi_{\mu\nu}$ is found to be

$$[K^{\mu\nu\alpha\beta} + m^2(\eta^{\mu\nu}\eta^{\alpha\beta} - \eta^{\mu(\alpha}\eta^{\beta)\nu})]\,\phi_{\alpha\beta} = 0. \tag{5.28}$$

A simpler form may be determined by taking the trace of this equation yielding

$$(2(\eta^{\alpha\beta}\Box - \partial^\alpha\partial^\beta) + 3m^2\eta^{\alpha\beta})\,\phi_{\alpha\beta} = 0. \tag{5.29}$$

On the other hand taking the divergence yields

$$m^2(\eta^{\alpha\beta}\partial^\mu - \eta^{\mu(\alpha}\partial^{\beta)})\,\phi_{\alpha\beta} = 0. \tag{5.30}$$

Combining the restrictions implied by these constraints we find

$$\eta^{\alpha\beta}\phi_{\alpha\beta}=0 \tag{5.31a}$$

$$\partial^{\alpha}\phi_{\alpha\beta}=0 \tag{5.31b}$$

$$(\square+m^2)\,\phi_{\alpha\beta}=0. \tag{5.31c}$$

The physics associated with these strictures is clear. The first two imply that no spin-zero or spin-one excitations are permitted. The third condition is that the remaining five spin-two degrees of freedom obey the Klein–Gordon equation.

Application of Eqs. (5.31a) and (5.31b) to the flat space-time limit of Eq. (5.6) (i.e., dropping the curvature-dependent terms) yields

$$[g^{\mu(\alpha}g^{\beta)\nu}(\square-2\Lambda+2\kappa p_0)-\kappa(\rho_0+p_0)(g^{\mu(\alpha}u^{\beta)}u^{\nu}+g^{\nu(\alpha}u^{\beta)}u^{\mu})]\,h_{\mu\nu}=0. \tag{5.32}$$

The procedure followed by Gross *et al.* [17] was to argue that only the 0, 0 component of $h_{\mu\nu}$ is relevant in the static limit, whereby the above equation becomes

$$(\square-2\Lambda-2\kappa\rho_0)\,h_{00}=0. \tag{5.33}$$

Thus the Jeans mass found in this way is

$$m_{\mathrm{J}}^2=-2\kappa\rho_0-2\Lambda \tag{5.34}$$

which agrees with [17] in the limit of vanishing cosmological constant. It is interesting also to note that there exists the possibility of stability—i.e., $m_{\mathrm{J}}{}^2>0$—in anti-de Sitter space when

$$\Lambda<-\kappa\rho_0. \tag{5.35}$$

However, this result should not be taken literally for a number of reasons. Most importantly, the background field solution

$$g^{(0)}_{\mu\nu}=\eta_{\mu\nu} \tag{5.36}$$

about which the metric was expanded is *not* a solution of the Einstein equation. This is clear since the existence of a non-zero thermal energy-momentum density *requires* a non-trivial metric solution $g_{\mu\nu}\neq\eta_{\mu\nu}$. In fact, of the four diagrams—Fig. 1—required in order to calculate the renormalized propagator, Gross *et al.* [17] evaluated only the vacuum polarization term, Fig. 1a. In particular, they omitted the tadpole terms in Fig. 1b. Such diagrams are infinite (this is obvious in momentum space since the massless graviton carries no four-momentum) and raise obvious questions about the consistency of this approach. The same problems beset the work of Kikuchi *et al.* [21]. Both calculations contain spurious contributions associated with the fact that

$$\delta S[\eta_{\mu\nu}+h_{\mu\nu}]-\delta S[\eta_{\mu\nu}]\neq 0 \tag{5.37}$$

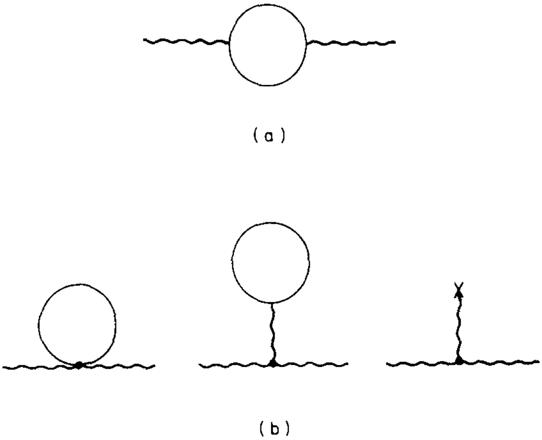

FIG. 1. Shown are diagrams which contribute to the graviton vacuum polarization. Diagram (a) was considered in Ref. [10]; those shown in (b) were not.

which means that their metric fluctuations are not consistent with the Einstein equations. Use of an appropriate metric solution $g_{\mu\nu}$ eliminates the tadpole infinities and allows a consistent calculation to be developed, as we shall show.

Thus, instead of Eq. (5.6), we use Eq. (5.10) as our starting point. Here any references to curvature have been eliminated by use of the Einstein equations, and the Jacobi equation—Eq. (5.5)—is satisfied, meaning we are expanding about a background metric $g_{\mu\nu}$ which satisfies Eq. (5.3a) and hence is a solution of the Einstein equations. The appropriate conditions which must be imposed upon the metric fluctuations may be found in the following manner. Write the equation of motion for $h_{\mu\nu}$ as

$$F^{\mu\nu\alpha\beta}h_{\alpha\beta}=0 \tag{5.38}$$

with

$$F^{\mu\nu\alpha\beta}=K^{\mu\nu\alpha\beta}+\Pi^{\mu\nu\alpha\beta}. \tag{5.39}$$

As before we must require

$$\begin{aligned} g_{\mu\nu}F^{\mu\nu\alpha\beta}h_{\alpha\beta}&=0\\ \nabla_\nu F^{\mu\nu\alpha\beta}h_{\alpha\beta}&=0. \end{aligned} \tag{5.40}$$

In addition we demand that

$$u_\nu F^{\mu\nu\alpha\beta}h_{\alpha\beta}=0, \tag{5.41}$$

where u_ν is a time-like four-vector.

However, the situation is not as simple as before where the spin-two projection operator $K^{\mu\nu\alpha\beta}$ was augmented by a simple mass term. Now, because of the co-ordinate invariance of the problem, no unique constraints on the metric fluctuation tensor $h_{\alpha\beta}$ are obtained. However, this is no longer true once a gauge is chosen. That this should be the case is clear from the analogous situation in electrodynamics wherein obtaining a consistent form for the propagator requires a specific gauge choice. In our case, we elect to use the harmonic co-ordinate condition

$$g^{\mu\nu}\Gamma^{\lambda}_{\mu\nu} = 0 \tag{5.42}$$

which is equivalent to

$$\nabla_{\mu} h^{\mu\nu} = \tfrac{1}{2}\nabla^{\nu} h^{\mu}{}_{\mu} + h^{\alpha\beta}\Gamma^{\nu}_{\alpha\beta}. \tag{5.43}$$

In Riemann normal co-ordinates this simplifies to

$$\nabla_{\mu} h^{\mu\nu} = \tfrac{1}{2}\nabla^{\nu} h^{\mu} \tag{5.44}$$

and the constraints become

$$\begin{aligned} g^{\alpha\beta} h_{\alpha\beta} &= 0 \\ \nabla^{\alpha} h_{\alpha\beta} &= 0 \\ u^{\alpha} h_{\alpha\beta} &= 0. \end{aligned} \tag{5.45}$$

The first two are simply the vanishing of the spin-zero and spin-one degrees of freedom, respectively. The latter corresponds to the requirement that the actual metric fluctuations (i.e., those satisfying the Einstein equations of motion) must be space-like. Using these constraints on $h_{\alpha\beta}$, we find

$$\begin{aligned} K^{\mu\nu\alpha\beta} h_{\alpha\beta} &= -g^{\mu(\alpha} g^{\beta)\nu} (\Box - 2\Lambda)\, h_{\alpha\beta} \\ \Pi^{\mu\nu\alpha\beta} h_{\alpha\beta} &= -g^{\mu(\alpha} g^{\beta)\nu} \kappa (p - \rho)\, h_{\alpha\beta} + \cdots. \end{aligned} \tag{5.46}$$

Thus, the effective Klein–Gordon equation obeyed by $h_{\alpha\beta}$ is given by

$$(\Box - 2\Lambda + \kappa(p - \rho) + \cdots)\, h_{\alpha\beta} = 0. \tag{5.47}$$

We can now read off the Jeans mass

$$m_J^2 = -\kappa(\rho - p) - 2\Lambda. \tag{5.48}$$

In the absence of a cosmological constant and in the matter dominated era wherein $\rho \gg p$ we find

$$m_J^2 \cong -\kappa\rho \tag{5.49}$$

which is exactly twice the classical Jeans result and half of the value found by Gross *et al.* [17]. It is interesting that proper inclusion of curvature effects exactly doubles the simple Newtonian result, just as in the calculation of the deflection of starlight [23, 26–28].

Inclusion of the effects of pressure and of the cosmological constant appear in intuitively reasonable ways. That is, one would suspect that the positive energy density associated with $\Lambda > 0$ would tend to increase the instabilities while the presence of the internal pressure would tend to go in the direction of stability. Naively speaking, spaces with

$$\Lambda > -\frac{\kappa}{2}(\rho - p) \tag{5.50}$$

appear to contain exponentially growing solutions, while anti-de Sitter spaces with

$$\Lambda < -\frac{\kappa}{2}(\rho - p) \tag{5.51}$$

actually tend to stability. However, comparison with the Einstein equations reveals that the time scale for growth of metric fluctuations is of the same order as the expansion rate of space-time. Equivalently, the wavelength of low frequency fluctuations is of the same scale as the horizon length. Obviously then, great care is required in order to properly interpret such results regarding the stability or instability of non-flat space-times.

We should emphasize that our result for the Jeans mass depends on Eqs. (5.10) and (5.38) being the appropriate ones for determining the stability of space-time. Equation (5.10) requires that $g_{\mu\nu}$ and $h_{\mu\nu}$ satisfy Eqs. (5.3) and (5.5). Subject to these restrictions we derived the equation of motion for $h_{\mu\nu}$, Eq. (5.38), from Eq. (5.10). This approach to stability analysis is local in nature. Hence, we expect this method to break down when the Jeans mass, as given by Eq. (5.48), is comparable to the inverse of the horizon size. In such a situation global considerations generally become important.

If global considerations are paramount one should work with Eqs. (5.3) directly. An exception to this caveat is when all of the global information can be encoded in local quantities, such as the background metric. This occurs, for example, in the case of the Robertson–Walker metric where the spherical symmetry determines the form of the metric. In such a situation we expect our result for the Jeans mass, Eq. (5.48), to remain valid even though the inverse horizon size and the Jeans mass are comparable. A treatment of fluctuations in a Robertson–Walker background is given in [28].

VI. Conclusion

We have examined the problem of spacetime at non-zero temperature. In order to do so, we developed an adiabatic approximation to the real-time thermal Green's function, for a free scalar field theory, correct to second order in the curvature, by using an expansion in terms of Riemann normal co-ordinates. This expansion is valid for all temperatures. We demonstrated how terms which break local Lorentz symmetry arise, due to the presence of the heat bath. This Green's function was then used, in the coincidence limit, in order to construct an effective Lagrangian for the classical gravity—hot quantum fluid system.

Our result for the effective Lagrangian is similar to that found by Nakazawa and Fukuyama [24] if one discards a spurious contribution proportional to

$$\ln(-g_{00})_{,i}{}^{i}$$

found by these authors. However, these authors worked only in the high temperature limit—$\beta m \ll 1$—and stopped their expansion at $O(R)$, whereas our work is valid for arbitrary temperature and is correct to $O(R^2)$.

We considered two applications of this result. The first was the evaluation of the energy-momentum tensor, by taking the variation of the effective action with respect to the metric. The resulting form was shown to be that for an imperfect fluid and the heat-flow vector and the viscosity tensor associated with non-equilibrium were evaluated. Our stress tensor agrees with that of Nakazawa and Fukuyama [24] except for certain temperature-dependent derivative terms that they omitted.

A second application was to the calculation of the tachyonic mass associated with metric fluctuations in a background gravitational field in the presence of a finite temperature fluid, i.e., the so-called Jeans mass. While our calculation is certainly not the first to address this problem in a field-theoretic framework, we believe that it is the first to do so self-consistently. The graviton mass was first examined in the pioneering paper of Gross *et al.* [17] and subsequently in a similar work by Kikuchi *et al.* [21]. However, neither of these papers employed a fully covariant framework. In addition, the background field utilized by these authors in order to study the metric fluctuation was flat space-time and was therefore not a solution of the Einstein equations. These workers attemped to deal with the resulting difficulties by omitting certain (infinite) tadpole contributions and working instead with the (finite) vacuum polarization diagram. However, it is apparent that flat space-time and a universe filled with a hot fluid are inconsistent with one another and that simply dropping possible infinities is a dangerous procedure. In our calculation the fact that the background field metric satisfies the Einstein equations is shown to eliminate these tadpole infinities. The tachyonic graviton mass squared found in our work

$$m_{\mathrm{J}}^2 = -\kappa\rho$$

is exactly twice the Newtonian value and half that found in the earlier calculation [17].

Our work has obvious use in modeling the early universe. Extending the results of this paper to more realistic field theories would prove particularly interesting in the study of inflation and symmetry breaking in curved space-time at finite temperature. Work on these topics is in progress.

APPENDIX 1

We collect here all of the identities involving derivatives of $\delta(p^2-m^2)$ that were used in deriving $\bar{G}(x, x')$. All of them can be obtained by using the chain rule and differentiating $(p^2-m^2)\,\delta(p^2-m^2)=0$,

$$\left(\frac{\partial}{\partial m^2}\right)^{n-1}\delta(p^2-m^2)=\frac{1}{n}(p^2-m^2)\left(\frac{\partial}{\partial m^2}\right)^n\delta(p^2-m^2) \tag{A.1}$$

$$\partial^\alpha\delta(p^2-m^2)=\frac{1}{2}(p^2-m^2)\,\partial^\alpha\frac{\partial}{\partial m^2}\delta(p^2-m^2) \tag{A.2}$$

$$\partial^\alpha\frac{\partial}{\partial m^2}\delta(p^2-m^2)=\frac{1}{3}(p^2-m^2)\,\partial^\alpha\left(\frac{\partial}{\partial m^2}\right)^2\delta(p^2-m^2) \tag{A.3}$$

$$\partial^\alpha\partial^\beta\delta(p^2-m^2)=\frac{1}{3}(p^2-m^2)\left[\partial^\alpha\partial^\beta-\eta^{\alpha\beta}\frac{\partial}{\partial m^2}\right]\frac{\partial}{\partial m^2}\delta(p^2-m^2) \tag{A.4}$$

$$\partial^\alpha\partial^\beta\frac{\partial}{\partial m^2}\delta(p^2-m^2)=\frac{1}{4}(p^2-m^2)\left[\partial^\alpha\partial^\beta-\frac{2}{3}\eta^{\alpha\beta}\frac{\partial}{\partial m^2}\right]\left(\frac{\partial}{\partial m^2}\right)^2\delta(p^2-m^2) \tag{A.5}$$

$$\partial^\alpha\partial^\beta\partial^\lambda\delta(p^2-m^2)$$
$$=\frac{1}{4}(p^2-m^2)\left[\partial^\alpha\partial^\beta\partial^\lambda-\frac{2}{3}(\eta^{\alpha\beta}\partial^\lambda+\eta^{\beta\lambda}\partial^\alpha+\eta^{\alpha\lambda}\partial^\beta)\frac{\partial}{\partial m^2}\right]\frac{\partial}{\partial m^2}\delta(p^2-m^2) \tag{A.6}$$

$$\partial^\alpha\partial^\beta\partial^\sigma\partial^\lambda\delta(p^2-m^2)=\frac{1}{5}(p^2-m^2)[\partial^\alpha\partial^\beta\partial^\sigma\partial^\lambda$$
$$-\frac{1}{4}(\eta^{\alpha\beta}\partial^\sigma\partial^\lambda+\eta^{\alpha\sigma}\partial^\beta\partial^\lambda+\eta^{\alpha\lambda}\partial^\beta\partial^\sigma+\eta^{\beta\sigma}\partial^\alpha\partial^\lambda+n^{\beta\lambda}\partial^\alpha\partial^\sigma+\eta^{\sigma\lambda}\partial^\alpha\partial^\beta)\frac{\partial}{\partial m^2}$$
$$+\frac{1}{3}(\eta^{\alpha\beta}\eta^{\sigma\lambda}+\eta^{\alpha\lambda}\eta^{\beta\sigma}+\eta^{\alpha\sigma}\eta^{\beta\lambda})\left(\frac{\partial}{\partial m^2}\right)^2\Bigg]\frac{\partial}{\partial m^2}\delta(p^2-m^2). \tag{A.7}$$

As an example of how Eqs. (A.1)–(A.7) are used we will go through the derivation of $\bar{G}_2(p)$ step-by-step. $\bar{G}_2(p)$ solves

$$(p^2-m^2)\,\bar{G}_2(p)=(\zeta-\tfrac{1}{6})\,R\bar{G}_0(p)-\tfrac{1}{3}[R^\mu{}_\alpha{}^\nu{}_\beta\,p_\mu p_\nu\partial^\alpha\partial^\beta-R_\alpha{}^\nu p_\nu\partial^\alpha]\,\bar{G}_0(p). \tag{2.13b}$$

$\bar{G}_0(p)$ is given by

$$\bar{G}_0(p)=\begin{pmatrix}1/(p^2-m^2+i\varepsilon) & 0\\ 0 & -1/(p^2-m^2-i\varepsilon)\end{pmatrix}+\delta(p^2-m^2)\,F(p). \tag{2.14}$$

Using Eq. (A.1) and $(p^2-m^2)^{-2}=(\partial/\partial m^2)(p^2-m^2)^{-1}$ we can write Eq. (2.10b) as

$$(p^2-m^2)\,\bar{G}_2(p)=(p^2-m^2)\left(\zeta-\frac{1}{6}\right)R\,\frac{\partial}{\partial m^2}\,\bar{G}_0(p)$$

$$-\frac{1}{3}\left[R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta}\,p_\mu p_\nu \partial^\alpha\partial^\beta - R_\alpha{}^\nu p_\nu \partial^\alpha\right]\bar{G}_0(p). \tag{A.8}$$

The terms in square brackets vanish for Lorentz-invariant functions so the zero-temperature piece of $\bar{G}_0(p)$ does not contribute to the bracketed terms. Therefore, we have

$$(p^2-m^2)\,\bar{G}_2(p)=(p^2-m^2)\left(\zeta-\frac{1}{6}\right)R\,\frac{\partial}{\partial m^2}\,\bar{G}_0(p)$$

$$-\frac{1}{3}\left[R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta}\,p_\mu p_\nu \partial^\alpha\partial^\beta - R_\alpha{}^\nu p_\nu \partial^\alpha\right]\delta(p^2-m^2)\,F(p). \tag{A.9}$$

Carrying out the differentiation gives

$$\begin{aligned}
&(p^2-m^2)\,\bar{G}_2(p)\\
&\quad=(p^2-m^2)\left(\zeta-\frac{1}{6}\right)R\,\frac{\partial}{\partial m^2}\,\bar{G}_0(p)\\
&\qquad-\frac{1}{3}\,R_\alpha{}^\nu\,p_\nu\left[\delta(p^2-m^2)\,\partial^\alpha F(p)+F(p)\,\partial^\alpha\delta(p^2-m^2)\right]\\
&\qquad-\frac{1}{3}\,R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta}\,p_\mu p_\nu\left[\delta(p^2-m^2)\,\partial^\alpha\partial^\beta F(p)+2\partial^{(\alpha}\delta(p^2-m^2)\,\partial^{\beta)}F(p)\right.\\
&\qquad\left.+F(p)\,\partial^\alpha\partial^\beta\delta(p^2-m^2)\right].
\end{aligned} \tag{A.10}$$

We now substitute Eqs. (A.2) and (A.4) for the derivatives of the delta functions. This yields

$$\begin{aligned}
&(p^2-m^2)\,\bar{G}_2(p)\\
&\quad=(p^2-m^2)\left(\zeta-\frac{1}{6}\right)R\,\frac{\partial}{\partial m^2}\,\bar{G}_0(p)\\
&\qquad+\frac{1}{3}\,(p^2-m^2)\,R_\alpha{}^\nu p_\nu\,\frac{\partial}{\partial m^2}\left[\delta(p^2-m^2)\,\partial^\alpha F(p)+\frac{1}{2}\,F(p)\,\partial^\alpha\delta(p^2-m^2)\right]\\
&\qquad-\frac{1}{3}\,(p^2-m^2)\,R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta}\,p_\mu p_\nu\,\frac{\partial}{\partial m^2}\left[\delta(p^2-m^2)\,\partial^\alpha\partial^\beta F(p)\right.\\
&\qquad\left.+\partial^{(\alpha}\delta(p^2-m^2)\,\partial^{\beta)}F(p)+\frac{1}{3}\,F(p)\left(\partial^\alpha\partial^\beta-\eta^{\alpha\beta}\,\frac{\partial}{\partial m^2}\right)\delta(p^2-m^2)\right].
\end{aligned} \tag{A.11}$$

Collecting terms gives

$$
\begin{aligned}
&(p^2-m^2)\,\bar{G}_2(p)\\
&\quad=(p^2-m^2)\left(\zeta-\frac{1}{6}\right)R\,\frac{\partial}{\partial m^2}\,\bar{G}_0(p)\\
&\qquad-\frac{1}{3}(p^2-m^2)[R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta}\,p_\mu p_\nu\partial^\alpha\partial^\beta-R_\alpha{}^\nu p_\nu\partial^\alpha]\frac{\partial}{\partial m^2}\,\delta(p^2-m^2)\,F(p)\\
&\qquad-\frac{1}{3}(p^2-m^2)\frac{\partial}{\partial m^2}\left\{\frac{1}{2}R_\alpha{}^\nu\,p_\nu F(p)\,\partial^\alpha\delta(p^2-m^2)\right.\\
&\qquad-R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta}\,p_\mu p_\nu\partial^{(\alpha}\delta(p^2-m^2)\,\partial^{\beta)}F(p)\\
&\qquad\left.-\frac{1}{3}R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta}\,p_\mu p_\nu\left(2F(p)\,\partial^\alpha\partial^\beta\delta(p^2-m^2)+\eta^{\alpha\beta}F(p)\frac{\partial}{\partial m^2}\,\delta(p^2-m^2)\right)\right\}.
\end{aligned}
\tag{A.12}
$$

The next step is to show that the last set of terms in brackets vanishes. To do this we express the derivatives $\delta(p^2-m^2)$ in terms of derivatives with respect to m^2 by using

$$\partial^\alpha\delta(p^2-m^2)=2p^\alpha\frac{\partial}{\partial p^2}\,\delta(p^2-m^2)=-2p^\alpha\frac{\partial}{\partial m^2}\,\delta(p^2-m^2)$$

$$\partial^\alpha\partial^\beta\delta(p^2-m^2)=-2\eta^{\alpha\beta}\frac{\partial}{\partial m^2}\,\delta(p^2-m^2)+4p^\alpha p^\beta\left(\frac{\partial}{\partial m^2}\right)^2\delta(p^2-m^2).$$

These give

$$
\begin{aligned}
&\frac{1}{2}R_\alpha{}^\nu p_\nu F(p)\,\partial^\alpha\delta(p^2-m^2)-R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta}\,p_\mu p_\nu\partial^{(\alpha}\delta(p^2-m^2)\,\partial^{\beta)}F(p)\\
&\qquad-\frac{1}{3}R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta}\,p_\mu p_\nu F(p)\left(2\partial^\alpha\partial^\beta\delta(p^2-m^2)+\eta^{\alpha\beta}\frac{\partial}{\partial m^2}\,\delta(p^2-m^2)\right)\\
&\quad=-R_\alpha{}^\nu p_\nu p^\alpha F(p)\frac{\partial}{\partial m^2}\,\delta(p^2-m^2)\\
&\qquad+2R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta}\,p_\mu p_\nu p^{(\alpha}\partial^{\beta)}F(p)\frac{\partial}{\partial m^2}\,\delta(p^2-m^2)\\
&\qquad-\frac{2}{3}R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta}\,p_\mu p_\nu F(p)\left(4p^\alpha p^\beta\frac{\partial}{\partial m^2}-2\eta^{\alpha\beta}\frac{\partial}{\partial m^2}\right)\delta(p^2-m^2)\\
&\qquad-\frac{1}{3}R^{\mu\nu}p_\mu p_\nu F(p)\frac{\partial}{\partial m^2}\,\delta(p^2-m^2)\\
&\quad=-\frac{8}{3}R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta}\,p_\mu p_\nu p^\alpha p^\beta F(p)\frac{\partial}{\partial m^2}\,\delta(p^2-m^2)\\
&\qquad+2R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta}\,p_\mu p_\nu p^{(\alpha}\partial^{\beta)}F(p)\frac{\partial}{\partial m^2}\,\delta(p^2-m^2).
\end{aligned}
\tag{A.13}
$$

Since $R_{\mu\alpha\nu\beta} = -R_{\alpha\mu\nu\beta} = -R_{\mu\alpha\beta\nu}$, this is zero. Thus we obtain that

$$
\begin{aligned}
&(p^2-m^2)\,\bar{G}_2(p)\\
&\quad=(p^2-m^2)\left(\zeta-\frac{1}{6}\right)R\,\frac{\partial}{\partial m^2}\,\bar{G}_0(p)\\
&\qquad-\frac{1}{3}\,(p^2-m^2)\,\frac{\partial}{\partial m^2}\,[R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta}\,p_\mu\,p_\nu\,\partial^\alpha\partial^\beta-R_\alpha{}^\nu p_\nu\,\partial^\alpha]\;\delta(p^2-m^2)\,F(p)\\
&\quad=(p^2-m^2)\left[\left(\zeta-\frac{1}{6}\right)R-\frac{1}{3}\,(R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta}\,p_\mu\,p_\nu\,\partial^\alpha\partial^\beta-R_\alpha{}^\nu p_\nu\,\partial^\alpha)\right]\frac{\partial}{\partial m^2}\,\bar{G}_0(p).
\end{aligned}
\tag{A.14}
$$

This yields the result quoted in Eq. (2.16). Note that the only things needed to go from Eq. (2.13b) to Eq. (A.14) were the chain rule and the permutation symmetry of $R_{\mu\nu\alpha\beta}$.

Appendix 2

The temperature-dependent functions in $L(\beta)$, Eq. (3.11), are given by

$$
\begin{aligned}
B_1&=(4\pi)^{-3/2}\,\frac{1}{\Gamma(3/2)}\int_m^\infty d\omega(\omega^2-m^2)^{1/2}\,\frac{\partial^2 n_{\mathrm{B}}(\omega)}{\partial\omega^2}\\
B_2&=(4\pi)^{-3/2}\,\frac{1}{\Gamma(3/2)}\int_m^\infty d\omega(\omega^2-m^2)^{1/2}\,\omega\,\frac{\partial^3 n_{\mathrm{B}}(\omega)}{\partial\omega^3}\\
C_1&=(4\pi)^{-3/2}\,\frac{1}{\Gamma(3/2)}\int_m^\infty d\omega(\omega^2-m^2)^{1/2}\,\omega\,\frac{\partial^2 n_{\mathrm{B}}(\omega)}{\partial\omega^2}\\
C_2&=\frac{1}{2}\,(4\pi)^{-3/2}\,\frac{1}{\Gamma(5/2)}\int_m^\infty d\omega(\omega^2-m^2)^{1/2}\,(4\omega^2-m^2)\,\frac{\partial^2 n_{\mathrm{B}}(\omega)}{\partial\omega^2}\\
D_1&=\frac{1}{2}\,(4\pi)^{-3/2}\,\frac{1}{\Gamma(5/2)}\int_m^\infty d\omega(\omega^2-m^2)^{1/2}\,(4\omega^2-m^2)\,n_{\mathrm{B}}(\omega)\\
I&=(4\pi)^{-3/2}\,\frac{1}{\Gamma(5/2)}\int_m^\infty d\omega(\omega^2-m^2)^{3/2}\,n_{\mathrm{B}}(\omega),
\end{aligned}
$$

where $n_{\mathrm{B}}(\omega)$ is the Bose–Einstein distribution.

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