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# A functional representation of potential surprise ordering

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## ABSTRACT

In the history of economic thought, Shackle was one of the representative critics about probability based economic theory. Specifically, he constructed his own concept of subjective uncertainty called potential surprise to replace probability. In 1980s, the potential surprise is axiomatized by Katzner as Kolmogorov-styled measure defined on the  $\sigma$ -field over the set of possible states. In this paper, potential surprise function is reconstructed as the functional representation of potential surprise ordering on the space of hypotheses about future called monad.

Key words: Shackle, uncertainty, probability

JEL classifications: B21; B50; D81

## 1. Introduction

In the history of economic thought, G.L.S. Shackle is a major critic of probability and expected utility theory (hereafter EUT). Shackle's objections to that theory mainly concerns the major properties which probability stands on, such as dependence on the repetitive experiment, distribution of probability values that sum to the unity, and additivity of probability for mutually independent events. Specifically, if the probability is construed in terms of frequency, then the probability calculus cannot be utilized when the choice is a single, unique activity because the probability value has meaning only when an experiment is to be repeated infinite numbers of times. But almost all economic decisions in reality are unique and irreversible. Besides, representing the possibility of an event having no supporting evidence with zero probability is not a suitable way to reflect ignorance on the part of the decision maker. The assignment of zero probability is relevant to the "knowledge" that the relative frequency of a specific event in a repetitive experiment is zero, and this is different from disbeliefs due to the lack of supporting information.

Not only that, if the probability is thought of as subjective, then for probability to be a meaningful concept, it is necessary that the decision maker has stable knowledge for all possible past and future outcomes. If not, for example, any potential change in the number of alternative outcomes in the future must alter the probability assigned to previously known outcomes. Furthermore, with maintaining the notion of additive probability, it is not possible to represent  $n > 2$  independent events whose realization and nonrealization are assessed as equally plausible with the identical probability value.<sup>1</sup> If additivity is preserved in such case, then the summation of probability  $n/2$  could be bigger than unity, in other words, the distributivity of probability is violated.

Hence, instead of the problematic use of probability as a basis for expected utility theory, Shackle constructed his own alternative theoretical framework explaining human decisions on the premise of historical time, ignorance of future events, and a non-probabilistic measure of uncertainty (Shackle 1954, 1969, 1972).<sup>2</sup>

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<sup>1</sup> P.29 in Shackle (1954).

<sup>2</sup> These arguments by Shackle are summarized in J.L. Ford (1999). Further detailed comparisons from technical and practical perspectives is stated in a later section.

However, although Shackle's criticism of probability was accepted by several Austrians and Post Keynesians, and even influenced several efforts to construct alternative approaches to decision making, *e.g.*, Ellsberg (1962) and Shafer (1976), his own approach has not gained much traction among decision theorists and microeconomists. As Zappia (2005) pointed out, Shackle ignored the trend in modern decision theory inspired by Savage's subjective expected utility theory (hereafter, SEUT) because in Shackle's view, there was no fundamental difference in Savage's probabilistic construction of SEUT from decision theory based on the frequentist approach to probability.<sup>3</sup> However the trend ignored by Shackle pushed decision theory into variations of the EUT model with increased technical sophistication in order to solve 'anomalies' that violated an axiom or prediction of EUT such as the Allais paradox (Allais 1953, Machina 1983), the common ratio effect (Allais 1953, Kahneman and Tversky 1979) and the Ellsberg paradox (1961).<sup>4</sup> In this process, Shackle's decision theory seems to have been forgotten.

There were several attempts to formalize Shackle's approach with technical language such as Levi (1979, 1980), Ford (1983), Ponsonnet (1996). Other authors argued that certain technical aspects of Shacklean theory can be related to various currents of non-standard decision theory such as evidence theory (Shafer, 1976) in Fioretti (2001), and possibility theory (Zadeh, 1978) in Prade and Yager (1994) and Klir (2002). However, these investigations are mainly partial formalizations of Shackle's individual decision steps, or alluding similarity to other currents of decision theory. As far as translating Shackle's ideas into the communicable technical language of modern decision theory is concerned, it is not apparent that these results were generally accepted as a comprehensive and coherent formalization of Shacklean decision theory. As Gorgescu-Roegen (1958) argued, the lack of a clear and sound axiomatic structure has undermined the potentiality of Shackle's insights and prevented Shackle's model from obtaining the attention it deserves.

However, contrary to the tendency described above, there was a movement at the University of Massachusetts at Amherst during the last quarter of the 20<sup>th</sup> century to rehabilitate Shacklean ideas. This movement which included, at a fundamental level, a new rigorous formalization of Shackle's decision theory which, most importantly, was expressed in terms of the common language of modern

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<sup>3</sup> P. 36 in Shackle (1954)

<sup>4</sup> Roughly speaking, since the 1980s, EUT has been generalized in two distinctive but closely related ways: Rank Dependent Expected Utility Theory by Quiggin (1982), Yaari (1987) etc.; and Choquet Expected Utility Theory based on the concept of "capacity function" formalized by Gilboa (1987) and Schmeidler (1989).

decision theory. In particular, economists such as R. Bausor, J. Crotty, D. Vickers and D. W. Katzner (hereafter, Amherst Methodologists Group: AMG) focused on Shackle's insights in seeking an alternative theoretic framework in sharp contrast to that of mainstream economics. Vickers (1978, 1987, 1994) employed Shacklean ideas in understanding firm behavior and financial decisions. Bausor (1982-83, 1984) focused on the issue of kaleidics and historical time experienced by actual economic agents. Crotty (1994) saw Shacklean theory as a framework for explaining the behavior of economic agents confronting a period of uncertainty in which the conventions that drove behavior in the pre-existing order had broken down. Interacting with these developments, Katzner provided a totally reformalized version of Shacklean decision theory (Katzner 1986-7, 1987-88, 1989-90) and extended the range of its application to simultaneous behavior (Katzner 1995), the demand for money (Katzner 2001), firm behavior (Katzner 1990-91), and macroeconomic phenomena (Katzner 1998). This series of studies was combined and expanded in a single book (Katzner 1998). Since Katzner nearly completed the task of reformalizing Shackle's approach, the reformalized theory deserves to be called Shackle-Katzner decision theory (hereafter, SKDT)<sup>5</sup>. That work furnishes Shacklean theory new room for interaction with modern decision theory and mainstream economics. The present work focuses on the issues raised by the AMG and specifically by Katzner's contribution. The purpose of it is to provide a reformalized frameworks for SKDT in terms of the language of order and utility theory.

For this task, in section 2, first we will briefly summarize the whole process of decision-making described in SKDT. In section 3, we will investigate topological and order structure from which functional representations of potential surprise orderings can be derived. Shackle's potential surprise function as introduced by Katzner (1986-87, 1987-88, 1998) emerges from it. As the basis from which these functional representations are obtained, a new space of unbreakable hypotheses named *monad* and the space of the scale for potential surprise called *degree space* will be introduced.

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<sup>5</sup> It is worth noting that there has been another major contribution to discussions of uncertainty that is irreducible to probabilistic risk. This tradition is called Knightian decision theory. Although it may include many different currents of non-Savagean expected utility (Nishimura and Ozaki 2017), the representative idea was formalized by Bewley (2001). Recent work on the implication of Knightian uncertainty in a general equilibrium framework (Rigotti and Shannon 2005, Bewley 2011, Ma 2015) has been based on Bewley's formalization. In the Shacklean context, Katzner achieved similar results to Bewley.

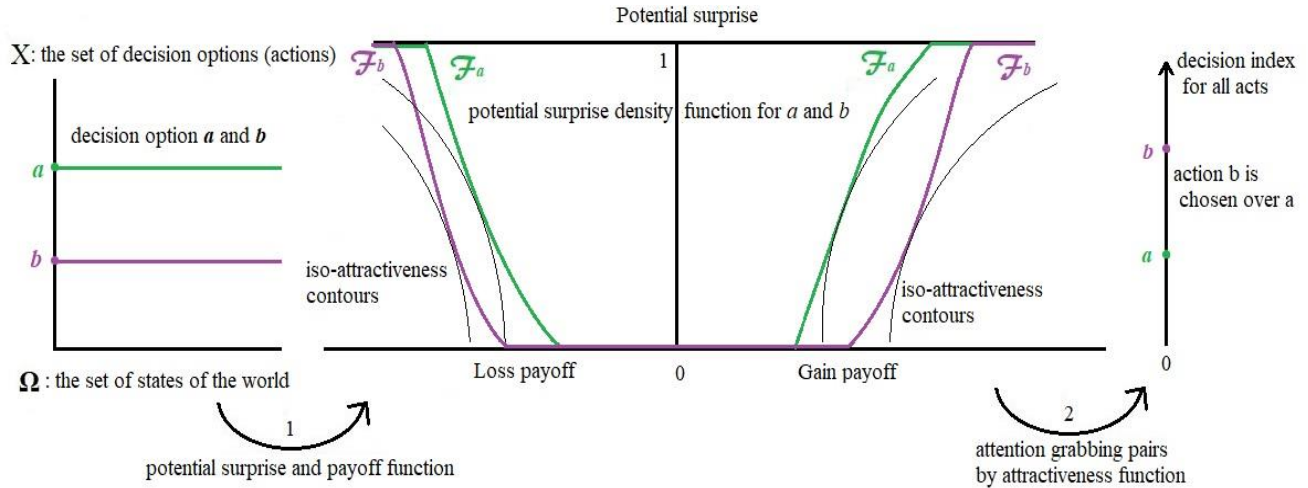
## 2. Choice under uncertainty in SKDT

In SKDT, decisions are made in a two-stage process. Encountering a problem of choice facing an uncertain future, the decision maker in stage 1 recognizes a set  $X$  of all available actions, a set  $\Omega$  of all imaginable future states of the world. A nonempty collection  $F^*$  of subsets of  $\Omega$  is called  $\sigma$ -field over  $\Omega$  if for any  $A$  in  $F^*$  and countable collection  $\{A_i \mid A_i \in F^*\}$ , it satisfies  $A^c$  in  $F^*$ ,  $\cup_i A_i$  in  $F^*$ , and  $\cap_i A_i$  in  $F^*$ . In Shackle's terminology, each element of  $F^*$  is called an *hypothesis*. Based on the recognition of available actions in  $X$  and hypotheses in  $F^*$ , the decision maker imagines (1) the degree of surprise he/she would feel now upon the future realization of an element of an hypotheses in  $F^*$  and (2) the future payoff summoned by the realization of a state of the world together with his/her chosen action from  $X$ . Here (1) is formalized as a potential surprise function defined on  $F^*$  into  $[0, 1]$  and (2) is represented in term of a payoff function (e.g. a utility or a profit function) defined on  $X \times \Omega$  to the real space  $R$ . This notion of potential surprise is the original concept conceived by Shackle (1954, 1969), and redefined in functional form by Katzner (1986-87, 1998).<sup>6</sup> The latter is a non-distributive, non-additive function, and its functional values indicate the degree of surprise called forth by the realization of a specific hypothesis in  $F^*$ .

At stage two, for each decision option  $x$  in  $X$  the decision maker is thought to focus on two pairs each consisting of a potential surprise and a payoff value that grabs his/her attention. One pair of a potential surprise and a payoff value is associated with possible "gains", the other pair with possible "losses." The determination of these pairs of values emerges from maximizing an attractiveness (ascendancy) function defined on  $[0, 1] \times R$  subject to a density function obtained from the potential surprise function. The pairs of values for each action are then evaluated in terms of a decision index or gambler preference map that expresses the decision maker's valuation on potential surprise and payoff. Here, it is worth noting that the attractiveness function excludes the less important baskets of payoff and potential surprise values generated by each action, and restricts the decision maker's focus to only the most attractive pairs of payoff and potential surprise values for each action. The decision index operates only on those selected pairs.

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<sup>6</sup> In Shackle (1969, p.79-85), a series of axiom was provided. For the rigorous functional form, see p. 46-59 of Katzner (1986-87, 1998).



[Figure 2.1]

Schematically, process of decision making in SKDT can be illustrated in Figure 2.1. In the left-hand diagram,  $a$  and  $b$  are two decision options in  $X$ . The diagram pointed to by curved arrow 1 indicates the determination of the two maximizing pairs of potential surprise and payoff values from decision options  $a$  and  $b$ . The domain of potential surprise function  $s$  is  $\Omega$ <sup>7</sup> and the domain of the payoff function  $u$  is the product space of the set of available acts  $X$  and the space of states of the world  $\Omega$ . This implies that the decision maker expresses the degree of uncertainty about states of the world in concrete numerical terms, and the options of choice in  $X$  do not influence the decision maker's valuation of potential surprise for each state of the world. The two wing-shaped curves in center diagram of the figure 2.1. indicates the potential surprise density functions  $\mathcal{F}_a$  and  $\mathcal{F}_b$  transposed to be defined over the conceivable possible payoffs resulting from actions  $a$  and  $b$  respectively. Iso-attractiveness curves for each act are elicited by attractiveness function, whose domain is the plane of payoff and potential surprise values, *i.e.*,  $\mathbf{R} \times [0, 1]$ . Here the locus of the potential surprise density function behaves as a constraint. Under such constraint, the attractiveness maximizing points for each act can be found at a tangency between a potential surprise locus and an iso-attractiveness contour. Among those points selected by the attractiveness maximizing process for each act, the decision maker makes comparisons (the right-hand diagram after arrow 2) according to their values in the decision index. Then, the pair with the maximal value in decision index determines which *act* is to be finally chosen.

<sup>7</sup> For composite hypotheses made of unions, negation and intersection of hypotheses, the domain is extended to the entire  $F^*$ .

### 3. Functional Representation of potential surprise ordering

In this section, we will formulate the potential surprise function in relation to underlying orderings. This will provide an alternative and illuminating way to think of potential surprise. Recall that  $\Omega$  is an incomplete set of states of the world and  $\mathbf{F}^*$  is a  $\sigma$ -field on  $\Omega$  containing uncountable unions and intersections of subsets of  $\Omega$ . In what follows, I will set out some of the basic ideas that are relevant and some of the fundamental propositions that can be proved in relation to the potential surprise function. Begin with the following additional definitions.

**Definition 3.1** For any  $A$  in  $\mathbf{F}^*$ , an hypothesis  $B$  in  $\mathbf{F}^*$  is called *rival* to  $A$  if  $A \cap B = \emptyset$ .

**Definition 3.2** An *exhaustive collection of rival hypotheses* is defined as a collection of hypotheses  $\{A_i\}$  such that:

- (i)  $A_1 = \emptyset$ .
- (ii)  $A_i$  is nonempty hypothesis for each  $i \neq 1$ .
- (iii) For all  $i \neq j$ ,  $A_i$  and  $A_j$  are rival hypotheses.
- (iv)  $\bigcup_i A_i = \Omega$ .

**Definition 3.3 Katzner (1998)** A potential surprise function on  $\mathbf{F}^*$  is a function  $s: \mathbf{F}^* \rightarrow [0, 1]$  such that:

- i) For all  $A$  in  $\mathbf{F}^*$ ,  $0 \leq s(A) \leq 1$ .
- ii) For any (possibly uncountable) collection  $\{A_i\}$  of nonempty subsets in  $\mathbf{F}^*$ ,  
 $s(\bigcup_i A_i) = \inf_i s(A_i)$ .
- iii) If  $\{A_i\}$  is an exhaustive set of rival hypothesis, then  $s(A_i) = 0$  for at least one  $i$ .

**Definition 3.4** Let  $s$  be a potential surprise function on  $\mathbf{F}^*$ . If an hypothesis  $A$  in  $\mathbf{F}^*$  has zero potential surprise value, *i.e.*  $s(A) = 0$ , then the hypothesis  $A$  is said to be *perfectly possible*. If  $s(A) = 1$ , then the hypothesis  $A$  is said to be *perfectly impossible*.

In addition, based on a concept opposite to that of potential surprise, the potential confirmation function is defined in SKDT as follows:

**Definition 3.3.A.** A potential confirmation function on  $\mathbf{F}^*$  is a function  $c: \mathbf{F}^* \rightarrow [0, 1]$  such that:

- i) For all  $A$  in  $\mathbf{F}^*$ ,  $0 \leq c(A) \leq 1$ .
- ii) For any collection  $\{A_i\}$  of nonempty subsets in  $\mathbf{F}^*$ ,  

$$c(\cup_i A_i) = \sup_i c(A_i).$$
- iii) If  $\{A_i\}$  is an exhaustive set of rival hypothesis, then  $c(A_i) = 1$  for at least one  $i$ .

The potential confirmation function was introduced by Katzner (1986-7, 1998). It is the analogue of probability in the SKDT context. The potential confirmation of an hypothesis  $A$  in  $\mathbf{F}^*$  is the degree of confidence the decision maker feels now were the future realization of an element of  $A$  to occur. Like potential surprise, potential confirmation is not founded on any knowledge of the future. It is therefore conceptually distinguished from probability. The potential confirmation function also has a different axiomatic base than the probability function. Because SKDT as originally presented is based on potential surprise, the remainder of this section is focused on it. We will return to potential confirmation at the end of the section. It should be noted here that in general, for all  $A$  in  $\mathbf{F}^*$ ,  $c(A) = 1 - s(A)$  does not necessarily hold.<sup>8</sup>

Our task here is reconstructing Katzner's potential surprise function above by finding exact mathematical conditions guaranteeing a representation of that function based on an underlying "original" ordering on  $\mathbf{F}^*$ . The reason for expressing potential surprise in terms of an underlying ordering is to provide additional insight into that concept in a more intuitive level. Although the usual concept of probability is reducible in terms of Lebesgue measure defined on the sample space, the construction of subjective probability has been established from the ordering relation defined in the decision space. Since the early development of EUT like Von Neumann and Morgenstein (1944), De Finetti (1936), and Anscombe and Aumann (1963), Savage (1954), the ordinal relation of probability is common starting point to derive the subjective probability and the expected utility. In addition, there is a large literature on *comparative probability* studying the ordering relation underlying probability which has added to an understanding of that concept (Fine 1973, Fishburn 1983a, 1983b) or qualitative probability theory (Krantz et al., 1971; Luce, 1967; Luce and Narens, 1978; Narens,

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<sup>8</sup> For detailed explanation, see Katzner (1998) p.62.

1980). Hence deriving the potential surprise function from an underlying order relation can constitute to an understanding of SKDT by giving more meaning to potential surprise.

However, before doing to this, we need to identify an appropriate subdomain  $F$  of  $F^*$  on which the initial order is defined. The extension of that order and its functional representation onto the full  $\sigma$ -field  $F^*$  will be conducted later.

Let  $\alpha \in \Omega$  be a state of the world. Then we denote an arbitrary nonempty subset of  $\Omega$  contained in  $F^*$  containing  $\alpha$  as  $A_\alpha$ , and the intersection of all such sets as  $A(\alpha) = \bigcap A_\alpha$ .

**Definition 3.5** A monad  $F$  on  $\Omega$  is a family of subsets of  $F^*$  having the form of  $F = \{A(\alpha) \neq \emptyset \mid \text{for each } \alpha \in \Omega\} \cup \{\emptyset\}$  where the intersection is taken over all sets  $A_\alpha$  for each  $\alpha \in \Omega$ .<sup>9</sup>

**Remark 3.6** For given  $\sigma$ -field  $F^*$ ,  $F$  is the collection of the smallest sets in  $F^*$  containing each  $\alpha$  in  $\Omega$ . Here the smallest subset of  $\Omega$  for each  $\alpha$  is uniquely determined. Suppose that we pick an arbitrary  $A(\alpha)$  from  $F$  with  $\alpha \neq \beta \in A(\alpha)$ , then we can show  $A(\alpha) = A(\beta)$ . Suppose not. Firstly, when  $A(\alpha) \subset A(\beta)$ . Then  $A(\beta)$  is not the smallest set including  $\beta$  because  $\beta$  in  $A(\alpha)$ . Secondly, when  $A(\alpha) \not\subset A(\beta)$  but any inclusion does not hold between  $A(\alpha)$  and  $A(\beta)$ , we can see that  $\beta \in A(\alpha) \cap A(\beta) \subset A(\beta)$ . This is contradiction because  $A(\beta)$  is not the smallest subset including  $\beta$ . Therefore  $A(\alpha) = A(\beta)$ . This means that each  $\alpha$  in  $\Omega$  corresponds to the unique element of  $F$ .

**Remark 3.7** For a given  $F^*$ ,  $F$  is uniquely determined. Conversely,  $F^*$  is also constructible from  $F$ . Each hypothesis in  $F^*$  is made of some union or complement or intersection of elements of  $F$ .

**Remark 3.8** The monad is the collection of all unit, that is to say, indecomposable hypotheses generating all composite hypotheses made by taking unions, intersections and complements among monad elements. Since no states of the world can be contained in  $\phi$ , this is interpretable as some unknown possibility beyond all current imaginable hypotheses listed in the monad. This is the reason why  $\phi$  is called as “*residual hypothesis*” in SKDT<sup>10</sup>.

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<sup>9</sup> It may seem that an exhaustive collection of rival hypothesis has a similar form to a monad  $F$ . But sets in an exhaustive collection of rival hypothesis is decomposable again into monad elements.

<sup>10</sup> See p.47 in Katzner (1998) and p. 49-50 in Shackle (1969).

**Remark 3.9** By the condition *ii*) of definition 3.3, it is easily verified that  $s(\Omega) = \inf_{A \in F} s(A)$ . But this does not imply that there exists an hypothesis  $A$  in  $F$  such that  $s(\Omega) = s(A)$ . For example, we can think  $\{s(A_i) | \text{for all } A_i \text{ in } F\}$  converging to  $s(\Omega) = \frac{\sqrt{2}}{100,000}$  but  $s(A_i) \subset \mathbf{Q} \cap [0,1]$ . This means that  $s(F) \subseteq s(F^*)$ . We will annihilate this kind of cases by introducing an appropriate assumption on mathematical property of the monad  $F$ .

### Representation of potential surprise ordering

Our task is to justify the existing conditions defining the potential surprise function in definition 3.3 by constructing an axiomatic system of the ‘potential surprise order’ defined on the set of hypotheses  $F^*$  rather than establishing a novel model describing the surprised response of a decision maker with respect to the realization of unexpected events. For this, we will refer to the existing outcomes of the theory of order and utility for the current context.

Suppose there is a binary relation  $\succsim_s$  on a monad  $F$  over  $\Omega$  and consider the following conditions:

**3.10** (Reflexive) For any  $A$  in  $F$ ,  $A \succsim_s A$ .

**3.11** (Transitive) For any  $A$ ,  $B$  and  $C$  in  $F$ ,  $A \succsim_s B$  and  $B \succsim_s C \Rightarrow A \succsim_s C$ .

**3.12** (Total) For any  $A$ ,  $B$  in  $F$ , either  $A \succsim_s B$  or  $B \succsim_s A$ .

**Definition 3.13** If a binary relation on  $F$  satisfies condition **3.10** and **3.11**, then it is called a *preorder of potential surprise on  $F$* . If a binary relation on  $F$  satisfies **3.10**, **3.11** and **3.12**, then it is called a *total preorder of potential surprise on  $F$* .

**Definition 3.14** For a subset  $D$  of the monad  $F$  and a total preorder  $\succsim_s$  of potential surprise on  $F$ , if  $A \succsim_s B$  and  $B \succsim_s A$  hold for any  $A$ ,  $B$  in  $D$ , then  $A$  and  $B$  are said to be *equivalent* and denoted by  $A \sim_s B$ . In addition, the subset  $D$  of  $F$  is said to be *an equivalence class in  $F$  over  $\sim_s$* , and the quotient set  $F/\sim_s$ , *i.e.* the set of all equivalence classes in  $F$  is called *the degree space of  $\succsim_s$* , and denoted by  $\bar{F} = F/\sim_s$ .

**Definition 3.15** If for any  $A$ ,  $B$  in  $F$ ,  $A \succsim_s B$  but not  $A \sim_s B$ , then it is called *strict total preorder of potential surprise on  $F$*  and denoted by  $\succ_s$ .

**Remark 3.16** Note that the equivalence class  $D$  of  $A$  and  $B$  over  $\sim_s$  is also denotable by  $[A]$  or  $[B]$ . Furthermore, for any distinct  $[A], [C]$  in  $\bar{F}$ , we can define the same total preorder  $\succsim_s$  of  $F$  onto  $\bar{F}$  as  $A \succsim_s C \Rightarrow [A] \succsim_s [C]$ . From now, the bracket  $[ ]$  will be omitted when the domain of order is specified as  $\bar{F}$ .

By the definition of equivalence class, the total preorder extended onto  $\bar{F}$  from  $F$  in remark 3.16 is antisymmetric. In other words, for any  $A, B$  in  $\bar{F}$  satisfying  $A \succsim_s B$  and  $B \succsim_s A$ ,  $A = B$  holds automatically. Then  $\succsim_s$  on  $\bar{F}$  is total order, *i.e.*, reflexive, transitive, total and antisymmetric.

**Definition 3.17** A total preorder  $\succsim_s$  of potential surprise on  $F$  is also called **a total order of potential surprise on  $\bar{F}$** .

Note that while elements of  $F, F^*$  are concrete hypotheses which can be ordered by the degree of potential surprise, the elements of  $\bar{F}$  can be understood as the degree itself of potential surprise. In actual reasoning process to make a decision, the degree of emotional response such as doubtfulness or confirmation on the realization of a possible future event is actually sensible by a decision maker in assigning some rank to each different hypothesis on possible future event. Of course, such degree is not measurable as a real number scale. We can see that the character of the set  $\bar{F}$  reflects this context.

Now consider an additional condition that permits the range of elements that can be compared under the relation  $\succsim_s$  to be extended to  $F^*$ . Before doing this, let's introduce the following definitions.

**Definition 3.18** If there exists a countable subset  $K$  of  $\bar{F}$ , such that for all  $[A],[B]$  in  $\bar{F} - K$  and there exists  $[C]$  in  $K$  such that  $[A] \succ_s [C] \succ_s [B]$ , then  $\bar{F}$  is called **order separable** and  $K$  is called **order dense** in  $\bar{F}$ .

**Definition 3.19** Let  $F$  be a monad defined on  $\Omega$  and  $\succsim_s$  be a total order of potential surprise on  $\bar{F}$ . For a subset  $K$  of  $\bar{F}$ , **a lower bound for  $K$**  is an element  $[A]$  in  $\bar{F}$ , for which  $[K] \succsim_s [A]$  for all  $[K]$  in  $K$ . If the set of all lower bound of  $K$  has the greatest element, then it is called **the greatest lower bound of  $K$**  or **infimum of  $K$**  and denoted by  $\inf K$ .

**3.20 (Reducibility)** For any hypothesis  $C \in F^*$  with  $C = \cup A_i$  where  $A_i$  in  $F$  for all  $i$  in some indexing set  $I$ ,  $[C] \sim_s \inf \{ [A_i] \text{ in } \bar{F} \mid i \text{ in } I \}$  holds.

As a simple illustration of reducibility for the finite case, for any event  $C \in \mathbf{F}^*$  with  $C = A \cup B$  where  $A, B \in \mathbf{F}$  and  $B \succsim_s A$ ,  $A \sim_s C$  holds. The intuition of reducibility is that as the coverage of future states in  $\Omega$  included in a union hypothesis is broader, the doubtfulness regarding the realization of that union hypothesis is as much low as the least doubtful hypothesis within it.

**Remark 3.21** It is trivial that the bigger a set in  $\mathbf{F}^*$  is, the lower the rank of the set in  $\mathbf{F}^*$  is. That is to say, for  $A, B$  in  $\mathbf{F}^*$ ,  $A \subset B \Rightarrow A \succsim_s B$ .

By virtue of the reducibility, we can extend the range of ordering on  $\mathbf{F}$  onto the entire  $\sigma$ -field  $\mathbf{F}^*$  whose elements can be constructed by unions of elements from Monad  $\mathbf{F}$ . Being different to the commodity space  $\mathbf{R}_+^n$  or the space of utility value  $\mathbf{R}$  in consumer theory,  $\mathbf{F}, \mathbf{F}^*$  and  $\bar{\mathbf{F}}$  are originated from the hypotheses of a decision maker on future events. Thus there is no guarantee that they have identical mathematical structures to the real space. Specifically, without imposing some mathematical restriction to  $\bar{\mathbf{F}}$  guaranteeing the existence of infimum, the axiom of reducibility over  $\succsim_s$  may fail. Then we are unable to exclude pathological cases like remark 3.9 because it is possible that the real value  $s(C) = \inf_i s(A_i)$  where  $C = \cup_i A_i, C \in \mathbf{F}^*$  and  $A_i \in \mathbf{F}$  as definition 3.3-ii) does not have its preimage in  $\mathbf{F}$ , i.e.  $s^{-1}(C) \notin \{A_i \in \mathbf{F} \mid s(C) = \inf_i s(A_i), C = \cup_i A_i\}$ . Thus, we cannot assign proper order of degree of potential surprise to the hypothesis  $C$ . To exclude mathematical complexity, we need to assume the greatest lower bound property of  $\bar{\mathbf{F}}$ , i.e., every non-empty subset of  $\bar{\mathbf{F}}$  with a lower bound has a greatest lower bound or infimum in  $\bar{\mathbf{F}}$ .

**3.22 (Completeness)**  $\bar{\mathbf{F}}$  has the greatest lower bound property.

Assuming completeness of  $\bar{\mathbf{F}}$  will be maintained through the end of current paper. The next definition is necessary in proving proposition 3.24 for the functional representation of total order of potential surprise on  $\bar{\mathbf{F}}$ .

**Definition 3.23** Let  $\succsim$  be a preorder on a set  $\mathbf{X}$ , and  $x, y$  elements of  $\mathbf{X}$ . The open interval  $(x, y) = \{z \in \mathbf{X} \mid y \succ z \succ x\}$  is called *a jump* if it is empty. Here,  $x$  and  $y$  are called *end points* of the jump  $(x, y)$ . Also,  $x$  is called an *immediate successor* of  $y$ , and  $y$  is an *immediate predecessor* of  $x$ .

While elements of  $\bar{F}$  means degrees of potential surprise so denotable by small letters a, b, c, elements of  $F, F^*$  are subsets of  $\Omega$  and usually denoted by A, B, C. In upcoming discussion, for simplicity, we will use small letters for elements of  $F, F^*$  when it is convenient.

**Proposition 3.24** *Functional representation of the total order of potential surprise on  $\bar{F}$*

Let  $\succsim_s$  be a total order of potential surprise on  $\bar{F}$ , i.e., a total preorder of potential surprise on  $F$ .

1) If  $\bar{F}$  is order separable, then there exists a real valued function  $s$  mapping  $F$  into  $\mathbf{R}$  representing  $\succsim_s$ , i.e.  $A \succsim_s B$  in  $F \Leftrightarrow s(A) \geq s(B)$ .

2) If  $\bar{F}$  is order separable and the reducibility holds, then there exists a real valued function  $s$  mapping  $F^*$  into  $\mathbf{R}$  representing  $\succsim_s$ , i.e.  $A \succsim_s B$  in  $F^* \Leftrightarrow s(A) \geq s(B)$ .

**Proof.**

1) Suppose that  $\bar{F}$  has a countable order dense subset  $K_1$ , and let  $K_2$  be the set of end points of all jumps of  $\bar{F}$ . As a claim, we will see  $K_2$  is countable. Let  $(x, y)$  be a jump in  $\bar{F}$ . Then by the definition of order separability,  $x \in K_1$  or  $y \in K_1$ . If  $x \notin K_1$ , then  $y$  is an immediate successor of  $x$ , and  $y \in K_1$ . If  $y \notin K_1$ , then  $x$  is an immediate predecessor of  $y$ , and  $x \in K_1$ . Since  $\succsim_s$  is total, there is 1-1 function from the set  $K_2$  of end points of all jumps into the countable set  $K_1$ . Then  $K_2$  is also countable.

Let  $K = K_1 \cup K_2 = \{k_1, k_2, k_3, \dots\}$  and define a function  $\delta: \bar{F} \times \bar{F} \rightarrow \{0, 1\}$  as follows.

$$\delta(x, y) = \begin{cases} 1 & \text{if } x \succ_s y \\ 0 & \text{otherwise} \end{cases} \text{ and define a function } r: \bar{F} \rightarrow \mathbf{R} \text{ as } r(a) = \sum_{n=1}^{\infty} \frac{\delta(a, k_n)}{2^n}.$$

For an arbitrary pair  $a, b$  with  $b \succsim_s a$  if for some  $n$ ,  $a \succ_s k_n$ , then  $b \succ_s k_n$ . Thus  $r(b) \geq r(a)$ .

Now let  $b \succ_s a$ . If  $(a, b)$  is a jump, then  $a \in F_2 \subset \bar{F}$ . Then there is some  $n$  with  $a = k_n$ , hence  $a \succ_s k_n$  does not hold. Thus  $r(b) > r(a)$ . When  $(a, b)$  is not a jump, then there exists  $c$  with  $b \succ_s c \succ_s a$ . If  $a \in K$ , then  $r(b) > r(a)$ . can be shown by the similar way just before. Suppose  $a \notin K$ . If  $c \in K$ , then  $c = k_n$  for some  $n$  where  $b \succ_s k_n$  but not  $c \succ_s k_n$ . Thus  $r(b) > r(a)$ . If  $c \notin K$ , then there exists  $k_n \in K$  such that  $c \succ_s k_n \succ_s a$  because  $a \notin K$  and the order separability. Thus  $b \succ_s k_n$  but  $a \succ_s k_n$  does not hold. Thus  $r(b) > r(a)$ .

Since the total order  $\succsim_s$  on  $\bar{F}$  is equivalent to the total preorder  $\succsim_s$  on  $F$ , the proposition is proven.

2) Pick an arbitrary  $A \in F^*$ . When  $A \in F$ , the existence of its functional representation preserving the total preorder on  $F$  was already shown in 1) of this proposition. If  $A \in F^* - F$ , then by the definition of Monad, there is a collection of hypotheses  $\{A_i \in F \mid A = \bigcup_i A_i \text{ for some indexing set } I\}$ . Then by reducibility,  $A \sim_s \inf \{A_i \in \bar{F} \mid A = \bigcup_i A_i\}$ . Also, by completeness of  $\bar{F}$ , there exists  $A_o \in \bar{F}$  such that  $A_i \succsim_s A_o$  for all  $i \in I$  and  $A_o \succsim_s B$  where  $B \in \bar{F}$  is a lower bound of  $\{A_i \in \bar{F} \mid A = \bigcup_i A_i\}$ . Then  $s(A_i) \geq s(A_o)$  and  $s(A_o) \geq s(B)$  where  $s(A_i) \geq s(B)$  for all  $i \in I$ . Hence  $s(A) = s(A_o) = \inf \{s(A_i)\}$ . In this manner, the order preserving functional representation of total preorder on  $F$  can be extended to the entire  $F^*$ . ■

Proposition 3-24 deals with only the existence of functional representations. In order to conduct conventional maximizing processes in the present context, it is necessary to have a continuous potential surprise function on. In consumer choice theory, in order to determine if a continuous utility representation of a consumer's preference ordering exists, it is enough to check the relative openness of upper and lower contour sets at points in the interior of the commodity space  $\mathbf{R}_+^n$ . However, since currently we are considering the space of hypotheses which may not be homeomorphic to Euclidean space, we need to consider topological conditions in general to ensure the existence of a continuous functional representation of potential surprise orderings.

**Definition 3.25** 1. Let  $\mathcal{R}$  be an arbitrary binary relation on  $F$  and  $\mathcal{R}'$  be a binary relation on  $F'$ . A function  $f$  is called an *order isomorphism* on  $F$  into  $F'$  if for any  $A, B$  in  $F$ ,  $A \mathcal{R} B$  if and only if  $f(A) \mathcal{R}' f(B)$ .

**Definition 3.26** Let  $(F, \succsim_s)$  be a totally preordered set, *i.e.*, reflexive, transitive and total. If for each  $A \in F$  the upper section  $[A, \infty) = \{B \mid B \succsim_s A\}$  and the lower section  $(-\infty, A] = \{B \mid A \succsim_s B\}$  are closed, then  $\succsim$  is said to be *continuous order*.

**Notation 3.27** Let  $F$  be a set and  $\mathcal{T}$  be a topology on  $F$ .<sup>11</sup> Then we will denote this topological space as  $(F, \mathcal{T})$ . If topology on  $F$  is not specified, we can simply denote  $F$  as a topological space in general.

**Definition 3.28** For a given topology  $\mathcal{T}$  on a set  $F$ , a **basis**  $\mathcal{B}$  for this topology  $\mathcal{T}$  is a collection of subsets of  $F$  (called basis element) such that *i*) for each  $x$  in  $F$ , there is at least one basis element  $B$  containing  $x$ , and *ii*) If  $x$  belongs to the intersection of two basis element  $B_1$  and  $B_2$ , then there is a basis element  $B_3$  containing  $x$  such that  $B_3 \subset B_1 \cap B_2$ .<sup>12</sup> A **subbasis**  $\mathcal{S}$  for the topology  $\mathcal{T}$  is a collection of subsets of  $F$  such that  $\mathcal{B} = \{B \mid B \text{ is the intersection of finitely many members of } \mathcal{S}\}$  is basis for  $\mathcal{T}$ .

**Definition 3.29** A topological space  $F$  is called **separable** if  $F$  has a countable dense subset; **second countable** if  $F$  has a countable basis for  $\mathcal{T}$ ; **connected** if  $F$  does not have any pairs of disjoint nonempty open subsets of  $F$  such that its union is  $F$ .

**Definition 3.30** Let  $\succsim$  be a total preorder on a set  $F$ . If the set of all strict upper section  $(A, \infty) = \{B \mid B \succ_s A\}$  and strict lower sections  $(\infty, x) = \{B \mid A \succ_s B\}$  is a subbasis for a topology on  $F$ , then this topology is called the **order topology**, and denoted by  $\mathcal{T}_{\succsim}$ .

**Definition 3.31** For arbitrary two topologies  $\mathcal{T}_1, \mathcal{T}_2$  on  $F$ , if  $\mathcal{T}_1 \supseteq \mathcal{T}_2$ , then  $\mathcal{T}_1$  is called finer than  $\mathcal{T}_2$  or  $\mathcal{T}_2$  is called coarser than  $\mathcal{T}_1$ .

Now we are ready to see the continuous representation of total order of potential surprise on  $\bar{F}$  and the extended one to  $F^*$ . The following two theorems are based on the standard outcomes of utility theory in Debreu (1964) and adjusted here for the context of the space of degree and hypotheses.

**Theorem 3.32** Let a degree space  $\bar{F}$  be a connected, separable topological space and  $\succsim_s$  a continuous total order of potential surprise on  $\bar{F}$ . If the condition 3.22 holds, then there exists a

<sup>11</sup> A topology on a set  $F$  is a collection  $\mathcal{T}$  of subsets of  $F$  having the following properties: 1)  $\emptyset$  and  $F$  are in  $\mathcal{T}$ . 2) The union of elements of any subcollection of  $\mathcal{T}$  is in  $\mathcal{T}$ . 3) The intersection of the elements of any finite subcollection of  $\mathcal{T}$  is in  $\mathcal{T}$ .

<sup>12</sup> We can define the topology  $\mathcal{T}$  generated by  $\mathcal{B}$  as follows: A subset  $U$  of  $F$  is said to be open in  $F$  if for each  $x \in F$ , there is a basis element  $B \in \mathcal{B}$  basis such that  $x \in B$  and  $B \subset U$ . Note that each basis element is itself an element of  $\mathcal{T}$ .

continuous real-valued order isomorphism  $s$  on  $\mathbf{F}^*$  to  $\mathbf{R}$  such that  $A \succ_s B \Leftrightarrow s(A) \geq s(B)$  and the condition *ii*) of definition 3.3 holds.

**Proof.**

The existence of functional representation on  $\bar{\mathbf{F}}$  and  $\mathbf{F}$  can be secured by Debreu (1964). Its extension to  $\mathbf{F}^*$  satisfying *ii*) of definition 3.3 can be shown as the proof for the 2) of proposition 3.24. ■

The connectedness and the separability of  $\bar{\mathbf{F}}$  in theorem 3.31 can be replaced by the second countability as the next theorem.

**Theorem 3.33** *Let  $\bar{\mathbf{F}}$  be a second countable topological space and  $\succ_s$  a continuous total order of potential surprise on  $\bar{\mathbf{F}}$ . If the condition 3.21 holds, then there exists a continuous real-valued order isomorphism  $s$  on  $\mathbf{F}^*$  to  $\mathbf{R}$  such that  $A \succ_s B \Leftrightarrow s(A) \geq s(B)$  and the condition *ii*) of definition 3.3 holds.*

**Proof.**

Let  $\mathbf{K} = \{K_1, K_2, K_3, \dots\}$  be a countable basis for the topology of  $\bar{\mathbf{F}}$ . For each  $a \in \bar{\mathbf{F}}$ , let  $L(a) = \{n \mid a \succ_s x, \text{ for all } x \in K_n\}$  and define a function  $r: \bar{\mathbf{F}} \rightarrow \mathbf{R}$  as  $r(a) = \sum_{n \in L(a)} \frac{1}{2^n}$ . If  $b \succ_s a$ , then  $L(a) \subseteq L(b)$ . Thus  $r(b) \geq r(a)$ . If  $b \succ_s a$ , then  $a \in (-\infty, b)$ . So there exists  $n$  such that  $a \in K_n \subset (-\infty, b)$ . Thus  $n \in L(b) - L(a)$  and  $r(b) > r(a)$ . The extension of order isomorphism to  $\mathbf{F}^*$  satisfying the condition *ii*) of definition 3.3 can be done by similar way so far. ■

Although the previous two theorems reflect standard results of topological generalization regarding order isomorphism, it seems unclear to expect topological properties of the hypothesis space  $\mathbf{F}^*$  and the degree space  $\bar{\mathbf{F}}$ . The following theorem presents looser condition than the previous two theorems.

**Theorem 3.34** *Suppose that  $\bar{\mathbf{F}}$  is order separable with respect to a total order  $\succ_s$  of potential surprise on  $\bar{\mathbf{F}}$ . If an arbitrary topology  $\tau$  of  $\bar{\mathbf{F}}$  is finer than its order topology  $\tau_{\succ}$  and the condition 3.22 holds, then there exists a continuous real-valued order isomorphism  $s$  on  $\mathbf{F}^*$  to  $[0, 1]$  such that the condition *ii*) of definition 3.3 holds.*

The proof of theorem 3.34 can be easily verified as following. When a topology of  $\bar{\mathbf{F}}$  is just order topology or finer than it, once if any two different degrees of potential surprise are comparable by

another degree between them, then such degrees can be represented by real numbers. So, the only information we need to discern is whether a topology on the degree space is finer than order topology. In fact, it is well-known that the order topology on a continuous total preordered space is the coarsest one. Thus theorem 3.34 is generalized statement enough to deal with arbitrary sorts of topologies given to the degree space.

So far, we have obtained the functional representation of a potential surprise ordering on  $F^*$  which satisfies only conditions *ii)* of definition 3.3. Now, to derive *i)* and *iii)* of definition 3.3, we need a condition that guarantees the existence of subset of  $\Omega$  having zero potential surprise value.

**Definition 3.35** Let  $F$  be a topological space and  $\succcurlyeq$  be a preorder on  $F$ . A subset  $K$  of  $F$  is called **decreasing** if for all  $x, y$  in  $F$ ,  $x$  in  $K$  and  $x \succcurlyeq y$  implies that  $y$  in  $K$ , **increasing** if  $x$  in  $K$  and  $y \succcurlyeq x$  implies that  $y$  in  $K$ .

**Definition 3.36** A topological space  $(F, \tau)$  is **normal** if for each pair  $K_1, K_2$  of disjoint closed subsets of  $F$ , there exist two disjoint open subsets  $M_1$  and  $M_2$  such that  $K_1 \subset M_1$  and  $K_2 \subset M_2$ .

By the theorem 3.38 below, the normality of monad  $F$  can guarantee the existence of a perfectly possible hypothesis in *iii)* of the definition 3.3. From this, we introduce another axiom for  $F$ .

**3.37 (Normality)** A monad  $F$  is normal.

### **Theorem 3.38 Nachbin Separation Theorem**

*Suppose that a monad  $F$  is a normal space and has a total preorder. If  $K_0$  and  $K_1$  are disjoint closed subsets of  $F$  such that  $K_0$  is decreasing and  $K_1$  is increasing, then  $K_0$  and  $K_1$  can be separated by a continuous monotone<sup>13</sup> function  $s: F \rightarrow [0, 1]$  such that  $s(K_0) = \{0\}$  and  $s(K_1) = \{1\}$ .<sup>14</sup>*

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<sup>13</sup> A function  $f$  from a preordered  $F$  to  $[0, 1]$  is called monotone increasing if for all pair  $a, b$  in  $F$  with  $a \succcurlyeq b$ ,  $f(a) \geq f(b)$  holds, and monotone decreasing if  $f(a) \geq f(b)$ . If a function  $f$  is monotone increasing or decreasing, then  $f$  is called monotone.

<sup>14</sup> For proof, see p.26 of Nachbin (1950).

Now by virtue of the normality, we can inject degrees of potential surprise right into  $[0, 1]$  and get some hint on the shape of potential surprise locus in the figure 2.1. Intuitively,  $\mathbf{K}_o$  is the subregion of the monad  $\mathbf{F}$ , which is interpretable as the set of “perfectly possible” hypotheses, *i.e.*, those satisfying  $s(A) = 0, A \in \mathbf{F}$ . And  $\mathbf{K}_I$  is the set of “perfectly impossible” hypothesis satisfying  $s(B) = 1, B \in \mathbf{F}$ . From theorem 3.38, between two extreme potential surprise values 0 and 1, the potential surprise value of each hypothesis is assigned, as Nachbin’s theorem says, through a monotone pattern. However, to get exact shape of potential surprise locus, we need to consider the relation between the potential surprise function and the payoff function. If the relation between potential surprise and payoff information is specified appropriately, then the shape of potential surprise locus can be drawn as in Figure 2.1.

### Potential surprise values of $\Omega$ and $\phi$

It remains to determine the function values assigned to  $s(\Omega)$  and  $s(\phi)$ . Since  $\Omega$  is the biggest set containing every element  $A$  of the monad  $\mathbf{F}$ , it follows that  $A \succ_s \Omega$  and  $s(\Omega) = \inf_{A \in \mathbf{F}} s(A)$  holds. Since the condition *iii*) of definition 3.3 was confirmed by the normality  $\mathbf{F}$  of theorem 3.38,  $s(\Omega) = 0$ . In other words, the decision maker trusts the current range  $\Omega$  of imaginable future states as perfectly possible.

On the other hand, the order of  $\phi$  is assigned initially by the total preorder  $\succ_s$  of potential surprise on the monad  $\mathbf{F}$ . As an example, suppose  $A \succ_s \phi$  for an arbitrary  $A$  in  $\mathbf{F}^*$ . Then, by theorems 3.24, 3.32, 3.33 or 3.34, we have  $s(A) > s(\phi)$ . However, since both  $A$  and  $\phi$  are elements of monad  $\mathbf{F}$ , in the current functional representation we get  $s(A) = s(A \cup \phi) = \inf\{s(A), s(\phi)\} = s(\phi)$ . This contradicts  $A \succ_s \phi$ . But if we exclude  $\phi$  in the definition of monad in order to derive the condition *ii*) of definition 3.3 with keeping the word ‘non-empty’, then we do not know how to assign the function value of  $S$  to  $\phi$  in  $\mathbf{F}^*$  because any information about the rank of  $\phi$  in the initial total preorder  $\succ_s$  on  $\mathbf{F}$  is not available. If we try to assign the rank of  $\phi$  separately, basically it is indifferent to the initial definition of monad, *i.e.*, including the residual hypothesis  $\phi$ . So, let’s introduce the following axiom.

**3.39 (self-confidence)** For any hypotheses  $A$  in  $\mathbf{F}^*$ ,  $\phi \succ_s A$ .<sup>15</sup>

This means that the residual hypothesis, that is to say, any hypothesis implying the realization of some currently unimaginable states is the most doubtful in comparison to any other currently imaginable hypotheses in  $\mathbf{F}^*$ . Intuitively, this is coherent to  $s(\Omega) = 0$  because both implies that a decision maker thinks the future state will be realized out of the current imaginable list.

**Definition 3.40** For a given  $\sigma$ -field  $\mathbf{F}^*$  over  $\Omega$ , if its monad  $\mathbf{F}$  is normal, the degree space  $\bar{\mathbf{F}}$  is complete, the binary relation  $\succ_s$  on  $\mathbf{F}$  is reflexive, transitive, total, and its extended binary relation to  $\mathbf{F}^*$  is reducible, then  $\succ_s$  is called the *potential surprise order* on  $\mathbf{F}^*$ .

Note that, in the previous definition, the anti-symmetric property of  $\bar{\mathbf{F}}$  is automatically satisfied because of the property of equivalence class  $\bar{\mathbf{F}} = \mathbf{F}/\sim_s$ . Thus  $\succ_s$  on  $\bar{\mathbf{F}}$  is total order.

**Theorem 3.41** Suppose that a monad  $\mathbf{F}$  is normal and  $\bar{\mathbf{F}}$  is order separable with respect to a potential surprise order on  $\mathbf{F}^*$ . If an arbitrary topology  $\mathcal{T}$  of  $\bar{\mathbf{F}}$  is finer than its order topology  $\mathcal{T}_{\succ_s}$ , then there exists a continuous potential surprise function  $s$  into  $[0, 1]$  preserving  $\succ_s$  on  $\mathbf{F}^*$ .

The theorem 3.41 is the restatement of theorem 3.34 with the new definition of potential surprise order on  $\mathbf{F}^*$  because the normality is a premise of definition 3.40.

**Potential confirmation order** on  $\mathbf{F}^*$  is defined in parallel to those of potential surprise. For a given  $\sigma$ -field  $\mathbf{F}^*$  over  $\Omega$ , if its monad  $\mathbf{F}$  is normal, the degree space  $\bar{\mathbf{F}}$  is complete, the binary relation  $\succ_c$  on  $\mathbf{F}$  is reflexive, transitive, total, and its extended binary relation to  $\mathbf{F}^*$  is reducible, then  $\succ_c$  is called the *potential confirmation order* on  $\mathbf{F}^*$ . Here the reducibility and self-confidence can be redefined as the followings.

- **Reducibility.** For any hypothesis  $C$  in  $\mathbf{F}^*$  with  $C = \cup A_i$  where  $A_i \in \mathbf{F}$  for all  $i$  in some indexing set  $I$ ,  $[C] \sim_c$  supremum or least upper bound of  $\{ [A_i] \text{ in } \bar{\mathbf{F}} \mid i \text{ in } I \}$ .

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<sup>15</sup> This condition is similar to nonnegativity and nondegeneracy out of five axioms of qualitative probability relation by De Finetti (1964) and Savage (1972).

- **Self-confidence.** For any hypotheses  $A$  in  $\mathbf{F}^*$ ,  $A \succ_c \phi$ .

The previous theorems for potential surprise function can be proved for potential confirmation function with the obvious modifications. The analogue of Theorem 3.41 in the potential confirmation context becomes as following:

**Theorem 3.42** *Suppose that a monad  $\mathbf{F}$  is normal and  $\bar{\mathbf{F}}$  is order separable with respect to a potential confirmation order on  $\mathbf{F}^*$ . If an arbitrary topology  $\mathcal{T}$  of  $\bar{\mathbf{F}}$  is finer than its order topology  $\mathcal{T}_{\succ_c}$ , then there is a continuous potential confirmation function  $c$  into  $[0, 1]$  preserving  $\succ_c$  on  $\mathbf{F}^*$ .*

#### 4. Closing remarks

In this paper, we have constructed an axiomatic system of ordering relation defined on the  $\sigma$ -field  $\mathbf{F}^*$  as the space of hypotheses in order to derive the functional form of potential surprise which was formalized by Katzner. For this task, we introduced monad  $\mathbf{F}$  as a kind of basis space of  $\mathbf{F}^*$  spanning all imaginable hypotheses by union operations in  $\mathbf{F}$ . In addition, to investigate necessary topological conditions, we introduced the degree space  $\bar{\mathbf{F}}$ , a quotient space of total preorder on monad  $\mathbf{F}$ . Potential surprise order and its functional representation on  $\mathbf{F}^*$  were constructed on such settings. From the outcomes of this paper, Shacklean potential surprise has also obtained broader theoretical spectrum to back up the concept.

However, the distinct features of Shacklean decision theory are not restricted to introducing just an alternative measure of subjective uncertainty, but also including noble way how to deal with such uncertainty measure in conjunction with the payoff information. Specifically, the payoff space in Shacklean decision theory is divided by the qualitative difference between gain and loss, so that it requires its corresponding functional steps to reflect the attitude of a decision maker with respect to diverging anticipation between potential gain and loss. While potential surprise negates probability, the next steps will be to replace the calculation of expected utility. This is the next research agenda.

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