On the Iwasawa Invariants of Nonordinary Modular Forms

ON THE IWASAWA INVARIANTS OF NONORDINARY MODULAR FORMS

A Dissertation Presented

by

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Dedication

In memory of Karen and Richard Gajek.

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ABSTRACT

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We extend known results on the behavior of Iwasawa invariants attached to Mazur-Tate elements for p-nonordinary modular forms of weight $k = 2$ to higher weight modular forms with $a_p = 0$. This is done by using a decomposition of the p-adic L-function due to R. Pollack in order to construct explicit lifts of Mazur-Tate elements to the full Iwasawa algebra. We then study the behavior of Iwasawa invariants upon projection to finite layers, allowing us to express the invariants of Mazur-Tate elements in terms of those coming from plus/minus p-adic L-functions. Our results combine with work of Pollack and Weston to relate the plus/minus and sharp/flat Iwasawa invariants attached to congruent pairs of modular forms at weights $p + 1$ and 2, respectively.

NOTATION

The labeling of theorems within the same chapter is 'section.number' and within different chapters is 'chapter.section.number'. For example, the proposition labelled 2.4.1 in Chapter II will be referred to as 'Proposition 2.4.1' within Chapter II itself, but will be called 'Proposition II.2.4.1' when referenced from other chapters.

If K is a field we write $\bar{K}$ for a fixed separable closure of K and $G_K = \mathrm{Gal}(\bar{K}/K)$. (All fields appearing in this thesis are perfect, thus there is no harm in replacing 'separable' with 'algebraic' in the previous sentence.) If K is a number field then we write $\mathcal{O}_K$ for the ring of integers of K. If K is a valued field with valuation v then by the *valuation ring of* K we mean all elements $x \in K$ with valuation $v(x) \geq 0$. For each prime p, let ord_p denote the unique valuation on $\bar{\mathbf{Q}}_p$ satisfying $\mathrm{ord}_p(p) = 1$. Write $\mathbf{C}_p$ for the completion of $\bar{\mathbf{Q}}_p$ with respect to the non-archimedean absolute value $|\cdot|_p$ corresponding to ord_p.

Fix throughout this thesis embeddings $\iota_p : \bar{\mathbf{Q}} \hookrightarrow \bar{\mathbf{Q}}_p$. For any number field $K/\mathbf{Q}$, the embedding ι_p determines a prime $\mathfrak{p}_K$ of K lying over p, which we usually call 'the prime of K corresponding to our fixed embedding $\bar{\mathbf{Q}} \hookrightarrow \bar{\mathbf{Q}}_p$'. (The valuation on $\bar{\mathbf{Q}}$ given by $v = \mathrm{ord}_p \, \iota_p$ makes $\bar{\mathbf{Q}}$ into a valued field with accompanying valuation ring $\mathcal{O}_v$ and maximal ideal $\mathfrak{p} \subseteq \mathcal{O}_v$. Now $\mathfrak{p} \cap O_K = \mathfrak{p}_K$ is a prime over p.)

For odd primes p, let $\omega : (\mathbf{Z}/p\mathbf{Z})^\times \to \mathbf{Z}_p^\times$ denote the odd character sending a mod p to the unique $(p-1)$st root of unity $\omega(a) \in \mathbf{Z}_p^\times$ satisfying $\omega(a) \equiv a \mod p$. By precomposing with reduction modulo p, we also view ω as a character on $\mathbf{Z}_p^\times$

and define $\langle\cdot\rangle : \mathbf{Z}_p^\times \to 1 + p\mathbf{Z}_p$ by $x \mapsto x\omega^{-1}(x)$.

For a congruence subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbf{Z})$, write $S_k(\Gamma)$ for the space of cuspidal modular forms $f(z) = \sum_{n\geq 1} a_n(f)q^n$, $q = e^{2\pi iz}$, of weight k and level Γ. For a modular form $f(z)$ and $\gamma = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \Gamma$ we write $(f|_k\gamma)(z) = (\det\gamma)^{k-1}(cz + d)^{-k}f(\gamma(z))$ for the usual slash operator on modular forms. For integers $N \geq 1$, write

$$\Gamma_0(N) = \{\gamma \in \mathrm{SL}_2(\mathbf{Z}) \mid \gamma \equiv \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \bmod N\},$$
$$\Gamma_1(N) = \{\gamma \in \mathrm{SL}_2(\mathbf{Z}) \mid \gamma \equiv \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right) \bmod N\}.$$

For $n \geq 1$ and $d \in (\mathbf{Z}/N\mathbf{Z})^\times$, let T_n and $\langle d\rangle$ denote the classical Hecke and diamond operators on modular forms, as defined in [8, Chapter 5]. For a Dirichlet character $\varepsilon : (\mathbf{Z}/N\mathbf{Z})^\times \to \mathbf{C}^\times$, let $S_k(N, \varepsilon) \subseteq S_k(\Gamma_1(N))$ denote the ε-eigenspace consisting of all cuspforms satisfying $\langle d\rangle f = \varepsilon(d)f$ for all $d \in (\mathbf{Z}/N\mathbf{Z})^\times$. We say that $f \in S_k(\Gamma_1(N))$ is an *eigenform* if f is an eigenvector for all Hecke and diamond operators and we say f is *normalized* if $a_1(f) = 1$. A normalized eigenform $f \in S_k(\Gamma_1(N))$ is called a *newform* if its Fourier coefficients at primes $p \nmid N$ do not arise as Fourier coefficients of modular forms at levels properly dividing N. For a normalized eigenform f, we write $K_f/\mathbf{Q}$ for the number field $\mathbf{Q}(\{a_n(f)\}_{n\geq 1})$.

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CHAPTER 1

INTRODUCTION

It has been known since at least the time of Euler that the classical Riemann zeta function, defined by $\zeta(s) = \sum n^{-s}$ on the halfplane $\Re(s) > 1$, takes on rational values at the negative integers (see [14, §2.1] for an overview of Euler's method). In the mid 19th century, while attempting to prove Fermat's Last Theorem, Kummer noticed an interesting relationship between the rational numbers $\zeta(-n) \in \mathbf{Q}$, $n \geq 0$, and the arithmetic structure of the fields $\mathbf{Q}(\zeta_p)$, where $\zeta_p \in \mathbf{C}$ is a primitive pth root of unity: if p appears in the numerator of one of the rational numbers $\zeta(-n)$ then the field $\mathbf{Q}(\zeta_p)$ does not have class number one.[1] The smallest such prime is 37, whose appearance in the value

$$\zeta(-31) = \frac{37 \cdot 683 \cdot 305065927}{2^6 \cdot 3 \cdot 5 \cdot 17}$$

tells us about (the failure of) prime factorization in the ring $\mathbf{Z}[\zeta_{37}]$.

This property – known as *Kummer's criterion*[2] – is an early result in what is now called *Iwasawa theory*, a beautiful topic bridging the complex analytic theory

[1] The *class number* is an invariant attached to a number field which is a rough measure of the failure of unique factorization in its ring of integers. If the field $\mathbf{Q}(\zeta_p)$ has class number larger than one then, unlike in the ring $\mathbf{Z}$, one cannot uniquely describe elements of $\mathbf{Z}[\zeta_p]$ by factoring into primes.

[2] In fact, Kummer's criterion is stronger than what we have stated here.

of L-functions (e.g., the Riemann zeta function) and the behavior of certain Galois modules (e.g., class groups of number fields, Selmer groups of elliptic curves). Iwasawa theory therefore comes in two flavors: there is an 'analytic' side, dealing with *p-adic L-functions* (analytic functions on the open p-adic unit disc interpolating classical L-values), and an 'algebraic' side, dealing with the asymptotic structure of modules X (such as those mentioned in the previous sentence) over the group algebra $\Lambda = \mathbf{Z}_p[[\Gamma]] \cong \mathbf{Z}_p[[T]]$, where Γ is a Galois group isomorphic to $\mathbf{Z}_p$. The analyticity of p-adic L-functions allow them to be viewed as power series $L_p(T) \in \bar{\mathbf{Q}}_p[[T]]$. Under nice enough conditions, one can define invariants:

- μ^{alg} and λ^{alg} measuring the $\mathbf{Z}_p$-torsion and rank of X, respectively.
- μ^{an} and λ^{an} measuring the divisibility of $L_p(T)$ by p and its number of roots as a function on the open p-adic unit disc, respectively.

So-called 'main conjectures' in Iwasawa theory (theorems in many cases – see [16], [23], [26], [37], [42]) predict that $\mu^{alg} = \mu^{an}$ and $\lambda^{alg} = \lambda^{an}$. Statements such as these allow us to translate analytic data coming from L-functions to arithmetic data on the algebraic side. For example, the $p = 37$-adic L-function associated to a certain Dirichlet character of modulus 37 has $\mu^{an} = 0$ and $\lambda^{an} = 1$ (see [15]). Using the classical 'Iwasawa main conjecture' in this setting (a theorem of Mazur-Wiles [26]), this implies that 37 divides the class number of *all* fields in the tower $\mathbf{Q}(\zeta_{37^n})$, $n \geq 0$, thereby extending the result of Kummer discussed above.

A modular form f is an analytic function on the complex upper half-plane satisfying certain symmetries (including periodicity) with respect to fractional linear transformations. One can use the coefficients appearing in the Fourier series $f(z) = \sum a_n q^n$, $q = e^{2\pi i z}$, of a modular form to define a complex L-function for f. If f is a cuspidal eigenform then after appropriate (and nontrivial) scaling, one can

p-adically interpolate certain algebraic values of this L-function to obtain a p-adic L-function $L_p(f)$ (see Chapter IV for an outline).

In contrast to the p-adic L-functions encountered in classical Iwasawa theory, the p-adic L-function of a modular form need not have p-adically bounded coefficients nor finitely many roots on the open unit disc (see [30] or [21] for examples). The analytic Iwasawa invariants in such cases are therefore not well-defined. This complication can be avoided by restricting to *p-ordinary*[3] modular forms, but in the *p-nonordinary* case it is conjectured – in fact, known in many cases – that one always has infinitely many roots. Amazingly, even in the nonordinary setting there are ways to get at a finite number of 'interesting roots'. One way to do this is by cleverly decomposing the p-adic L-function in order to separate the (infinitely many) 'trivial' zeroes from the (finitely many) zeroes with arithmetic significance. This was done for eigenforms $f \in S_k(N, \varepsilon)$ with $a_p = 0$ by Pollack [30], where he defined functions $L_p^{\pm}(f)$ (finitely many zeros) and $\log_{p,k}^{\pm}$ (infinitely many zeros) such that if α is a root of the polynomial $X^2 + \varepsilon(p)p^{k-1}$ then

$$L_p(f) = L_p^+(f)\log_{p,k}^+ + \alpha L_p^-(f)\log_{p,k}^- . \tag{1.1}$$

Other decompositions also exist – see [38] and [21], for example.

Another way to define Iwasawa invariants in the p-nonordinary setting is to consider Mazur-Tate elements – so-called 'finite-layer' analogues of p-adic L-functions. Mazur-Tate elements $\theta_n(f)$ reside in the nth-layer quotient $\Lambda_n = \Lambda/((1+T)^{p^n} - 1)$ of the classical Iwasawa algebra and, like p-adic L-functions, they interpolate critical values of the complex L-function of f. In the ordinary case, one can show that (under some assumptions) the Iwasawa invariants coming from Mazur-Tate elements agree with those of the associated p-adic L-function. In the nonordinary

[3] Assuming $a_p \in \mathbf{Z}$, ordinary and nonordinary refer to when $p \nmid a_p$ and $p \mid a_p$, respectively.

case, we only have a complete picture for weight two modular forms: results of Kurihara [19] and Perrin-Riou [29] show that if f is p-nonordinary of weight $k = 2$ then

$$\lambda\big(\theta_n(f)\big) = q_n + \lambda^{\pm}(f), \qquad n \gg 0, \tag{1.2}$$

where q_n is an explicit finite sum (unbounded in n) of powers of p and the constants $\lambda^{\pm}(f) \geq 0$, whose sign depends on the parity of n, are related to invariants coming decompositions of the p-adic L-function. Thus, in both the ordinary case (of any weight $k \geq 2$) and the nonorindary case (at weight $k = 2$), the sequence of Mazur-Tate elements yields information about the p-adic L-function.

In this thesis, we consider whether a similar relationship holds for higher weight modular forms, where only partial results are currently known (see [35]). In order to use Pollack's concrete decomposition (1.1) of the p-adic L-function, we focus on higher weight modular forms with $a_p = 0$ and show that, as in the weight two case, the sequence of Mazur-Tate elements can be used to recover the Iwasawa invariants attached to the p-adic L-function. A special case of our main theorem (compare with Theorem VII.7.2.1) is as follows.

Theorem. *Let f be a newform of weight k and level $\Gamma_1(N)$. Assume p is an odd prime not dividing N and that $a_p(f) = 0$. There are explicit constants $\nu_k^{\pm}, \iota_k \in \mathbf{Z}_{\geq 0}$, depending only on p and k, such that if*

(i) $k \not\equiv p + 2 \bmod (p^2 - 1)$, or

(ii) $k \equiv p + 2 \bmod (p^2 - 1)$ and $\lambda\big(L_p^{\pm}(f)\big) < \nu_k^{\pm}$,

then for $n \gg 0$ we have

$$\begin{aligned} \mu\big(\theta_n(f)\big) &= \mu\big(L_p^{\pm}(f)\big) + \iota_k \\ \lambda\big(\theta_n(f)\big) &= (k-1)q_n - \iota_k(p^n - p^{n-1}) + \lambda\big(L_p^{\pm}(f)\big), \end{aligned}$$

where $\pm = \operatorname{sgn}(-1)^{n+1}$.

The constant ι_k vanishes at $k = 2$, thus the above result is a natural extension of (1.2). We also show the following:

(a) (Proposition IV.4.3.1 and Remark IV.4.3.9) The p-adic valuation of the algebraic parts of the complex L-function for modular forms f as in the theorem can be computed explicitly in terms of Mazur-Tate elements.

(b) (Corollary VII.7.3.1) The difference between the Iwasawa invariants of Mazur-Tate elements for pairs of newforms (both with $a_p = 0$) with congruent weights modulo $p^2 - 1$ are stable over n of fixed parity.

(c) (Corollary VII.7.3.9) The plus/minus invariants of a newform f with $a_p(f) = 0$ and weight $k = p + 1$ can be computed in terms of *chromatic* Iwasawa invariants attached to a weight two modular form g that is congruent to f modulo p.

(d) (Proposition VII.7.2.15) Under some assumptions on $L_p^{\pm}(f)$, the behavior of Mazur-Tate elements (for f as in the theorem) can also be determined in the case $k \equiv p + 2 \bmod (p^2 - 1)$ and $\lambda\big(L_p^{\pm}(f)\big) \geq \nu_k^{\pm}$.

The proof of the theorem above is roughly as follows. First, we extend a result of Pollack [30] and construct lifts of Mazur-Tate elements to the full Iwasawa algebra. These lifts are explicit – they are built out of certain products of cyclotomic polynomials and incorporate $L_p^{\pm}$ – and, in particular, their Iwasawa invariants are easy to compute. We then consider the behavior of Iwasawa invariants upon projection to layer n, the idea being to compute the invariants of the Mazur-Tate elements in terms of the known invariants of the lifts. This step involves a careful study of the p-adic properties of remainders of power series upon division by the polynomial $(1+T)^{p^n} - 1$. The rest of the proof is broken into two parts: weights $2 \leq k \leq p+2$,

where we show that the Iwasawa invariants of the lifts are stable under projection, and weights $k \geq p+3$, where the invariants of the lifts are no longer stable under projection.

The structure of this thesis is as follows:

Chapter II. We review the definition of classical Iwasawa invariants and give two (equivalent) characterizations of Iwasawa invariants at finite layers. We also briefly discuss the asymptotic behavior of these invariants upon projection to layer n.

Chapter III. We follow Pollack-Weston [35] and construct Mazur-Tate elements for modular symbols. We slightly generalize their approach in order to obtain Mazur-Tate elements for each critical L-value of a modular form.

Chapter IV. We briefly review the construction of p-adic L-functions via locally analytic p-adic distributions and summarize Pollack's decomposition of the p-adic L-function for modular forms with $a_p = 0$. The remainder of the chapter focuses on proving (a) above.

Chapter V. This short chapter contains the construction of our lifts of Mazur-Tate elements to the full Iwasawa algebra.

Chapter VI. We study the behavior of Iwasawa invariants upon projection to layer n for a certain class of power series. This chapter is the logical continuation of Chapter II.§2.4 but from a somewhat different perspective.

Chapter VII. This chapter contains all of our main results. We begin with a summary of known results on the Iwasawa invariants of Mazur-Tate elements. We then combine the results of Chapters V and VI to prove (a more general

version of) the theorem stated above. The rest of the chapter focuses on establishing (b)-(d).

Appendix A. We briefly outline our method for computing Mazur-Tate elements.

Appendix B. We provide tables containing the invariants $\lambda(\theta_n(f))$ and $\lambda(L_p^{\pm}(f))$ at $p = 3, 5, 7$ for various newforms f of weights $2 \leq k \leq 14$ with $a_p(f) = 0$.

CHAPTER 2

IWASAWA INVARIANTS

2.1 Classical Iwasawa invariants

Fix a prime p and let K be a finite extension of $\mathbf{Q}_p$ with valuation ring $\mathcal{O} \subseteq K$. Choose a uniformizer $\varpi \in K$ and write $\Lambda = \mathcal{O}[[T]]$. Associated to any $F = \sum_{i=0}^{\infty} a_i T^i \in \Lambda$ are *Iwasawa invariants* $\mu(F)$ and $\lambda(F)$ defined by

$$\mu(F) = \min\{\mathrm{ord}_p(a_i) \mid i \geq 0\} \in e^{-1}\mathbf{Z},$$
$$\lambda(F) = \min\{\, i \mid \mathrm{ord}_p(a_i) = \mu(F)\} \in \mathbf{Z}_{\geq 0},$$

where $e = 1/\mathrm{ord}_p(\varpi)$ is the absolute ramification index of K. By allowing for negative μ-invariants, these definitions extend to any power series $F \in K[[T]]$ with p-adically bounded coefficients and therefore to elements of $\Lambda_K = K \otimes_{\mathcal{O}} \Lambda$.

The following theorem says that elements $F \in \Lambda_K$ are *almost* polynomials, in the sense that they differ from a polynomial by (at worst) a unit in Λ. Recall that a polynomial $D \in \mathcal{O}[T]$ is called *distinguished* if $D \equiv T^{\deg D} \bmod \varpi$.

Theorem 2.1.1 (Weierstrass Preparation Theorem)**.** *For any $F \in \Lambda_K$ there is a unique decomposition*

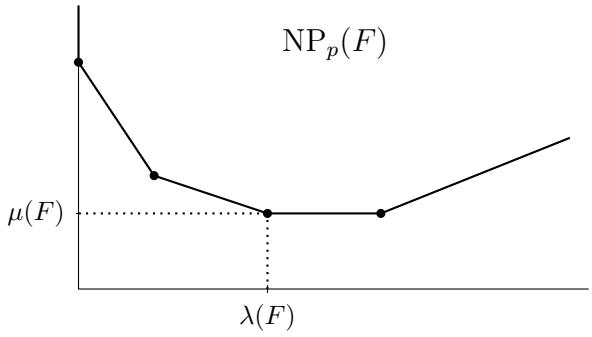
$$F = p^{\mu(F)} DU, \tag{2.1}$$

where $U \in \Lambda^\times$ and $D \in \Lambda$ is a distinguished polynomial of degree $\lambda(F)$.

Proof. See [43, Theorem 7.3]. □

Power series $F \in \Lambda_K$ define functions on the open $\mathbf{C}_p$-unit disc (this follows easily from the boundedness of their coefficients) and we can therefore ask about their number of roots on this disc. From the Weierstrass Preparation Theorem, we know that this number is finite and is exactly equal to $\lambda(F)$.

Remark 2.1.2. Recall that the *p-adic Newton polygon* of $F = \sum a_i T^i \in K[[T]]$, which we denote $\mathrm{NP}_p(F)$, is the unbounded region in $\mathbf{R}^2$ determined by the lower convex hull of the points $\{(i, \mathrm{ord}_p\, a_i) \mid i \geq 0\}$. The Iwasawa invariants of F can be understood in terms of its Newton polygon: the μ-invariant is simply the distance from the x-axis to the polygon and the λ-invariant is the sum of the lengths[1] of all segments of $\mathrm{NP}_p(F)$ having negative slope, as illustrated in the diagram below.



Using the theory of Newton polygons (see [11] or [27] for an introduction) this interpretation provides another way to see that the λ-invariant is equal to the number of roots on the open unit disc. The length of a segment of $\mathrm{NP}_p(F)$ is equal to the number of roots of F in $\mathbf{C}_p$ having valuation equal to the negative of the slope of that segment. Thus a segment of length n with negative slope signifies

[1]The *length of a segment* of a Newton polygon is by definition the length of an edge upon projection to the x-axis. For example, the polygon $\mathrm{NP}_p(X - p)$ has one segment of length 1.

n (not necessarily distinct) roots having positive valuation (hence absolute value < 1).

Lemma 2.1.3. *Fix $x \in K$ and let $F, G \in \Lambda_K$.*

(1) $\lambda(FG) = \lambda(F) + \lambda(G)$.

(2) $\mu(FG) = \mu(F) + \mu(G)$.

(3) $\lambda(xF) = \lambda(F)$.

(4) $\mu(xF) = \mathrm{ord}_p(x) + \mu(F)$.

Proof. Parts (1) and (2) follow easily from the uniqueness part of the Weierstrass Preparation Theorem, and parts (3) and (4) are special cases of (1) and (2), respectively. □

Remark 2.1.4. The Mazur-Tate elements and p-adic L-functions that we will define in the following chapters depend upon a choice of complex periods. We will fix such a choice in Chapter III, however Lemma 2.1.3 shows that only our μ-invariants might be affected by this choice.

The following simple corollary of the Weierstrass Preparation Theorem allows us to recover the Iwasawa invariants of a function by computing valuations. In particular, it says that for any fixed $F \in \Lambda_K$ we can always find some $x \in \mathcal{O}_{\mathbf{C}_p} = \{x \in \mathbf{C}_p \mid \mathrm{ord}_p\, x \geq 0\}$ so that the valuation of $F(x)$ can be written in terms of the Iwasawa invariants of F.

Corollary 2.1.5. *Fix $F \in \Lambda_K$ and let $x \in \mathcal{O}_{\mathbf{C}_p}$ be such that $\mathrm{ord}_p\, x < \frac{1}{e\lambda(F)}$. Then*

$$\mathrm{ord}_p\, F(x) = \mu(F) + \lambda(F)\,\mathrm{ord}_p\, x.$$

Proof. Set $\lambda = \lambda(F)$ and $\mu = \mu(F)$ and let $F = p^\mu DU$ be the Weierstrass decomposition of F. Write $D = T^\lambda + \varpi D_0$ for some polynomial $D_0 \in \mathcal{O}[T]$ of degree $< \lambda$.

Observe that

$$\operatorname{ord}_p D(x) = \min(\lambda \operatorname{ord}_p x, 1/e + \operatorname{ord}_p D_0(x)) = \lambda \operatorname{ord}_p x$$

since $\lambda \operatorname{ord}_p x < 1/e \leq 1/e + \operatorname{ord}_p D_0(x)$. The result now follows since $\operatorname{ord}_p F(x) = \mu + \operatorname{ord}_p D(x)$. □

2.2 Finite-layer invariants via the division algorithm

The Iwasawa algebra is equipped with a division algorithm with respect to distinguished polynomials.

Proposition 2.2.1. *Let $D \in \mathcal{O}[T]$ be a distinguished polynomial. For any $F \in \Lambda$ (resp., $\in \Lambda_K$) we can uniquely write*

$$F = QD + R \tag{2.2}$$

for some power series $Q \in \Lambda$ (resp., $\in \Lambda_K$) and a polynomial $R \in \mathcal{O}[T]$ (resp., $\in K[T]$) of degree $< \deg D$.

Proof. See [43, Proposition 7.2] for a slightly more general statement. □

For $n \geq 1$, define $\omega_n = (1+T)^{p^n} - 1$ and $\Lambda_n = \mathcal{O}[T]/(\omega_n)$. Since ω_n is distinguished, the above proposition provides ring homomorphisms $\pi_n : \Lambda \twoheadrightarrow \Lambda_n$ given by writing $F = Q_n\omega_n + F_n$ as in (2.2) and defining $\pi_n(F) = F_n \bmod \omega_n$. These maps piece together to give an isomorphism of topological rings:

$$\Lambda \to \varprojlim \mathcal{O}[T]/(\omega_n), \qquad F \mapsto (\pi_n(F)). \tag{2.3}$$

(See [43] for the precise details.) We can now define Iwasawa invariants in Λ_n.

Definition 2.2.2. By the division algorithm, we have for each class $\bar{F} \in \Lambda_n$ a unique polynomial $F \in \Lambda$ of degree $< p^n$ mapping to $\bar{F}$ under the natural projection $\pi_n : \Lambda \to \Lambda_n$. Define $\lambda(\bar{F}) = \lambda(F)$ and $\mu(\bar{F}) = \mu(F)$.

As in §2.1, this definition extends to $\Lambda_{K,n} = \Lambda_K/\omega_n\Lambda_K = K[T]/(\omega_n)$ by allowing for negative μ-invariants.

Example 2.2.3. Since $T^{p^n} \equiv -\sum_{i=1}^{p^n-1} \binom{p^n}{i} T^i \bmod \omega_n$ we have

$$\mu\big(\pi_n(T^{p^n})\big) = \mu\big(\textstyle\sum_{i=1}^{p^n-1} \binom{p^n}{i} T^i\big) = 1$$
$$\lambda\big(\pi_n(T^{p^n})\big) = \lambda\big(\textstyle\sum_{i=1}^{p^n-1} \binom{p^n}{i} T^i\big) = p^{n-1}.$$

(Thus, π_n need not preserve Iwasawa invariants.)

2.3 Finite-layer invariants via augmentation ideals

Let Γ_n be an inverse system of finite groups, each having order p^n. For $m \geq n$ we have *restriction* maps (algebra homomorphisms) $\mathrm{res}_n^m : \mathcal{O}[\Gamma_m] \to \mathcal{O}[\Gamma_n]$ induced by the transition maps $\Gamma_m \to \Gamma_n$. We also have (additive) *corestriction* maps, defined for $\sigma \in \Gamma_n$ by

$$\mathrm{cor}_n^m : \mathcal{O}[\Gamma_n] \to \mathcal{O}[\Gamma_m], \quad \sigma \mapsto \sum \tau,$$

where the sum is over all $\tau \in \Gamma_m$ with $\mathrm{res}_n^m(\tau) = \sigma$.

Fix a compatible system of generators $\gamma_n \in \Gamma_n$. Since the isomorphisms

$$\mathcal{O}[\Gamma_n] \xrightarrow{\cong} \Lambda_n \quad \gamma_n \mapsto 1 + T \bmod \omega_n$$

respect restriction maps on both sides, (2.3) yields an isomorphism of topological groups:

$$\varprojlim \mathcal{O}[\Gamma_n] \cong \mathcal{O}[[T]], \quad \gamma = (\gamma_n) \mapsto 1 + T.$$

In [35] and [31], finite-layer invariants are defined intrinsically (i.e., without resorting to the definition in Λ) in terms of the group algebras $\mathcal{O}[\Gamma_n]$ as follows.

Definition 2.3.1. Define the μ-invariant of $\theta = \sum_\sigma c_\sigma \sigma \in \mathcal{O}[\Gamma_n]$ by

$$\mu(\theta) = \min_{\sigma \in \Gamma_n} \mathrm{ord}_p(c_\sigma).$$

Let $k = \mathcal{O}/(\varpi)$ be the residue field of K and let I_n be the augmentation ideal of $k[\Gamma_n]$. Thus, I_n is the ideal generated by the image of $\gamma_n - 1$ in $k[\Gamma_n]$. Let $\theta' = p^{-\mu(\theta)}\theta \in \mathcal{O}[\Gamma_n]$. Since θ' has nonzero image under the natural reduction map $\overline{(\cdot)} : \mathcal{O}[\Gamma_n] \to k[\Gamma_n]$ and all ideals of $k[\Gamma_n]$ are powers of I_n, we can now define

$$\lambda(\theta) = \mathrm{ord}_{I_n} \overline{\theta'} := \max\{j \mid \overline{\theta'} \in I_n^j\} \in \{0, 1, \ldots, p^n - 1\}.$$

Proposition 2.3.2. *Fix $\theta \in \mathcal{O}[\Gamma_n]$ and let $F \in \Lambda$ be the unique polynomial of degree $< p^n$ mapping to θ under the composition*

$$\Lambda \twoheadrightarrow \Lambda_n \xrightarrow{\cong} \mathcal{O}[\Gamma_n], \quad T \mapsto \gamma_n - 1,$$

Then $\lambda(\theta) = \lambda(F)$ and $\mu(\theta) = \mu(F)$.

Proof. Write $\theta = p^{\mu(\theta)}\theta'$ for some $\theta' \in \mathcal{O}[\Gamma_n]$. Let $F_{\theta'} \in \mathcal{O}[T]$ be a representative of the image of θ' in $\mathcal{O}[T]/(\omega_n)$, so $F \equiv p^{\mu(\theta)} F_{\theta'} \bmod \omega_n$. By the division algorithm, we can choose $F_{\theta'}$ such that $\deg F_{\theta'} < p^n$, in which case the degree of F forces the equality $F = p^{\mu(\theta)} F_{\theta'}$ in $\mathcal{O}[T]$. Since $\theta' \equiv (\gamma_n - 1)^{\lambda(\theta)}\theta'' \bmod \varpi$ for some $\theta'' \in \mathcal{O}[\Gamma_n]$, the commutativity of the diagram

$$\begin{array}{ccc} \mathcal{O}[G_n] & \longrightarrow & \Lambda_n \\ \downarrow & & \downarrow \\ k[G_n] & \longrightarrow & k[T]/(\omega_n), \end{array}$$

where the horizontal maps are $\gamma_n \mapsto 1 + T$ and the vertical maps are reduction mod ϖ, implies that $F_{\theta'} \equiv T^{\lambda(\theta)} F_{\theta''} \bmod \varpi$ for some $F_{\theta''} \in \mathcal{O}[T]$ of degree $< p^n$.

Hensel's lemma (see [27, 4.6]) now gives a factorization $F_{\theta'} = DU$ in $\mathcal{O}[T]$, where $D, U \in \mathcal{O}[T]$ are such that $\deg D = \lambda(\theta)$, $D \equiv T^{\lambda(\theta)} \bmod \varpi$ (so D is distinguished), and $U \equiv F_{\theta''} \bmod \varpi$ (so U is a unit in Λ since the constant term of $F_{\theta''}$ does not vanish mod ϖ by maximality of $\lambda(\theta)$). Putting this all together, we have written $F = p^{\mu(\theta)}DU$ exactly in the form (2.1). By uniqueness, it follows that $\mu(\theta) = \mu(F)$ and $\lambda(\theta) = \lambda(F)$. □

Corollary 2.3.3. *The Iwasawa invariants defined in Λ_n and $\mathcal{O}[\Gamma_n]$ agree.*

2.4 Behavior under projection to large layers

For fixed $F \in \Lambda$, what can we say about the relationship between the Iwasawa invariants of F and those of $\pi_n(F) \in \Lambda_n$? Asymptotically, the answer is trivial: we must have $\lambda(\pi_n(F)) \to \lambda(F)$ and $\mu(\pi_n(F)) \to \mu(F)$ as $n \to \infty$ since $F = \lim \pi_n(F)$. The following useful result (stated in [31], for example) clarifies this relationship further by specifying the minimum layer n needed in order to guarantee equality between invariants in Λ and Λ_n. The proof essentially follows from the Weierstrass Preparation Theorem, but we offer a different proof in Chapter VI.

Proposition 2.4.1. *Fix $n \geq 0$ and $F \in \Lambda_K$. If $\lambda(F) < p^n$ then the Iwasawa invariants of F coincide with those of $\pi_n(F)$.*

Proof. Note that for any $x \in K$ we have

$$\mu(\pi_n(xF)) = \mathrm{ord}_p(x) + \mu(\pi_n(F)) \qquad \lambda(\pi_n(xF)) = \lambda(\pi_n(F)). \tag{2.4}$$

Thus we need only prove the result for functions $F \in \Lambda$ with $\mu(F) = 0$. (Explicitly, suppose it holds for trivial μ-invariant. Write $F = p^{\mu(F)}F_0$ with $\mu(F_0) = 0$. Then

by (2.4), $\mu(\pi_n(F)) = \mu(F) + \mu(\pi_n(F_0)) = \mu(F) + \mu(F_0) = \mu(F)$.) The division algorithm in Λ gives a unique decomposition

$$F = \omega_n Q_n + F_n \tag{2.5}$$

for some $Q_n \in \Lambda$ and a polynomial $F_n \in \Lambda$ of degree $< p^n$. By definition, $\mu(\pi_n(F)) = \mu(F_n)$ and $\lambda(\pi_n(F)) = \lambda(F_n)$. If $\mu(F_n)$ were positive then (2.5) implies

$$F \equiv T^{p^n} Q_n \bmod \varpi,$$

which contradicts the fact that $\lambda(F) < p^n$. Hence we must have $\mu(F_n) = 0$. Combining (2.5) with the Weierstrass decomposition $F = (T^{\lambda(F)} + \varpi F_0)U$ yields

$$F_n \equiv T^{\lambda(F)}(U - T^{p^n - \lambda(F)} Q_n) \bmod \varpi.$$

As $\lambda(F) < p^n$ and U is a unit, the function $U - T^{p^n - \lambda(F)} Q_n$ must also be a unit. It follows that $\lambda(F_n) = \lambda(F)$. □

Remark 2.4.2. If $\lambda(F) \geq p^n$ then the Iwasawa invariants of F may differ from those of $\pi_n(F)$ – see example 2.2.3 above. This phenomena will be explored further in Chapter VI.

For any $U \in \Lambda^\times$ and $F \in \Lambda$, it is obvious that UF has the same Iwasawa invariants as F. Using the proposition above, we can show that the same is true at layer n.

Proposition 2.4.3. *If $F \in \Lambda$ and $U \in \Lambda^\times$ then the Iwasawa invariants of $\pi_n(UF)$ coincide with those of $\pi_n(F)$.*

Proof. Use the division algorithm to write $F = \omega_n Q_n + F_n$ with $\deg F_n < p^n$. Then

$$\lambda(\pi_n(F)) = \lambda(F_n) = \lambda(UF_n) = \lambda(\pi_n(UF_n)) = \lambda(\pi_n(UF)),$$

where the third equality follows from Proposition 2.4.1 (because $\lambda(UF_n) = \lambda(F_n) \leq \deg F_n < p^n$) and the final equality is because $UF \equiv UF_n \mod \omega_n$. The same string of equalities holds for μ-invariants. $\square$

CHAPTER 3

MAZUR-TATE ELEMENTS

3.1 Modular symbols

3.1.1 Basic definitions

Let $V_g(R) = \mathrm{Sym}^g(R^2)$ be the algebra of degree g homogeneous polynomials in the variables X and Y over a commutative ring R. There is a right action of $\mathrm{GL}_2(R)$ on $V_g(R)$ given by

$$P(X,Y) \mid \gamma = P((X,Y)\gamma^*) = P(dX - cY, -bX + aY)$$

where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $\gamma^* = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$. Writing $\Delta_0 = \mathrm{Div}^0(\mathbf{P}^1(\mathbf{Q}))$, define an action of the semigroup $M_2(\mathbf{Z})$ of 2×2 integer matrices on $\xi \in \mathrm{Hom}_{\mathbf{Z}}(\Delta_0, V_g(R))$ by

$$(\xi \mid \gamma)(D) = \xi(\gamma D) \mid \gamma,$$

where the action of $\gamma \in M_2(\mathbf{Z})$ on Δ_0 is by fractional linear transformations.

Definition 3.1.1. The R-module of *modular symbols of weight g* over a fixed congruence subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbf{Z})$ is defined by

$$\mathrm{Symb}_\Gamma(V_g(R)) = \mathrm{Hom}_\Gamma(\Delta_0, V_g(R)) = \{\xi : \Delta_0 \to V_g(R) \mid \xi|\gamma = \xi \text{ for all } \gamma \in \Gamma\}.$$

Remark 3.1.2. The space of modular symbols has a natural interpretation in terms of compact cohomology groups – see [1] for more on this point of view.

If Γ is normalized by the matrix $\iota = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ then the map $\xi \mapsto \xi|\iota$ defines an involution on $\mathrm{Symb}_\Gamma(V_g(R))$. If, in addition, 2 is invertible in R then the orthogonal idempotents for ι give a decomposition

$$\mathrm{Symb}_\Gamma(V_g(R)) = \mathrm{Symb}_\Gamma^+(V_g(R)) \oplus \mathrm{Symb}_\Gamma^-(V_g(R)), \tag{3.1}$$

where the direct summands are the $\pm$-eigenspaces for ι. Accordingly, we can uniquely decompose a modular symbol $\xi = \xi^+ + \xi^-$, where $\xi^\pm = \frac{\xi \pm \xi|\iota}{2}$.

3.1.2 Hecke operators

Let $\Gamma = \Gamma_1(N)$ and fix $\alpha \in \mathrm{GL}_2^+(\mathbf{Q})$ with integer entries. Writing $\Gamma\alpha\Gamma$ as a finite disjoint union $\coprod \Gamma\beta_i$ for some $\beta_i \in \Gamma\alpha\Gamma$ (this can be done by [8, Lemma 5.1.2]) one can define a double-coset action on modular symbols via

$$\xi \mid [\Gamma\alpha\Gamma] = \sum \xi \mid \beta_i \in \mathrm{Symb}_\Gamma(V_g(R)).$$

This makes $\mathrm{Symb}_\Gamma(V_g(R))$ into a module over the classical Hecke algebra $\mathbf{Z}[T_n, \langle d \rangle]$: for $d \in (\mathbf{Z}/N\mathbf{Z})^\times$ and a prime p, one defines

$$T_p : \xi \mapsto \xi \mid [\Gamma \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right) \Gamma],$$

$$\langle d \rangle : \xi \mapsto \xi \mid [\Gamma\alpha\Gamma] = \xi \mid \alpha,$$

where $\alpha \in \Gamma_0(N)$ is any matrix whose lower right entry is congruent to d mod N.

3.1.3 Mazur-Tate elements for a modular symbol

Fix a modular symbol $\xi \in \mathrm{Symb}_\Gamma(V_g(R))$ and an integer $0 \le j \le g$. For coprime integers a and m with $1 \le a < m$, or for the pair $m = 1$ and $a = 0$, define the *modular elements* $\left[\frac{a}{m}\right]_{\xi,j} \in R$ by the equation

$$\left(\xi \mid \left(\begin{smallmatrix} 1 & a \\ 0 & m \end{smallmatrix}\right)\right)(\{\infty\} - \{0\}) = \sum_{j=0}^{g} \left[\frac{a}{m}\right]_{\xi,j} X^j Y^{g-j}. \tag{3.2}$$

Definition 3.1.3. The *Mazur-Tate element* for the tuple (ξ, m, j) is defined by

$$\mathcal{M}_{m,j}(\xi) = \sum_{a \in (\mathbf{Z}/m\mathbf{Z})^\times} \left[\frac{a}{m}\right]_{\xi,j} \cdot \sigma_a \in R[\mathcal{G}_m],$$

where $\mathcal{G}_m = \mathrm{Gal}(\mathbf{Q}(\zeta_m)/\mathbf{Q})$ and $\sigma_a : \zeta_m \mapsto \zeta_m^a$.

Remark 3.1.4. We will occasionally write $\xi(D)_{(X,Y)}$ for the homogeneous polynomial $\xi(D) \in V_g(R)$.

Remark 3.1.5. The Mazur-Tate elements at $j = 0$ and prime powers $m = p^n$ recover those discussed by Pollack-Weston [35]: we have

$$\mathcal{M}_{p^n,0}(\xi) = \sum_{a \in (\mathbf{Z}/p^n\mathbf{Z})^\times} \left[\frac{a}{p^n}\right]_{\xi,0} \cdot \sigma_a = \sum_{a \in (\mathbf{Z}/p^n\mathbf{Z})^\times} \xi(\{\infty\} - \{a/p^n\})_{(X,Y)=(0,1)} \cdot \sigma_a.$$

3.1.4 Modular symbols over a DVF

We now assume that R is a discrete valuation ring with field of fractions K and associated nonarchimedean absolute value $|\cdot|_K : K \to \mathbf{R}$. For this section only, we write $|\cdot| = |\cdot|_K$. Define a norm $||\cdot||$ on $V_g(K)$ as follows. For any $F \in V_g(K)$ we can write

$$F = \sum_{j=0}^{g} a_j(F) X^j Y^{g-j}, \quad a_j(F) \in K,$$

and we define

$$||F|| := \max_{0 \le j \le g} |a_j(F)|.$$

Lemma 3.1.6. *Let $F, G \in V_g(K)$ and $x \in K$.*

(1) If $||F|| = 0$ then $F = 0$.

(2) $||xF|| = |x| \cdot ||F||$.

(3) $||F + G|| \le \max(||F||, ||G||)$.

(4) $||F \mid \gamma|| \le ||F||$ for all $\gamma \in M_2(\mathbf{Z})$.

Proof. If $||F|| = 0$ then all coefficients of F have absolute value ≤ 0, hence they must all be zero, which proves (1). Part (2) is obvious. For (3), we have

$$||F+G|| = \max_{0\leq j\leq g} |a_j(F)+a_j(G)| \leq \max_{0\leq j\leq g} \big(\max(|a_j(F)|, |a_j(G)|)\big) = \max\big(||F||, ||G||\big).$$

For (4), writing $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ we have $F \mid \gamma = \sum a_j(dX - cY)^j(-bX + aY)^{g-j}$ and since $||(dX - cY)^j(-bX + aY)^{g-j}|| \leq 1$ (because $|\mathbf{Z}| \leq 1$), the result follows from (3) and (2). □

Corollary 3.1.7. *If $\xi \in \mathrm{Symb}_\Gamma(V_g(K))$ then $||\xi(\alpha D)|| \leq ||\xi(D)||$ for all $D \in \Delta_0$ and $\alpha \in \mathbf{Z}[\Gamma]$.*

Proof. Recalling that Δ_0 is finitely generated as a $\mathbf{Z}[\Gamma]$-module (see [32, 2.4]), write α as a finite sum $\alpha = \sum n_i\gamma_i$, where $n_i \in \mathbf{Z}$ and $\gamma_i \in \Gamma$. Then the previous lemma and the fact that $\xi(\gamma_i D) = \xi(D) \mid \gamma_i^{-1}$ for all i (this is the modular symbol condition) imply

$$||\xi(\alpha D)|| \leq \max_i ||n_i\xi(\gamma_i D)|| \leq \max_i ||\xi(\gamma_i D)|| = \max_i ||\xi(D) \mid \gamma_i^{-1}|| \leq ||\xi(D)||.$$

□

We can extend $||\cdot||$ to $\mathrm{Symb}_\Gamma(V_g(K))$ by setting

$$||\xi|| := \max_{D\in\Delta_0} ||\xi(D)||.$$

Taking the maximum as opposed to the supremum is justified by the previous lemma. Explicitly, fixing a generating set $\{D_1, \cdots, D_d\}$ for Δ_0 as a $\mathbf{Z}[\Gamma]$-module and writing $D \in \Delta_0$ as an appropriate finite sum $\sum \alpha_i D_i$, $\alpha_i \in \mathbf{Z}[\Gamma]$, Lemmas 3.1.6 and 3.1.7 give

$$||\xi(D)|| \leq \max_i ||\xi(\alpha_i D_i)|| \leq \max_{1\leq i\leq d} ||\xi(D_i)||.$$

Taking the maximum over all $D \in \Delta_0$, we see that

$$||\xi|| = \max_{1\leq i\leq d} ||\xi(D_i)||.$$

Definition 3.1.8. We say that $\xi \in \mathrm{Symb}_\Gamma(V_g(K))$ is *cohomological* if $||\xi|| = 1$.

Concretely, ξ is cohomological if both of the following hold:

- For every divisor $D \in \Delta_0$ we can write $\xi(D) = \sum_{j=0}^{g} a_j(\xi, D) X^j Y^{g-j}$ with all $a_j(\xi, D) \in R$.
- There is some divisor $D_0 \in \Delta_0$ such that at least one of $a_0(\xi, D_0), \cdots, a_g(\xi, D_0)$ is a unit in R.

We can associate to any nonzero K-valued symbol ξ a unique (up to scaling by R-units) cohomological symbol as follows. Recall that if $v : K \to \mathbf{Z} \cup \{\infty\}$ is the valuation on K then there is some real number $c > 1$ such that $|x| = c^{-v(x)}$. Hence $||\xi|| = c^{v_\xi}$ for some unique integer v_ξ and if ϖ is a uniformizer in K then the desired cohomological symbol is $\varpi^{v_\xi}\xi$.

Remark 3.1.9. If ξ is cohomological then the modular elements $[a/m]_{\xi,j}$ lie in R since $||\big(\xi \mid \left(\begin{smallmatrix} 1 & a \\ 0 & m \end{smallmatrix}\right)\big)(\{\infty\} - \{0\})|| \leq ||\xi(\{\infty\} - \{a/m\})|| \leq 1$ by Lemma 3.1.6.

3.2 Mazur-Tate elements of modular forms

Fix a congruence subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbf{Z})$ and a cuspform $f \in S_k(\Gamma)$.

3.2.1 Modular symbol attached to a modular form

Consider the map $\xi_f \in \mathrm{Hom}(\Delta_0, V_{k-2}(\mathbf{C}))$ defined by

$$\xi_f(\{r\} - \{s\}) = 2\pi i \int_s^r f(z)(zX + Y)^{k-2} dz,$$

where the integral is over the geodesic in the complex upper half-plane connecting r and s. One can check using a change of variables that for any $\gamma \in \mathrm{GL}_2^+(\mathbf{Q})$ we

have $\xi_f|\gamma = \xi_{f|_k\gamma}$ This shows that ξ_f inherits Γ-invariance from f and we therefore have a modular symbol $\xi_f \in \mathrm{Symb}_\Gamma(V_{k-2}(\mathbf{C}))$.

3.2.2 Complex Mazur-Tate elements

From now on we take $\Gamma = \Gamma_1(N)$ and assume that $f = \sum a_n q^n \in S_k(N, \varepsilon)$ is a newform with associated number field $K_f = \mathbf{Q}(\{a_n\})$. Since $\Gamma_1(N)$ is normalized by $\iota = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, we can write $\xi_f = \xi_f^+ + \xi_f^-$ in accordance with decomposition (3.1). The following theorem describes complex *periods* $\Omega_f^\pm \in \mathbf{C}$ – whose existence was seemingly first established by Shimura [36, Theorem 1] as a statement about L-values – that allow us to view the $\mathbf{C}$-valued $\pm$-modular symbols for f algebraically.

Theorem 3.2.1. *There exist* $\Omega_f^\pm \in \mathbf{C}$ *such that* $\xi_f^\pm/\Omega_f^\pm \in \mathrm{Symb}^\pm_{\Gamma_1(N)}(V_{k-2}(K_f))$.

Proof. This is proved from the point of view of *period polynomials* in [28, Proposition 5.11]. For the $\varepsilon = 1$ case, see [12, 3.5.4] for a statement and [14] for a proof. (See also [44] for an overview of this result.) We translate the result of [28] into the language of modular symbols in Appendix A. □

Definition 3.2.2. Define the K_f-valued modular symbol $\varphi_f^\pm = \xi_f^\pm/\Omega_f^\pm$ and write $[a/m]^\pm_{f,j} = [a/m]_{\varphi_f^\pm, j} \in K_f$ for the coefficients of the Mazur-Tate elements $\mathcal{M}_{m,j}(\varphi_f^\pm)$.

In terms of integrals, we have

$$\left[\frac{a}{m}\right]^\pm_{f,j} = \frac{\pi i}{\Omega_f^\pm}\binom{k-2}{j}\left(\int_{a/m}^{i\infty} f(z)(mz-a)^j dz \pm (-1)^{k-j}\int_{-a/m}^{i\infty} f(z)(mz+a)^j dz\right).$$

To see this, let $\gamma = \left(\begin{smallmatrix} 1 & a \\ 0 & m \end{smallmatrix}\right)$, $D = \{\infty\} - \{0\}$, and note that

$$\begin{aligned}\big(\varphi_f^\pm \mid \gamma\big)(D) &= \frac{(\xi_f|\gamma)(D) \pm (\xi_f|\iota\gamma)(D)}{2\Omega_f^\pm} \\ &= \frac{1}{2\Omega_f^\pm}\Big(\xi_f(\{\infty\} - \{a/m\})_{(mX,-aX+Y)} \pm \xi_f(\{\infty\} - \{-a/m\})_{(mX,aX-Y)}\Big).\end{aligned}$$

Setting $\xi_{f,j}(a,m) = 2\pi i\binom{k-2}{j}\int_{a/m}^{i\infty} f(z)(mz-a)^j dz$, we have

$$\xi_f\big(\{\infty\}-\{a/m\}\big)_{(mX,-aX+Y)} = \sum_{j=0}^{k-2}\xi_{f,j}(a,m)X^jY^{k-2-j}$$

$$\xi_f(\{\infty\}-\{-a/m\})_{(mX,aX-Y)} = \sum_{j=0}^{k-2}(-1)^{k-2-j}\xi_{f,j}(-a,m)X^jY^{k-2-j},$$

thus the claimed integral representation is

$$\left[\frac{a}{m}\right]_{f,j}^{\pm} = \frac{\xi_{f,j}(a,m)\pm(-1)^{k+j}\xi_{f,j}(-a,m)}{2\Omega_f^{\pm}} \in K_f. \tag{3.3}$$

3.2.3 Interpolation property

Let $\pm = \operatorname{sgn}(-1)^{k+j}$ and observe that for all $0 \le j \le k-2$,

$$[0/1]_{f,j}^{\pm} = \binom{k-2}{j}\frac{2\pi i}{\Omega_f^{\pm}}\int_0^{i\infty} f(z)z^j dz = \binom{k-2}{j}\frac{j!}{(-2\pi i)^j\Omega_f^{\pm}}L(f,j+1)$$

by (3.3) and [8, §5.10]. Equivalently, one can write

$$\chi\big(\mathcal{M}_{1,j}^{\pm}(f)\big) = \binom{k-2}{j}\frac{j!}{(-2\pi i)^j\Omega_f^{\pm}}L(f,j+1),$$

where χ is the unique character of modulus 1. More generally, the following result shows that Mazur-Tate elements interpolate the critical L-values of twists of f.

Proposition 3.2.3. *Fix a primitive character $\chi : (\mathbf{Z}/m\mathbf{Z})^\times \to \mathbf{C}^\times$. Then*

$$\chi\big(\mathcal{M}_{m,j}^{\pm}(f)\big) = \begin{cases} \binom{k-2}{j}\frac{j!m^j\tau(\chi)}{\Omega_f^{\pm}(-2\pi i)^j}\cdot L(f_{\chi^{-1}},j+1)\in K_f(\chi) & \text{if } \pm = \operatorname{sgn}\chi(-1)(-1)^{k-j},\\ 0 & \text{otherwise,}\end{cases}$$

where $K_f(\chi) = K_f(\operatorname{im}\chi)$, $\tau(\chi) = \sum_{a\in(\mathbf{Z}/m\mathbf{Z})^\times}\chi(a)\zeta_m^a$ is a classical Gauss sum, and $f_\chi = \sum\chi(n)a_nq^n$ denotes the twist of f by χ.

Proof. By [25, Proposition 3] we know that the values $\xi_{f,j}(a,m)$ depend only on $a \bmod m$, so

$$\sum_{a\in(\mathbf{Z}/m\mathbf{Z})^\times}\chi(a)\xi_{f,j}(-a,m) = \chi(-1)\sum_{a\in(\mathbf{Z}/m\mathbf{Z})^\times}\chi(a)\xi_{f,j}(a,m).$$

Using (3.3) it follows that

$$\chi\big(\mathcal{M}^{\pm}_{m,j}(f)\big) = \left(\frac{1 \pm \chi(-1)(-1)^{k-j}}{2\Omega_f^{\pm}}\right) \sum_{a \in (\mathbf{Z}/m\mathbf{Z})^\times} \chi(a)\xi_{f,j}(a,m)$$

$$= \begin{cases} \frac{1}{\Omega_f^{\pm}} \sum_{a \in (\mathbf{Z}/m\mathbf{Z})^\times} \chi(a)\xi_{f,j}(a,m) & \text{if } \pm = \operatorname{sgn} \chi(-1)(-1)^{k-j}, \\ 0 & \text{otherwise.} \end{cases}$$

The result now follows from [25, 8.6], which states that

$$\sum_{a \in (\mathbf{Z}/m\mathbf{Z})^\times} \chi(a)\xi_{f,j}(a,m) = -\binom{k-2}{j} \frac{j! m^j \tau(\chi)}{(-2\pi i)^j} \cdot L(f_{\chi^{-1}}, j+1).$$

□

Remark 3.2.4. Another way to phrase the above proposition is that

$$\sum_{a \in (\mathbf{Z}/m\mathbf{Z})^\times} \chi(a)\big(\varphi_f \mid \left(\begin{smallmatrix} 1 & a \\ 0 & m \end{smallmatrix}\right)\big)(\{\infty\} - \{0\}) = \sum_{j=0}^{k-2} C_j L(f_{\chi^{-1}}, j+1) X^j Y^{k-2-j}$$

where $\varphi_f = \varphi_f^+ + \varphi_f^-$ and $C_j = \binom{k-2}{j} \frac{j! m^j \tau(\chi)}{\Omega_f^*(-2\pi i)^j}$ with $* = \operatorname{sgn} \chi(-1)(-1)^{k-j}$.

Remark 3.2.5. This interpolation property is similar to one enjoyed by classical Stickelberger elements. Recall that the Stickelberger element of the field $\mathbf{Q}(\zeta_m)$ is

$$\vartheta_m = \frac{1}{m} \sum_{a \in (\mathbf{Z}/m\mathbf{Z})^\times} a \cdot \sigma_a^{-1} \in \mathbf{Q}[\mathcal{G}_m],$$

and if χ is a primitive Dirichlet character mod m we have

$$\chi(\vartheta_m) = -L(\chi^{-1}, 0)$$

by [43, Theorem 4.2]. Thus Mazur-Tate elements can be viewed analogues of Stickelberger elements.

3.2.4 3-term relation

Consider the following group homomorphisms between layers $n \mid m$:

$$\mathrm{Res}_n^m : K_f[\mathcal{G}_m] \to K_f[\mathcal{G}_n], \quad \sigma \mapsto \sigma|_{\mathbf{Q}(\zeta_n)},$$
$$\mathrm{Cor}_m^n : K_f[\mathcal{G}_n] \to K_f[\mathcal{G}_m], \quad \sigma \mapsto \textstyle\sum_{\mathrm{Res}_n^m(\tau)=\sigma} \tau.$$

Concretely, for $b \in (\mathbf{Z}/m\mathbf{Z})^\times$ and $a \in (\mathbf{Z}/n\mathbf{Z})^\times$ the value of these maps on elements of the respective Galois groups $\mathcal{G}_m$ and $\mathcal{G}_n$ are

$$\mathrm{Res}_n^m(\sigma_b) = \sigma_{b \bmod n}, \qquad \mathrm{Cor}_m^n(\sigma_a) = \textstyle\sum_{b \equiv a \bmod n} \sigma_b.$$

Proposition 3.2.6. *For primes $p \mid m$ we have*

$$\mathrm{Res}_m^{pm}\left(\mathcal{M}^{\pm}_{pm,j}(f)\right) = a_p \mathcal{M}^{\pm}_{m,j}(f) - \varepsilon(p) p^{k-2} \mathrm{Cor}_m^{m/p}\left(\mathcal{M}^{\pm}_{m/p,j}(f)\right).$$

Proof. Fix $a \in (\mathbf{Z}/m\mathbf{Z})^\times$. From [25, 4.2] and equation (3.3) we have

$$\sum_{\substack{b \in (\mathbf{Z}/pm\mathbf{Z})^\times \\ b \equiv a \bmod m}} \left[\frac{b}{pm}\right]^{\pm}_{f,j} \cdot \sigma_a = \left(a_p \left[\frac{a}{m}\right]^{\pm}_{f,j} - \varepsilon(p) p^{k-2} \left[\frac{a}{m/p}\right]^{\pm}_{f,j}\right) \cdot \sigma_a \in K_f[\mathcal{G}_m].$$

Summing over all $a \in (\mathbf{Z}/m\mathbf{Z})^\times$ now gives the result. □

3.2.5 p-adic Mazur-Tate elements

Fix an odd prime p and let $K/\mathbf{Q}_p$ be the completion of K_f at the prime over p determined by our chosen embedding $\bar{\mathbf{Q}} \hookrightarrow \bar{\mathbf{Q}}_p$. Write $\mathcal{O} \subseteq K$ for the associated valuation ring. With respect to this embedding, we may view the K_f-valued modular symbols $\varphi_f^{\pm}$ as being K-valued. Scaling $\varphi_f^{\pm}$ as necessary (see §3.1.4), we assume from now on that $\varphi_f^{\pm}$ is a cohomological symbol. Thus, for every divisor $D \in \Delta_0$ we may write

$$\varphi_f^{\pm}(D) = \sum_{j=0}^{k-2} a_j(\varphi_f^{\pm}, D) X^j Y^{k-2-j}, \qquad a_j(\varphi_f^{\pm}, D) \in \mathcal{O}$$

and there is at least one divisor D_0 such that at least one of the values $a_0(\varphi_f^{\pm}, D_0)$, ..., $a_{k-2}(\varphi_f^{\pm}, D_0)$ is a unit. Fixing such a period, our symbols $\varphi_f^{\pm}$ are now unique up to scaling by elements of K_f whose image under our embedding is a unit in $\mathcal{O}$.

Definition 3.2.7. The *nth-layer p-adic Mazur-Tate element* for f is defined by

$$\vartheta_{n,j}^{\pm}(f) = \mathcal{M}_{p^n,j}(\varphi_f^{\pm}) \in \mathcal{O}[\mathcal{G}_{p^n}].$$

We have a canonical decomposition $\mathcal{G}_{p^{n+1}} = \Delta \times \Gamma_n$ (recall $p > 2$) where Δ is cyclic of order $p-1$ and Γ_n is cyclic of order p^n. Thus $\mathcal{O}[\mathcal{G}_{p^{n+1}}] = \mathcal{O}[\Delta \times \Gamma_n]$ is a $\mathbf{Z}_p[\Delta]$-module, and it decomposes via orthogonal idempotents into a sum of eigenspaces for the action of Δ.

Definition 3.2.8. Fix $\psi : \Delta \to \mathbf{Z}_p^{\times}$ and let $\varepsilon_\psi = (p-1)^{-1} \sum_{\delta \in \Delta} \psi(\delta)\delta^{-1} \in \mathbf{Z}_p[\Delta]$ be the corresponding orthogonal idempotent. Define $\theta_{n,j}^{\psi}(f) \in \mathcal{O}[\Gamma_n]$ by the equation

$$\varepsilon_\psi \vartheta_{n,j}^{*}(f) = \theta_{n,j}^{\psi}(f)\varepsilon_\psi,$$

where $* = \operatorname{sgn}\psi(-1)(-1)^{k+j}$.

Explicitly, decomposing $\sigma_a = \delta(a)\gamma_n(a)$ where $\delta(a) \in \Delta$ and $\gamma_n(a) \in \Gamma_n$, we have

$$\theta_{n,j}^{\psi}(f) = \sum_{a \in (\mathbf{Z}/p^{n+1}\mathbf{Z})^{\times}} \psi\big(\delta(a)\big) \left[\frac{a}{p^{n+1}}\right]_{f,j}^{\operatorname{sgn}\psi(-1)(-1)^{k+j}} \cdot \gamma_n(a).$$

As in §3.2.4, we have (co)restriction maps on $\mathcal{O}[\Gamma_n]$: for $m \leq n$ define

$$\operatorname{res}_m^n := \operatorname{Res}_{p^{m+1}}^{p^{n+1}} \big|_{\Gamma_n} : \mathcal{O}[\Gamma_n] \to \mathcal{O}[\Gamma_m]$$
$$\operatorname{cor}_n^m := \operatorname{Cor}_{p^{m+1}}^{p^{m+1}} \big|_{\Gamma_m} : \mathcal{O}[\Gamma_m] \to \mathcal{O}[\Gamma_n]$$

The 3-term relation of section §3.2.4 on p-adic Mazur-Tate elements reads as follows.

Corollary 3.2.9. $\operatorname{res}_n^{n+1}\left(\theta_{n+1,j}^{\psi}(f)\right) = a_p\theta_{n,j}^{\psi}(f) - \varepsilon(p)p^{k-2}\operatorname{cor}_n^{n-1}\left(\theta_{n-1,j}^{\psi}(f)\right)$.

Proof. The character ψ gives rise to the following map

$$\mathcal{O}[\Delta \times \Gamma_n] \to \mathcal{O}[\Gamma_n], \quad \sum c_a \delta(a)\gamma(a) \mapsto \sum c_a \psi(\delta(a))\gamma(a),$$

which we denote $\mathcal{M} \mapsto \mathcal{M}^\psi$. Since $\theta^\psi_{n,j}(f) = \left(\mathcal{M}^*_{p^{n+1},j}(f)\right)^\psi$, where $* = \operatorname{sgn}\psi(-1)(-1)^{k+j}$, applying this map to Proposition 3.2.6 and using the relations

$$\operatorname{Res}^{p^{n+2}}_{p^{n+1}}(\mathcal{M})^\psi = \operatorname{res}^{n+1}_n(\mathcal{M}^\psi),$$
$$\operatorname{Cor}^{p^{n+1}}_{p^{n+2}}(\mathcal{M})^\psi = \operatorname{cor}^n_{n+1}(\mathcal{M}^\psi),$$

gives the result. □

We close this chapter with an interpretation of $\theta^\psi_{n,j}(f)$ in terms of polynomials. Viewing ψ as a character on $(\mathbf{Z}/p^{n+1}\mathbf{Z})^\times$ (precompose with reduction mod p) and fixing a generator γ of Γ_n, we can write

$$\theta^\psi_{n,j}(f) = \sum_{a \in (\mathbf{Z}/p^{n+1}\mathbf{Z})^\times} \psi(a) \left[\frac{a}{p^{n+1}}\right]^{\operatorname{sgn}\psi(-1)(-1)^{k+j}}_{f,j} \gamma^{\log_\gamma(a)}$$

where $0 \le \log_\gamma(a) \le p^n - 1$ is the unique integer for which $a \equiv \omega(a)(1+p)^{\log_\gamma(a)} \bmod p^{n+1}$. Now

$$\theta^\psi_{n,j}(f,T) = \sum_{a \in (\mathbf{Z}/p^{n+1}\mathbf{Z})^\times} \psi(a) \left[\frac{a}{p^{n+1}}\right]^{\operatorname{sgn}\psi(-1)(-1)^{k+j}}_{f,j} (1+T)^{\log_\gamma(a)} \in \mathcal{O}[T],$$

is the unique polynomial of degree $< p^n$ mapping to $\theta^\psi_{n,j}(f)$ under the composition

$$\mathcal{O}[T] \twoheadrightarrow \mathcal{O}[T]/(\omega_n) \xrightarrow{\cong} \mathcal{O}[\Gamma_n], \quad T \mapsto \gamma - 1.$$

CHAPTER 4

p-ADIC L-FUNCTIONS

The goal of this chapter is to define the p-adic L-function associated to a modular form f and discuss its properties in the case $a_p(f) = 0$. The only original material in this chapter is §4.3.2-4.3.3, where we compute the p-adic valuation of the algebraic parts of the L-function for twists of such modular forms.

4.1 p-adic distributions

4.1.1 Locally analytic distributions

Let $\mathcal{A}$ be the space of locally analytic functions on $\mathbf{Z}_p^\times$ with values in $\mathbf{C}_p$, and write $\mathcal{A}^h \subseteq \mathcal{A}$ for the subspace of functions which are locally polynomials of degree $\leq h$, where h is some fixed nonnegative real number. Thus, if $\psi \in \mathcal{A}^h$ then for any $a \in \mathbf{Z}_p^\times$ we can write

$$\psi\mid_{a+p^n\mathbf{Z}_p}(x) = \sum_{i=0}^{\lfloor h \rfloor} \frac{\psi^{(i)}(a)}{i!}(x-a)^i.$$

Note that $\mathcal{A}^0$ is the set of locally constant functions on $\mathbf{Z}_p^\times$.

A linear map $\mu : \mathcal{A} \to \mathbf{C}_p$ is called a *locally analytic distribution* on $\mathbf{Z}_p^\times$. A linear map $\mu : \mathcal{A}^h \to \mathbf{C}_p$ satisfying the growth condition

$$\sup_{a \in \mathbf{Z}_p^\times} |\mu\big((x-a)^i \cdot \mathbb{1}_{a+p^n\mathbf{Z}_p}\big)| = O(p^{n(h-i)}),$$

for all $0 \leq i \leq h$ is called an *h-admissible measure* on $\mathbf{Z}_p^\times$.

Theorem 4.1.1 (Visik [41], Amice-Vélu [2])**.** *Any h-admissible measure μ can be extended to a locally analytic distribution.*

The rough idea behind this theorem is as follows. Fixing an h-admissible measure μ, we want to define a locally analytic distribution $\tilde{\mu}$ that agrees with μ on $\mathcal{A}^h$. Since μ is only defined on polynomials of degree $\leq h$, one way to define the local measure of a function $\psi \in \mathcal{A}$ is to ignore the higher order terms in the power series expansion of ψ on a cover of $\mathbf{Z}_p^\times$. In other words, define

$$\tilde{\mu}(\psi \mid_{a+p^n\mathbf{Z}_p}) := \sum_{i=0}^{\lfloor h \rfloor} \frac{\psi^{(i)}(a)}{i!} \mu((x-a)^i \mathbb{1}_{a+p^n\mathbf{Z}_p}).$$

To obtain the measure of ψ over all of $\mathbf{Z}_p^\times$, define the 'generalized' Riemann sum[1]:

$$S_n(\psi) := \sum_{a \in (\mathbf{Z}/p^n\mathbf{Z})^\times} \tilde{\mu}(\psi \mid_{a+p^n\mathbf{Z}_p}).$$

and set

$$\tilde{\mu}(\psi) = \int \psi d\tilde{\mu} := \lim_{n\to\infty} S_n(\psi)$$

Of course, one must check this definition is independent of the chosen cover of $\mathbf{Z}_p^\times$ and that the limit $\lim_{n\to\infty} S_n(\psi)$ actually exists. See [41] for these technical details.

Remark 4.1.2. Notice that $\tilde{\mu}$ is completely determined by the values $\mu(x^i \mathbb{1}_{a+p^n\mathbf{Z}_p})$, where $0 \leq i \leq \lfloor h \rfloor$ and $n \geq 1$. Since we can write the indicator functions $\mathbb{1}_{a+p^n\mathbf{Z}_p}$ in terms of characters on $(\mathbf{Z}/p^n\mathbf{Z})^\times$ – for example,

$$\frac{1}{p^n - p^{n-1}} \sum_{\chi \in \widehat{(\mathbf{Z}/p^n\mathbf{Z})^\times}} \chi = \mathbb{1}_{1+p^n\mathbf{Z}_p}.$$

[1]If $h = 0$ then this is a true Riemann sum, for in that case

$$S_n(\psi) = \sum_{a \in (\mathbf{Z}/p^n\mathbf{Z})^\times} \psi(a)\mu(\mathbb{1}_{a+p^n\mathbf{Z}_p}).$$

– the distribution $\tilde{\mu}$ is also uniquely determined by the values $\mu(x^i\chi)$ for $0 \leq i \leq \lfloor h \rfloor$ and χ ranging over characters on $(\mathbf{Z}/p^n\mathbf{Z})^\times$ for all n.

4.1.2 The function $\log_\gamma$

For the remainder of this thesis we let γ denote a fixed topological generator of $1 + p\mathbf{Z}_p \subseteq \mathbf{Z}_p^\times$. Recall the usual p-adic logarithm $\log_p$ gives an isomorphism of topological groups $(1 + p^n\mathbf{Z}_p, \times) \to (p^n\mathbf{Z}_p, +)$, $n \geq 1$, with inverse exp (see [27, Proposition II.5.5]). Thus, the function

$$\log_\gamma := \frac{\log_p}{\log_p(\gamma)},$$

being a scaling of the p-adic logarithm by an element of valuation $\mathrm{ord}_p \log_p(\gamma) = 1$, gives an isomorphism $1 + p^{n+1}\mathbf{Z}_p \to p^n\mathbf{Z}_p$, $n \geq 0$, with inverse $x \mapsto \gamma^x = \exp(x \log_p(\gamma))$. In particular, we have the identities

$$\gamma^{\log_\gamma(x)} = \langle x \rangle, \qquad x \in \mathbf{Z}_p^\times,$$
$$\log_\gamma(\gamma^x) = x, \qquad x \in \mathbf{Z}_p.$$

The definition of $\log_\gamma$ given here is compatible with the function of the same name defined at the end of Chapter III: if $\alpha \in \mathbf{Z}_p^\times$ is any lift of some fixed $a \in (\mathbf{Z}/p^{n+1}\mathbf{Z})^\times$ then $a \equiv \omega(\alpha)\langle\alpha\rangle \equiv \omega(\alpha)\gamma^{\log_\gamma(\alpha)} \bmod p^{n+1}$, thus $\log_\gamma(a)$ (from Chapter III) is equal to $\log_\gamma(\alpha) \bmod p^n$ (which is independent of the lift).

4.1.3 Analyticity

A continuous character p-adic character $\chi : \mathbf{Z}_p^\times \to \mathbf{C}_p^\times$ is called *tame* if it factors through a power of ω and is called *wild* otherwise. Any continuous character $\chi : \mathbf{Z}_p^\times \to \mathbf{C}_p^\times$ has a unique decomposition $\chi = \psi\chi_u$, for some tame character ψ and

some $u \in \mathbf{C}_p$ with $|u-1| < 1$, where χ_u $(= u^{\log_\gamma(\cdot)})$ is the wild character defined by

$$\mathbf{Z}_p^\times \xrightarrow{x \mapsto \langle x \rangle} 1 + p\mathbf{Z}_p \xrightarrow{\gamma \mapsto u} \mathbf{C}_p^\times.$$

Explicitly, given a continuous character χ, the above decomposition is $\chi = \psi\chi_{\chi(\gamma)}$ where ψ is the tame character $x \mapsto \chi(\omega(x))$. This is because

$$\chi(x) = \chi(\omega(x)\langle x \rangle) = \psi(x)\chi(\langle x \rangle) = \psi(x)\chi(\gamma)^{\log_\gamma x} = \psi(x)\chi_{\chi(\gamma)}(x),$$

where the third equality uses the continuity of χ.

For a fixed tame character ψ, Visik [41] and Amice-Vélu [2] have shown that an h-admissible measure μ is analytic at $u = 1$ in the sense that the function

$$\{u \in \mathbf{C}_p \mid |u-1| < 1\} \to \mathbf{C}_p, \qquad u \mapsto \mu(\psi\chi_u)$$

can be expanded about $u = 1$ to obtain a power series

$$\mu(\psi, T) := \mu(\psi\chi_u) \in L[[T]], \quad T = u - 1,$$

which converges on the open unit disc in $\mathbf{C}_p$. (Here $L/\mathbf{Q}_p$ is any field containing all values of μ.) Following [34], this power series can be obtained explicitly as follows

$$\begin{aligned}
\int_{\mathbf{Z}_p^\times} \psi\chi_u(x)d\mu(x) &= \int_{\mathbf{Z}_p^\times} \psi(x)u^{\log_\gamma(x)}d\mu(x) \\
&= \int_{\mathbf{Z}_p^\times} \psi(x)(1+T)^{\log_\gamma(x)}d\mu(x) \\
&= \int_{\mathbf{Z}_p^\times} \psi(x)\sum_{n\geq 0}\binom{\log_\gamma(x)}{n}T^n d\mu(x) \\
&= \sum_{n\geq 0}\left(\int_{\mathbf{Z}_p^\times}\psi(x)\binom{\log_\gamma(x)}{n}d\mu(x)\right)T^n.
\end{aligned}$$

In particular, this tells us that the constant term of the power series associated to μ is simply the total measure of ψ: we have

$$\mu(\psi, 0) = \int_{\mathbf{Z}_p^\times} \psi d\mu.$$

Summarizing the discussion above, we have the following.

Theorem 4.1.3. *[Visik [41], Amice-Vélu [2]] Associated to an h-admissible measure $\mu \in \mathcal{A}^h$ and a fixed tame character ψ is a power series $\mu(\psi, T) \in L[[T]]$ converging on the open unit disc in $\mathbf{C}_p$.*

4.2 p-adic L-functions of modular forms

For the remainder of this chapter, we fix a newform $f = \sum a_n q^n \in S_k(N, \varepsilon)$ and assume always that $p \nmid N$. Associated to f are *period integrals* defined by

$$\phi_f(P, r) = 2\pi i \int_r^{i\infty} f(z)P(z)dz,$$

where $P \in \mathbf{C}[z]$ has degree $\leq k-2$ and $r \in \mathbf{Q}$. Define the $\pm$-parts of these integrals by

$$\lambda_f^{\pm}(P; a, m) := \frac{\phi_f(P(mz-a), a/m) \pm (-1)^{k-\deg P}\phi_f(P(mz+a), -a/m)}{2\Omega_f^{\pm}},$$

where $\Omega_f^{\pm}$ is the complex period specified in Chapter III.3.2.5 making the modular symbol $\varphi_f^{\pm}$ cohomological. Since ϕ_f is linear in P, observe that $\lambda_f^{\pm}(P; a, m)$ can be computed in terms of the values $\lambda_f^{\pm}(z^j; a, m)$ for $0 \leq j \leq k-2$ and that these values can in turn be computed in terms of $\varphi_f^{\pm}$. In particular, we see that $\lambda_f^{\pm}(P; a, m)$ takes values in K_f and, via our fixed embedding $\bar{\mathbf{Q}} \hookrightarrow \bar{\mathbf{Q}}_p$, we henceforth consider $\lambda_f^{\pm}(P; a, m)$ as being K-valued. In particular, our definitions yield

$$\binom{k-2}{j}\lambda_f^{\pm}(z^j; a, m) = \left[\frac{a}{m}\right]_{f,j}^{\pm} \in \mathcal{O}. \tag{4.1}$$

Now fix an *allowable* root α of $X^2 - a_p X + \varepsilon(p)p^{k-1}$ (i.e, one having valuation $\mathrm{ord}_p(\alpha) < k-1$). By [30, Proposition 2.8], the map

$$\mu_{f,\alpha}^{\pm}(P \cdot \mathbb{1}_{a+p^n\mathbf{Z}_p}) := \frac{1}{\alpha^n}\left(\lambda_f^{\pm}(P; a, p^n) - \frac{\varepsilon(p)p^{k-2}}{\alpha}\lambda_f^{\pm}(P; a, p^{n-1})\right) \in K(\alpha),$$

defines an $h = \mathrm{ord}_p(\alpha)$-admissible measure. By Theorem 4.1.1, $\mu_{f,\alpha}^{\pm}$ extends to a locally analytic p-adic distrubution, which we also denote $\mu_{f,\alpha}^{\pm}$.

Definition 4.2.1. The p-adic L-function for the pair (f, α) is defined by

$$L_p(f, \alpha, \chi) := \mu_{f,\alpha}^{*}(\chi),$$

where $\chi \in \mathrm{Hom}_{cts.}(\mathbf{Z}_p^{\times}, \mathbf{C}_p^{\times})$ and $* = \mathrm{sgn}(\chi(-1)(-1)^k)$.

By Theorem 4.1.3, for each tame character ψ we have an associated power series

$$L_p(f, \alpha, \psi, T) := L_p(f, \alpha, \psi\chi_u) \in K(\alpha)[[T]], \quad T = u - 1,$$

which converges on the open unit disc in $\mathbf{C}_p$.

4.2.1 Interpolation property

Proposition 4.2.2. *If $\chi : (\mathbf{Z}/p^n\mathbf{Z})^{\times} \to \mathbf{Z}_p^{\times}$ is primitive and nontrivial then for all $0 \leq j \leq k-2$ we have*

$$L_p(f, \alpha, x^j\chi) = \frac{1}{\alpha^n} \frac{j! p^{nj} \tau(\chi)}{\Omega_f^{\pm}(-2\pi i)^j} \cdot L(f_{\chi^{-1}}, j+1),$$

where $\pm = \mathrm{sgn}(\chi(-1)(-1)^{k-j})$.

Proof. This is essentially [25, Proposition 14], but the details are as follows. Viewing χ as a character on $\mathbf{Z}_p^{\times}$, equation (4.1) and the definitions give

$$\begin{aligned} L_p(f, \alpha, x^j\chi) &= \int_{\mathbf{Z}_p^{\times}} x^j \chi d\mu_{f,\alpha}^{\pm} \\ &= \sum_{a \in (\mathbf{Z}/p^n\mathbf{Z})^{\times}} \chi(a) \int_{a+p^n\mathbf{Z}_p} x^j d\mu_{f,\alpha}^{\pm} \\ &= \frac{1}{\alpha^n} \binom{k-2}{j}^{-1} \sum_{a \in (\mathbf{Z}/p^n\mathbf{Z})^{\times}} \chi(a) \left(\left[\frac{a}{p^n} \right]_{f,j}^{\pm} - \frac{\varepsilon(p) p^{k-2}}{\alpha} \left[\frac{a}{p^{n-1}} \right]_{f,j}^{\pm} \right). \end{aligned}$$

The sum $\sum_a \chi(a)[a/p^n]^{\pm}_{f,j}$ is simply the Mazur-Tate element $\vartheta^{\pm}_{n,j}(f)$ evaluated at χ and by Proposition III.3.2.3 this recovers the correct L-values:

$$\sum_{a\in(\mathbf{Z}/p^n\mathbf{Z})^\times} \chi(a)\left[\frac{a}{p^n}\right]^{\pm}_{f,j} = \chi(\vartheta^{\pm}_{n,j}(f)) = \binom{k-2}{j}\frac{j!p^{nj}\tau(\chi)}{\Omega^{\pm}_f(-2\pi i)^j}\cdot L(f_{\chi^{-1}}, j+1).$$

Thus we need only show that $\sum_a \chi(a)[a/p^{n-1}]^{\pm}_{f,j} = 0$. Since χ is nontrivial and primitive there is some $a' \in (\mathbf{Z}/p^n\mathbf{Z})^\times$ with $a' \equiv 1 \bmod p^{n-1}$ such that $\chi(a') \neq 1$. Fixing $b \in (\mathbf{Z}/p^{n-1}\mathbf{Z})^\times$, we now have

$$\chi(a')\sum_{\substack{a\in(\mathbf{Z}/p^n\mathbf{Z})^\times \\ a\equiv b \bmod p^{n-1}}} \chi(a) = \sum_{\substack{a\in(\mathbf{Z}/p^n\mathbf{Z})^\times \\ a\equiv b \bmod p^{n-1}}} \chi(aa') = \sum_{\substack{a\in(\mathbf{Z}/p^n\mathbf{Z})^\times \\ a\equiv b \bmod p^{n-1}}} \chi(a),$$

hence $(\chi(a')-1)\sum_{a\equiv b \bmod p^{n-1}} \chi(a) = 0$ and therefore $\sum_{a\equiv b \bmod p^{n-1}} \chi(a) = 0$. Because $[a/p^{n-1}]^{\pm}_{f,j} = \lambda^{\pm}_f(x^j; a, p^{n-1})$ depends only on $a \bmod p^{n-1}$ (see [25]), we have

$$\sum_{a\in(\mathbf{Z}/p^n\mathbf{Z})^\times} \chi(a)[a/p^{n-1}]^{\pm}_{f,j} = \sum_{b\in(\mathbf{Z}/p^{n-1}\mathbf{Z})^\times} [b/p^{n-1}]^{\pm}_{f,j} \sum_{\substack{a\in(\mathbf{Z}/p^n\mathbf{Z})^\times \\ a\equiv b \bmod p^{n-1}}} \chi(a) = 0.$$

□

4.2.2 Ordinary vs. nonordinary

Using our fixed embedding $\iota_p : \bar{\mathbf{Q}} \hookrightarrow \bar{\mathbf{Q}}_p$, we may view the Fourier coefficients of f as lying in $\bar{\mathbf{Q}}_p$. This allows us to make the following definition.

Definition 4.2.3. *We say that f is p-*ordinary *(resp. p-*nonordinary*) at ι_p if* $\mathrm{ord}_p(\iota_p(a_p)) = 0$ *(resp.,* $\mathrm{ord}_p(\iota_p(a_p)) > 0$*).*

If $\mathfrak{p}$ is the prime of K_f corresponding to ι_p then an equivalent definition is to say that f is ordinary (resp. nonordinary) at $\mathfrak{p}$ if $a_p \not\equiv 0 \bmod \mathfrak{p}$ (resp. $a_p \equiv 0 \bmod \mathfrak{p}$).

Notational convention. *We henceforth omit the embedding ι_p and simply write '$\mathrm{ord}_p\, x$' instead of '$\mathrm{ord}_p\, \iota_p(x)$' whenever $x \in \bar{\mathbf{Q}}$. Thus, the phrase "f is p-(non)ordinary" is always in reference to our chosen embedding.*

If f is ordinary at p then there is a unique allowable root $\alpha \in \mathcal{O}^\times$ of the Hecke polynomial $X^2 - a_p X + \varepsilon(p)p^{k-2}$ (seen easily by looking at the Newton polygon, for example). Hence, in the ordinary case we have a unique $h = 0$-admissible (i.e., bounded) measure $\mu_{f,\alpha}^{\pm}$ taking values in $\mathcal{O}$ and therefore the associated p-adic L-function $L_p(f, \psi, T) := L_p(f, \alpha, \psi, T)$ has integral coefficients. In the nonordinary case, both roots of the Hecke polynomial are allowable (although neither is a unit!), so in this case we can construct a p-adic L-function at either root but our coefficients need not be integral. Summarizing, for an allowable root α, we can say the following:

$$L_p(f, \alpha, \psi, T) \in \begin{cases} \mathcal{O}[[T]] & \text{if } f \text{ is } p\text{-ordinary,} \\ K(\alpha)[[T]] & \text{if } f \text{ is } p\text{-nonordinary.} \end{cases} \tag{4.2}$$

Remark 4.2.4. In the p-ordinary case, we may now consider the Iwasawa invariants $\lambda(L_p(f, \psi, T))$ and $\mu(L_p(f, \psi, T))$ attached to the p-adic L-function. In the p-nonordinary case, the coefficients of $L_p(f, \alpha, \psi, T)$ need not be bounded (in fact, they are known to be unbounded in many instances – see [30] or [21]), thus we have no obvious Iwasawa invariants available. The following section outlines a method to extract meaningful Iwasawa invariant in the special case where $a_p = 0$.

4.3 p-adic L-functions of modular forms: case $a_p = 0$

Assume from now on that $a_p(f) = 0$. The goal of this section is to introduce Pollack's decomposition of the p-adic L-function for f and to use this decomposition

to compute the valuation of certain algebraic L-values. Specifically, we prove the following generalization of [30, Proposition 6.9] and [38].

Proposition 4.3.1. *If χ is a character on $\mathbf{Z}_p^\times$ of order p^n then for $n \gg 0$ we have*

$$\mathrm{ord}_p\left(\frac{j!p^{j(n+1)}\tau(\chi)}{\Omega_f^{\pm}(-2\pi i)^j}L(f_{\chi^{-1}},j+1)\right) = \mu(L_{p,\psi\omega^j}^{-\varepsilon_n}) + \frac{(k-1)q_n + \lambda(L_{p,\psi\omega^j}^{-\varepsilon_n})}{p^n - p^{n-1}},$$

where $L_p^{\pm} = L_p^{\pm}(f)$ are Pollack's plus/minus p-adic L-functions (defined below), q_n is a certain finite sum of powers of p (see (4.4)), and ψ is the tame part of χ.

4.3.1 Pollack decomposition of p-adic L-function

Regardless of whether or not $a_p = 0$, we can define the power series

$$G_\psi^+(f,T) = \frac{L_p(f,\alpha_1,\psi,T) + L_p(f,\alpha_2,\psi,T)}{2} \in K[[T]],$$
$$G_\psi^-(f,T) = \frac{L_p(f,\alpha_1,\psi,T) - L_p(f,\alpha_2,\psi,T)}{2\alpha_1} \in K[[T]],$$

where α_1, α_2 are the roots of $X^2 - a_pX + \varepsilon(p)p^{k-1}$, and obtain a decomposition

$$L_p(f,\alpha,\psi,T) = G_\psi^+(f,T) + \alpha G_\psi^-(f,T), \qquad \alpha = \alpha_1, \alpha_2.$$

As we are assuming $a_p = 0$, there is strong symmetry between the roots α_1 and α_2 (one being the negative of the other) and by using this symmetry together with the interpolation property for the p-adic L-function, Pollack [30] was able to refine the above decomposition by determining (and removing) all but finitely many roots of $G_\psi^{\pm}$, allowing for the definition of functions $L_p^{\pm}$ with bounded coefficients. Explicitly, define the *half-logarithms of weight k* by

$$\log_k^{\pm}(T) = p^{-(k-1)}\prod_{j=0}^{k-2}\prod_{\substack{n=1\\ \varepsilon_n=\pm}}^{\infty}\frac{\Phi_n\big(\gamma^{-j}(1+T)\big)}{p} \in \mathbf{Q}_p[[T]],$$

where $\varepsilon_n = \mathrm{sgn}(-1)^n$ and $\Phi_n(T) = \sum_{i=0}^{p-1} T^{ip^{n-1}}$ is the p^nth cyclotomic polynomial.

Proposition 4.3.2. *The function* $\log_k^{\pm}$ *converges on the open unit disc in* $\mathbf{C}_p$ *and its set* $Z^{\pm}$ *of zeroes is given by*

$$Z^{\pm} = \{\gamma^j \zeta_n - 1 \mid \varepsilon_n = \pm,\, n \geq 1,\, 0 \leq j \leq k-2\},$$

where ζ_n *denotes a primitive* p^n*th root of unity.*

Proof. See [30, Lemma 4.1]. □

The interpolation property for the p-adic L-function forces $G_{\psi}^{\pm}(f)$ to also have roots on the set $Z^{\pm}$, allowing us to define the *plus/minus p-adic L-functions* by

$$L_{p,\psi}^{\pm}(f,T) = \frac{G_{\psi}^{\pm}(f,T)}{\log_k^{\pm}(T)}.$$

Theorem 4.3.3 (Pollack)**.** *The functions* $L_{p,\psi}^{\pm}(f)$ *lie in* $K \otimes \mathcal{O}[[T]]$ *and we have*

$$L_p(f, \alpha_i, \psi, T) = \log_k^{+}(T) L_{p,\psi}^{+}(f,T) + \alpha_i \log_k^{-}(T) L_{p,\psi}^{-}(f,T).$$

Proof. See [30, Theorem 5.1]. □

Remark 4.3.4. While there are no Iwasawa invariants attached to the p-adic L-function $L_p(f, \alpha, \psi)$ itself, since the plus/minus p-adic L-functions have bounded coefficients we can now consider the *pairs* of invariants $\lambda(L_{p,\psi}^{\pm}(f))$ and $\mu(L_{p,\psi}^{\pm}(f))$.

4.3.2 Valuation of half-logarithms

In this section we compute the p-adic valuations of Pollack's half-logarithms at the values $\gamma^j \zeta_n - 1$. We begin with an elementary lemma.

Lemma 4.3.5. *For any* $u \in 1 + p\mathbf{Z}_p$ *and* $n \geq 0$ *we have*

$$\operatorname{ord}_p\left(\Phi_m(u\zeta_n)\right) = \begin{cases} \frac{p^m - p^{m-1}}{p^n - p^{n-1}} & \text{if } m < n, \\ 1 & \text{if } m > n, \end{cases}$$

Proof. Suppose $1 \leq m < n$. Observe that

$$\Phi_m(u\zeta_n) = \frac{(u\zeta_n)^{p^m} - 1}{(u\zeta_n)^{p^{m-1}} - 1} = u^{(p-1)p^{m-1}} \left(\frac{\zeta_{n-m} - u^{-p^m}}{\zeta_{n-m+1} - u^{-p^{m-1}}} \right).$$

Note that $u^{-p^m} \in 1 + p^{m+1}\mathbf{Z}_p$, so we can write $u^{-p^m} = 1 + p^{m+1}u'$ for some $u' \in \mathbf{Z}_p$. Hence,

$$\begin{aligned} \operatorname{ord}_p(\zeta_{n-m} - u^{-p^m}) &= \operatorname{ord}_p(\zeta_{n-m} - 1 - p^{m+1}u') \\ &= \min\big(\operatorname{ord}_p(\zeta_{n-m} - 1), \operatorname{ord}_p(p^{m+1}u')\big) \\ &= \frac{1}{p^{n-m} - p^{n-m-1}}, \end{aligned}$$

and the claimed valuation of $\Phi_m(u\zeta_n)$ follows. Now suppose $m > n$. Then $\zeta_n^{p^{m-1}} = 1$ and we have

$$\Phi_m(u\zeta_n) = 1 + \sum_{t=1}^{p-1} u^{tp^{m-1}} \equiv 1 + (p-1) \bmod p^m \equiv p \bmod p^m,$$

because $u^{tp^{m-1}} \in 1 + p^m\mathbf{Z}_p$ for $1 \leq t \leq p-1$. Thus $\Phi_m(u\zeta_n) \in p + p^m\mathbf{Z}_p$ and it follows that $\operatorname{ord}_p\big(\Phi_m(u\zeta_n)\big) = 1$ for $m \geq 2$. If $m = 1$ (hence $n = 0$) one can check directly that $\Phi_1(u) = p \bmod p^2$. □

Define the polynomials

$$\omega_{n,j}^{\pm} = \prod_{\substack{i=1 \\ \varepsilon_i = \pm}}^{n} \Phi_i(\gamma^{-j}(1+T)), \qquad \log_{k,n}^{\pm} = \prod_{j=0}^{k-2} \omega_{n,j}^{\pm} \in \mathbf{Z}_p[T].$$

Notice that we can recover the usual half-logarithm as limits of $\log_{k,n}^{\pm}$: as functions converging on the open unit disc we have

$$\lim_{n\to\infty} p^{-(k-1)(1+\delta_n^{\pm})} \log_{k,n}^{\pm} = \log_k^{\pm},$$

where $\delta_n^+ = \lfloor n/2 \rfloor$ and $\delta_n^- = \lfloor (n+1)/2 \rfloor$. We call the functions $\log_{k,n}^{\pm}$ *half-logarithms at layer* n, since (as we will see in the next chapter) they can be used to describe the behavior of $\log_k^{\pm}$ modulo $\omega_n = (1+T)^{p^n} - 1$.

To deal with the tail-end of the half-logarithms, we also define

$$u_{n,j}^{\pm}(T) = \prod_{\substack{i=n+1 \\ \varepsilon_i=\pm}}^{\infty} \frac{\Phi_i\big(\gamma^{-j}(1+T)\big)}{p}, \qquad v_{k,n}^{\pm}(T) = \prod_{j=0}^{k-2} u_{n,j}^{\pm}(T) \in \mathbf{Q}_p[[T]]$$

and note the relation

$$\log_k^{\pm} = p^{-(k-1)(1+\delta_n^{\pm})} \log_{k,n}^{\pm} v_{k,n}^{\pm}. \tag{4.3}$$

In particular, $\log_{k,n}^{\pm}$ divides $\log_k^{\pm}$ in $\mathbf{Q}_p[[T]]$ and since the latter function converges on the open unit disc, $v_{k,n}^{\pm}$ must also converge on this disc.

Lemma 4.3.6. *For any $1 \leq m \leq n$ and $j \in \mathbf{Z}$ we have*

$$v_{k,n}^{\pm}(\gamma^j \zeta_m - 1) = v_{k,n}^{\pm}(\gamma^j - 1) \in \mathbf{Z}_p^{\times}.$$

Proof. The equality follows directly from the fact that $\Phi_i(x\zeta_m) = \Phi_i(x)$ for all $i \geq n+1$. To show these values are units, note that for any $x \in 1 + p\mathbf{Z}_p$ and $i \geq 1$ we have $\operatorname{ord}_p \Phi_i(x) = 1$ by Lemma 4.3.5. Thus $u_{n,j'}^{\pm}(\gamma^j - 1)$ are units for any $j, j' \in \mathbf{Z}$, and hence $v_{k,n}^{\pm}(\gamma^j - 1)$ must also be units. □

For convenience, we now define the *Kurihara terms*

$$q_n = \begin{cases} p^{n-1} - p^{n-2} + \cdots + p - 1 & n \geq 2 \text{ even}, \\ p^{n-1} - p^{n-2} + \cdots + p^2 - p & n \geq 2 \text{ odd}, \\ 0 & n \leq 1. \end{cases} \tag{4.4}$$

Lemma 4.3.7. *For $n \geq 1$ we have*

$$\operatorname{ord}_p\left(\log_k^{+}(\gamma^j \zeta_n - 1)\right) = \begin{cases} (k-1)\left(\frac{q_n}{p^n - p^{n-1}} - \frac{n+1}{2}\right) & n \text{ odd} \\ \infty & n \text{ even} \end{cases}$$

$$\operatorname{ord}_p\left(\log_k^{-}(\gamma^j \zeta_n - 1)\right) = \begin{cases} (k-1)\left(\frac{q_n}{p^n - p^{n-1}} - \frac{n+2}{2}\right) & n \text{ even} \\ \infty & n \text{ odd.} \end{cases}$$

Proof. The infinite valuations follow from the fact that $\log_{p,k}^{+}$ (resp., $\log_{p,k}^{-}$) vanishes at $\gamma^j\zeta_n - 1$ for n even (resp., n odd). For the other cases, Lemma 4.3.6 and (4.3) give

$$\operatorname{ord}_p \log_k^{-\varepsilon_n}(\gamma^j\zeta_n - 1) = -(k-1)(1+\delta_n^{-\varepsilon_n}) + \operatorname{ord}_p(\log_{k,n}^{-\varepsilon_n}(\gamma^j\zeta_n - 1)).$$

It now suffices to compute the valuation of $\Phi_i(\gamma^{j-j'}\zeta_n)$ for $0 \le j' \le k-2$. But this was done in Lemma 4.3.5 above, which we apply to each factor in $\omega_{n,j'}^{\pm}$ to give

$$\operatorname{ord}_p \omega_{n,j'}^{-\varepsilon_n}(\gamma^j\zeta_n - 1) = \sum_{\substack{i=1 \\ \varepsilon_i = -\varepsilon_n}}^{n} \frac{p^i - p^{i-1}}{p^n - p^{n-1}} = \frac{q_n}{p^n - p^{n-1}}.$$

It follows that $\operatorname{ord}_p \log_{k,n}^{-\varepsilon_n}(\gamma^j\zeta_n - 1) = (k-1)q_n/(p^n - p^{n-1})$ and we are done. □

4.3.3 Valuation of algebraic parts of L-values

Proof of Proposition 4.3.1. Writing $\chi = \psi\chi_{\zeta_n}$, where $\zeta_n = \chi(\gamma)$ and $\psi = \chi \circ \omega$, we decompose $x^j\chi = \psi\omega^j \cdot \chi_{\gamma^j\zeta_n}$ into its tame and wild parts. By Proposition 4.2.2, it suffices[2] to compute the valuation of

$$\begin{aligned}\alpha^{n+1}L_p(f,\alpha,x^j\chi) &= \alpha^{n+1}L_p(f,\alpha,\psi\omega^j,\gamma^j\zeta_n - 1) \\ &= \begin{cases} \alpha^{n+2}L_{p,\psi\omega^j}^{-}(f,\gamma^j\zeta_n - 1)\log_k^{-}(\gamma^j\zeta_n - 1) & n \text{ even}, \\ \alpha^{n+1}L_{p,\psi\omega^j}^{+}(f,\gamma^j\zeta_n - 1)\log_k^{+}(\gamma^j\zeta_n - 1) & n \text{ odd}, \end{cases}\end{aligned}$$

where the latter equality is due to the Pollack decomposition and the known roots of the half-logarithms $\log_k^{\pm}$. We already know the valuation of the half-logarithms from Lemma 4.3.7. We also know that $\operatorname{ord}_p(\alpha) = (k-1)/2$ since α is a root of $X^2 + \varepsilon(p)p^{k-1}$. Thus, we need only consider the contribution coming from the plus/minus p-adic L-functions.

[2]Since χ has order p^n we know that $\chi(\gamma^{p^n}) = 1$ and hence χ is trivial on $\gamma^{p^n}\mathbf{Z}_p = 1 + p^{n+1}\mathbf{Z}_p$. Thus χ defines a primitive Dirichlet character of conductor p^{n+1} and we can use Proposition 4.2.2.

Write $\mu^{\pm} = \mu(L^{\pm}_{p,\psi\omega^j}(f))$ and $\lambda^{\pm} = \lambda(L^{\pm}_{p,\psi\omega^j}(f))$. Consider the Weierstrass decomposition $L^{\pm}_{p,\psi\omega^j}(f) = p^{\mu^{\pm}} D^{\pm} U^{\pm}$ where $D^{\pm}$ is distinguished of degree $\lambda^{\pm}$ and $U^{\pm} \in \mathcal{O}[[T]]^{\times}$. We can take $n \gg 0$ so that $\lambda^{\pm} < (p^n - p^{n-1})/e$, where e is the ramification index of $K/\mathbf{Q}_p$, so after checking[3] that $\mathrm{ord}_p(\gamma^j\zeta_n - 1) = 1/(p^n - p^{n-1})$, Lemma II.2.1.5 yields

$$\mathrm{ord}_p\, L^{\pm}_{p,\psi\omega^j}(f, \gamma^j\zeta_n - 1) = \mu^{\pm} + \frac{\lambda^{\pm}}{p^n - p^{n-1}},$$

and we are done. □

Remark 4.3.8. The above proposition is well-known in the case of rational elliptic curves with good supersingular reduction at p. In this case, it can be used (see [30], [38], or [29]) to compute the growth of Shafarevich-Tate groups (specifically, their analytic size as predicted by the Birch and Swinnerton-Dyer conjecture) along the cyclotomic $\mathbf{Z}_p$-extension of $\mathbf{Q}$. Assuming the Bloch-Kato conjecture [4], one might expect the valuations computed above to be related to the size of Shafarevich-Tate groups for certain twists of the p-adic Galois representation associated to f.

Remark 4.3.9. Under the assumptions of Theorem VII.7.2.1, we also have

$$\mathrm{ord}_p\left(\binom{k-2}{j}\frac{j!p^{j(n+1)}\tau(\chi)}{\Omega_f^{\pm}(-2\pi i)^j}L(f_{\chi^{-1}}, j+1)\right) = \mu\big(\theta^{\psi}_{n,j}(f)\big) + \frac{\lambda\big(\theta^{\psi}_{n,j}(f)\big)}{p^n - p^{n-1}}$$

for $n \gg 0$. This is due to the equality (a consequence of Theorem VII.7.2.1):

$$\lambda(L^{-\varepsilon_n}_{p,\psi\omega^j}) + \mu(L^{-\varepsilon_n}_{p,\psi\omega^j})(p^n - p^{n-1}) = \lambda(\theta^{\psi}_{n,j}) + \mu(\theta^{\psi}_{n,j})(p^n - p^{n-1}) - (k-1)q_n - \mathrm{ord}_p\tbinom{k-2}{j}.$$

[3]Setting $u = \gamma^j \in 1 + p\mathbf{Z}_p$, we have $\mathrm{ord}_p(u\zeta_n - 1) = \mathrm{ord}_p(\zeta_n - u^{-1})$. The rest follows by writing $u^{-1} = 1 + p\beta$ for some $\beta \in \mathbf{Z}_p$.

CHAPTER 5

LIFTS OF MAZUR-TATE ELEMENTS

Fix a newform $f = \sum a_n q^n \in S_k(N, \varepsilon)$, an odd prime $p \nmid N$, a tame character ψ, and an integer $0 \leq j \leq k-2$. Assume $a_p = 0$. Let K and $\mathcal{O}$ be as in §III.3.2.5 and write Λ, Λ_K, Λ_n, $\Lambda_{K,n}$ for the associated Iwasawa algebras. In this chapter we construct lifts of the Mazur-Tate elements $\theta^{\psi}_{n,j}(f) \in \Lambda_n$ to the full Iwasawa algebra Λ_K. The lifts obtained incorporate Pollack's plus/minus p-adic L-functions and the layer-n half-logarithms of the previous chapter and they are central to our method (see Chapter VII) for computing Iwasawa invariants of Mazur-Tate elements. Indeed, combined with results of Chapter II, the existence of these lifts alone is already enough to determine the Iwasawa invariants at weights $k \leq p+1$ – see §VII.7.2.1 for an outline of this special case. The main theorem of this chapter is as follows.

Theorem 5.0.1. *For all $n \geq 0$ there are explicit units $u_{n,j} \in \mathbf{Z}_p[\varepsilon(p)]^{\times}$ such that*

$$\theta^{\psi}_{n,j}(f) \equiv u_{n,j}\tbinom{k-2}{j}\eta^j\big(L^{-\varepsilon_n}_{p,\psi\omega^j}(f)\log^{-\varepsilon_n}_{k,n}\big) \bmod \omega_n,$$

where $\eta \in \mathrm{Aut}(\Lambda_K)$ is the map $F(T) \mapsto F\big(\gamma(1+T)-1\big)$.

5.1 Riemann sums

For this section only, we drop the assumption that $a_p = 0$. Fix $n \geq 0$, let α_1,α_2 be roots of the polynomial $X^2 - a_p X + \varepsilon(p)p^{k-1}$, let $* = \operatorname{sgn}\psi(-1)(-1)^k$, and define

$$L_{n,j}(f, \alpha_i, \psi, T) = \binom{k-2}{j} \sum_{a \in (\mathbf{Z}/p^{n+1}\mathbf{Z})^\times} \psi\omega^{-j}(a)\mu^*_{f,\alpha_i}(x^j; a+p^{n+1}\mathbf{Z}_p)(1+T)^{\log_\gamma(a)} \in K(\alpha_i)[T].$$

These polynomials are related to Riemann sums as follows: if χ is a character on $\mathbf{Z}_p^\times$ of order p^n and tame part $\psi\omega^{-j}$ then

$$\tbinom{k-2}{j} S_{n+1}(\mu^*_{f,\alpha_i}, x^j\chi) = L_{n,j}(f, \alpha_i, \psi, \chi(\gamma) - 1),$$

where $S_n(\mu, \cdot)$ denotes the nth generalized Riemann sum associated to an h-admissible measure μ (see §IV.4.1.1). This equation follows from the fact that the character $\chi = \psi\omega^{-j}\chi_u$, where $u = \chi(\gamma) \in \mu_{p^n}$, is constant on $1 + p^{n+1}\mathbf{Z}_p = \gamma^{p^n\mathbf{Z}_p}$ hence

$$S_{n+1}(\mu^*_{f,\alpha_i}, x^j\chi) = \sum_{a \in (\mathbf{Z}/p^{n+1}\mathbf{Z})^\times} \psi\omega^{-j}(a)u^{\log_\gamma(a)} \int_{a+p^{n+1}\mathbf{Z}_p} x^j d\mu^*_{f,\alpha_i}.$$

Given this, it is not so surprising that we can use the polynomials $L_{n,j}$ to evaluate the p-adic L-function at characters of finite order (i.e., at values $\gamma^j\zeta_m - 1$ where ζ_m is a primitive p^mth root of unity).

Proposition 5.1.1. *If $0 \leq m \leq n$ then*

$$\tbinom{k-2}{j} L_p(f, \alpha_i, \psi, \gamma^j\zeta_m - 1) = L_{n,j}(f, \alpha_i, \psi, \zeta_m - 1).$$

Proof. Fix $i \in \{1, 2\}$ and set $\alpha = \alpha_i$. By definition, evaluating $L_p(f, \alpha, \psi, T)$ at $T = \gamma^j\zeta_m - 1$ amounts to integrating the character $\psi\chi_{\gamma^j\zeta_m}$. Since $\chi_\gamma(x) = \gamma^{\log_\gamma(x)} = \langle x \rangle = x\omega^{-1}(x)$ we can write $\chi_{\gamma^j\zeta_m} = \omega^{-j}\chi_{\zeta_m}x^j$. Noting that $\psi\omega^{-j}\chi_{\zeta_m}$ is constant

(in fact, trivial) on $1 + p^{m+1}\mathbf{Z}_p$ yields

$$\begin{aligned} L_p(f, \alpha, \psi, \gamma^j\zeta_m - 1) &= L_p(f, \alpha, \psi\omega^{-j}\chi_{\zeta_m}x^j) \\ &= \sum_{a \in (\mathbf{Z}/p^{m+1}\mathbf{Z})^\times} \psi\omega^{-j}\chi_{\zeta_m}(a)\mu^*_{f,\alpha}(x^j; a + p^{m+1}\mathbf{Z}_p) \\ &= \sum_{a \in (\mathbf{Z}/p^{m+1}\mathbf{Z})^\times} \psi\omega^{-j}(a)\mu^*_{f,\alpha}(x^j; a + p^{m+1}\mathbf{Z}_p)\zeta_m^{\log_\gamma(a)} \\ &= \tbinom{k-2}{j}^{-1} L_{m,j}(\alpha, \psi, \zeta_m - 1), \end{aligned}$$

where $* = \operatorname{sgn}(\psi\omega^{-j}\chi_{\zeta_m}x^j(-1)^k) = \operatorname{sgn}(\psi(-1)(-1)^k)$ and $\chi_{\zeta_m}(a)$ denotes the value of χ_{ζ_m} on the coset in $\mathbf{Z}_p^\times/(1+p^{m+1}\mathbf{Z}_p) \cong (\mathbf{Z}/p^{m+1}\mathbf{Z})^\times$ corresponding to a. It then follows from Lemma 5.1.2 bellow that $L_{m,j}(\alpha, \psi, \zeta_m - 1) = L_{n,j}(f, \alpha, \psi, \zeta_m - 1)$ for all $0 \le m \le n$. □

Lemma 5.1.2. *Let μ be any finitely additive function on subsets of $\mathbf{Z}_p^\times$ and let $L_n(T) = \sum_{a \in (\mathbf{Z}/p^{n+1}\mathbf{Z})^\times} \mu(a + p^{n+1}\mathbf{Z}_p)(1+T)^{\log_\gamma(a)}$. For all $0 \le m \le n$ we have $L_n(\zeta_m - 1) = L_m(\zeta_m - 1)$*

Proof. Since $a \equiv b \bmod p^{m+1}$ implies $\log_\gamma(a) \equiv \log_\gamma(b) \bmod p^m$ and

$$\coprod_{\substack{a \in (\mathbf{Z}/p^{n+1}\mathbf{Z})^\times \\ a \equiv b \bmod p^{m+1}}} a + p^{n+1}\mathbf{Z}_p = b + p^{m+1}\mathbf{Z}_p,$$

we have

$$\begin{aligned} L_n(\zeta_m - 1) &= \sum_{a \in (\mathbf{Z}/p^{n+1}\mathbf{Z})^\times} \mu(a + p^{n+1}\mathbf{Z}_p)\zeta_m^{\log_\gamma(a)} \\ &= \sum_{b \in (\mathbf{Z}/p^{m+1}\mathbf{Z})^\times} \sum_{\substack{a \in (\mathbf{Z}/p^{n+1}\mathbf{Z})^\times \\ a \equiv b \bmod p^{m+1}}} \mu(a + p^{n+1}\mathbf{Z}_p)\zeta_m^{\log_\gamma(b)} \\ &= \sum_{b \in (\mathbf{Z}/p^{m+1}\mathbf{Z})^\times} \mu(b + p^{m+1}\mathbf{Z}_p)\zeta_m^{\log_\gamma(b)} \\ &= L_m(\zeta_m - 1). \end{aligned}$$

□

Defining

$$G^+_{n,j}(\psi,T) = \frac{L_{n,j}(\alpha_1,\psi,T)+L_{n,j}(\alpha_2,\psi,T)}{2} \in K[T],$$
$$G^-_{n,j}(\psi,T) = \frac{L_{n,j}(\alpha_1,\psi,T)+L_{n,j}(\alpha_2,\psi,T)}{2\alpha_1} \in K[T],$$

we have a decomposition

$$L_{n,j}(\alpha,\psi,T) = G^+_{n,j}(\psi,T) + \alpha G^-_{n,j}(\psi,T), \qquad \alpha = \alpha_1, \alpha_2.$$

5.2 Relation to Mazur-Tate elements

For the remainder of this chapter we assume that $a_p = 0$. Thus $\alpha := \alpha_1 = \sqrt{-\varepsilon(p)p^{k-1}}$ and $\alpha_2 = -\alpha$. This symmetry between the roots allows us to compute $G^{\pm}_{n,j}$ explicitly.

Proposition 5.2.1. *Let* $* = \operatorname{sgn}\psi(-1)(-1)^k$. *Then*

$$G^+_{n,j}(\psi,T) = \begin{cases} \dfrac{1}{\alpha^{n+1}} \displaystyle\sum_{a\in(\mathbf{Z}/p^{n+1}\mathbf{Z})^\times} \psi\omega^{-j}(a)\left[\frac{a}{p^{n+1}}\right]^*_{f,j}(1+T)^{\log_\gamma(a)} & n \text{ odd,} \\ \dfrac{-\varepsilon(p)p^{k-2}}{\alpha^{n+2}} \displaystyle\sum_{a\in(\mathbf{Z}/p^{n+1}\mathbf{Z})^\times} \psi\omega^{-j}(a)\left[\frac{a}{p^{n}}\right]^*_{f,j}(1+T)^{\log_\gamma(a)} & n \text{ even.} \end{cases}$$

$$G^-_{n,j}(\psi,T) = \begin{cases} \dfrac{-\varepsilon(p)p^{k-2}}{\alpha^{n+3}} \displaystyle\sum_{a\in(\mathbf{Z}/p^{n+1}\mathbf{Z})^\times} \psi\omega^{-j}(a)\left[\frac{a}{p^{n}}\right]^*_{f,j}(1+T)^{\log_\gamma(a)} & n \text{ odd,} \\ \dfrac{1}{\alpha^{n+2}} \displaystyle\sum_{a\in(\mathbf{Z}/p^{n+1}\mathbf{Z})^\times} \psi\omega^{-j}(a)\left[\frac{a}{p^{n+1}}\right]^*_{f,j}(1+T)^{\log_\gamma(a)} & n \text{ even.} \end{cases}$$

In particular, $G^{\pm}_{n,j}(\psi,T) \in K[T]$.

Proof. From equation IV.(4.1) we know that

$$\binom{k-2}{j}\mu^{\pm}_{f,\alpha}(x^j, a+p^{n+1}\mathbf{Z}_p) = \frac{1}{\alpha^{n+1}}\left(\left[\frac{a}{p^{n+1}}\right]^{\pm}_{f,j} - \frac{\varepsilon(p)p^{k-2}}{\alpha}\left[\frac{a}{p^n}\right]^{\pm}_{f,j}\right).$$

Since $\alpha_1 = -\alpha_2$ this implies

$$\tbinom{k-2}{j}\big(\mu^*_{f,\alpha_1}(x^j; a+p^{n+1}\mathbf{Z}_p) + \mu^*_{f,\alpha_2}(x^j; a+p^{n+1}\mathbf{Z}_p)\big) = \begin{cases} \frac{2}{\alpha^{n+1}}\left[\frac{a}{p^{n+1}}\right]^*_{f,j} & n \text{ odd}, \\ \frac{-2\varepsilon(p)p^{k-2}}{\alpha^{n+2}}\left[\frac{a}{p^n}\right]^*_{f,j} & n \text{ even}, \end{cases}$$

$$\tbinom{k-2}{j}\big(\mu^*_{f,\alpha_1}(x^j; a+p^{n+1}\mathbf{Z}_p) - \mu^*_{f,\alpha_2}(x^j; a+p^{n+1}\mathbf{Z}_p)\big) = \begin{cases} \frac{-2\varepsilon(p)p^{k-2}}{\alpha^{n+2}}\left[\frac{a}{p^n}\right]^*_{f,j} & n \text{ odd}, \\ \frac{2}{\alpha^{n+1}}\left[\frac{a}{p^{n+1}}\right]^*_{f,j} & n \text{ even}. \end{cases}$$

The desired equalities now follow from the definition of $G^{\pm}_{n,j}(\psi, T)$. The final claim is because $\alpha = \sqrt{-\varepsilon(p)p^{k-1}}$ (thus $\alpha^{n+2} \in K$ for n even and $\alpha^{n+1} \in K$ for n odd) and $[a/p^n]^*_{f,j} \in \mathcal{O}$. □

Corollary 5.2.2. $\theta^{\psi}_{n,j}(f,T) = \begin{cases} \alpha^{n+1}G^+_{n,j}(\psi\omega^j, T) & n \textit{ odd}, \\ \alpha^{n+2}G^-_{n,j}(\psi\omega^j, T) & n \textit{ even}. \end{cases}$

Proof. By definition,

$$\theta^{\psi}_{n,j}(f,T) = \sum_{a \in (\mathbf{Z}/p^{n+1}\mathbf{Z})^\times} \psi(a)\left[\frac{a}{p^{n+1}}\right]^{\operatorname{sgn}\psi(-1)(-1)^{k+j}}_{f,j} (1+T)^{\log_\gamma(a)}.$$

Suppose n is odd (the even case is the same). By the previous proposition,

$$\alpha^{n+1}G^+_{n,j}(\psi\omega^j, T) = \sum_{a \in (\mathbf{Z}/p^{n+1}\mathbf{Z})^\times} \psi(a)\left[\frac{a}{p^{n+1}}\right]^{*}_{f,j} (1+T)^{\log_\gamma(a)}$$

where $* = \operatorname{sgn}(\psi\omega^j(-1)(-1)^k) = \operatorname{sgn}\psi(-1)(-1)^{k+j}$, and we are done. □

We now use Pollack's decomposition of the p-adic L-function for f to determine explicit roots of the polynomials $G^{\pm}_{n,j}$.

Proposition 5.2.3. *Let $G^{\pm}_{\psi}$ be as in §IV.4.3.1. Then*

$$\tbinom{k-2}{j}G^{\pm}_{\psi}(\gamma^j\zeta_m - 1) = G^{\pm}_{n,j}(\psi, \zeta_m - 1)$$

for $0 \le m \le n$. In particular, $G^{\pm}_{n,j}(\psi, 0) = \binom{k-2}{j}G^{\pm}_{\psi}(\gamma^j - 1)$ and $G^{\pm}_{n,j}(\psi, \zeta_m - 1) = 0$ for all $1 \le m \le n$ with $\varepsilon_m = \pm$.

Proof. Using the definitions and Proposition 5.1.1 we have

$$\begin{aligned}\tbinom{k-2}{j} G_\psi^+(\gamma^j\zeta_m - 1) &= \binom{k-2}{j}\left(\frac{L_p(f,\alpha_1,\psi,\gamma^j\zeta_m - 1) + L_p(f,\alpha_2,\psi,\gamma^j\zeta_m - 1)}{2}\right) \\ &= \frac{L_{n,j}(\alpha_1,\psi,\zeta_m - 1) + L_{n,j}(\alpha_2,\psi,\zeta_m - 1)}{2} \\ &= G_{n,j}^+(\psi,\zeta_m - 1),\end{aligned}$$

with an analogous computation for G_ψ^-. The second statement follows from the first by setting $m = 0$, and the final statement is a consequence of the vanishing properties of $G_\psi^\pm = L_{p,\psi}^\pm \log_k^\pm$ coming from Proposition IV.4.3.2. □

5.3 Proof of Theorem 5.0.1

Proof of Theorem 5.0.1. Define the units $u_{n,j} \in \mathbf{Z}_p[\varepsilon(p)]^\times$ by

$$u_{n,j} = \begin{cases} (-\varepsilon(p))^{\frac{n+1}{2}} v_{k,n}^+(\gamma^j - 1) & n \text{ odd}, \\ (-\varepsilon(p))^{\frac{n+2}{2}} v_{k,n}^-(\gamma^j - 1) & n \text{ even}, \end{cases}$$

where $v_{k,n}^\pm$ are as defined in §IV.4.3.2. (We know these are units by Lemma IV.4.3.6.) The roots of $\omega_n^\pm := \prod_{m=1,\,\varepsilon_m=\pm}^n \Phi_m(1+T)$ are $\zeta_m - 1$ for $1 \le m \le n$ with $\varepsilon_m = \pm$, hence Proposition 5.2.3 implies that we can find $H_{n,j}^\pm(\psi,T) \in K[T]$ such that

$$G_{n,j}^\pm(\psi,T) = \omega_n^\pm(T) H_{n,j}^\pm(\psi,T).$$

Thus,

$$\theta_{n,j}^\psi(f,T) = \begin{cases} \alpha^{n+1}\omega_n^+(T) H_{n,j}^+(\psi\omega^j,T) & n \text{ odd}, \\ \alpha^{n+2}\omega_n^-(T) H_{n,j}^-(\psi\omega^j,T) & n \text{ even}, \end{cases} \tag{5.1}$$

by Corollary 5.2.2.

Consider that for any $F \in \Lambda_K$ and $i \in \mathbf{Z}$ we have

$$\eta^i(F)\big(T\big) = F(\gamma^i(1+T)-1) \tag{5.2}$$

One can use this to check that $\eta^j(\omega_{n,j}^{\pm}) = \omega_{n,0}^{\pm} = \omega_n^{\pm}$, where $\omega_{n,j}^{\pm}$ are as in §IV.4.3.2, and therefore

$$\eta^j(\log_{k,n}^{\pm}) = \omega_n^{\pm} \cdot \eta^j\Bigg(\prod_{\substack{j'=0\\ j'\neq j}}^{k-2} \omega_{n,j'}^{\pm}\Bigg). \tag{5.3}$$

Since $\omega_n = T\omega_n^+\omega_n^-$, equations (5.1) and (5.3) show that it suffices to establish the following for any tame character ψ:

$$\alpha^{n+1}H_{n,j}^+(\psi,T) \equiv u_{n,j}\tbinom{k-2}{j}\eta^j\Bigg(L_{p,\psi}^+\prod_{\substack{j'=0\\ j'\neq j}}^{k-2}\omega_{n,j'}^+\Bigg) \bmod T\omega_n^-, \qquad \text{(for odd } n\text{)} \tag{5.4}$$

$$\alpha^{n+2}H_{n,j}^-(\psi,T) \equiv u_{n,j}\tbinom{k-2}{j}\eta^j\Bigg(L_{p,\psi}^-\prod_{\substack{j'=0\\ j'\neq j}}^{k-2}\omega_{n,j'}^-\Bigg) \bmod T\omega_n^+, \qquad \text{(for even } n\text{)} \tag{5.5}$$

To prove these congruences, we need only check that the left and right sides agree at all roots of the modulus $T\omega_n^-$ (resp., $T\omega_n^+$), i.e., at the values $T = 0$ and $T = \zeta_m - 1$ for all odd m with $1 \le m \le n$ (resp., all even m with $1 \le m \le n$). We now prove (5.4). Fix n odd and let $m = 0$ or take $1 \le m \le n$ with m odd. Then

$$\begin{aligned}
\alpha^{n+1}\tbinom{k-2}{j}^{-1}H_{n,j}^+(\psi,\zeta_m-1) &= \alpha^{n+1}\tbinom{k-2}{j}^{-1}\omega_n^+(\zeta_m-1)^{-1}G_{n,j}^+(\psi,\zeta_m-1)\\
&= \alpha^{n+1}\omega_n^+(\zeta_m-1)^{-1}G_{\psi}^+(\gamma^j\zeta_m-1)\\
&= \alpha^{n+1}\omega_n^+(\zeta_m-1)^{-1}L_{p,\psi}^+(\gamma^j\zeta_m-1)\log_k^+(\gamma^j\zeta_m-1)\\
&= \alpha^{n+1}\omega_n^+(\zeta_m-1)^{-1}\eta^j\big(L_{p,\psi}^+\log_k^+\big)(\zeta_m-1)\\
&= \alpha^{n+1}p^{-(k-1)(\frac{n+1}{2})}\omega_n^+(\zeta_m-1)^{-1}\eta^j\Bigg(L_{p,\psi}^+v_{k,n}^+\log_{k,n}^+\Bigg)(\zeta_m-1)\\
&= (-\varepsilon(p))^{\frac{n+1}{2}}v_{k,n}^+(\gamma^j\zeta_m-1)\eta^j\Bigg(L_{p,\psi}^+\textstyle\prod_{j'=0,j'\neq j}^{k-2}\omega_{n,j'}^+\Bigg)(\zeta_m-1)\\
&= u_{n,j}\eta^j\Bigg(L_{p,\psi}^+\textstyle\prod_{j'=0,j'\neq j}^{k-2}\omega_{n,j'}^+\Bigg)(\zeta_m-1)
\end{aligned}$$

where the first and third equalities are by definition, the second is from Proposition 5.2.3, the fourth is from (5.2), the fifth is from IV.(4.3), the sixth is because $\alpha = \sqrt{-\varepsilon(p)p^{k-1}}$, and the final equation follows from Lemma IV.4.3.6. This proves Theorem 5.0.1 for odd n. The proof of (5.5) is exactly the same: fix n even and take $0 \leq m \leq n$ with m even, then replace all instances of "$n+1$" with "$n+2$" and the superscripts "+" with "−" in the above string of equalities. $\square$

CHAPTER 6

FINITE-LAYER IWASAWA INVARIANTS

In this chapter we assume the notation of Chapter II and let K denote a fixed finite extension of $\mathbf{Q}_p$ with valuation ring $\mathcal{O}$. In §II.2.4, it was shown that the Iwasawa invariants of power series $F \in \Lambda_K$ are preserved upon projection to layer n as long as $\lambda(F) < p^n$. In this chapter, we consider what happens to the nth-layer Iwasawa invariants when $\lambda(F) \geq p^n$. This will allow us to compute the Iwasawa invariants of the Mazur-Tate elements for higher weight modular forms with $a_p = 0$ using the lifts obtained in the previous chapter.

The main idea is to derive an explicit formula for the remainder of a power series upon division by $\omega_n = (1+T)^{p^n} - 1$ and then study the p-adic properties of the coefficients of this remainder. More precisely, we consider the unique sequence of integers $(c_N^{(i)})_{N \geq 0}$, $0 \leq i \leq p^n - 1$, satisfying

$$T^{p^n+N} \equiv \sum_{i=0}^{p^n-1} c_N^{(i)} T^i \bmod \omega_n, \tag{6.1}$$

and observe that the image of F under the projection $\pi_n : \Lambda_K \to \Lambda_{K,n} = K[T]/(\omega_n)$ is uniquely determined by this sequence together with the coefficients of F: if $F = \sum_{i \geq 0} a_i T^i$ we have (see Proposition 6.2.2 below)

$$F \equiv \sum_{i=0}^{p^n-1} \left(a_i + \sum_{N=0}^{\infty} a_{p^n+N} c_N^{(i)} \right) T^i \bmod \omega_n. \tag{6.2}$$

As placeholders for the layer-n half-logarithms of the previous chapter (specifically, those with large weights), we consider a particular class of functions (see §6.3) that we call *p-large at n*. Roughly speaking, these are power series $F \in \Lambda_K$ with $\lambda(F) \geq p^n$ and large p-divisibility in their first $\lambda(F) - 1$ coefficients. 'Large' here means that the valuations of these initial coefficients are taken high enough so that, by (6.2), the behavior of $\pi_n(F)$ is more-or-less controlled by the integers $c_N^{(i)}$, whose p-adic properties can be described relatively concretely. Our main result explicitly computes the nth-layer invariants of such functions.

Theorem 6.0.1. *If $F \in \Lambda_K$ is p-large at $n \geq 1$ then*

$$\mu(\pi_n(F)) = \left\lfloor \frac{\lambda(F) - p^n}{p^n - p^{n-1}} \right\rfloor + \mu(F) + 1,$$
$$\lambda(\pi_n(F)) = p^{n-1} + r(F)_n$$

where $r(F)_n \equiv \lambda(F) - p^n \bmod (p^n - p^{n-1})$. In particular, we have the relation

$$\lambda(F) + \mu(F)(p^n - p^{n-1}) = \lambda(\pi_n(F)) + \mu(\pi_n(F))(p^n - p^{n-1}). \tag{6.3}$$

Remark 6.0.2. The theorem says that if F is p-large at n then the μ-invariant of the nth-layer projection will always be positive. This fact is clarified by the relation (6.3): while $\lambda(F)$ can be made as large as one likes, $\lambda(\pi_n(F))$ is always bounded by p^n (by definition), thus a large λ-invariant in Λ_K is passed down to a large μ-invariant in $\Lambda_{K,n}$.

Remark 6.0.3. Several behaviors can occur for functions which are not p-large at n – see the end of §6.4 for some examples. In general, if the valuations of the coefficients of $F = \sum a_i T^i$ are small in comparison to the valuations of $a_{p^n+N} c_N^{(i)}$ then (6.2) suggests that the nth-layer invariants should be controlled by the first $p^n - 1$ coefficients of F. In this case, the nth-layer invariants may be unrelated to the original Iwasawa invariants.

6.1 ω_n-division terms

Fix $n \geq 1$ and define the sequence of *ω_n-division terms* $c_N^{(i)}$ for $N \geq 0$ and $i \in \mathbf{Z}_{\leq p^n-1}$ recursively by the equations

$$c_0^{(i)} = \begin{cases} -\binom{p^n}{i} & \text{if } 1 \leq i \leq p^n - 1, \\ 0 & \text{if } i \leq 0, \end{cases} \tag{6.4}$$

$$c_N^{(i)} = c_{N-1}^{(i-1)} + c_{N-1}^{(p^n-1)} c_0^{(i)}, \quad \text{for } N \geq 1. \tag{6.5}$$

The recursion relation easily implies

$$c_N^{(i)} = 0, \quad i \leq 0, \tag{6.6}$$

We now check that (6.1) holds.

Lemma 6.1.1. $T^{p^n+N} \equiv \sum_{i=0}^{p^n-1} c_N^{(i)} T^i \mod \omega_n$

Proof. The lemma holds for $N = 0$ since $T^{p^n} = \omega_n - \sum_{i=1}^{p^n-1} \binom{p^n}{i} T^i$. Now suppose it holds for some $N \geq 0$. Then

$$\begin{aligned} T^{p^n+N+1} &\equiv T \cdot \sum_{i=0}^{p^n-1} c_N^{(i)} T^i \\ &\equiv \sum_{i=2}^{p^n-1} c_N^{(i-1)} T^i + c_N^{(p^n-1)} T^{p^n} \\ &\equiv c_N^{(p^n-1)} c_0^{(1)} T + \sum_{i=2}^{p^n-1} \left(c_N^{(i-1)} + c_N^{(p^n-1)} c_0^{(i)} \right) T^i \\ &\equiv \sum_{i=0}^{p^n-1} c_{N+1}^{(i)} T^i \mod \omega_n. \end{aligned}$$

□

The remainder of this section is devoted to understanding some p-adic properties of the ω_n-division terms.

Lemma 6.1.2. *Fix $N \geq 0$.*

(1) $\operatorname{ord}_p c_N^{(i)} \geq 1$.

(2) If $i \geq N+1$ then $\operatorname{ord}_p c_N^{(i)} = n - \operatorname{ord}_p(i-N)$.

(3) If $i \leq N$ then $c_N^{(i)} = \sum_{j=0}^{i-1} c_{N-1-j}^{(p^n-1)} c_0^{(i-j)}$.

Proof. (1). The case $i \leq 0$ follows from (6.6). For the remaining cases $1 \leq i \leq p^n - 1$, the well-known identity

$$\operatorname{ord}_p \binom{p^n}{i} = n - \operatorname{ord}_p(i), \tag{6.7}$$

proves the result for $N = 0$. Now suppose it holds for some $N \geq 0$. Then

$$\operatorname{ord}_p c_{N+1}^{(i)} = \operatorname{ord}_p(c_N^{(i-1)} + c_N^{(p^n-1)} c_0^{(i)}) \geq \min(\operatorname{ord}_p c_N^{(i-1)}, \operatorname{ord}_p c_N^{(p^n-1)} + \operatorname{ord}_p c_0^{(i)}) \geq 1.$$

(2). If $N = 0$ then the result follows from (6.7). Suppose it holds for some $N \geq 0$ and fix $i \geq N+2$. Then $\operatorname{ord}_p c_N^{(i-1)} = n - \operatorname{ord}_p(i-1-N)$ and $\operatorname{ord}_p(c_N^{(p^n-1)} c_0^{(i)}) = 2n - \operatorname{ord}_p(p^n-1-N) - \operatorname{ord}_p i$. Since $i \leq p^n - 1$ one can check that $n - \operatorname{ord}_p(i-1-N) < 2n - \operatorname{ord}_p(p^n - 1 - N) - \operatorname{ord}_p i$. Therefore, the recursion relation (6.5) yields

$$\operatorname{ord}_p c_{N+1}^{(i)} = \min\left(\operatorname{ord}_p c_N^{(i-1)}, \operatorname{ord}_p(c_N^{(p^n-1)} c_0^{(i)})\right) = n - \operatorname{ord}_p(i - (N+1)).$$

(3). Fix $i \leq N$ and apply the recursion relation i times:

$$\begin{aligned} c_N^{(i)} &= c_{N-1}^{(i-1)} + c_{N-1}^{(p^n-1)} c_0^{(i)} \\ &= (c_{N-2}^{(i-2)} + c_{N-2}^{(p^n-1)} c_0^{(i-1)}) + c_{N-1}^{(p^n-1)} c_0^{(i)} \\ &\vdots \\ &= c_{N-i}^{(0)} + \sum_{j=0}^{i} c_{N-(j+1)}^{(p^n-1)} c_0^{(i-j)}. \end{aligned}$$

Now the result follows from (6.6). □

Definition 6.1.3. Let

$$r(N) = r(N)_n = N \bmod (p^n - p^{n-1})$$
$$q(N) = q(N)_n = \left\lfloor \frac{N}{p^n - p^{n-1}} \right\rfloor$$

denote the remainder and quotient, respectively, of an integer N upon division by $p^n - p^{n-1}$, so we can uniquely write $N = r(N) + q(N)(p^n - p^{n-1})$.

Proposition 6.1.4. *For all $N \geq 0$ we have*

$$\operatorname{ord}_p c_N^{(i)} \begin{cases} \geq q(N) + 2 & \text{if } 0 \leq i \leq r(N) + p^{n-1} - 1 \\ = q(N) + 1 & \text{if } i = r(N) + p^{n-1} \\ \geq q(N) + 1 & \text{if } r(N) + p^{n-1} + 1 \leq i \leq p^n - 1. \end{cases}$$

In particular, we always have $\operatorname{ord}_p c_N^{(i)} \geq q(N) + 1$.

Proof. If $N = 0$ then the first inequality and the middle equality follow from Lemma 6.1.2.2, and the last inequality follows from Lemma 6.1.2.1. Now consider the following claims.

Claim 1. *Fix $M \geq 0$ and suppose the proposition holds for $N = M(p^n - p^{n-1})$. Then it also holds for all N with $q(N) = M$.*

Claim 2. *If the proposition holds for all N with $q(N) = M$ then it also holds for $N = (M + 1)(p^n - p^{n-1})$.*

It suffices to establish the above claims. The idea is to partition the integers into multiples of $p^n - p^{n-1}$ and use the claims to induct over these partitions. Precisely, let S_M be the set of all N with $q(N) = M$. Since we already know the proposition holds at $N = 0$, Claim 1 implies it holds on the set S_0. Now suppose the proposition holds on S_M for some $M \geq 0$. Then Claim 2 implies it also holds at $N = (M + 1)(p^n - p^{n-1})$, and then Claim 1 implies it holds on all of S_{M+1}.

Proof of Claim 1. By assumption, the proposition is true at $N = M(p^n - p^{n-1})$. Taking this as our base-case, assume that the proposition holds for some $N \in S_M$ with $N+1$ also in S_M. We show the proposition also holds at $N+1$. Since $q(N) = q(N+1) = M$ we know that $r(N+1) = r(N) + 1$, so we must show

$$\operatorname{ord}_p c_{N+1}^{(i)} \begin{cases} \geq M+2 & \text{if } 0 \leq i \leq r(N) + p^{n-1} \\ = M+1 & \text{if } i = r(N) + p^{n-1} + 1 \\ \geq M+1 & \text{if } r(N) + p^{n-1} + 2 \leq i \leq p^n - 1. \end{cases}$$

Our inductive hypothesis tells us that

$$\operatorname{ord}_p c_N^{(i-1)} \begin{cases} \geq M+2 & \text{if } 1 \leq i \leq r(N) + p^{n-1} \\ = M+1 & \text{if } i = r(N) + p^{n-1} + 1 \\ \geq M+1 & \text{if } r(N) + p^{n-1} + 2 \leq i \leq p^n, \end{cases} \tag{6.8}$$

and combining this with Lemma 6.1.2.1 gives for all $0 \leq i \leq p^n - 1$,

$$\operatorname{ord}_p(c_N^{(p^n-1)} c_0^{(i)}) \geq M+2. \tag{6.9}$$

The proposition for $N+1$ now follows by taking valuations in the recursion relation

$$c_{N+1}^{(i)} = c_N^{(i-1)} + c_N^{(p^n-1)} c_0^{(i)}.$$

Proof of Claim 2. For $0 \leq j \leq p^n - p^{n-1} - 1$, let $N_j \in S_M$ denote the unique integer with $r(N_j) = p^n - p^{n-1} - 1 - j$. By assumption, the proposition holds for all N_j. We claim that it also holds for $N = (M+1)(p^n - p^{n-1})$, i.e., we will show that

$$\operatorname{ord}_p c_{(M+1)(p^n-p^{n-1})}^{(i)} \begin{cases} \geq M+3 & \text{if } 0 \leq i \leq p^{n-1} - 1, \\ = M+2 & \text{if } i = p^{n-1}, \\ \geq M+2 & \text{if } p^{n-1} + 1 \leq i \leq p^n - 1. \end{cases} \tag{6.10}$$

Since the proposition holds for N_0, we know

$$\operatorname{ord}_p c_{N_0}^{(i-1)} \begin{cases} \geq M+2 & \text{if } i \leq p^n - 1, \\ = M+1 & \text{if } i = p^n. \end{cases} \tag{6.11}$$

We can say even more: taking $i \leq p^{n-1}$ forces $N_0 \geq i-1$ so we can apply Lemma 6.1.2.3 to obtain

$$c_{N_0}^{(i-1)} = \sum_{j=0}^{i-2} \underbrace{c_{N_{j+1}}^{(p^n-1)}}_{\operatorname{ord}_p \geq M+1} \underbrace{c_0^{(i-j-1)}}_{\operatorname{ord}_p \geq 2},$$

where the leftmost valuation (indicated with brackets) is because the proposition holds for all N_j and the rightmost valuation follows from Lemma 6.1.2.2. This shows that $\operatorname{ord}_p c_{N_0}^{(i-1)} \geq M+3$ for all $i \leq p^{n-1}$. Since the proposition holds for $N = 0$, the equality in (6.11) implies

$$\operatorname{ord}_p(c_{N_0}^{(p^n-1)} c_0^{(i)}) = M + 1 + \operatorname{ord}_p c_0^{(i)} \begin{cases} \geq M+3 & \text{if } 0 \leq i \leq p^{n-1} - 1 \\ = M+2 & \text{if } i = p^{n-1}, \\ \geq M+2 & \text{if } p^{n-1}+1 \leq i \leq p^n - 1. \end{cases} \tag{6.12}$$

The proposition for $N = (M+1)(p^n - p^{n-1})$ now follows from taking valuations in the recursion relation $c_{(M+1)(p^n-p^{n-1})}^{(i)} = c_{N_0}^{(i-1)} + c_{N_0}^{(p^n-1)} c_0^{(i)}$. □

6.2 Some corollaries of Proposition 6.1.4

In this section we show how Proposition 6.1.4 trivially implies Theorem 6.0.1 in the special case of functions $F = T^{p^n+N}$. We then establish equation (6.2), which says that $\pi_n(F)$ can be computed explicitly in terms of the coefficients of F and the ω_n-division terms, and derive another proof of Proposition II.2.4.1.

Corollary 6.2.1. *We have*

$$\mu(\pi_n(T^{p^n+N})) = q(N) + 1$$
$$\lambda(\pi_n(T^{p^n+N})) = r(N) + p^{n-1}$$

Proof. Lemma 6.1.1 gives $T^{p^n+N} \equiv W_N \bmod \omega_n$, where $W_N = \sum_{i=0}^{p^n-1} c_N^{(i)} T^i$, and Proposition 6.1.4 says exactly that $\mu(W_N) = q(N) + 1$ and $\lambda(W_N) = r(N) + p^{n-1}$, which proves the result. □

Now fix a power series $F = \sum_{i\geq 0} a_i T^i \in \Lambda_K$.

Corollary 6.2.2. *We have*

$$F \equiv \sum_{i=0}^{p^n-1} \left(a_i + \sum_{N=0}^{\infty} a_{p^n+N} c_N^{(i)} \right) T^i \bmod \omega_n.$$

Proof. Lemma 6.1.1 gives the result for functions $F = T^{p^n+N}$ with $N \geq 0$. For the general case, fix $F = \sum_{i\geq 0} a_i T^i \in \Lambda_K$. By Proposition 6.1.4, the ω_n-division terms have unbounded valuations as $N \to \infty$, hence the sum $\sum_{N=0}^{\infty} a_{p^n+N} c_N^{(i)}$ converges. The result now follows by writing

$$F = \sum_{i=0}^{p^n-1} a_i T^i + \sum_{N=0}^{\infty} a_{p^n+N} T^{p^n+N},$$

reducing modulo ω_n, and applying Lemma 6.1.1 to each term T^{p^n+N}. □

Corollary 6.2.3. *If $\lambda(F) < p^n$ then*

$$\mu(\pi_n(F)) = \mu(F),$$
$$\lambda(\pi_n(F)) = \lambda(F).$$

Proof. Write $\lambda = \lambda(F)$ and $\mu = \mu(F)$. As in the proof of Proposition II.2.4.1, we may assume that $\mu = 0$. Proposition 6.2.2 shows that $F \equiv \sum_{i=0}^{p^n-1} (a_i + s_i) T^i \bmod \omega_n$ where $s_i = \sum_{N\geq 0} a_{p^n+N} c_N^{(i)}$, and Proposition 6.1.4 tells us that $\operatorname{ord}_p c_N^{(i)} \geq 1$ for all

i, so $\mathrm{ord}_p(s_i) \geq 1$. Hence, if $\lambda = 0$ then a_0 is a unit and the Iwasawa invariants of $\pi_n(F)$ are trivial. If $\lambda > 0$ then $\mathrm{ord}_p\, a_\lambda = 0$ and $\mathrm{ord}_p\, a_i > 0$ for $0 \leq i \leq \lambda - 1$, in which case

$$\mathrm{ord}_p(a_i + s_i) \begin{cases} > 0 & \text{if } 0 \leq i \leq \lambda - 1 \\ = 0 & \text{if } i = \lambda, \end{cases}$$

which proves that $\lambda(\pi_n(F)) = \lambda$ and $\mu(\pi_n(F)) = 0$. □

The following table illustrates the behaviors described in Corollaries 6.2.1 and 6.2.3 above.

F	$\mu(F)$	$\lambda(F)$	$\mu(\pi_1(F))$	$\lambda(\pi_1(F))$
T	0	1	0	1
T^2	0	2	0	2
T^3	0	3	1	1
T^4	0	4	1	2
T^5	0	5	2	1
T^6	0	6	2	2
T^7	0	7	3	1
T^8	0	8	3	2

Table 6.1: Comparison of invariants in Λ and Λ_1 at $p = 3$.

6.3 p-large functions

For fixed $F \in \Lambda_K$ and $n \geq 1$, define the integers

$$N_F = \lambda(F) - p^n,$$

$$q(F) = q(N_F)_n,$$

$$r(F) = r(N_F)_n.$$

Definition 6.3.1. A function $F = \sum a_i T^i \in \Lambda_K$ is *p-large at n* if

(i) $\lambda(F) \geq p^n$,

(ii) $\operatorname{ord}_p a_i \geq \begin{cases} \mu(F) + q(F) + 2 & \text{for } 0 \leq i \leq r(F) + p^{n-1}, \\ \mu(F) + q(F) + 1 & \text{for } r(F) + p^{n-1} + 1 \leq i \leq p^n - 1, \\ \mu(F) + \left\lceil \frac{\lambda(F)+1-i}{p^n - p^{n-1}} \right\rceil & \text{for } p^n \leq i \leq \lambda(F) - 1. \end{cases}$

Example 6.3.2. $F = T^{p^n}$ is p-large at n, while $F = T^{p^n} + p$ is not.

Remark 6.3.3. It is clear that p-largeness is preserved by scaling: since $\lambda(xF) = \lambda(F)$ and $\mu(xF) = \operatorname{ord}_p x + \mu(F)$ for any $x \in K$, if F is p-large at n then so is xF.

6.4 Proof of Theorem 6.0.1

Fix $F = \sum_{i \geq 0} a_i T^i \in \Lambda_K$ and write $\mu = \mu(F)$, $\lambda = \lambda(F)$, $q = q(F)$, and $r = r(F)$.

Lemma 6.4.1. *Assume $\mu = 0$. If either*

(i) $\lambda = p^n$, or

(ii) $\lambda \geq p^n + 1$ and $\operatorname{ord}_p a_i \geq \left\lceil \frac{\lambda+1-i}{p^n - p^{n-1}} \right\rceil$ for all $p^n \leq i \leq \lambda - 1$,

then $\operatorname{ord}_p \left(\sum_{N \geq 0} a_{p^n+N} c_N^{(i)} \right) \begin{cases} \geq q + 2 & \text{if } 0 \leq i \leq r + p^{n-1} - 1, \\ = q + 1 & \text{if } i = r + p^{n-1}, \\ \geq q + 1 & \text{if } r + p^{n-1} + 1 \leq i \leq p^n - 1. \end{cases}$

Proof. Consider the following claim.

Claim. *If $N \neq N_F$ then*

(1) $\operatorname{ord}_p a_{p^n+N} c_N^{(i)} \geq q+2$ *for* $0 \leq i \leq r+p^{n-1}$,

(2) $\operatorname{ord}_p a_{p^n+N} c_N^{(i)} \geq q+1$ *for* $r+p^{n-1}+1 \leq i \leq p^n-1$.

Assuming the claim, the lemma is proved as follows. Define

$$s_i := \sum_{N \geq 0} a_{p^n+N} c_N^{(i)} = a_\lambda c_{N_F}^{(i)} + \sum_{N \neq N_F} a_{p^n+N} c_N^{(i)}.$$

By Proposition 6.1.4 and the fact that a_λ is a unit (recall we are assuming $\mu = 0$) we have

$$\operatorname{ord}_p a_\lambda c_{N_F}^{(i)} = \operatorname{ord}_p c_{N_F}^{(i)} \begin{cases} \geq q+2 & \text{if } 0 \leq i \leq r+p^{n-1}-1 \\ = q+1 & \text{if } i = r+p^{n-1} \\ \geq q+1 & \text{if } r+p^{n-1}+1 \leq i \leq p^n-1. \end{cases} \tag{6.13}$$

It now follows from the Claim that

$$\operatorname{ord}_p s_i \geq \min_{N \geq 0} \left(\operatorname{ord}_p a_{p^n+N} c_N^{(i)} \right) \geq \begin{cases} q+2 & \text{if } 0 \leq i \leq r+p^{n-1}-1, \\ q+1 & \text{if } r+p^{n-1}+1 \leq i \leq p^n-1, \end{cases}$$

thus we need only consider the valuation of s_i at $i = r+p^{n-1}$. In this case, we have

$$\operatorname{ord}_p s_{r+p^{n-1}} = \min \left(\operatorname{ord}_p a_\lambda c_{N_F}^{(r+p^{n-1})}, \operatorname{ord}_p \sum_{N \neq N_F} a_{p^n+N} c_N^{(r+p^{n-1})} \right) = q+1$$

since the valuation of $a_\lambda c_{N_F}^{(r+p^{n-1})}$ is exactly $q+1$ by (6.13) and the valuation of the sum is $\geq q+2$ by the first part of the Claim. Thus it remains to prove the claim.

Proof of the Claim. Suppose first that $N \geq N_F + 1$, so either $q(N) = q$ or $q(N) \geq q+1$. If the latter is true then Proposition 6.1.4 implies

$$\operatorname{ord}_p c_N^{(i)} \geq \begin{cases} q+3 & \text{if } 0 \leq i \leq r(N)+p^{n-1}-1, \\ q+2 & \text{if } r(N)+p^{n-1} \leq i \leq p^n-1. \end{cases}$$

Thus for any $0 \leq i \leq p^n - 1$ we have $\operatorname{ord}_p a_{p^n+N} c_N^{(i)} \geq \operatorname{ord}_p c_N^{(i)} \geq q+2$, which proves (i)-(ii) in the case $q(N) \geq q+1$. On the other hand, if $q(N) = q$ then again by Proposition 6.1.4 we have $\operatorname{ord}_p a_{p^n+N} c_N^{(i)} \geq q+1$ for any $0 \leq i \leq p^n - 1$, which proves (ii). For (i), fix $0 \leq i \leq r + p^{n-1}$ and note that since $N \geq N_F + 1$ and $q(N) = q$ we must have $r(N) \geq r+1$. So $i \leq r(N) + p^{n-1} - 1$, and Proposition 6.1.4 implies $\operatorname{ord}_p a_{p^n+N} c_N^{(i)} \geq q+2$ as needed.

We now suppose $N \leq N_F - 1$. Then our hypotheses on the valuations of a_i give

$$\begin{aligned}
\operatorname{ord}_p a_{p^n+N} &\geq \left\lceil \frac{\lambda + 1 - (p^n + N)}{p^n - p^{n-1}} \right\rceil \\
&= \left\lceil \frac{\big(p^n + r + q(p^n - p^{n-1})\big) - \big(p^n + r(N) + q(N)(p^n - p^{n-1})\big) + 1}{p^n - p^{n-1}} \right\rceil \\
&= q - q(N) + \left\lceil \frac{r - r(N) + 1}{p^n - p^{n-1}} \right\rceil \\
&= q - q(N) + \begin{cases} 1 & \text{if } r \geq r(N), \\ 0 & \text{if } r < r(N). \end{cases}
\end{aligned} \tag{6.14}$$

Hence for any $0 \leq i \leq p^n - 1$ Proposition 6.1.4 gives

$$\begin{aligned}
\operatorname{ord}_p a_{p^n+N} c_N^{(i)} &\geq \operatorname{ord}_p a_{p^n+N} + (q(N)+1) \\
&\geq \left\{ \begin{array}{ll} q - q(N) + 1 & \text{if } r \geq r(N) \\ q - q(N) & \text{if } r < r(N) \end{array} \right\} + q(N) + 1 \\
&= \begin{cases} q+2 & \text{if } r \geq r(N) \\ q+1 & \text{if } r < r(N). \end{cases}
\end{aligned} \tag{6.15}$$

This proves property (ii) for $N \leq N_F - 1$. For (i), fix $0 \leq i \leq r + p^{n-1}$. If $r \geq r(N)$ then (6.15) shows $\operatorname{ord}_p a_{p^n+N} c_N^{(i)} \geq q+2$. On the other hand, if $r < r(N)$ then $i \leq r(N) + p^{n-1} - 1$ and Proposition 6.1.4 together with (6.14) yield $\operatorname{ord}_p a_{p^n+N} c_N^{(i)} \geq (q - q(N)) + (q(N)+2) = q+2$ and we are done. □

Proof of Theorem 6.0.1. Suppose F is p-large at n. We must show that

$$\mu(\pi_n(F)) = q + \mu + 1,$$
$$\lambda(\pi_n(F)) = r + p^{n-1}.$$

We may assume that $\mu = 0$ (use II.(2.4)). From Proposition 6.2.2, the Iwasawa invariants of $\pi_n(F)$ are equal to those of the polynomial $\sum_{i=0}^{p^n-1}(a_i + s_i)T^i$, where s_i is as in the (proof of) the previous lemma. Setting $b_i = a_i + s_i$, Lemma 6.4.1 and the assumption that F is p-large at n imply

$$\operatorname{ord}_p(b_i) \geq \min(\operatorname{ord}_p a_i, \operatorname{ord}_p s_i) \geq \begin{cases} q+2 & \text{if } 0 \leq i \leq r + p^{n-1} - 1 \\ q+1 & \text{if } r + p^{n-1} + 1 \leq i \leq p^n - 1, \end{cases}$$

$$\operatorname{ord}_p(b_{r+p^{n-1}}) = \min(\operatorname{ord}_p a_{r+p^{n-1}}, q+1) = q+1.$$

The desired nth-layer invariants now follow. The relation (6.3) is easily seen by writing $\lambda = p^n + r + q(p^n - p^{n-1})$ and then using the formulae for the Iwasawa invariants of $\pi_n(F)$ just proven. □

Example 6.4.2. The above theorem can fail for functions which are not p-large at n. One extreme example comes by considering $F = \omega_n$, which is not p-large at n since its p^{n-1}st coefficient has a valuation of 1 (instead of being ≥ 2). In this case we have $\lambda(\omega_n) = p^n$ and $\mu(\omega_n) = 0$, but since $\pi_n(\omega_n) = 0$, both of the nth-layer Iwasawa invariants vanish. However, forcing an extra multiple of p at the p^{n-1}st coefficient turns this nonexample into an example. That is, define the polynomial ω_n' to have the same coefficients as $\omega_n = T^{p^n} + \sum_{i=1}^{p^n-1} \binom{p^n}{i} T^i$ except we replace the p^{n-1}st coefficient by the value $p \cdot \binom{p^n}{p^{n-1}}$. Then ω_n' is p-large at n and one checks that $\mu(\pi_n(\omega_n')) = 1$ and $\lambda(\pi_n(\omega_n')) = p^{n-1}$.

Example 6.4.3. Theorem 6.0.1 appears not to be as strong as possible in that the conclusion might still be satisfied by functions which are not p-large at n.

Although the theorem still works when $\mathcal{O}$ is ramified, the following example suggests a finer definition of p-largeness might be better suited to this case. Consider $F = 3\sqrt{3} + T^3 \in \mathcal{O}[T]$, where $\mathcal{O} = \mathbf{Z}_3[\sqrt{3}]$. According to our definition, this function is not 3-large at $n = 1$ since the valuation of its constant term is not great than or equal to 2. Nevertheless, since

$$F \equiv 3\sqrt{3} - 3T - 3T^2 \bmod \omega_1$$

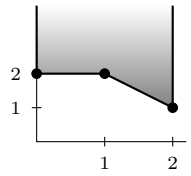
we have $\mu(\pi_1(F)) = 1$ and $\lambda(\pi_1(F)) = 1$, hence F satisfies the conclusion of the previous theorem.

6.5 Newton polygon interpretation

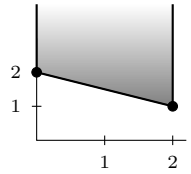
In this section we reinterpret Theorem 6.0.1 in terms of the p-adic Newton polygon (see Remark II.2.1.2 for the definition). If λ and μ are the Iwasawa invariants of some $F \in \Lambda_K$, the idea is to use the lower bounds appearing in the definition of p-largeness to construct a region $\mathcal{R}_{\lambda,\mu} \subseteq \mathbf{R}^2$ with the property that, if the Newton polygon of F is contained entirely in this region then we will know that F is p-large at n. One simple candidate for $\mathcal{R}_{\lambda,\mu}$ is to take the upper convex hull of the points (i, v), where v ranges over the aforementioned lower bounds for the valuations of the coefficients of F. This does work, but it turns out a slightly looser definition is possible and is in fact necessary for our specific application of these ideas to the nth-layer half-logarithms carried out in the following chapter.

Before introducing any new definitions, we discuss a simple example. Let $p = 3$ and consider what is required for a monic degree 3 polynomial of the form $F = a_0 + a_1T + a_2T^2 + T^3 \in \mathbf{Z}_3[T]$ (so $\lambda = 3$ and $\mu = 0$) to be 3-large at $n = 1$: we need $\operatorname{ord}_p a_i \geq 2$ for $i = 0, 1$ and $\operatorname{ord}_p(a_3) \geq 1$. In other words, as long as the Newton

polygon for the truncation $a_0 + a_1T + a_2T^2$ is contained in or on the boundary of the region below, we will know that F is 3-large at $n = 1$.



However, since there are no points (that could occur as valuations of coefficients of F) in the interior of the rectangle with vertices $(0, 1)$, $(0, 2)$, $(2, 1)$, and $(2, 2)$, we also see that F is 3-large at $n = 1$ if its Newton polygon is contained in the slightly larger region pictured below.



This latter region above is exactly $\mathcal{R}_{3,0}$ (at $p = 3$ and $n = 1$). To generalize this idea, we make the following assumption for the remainder of this chapter.

Assumption. $K/\mathbf{Q}_p$ is unramified.

This assumption is needed because if F has coefficients in a ring with nontrivial ramification then our "smoothing out" process above might introduce additional points which, should they occur as vertices of the Newton polygon for F, then F might not be p-large. For example, if F as above has coefficients in a ring with ramification index $e = 2$ then the point $(1, 3/2)$ would lie on the boundary of the second region, but a polynomial whose Newton polygon has this point as a vertex would not be 3-large at $n = 1$. This could possibily be remedied with a modified definition of p-largeness (one which might encompass, for example, the polynomial given in Example 6.4.3).

6.5.1 Strictly p-large functions

Fix $n \geq 1$, define $t_n = p^n - p^{n-1}$, and recall that for $N \geq 0$ we write

$$N = r(N) + q(N)t_n$$

where $q(N)$ and $r(N)$ are as in the previous section. Let $\mathcal{T}_N(F) = \sum_{i=0}^{N} a_i T^i$ denote the Nth-degree polynomial truncation of a power series $F = \sum_{i \geq 0} a_i T^i$.

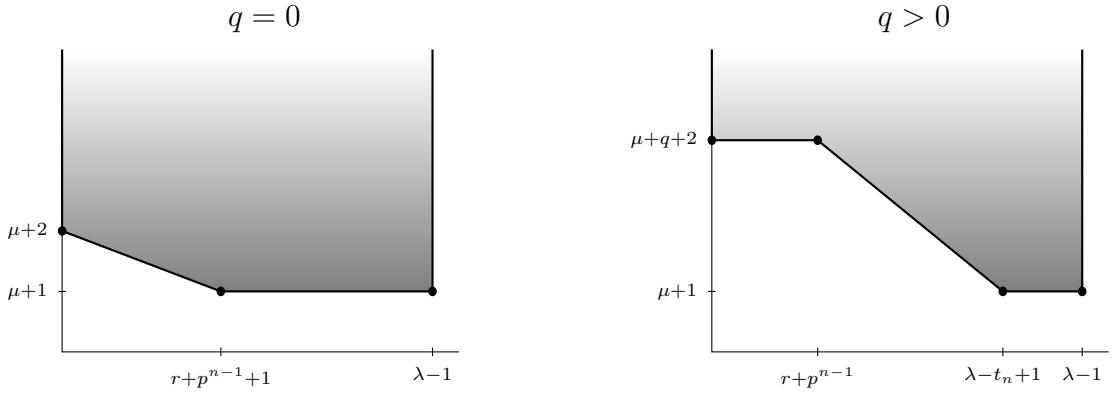
Definition 6.5.1. Let $\lambda, \mu \in \mathbf{Z}$ be such that $\lambda \geq p^n$. Set $q = q(\lambda - p^n)_n$ and $r = r(\lambda - p^n)_n$. If $q = 0$, define the region $\mathcal{R}_{\lambda,\mu} \subseteq \mathbf{R}^2$ to be the lower convex hull of the three points $(0, \mu + 2), (r + p^{n-1} + 1, \mu + 1)$, and $(\lambda - 1, \mu + 1)$. If $q > 0$, let $\mathcal{R}_{\lambda,\mu}$ be the region lying on or above the graph of the piecewise linear function

$$f(x) = \begin{cases} \mu + q + 2, & 0 \leq x \leq r + p^{n-1}, \\ \mu + q + 2 - \frac{(q+1)(x-r-p^{n-1})}{qt_n+1}, & r + p^{n-1} \leq x \leq \lambda - t_n + 1, \\ \mu + 1, & \lambda - t_n + 1 \leq x \leq \lambda - 1. \end{cases}$$

Definition 6.5.2. The function $F \in \Lambda_K$ is *strictly p-large at n* if

(i) $\lambda(F) \geq p^n$, and

(ii) $\mathcal{R}_{\lambda(F),\mu(F)} \supseteq \mathrm{NP}_p(\mathcal{T}_{\lambda(F)-1}(F))$.



Lemma 6.5.3. *For any $x \in K$, if F is strictly p-large at n then so is xF.*

Proof. The Newton polygon for xF is identical to that of F, except that it is shifted vertically by $\operatorname{ord}_p x$. Hence if $\mathcal{R}_{\lambda(F),\mu(F)}$ contains $\mathrm{NP}_p(\mathcal{T}_{\lambda(F)-1}(F))$ then $\mathcal{R}_{\lambda(xF),\mu(xF)} = \mathcal{R}_{\lambda(F),\operatorname{ord}_p(x)+\mu(F)}$ will contain $\mathrm{NP}_p(\mathcal{T}_{\lambda(xF)-1}(xF)) = \mathrm{NP}_p(\mathcal{T}_{\lambda(F)-1}(xF))$. □

6.5.2 Strictly p-large implies p-large

Fix $n \geq 1$, $F \in \Lambda_K$, and let $\lambda = \lambda(F)$, $\mu = \mu(F)$, $q = q(F)_n$, and $r = r(F)_n$. Write $t = t_n$ and $s = s_n := r + p^n$. We begin with an equivalent characterization of p-largeness.

Lemma 6.5.4. *Assume $\lambda \geq p^n$.*

(1) If $q = 0$ then F is p-large at n if and only if

$$\operatorname{ord}_p a_i \geq \begin{cases} \mu + 2 & i \in [0, r + p^{n-1}], \\ \mu + 1 & i \in [r + 1 + p^{n-1}, \lambda - 1], \end{cases} \tag{6.16}$$

(2) If $q > 0$ then F is p-large at n if and only if

$$\operatorname{ord}_p a_i \geq \begin{cases} \mu + q + 2 & i \in [0, r + p^{n-1}], \\ \mu + q + 1 & i \in [r + 1 + p^{n-1}, s], \\ \mu + q & i \in s + [1, t], \\ \mu + q - 1 & i \in s + [1 + t, 2t], \\ \vdots & \vdots \\ \mu + 3 & i \in s + [1 + (q-3)t, (q-2)t], \\ \mu + 2 & i \in s + [1 + (q-2)t, (q-1)t], \\ \mu + 1 & i \in s + [1 + (q-1)t, qt - 1]. \end{cases} \tag{6.17}$$

Proof. In both cases we may assume $\mu = 0$. Since $[0, p^n - 1] \subseteq [0, s]$, there is nothing to prove in either case for $i \in [0, p^n - 1]$. If $i \in [p^n, s]$ then we can write $i = p^n + a$ for some $0 \leq a \leq s - p^n = r$. Since $\lambda = s + qt$, the equivalence for (1) and (2) now follows from

$$\left\lceil \frac{\lambda + 1 - i}{t} \right\rceil = q + \left\lceil \frac{r + 1 - a}{t} \right\rceil = q + 1.$$

To finish proving (2), we assume $i \in I_j$ for some $0 \leq j \leq q - 1$, where

$$I_j := \begin{cases} s + [1 + jt, (j+1)t] & j \neq q - 1 \\ s + [1 + (q-1)t, qt - 1] & j = q - 1. \end{cases}$$

We can write $i = s + 1 + jt + a$ for some $0 \leq a \leq t - 1$ (if $j = q - 1$, one must take $a \leq t - 2$). In particular, for any j we have $\lambda + 1 - i = (q - j)t - a$ and

$$\left\lceil \frac{\lambda + 1 - i}{t} \right\rceil = q - j + \left\lceil \frac{-a}{t} \right\rceil = q - j - \left\lfloor \frac{a}{t} \right\rfloor = q - j.$$

The equivalence follows. □

Proposition 6.5.5. *If F is strictly p-large at n then F is also p-large at n.*

Proof. Assume that F is strictly p-large at n and write $F = p^\mu F_0$ so that $\mu(F_0) = 0$. Note that F_0 is also strictly p-large by Lemma 6.5.3. If the proposition holds for F_0 then Remark 6.3.3 implies that F itself is also p-large at n. Thus, it suffices to prove the result assuming $\mu = 0$.

The proposition becomes somewhat obvious after drawing some pictures, thus we give both a 'picture proof' and a rigorous proof below.

Picture proof. We consider the cases $q = 0$ and $q > 0$ separately.

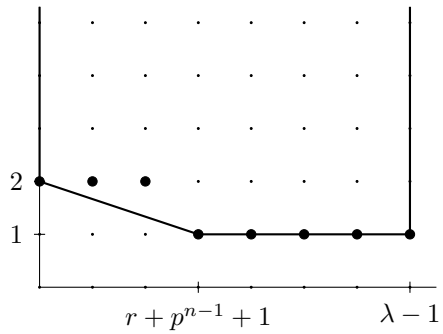


Figure 6.1: ($q = 0$) The y-coordinates of the bold points represent the lowest possible valuations (as given in Lemma 6.5.4.1) attainable by the coefficients of F in order for F to be p-large at n. Thus, if the Newton polygon for F is contained entirely within the region $\mathcal{R}_{\lambda,0}$ (outlined in bold) then clearly the vertices of the Newton polygon must lie on or above the bold points, meaning that F must be p-large at n.

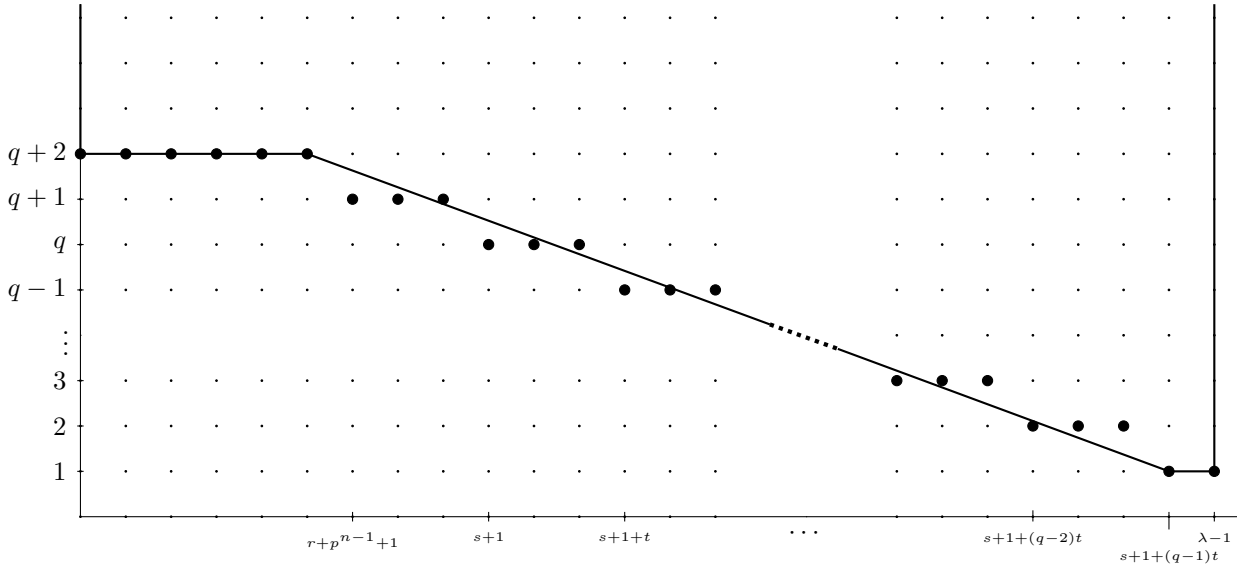


Figure 6.2: ($q > 0$) As in the previous case, the bold points represent the lowest possible valuations (given in Lemma 6.5.4.2) permissible for F to be p-large at n. Notice that if the Newton polygon for F is contained in $\mathcal{R}_{\lambda,0}$ (outlined) then all vertices of the Newton polygon lie on or above the bold points, signifying that F must be p-large at n.

Rigorous proof. Let $y = f(x)$ be the piecewise linear function describing the lower edges of $\mathcal{R}_{\lambda,\mu=0}$. Since F is strictly p-large at n, we know that

$$\operatorname{ord}_p a_i \geq f(i) \tag{6.18}$$

for all $0 \leq i \leq \lambda - 1$.

Case $q = 0$. We have

$$f(x) = \begin{cases} 2 - \frac{x}{r+p^{n-1}+1} & 0 \leq x \leq r + p^{n-1} + 1, \\ 1 & r + p^{n-1} + 1 \leq x \leq \lambda - 1. \end{cases}$$

Suppose that F is not p-large at n. Then either $\operatorname{ord}_p(a_i) < 2$ for some $0 \leq i \leq r + p^{n-1}$, or $\operatorname{ord}_p(a_i) < 1$ for some $r + p^{n-1} + 1 \leq i \leq \lambda - 1$. The latter clearly contradicts inequality (6.18), and the former does as well: if $\operatorname{ord}_p(a_i) < 2$ then (since K is unramified) $\operatorname{ord}_p a_i \leq 1 < f(i)$ for $0 \leq i \leq r + p^{n-1}$.

Case $q > 0$. In this case,

$$f(x) = \begin{cases} q + 2 & 0 \leq x \leq r + p^{n-1}, \\ q + 2 - \frac{(q+1)(x-r-p^{n-1})}{qt+1}, & r + p^{n-1} \leq x \leq \lambda - t + 1, \\ 1 & \lambda - t + 1 \leq x \leq \lambda - 1. \end{cases}$$

Again, we suppose that F is not p-large at n. Then, by Lemma 6.5.4 (and using the notation of its proof) we know one of the following holds:

(1) $\operatorname{ord}_p a_i < q + 2$ for some $i \in [0, r + p^{n-1}]$,

(2) $\operatorname{ord}_p a_i < q + 1$ for some $i \in [r + p^{n-1} + 1, s]$, or

(3) $\operatorname{ord}_p a_i < q - j$ for some $i \in I_j$ and $0 \leq j \leq q - 1$.

Clearly (1) contradicts (6.18). Assuming (2), we have $\operatorname{ord}_p a_i \leq q$ (K is unramified) for some $i \in [r + p^{n-1} + 1, s]$. But for i in this range it is easy to check that $f(i) > q \geq \operatorname{ord}_p a_i$, contradicting (6.18). Finally, if (3) holds for some $0 \leq j \leq q-1$

then there is some $i \in I_j$ for which $\operatorname{ord}_p a_i \leq q-(j+1)$. If $j = q-1$ then $i \in I_j = [\lambda - t + 1, \lambda - 1]$ and we have $\operatorname{ord}_p a_i = 0 < 1 = f(i)$, contradicting (6.18). If $j \leq q-2$ then, writing $i = s + 1 + jt + a$ for some $0 \leq a \leq t-1$, we have

$$\begin{aligned} f(i) &= q + 2 - \frac{(q+1)(a+1+(j+1)t)}{qt+1} \\ &\geq q + 2 - \frac{(q+1)(t+(j+1)t)}{qt+1} \\ &= q + 2 - \frac{t(q+1)(j+2)}{qt+1} \\ &= q - j - \frac{(j+2)(t-1)}{qt+1} \\ &> q - j - 1 \end{aligned}$$

where the last inequality follows from the fact that $j \leq q-2$. But now $\operatorname{ord}_p a_i \leq q-j-1 < f(i)$, contradicting (6.18). □

Remark 6.5.6. A p-large function need not be strictly p-large. For instance, with regard to the figures above, one need only choose a polynomial whose Newton polygon has a vertex at one of the black (allowable) dots lying *below* the region $\mathcal{R}_{\lambda,\mu}$ but with all other vertices contained in (or on the boundary of) $\mathcal{R}_{\lambda,\mu}$. For example, the polynomial $F = 3^3 + 3^2T^4 + 3T^{14} + T^{15}$ is 3-large at $n = 2$ but not strictly 3-large. Since $\lambda(F) = 3^2 + (3^2 - 3)$, we have $r = 0$, $q = 1$, $s = 9$, and we draw $\mathcal{R}_{15,0}$ (in black) and $\mathrm{NP}_3(\mathcal{T}_{14}(F))$ (dotted) in the following figure.

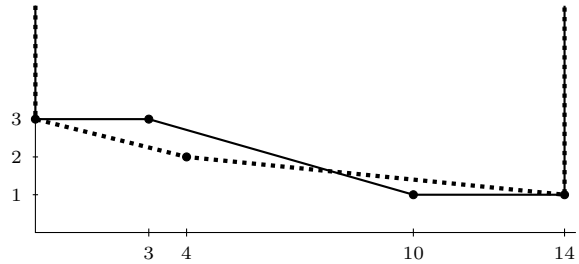


Figure 6.3: A polynomial that is 3-large but not strictly 3-large at $n = 2$.

The following simple but useful corollary shows that in order to prove F is p-large at n it suffices to find a suitable polynomial G – one which is known to be strictly p-large at n and has the same λ-invarinat of F – whose Newton polygon provides a lower bound for that of F, in the sense of the following corollary.

Corollary 6.5.7. *Let $F, G \in \Lambda_K$ be such that $\lambda(F) = \lambda(G)$. If G is strictly p-large at n and $\mathrm{NP}_p(\mathcal{T}_{\lambda-1}(G)) \supseteq \mathrm{NP}_p(\mathcal{T}_{\lambda-1}(F))$ then F is p-large at n.*

Proof. This is obvious. Letting λ be the λ-invariant of F (and G), since G is strictly p-large and the Newton polygon of $\mathcal{T}_{\lambda-1}(G)$ contains that of $\mathcal{T}_{\lambda-1}(F)$, we know that F is also strictly p-large. The fact that F is actually p-large is Proposition 6.5.5. $\square$

CHAPTER 7

IWASAWA INVARIANTS OF MAZUR-TATE ELEMENTS

In this chapter we apply the results of Chapters V and VI to compute the Iwasawa invariants of Mazur-Tate elements for modular forms with $a_p = 0$. We then discuss applications of this result to the plus/minus invariants of the associated p-adic L-functions. Before stating our main theorem, we review some known results on the Iwasawa invariants of Mazur-Tate elements in general.

7.1 Known results

Fix an odd prime p, a newform $f = \sum a_n q^n \in S_k(N, \varepsilon)$ with $p \nmid N$, and a tame character ψ. Let $K/\mathbf{Q}_p$ be the field associated to f in §III.3.2.5, let $\mathcal{O}$ be its valuation ring, and write Λ, Λ_K, Λ_n, and $\Lambda_{K,n}$ for the associated Iwasawa algebras and their nth-layer quotients.

7.1.1 Ordinary case

Suppose that f is ordinary at p and let $\bar{\rho}_f : G_{\mathbf{Q}} \to \mathrm{GL}_2(\bar{\mathbf{F}}_p)$ denote the residual representation associated to f (see [7] and [5]). (This representation depends on our fixed embedding $\bar{\mathbf{Q}} \hookrightarrow \bar{\mathbf{Q}}_p$.) Write $L_p(f, \psi) \in \mathcal{O}[[T]]$ for the (bounded) p-adic L-function attached to f and let $\theta_n^{\psi}(f)$ denote the Mazur-Tate elements for f at

$j = 0$. The following proposition says that, under some assumptions, the Iwasawa invariants of the p-adic L-function can be obtained from the sequence of Mazur-Tate elements.

Proposition 7.1.1. *If $\bar{\rho}_f$ is irreducible and $\mu(L_p(f,\psi)) = 0$ then for $n \gg 0$,*

$$\mu\big(\theta_n^{\psi}(f)\big) = 0,$$
$$\lambda\big(\theta_n^{\psi}(f)\big) = \lambda\big(L_p(f,\psi)\big).$$

Proof. See [35, Proposition 3.7]. □

Remark 7.1.2. It is a well-known conjecture of Greenberg that $\bar{\rho}_f$ being irreducible should imply $\mu = 0$, thus we can conjecturally drop one assumption in the above proposition. If $\bar{\rho}_f$ is reducible then $\lambda\big(\theta_n^{\psi}(f)\big)$ need not converge – see [35, §3.3] for an example and discussion of this phenomenon.

Remark 7.1.3. Pollack-Weston [35] consider only the Mazur-Tate elements at $j = 0$ (i.e., those constructed using the coefficient of Y^{k-2} in the modular symbol for f), hence the restriction to this j-value above. We suspect an analogous result holds for all $0 \leq j \leq k-2$ but have not yet carried out this generalization. The idea should be to use $\theta_{n,j}^{\psi}(f)$ to construct (possibly some twist of) the p-adic L-function for f at ψ and then apply the same arguments of [35].

7.1.2 Nonordinary case at weight $k = 2$

Kurihara [19] and Perrin-Riou [29] computed the Iwasawa invariants of Mazur-Tate elements attached to rational elliptic curves with good supersingular reduction at p. Via the modularity theorem, their results cover the case of rational newforms that are p-nonordinary of weight $k = 2$ and good level $\Gamma_0(N)$ at p. As outlined

in [35], the proofs of Kurihara and Perrin-Riou extend naturally to weight two modular forms over more general number fields, the main tool being the fact that Mazur-Tate elements satisfy a 3-term relation with respect to (co)restriction maps. Specifically, recall from Corollary III.3.2.9 that

$$\mathrm{res}_n^{n+1}\left(\theta^{\psi}_{n+1,j}(f)\right) = a_p\theta^{\psi}_{n,j}(f) - \varepsilon(p)p^{k-2}\,\mathrm{cor}_n^{n-1}\left(\theta^{\psi}_{n-1,j}(f)\right).$$

If f is p-nonordinary then over $\mathcal{O}/(\varpi)[\Gamma_n]$ this equation reads

$$\mathrm{res}_n^{n+1}\left(\theta^{\psi}_{n+1,j}(f)\right) = \begin{cases} -\varepsilon(p)\,\mathrm{cor}_n^{n-1}\left(\theta^{\psi}_{n-1}(f)\right) & k=2, \\ 0 & k>2. \end{cases}$$

One can explicitly write down the behavior of Iwasawa invariants with respect to (co)restriction maps (see [31]), and at weight $k=2$, these behaviors combine with the relation above to give the following result.

Theorem 7.1.4 (Kurihara, Perrin-Riou)**.** *If f is p-nonordinary of weight $k=2$ then the sequence $\mu(\theta^{\psi}_n(f))$ stabilizes over n of fixed parity. If $\mu(\theta^{\psi}_n(f))$ stabilizes over all n then there exist nonnegative integers $\lambda^{\pm}(f,\psi)$ such that for $n \gg 0$,*

$$\lambda(\theta^{\psi}_n(f,\psi)) = q_n + \lambda^{-\varepsilon_n}(f,\psi),$$

where q_n is the Kurihara term defined in IV.(4.4)*.*

Proof. See [35], [29], or [19]. □

Remark 7.1.5. Let $\mu^-(f,\psi)$ (resp., $\mu^+(f,\psi)$) be the limit of $\mu(\theta_{2n}(f,\psi))$ (resp., $\mu(\theta_{2n+1}(f,\psi))$) as $n \to \infty$. It is conjectured by Perrin-Riou that $\mu^+(f,\psi) = \mu^-(f,\psi) = 0$, thus the above result is conjecturally true without the assumption on μ-invariants.

Remark 7.1.6. The invariants $\lambda^{\pm}(f,\psi)$ have the opposite parity of those defined by Pollack and Weston in [35]. Specifically, if $\lambda^{\pm}_{PW}(f,\psi)$ are the invariants given in [35, Theorem 4.1] then

$$\lambda^{+}(f,\psi) = \lambda^{-}_{PW}(f,\psi), \qquad \lambda^{-}(f,\psi) = \lambda^{+}_{PW}(f,\psi).$$

After discussions with Pollack and Weston, it appears their choice of parity is a typo. The problem is that defining the plus/minus invariants coming from Mazur-Tate elements as in [35] combines with results of Sprung [38] to give a contradiction, which we now outline for the interested reader.

Assume $\psi = 1$ and let f be as in the previous theorem. Write $\lambda^{\pm}, \mu^{\pm}$ for the plus/minus invariants defined in the theorem above and let $\lambda^{*}_{PW} = \lambda^{-*}$, $* \in \{\pm\}$, be the invariants defined in [35]. Sprung [38] has defined (see §7.3.2.1 below for a summary) *chromatic* p-adic L-functions $L_p^{\sharp/\flat} \in \Lambda$ for f, which recover Pollack's functions $L_p^{\pm}$ when $a_p = 0$. Assuming $\mu(L_p^{\sharp}) = \mu(L_p^{\flat})$ (conjecturally true), [38, Corollary 8.9] says that $\lambda^{+}_{PW} = \lambda(L_p^{\sharp})$ and $\lambda^{-}_{PW} = \lambda(L_p^{\flat})$. Now suppose that $a_p(f) = 0$. Then $L_p^{\sharp} = L_p^{+}$ and $L_p^{\flat} = L_p^{-}$ (up to units) and we obtain

$$\lambda^{+}_{PW} = \lambda(L_p^{+}), \qquad \lambda^{-}_{PW} = \lambda(L_p^{-}).$$

But, using Proposition 7.1.9 below, one can also deduce

$$\lambda^{+}_{PW} = \lambda(L_p^{-}) \qquad \lambda^{-}_{PW} = \lambda(L_p^{+}),$$

and obviously the two previous pairs of equations cannot be both be true in general.

Sprung uses the same parity definition as Pollack-Weston, however the proof of his [38, Corollary 8.9] relies on work of Pollack-Greenberg-Iovita [33], which uses the parity of Theorem 7.1.4 above. Thus, if Definition 8.8 in [38] is replaced with that of Theorem 7.1.4 then all invariants (and the proof of [38, Corollary 8.9]) make

sense, in that we have $\lambda^+ = \lambda(L_p^{\#}) = \lambda(L_p^+)$ and $\lambda^- = \lambda(L_p^{\flat}) = \lambda(L_p^-)$, and the same for μ-invariants.

7.1.3 Weight $k = 2$ and $a_p = 0$: A different approach

In this section we assume that f has rational Fourier coefficients and $a_p = 0$. Write θ_n and $L_p^{\pm}$ for the Mazur-Tate elements and plus/minus p-adic L-functions for f at the character $\psi = 1$, respectively. Let $\mu^{\pm}$ and $\lambda^{\pm}$ denote the Iwasawa invariants associated to θ_n in Theorem 7.1.4. We now show, using the notation of Chapter V, how a result of Pollack describing lifts of θ_n can be used to deduce the behavior of Kurihara and Perrin-Riou described in the Theorem 7.1.4. (This section therefore foreshadows the idea for higher weights.)

Proposition 7.1.7. $\theta_n \equiv \omega_n^{-\varepsilon_n} L_p^{-\varepsilon_n} \bmod \omega_n$,

Proof. See [30, Proposition 6.18]. □

Remark 7.1.8. Notice that $\log_{k=2,n}^{\pm} = \omega_{n,j=0}^{\pm} = \omega_n^{\pm}$, so Theorem V.5.0.1 can be viewed as a generalization of the above proposition to higher weights.

Corollary 7.1.9. *For $n \gg 0$,*

$$\mu(\theta_n) = \mu(L_p^{-\varepsilon_n}),$$
$$\lambda(\theta_n) = q_n + \lambda(L_p^{-\varepsilon_n}).$$

In particular, $\mu^{\pm} = \mu(L_p^{\pm})$ and $\lambda^{\pm} = \lambda(L_p^{\pm})$.

Proof. Proposition 7.1.7 says that $\pi_n\left(\omega_n^{-\varepsilon_n} L_p^{-\varepsilon_n}\right) = \theta_n$. Since $\lambda(\Phi_n(1+T)) = p^n - p^{n-1}$, we have $\lambda(\omega_n^{-\varepsilon_n} L_p^{-\varepsilon_n}) = q_n + \lambda(L_p^{-\varepsilon_n})$. The sequence $p^n - q_n$ is unbounded, thus we may take n large enough so that $\lambda(\omega_n^{-\varepsilon_n} L_p^{-\varepsilon_n}) < p^n$ holds, and the result now follows from Proposition II.2.4.1. The final statement follows by comparing the formulae just proven with those in Theorem 7.1.4. □

Remark 7.1.10. The above corollary can be viewed as an analogue of Proposition 7.1.1 above. It says that even in the nonordinary case, the (pairs of) Iwasawa invariants coming from the p-adic L-function can be recovered by computing Mazur-Tate elements.

Example 7.1.11. Consider the elliptic curve $E/\mathbf{Q}$ with Weierstrass model

$$y^2 + y = x^3 + x^2.$$

(This curve has Cremona label `43a1`.) By checking the L-functions and Modular Forms Database (LMFDB) [20], we know that $\mu(L_5(E)) = 0$ and that $E[5]$ is irreducible as a $G_{\mathbf{Q}}$-module (the latter follows from the fact that E has no rational 5-isogenies). The newform $f \in S_2(\Gamma_0(43))$ corresponding to E has $a_5 = -4$ and $a_7 = 0$. Using the algebra package Magma, we compute the following table.

n	0	1	2	3	4	5
$\lambda_{p=5}(\theta_n)$	0	1	1	1	1	1
$\lambda_{p=7}(\theta_n)$	0	0+1	6+3	42+1	300+3	2100+ 1

Table 7.1: Invariants $\lambda(\theta_n(E))$ for $0 \leq n \leq 5$ at the primes $p = 5$ and 7. The invariants in the second row are written in the form $q_n + \lambda^{\pm}$.

From Propositions 7.1.1 and 7.1.9, this table suggests that

$$\lambda(L_5(f)) = 1, \quad \lambda(L_7^+(f)) = 1, \quad \lambda(L_7^-(f)) = 3.$$

7.1.4 Higher weights: Serre weight two case

In contrast to the single behavior seen in the λ-invariants of Mazur-Tate elements for p-nonordinary forms of weight $k = 2$, the invariants coming from nonordinary forms at weights $k > 2$ can exhibit a variety of behaviors, as seen in the following table.

$k(\bar{\rho}_f)$	N	$\mathrm{ord}_p(a_p)$	0	1	2	3	4	5	label
6	11	1	0	1	3	9	27	81	`G0N11k6A`
2	65	1	0	0	7	19	55	163	`G0N65k6A`
6	17	2	0	0	5	21	47	147	`G0N17k6A`
6	26	∞	0	1	5	17	47	143	`G0N26k6A`

Table 7.2: Invariants $\lambda(\theta_n)$, $0 \leq n \leq 5$, at $p = 3$ attached to various 3-nonordinary rational newforms f of weight $k = 6$ and level $\Gamma_0(N)$. The leftmost column gives the Serre weight (see below) and the rightmost column contains the Magma label (see Appendix B) for the modular form.

Pollack-Weston [35] characterized the Iwasawa invariants of Mazur-Tate elements coming from many higher weight modular forms of *Serre weight two*. The Serre weight of a modular form f is the weight $k(\bar{\rho}_f)$ of its mod p representation $\bar{\rho}_f$ (assumed to be irreducible) as predicted by Serre's modularity conjecture (a theorem of Khare and Wintenberger [17] – see [9] for an overview). It is defined in terms of the structure of $\bar{\rho}_f|_{G_{\mathbf{Q}_p}}$, but using [9, Theorem 1.12], we note that the mod p representation $\bar{\rho}_f$ attached to an eigenform $f \in S_k(N, \varepsilon)$ with $p \nmid N$ will have Serre weight two whenever we can find an eigenform $g \in S_2(N', \varepsilon')$, $p \nmid N'$, satisfying $\bar{\rho}_f \cong \bar{\rho}_g$. Such pairs f and g are said to be *congruent modulo* p.[1]

The advantage of considering modular forms of Serre weight two is that one can hope to apply a strong enough version of 'mod p multiplicity one' to deduce information about the higher weight case by reducing it (via a congruence modulo p) to the weight two case, where the Iwasawa theory is better understood. One of the main results of Pollack and Weston is as follows.

[1]More generally, pairs $f \in S_{k_f}(\Gamma_1(N_f))$ and $g \in S_{k_g}(\Gamma_1(N_g))$ are congruent modulo p if their residual representations at p (with respect to our choice $\bar{\mathbf{Q}} \hookrightarrow \bar{\mathbf{Q}}_p$) are isomorphic. This property can also be characterized somewhat more concretely using the fact that $\bar{\rho}_f(\mathrm{Frob}_\ell) = a_\ell(f)$ for primes $\ell \nmid pN_f$. Let $L/\mathbf{Q}$ be the smallest field containing all Fourier coefficients of f and g and let $\mathfrak{p} \subseteq \mathcal{O}_L$ be the unique prime over p specified by our fixed embedding $\bar{\mathbf{Q}} \hookrightarrow \bar{\mathbf{Q}}_p$. Then f is congruent to g modulo p if $a_\ell(f) \equiv a_\ell(g) \bmod \mathfrak{p}$ for all but finitely many primes ℓ.

Theorem 7.1.12. *Suppose $f \in S_k(\Gamma_0(N))$ is a p-nonordinary eigenform having the following properties:*

(i) $\bar{\rho}_f$ is irreducible,

(ii) $2 < k < p^2 + 1$,

(iii) $k(\bar{\rho}_f) = 2$,

(iv) $\bar{\rho}_f|_{G_{\mathbf{Q}_p}}$ is not decomposable.

Then there is an eigenform $g \in S_2(\Gamma_0(N))$ which is congruent to f modulo p and such that if $\bar{\rho}_f|_{G_{\mathbf{Q}_p}}$ is reducible (resp., irreducible) then

(1) $\mu(\theta_n^{\psi}(f)) = 0$ for $n \gg 0 \iff \mu(L_p^{\psi}(g)) = 0$ (resp. $\mu^{\pm}(g, \psi) = 0$), and

(2) if either equivalent condition in (1) holds then for $n \gg 0$ we have

$$\lambda(\theta_n^{\psi}(f)) = p^n - p^{n-1} + \begin{cases} \lambda(L_p^{\psi}(g)) & \text{if } \bar{\rho}_f|_{G_{\mathbf{Q}_p}} \text{ is reducible,} \\ q_{n-1} + \lambda^{\varepsilon_n}(g, \psi) & \text{if } \bar{\rho}_f|_{G_{\mathbf{Q}_p}} \text{ is irreducible.} \end{cases}$$

Proof. See Theorem 5.1 and Corollary 5.3 in [35]. (Note especially that g is ordinary at p if and only if $\bar{\rho}_g|_{G_{\mathbf{Q}_p}} \cong \bar{\rho}_f|_{G_{\mathbf{Q}_p}}$ is reducible – see Remark 5.2.1, op. cit.) □

Remark 7.1.13. When g is nonordinary, one can replace the invariants $\lambda^{\pm}(g, \psi)$ coming from the sequence of Mazur-Tate elements for g by the invariants of Sprung's [38] chromatic p-adic L-functions – see Proposition 7.3.8 below.

The follow two examples illustrate both behaviors described in the theorem above.

Example 7.1.14. Consider the 3-nonordinary newform (second entry in the table above):

$$f = q + 5q^2 + 6q^3 - 7q^4 - 25q^5 + \cdots \in S_6(\Gamma_0(65)).$$

One can check (in Magma) that $\bar{\rho}_f$ is irreducible and that f is congruent modulo 3 to the the 3-ordinary form

$$g = q - q^2 - 2q^3 - q^4 - q^5 + \cdots \in S_2(\Gamma_0(65)).$$

To use Theorem 7.1.12, it therefore remains to check that $\bar{\rho}_g|_{G_{\mathbf{Q}_p}}$ is not decomposable. This can be accomplished using Gross's tameness criterion [13], which says that $\bar{\rho}_g|_{G_{\mathbf{Q}_p}}$ is decomposable if and only if there exists a normalized eigenform g' (a so-called *companion form*) of level $\Gamma_0(65)$ and weight $k' = p + 1 - k(g) = 2$ satisfying $n^2 a_n(g') \equiv n a_n(g) \bmod 3$ for all $n \geq 1$. There are only finitely many possibilities for g' and one can check in Magma that no such modular forms exist. We now check[2] LMFDB to see that $\mu(L_3(g)) = 0$ and $\lambda(L_3(g)) = 1$, and use Theorem 7.1.12 to conclude that

$$\lambda(\theta_n(f)) = 3^n - 3^{n-1} + 1, \quad n \gg 0.$$

By the table above, we see this equation actually holds for $n \geq 2$.

Example 7.1.15. Consider the 3-nonordinary newform

$$f = q - 2q^2 + 3q^3 + 4q^4 + 11q^5 + \cdots \in S_4(\Gamma_0(26)).$$

Since f has weight $4 = p + 1$, a result of Edixhoven [9] (see [35, Remark 5.2.5]) implies that $\bar{\rho}_f|_{G_{\mathbf{Q}_p}}$ is irreducible of Serre weight two. One can check that f is congruent modulo 3 to the 3-nonordinary newform

$$g = q + q^2 - 3q^3 + q^4 - q^5 + \cdots \in S_2(\Gamma_0(26)),$$

and by looking at LMFDB, we know that $\mu(L_3^{\sharp/\flat}(g)) = \lambda(L_3^{\sharp/\flat}(g)) = 0$, where $L_3^{\sharp/\flat}(g)$ are Sprung's chromatic p-adic L-functions (see §7.3.2.1 below). It now follows from Theorem 7.1.12 (and Proposition 7.3.8 below) that

$$\lambda(\theta_n(f)) = 3^n - 3^{n-1} + q_{n-1}, \quad n \gg 0.$$

[2]Note that the λ-invariant of $L_3(g)$ can also be computed using Mazur-Tate elements via Proposition 7.1.1

7.2 Higher weights: $a_p = 0$ case

The goal of this section is to describe the behavior of Mazur-Tate elements attached to modular forms with $a_p = 0$. Thus, fix for the rest of this section a newform $f = \sum a_n q^n \in S_k(N, \varepsilon)$ with $a_p = 0$, an integer $0 \leq j \leq k-2$, and write $\theta_{n,j}^{\psi}$ and $L_{p,\psi}^{\pm}$ for the Mazur-Tate elements and plus/minus p-adic L-functions associated to f.

7.2.1 Statement of main theorem and sketch of proof

Define the integers

$$\nu_k = \left\lfloor \frac{k-p-2}{p^2-1} \right\rfloor, \qquad \nu_k^- = \nu_k(p-1)+1, \qquad \nu_k^+ = p\nu_k^-,$$

$$\iota_k = \begin{cases} 0 & \text{if } 2 \leq k \leq p+1, \\ \left\lfloor \frac{p(k-2)-1}{p^2-1} \right\rfloor & \text{if } k > p+1 \text{ and } k \not\equiv p+2 \bmod (p^2-1), \\ \frac{p(k-2)-1}{p^2-1} - 1 & \text{if } k > p+1 \text{ and } k \equiv p+2 \bmod (p^2-1). \end{cases}$$

The main result of this chapter is as follows.

Theorem 7.2.1. *Suppose that either*

(i) $k \not\equiv p+2 \bmod (p^2-1)$, *or*

(ii) $k \equiv p+2 \bmod (p^2-1)$ *and* $\lambda\big(L_{p,\psi\omega^j}^{\pm}\big) < \nu_k^{\pm}$.

Then for $n \gg 0$ *we have*

$$\mu\big(\theta_{n,j}^{\psi}\big) = \mu\big(L_{p,\psi\omega^j}^{-\varepsilon_n}\big) + \iota_k + \operatorname{ord}_p\tbinom{k-2}{j}$$

$$\lambda(\theta_{n,j}^{\psi}) = (k-1)q_n - \iota_k(p^n - p^{n-1}) + \lambda\big(L_{p,\psi\omega^j}^{-\varepsilon_n}\big).$$

Remark 7.2.2. Notice that $\iota_k = 0$ at $k = 2$, thus the above theorem generalizes Proposition 7.1.9.

Remark 7.2.3. In §7.2.4, we present a conjectural description of the Iwasawa invariants in the case $k \equiv p+2 \bmod (p^2-1)$ and $\lambda\big(L_{p,\psi\omega^j}^{\pm}\big) \geq \nu_k^{\pm}$.

To give a sense of the ideas involved, we now sketch the proof for $\theta_n = \theta_{n,j=0}^{\psi=1}$.

Sketch of proof for small weights. Define $\Theta_n := u_n L_p^{-\varepsilon_n} \log_{k,n}^{-\varepsilon_n} \in \Lambda_K$, where $u_n = u_{n,j=0}$ are the units described in Theorem V.5.0.1. By Proposition II.2.4.1, if $\lambda(\Theta_n) < p^n$ then

$$\begin{aligned}(k-1)q_n + \lambda\big(L_p^{-\varepsilon_n}\big) &= \lambda(\Theta_n) && \text{(simple computation)}\\ &= \lambda\big(\pi_n(\Theta_n)\big) && \text{(Proposition II.2.4.1)}\\ &= \lambda\big(\theta_n\big), && \text{(Theorem V.5.0.1)}\end{aligned}$$

where the same equalities hold for μ after replacing the first with $\mu\big(L_p^{-\varepsilon_n}\big) = \mu(\Theta_n)$. Thus we need only determine when $(k-1)q_n + \lambda\big(L_p^{-\varepsilon_n}\big) < p^n$ holds. The theorem for $k \leq p+2$ now follows from the fact that

$$p^n - (k-1)q_n = \begin{cases} \text{unbounded} & 2 \leq k \leq p+1, \\ 1 \text{ or } p & k = p+2. \end{cases}$$

Sketch of proof for large weights. Let $\ell_{k,n} \in \mathbf{Z}_p[T]$ be the unique polynomial of degree $< p^n$ for which $\log_{k,n}^{-\varepsilon_n} \equiv \ell_{k,n} \bmod \omega_n$. By Theorem V.5.0.1, we see that $\Theta_n := u_n L_p^{-\varepsilon_n} \ell_{k,n}$ also lift θ_n. Assuming once again that we already know $\lambda(\Theta_n) < p^n$, we obtain

$$\begin{aligned}\lambda(\ell_{k,n}) + \lambda\big(L_p^{-\varepsilon_n}\big) &= \lambda(\Theta_n)\\ &= \lambda\big(\pi_n(\Theta_n)\big) && \text{(Proposition II.2.4.1)}\\ &= \lambda\big(\theta_n\big). && \text{(Theorem V.5.0.1)}\end{aligned}$$

The exact same equalities hold for μ. Thus it suffices to

(A) Compute the Iwasawa invariants of $\ell_{k,n}$.

(B) Check that $\lambda(\Theta_n) = \lambda(\ell_{k,n}) + \lambda\big(L_p^{-\varepsilon_n}\big) < p^n$ for $n \gg 0$.

For (A), recall that functions F which are p-large at n satisfy the relation (see VI.(6.3)):

$$\lambda(F) + \mu(F)(p^n - p^{n-1}) = \lambda(\pi_n(F)) + \mu(\pi_n(F))(p^n - p^{n-1}).$$

We can show (up to units in $\mathbf{Z}_p[[T]]$) that the nth-layer half-logarithms $\log_{k,n}^{-\varepsilon_n}$ are p-large at n (see Lemma 7.2.11). For these functions, the above relation reads

$$(k-1)q_n + 0 \cdot (p^n - p^{n-1}) = \lambda(\ell_{k,n}) + \mu(\ell_{k,n}) \cdot (p^n - p^{n-1}).$$

We then check that $\mu(\ell_{k,n})$ converges in n to ι_k. Putting this all together, for large n we have

$$\lambda(\ell_{k,n}) = (k-1)q_n - \iota_k(p^n - p^{n-1})$$

For (B), we need only determine when the following holds:

$$\lambda\big(L_p^{-\varepsilon_n}\big) < p^n - (k-1)q_n + \iota_k(p^n - p^{n-1}).$$

The theorem then follows by checking that the right-hand-side is unbounded in n if $k \not\equiv p+2 \bmod (p^2-1)$ and is otherwise exactly equal to $\nu_k^{\pm}$ (depending on the parity of n). □

7.2.2 Weights $k \leq p+2$

First, we show that the automorphism $\eta : \Lambda_K \to \Lambda_K$ defined by $F(T) \mapsto F(\gamma(1+T)-1)$ preserves Iwasawa invariants.

Lemma 7.2.4. *For any $F \in \Lambda_K$ and $i \in \mathbf{Z}$ we have $\lambda(\eta^i(F)) = \lambda(F)$ and $\mu(\eta^i(F)) = \mu(F)$.*

Proof. Let λ and μ be the invariants for F. Consider the Weierstrass decomposition $F = p^\mu(T^\lambda + \varpi F_0)U$, where F_0 is a polynomial of degree $< \lambda$ and $U \in \Lambda^\times$. We need only show that $\eta^i(T^\lambda + \varpi F_0)$ is distinguished and that $\eta^i(U)$ is a unit, in which case the Weierstrass decomposition for $\eta^i(F)$ is simply $p^\mu \eta^i(T^\lambda + \varpi F_0)\eta^i(U)$. Since $\gamma^i \equiv 1 \bmod \varpi$ we have

$$\eta^i(T^\lambda + \varpi F_0) \equiv \big(\gamma^i(1+T) - 1\big)^\lambda \equiv T^\lambda \bmod \varpi.$$

Hence η^i preserves distinguished polynomials. Furthermore, writing $U = \sum_{n\geq 0} u_n T^n$ we see that

$$\operatorname{ord}_p \eta^i(U)(0) = \operatorname{ord}_p U(\gamma^i - 1) = \min_n \operatorname{ord}_p \big(u_n(\gamma^i - 1)^n\big) = \operatorname{ord}_p u_0 = 0$$

because $\operatorname{ord}_p(\gamma^i - 1) \geq 1$. □

Example 7.2.5. Since $\eta^i\big(\Phi_n(\gamma^{-i}(1+T))\big) = \Phi_n(1+T)$, Lemma 7.2.4 implies that the Iwasawa invariants of $\Phi_n(\gamma^{-i}(1+T))$ agree with those of $\Phi_n(1+T)$. Thus $\mu(\omega^\pm_{n,i}) = 0$ and because $\lambda(\Phi_n(1+T)) = p^n - p^{n-1}$, we see that

$$\lambda(\omega^+_{n,i}) = \sum_{\substack{m=1 \\ m \text{ even}}}^{n} \lambda(\Phi_m(1+T)) = \begin{cases} q_{n+1} & n \text{ even}, \\ q_n & n \text{ odd}, \end{cases}$$

$$\lambda(\omega^-_{n,i}) = \sum_{\substack{m=1 \\ m \text{ odd}}}^{n} \lambda(\Phi_m(1+T)) = \begin{cases} q_n & n \text{ even}, \\ q_{n+1} & n \text{ odd}, \end{cases}$$

In particular, we have $\mu(\log^\pm_{k,n}) = 0$ and $\lambda(\log^{-\varepsilon_n}_{k,n}) = (k-1)q_n$.

Lemma 7.2.6.

(1) If $k \leq p+1$ then $(k-1)q_n < p^n$ for $n \gg 0$ and $\lim_{n\to\infty}(p^n - (k-1)q_n) = \infty$.

(2) If $k = p+2$ then for all $n \geq 0$, $p^n - (k-1)q_n = \begin{cases} 1 & n \text{ even}, \\ p & n \text{ odd}. \end{cases}$

(3) If $k \geq p+3$ then $(k-1)q_n \geq p^n$ for $n \gg 0$ and $\lim_{n\to\infty}(p^n - (k-1)q_n) = -\infty$.

Proof. Obvious from the definitions. □

Proposition 7.2.7. *Suppose that either*

(i) $2 \leq k \leq p+1$, *or*

(ii) $k = p+2$, $\lambda\big(L^{-}_{p,\psi\omega^j}\big) = 0$, *and* $\lambda\big(L^{+}_{p,\psi\omega^j}\big) \leq p-1$.

Then for $n \gg 0$ *we have*

$$\mu(\theta^{\psi}_{n,j}) = \mu\big(L^{-\varepsilon_n}_{p,\psi\omega^j}\big) + \mathrm{ord}_p\tbinom{k-2}{j}$$
$$\lambda(\theta^{\psi}_{n,j}) = (k-1)q_n + \lambda\big(L^{-\varepsilon_n}_{p,\psi\omega^j}\big).$$

Proof. Recall from Example 7.2.5 that $\mu(\log^{\pm}_{k,n}) = 0$ and $\lambda(\log^{-\varepsilon_n}_{k,n}) = (k-1)q_n$, thus Lemma 7.2.4 gives

$$\mu\big(\eta^j(L^{-\varepsilon_n}_{p,\psi\omega^j}\log^{-\varepsilon_n}_{k,n})\big) = \mu\big(L^{-\varepsilon_n}_{p,\psi\omega^j}\big),$$
$$\lambda\big(\eta^j(L^{-\varepsilon_n}_{p,\psi\omega^j}\log^{-\varepsilon_n}_{k,n})\big) = (k-1)q_n + \lambda\big(L^{-\varepsilon_n}_{p,\psi\omega^j}\big).$$

By Lemma 7.2.6, we can take n large enough so that $(k-1)q_n + \lambda\big(L^{-\varepsilon_n}_{p,\psi\omega^j}\big) < p^n$, allowing us to apply Proposition II.2.4.1 to the lifts in Theorem V.5.0.1 and complete the proof. □

7.2.3 Weights $k \geq p+3$

7.2.3.1 Newton polygons of nth-layer half-logarithms

Using results of the previous section, we have the following for weights $k \leq p+2$:

$$\mu\big(\pi_n(\eta^j\log^{-\varepsilon_n}_{k,n})\big) = 0,$$
$$\lambda\big(\pi_n(\eta^j\log^{-\varepsilon_n}_{k,n})\big) = (k-1)q_n.$$

This is due to the fact that $\lambda(\eta^j\log^{-\varepsilon_n}_{k,n}) = (k-1)q_n < p^n$ for these smaller weights, thus Proposition II.2.4.1 applies. At larger weights, the inequality in the previous

sentence no longer holds (see Lemma 7.2.6), thus another method is needed to determine these nth-layer invariants. The goal of this section is to prove the following proposition.

Proposition 7.2.8. *If $k \geq p+3$ then for $n \gg 0$ we have*

$$\mu\big(\pi_n(\eta^j \log_{k,n}^{-\varepsilon_n})\big) = \iota_k,$$
$$\lambda\big(\pi_n(\eta^j \log_{k,n}^{-\varepsilon_n})\big) = (k-1)q_n - \iota_k(p^n - p^{n-1}).$$

The idea behind the proof is to define a collection of polynomials $F_{k,n}$ which are p-large at n and have the same nth-layer Iwasawa invariants as $\log_{k,n}^{-\varepsilon_n}$, and then use Theorem VI.6.0.1 to compute the nth-layer invariants of $F_{k,n}$.

We begin with some preliminary remarks. For any $i \geq 0$, recall that $\omega_{n,i}^{-\varepsilon_n}$ has λ-invariant q_n and trivial μ-invariant. Since the map η^j respects Iwasawa invariants, the same is true of the polynomials $\eta^j(\omega_{n,i}^{-\varepsilon_n})$. Thus, by the Weierstrass Preparation Theorem, we may write

$$\eta^j(\omega_{n,i}^{-\varepsilon_n}) = (T^{q_n} + pW_i)U_i,$$

for some $W_i \in \mathbf{Z}_p[T]$ of degree $< q_n$ and units $U_i \in \mathbf{Z}_p[[T]]^\times$, both clearly dependent on n and j. Therefore,

$$\eta^j(\log_{k,n}^{-\varepsilon_n}) = U\prod_{i=0}^{k-2}(T^{q_n} + pW_i).$$

where $U \in \mathbf{Z}_p[[T]]^\times$ is the product of all the U_i. Since finite-layer Iwasawa invariants are stable under multiplication by unital power series (see Proposition II.2.4.3), we define

$$F_{k,n} := \prod_{i=0}^{k-2}(T^{q_n} + pW_i),$$

and observe that

$$\begin{aligned}\mu\big(\pi_n(\eta^j \log_{k,n}^{-\varepsilon_n})\big) &= \mu\big(\pi_n(F_{k,n})\big)\\ \lambda\big(\pi_n(\eta^j \log_{k,n}^{-\varepsilon_n})\big) &= \lambda\big(\pi_n(F_{k,n})\big).\end{aligned} \tag{7.1}$$

Thus, it suffices to compute the n-th layer invariants of $F_{k,n}$.

Lemma 7.2.9. *Fix integers $m \geq 1$, $q \geq 0$, and let $W_1, \ldots, W_m \in \mathbf{Z}_p[T]$. Then*

$$\prod_{i=1}^{m}(T^q + pW_i) = \sum_{i=0}^{m} p^i T^{(m-i)q} e_i(W_1, \cdots W_m),$$

where $e_i(W_1, \cdots W_m)$ denotes the i^{th} elementary symmetric polynomial in the m variables $W_1, \ldots, W_m$.

Proof. The claim is clearly true if $m = 1$ since $e_0(W_1) = 1$ and $e_1(W_1) = W_1$. Proceeding via induction, suppose it is true for some $m \geq 1$. Writing $e_i^{(m)} := e_i(W_1, \cdots W_m)$, we now have

$$\begin{aligned}
\prod_{i=1}^{m+1}(T^q + pW_i) &= (T^q + pW_{m+1}) \prod_{i=1}^{m}(T^q + pW_i) \\
&= (T^q + pW_{m+1}) \cdot \sum_{i=0}^{m} p^i T^{(m-i)q} e_i^{(m)} \\
&= T^{(m+1)q} e_0^{(m)} + \sum_{i=1}^{m} p^i T^{(m+1-i)q} \left(e_i^{(m)} + e_{i-1}^{(m)} W_{m+1}\right) + p^{m+1} e_m^{(m)} W_{m+1} \\
&= \sum_{i=0}^{m+1} p^i T^{(m+1-i)q} e_i^{(m+1)},
\end{aligned}$$

where the final equality follows from the well-known [45] recursion relation $e_i^{(m+1)} = e_i^{(m)} + e_{i-1}^{(m)} W_{m+1}$. □

The above lemma allows us to write

$$F_{k,n} = \sum_{i=0}^{k-1} p^{k-1-i} T^{iq_n} e_{k-1-i}(W_0, \cdots W_{k-2}). \tag{7.2}$$

Let us now introduce the polynomials

$$G_{k,n} := p^{k-1} + pT^{(k-2)q_n} + pT^{(k-1)q_n - 1} + T^{(k-1)q_n}.$$

These polynomials have the same Iwasawa invariants as $F_{k,n}$ and their simple Newton polygons will be used to 'bound' the Newton polygon of $F_{k,n}$, in the sense of the following lemma.

Lemma 7.2.10. $\mathrm{NP}_p\left(\mathcal{T}_{(k-1)q_n-1}G_{k,n}\right) \supseteq \mathrm{NP}_p\left(\mathcal{T}_{(k-1)q_n-1}F_{k,n}\right)$.

Proof. Letting $e_i = e_i(W_0, \cdots W_{k-2})$, equation (7.2) implies

$$\begin{aligned}
F_{k,n} &\equiv T^{q_n}\Big(\sum_{i=1}^{k-1} p^{k-1-i}T^{(i-1)q_n}e_{k-1-i}\Big) \bmod p^{k-1} \\
&\equiv T^{2q_n}\Big(\sum_{i=2}^{k-1} p^{k-1-i}T^{(i-2)q_n}e_{k-1-i}\Big) \bmod p^{k-2} \\
&\equiv T^{3q_n}\Big(\sum_{i=3}^{k-1} p^{k-1-i}T^{(i-3)q_n}e_{k-1-i}\Big) \bmod p^{k-3} \\
&\vdots \\
&\equiv T^{(k-2)q_n}\Big(\sum_{i=k-2}^{k-1} p^{k-1-i}T^{(i-k+2)q_n}e_{k-1-i}\Big) \bmod p^2 \\
&\equiv T^{(k-1)q_n} \bmod p.
\end{aligned}$$

Writing $F_{k,n} = \sum_{i=0}^{(k-1)q_n} a_i T^i$, these congruences imply the respective inequalities:

$$\operatorname{ord}_p a_i \geq \begin{cases} k-1 & \text{for } 0 \leq i \leq q_n - 1, \\ k-2 & \text{for } q_n \leq i \leq 2q_n - 1, \\ k-3 & \text{for } 2q_n \leq i \leq 3q_n - 1, \\ \vdots & \vdots \\ 2 & \text{for } (k-3)q_n \leq i \leq (k-2)q_n - 1, \\ 1 & \text{for } (k-2)q_n \leq i \leq (k-1)q_n - 1. \end{cases}$$

Drawing the Newton polygon (see the figure below) for $\mathcal{T}_{(k-1)q_n-1}(G_{k,n}) = p^{k-1} + pT^{(k-2)q_n} + pT^{(k-1)q_n-1}$, we see that it contains all points (i, v) (indicated with bold points in the figure), where v ranges over the lower bounds for $\operatorname{ord}_p a_i$ above and $0 \leq i \leq (k-1)q_n - 1$. The Newton polygon for $\mathcal{T}_{(k-1)q_n-1}(F_{k,n})$, whose vertices lie on or above the points (i, v), must therefore be contained in that of $\mathcal{T}_{(k-1)q_n-1}(G_{k,n})$. □

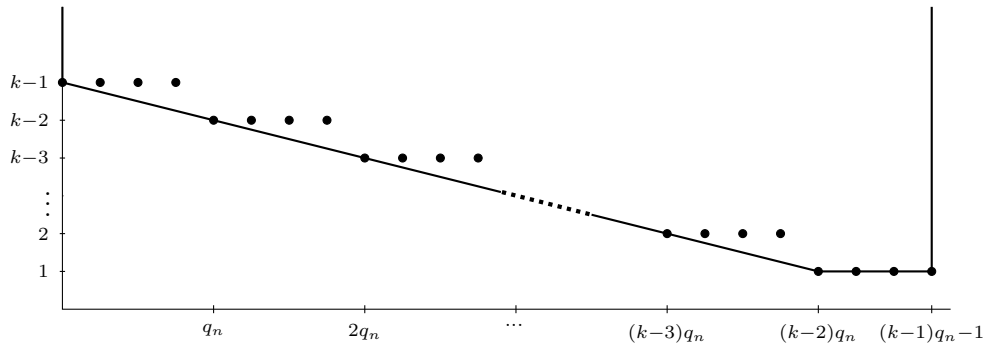


Figure 7.1: The outlined region is the Newton polygon of $\mathcal{T}_{(k-1)q_n-1}(G_{k,n})$. The above lemma shows that the Newton polygon of $\mathcal{T}_{(k-1)q_n-1}(F_{k,n})$ has vertices on or above the bold points.

Lemma 7.2.11. *$F_{k,n}$ is p-large at all $n \gg 0$.*

Proof. By Corollary VI.6.5.7 and the previous lemma, it suffices to show that $G_{k,n}$ is strictly p-large at $n \gg 0$. Let $\lambda_n := \lambda(F_{k,n}) = \lambda(G_{k,n}) = (k-1)q_n$ and write $\lambda_n = p^n + R_n + Q_n t_n$ where $t_n = p^n - p^{n-1}$, $R_n = \lambda_n - p^n \bmod t_n$, and $Q_n = \lfloor \frac{\lambda_n - p^n}{t_n} \rfloor$. We must prove that $\mathrm{NP} = \mathrm{NP}_p\left(\mathcal{T}_{\lambda_n - 1} G_{k,n}\right)$ lies within the region $\mathcal{R} = \mathcal{R}_{\lambda_n,0}$ (defined in Chapter VI), as seen in the following diagram.

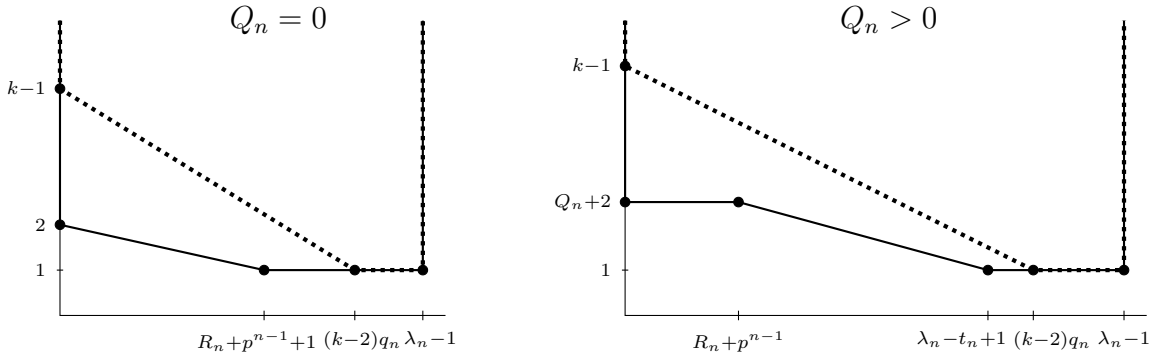


Figure 7.2: The Newton polygon of $\mathcal{T}_{\lambda_n-1}(G_{k,n})$ (dotted) and the region $\mathcal{R}_{\lambda_n,0}$ (bold).

It suffices to show the following:

(i) The middle vertex of NP lies along the rightmost (horizontal) edge of $\mathcal{R}$.

(ii) The the unique nonzero (and non-infinite) slope of NP is less than or equal to the unique nonzero (and non-infinite) slope of $\mathcal{R}$.

Case $Q_n = 0$. The region $\mathcal{R}$ has vertices at $(0,2)$, $(R_n + p^{n-1} + 1, 1)$, and $(\lambda_n - 1, 1)$. To show (ii), we must establish that $-\frac{1}{q_n} \leq -\frac{1}{R_n + p^{n-1}+1}$, and this follows easily from the fact that $q_n < p^{n-1} \leq p^{n-1} + R_n + 1$ for $n \geq 0$. For (i), we must show that

$$R_n + p^{n-1} + 1 \leq (k-2)q_n, \qquad n \gg 0. \tag{7.3}$$

If $k \geq p+4$ then $(R_n + p^{n-1} + 1)/q_n \leq p^n/q_n \to p+1$ as $n \to \infty$, hence certainly we can take n large enough so that $(R_n + p^{n-1} + 1)/q_n \leq p+2 \leq k-2$. If $k = p+3$ then the identity

$$(p+1)q_n = \begin{cases} p^n - 1 & n \text{ even}, \\ p^n - p & n \text{ odd}, \end{cases} \tag{7.4}$$

shows that $R_n = q_n - 1$ or $q_n - p$ depending on whether n is even or odd, respectively. Thus equation (7.3) reads

$$\left\{ \begin{array}{ll} q_n + p^{n-1} & n \text{ even} \\ q_n + p^{n-1} - (p-1) & n \text{ odd} \end{array} \right\} \leq \left\{ \begin{array}{ll} p^n - 1 & n \text{ even} \\ p^n - p & n \text{ odd} \end{array} \right\},$$

and this follows from the fact that $q_n \leq t_n - 1$ for $n \geq 1$.

Case $Q_n > 0$. Now $\mathcal{R}$ has vertices at $(0, Q_n + 2)$, $(R_n + p^{n-1}, Q_n + 2)$, $(\lambda_n - t_n + 1, 1)$, and $(\lambda_n - 1, 1)$. Since $q_n \leq t_n - 1$, we know that $\lambda_n - t_n + 1 \leq (k-2)q_n$ for $n \geq 1$, thus (i) holds. For (ii), we need to show that $-\frac{1}{q_n} \leq -\frac{Q_n+1}{Q_n t_n + 1}$, and this is equivalent to checking that $(q_n - 1)/(t_n - q_n) \leq Q_n$, which is true since $(q_n - 1)/(t_n - q_n) < 1$ for $n \gg 0$. □

Proof of Proposition 7.2.8. Take n large enough so that $F_{k,n}$ is p-large at n. Then since $\lambda(\eta^j(\log_{k,n}^{-\varepsilon_n})) = \lambda(F_{k,n}) = (k-1)q_n$, Theorem VI.6.0.1 and (7.1) yield

$$\mu(\pi_n(\eta^j(\log_{k,n}^{-\varepsilon_n}))) = \mu(\pi_n(F_{k,n})) = \left\lfloor \frac{(k-1)q_n - p^n}{t_n} \right\rfloor + 1.$$

Now observe that $q_n/t_n \nearrow p/(p^2-1)$ (i.e., increases toward) as $n \to \infty$. Hence

$$\lim_{n\to\infty} \frac{(k-1)q_n - p^n}{t_n} + 1 = \frac{p(k-1)}{p^2-1} - \frac{p}{p-1} + 1 = \frac{p(k-2)-1}{p^2-1},$$

and this value is an integer if any only if $k \equiv p+2 \bmod (p^2-1)$. Thus, if $k \not\equiv p+2 \bmod (p^2-1)$ then the floor function is continuous at $\frac{p(k-2)-1}{p^2-1}$ and we have

$$\lim_{n\to\infty} \left\lfloor \frac{(k-1)q_n - p^n}{t_n} + 1 \right\rfloor = \begin{cases} \left\lfloor \frac{p(k-2)-1}{p^2-1} \right\rfloor & \text{if } k \not\equiv p+2 \bmod (p^2-1), \\ \frac{p(k-2)-1}{p^2-1} - 1 & \text{if } k \equiv p+2 \bmod (p^2-1), \end{cases}$$

where the second equality is because $\frac{(k-1)q_n - p^n}{t_n} + 1 \leq \frac{p(k-2)-1}{p^2-1}$ for all $n \gg 0$. Thus $\mu(\pi_n(\eta^j \log_{k,n}^{-\varepsilon_n})) = \iota_k$ for $n \gg 0$, and the λ-invariant follows from the fact that $\lambda(\pi_n(F_{k,n})) = \lambda(F_{k,n}) - \iota_k t_n$ by VI.(6.3). □

7.2.3.2 Another description of $\lambda(\eta^j \log_{k,n}^{-\varepsilon_n})$

In the previous section we were able to compute the nth-layer λ-invariants of $\eta^j(\log_{k,n}^{-\varepsilon_n})$ without explicitly computing the sequence

$$R_n = r(\eta^j \log_{k,n}^{\pm})_n = \left((k-1)q_n - p^n\right) \bmod (p^n - p^{n-1})$$

appearing in Theorem VI.6.0.1. This was done by instead computing the nth-layer μ-invariants and then using the relation VI.(6.3). Though not necessary to finish proving Theorem 7.2.1, computing this sequence will prove useful later on.

Define modified Kurihara terms $q'_n \in \mathbf{Z}$ by $pq_n \equiv q'_n \bmod (p^n - p^{n-1})$. Explicitly,

$$q'_n = \begin{cases} p^{n-2} - p^{n-3} + \cdots + p^2 - p & n > 3 \text{ even}, \\ p^{n-2} - p^{n-3} + \cdots + p^3 - p^2 & n > 3 \text{ odd}, \end{cases}$$

and $q'_n = 0$ for $n \leq 3$.

Lemma 7.2.12. *Let $l = k-p-2 \bmod (p^2-1)$ and define $a = l \bmod p$, $b = \lfloor l/p \rfloor$. Then for $n \gg 0$ we have*

$$R_n = \begin{cases} aq_n + bq'_n - \nu^{-\varepsilon_n} & \text{if } l > 0, \\ p^n - p^{n-1} - \nu^{-\varepsilon_n} & \text{if } l = 0, \end{cases}$$

where $\nu^{\pm} = \nu_k^{\pm}$ are the integers defined in section §7.2.1.

Proof. Set $m = k - p - 2 \geq 1$ (recall $k \geq p+3$) and use (7.4) to write

$$R_n \equiv \left\{ \begin{array}{ll} mq_n - 1, & n \text{ even} \\ mq_n - p, & n \text{ odd} \end{array} \right\} \bmod (p^n - p^{n-1}). \tag{7.5}$$

Since $m = l + \nu(p^2-1)$, where $\nu = \nu_k$, and

$$(p^2-1)q_n = \begin{cases} p^{n+1} - p^n - (p-1) & n \text{ even}, \\ p^{n+1} - p^n - (p^2-p) & n \text{ odd}, \end{cases} \tag{7.6}$$

notice that

$$mq_n \equiv lq_n - \left\{ \begin{array}{ll} \nu(p-1), & n \text{ even} \\ \nu(p^2-p), & n \text{ odd} \end{array} \right\} \bmod (p^n - p^{n-1}). \tag{7.7}$$

If $l = 0$ then (7.5) gives $R_n \equiv -\nu^{-\varepsilon_n} \equiv p^n - p^{n-1} - \nu^{-\varepsilon_n} \bmod (p^n - p^{n-1})$ and the proposition follows by taking n large enough so that $\nu^{\pm} < p^n - p^{n-1}$. Now suppose $l > 0$ and write $l = a + pb$. Then $lq_n \equiv aq_n + bq'_n \bmod (p^n - p^{n-1})$, thus (7.5) and (7.7) imply $R_n \equiv aq_n + bq'_n - \nu^{-\varepsilon_n} \bmod (p^n - p^{n-1})$. As $a, b \leq p-1$ (the bound on b follows from $l < p^2 - 1$) we know that $aq_n + bq'_n = O(p^{n-1})$, hence we can n large enough so that $aq_n + bq'_n - \nu^{-\varepsilon_n} < p^n - p^{n-1}$ and we are done. □

Corollary 7.2.13. *For $n \gg 0$,*

$$\lambda\big(\pi_n(\eta^j \log_{k,n}^{-\varepsilon_n})\big) = p^{n-1} + R_n = \begin{cases} p^{n-1} + aq_n + bq'_n - \nu^{-\varepsilon_n} & \text{if } k \not\equiv p+2 \bmod (p^2-1), \\ p^n - \nu^{-\varepsilon_n} & \text{if } k \equiv p+2 \bmod (p^2-1). \end{cases}$$

Proof. From (7.1) and the fact that $F_{k,n}$ are p-large at $n \gg 0$, we apply Theorem VI.6.0.1 to obtain

$$\lambda\big(\pi_n(\eta^j \log_{k,n}^{-\varepsilon_n})\big) = \lambda(\pi_n(F_{k,n})) = p^{n-1} + r(F_{k,n})_n = p^{n-1} + R_n.$$

Now use the previous lemma. □

7.2.3.3 Proof of Theorem 7.2.1 for weights $k \geq p+3$

Proposition 7.2.14. *Let $k \geq p+3$ and suppose that either*

(i) $k \not\equiv p+2 \bmod (p^2-1)$, or

(ii) $k \equiv p+2 \bmod (p^2-1)$ and $\lambda\big(L_{p,\psi\omega^j}^{\pm}\big) < \nu^{\pm}$.

Then for $n \gg 0$ we have

$$\begin{aligned}
\mu(\theta_{n,j}^{\psi}) &= \mu\big(L_p^{-\varepsilon_n}\big) + \iota + \operatorname{ord}_p\tbinom{k-2}{j} \\
\lambda(\theta_{n,j}^{\psi}) &= (k-1)q_n - \iota(p^n - p^{n-1}) + \lambda\big(L_p^{-\varepsilon_n}\big) \\
&= \begin{cases} p^{n-1} + aq_n + bq_n' - \nu^{-\varepsilon_n} + \lambda\big(L_{p,\psi\omega^j}^{-\varepsilon_n}\big) & \text{if (i),} \\ p^n - \nu^{-\varepsilon_n} + \lambda\big(L_{p,\psi\omega^j}^{-\varepsilon_n}\big) & \text{if (ii),} \end{cases}
\end{aligned}$$

where $\iota = \iota_k$ and a,b are the integers (dependent on k) defined in Proposition 7.2.12.

Proof. Let $\ell_{k,n}^{(j)} \in \mathbf{Z}_p[T]$ be the unique polynomial of degree $< p^n$ for which $\eta^j(\log_{k,n}^{-\varepsilon_n}) \equiv \ell_{k,n}^{(j)} \bmod \omega_n$. By Theorem V.5.0.1, we have

$$\theta_{n,j}^{\psi} = \pi_n\big(\tbinom{k-2}{j}\eta^j(L_{p,\psi\omega^j}^{-\varepsilon_n}\log_{k,n}^{\pm})\big) = \tbinom{k-2}{j}\pi_n\big(\eta^j(L_{p,\psi\omega^j}^{-\varepsilon_n})\ell_{k,n}^{(j)}\big),$$

modulo units in $\mathbf{Z}_p[\varepsilon(p)]$. Hence, if $\Theta_n := \eta^j(L_{p,\psi\omega^j}^{-\varepsilon_n})\ell_{k,n}^{(j)}$ then

$$\begin{aligned}
\mu(\theta_{n,j}^{\psi}) &= \mu(\pi_n(\Theta_n)) + \operatorname{ord}_p\tbinom{k-2}{j} \\
\lambda(\theta_{n,j}^{\psi}) &= \lambda(\pi_n(\Theta_n)).
\end{aligned} \tag{7.8}$$

Let $\lambda^{\pm}$ and $\mu^{\pm}$ denote the Iwasawa invariants of $L_{p,\psi\omega^j}^{\pm}$ and note that

$$\lambda(\Theta_n) = \lambda^{-\varepsilon_n} + \lambda(\ell_{k,n}) = \lambda^{-\varepsilon_n} + \lambda\big(\pi_n(\eta^j \log_{k,n}^{-\varepsilon_n})\big)$$

where the first equality is due to Lemma 7.2.4, the second is by definition, and both these equalities also hold for μ-invariants. Combined with Proposition 7.2.8 and Corollary 7.2.13, we now have for $n \gg 0$:

$$\begin{aligned} \mu(\Theta_n) &= \mu^{-\varepsilon_n} + \iota \\ \lambda(\Theta_n) &= \lambda^{-\varepsilon_n} + (k-1)q_n - \iota(p^n - p^{n-1}) \\ &= \lambda^{-\varepsilon_n} + p^{n-1} + R_n \end{aligned} \tag{7.9}$$

where $R_n = r(\eta^j \log_{k,n}^{-\varepsilon_n})_n$ is the sequence computed in the previous section. Observe that

$$\lambda(\Theta_n) < p^n \iff \lambda^{-\varepsilon_n} < p^n - p^{n-1} - R_n.$$

From Proposition 7.2.12, it is easy to see that $p^n - p^{n-1} - R_n$ is unbounded precisely when $k \not\equiv p+2 \bmod (p^2-1)$, and is otherwise equal to $\nu^{-\varepsilon_n}$ for $n \gg 0$. Hence, if $k \not\equiv p+2 \bmod (p^2-1)$ (or $k \equiv p+2 \bmod (p^2-1)$ and $\lambda(L_{p,\psi\omega^j}^{\pm}) < \nu^{\pm}$) then Proposition II.2.4.1 implies $\lambda(\pi_n(\Theta_n)) = \lambda(\Theta_n)$ and $\mu(\pi_n(\Theta_n)) = \mu(\Theta_n)$. The result now follows from (7.8), (7.9), and the description of R_n given in Proposition 7.2.12. □

7.2.4 The missing weights

We now consider the case where $k \equiv p+2 \bmod (p^2-1)$ and $\lambda\big(L_{p,\psi\omega^j}^{\pm}\big) \geq \nu_k^{\pm}$ under the following assumption.

Assumption. $\eta^j(L_{p,\psi\omega^j}^{\pm} \log_{k,n}^{\pm})$ is p-large for $n \gg 0$.

Under this assumption, Theorem VI.6.0.1 (see also Remark VI.6.3.3) implies

$$\mu(\theta^{\psi}_{n,j}) = \mu(\pi_n(\eta^j(L_{p,\psi\omega^j}^{-\varepsilon_n}\log_{k,n}^{-\varepsilon_n})) = \mu^{-\varepsilon_n} + \left\lfloor \tfrac{(k-1)q_n - p^n}{p^n - p^{n-1}} \right\rfloor + 1 + \mathrm{ord}_p\binom{k-2}{j} \quad (7.10)$$

$$\lambda(\theta^{\psi}_{n,j}) = \lambda(\pi_n(\eta^j(L_{p,\psi\omega^j}^{-\varepsilon_n}\log_{k,n}^{-\varepsilon_n})) = p^{n-1} + r(\eta^j(L_{p,\psi\omega^j}^{-\varepsilon_n}\log_{k,n}^{-\varepsilon_n}))_n, \quad (7.11)$$

where $\mu^{\pm} = \mu\big(L^{\pm}_{p,\psi\omega^j}\big)$. By (the proof of) Proposition 7.2.8, the μ-invariants obtained here are asymptotically no different from those considered in the previous section: we have $\mu(\theta^{\psi}_{n,j}) = \mu^{-\varepsilon_n} + \iota_k + \mathrm{ord}_p\binom{k-2}{j}$ for $n \gg 0$. Now set $\lambda^{\pm} = \lambda\big(L^{\pm}_{p,\psi\omega^j}\big)$ and $R_n = r(\eta^j \log_{k,n}^{-\varepsilon_n})_n$ as before. The definitions and Proposition 7.2.12 imply

$$\begin{aligned} r(\eta^j(L_{p,\psi\omega^j}^{-\varepsilon_n}\log_{k,n}^{-\varepsilon_n}))_n &\equiv \lambda\big(\eta^j(L_{p,\psi\omega^j}^{-\varepsilon_n}\log_{k,n}^{-\varepsilon_n})\big) - p^n \bmod (p^n - p^{n-1}) \\ &\equiv \lambda^{-\varepsilon_n} + \lambda(\log_{k,n}^{-\varepsilon_n}) - p^n \bmod (p^n - p^{n-1}) \\ &\equiv \lambda^{-\varepsilon_n} + R_n \bmod (p^n - p^{n-1}) \\ &\equiv \lambda^{-\varepsilon_n} - \nu_k^{-\varepsilon_n} \bmod (p^n - p^{n-1}). \end{aligned}$$

Since we are assuming $\lambda^{\pm} \geq \nu_k^{\pm}$, we can now take n large enough so that

$$r(\eta^j(L_{p,\psi\omega^j}^{-\varepsilon_n}\log_{k,n}^{-\varepsilon_n}))_n = \lambda^{-\varepsilon_n} - \nu_k^{-\varepsilon_n}.$$

This combines with (7.10) to give $\lambda(\theta^{\psi}_{n,j}) = p^{n-1} - \nu_k^{-\varepsilon_n} + \lambda^{-\varepsilon_n}$ Thus, the full result for $k \equiv p+2 \bmod (p^2-1)$ reads as follows.

Proposition 7.2.15. *If $k \equiv p+2 \bmod (p^2-1)$ then*

$$\lambda(\theta^{\psi}_{n,j}) = \begin{cases} p^n - \nu_k^{-\varepsilon_n} + \lambda\big(L_{p,\psi\omega^j}^{-\varepsilon_n}\big) & \text{if } \lambda\big(L^{\pm}_{p,\psi\omega^j}\big) < \nu_k^{\pm}, \\ p^{n-1} - \nu_k^{-\varepsilon_n} + \lambda\big(L_{p,\psi\omega^j}^{-\varepsilon_n}\big) & \text{if } \lambda\big(L^{\pm}_{p,\psi\omega^j}\big) \geq \nu_k^{\pm} \text{ and the Assumption holds.} \end{cases}$$

The behaviors for $k \equiv p+2 \bmod (p^2-1)$ are therefore identical except that in one case we have $O(p^{n-1})$ growth instead of $O(p^n)$. In the table below we have computed the sequence of Mazur-Tate elements for various newforms with

$a_3 = 0$ and, assuming they follow the above patterns, we have written down the corresponding invariants of $L_3^{\pm}$. See Appendix B for more examples.

k	N	$\lambda(L_p^+)$	$\lambda(L_p^-)$	0	1	2	3	4	5	6	label
5	40	4?	2?	0	1	4	10	28	82	244	`G1N40k5F`
5	40	2?	0?	0	2	8	26	80	242	728	`G1N40k5G`
5	52	3?	1?	0	1	3	9	27	81	243	`G1N52k5G`
13	31	9?	3?	0	1	3	9	27	81	243	`G1N31k13D`
13	40	10?	4?	0	2	4	10	28	82	244	`G1N40k13E`
13	40	8?	2?	0	2	8	26	80	242	728	`G1N40k13F`

Table 7.3: Invariants $\lambda(\theta_{n,j=0}^{\psi=1}(f))$, $0 \leq n \leq 6$, attached to $p = 3$-nonordinary rational newforms $f \in S_k(\Gamma_1(N))$ with $a_3 = 0$. Note that $\nu_5^+ = 3$, $\nu_5^- = 1$, and $\nu_{13}^+ = 9$, $\nu_{13}^- = 3$.

In all our computations, we have only observed two behaviors for the λ-invariant at weights $k \equiv p+2 \bmod (p^2-1)$. Given this, one is led to wonder whether there exist instances where $\lambda(L_p^+) < \nu_k^+$ but $\lambda(L_p^-) \geq \nu_k^-$, or vice versa, since such a case would not be covered by the proposition above. Of course, it is possible that such examples *do* exist and just happen to follow the same behavior for some other reason.

Question 7.2.16. *If $f \in S_k(N, \varepsilon)$, $p \nmid N$, is a newform with $a_p = 0$ and $k \equiv p+2 \bmod (p^2-1)$, is it true that exactly one of $\lambda\big(L_p^{\pm}(f)\big) < \nu_k^{\pm}$ or $\lambda\big(L_p^{\pm}(f)\big) \geq \nu_k^{\pm}$ holds?*

7.3 Relating Iwasawa invariants at different weights

7.3.1 Weights congruent modulo $p^2 - 1$

Fix newforms $f \in S_k(\Gamma_1(N))$, $g \in S_{k'}(\Gamma_1(N'))$, where $p \nmid NN'$, and assume $a_p(f) = a_p(g) = 0$. Choose $0 \leq j \leq k-2$, $0 \leq j' \leq k'-2$, and tame characters

$\psi, \psi' : \mathbf{Z}_p^\times \to \mathbf{C}_p^\times$. If the weights of f and g are both ≥ 3 and agree modulo p^2-1 we can use the results of the previous section to show that the difference between the λ-invariants of the Mazur-Tate elements for these modular forms is constant over n of fixed parity (this is not true in general). Let $\nu_k^\pm$ and $\nu_{k'}^\pm$ denote the integers defined in the previous chapter associated to f and g, respectively.

Corollary 7.3.1. *Suppose without loss of generality that $k \geq k'$ and that both weights are ≥ 3. If either of the following hold:*

(i) $k \equiv k' \not\equiv p+2 \bmod (p^2-1)$, or

(ii) $k \equiv k' \equiv p+2 \bmod (p^2-1)$, $\lambda\big(L_{p,\psi\omega^j}^\pm(f)\big) < \nu_k^\pm$, and $\lambda\big(L_{p,\psi'\omega^{j'}}^\pm(g)\big) < \nu_{k'}^\pm$,

then for $n \gg 0$ we have

$$\mu\big(\theta_{n,j}^{\psi}(f)\big) - \mu\big(\theta_{n,j'}^{\psi'}(g)\big) = \mu\big(L_{p,\psi\omega^j}^{-\varepsilon_n}(f)\big) - \mu\big(L_{p,\psi'\omega^{j'}}^{-\varepsilon_n}(g)\big) + \iota_k - \iota_{k'} + \operatorname{ord}_p\tbinom{k-2}{j} - \operatorname{ord}_p\tbinom{k'-2}{j'},$$

$$\lambda\big(\theta_{n,j}^{\psi}(f)\big) - \lambda\big(\theta_{n,j'}^{\psi'}(g)\big) = \begin{cases} \lambda\big(L_{p,\psi\omega^j}^{-}(f)\big) - \lambda\big(L_{p,\psi'\omega^{j'}}^{-}(g)\big) - \frac{k-k'}{p+1} & n \text{ even,} \\ \lambda\big(L_{p,\psi\omega^j}^{+}(f)\big) - \lambda\big(L_{p,\psi'\omega^{j'}}^{+}(g)\big) - \frac{p(k-k')}{p+1} & n \text{ odd.} \end{cases}$$

Remark 7.3.2. The congruence $k \equiv k' \bmod (p^2-1)$ is also telling us concrete information about the local structure of the mod p representations of f and g. See [3, Theorem 4.2.1].

Remark 7.3.3. Under the assumptions of the corollary, we have

$$\lambda\big(\theta_{n,j}^{\psi}(f)\big) - \lambda\big(\theta_{n,j'}^{\psi'}(g)\big) \equiv \lambda\big(L_{p,\psi\omega^j}^{-\varepsilon_n}(f)\big) - \lambda\big(L_{p,\psi'\omega^{j'}}^{-\varepsilon_n}(g)\big) \bmod (p-1).$$

Proof. The equation for μ-invariants follows directly from Theorem 7.2.1. Turning our attention to λ-invariants, one can check that $k \equiv k \bmod (p^2-1)$ implies $\nu_k - \nu_{k'} = (k-k')/(p^2-1)$ thus it suffices to show that

$$\lambda\big(\theta_{n,j}^{\psi}(f)\big) - \lambda\big(\theta_{n,j'}^{\psi'}(g)\big) = \lambda\big(L_{p,\psi\omega^j}^{-\varepsilon_n}(f)\big) - \lambda\big(L_{p,\psi'\omega^{j'}}^{-\varepsilon_n}(g)\big) - (\nu_k^{-\varepsilon_n} - \nu_{k'}^{-\varepsilon_n}).$$

The case $k = k'$ is a trivial consequence of Theorem 7.2.1, so assume $k > k'$. First, suppose both k and k' are $\geq p+3$. Let l, a, b and l', a', b' be the integers defined in Proposition 7.2.12 corresponding to f and g, respectively. Since $k \equiv k' \bmod (p^2-1)$ we know that $l = l'$ and therefore $a = a'$, $b = b'$, and we are done by Proposition 7.2.14.

Now consider the case $k' = p+2$ (so $k \geq p+3$ by necessity). Applying Proposition 7.2.7 to g yields

$$\lambda\big(\theta^{\psi}_{n,j}(g)\big) = (p+1)q_n - \lambda\big(L^{-\varepsilon_n}_{p,\psi'\omega^{j'}}(g)\big) = p^n - \nu^{-\varepsilon_n}_{p+2} + \lambda\big(L^{-\varepsilon_n}_{p,\psi'\omega^{j'}}(g)\big).$$

Comparing this with the λ-invariants for f given in Proposition 7.2.14, we have the result.

Finally, assume $3 \leq k' \leq p+1$ (again, $k \geq p+3$ by necessity). Then $l = k-p-2 \equiv k'-p-2 \bmod (p^2-1)$ and the bounds on k' imply $l = k'-p-2+(p^2-1) = k'+p^2-p-3$. Hence $a = k'-3$ and $b = p-1$ (since p is odd and $k' \neq 2$). Applying Proposition 7.2.14 to f we have

$$\begin{aligned}\lambda\big(\theta^{\psi}_{n,j}(f)\big) &= p^{n-1} + (k'-3)q_n + (p-1)q'_n - \nu_k^{-\varepsilon_n} + \lambda\big(L^{-\varepsilon_n}_{p,\psi\omega^j}(f)\big) \\ &= (k'-1)q_n + \lambda\big(L^{-\varepsilon_n}_{p,\psi\omega^j}(f)\big) - (\nu_k^{-\varepsilon_n} - \nu_{k'}^{-\varepsilon_n}).\end{aligned}$$

(Write $q'_n = p^{n-1} - q_n - \star$ where $\star = 1$ or p if n is even or odd, respectively, and note that $\nu_{k'} = -1$, so $\nu^+_{k'} = 2-p$ and $\nu^-_{k'} = 2p-p^2$.) Comparing this with the equation for $\lambda\big(\theta^{\psi}_{n,j}(g)\big)$ given in Proposition 7.2.7 finishes the proof. □

If g has weight $k' = 2$ then the above corollary no longer holds. In particular, the difference between the λ-invariants of the Mazur-Tate elements in this case is no longer bounded, but grows in a specific way as shown below.

Corollary 7.3.4. *If $k' = 2$ and $k \equiv 2 \bmod (p^2-1)$ then for $n \gg 0$ we have*

$$\mu\big(\theta^{\psi}_{n,j}(f)\big) - \mu\big(\theta^{\psi'}_{n,j'}(g)\big) = \mu\big(L^{-\varepsilon_n}_{p,\psi\omega^j}(f)\big) - \mu\big(L^{-\varepsilon_n}_{p,\psi'\omega^{j'}}(g)\big) + \iota_k + \operatorname{ord}_p\tbinom{k-2}{j}$$

$$\lambda\big(\theta^{\psi}_{n,j}(f)\big) - \lambda\big(\theta^{\psi'}_{n,j'}(g)\big) = p^n - p^{n-1} + \begin{cases} \lambda\big(L^{-}_{p,\psi\omega^j}(f)\big) - \lambda\big(L^{-}_{p,\psi'\omega^{j'}}(g)\big) - \frac{k-2}{p+1} & n \text{ even,} \\ \lambda\big(L^{+}_{p,\psi\omega^j}(f)\big) - \lambda\big(L^{+}_{p,\psi'\omega^{j'}}(g)\big) - \frac{p(k-2)}{p+1} & n \text{ odd.} \end{cases}$$

Proof. The relation between μ-invariants follows from Theorem 7.2.1. Letting $\lambda^{\pm}(f)$ and $\lambda^{\pm}(g)$ be the λ-invariants of $L^{\pm}_{p,\psi\omega^j}(f)$ and $L^{\pm}_{p,\psi'\omega^{j'}}(g)$, the same theorem implies

$$\begin{aligned} \lambda\big(\theta^{\psi}_{n,j}(f)\big) - \lambda\big(\theta^{\psi'}_{n,j'}(g)\big) &= (k-2)q_n - \iota_k(p^n - p^{n-1}) + \lambda^{-\varepsilon_n}(f) - \lambda^{-\varepsilon_n}(g) \\ &= p^n - p^{n-1} + (k-2)\left(q_n - \frac{p^{n+1}-p^n}{p^2-1}\right) + \lambda^{-\varepsilon_n}(f) - \lambda^{-\varepsilon_n}(g), \end{aligned}$$

where the second equality follows from the fact that $\iota_k = p(k-2)/(p^2-1) - 1$ (since $k \equiv 2 \bmod (p^2-1)$). Using (7.6), this expression simplifies as claimed. □

We encourage the reader to browse through the tables in Appendix B to see examples of the previous two corollaries.

7.3.1.1 A conjecture on λ-invariants for pairs of p-congruent modular forms

Let $L^{\pm}_p(f)$ (resp., $L^{\pm}_p(g)$) and $\theta_n(f)$ (resp., $\theta_n(g)$) denote the plus/minus p-adic L-functions and Mazur-Tate elements at $\psi = \psi' = 1$ and $j = j' = 0$ for the newforms f (resp., g) as in the previous section. Suppose that $k \equiv k' \bmod (p^2 - 1)$. By the corollaries above, knowing precisely when $\lambda(\theta_n(f)) - \lambda(\theta_n(g))$ vanishes (or equals $p^n - p^{n-1}$ if $k' = 2$) would force a relation between the λ-invariants of the associated plus/minus p-adic L-functions. Computations suggest that this difference vanishes often, and in particular, it appears to be zero when f and g are congruent modulo p.

k	N	$\lambda(L_p^+)$	$\lambda(L_p^-)$	0	1	2	3	4	5	6	label
2	32	0	0	0	0	2	6	20	60	182	G0N32k2A
10	32	6	2	0	2	8	24	74	222	668	G0N32k10A
18	32	12	4	0	2	8	24	74	222	668	G0N32k18A
3	7	1	0	0	1	4	13	40	121	364	G1N7k3A
11	7	7	2	0	1	4	13	40	121	364	G1N7k11B
19	7	13	4	0	1	4	13	40	121	364	G1N7k19B
4	32	2	0	0	2	6	20	60	182	546	G0N32k4A
12	32	8	2	0	2	6	20	60	182	546	G0N32k12A
20	32	14	4	0	2	6	20	60	182	546	G0N32k20A
5	4	2?	0?	0	2	8	26	80	242	728	G1N4k5A
13	4	8?	2?	0	2	8	26	80	242	728	G1N4k13A
21	4	14?	4?	0	2	8	26	80	242	728	G1N4k21A

Table 7.4: Invariants $\lambda(\theta_n(f))$, $0 \leq n \leq 6$, attached to rational newforms f of level $\Gamma_0(N)$ with $a_3 = 0$. Forms in the same block are congruent modulo 3.

Conjecture 7.3.5. *Let $f \in S_k(N, \varepsilon)$, $g \in S_{k'}(N, \varepsilon)$, $p \nmid N$, be newforms with $a_p(f) = a_p(g) = 0$ and $k \equiv k' \bmod (p^2 - 1)$. If f and g are congruent mod p then*

$$\lambda(L_p^-(f)) - \lambda(L_p^-(g)) = \frac{k - k'}{p+1},$$
$$\lambda(L_p^+(f)) - \lambda(L_p^+(g)) = \frac{p(k - k')}{p+1}.$$

7.3.2 Weights 2 and $p+1$

7.3.2.1 Sprung's chromatic p-adic L-functions

For newforms $f \in S_k(N, \varepsilon)$ with $a_p = 0$, recall from Chapter IV that we have a decomposition of the p-adic L-function into bounded and unbounded parts:

$$L_p(f, \alpha, \psi) = L_{p,\psi}^+(f) \log_k^+ + \alpha L_{p,\psi}^-(f) \log_k^-,$$

where $\log_k^{\pm} \in \mathbf{Q}_p[[T]]$ is an explicit infinite product of cyclotomic polynomials and α is a root $X^2 + \varepsilon(p)p^{k-1}$. For p-nonordinary forms f of weight $k = 2$, it turns out an analogous decomposition exists even when a_p does not vanish.

Theorem 7.3.6 (Sprung)**.** *If $k = 2$ and f is nonordinary at p then there are functions $\log_\alpha^{\sharp/\flat} \in K(\alpha)[[T]]$ which converge on the open unit disc in $\mathbf{C}_p$ and functions $L_{p,\psi}^{\sharp/\flat}(f,\psi) \in \Lambda$ such that*

$$L_p(f,\alpha,\psi) = L_{p,\psi}^{\sharp}(f)\log_\alpha^{\sharp} + L_{p,\psi}^{\flat}(f)\log_\alpha^{\flat}\,.$$

Proof. See [38, Theorem 2.14]. □

Remark 7.3.7. Using p-adic Hodge theory and the theory of Wach modules (see [3]), Lei, Loeffler, and Zerbes [21] obtain a very general decomposition of the p-adic L-function for nonordinary modular forms of higher weights.

We now have two sets of Iwasawa invariants associated to p-nonordinary modular forms of weight two: those coming from the bounded parts of the p-adic L-function and those determined by the sequence of Mazur-Tate elements. Under mild (i.e., conjecturally true) hypotheses, these invariants turn out to be the same.

Proposition 7.3.8. *Using the notation of Theorem 7.1.4, if $\mu(L_{p,\psi}^{\sharp}(f)) = \mu(L_{p,\psi}^{\flat}(f))$ then*

$$\mu(L_{p,\psi}^{\sharp}(f)) = \mu^+(f,\psi) \qquad \lambda(L_{p,\psi}^{\sharp}(f)) = \lambda^+(f,\psi)$$
$$\mu(L_{p,\psi}^{\flat}(f)) = \mu^-(f,\psi) \qquad \lambda(L_{p,\psi}^{\flat}(f)) = \lambda^-(f,\psi).$$

Proof. See [38, Corollary 8.9] (and Remark 7.1.6 above). □

7.3.2.2 Plus/minus λ-invariants of Serre weight two modular forms

We now assume that f has weight $p+1$, level $\Gamma_0(N)$, and that $a_p(f) = 0$. By [9] (see also Remark 5.2.5 in [35]), we know that $k(\bar{\rho}_f) = 2$ and $\bar{\rho}_f|_{G_{\mathbf{Q}_p}}$ is irreducible. Let g be the weight two modular form that is congruent to f modulo p, as specified in Theorem 7.1.12. Assuming that either μ-invariants of f or g vanish, Theorem

7.1.12 gives a description of the λ-invariants for $\theta_n^{\psi}(f) = \theta_{n,j=0}^{\psi}(f)$ in terms of g: we have

$$\lambda(\theta_n^{\psi}(f)) = \begin{cases} pq_n + \lambda(L_{p,\psi}^{\sharp}(g)) & n \text{ even}, \\ pq_n + p - 1 + \lambda(L_{p,\psi}^{\flat}(g)) & n \text{ odd}. \end{cases}$$

But we can also use Theorem 7.2.1 to compute that $\lambda(\theta_n^{\psi}(f)) = pq_n + \lambda(L_{p,\psi}^{-\varepsilon_n}(f))$. Comparing these two equations immediately forces the following relation between λ-invariants for the p-adic L-functions of f and g.

Corollary 7.3.9. *If either* $\mu\big(L_{p,\psi}^{\pm}(f)\big) = 0$ *or* $\mu(L_{p,\psi}^{\sharp/\flat}(g)) = 0$ *then*

$$\lambda(L_{p,\psi}^{-}(f)) = \lambda(L_{p,\psi}^{\sharp}(g)),$$
$$\lambda(L_{p,\psi}^{+}(f)) = p - 1 + \lambda(L_{p,\psi}^{\flat}(g)).$$

The following table illustrates this behavior at $p = 3$.

k	N	$\lambda(L_p^+)$	$\lambda(L_p^-)$	0	1	2	3	4	5	6	label
2	32	0	0	0	0	2	6	20	60	182	G0N32k2A
2	154	2	2	0	2	4	8	22	62	184	G0N154k2C
2	256	1	7	0	1	0	7	27	61	189	G0N256k2A
4	32	2+0	0	0	2	6	20	60	182	546	G0N32k4A
4	154	2+2	2	0	2	8	22	62	184	548	G0N154k4D
4	256	2+7	1	0	1	7	9	61	189	547	G0N256k4B

Table 7.5: Invariants $\lambda(\theta_n(f))$, $0 \leq n \leq 6$, attached to rational newforms f of level $\Gamma_0(N)$ with $a_3 = 0$. Forms of the same level are congruent mod 3.

We now briefly outline how the corollary above combines with Conjecture 7.3.5, allowing one to compute the plus/minus invariants of Serre weight two forms at weights higher than $p + 1$ (in particular, those outside the range $2 < k < p^2 + 1$ of Pollack-Weston). Consider the following setup, assuming the above conjecture. Let f and g be a pair as in this section, thus f has weight $k_f = p + 1$, $a_p(f) = 0$, and $\bar{\rho}_f \cong \bar{\rho}_g$ at p. Suppose there exists another form h with $a_p(h) = 0$ of weight

$k_h \equiv k_f \equiv p+1 \bmod (p^2-1)$ which is also congruent to f (and therefore g) mod p. Corollary 7.3.9 relates the plus/minus invariants of g and f, and Conjecture 7.3.5 relates those of f and and h. Combining these, we find that:

$$\lambda(L_p^-(h)) \stackrel{?}{=} \frac{k_h}{p+1} - 1 + \lambda(L_p^\sharp(g)), \tag{7.12}$$

$$\lambda(L_p^+(h)) \stackrel{?}{=} \frac{pk_h}{p+1} - 1 + \lambda(L_p^\flat(g)). \tag{7.13}$$

Example 7.3.10. Let g, f, and h be the newforms of level $\Gamma_0(32)$ and weights 2, 4, and 12, respectively, given in Table 7.4 above. The Fourier coefficient at $p = 3$ vanishes for all three of these forms, and one can check that $\bar{\rho}_g \cong \bar{\rho}_f \cong \bar{\rho}_h$. Using the plus/minus invariants computed in Table 7.4, we see that (7.12) and (7.13) are true for this triplet of newforms.

A P P E N D I X A

COMPUTING MAZUR-TATE ELEMENTS

In this appendix, we briefly summarize our method for computing the sequence of Mazur-Tate elements for a fixed newform $f \in S_k(N, \varepsilon)$. The relevant materials for this section are mainly [39], [40], and [28].

A.0.1 Homological modular symbols

Let $\mathcal{S}_k = \mathcal{S}_k(N, \varepsilon)$ be the finite-dimensional complex vector of cuspidal (homological) modular symbols (see [39, Chapter 2.2] for the precise definition). This space is generated by the formal symbols $X^j Y^{k-2-j}\{s, r\}$, where $0 \leq j \leq k-2$ and $s, r \in \mathbf{P}^1(\mathbf{Q})$, modulo various relations. There is a left action of $\mathrm{GL}_2(\mathbf{Q})$ on these formal symbols, given by

$$\gamma.F(X,Y)\{s,r\} = F(dX - bY, -cX + aY)\{\gamma(s), \gamma(r)\},$$

where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $F(X,Y) \in V_{k-2}(\mathbf{C})$. An element $\gamma \in \Gamma_0(N)$ with top left entry a acts on $\mathcal{S}_k$ as multiplication by $\varepsilon(\gamma) := \varepsilon(a)$. As in Chapter III, the matrix $\iota = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ defines an involution on $\mathcal{S}_k$ and we have a decomposition $\mathcal{S}_k = \mathcal{S}_k^+ \oplus \mathcal{S}_k^-$ into $\pm$-eigenspaces.

Consider the map

$$\Phi_f : \mathcal{S}_k \to \mathbf{C}, \quad F\{s,r\} \mapsto 2\pi i \int_s^r f(z)F(z,1)dz,$$

and let $\Phi_f^{\pm} = \Phi_f \circ (\frac{1 \pm \iota}{2})$ denote the composition of Φ_f with projection onto $\mathcal{S}_k^{\pm}$. The modular symbols $\xi_f^{\pm}$ associated to f in Chapter III can be recovered from Φ_f, in the sense that

$$(\xi_f^{\pm} \mid \gamma)(\{r\} - \{s\}) = \sum_{j=0}^{k-2} \binom{k-2}{j} \Phi_f^{\pm}(\gamma . X^j Y^{k-2-j}\{s, r\}) X^j Y^{k-2-j}, \qquad \text{(A.1)}$$

for $\gamma \in M_2(\mathbf{Z})$. (Use the fact that $\Phi_f(\gamma . X^j Y^{k-2-j}\{s, r\}) = 2\pi i \int_s^r (f|_k \gamma)(z) z^j dz$.)

A.0.2 Period polynomials

Let $\mathbf{C}[x]_d$ denote the space of complex polynomials having degree $\leq d$, which we endow with a right action of $\mathrm{GL}_2(\mathbf{Z})$ by setting $F(x)|\gamma = F(\gamma(x))(cx + d)^d$. Define a right action of $\mathrm{SL}_2(\mathbf{Z})$ on $P \in \mathrm{Hom}(\Gamma_1(N)\backslash \mathrm{SL}_2(\mathbf{Z}), \mathbf{C}[x]_{k-2})$ by

$$P|\gamma(A) = P(A\gamma^{-1}) \mid \gamma.$$

The space of *period polynomials* for $\Gamma_1(N)$ is a certain quotient of $\mathrm{Hom}(\Gamma_1(N)\backslash \mathrm{SL}_2(\mathbf{Z}), \mathbf{C}[x]_{k-2})$ by explicit relations coming from this action (see [28] for the precise definition and other interpretations). The period polynomials associated to f consist of elements in the image of the map $P_f : \Gamma_1(N)\backslash \mathrm{SL}_2(\mathbf{Z}) \to \mathbf{C}[x]_{k-2}$ defined by

$$P_f(A) = 2\pi i \int_0^{i\infty} (f \mid A(z))(z - x)^{k-2} dz,$$

where the coset $A = \Gamma_1(N)\gamma$ acts by $(f|A)(z) = (f|\gamma)(z) = (cz + d)^{-k} f(\gamma z)$, which makes sense since f is $\Gamma_1(N)$-invariant.

Fix a set of representatives $\{\gamma_i\}_{1 \leq i \leq d}$ for $\Gamma_0(N)\backslash \mathrm{SL}_2(\mathbf{Z})$. If $\alpha \in \mathrm{SL}_2(\mathbf{Z})$ we can write $\alpha = \beta\gamma_i$ for some $\beta \in \Gamma_0(N)$ and we find that

$$P_f(\Gamma_1(N)\alpha) = \varepsilon(\beta) P_f(\Gamma_1(N)\gamma_j),$$

since $f|\beta = \varepsilon(\beta) f$. Thus P_f is determined by its values on a set of representatives for $\Gamma_0(N)\backslash \mathrm{SL}_2(\mathbf{Z})$.

Note that the coefficients in the sum $P_f(\Gamma_1(N)\gamma_i) = \sum_{j=0}^{k-2} c_j(\gamma_i)x^j$, are given by

$$c_j(\gamma_i) = (-1)^j \binom{k-2}{j} 2\pi i \int_0^{i\infty} (f \mid \gamma_i)(z) z^{k-2-j} dz.$$

These values can be expressed in terms of Φ_f: one checks that

$$c_{k-2-j}(\gamma_i) = (-1)^{k-j} \binom{k-2}{j} \Phi_f(\gamma_i . X^j Y^{k-2-j}\{0,\infty\}). \tag{A.2}$$

The matrix ι also defines an involution on the space of period polynomials (see [28]), so we have a unique decomposition $P_f = P_f^+ + P_f^-$.

Theorem A.0.1. *There are nonzero complex numbers $\Omega_f^\pm \in \mathbf{C}$ such that all values of $P_f^\pm/\Omega_f^\pm$ have coefficients lying in K_f.*

Proof. See [28, Prop. 5.11]. □

Corollary A.0.2. *The modular symbol $\xi_f^\pm/\Omega_f^\pm$ is K_f-valued.*

Proof. Note that Δ_0 is finitely generated as a $\mathbf{Z}[\Gamma_0(N)]$-module by the divisors $\{\gamma_i(\infty)\} - \{\gamma_i(0)\}$ for $1 \le i \le d$ (this is essentially *Manin's trick* – see [32]). Thus $\xi_f^\pm$ is uniquely determined by its values on these divisors and we have

$$\begin{aligned} X^{k-2} P_f^\pm(\Gamma_1(N)\gamma_i)(-Y/X) &= \xi_{f|\gamma_i}^\pm(\{\infty\} - \{0\})_{(X,Y)} \\ &= \xi_f^\pm\big(\{\gamma_i(\infty)\} - \{\gamma_i(0)\}\big)_{(X,Y)\gamma_i^{-1}}. \end{aligned}$$

Letting $\Omega_f^\pm$ be the periods in the previous theorem, we know that all coefficients of $\xi_f^\pm/\Omega_f^\pm\big(\{\gamma_i(\infty)\} - \{\gamma_i(0)\}\big)_{(X,Y)\gamma_i^{-1}}$ lie in K_f. This proves the result, since if $\gamma \in \mathrm{SL}_2(\mathbf{Z})$ and $G(X,Y) = F((X,Y)\gamma^{-1}) \in K_f[X,Y]$ then $F(X,Y) = G((X,Y)\gamma)$ also has coefficients in $K_f[X,Y]$. □

Pasol-Popa [28] have implemented an algorithm to numerically compute the coefficients, call them $c_j^\pm(\gamma_i) \in K_f$, of $P_f^\pm/\Omega_f^\pm(\Gamma_1(N)\gamma_i)$. We can use them to

compute Mazur-Tate elements as follows. By (A.2) we have

$$c^{\pm}_{k-2-j}(\gamma_i) = C_f(-1)^{k-j}\binom{k-2}{j}\Phi^{\pm}_f(\gamma_i.X^jY^{k-2-j}\{0,\infty\}). \tag{A.3}$$

where $C_f \in K_f$ is some nonzero constant independent of i and j. Recall that the set of Manin symbols $[X^jY^{k-2-j},\gamma_i] := \gamma_i.X^jY^{k-2-j}\{0,\infty\}$ generate $\mathcal{S}_k$ over $\mathbf{Z}[\varepsilon]$ (see [40, Proposition 8.2]). Fix coprime integers a and m with $1 \le a < m$ and let $\gamma = \left(\begin{smallmatrix} 1 & a \\ 0 & m \end{smallmatrix}\right)$. Then for $0 \le j_0 \le k-2$ we have

$$\begin{aligned}
\binom{k-2}{j_0}^{-1}\left[\frac{a}{m}\right]^{\pm}_{f,j_0} &= \left(\text{coeff. of } X^{j_0}Y^{k-2-j_0} \text{ in } (\varphi^{\pm}_f \mid \gamma)(\{\infty\}-\{0\})\right)\cdot\binom{k-2}{j_0}^{-1} \\
&= \Phi^{\pm}_f(\gamma.X^{j_0}Y^{k-2-j_0}\{0,\infty\}) \\
&= \sum_{i=1}^{d}\sum_{j=0}^{k-2} n_{ij}\Phi^{\pm}_f(\gamma_i.X^jY^{k-2-j}\{0,\infty\}) \quad (n_{ij}\in\mathbf{Z}[\varepsilon] \text{ depend on } j_0) \\
&= C_f\sum_{i=1}^{d}\sum_{j=0}^{k-2} n_{ij}(-1)^{k-j}\binom{k-2}{j}^{-1} c^{\pm}_{k-2-j}(\gamma_i),
\end{aligned}$$

where the second equality is from (A.1), the third comes from writing $\gamma.X^{j_0}Y^{k-2-j_0}\{0,\infty\}$ as a sum of Manin symbols, and the fourth is from (A.3).

Remark A.0.3. To check that the above method works, we can verify (in Magma) that the Mazur-Tate elements obtained satisfy the 3-term relation discussed in Chapter III, which is written in polynomial form as follows:

$$\theta^{\psi}_{n+1,j}(f,T) \equiv a_p\theta^{\psi}_{n,j}(f,T) - \varepsilon(p)p^{k-2}\Phi_n(1+T)\theta^{\psi}_{n-1,j}(f,T) \bmod \omega_n.$$

We have also checked that the Iwasawa invariants computed from these Mazur-Tate elements agree with examples and results of Pollack-Weston [35].

Remark A.0.4. To compute the correct μ-invariants, the Mazur-Tate elements obtained from the above procedure must be scaled by appropriate cohomological periods for f as outlined in Chapter III. We have not yet implemented such a procedure.

APPENDIX B

TABLES OF λ-INVARIANTS

The following tables contain the invariants $\lambda(\theta_n) = \lambda(\theta_{n,j=0}^{\psi=1}(f))$ and $\lambda^{\pm} = \lambda(L_p^{\pm}(f))$ at the primes $p = 3, 5, 7$ for all newforms $f \in S_k(N, \varepsilon)$, $p \nmid N$, with $a_p(f) = 0$, degree $d = [K_f : \mathbf{Q}]$, and weight $2 \leq k \leq 14$ under the following constraints:

(a) If k is even, $d = 1$, and $\varepsilon = 1$, we include all newforms of level $N \leq 300$.

(b) If either (i) k is even and $2 \leq d \leq 6$ or (ii) k is odd and $1 \leq d \leq 6$, then we include all newforms of level $N \leq 40$.

We include $\lambda(\theta_n)$ for $n \leq 6$ at $p = 3$ and $n \leq 5$ otherwise. The invariants $\lambda^{\pm}$ at weights $k \not\equiv p+2 \bmod (p^2-1)$ are then determined by Theorem VII.7.2.1. In order to apply Theorem VII.7.2.1 at weights $k \equiv p + 2 \bmod (p^2 - 1)$, we would need to know *a priori* whether $\lambda^{\pm}$ satisfy certain bounds. We have not been able to verify these bounds independently, thus at these weights we give the plus/minus invariants suggested by the sequence $\lambda(\theta_n)$, assuming it follows one of the behaviors indicated in Proposition VII.7.2.15. These invariants are followed by a question mark to indicate that they are conjectural.

The column with the heading '$d, \mathfrak{p}$' contains the pair of integers d and n_p, respectively, where n_p refers to the prime ideal used in our computations and is

defined as follows. If $d = 1$, we define $n_p = 1$. If $d > 1$, let n_p be the index of a fixed prime $\mathfrak{p}$ of K_f as it occurs in the decomposition $p\mathcal{O}_{K_f} = \prod_{i=1}^{r} \mathfrak{p}_i^{e_i}$, ordered according to the Magma function `Decomposition(O,p)`, where `O` is the ring of integers $\mathcal{O}_{K_f}$. If $n_p = 1$ we omit it from the table.

The column titled 'label' contains the Magma label for the modular form, which can be recovered using the command `Newform("label")`.

We note that rational newforms f of weight $k = 2$ (therefore of type (a) above) correspond to isogeny classes of rational elliptic curves whose plus/minus invariants are currently available on LMFDB. Thus, many of the weight two invariants in these tables are known. For all higher weight forms (and those of weight two with $d > 1$), the plus/minus invariants are not currently available on LMFDB.

These computations were carried out on the computer algebra system Magma. The invariants attached to all modular forms of type (a) were computed using code written by Pollack and Weston, which relies on the implementation of modular symbols in Magma due to William Stein. The invariants attached to modular forms of type (b) were computed using code written by the author based on the material in Appendix A. This code relies heavily on a package of Pasol-Popa [28] for computing the period polynomials of modular forms. The programs for type (a) modular forms run significantly faster, thus we are able to include more data in this case.

B.1 $p = 3$

Table B.1: λ-invariants at $p = 3$.

k	N	$d, \mathfrak{p}$	λ^+	λ^-	0	1	2	3	4	5	6	label
2	17	1	0	0	0	0	2	6	20	60	182	G0N17k2A
2	20	2	0	0	0	0	2	6	20	60	182	G1N20k2B
2	26	2	0	0	0	0	2	6	20	60	182	G1N26k2C
2	28	2	0	0	0	0	2	6	20	60	182	G1N28k2C
2	32	1	0	0	0	0	2	6	20	60	182	G0N32k2A
2	34	2	0	0	0	0	2	6	20	60	182	G1N34k2C
2	37	2	1	5	0	1	7	7	25	61	187	G1N37k2G
2	40	1	0	0	0	0	2	6	20	60	182	G0N40k2A
2	46	1	0	0	0	0	2	6	20	60	182	G0N46k2A
2	49	1	0	0	0	0	2	6	20	60	182	G0N49k2A
2	52	1	0	0	0	0	2	6	20	60	182	G0N52k2A
2	55	1	0	0	0	0	2	6	20	60	182	G0N55k2A
2	56	1	0	0	0	0	2	6	20	60	182	G0N56k2B
2	62	1	0	0	0	0	2	6	20	60	182	G0N62k2A
2	64	1	0	0	0	0	2	6	20	60	182	G0N64k2A
2	70	1	0	0	0	0	2	6	20	60	182	G0N70k2A
2	73	1	0	0	0	0	2	6	20	60	182	G0N73k2A
2	80	1	0	0	0	0	2	6	20	60	182	G0N80k2A
2	91	1	1	7	0	1	0	7	27	61	189	G0N91k2A
2	94	1	0	0	0	0	2	6	20	60	182	G0N94k2A
2	109	1	0	0	0	0	2	6	20	60	182	G0N109k2A
2	112	1	0	0	0	0	2	6	20	60	182	G0N112k2B
2	115	1	0	0	0	0	2	6	20	60	182	G0N115k2A
2	124	1	0	0	0	0	2	6	20	60	182	G0N124k2A
2	130	1	0	0	0	0	2	6	20	60	182	G0N130k2C
2	142	1	0	0	0	0	2	6	20	60	182	G0N142k2B
2	145	1	1	7	0	1	0	7	27	61	189	G0N145k2A
2	154	1	1	1	0	1	3	7	21	61	183	G0N154k2A
2	154	1	2	2	0	2	4	8	22	62	184	G0N154k2C
2	161	1	0	0	0	0	2	6	20	60	182	G0N161k2A
2	163	1	5	1	0	0	3	11	21	65	183	G0N163k2A
2	179	1	0	0	0	0	2	6	20	60	182	G0N179k2A
2	182	1	0	0	0	0	2	6	20	60	182	G0N182k2C
2	184	1	0	0	0	0	2	6	20	60	182	G0N184k2B
2	187	1	0	0	0	0	2	6	20	60	182	G0N187k2A
2	194	1	0	0	0	0	2	6	20	60	182	G0N194k2A

Table B.2: λ-invariants at $p = 3$.

k	N	$d, \mathfrak{p}$	λ^+	λ^-	0	1	2	3	4	5	6	label
2	197	1	1	1	0	1	3	7	21	61	183	G0N197k2A
2	200	1	0	0	0	0	2	6	20	60	182	G0N200k2C
2	202	1	0	0	0	0	2	6	20	60	182	G0N202k2A
2	205	1	1	1	0	1	3	7	21	61	183	G0N205k2C
2	208	1	0	0	0	0	2	6	20	60	182	G0N208k2B
2	215	1	1	1	0	1	3	7	21	61	183	G0N215k2A
2	221	1	0	0	0	0	2	6	20	60	182	G0N221k2B
2	238	1	5	1	0	0	3	11	21	65	183	G0N238k2A
2	238	1	0	0	0	0	2	6	20	60	182	G0N238k2E
2	244	1	1	1	0	1	3	7	21	61	183	G0N244k2A
2	248	1	1	5	0	1	7	7	25	61	187	G0N248k2C
2	254	1	1	3	0	1	5	7	23	61	185	G0N254k2A
2	254	1	2	4	0	2	6	8	24	62	186	G0N254k2B
2	256	1	1	7	0	1	0	7	27	61	189	G0N256k2A
2	256	1	0	0	0	0	2	6	20	60	182	G0N256k2C
2	259	1	2	6	0	2	8	8	26	62	188	G0N259k2A
2	262	1	1	1	0	1	3	7	21	61	183	G0N262k2A
2	265	1	1	1	0	1	3	7	21	61	183	G0N265k2A
2	269	1	1	1	0	1	3	7	21	61	183	G0N269k2A
2	272	1	1	1	0	1	3	7	21	61	183	G0N272k2D
2	274	1	1	1	0	1	3	7	21	61	183	G0N274k2A
2	274	1	1	1	0	1	3	7	21	61	183	G0N274k2B
2	275	1	1	1	0	1	3	7	21	61	183	G0N275k2A
2	289	1	1	1	0	1	3	7	21	61	183	G0N289k2A
2	290	1	5	1	0	1	3	11	21	65	183	G0N290k2A
2	298	1	3	1	0	0	3	9	21	63	183	G0N298k2A
3	7	1	1	0	0	1	4	13	40	121	364	G1N7k3A
3	16	1	1	0	0	1	4	13	40	121	364	G1N16k3A
3	19	1	1	0	0	1	4	13	40	121	364	G1N19k3C
3	20	2	1	0	0	1	4	13	40	121	364	G1N20k3E
3	26	2	1	0	0	1	4	13	40	121	364	G1N26k3C
3	31	3	1	0	0	1	4	13	40	121	364	G1N31k3F
3	40	1	1	0	0	1	4	13	40	121	364	G1N40k3E
3	40	1	1	0	0	1	4	13	40	121	364	G1N40k3F

Table B.3: λ-invariants at $p = 3$.

k	N	$d, \mathfrak{p}$	λ^+	λ^-	0	1	2	3	4	5	6	label
4	20	2	2	0	0	2	6	20	60	182	546	G1N20k4B
4	28	2	2	0	0	2	6	20	60	182	546	G1N28k4E
4	32	1	2	0	0	2	6	20	60	182	546	G0N32k4A
4	49	1	2	0	0	2	6	20	60	182	546	G0N49k4D
4	64	1	2	0	0	2	6	20	60	182	546	G0N64k4A
4	154	1	4	2	0	2	8	22	62	184	548	G0N154k4D
4	254	1	5	1	0	1	7	23	61	185	547	G0N254k4A
4	256	1	2	0	0	2	6	20	60	182	546	G0N256k4A
4	256	1	9	1	0	1	7	9	61	189	547	G0N256k4B
4	290	1	3	5	0	1	5	21	65	183	551	G0N290k4A
5	4	1	2?	0?	0	2	8	26	80	242	728	G1N4k5A
5	7	1	2?	0?	0	2	8	26	80	242	728	G1N7k5B
5	19	1	2?	0?	0	2	8	26	80	242	728	G1N19k5C
5	31	3	2?	0?	0	2	8	26	80	242	728	G1N31k5D
5	40	1	4?	2?	0	1	4	10	28	82	244	G1N40k5F
5	40	1	2?	0?	0	2	8	26	80	242	728	G1N40k5G
6	20	2	6	1	0	1	5	18	47	144	425	G1N20k6B
6	26	1	5	1	0	1	5	17	47	143	425	G0N26k6A
6	28	2	4	1	0	2	5	16	47	142	425	G1N28k6F
6	32	1	7	1	0	1	5	19	47	145	425	G0N32k6A
6	49	1	4	2	0	2	6	16	48	142	426	G0N49k6B
6	64	1	4	4	0	2	8	16	50	142	428	G0N64k6A
6	208	1	4	8	0	2	6	16	54	142	432	G0N208k6A
6	256	1	9	1	0	1	5	21	47	147	425	G0N256k6A
6	256	1	4	2	0	2	6	16	48	142	426	G0N256k6B
7	7	1	4	1	0	2	7	22	67	202	607	G1N7k7C
7	16	1	4	1	0	2	7	22	67	202	607	G1N16k7A
7	19	1	4	1	0	2	7	22	67	202	607	G1N19k7C
7	20	2	4	1	0	2	7	22	67	202	607	G1N20k7E
7	31	1	4	1	0	2	7	22	67	202	607	G1N31k7D
7	31	2	4	1	0	2	7	22	67	202	607	G1N31k7E
7	40	1	4	1	0	2	7	22	67	202	607	G1N40k7E
7	40	1	4	1	0	2	7	22	67	202	607	G1N40k7F
8	20	2	5	4	0	1	6	11	36	101	306	G1N20k8C
8	28	2	5	2	0	1	4	11	34	101	304	G1N28k8E
8	32	1	5	5	0	1	7	11	37	101	307	G0N32k8A
8	49	1	6	2	0	2	4	12	34	102	304	G0N49k8B
8	64	1	8	2	0	2	4	14	34	104	304	G0N64k8A
8	256	1	5	7	0	1	3	11	39	101	309	G0N256k8A
8	256	1	6	2	0	2	4	12	34	102	304	G0N256k8B

Table B.4: λ-invariants at $p = 3$.

k	N	$d, \mathfrak{p}$	λ^+	λ^-	0	1	2	3	4	5	6	label
9	4	1	5	1	0	1	5	17	53	161	485	G1N4k9A
9	7	1	5	1	0	1	5	17	53	161	485	G1N7k9B
9	19	1	5	1	0	1	5	17	53	161	485	G1N19k9C
9	31	3	6	2	0	2	6	18	54	162	486	G1N31k9D
9	40	1	5	1	0	1	5	17	53	161	485	G1N40k9E
9	40	1	7	3	0	1	7	19	55	163	487	G1N40k9F
10	20	2	6	2	0	2	8	24	74	222	668	G1N20k10C
10	28	2	6	2	0	2	8	24	74	222	668	G1N28k10E
10	32	1	6	2	0	2	8	24	74	222	668	G0N32k10A
10	49	1	6	2	0	2	8	24	74	222	668	G0N49k10A
10	64	1	6	2	0	2	8	24	74	222	668	G0N64k10A
10	256	1	7	9	0	1	3	25	27	223	675	G0N256k10A
10	256	1	6	2	0	2	8	24	74	222	668	G0N256k10B
11	7	1	7	2	0	1	4	13	40	121	364	G1N7k11B
11	16	1	7	2	0	1	4	13	40	121	364	G1N16k11A
11	19	1	7	2	0	1	4	13	40	121	364	G1N19k11C
11	20	2	7	2	0	1	4	13	40	121	364	G1N20k11E
11	31	3	7	2	0	1	4	13	40	121	364	G1N31k11D
11	40	1	7	2	0	1	4	13	40	121	364	G1N40k11E
11	40	1	7	2	0	1	4	13	40	121	364	G1N40k11F
12	20	2	8	2	0	2	6	20	60	182	546	G1N20k12C
12	28	2	8	2	0	2	6	20	60	182	546	G1N28k12E
12	32	1	8	2	0	2	6	20	60	182	546	G0N32k12A
12	49	1	8	2	0	2	6	20	60	182	546	G0N49k12B
12	64	1	8	2	0	2	6	20	60	182	546	G0N64k12A
12	256	1	8	2	0	2	6	20	60	182	546	G0N256k12A
12	256	1	15	3	0	1	7	9	61	189	547	G0N256k12B
13	4	1	8?	2?	0	2	8	26	80	242	728	G1N4k13A
13	7	1	8?	2?	0	2	8	26	80	242	728	G1N7k13B
13	19	1	8?	2?	0	2	8	26	80	242	728	G1N19k13C
13	31	1	9?	3?	0	1	3	9	27	81	243	G1N31k13D
13	31	2	9?	3?	0	1	3	9	27	81	243	G1N31k13E
13	40	1	10?	4?	0	2	4	10	28	82	244	G1N40k13E
13	40	1	8?	2?	0	2	8	26	80	242	728	G1N40k13F
14	20	2	12	3	0	1	5	18	47	144	425	G1N20k14C
14	28	2	10	3	0	2	5	16	47	142	425	G1N28k14E
14	32	1	13	3	0	1	5	19	47	145	425	G0N32k14A
14	49	1	10	4	0	2	6	16	48	142	426	G0N49k14A
14	64	1	10	6	0	2	8	16	50	142	428	G0N64k14A
14	256	1	10	4	0	2	6	16	48	142	426	G0N256k14A
14	256	1	15	3	0	1	5	21	47	147	425	G0N256k14B

B.2 $p = 5$

Table B.5: λ-invariants at $p = 5$.

k	N	$d, \mathfrak{p}$	λ^+	λ^-	0	1	2	3	4	5	label
2	14	1	0	0	0	0	4	20	104	520	G0N14k2A
2	18	2	0	0	0	0	4	20	104	520	G1N18k2A
2	21	2	0	0	0	0	4	20	104	520	G1N21k2B
2	24	2	0	0	0	0	4	20	104	520	G1N24k2C
2	27	1	0	0	0	0	4	20	104	520	G0N27k2A
2	28	2	0	1	0	0	5	20	105	520	G1N28k2C
2	34	1	0	0	0	0	4	20	104	520	G0N34k2A
2	36	1	0	0	0	0	4	20	104	520	G0N36k2A
2	37	1	0	0	0	0	4	20	104	520	G0N37k2B
2	38	1	0	0	0	0	4	20	104	520	G0N38k2A
2	38	2	0	0	0	0	4	20	104	520	G1N38k2D
2	39	4	0	0	0	0	4	20	104	520	G1N39k2C
2	39	4,2	0	0	0	0	4	20	104	520	G1N39k2C
2	39	2	0	1	0	0	5	20	105	520	G1N39k2I
2	49	1	0	0	0	0	4	20	104	520	G0N49k2A
2	53	1	1	1	0	1	5	21	105	521	G0N53k2A
2	66	1	0	0	0	0	4	20	104	520	G0N66k2A
2	69	1	0	0	0	0	4	20	104	520	G0N69k2A
2	84	1	0	0	0	0	4	20	104	520	G0N84k2B
2	92	1	0	0	0	0	4	20	104	520	G0N92k2A
2	94	1	0	0	0	0	4	20	104	520	G0N94k2A
2	98	1	0	0	0	0	4	20	104	520	G0N98k2A
2	102	1	0	0	0	0	4	20	104	520	G0N102k2B
2	106	1	0	0	0	0	4	20	104	520	G0N106k2C
2	108	1	0	0	0	0	4	20	104	520	G0N108k2A
2	112	1	0	0	0	0	4	20	104	520	G0N112k2C
2	114	1	0	0	0	0	4	20	104	520	G0N114k2A
2	114	1	0	0	0	0	4	20	104	520	G0N114k2C
2	126	1	0	0	0	0	4	20	104	520	G0N126k2B
2	136	1	0	0	0	0	4	20	104	520	G0N136k2A
2	138	1	0	0	0	0	4	20	104	520	G0N138k2B
2	141	1	0	0	0	0	4	20	104	520	G0N141k2B
2	142	1	0	0	0	0	4	20	104	520	G0N142k2D
2	144	1	0	0	0	0	4	20	104	520	G0N144k2B
2	152	1	0	0	0	0	4	20	104	520	G0N152k2B
2	156	1	0	0	0	0	4	20	104	520	G0N156k2B

Table B.6: λ-invariants at $p = 5$.

k	N	$d, \mathfrak{p}$	λ^+	λ^-	0	1	2	3	4	5	label
2	162	1	0	0	0	0	4	20	104	520	G0N162k2B
2	162	1	0	0	0	0	4	20	104	520	G0N162k2C
2	172	1	1	1	0	1	5	21	105	521	G0N172k2A
2	182	1	0	0	0	0	4	20	104	520	G0N182k2B
2	182	1	0	0	0	0	4	20	104	520	G0N182k2E
2	184	1	0	0	0	0	4	20	104	520	G0N184k2B
2	184	1	0	0	0	0	4	20	104	520	G0N184k2C
2	197	1	7	1	0	0	5	27	105	527	G0N197k2A
2	198	1	0	0	0	0	4	20	104	520	G0N198k2A
2	198	1	0	0	0	0	4	20	104	520	G0N198k2D
2	198	1	0	0	0	0	4	20	104	520	G0N198k2E
2	201	1	1	1	0	1	5	21	105	521	G0N201k2B
2	207	1	1	3	0	1	7	21	107	521	G0N207k2A
2	214	1	0	0	0	0	4	20	104	520	G0N214k2C
2	222	1	0	0	0	0	4	20	104	520	G0N222k2D
2	222	1	0	0	0	0	4	20	104	520	G0N222k2E
2	224	1	1	1	0	1	5	21	105	521	G0N224k2A
2	224	1	0	0	0	0	4	20	104	520	G0N224k2B
2	238	1	0	0	0	0	4	20	104	520	G0N238k2C
2	243	1	1	1	0	1	5	21	105	521	G0N243k2A
2	243	1	0	0	0	0	4	20	104	520	G0N243k2B
2	252	1	0	0	0	0	4	20	104	520	G0N252k2B
2	254	1	0	0	0	0	4	20	104	520	G0N254k2C
2	256	1	1	1	0	1	5	21	105	521	G0N256k2B
2	256	1	0	0	0	0	4	20	104	520	G0N256k2D
2	262	1	1	1	0	1	5	21	105	521	G0N262k2A
2	264	1	0	0	0	0	4	20	104	520	G0N264k2D
2	267	1	0	0	0	0	4	20	104	520	G0N267k2B
2	272	1	1	3	0	1	7	21	107	521	G0N272k2A
2	272	1	0	0	0	0	4	20	104	520	G0N272k2C
2	274	1	5	1	0	0	5	25	105	525	G0N274k2A
2	291	1	3	1	0	3	5	23	105	523	G0N291k2A
2	296	1	1	1	0	1	5	21	105	521	G0N296k2B
3	7	1	1	0	0	1	8	41	208	1041	G1N7k3A
3	8	1	1	0	0	1	8	41	208	1041	G1N8k3A
3	12	1	1	0	0	1	8	41	208	1041	G1N12k3B
3	21	2	1	0	0	1	8	41	208	1041	G1N21k3E
3	23	3	1	0	0	1	8	41	208	1041	G1N23k3B

Table B.7: λ-invariants at $p = 5$.

k	N	$d, \mathfrak{p}$	λ^+	λ^-	0	1	2	3	4	5	label
3	27	1	1	0	0	1	8	41	208	1041	G1N27k3C
3	32	1	1	0	0	1	8	41	208	1041	G1N32k3C
3	39	2	1	0	0	1	8	41	208	1041	G1N39k3D
3	39	2	1	0	0	1	8	41	208	1041	G1N39k3G
4	9	1	2	0	0	2	12	62	312	1562	G0N9k4A
4	21	2	2	0	0	2	12	62	312	1562	G1N21k4G
4	24	2	2	0	0	2	12	62	312	1562	G1N24k4C
4	28	2	2	0	0	2	12	62	312	1562	G1N28k4E
4	39	4	2	0	0	2	12	62	312	1562	G1N39k4H
4	39	4,2	2	0	0	2	12	62	312	1562	G1N39k4H
4	49	1	2	0	0	2	12	62	312	1562	G0N49k4D
4	66	1	2	0	0	2	12	62	312	1562	G0N66k4B
4	108	1	2	0	0	2	12	62	312	1562	G0N108k4A
4	108	1	2	0	0	2	12	62	312	1562	G0N108k4B
4	132	1	2	0	0	2	12	62	312	1562	G0N132k4B
4	144	1	2	0	0	2	12	62	312	1562	G0N144k4A
4	174	1	2	0	0	2	12	62	312	1562	G0N174k4C
4	182	1	2	0	0	2	12	62	312	1562	G0N182k4D
4	198	1	2	0	0	2	12	62	312	1562	G0N198k4A
4	222	1	2	0	0	2	12	62	312	1562	G0N222k4B
4	224	1	2	0	0	2	12	62	312	1562	G0N224k4A
4	224	1	2	0	0	2	12	62	312	1562	G0N224k4B
4	243	1	2	0	0	2	12	62	312	1562	G0N243k4A
4	243	1	2	0	0	2	12	62	312	1562	G0N243k4B
4	256	1	2	0	0	2	12	62	312	1562	G0N256k4G
4	256	1	2	0	0	2	12	62	312	1562	G0N256k4H
4	273	1	2	0	0	2	12	62	312	1562	G0N273k4C
5	7	1	3	0	0	3	16	83	416	2083	G1N7k5B
5	8	1	3	0	0	3	16	83	416	2083	G1N8k5A
5	12	1	3	0	0	3	16	83	416	2083	G1N12k5B
5	21	2	3	0	0	3	16	83	416	2083	G1N21k5E
5	23	3	3	0	0	3	16	83	416	2083	G1N23k5B
5	27	1	3	0	0	3	16	83	416	2083	G1N27k5C
5	32	1	3	0	0	3	16	83	416	2083	G1N32k5D
5	39	2	3	0	0	3	16	83	416	2083	G1N39k5D
5	39	2	3	0	0	3	16	83	416	2083	G1N39k5G

Table B.8: λ-invariants at $p = 5$.

k	N	$d, \mathfrak{p}$	λ^+	λ^-	0	1	2	3	4	5	label
6	21	2	4	0	0	4	20	104	520	2604	G1N21k6E
6	24	2	4	0	0	4	20	104	520	2604	G1N24k6E
6	27	1	4	0	0	4	20	104	520	2604	G0N27k6A
6	28	2	5	0	0	1	20	105	520	2605	G1N28k6F
6	36	1	4	0	0	4	20	104	520	2604	G0N36k6A
6	39	4	4	0	0	4	20	104	520	2604	G1N39k6E
6	39	4,2	4	0	0	4	20	104	520	2604	G1N39k6E
6	49	1	4	0	0	4	20	104	520	2604	G0N49k6B
6	108	1	4	0	0	4	20	104	520	2604	G0N108k6A
6	144	1	4	0	0	4	20	104	520	2604	G0N144k6A
6	243	1	4	0	0	4	20	104	520	2604	G0N243k6A
6	243	1	5	1	0	1	21	105	521	2605	G0N243k6B
6	256	1	4	0	0	4	20	104	520	2604	G0N256k6C
6	256	1	5	1	0	1	21	105	521	2605	G0N256k6D
7	3	1	4?	0?	0	4	24	124	624	3124	G1N3k7A
7	7	1	4?	0?	0	4	24	124	624	3124	G1N7k7C
7	8	1	4?	0?	0	4	24	124	624	3124	G1N8k7A
7	23	1	5?	1?	0	1	5	25	125	625	G1N23k7B
7	23	2	4?	0?	0	4	24	124	624	3124	G1N23k7C
7	23	2,2	5?	1?	0	1	5	25	125	625	G1N23k7C
7	32	1	6?	2?	0	2	6	26	126	626	G1N32k7D
7	39	2	5?	1?	0	1	5	25	125	625	G1N39k7G
8	21	2	6	1	0	2	9	46	229	1146	G1N21k8E
8	24	2	6	1	0	2	9	46	229	1146	G1N24k8E
8	27	1	6	1	0	2	9	46	229	1146	G0N27k8A
8	28	2	6	1	0	2	9	46	229	1146	G1N28k8E
8	36	1	6	1	0	2	9	46	229	1146	G0N36k8A
8	39	4	6	1	0	2	9	46	229	1146	G1N39k8E
8	39	4,2	6	2	0	2	10	46	230	1146	G1N39k8E
8	49	1	10	1	0	1	9	50	229	1150	G0N49k8B
8	66	1	6	1	0	2	9	46	229	1146	G0N66k8A
8	108	1	6	1	0	2	9	46	229	1146	G0N108k8A
8	144	1	6	1	0	2	9	46	229	1146	G0N144k8A
8	198	1	6	1	0	2	9	46	229	1146	G0N198k8D
8	243	1	10	1	0	1	9	50	229	1150	G0N243k8A
8	243	1	6	1	0	2	9	46	229	1146	G0N243k8B
8	256	1	6	1	0	2	9	46	229	1146	G0N256k8C
8	256	1	6	1	0	2	9	46	229	1146	G0N256k8D

Table B.9: λ-invariants at $p = 5$.

k	N	$d, \mathfrak{p}$	λ^+	λ^-	0	1	2	3	4	5	label
9	7	1	7	1	0	3	13	67	333	1667	G1N7k9B
9	8	1	7	1	0	3	13	67	333	1667	G1N8k9A
9	12	1	8	1	0	4	13	68	333	1668	G1N12k9B
9	21	2	7	1	0	3	13	67	333	1667	G1N21k9D
9	23	3	7	1	0	3	13	67	333	1667	G1N23k9B
9	27	1	7	1	0	3	13	67	333	1667	G1N27k9C
9	32	1	7	1	0	3	13	67	333	1667	G1N32k9D
9	39	2	7	1	0	3	13	67	333	1667	G1N39k9D
9	39	2	7	1	0	3	13	67	333	1667	G1N39k9G
10	9	1	9	1	0	1	17	89	437	2189	G0N9k10A
10	21	2	8	1	0	4	17	88	437	2188	G1N21k10H
10	24	2	8	1	0	4	17	88	437	2188	G1N24k10F
10	28	2	8	1	0	4	17	88	437	2188	G1N28k10E
10	39	4	8	1	0	4	17	88	437	2188	G1N39k10H
10	39	4,2	8	1	0	4	17	88	437	2188	G1N39k10H
10	49	1	8	2	0	4	18	88	438	2188	G0N49k10A
10	108	1	11	1	0	3	17	91	437	2191	G0N108k10A
10	108	1	8	2	0	4	18	88	438	2188	G0N108k10B
10	144	1	8	2	0	4	18	88	438	2188	G0N144k10A
10	243	1	8	4	0	4	20	88	440	2188	G0N243k10A
10	243	1	9	1	0	1	17	89	437	2189	G0N243k10B
10	256	1	11	1	0	3	17	91	437	2191	G0N256k10C
10	256	1	8	4	0	4	20	88	440	2188	G0N256k10D
11	7	1	8	1	0	4	21	108	541	2708	G1N7k11B
11	8	1	8	1	0	4	21	108	541	2708	G1N8k11A
11	12	1	8	1	0	4	21	108	541	2708	G1N12k11B
11	21	2	8	1	0	4	21	108	541	2708	G1N21k11D
11	23	3	8	1	0	4	21	108	541	2708	G1N23k11B
11	27	1	8	1	0	4	21	108	541	2708	G1N27k11C
11	32	1	8	1	0	4	21	108	541	2708	G1N32k11D
11	39	2	8	1	0	4	21	108	541	2708	G1N39k11D
11	39	2	8	1	0	4	21	108	541	2708	G1N39k11G
12	21	2	9	2	0	1	6	29	146	729	G1N21k12F
12	24	2	9	2	0	1	6	29	146	729	G1N24k12F
12	27	1	9	2	0	1	6	29	146	729	G0N27k12A
12	28	2	9	2	0	1	6	29	146	729	G1N28k12E
12	39	4	9	2	0	1	6	29	146	729	G1N39k12E
12	39	4,2	10	2	0	2	6	30	146	730	G1N39k12E

Table B.10: λ-invariants at $p = 5$.

k	N	$d, \mathfrak{p}$	λ^+	λ^-	0	1	2	3	4	5	label
12	36	1	9	2	0	1	6	29	146	729	G0N36k12A
12	49	1	9	6	0	1	10	29	150	729	G0N49k12B
12	108	1	9	2	0	1	6	29	146	729	G0N108k12A
12	144	1	9	2	0	1	6	29	146	729	G0N144k12A
12	243	1	9	6	0	1	10	29	150	729	G0N243k12A
12	243	1	9	2	0	1	6	29	146	729	G0N243k12B
12	256	1	9	2	0	1	6	29	146	729	G0N256k12C
12	256	1	9	2	0	1	6	29	146	729	G0N256k12D
13	3	1	10	2	0	2	10	50	250	1250	G1N3k13A
13	7	1	10	2	0	2	10	50	250	1250	G1N7k13B
13	8	1	10	2	0	2	10	50	250	1250	G1N8k13A
13	23	1	10	3	0	2	11	50	251	1250	G1N23k13B
13	23	2	10	2	0	2	10	50	250	1250	G1N23k13C
13	23	2,2	11	2	0	3	10	51	250	1251	G1N23k13C
13	39	2	10	2	0	2	10	50	250	1250	G1N39k13I
14	21	2	11	2	0	3	14	71	354	1771	G1N21k14E
14	24	2	11	2	0	3	14	71	354	1771	G1N24k14F
14	27	1	11	7	0	3	19	71	359	1771	G0N27k14A
14	28	2	11	2	0	3	14	71	354	1771	G1N28k14E
14	36	1	12	2	0	4	14	72	354	1772	G0N36k14A
14	49	1	12	2	0	4	14	72	354	1772	G0N49k14A
14	108	1	11	3	0	3	15	71	355	1771	G0N108k14A
14	144	1	12	2	0	4	14	72	354	1772	G0N144k14A

B.3 $p = 7$

Table B.11: λ-invariants at $p = 7$.

k	N	$d, \mathfrak{p}$	λ^+	λ^-	0	1	2	3	4	5	label
2	13	2	0	0	0	0	6	42	300	2100	G1N13k2A
2	13	2,2	0	0	0	0	6	42	300	2100	G1N13k2A
2	15	1	0	0	0	0	6	42	300	2100	G0N15k2A
2	20	2	0	0	0	0	6	42	300	2100	G1N20k2B
2	24	1	0	0	0	0	6	42	300	2100	G0N24k2A
2	24	2	0	2	0	0	8	42	302	2100	G1N24k2C
2	32	1	0	0	0	0	6	42	300	2100	G0N32k2A
2	33	2	0	1	0	0	7	42	301	2100	G1N33k2E
2	34	2	0	0	0	0	6	42	300	2100	G1N34k2C
2	34	2	0	1	0	0	7	42	301	2100	G1N34k2E
2	36	2	0	1	0	0	7	42	301	2100	G1N36k2D
2	43	1	1	3	0	1	9	43	303	2101	G0N43k2A
2	45	1	0	0	0	0	6	42	300	2100	G0N45k2A
2	48	1	0	0	0	0	6	42	300	2100	G0N48k2A
2	55	1	0	0	0	0	6	42	300	2100	G0N55k2A
2	57	1	0	0	0	0	6	42	300	2100	G0N57k2B
2	62	1	0	0	0	0	6	42	300	2100	G0N62k2A
2	64	1	0	0	0	0	6	42	300	2100	G0N64k2A
2	72	1	0	0	0	0	6	42	300	2100	G0N72k2A
2	75	1	0	0	0	0	6	42	300	2100	G0N75k2B
2	94	1	0	0	0	0	6	42	300	2100	G0N94k2A
2	102	1	0	0	0	0	6	42	300	2100	G0N102k2C
2	106	1	3	1	0	3	7	45	301	2103	G0N106k2A
2	113	1	0	0	0	0	6	42	300	2100	G0N113k2A
2	114	1	0	0	0	0	6	42	300	2100	G0N114k2B
2	120	1	0	0	0	0	6	42	300	2100	G0N120k2B
2	121	1	1	1	0	1	7	43	301	2101	G0N121k2A
2	129	1	0	0	0	0	6	42	300	2100	G0N129k2B
2	130	1	0	0	0	0	6	42	300	2100	G0N130k2C
2	136	1	0	0	0	0	6	42	300	2100	G0N136k2A
2	138	1	0	0	0	0	6	42	300	2100	G0N138k2C
2	141	1	0	0	0	0	6	42	300	2100	G0N141k2C
2	142	1	0	0	0	0	6	42	300	2100	G0N142k2B
2	144	1	0	0	0	0	6	42	300	2100	G0N144k2A
2	155	1	1	1	0	1	7	43	301	2101	G0N155k2A
2	158	1	0	0	0	0	6	42	300	2100	G0N158k2D

Table B.12: λ-invariants at $p = 7$.

k	N	$d, \mathfrak{p}$	λ^+	λ^-	0	1	2	3	4	5	label
2	171	1	0	0	0	0	6	42	300	2100	G0N171k2A
2	174	1	0	0	0	0	6	42	300	2100	G0N174k2B
2	178	1	0	0	0	0	6	42	300	2100	G0N178k2A
2	192	1	0	0	0	0	6	42	300	2100	G0N192k2B
2	192	1	0	0	0	0	6	42	300	2100	G0N192k2D
2	195	1	0	0	0	0	6	42	300	2100	G0N195k2C
2	201	1	3	1	0	3	7	45	301	2103	G0N201k2B
2	204	1	0	0	0	0	6	42	300	2100	G0N204k2B
2	206	1	0	0	0	0	6	42	300	2100	G0N206k2A
2	212	1	0	0	0	0	6	42	300	2100	G0N212k2A
2	220	1	0	0	0	0	6	42	300	2100	G0N220k2B
2	222	1	0	0	0	0	6	42	300	2100	G0N222k2A
2	225	1	0	0	0	0	6	42	300	2100	G0N225k2C
2	226	1	1	1	0	1	7	43	301	2101	G0N226k2A
2	228	1	0	0	0	0	6	42	300	2100	G0N228k2A
2	240	1	0	0	0	0	6	42	300	2100	G0N240k2B
2	240	1	0	0	0	0	6	42	300	2100	G0N240k2D
2	248	1	0	0	0	0	6	42	300	2100	G0N248k2B
2	249	1	1	1	0	1	7	43	301	2101	G0N249k2B
2	254	1	0	0	0	0	6	42	300	2100	G0N254k2B
2	256	1	1	1	0	1	7	43	301	2101	G0N256k2A
2	256	1	3	1	0	3	7	45	301	2103	G0N256k2B
2	256	1	0	0	0	0	6	42	300	2100	G0N256k2C
2	256	1	0	0	0	0	6	42	300	2100	G0N256k2D
2	264	1	0	0	0	0	6	42	300	2100	G0N264k2A
2	272	1	1	1	0	1	7	43	301	2101	G0N272k2A
2	274	1	1	1	0	1	7	43	301	2101	G0N274k2C
2	275	1	1	1	0	1	7	43	301	2101	G0N275k2A
2	282	1	0	0	0	0	6	42	300	2100	G0N282k2A
2	288	1	1	1	0	1	7	43	301	2101	G0N288k2A
2	288	1	0	0	0	0	6	42	300	2100	G0N288k2B
2	288	1	0	0	0	0	6	42	300	2100	G0N288k2D
3	8	1	1	0	0	1	12	85	600	4201	G1N8k3A
3	11	1	1	0	0	1	12	85	600	4201	G1N11k3B
3	15	1	1	0	0	1	12	85	600	4201	G1N15k3C
3	15	1	1	0	0	1	12	85	600	4201	G1N15k3D
3	16	1	1	0	0	1	12	85	600	4201	G1N16k3A
3	20	2	1	0	0	1	12	85	600	4201	G1N20k3E
3	23	3	1	0	0	1	12	85	600	4201	G1N23k3B
3	23	3,2	1	0	0	1	12	85	600	4201	G1N23k3B

Table B.13: λ-invariants at $p = 7$.

k	N	$d, \mathfrak{p}$	λ^+	λ^-	0	1	2	3	4	5	label
3	32	1	1	0	0	1	12	85	600	4201	G1N32k3C
3	36	1	1	0	0	1	12	85	600	4201	G1N36k3A
3	36	1	1	0	0	1	12	85	600	4201	G1N36k3B
3	39	2	1	0	0	1	12	85	600	4201	G1N39k3I
3	39	2	1	0	0	1	12	85	600	4201	G1N39k3J
4	20	2	2	0	0	2	18	128	900	6302	G1N20k4B
4	24	2	2	0	0	2	18	128	900	6302	G1N24k4C
4	32	1	2	0	0	2	18	128	900	6302	G0N32k4A
4	33	2	2	0	0	2	18	128	900	6302	G1N33k4I
4	36	2	2	0	0	2	18	128	900	6302	G1N36k4E
4	64	1	2	0	0	2	18	128	900	6302	G0N64k4A
4	120	1	2	0	0	2	18	128	900	6302	G0N120k4C
4	121	1	2	0	0	2	18	128	900	6302	G0N121k4A
4	135	1	2	0	0	2	18	128	900	6302	G0N135k4C
4	135	1	2	0	0	2	18	128	900	6302	G0N135k4D
4	222	1	2	0	0	2	18	128	900	6302	G0N222k4C
4	240	1	2	0	0	2	18	128	900	6302	G0N240k4A
4	256	1	2	0	0	2	18	128	900	6302	G0N256k4A
4	256	1	2	0	0	2	18	128	900	6302	G0N256k4B
4	256	1	2	0	0	2	18	128	900	6302	G0N256k4G
4	256	1	2	0	0	2	18	128	900	6302	G0N256k4H
4	288	1	2	0	0	2	18	128	900	6302	G0N288k4C
4	288	1	2	0	0	2	18	128	900	6302	G0N288k4D
4	288	1	2	0	0	2	18	128	900	6302	G0N288k4K
5	4	1	3	0	0	3	24	171	1200	8403	G1N4k5A
5	8	1	3	0	0	3	24	171	1200	8403	G1N8k5A
5	11	1	3	0	0	3	24	171	1200	8403	G1N11k5B
5	15	1	3	0	0	3	24	171	1200	8403	G1N15k5C
5	15	1	3	0	0	3	24	171	1200	8403	G1N15k5D
5	23	3	3	0	0	3	24	171	1200	8403	G1N23k5B
5	23	3,2	3	0	0	3	24	171	1200	8403	G1N23k5B
5	32	1	3	0	0	3	24	171	1200	8403	G1N32k5D
5	36	1	3	0	0	3	24	171	1200	8403	G1N36k5A
5	39	1	3	0	0	3	24	171	1200	8403	G1N39k5I
5	39	1	3	0	0	3	24	171	1200	8403	G1N39k5J
5	39	2	3	0	0	3	24	171	1200	8403	G1N39k5K
5	39	2,2	3	0	0	3	24	171	1200	8403	G1N39k5K

Table B.14: λ-invariants at $p = 7$.

k	N	$d, \mathfrak{p}$	λ^+	λ^-	0	1	2	3	4	5	label
6	20	2	4	0	0	4	30	214	1500	10504	G1N20k6B
6	24	2	4	0	0	4	30	214	1500	10504	G1N24k6E
6	32	1	4	0	0	4	30	214	1500	10504	G0N32k6A
6	33	2	4	0	0	4	30	214	1500	10504	G1N33k6I
6	36	2	4	0	0	4	30	214	1500	10504	G1N36k6E
6	64	1	4	0	0	4	30	214	1500	10504	G0N64k6A
6	121	1	4	0	0	4	30	214	1500	10504	G0N121k6A
6	256	1	4	0	0	4	30	214	1500	10504	G0N256k6A
6	256	1	4	0	0	4	30	214	1500	10504	G0N256k6B
6	256	1	4	0	0	4	30	214	1500	10504	G0N256k6C
6	256	1	4	0	0	4	30	214	1500	10504	G0N256k6D
6	288	1	4	0	0	4	30	214	1500	10504	G0N288k6G
6	288	1	4	0	0	4	30	214	1500	10504	G0N288k6H
6	288	1	4	0	0	4	30	214	1500	10504	G0N288k6I
7	8	1	5	0	0	5	36	257	1800	12605	G1N8k7A
7	11	1	5	0	0	5	36	257	1800	12605	G1N11k7B
7	15	1	5	0	0	5	36	257	1800	12605	G1N15k7C
7	15	1	5	0	0	5	36	257	1800	12605	G1N15k7D
7	16	1	5	0	0	5	36	257	1800	12605	G1N16k7A
7	20	2	5	0	0	5	36	257	1800	12605	G1N20k7E
7	23	1	5	0	0	5	36	257	1800	12605	G1N23k7B
7	23	2	5	0	0	5	36	257	1800	12605	G1N23k7C
7	32	1	5	0	0	5	36	257	1800	12605	G1N32k7D
7	36	1	5	0	0	5	36	257	1800	12605	G1N36k7A
7	36	1	5	0	0	5	36	257	1800	12605	G1N36k7B
7	39	2	5	0	0	5	36	257	1800	12605	G1N39k7H
7	39	2	5	0	0	5	36	257	1800	12605	G1N39k7I
8	20	2	6	0	0	6	42	300	2100	14706	G1N20k8C
8	24	2	8	0	0	2	42	302	2100	14708	G1N24k8E
8	32	1	6	0	0	6	42	300	2100	14706	G0N32k8A
8	33	2	7	0	0	1	42	301	2100	14707	G1N33k8I
8	36	2	7	0	0	1	42	301	2100	14707	G1N36k8F
8	64	1	6	0	0	6	42	300	2100	14706	G0N64k8A
8	121	1	7	1	0	1	43	301	2101	14707	G0N121k8A
8	256	1	7	1	0	1	43	301	2101	14707	G0N256k8A
8	256	1	6	0	0	6	42	300	2100	14706	G0N256k8B
8	256	1	6	0	0	6	42	300	2100	14706	G0N256k8C
8	256	1	7	3	0	1	45	301	2103	14707	G0N256k8D
8	288	1	6	0	0	6	42	300	2100	14706	G0N288k8A
8	288	1	7	1	0	1	43	301	2101	14707	G0N288k8D
8	288	1	6	0	0	6	42	300	2100	14706	G0N288k8E

Table B.15: λ-invariants at $p = 7$.

k	N	$d, \mathfrak{p}$	λ^+	λ^-	0	1	2	3	4	5	label
9	4	1	6?	0?	0	6	48	342	2400	16806	G1N4k9A
9	8	1	6?	0?	0	6	48	342	2400	16806	G1N8k9A
9	11	1	6?	0?	0	6	48	342	2400	16806	G1N11k9B
9	15	1	8?	2?	0	2	8	50	344	2402	G1N15k9C
9	15	1	6?	0?	0	6	48	342	2400	16806	G1N15k9D
9	23	3	6?	0?	0	6	48	342	2400	16806	G1N23k9B
9	23	3,2	7?	1?	0	1	7	49	343	2401	G1N23k9B
9	32	1	7?	1?	0	1	7	49	343	2401	G1N32k9D
9	36	1	8?	2?	0	2	8	50	344	2402	G1N36k9A
9	39	1	7?	1?	0	1	7	49	343	2401	G1N39k9I
9	39	1	7?	1?	0	1	7	49	343	2401	G1N39k9J
9	39	2	6?	0?	0	6	48	342	2400	16806	G1N39k9K
9	39	2,2	8?	2?	0	2	8	50	344	2402	G1N39k9K
10	20	2	8	1	0	2	13	92	643	4502	G1N20k10C
10	24	2	8	1	0	2	13	92	643	4502	G1N24k10F
10	32	1	8	2	0	2	14	92	644	4502	G0N32k10A
10	33	2	8	1	0	2	13	92	643	4502	G1N33k10H
10	36	2	8	1	0	2	13	92	643	4502	G1N36k10F
10	64	1	8	1	0	2	13	92	643	4502	G0N64k10A
10	121	1	8	1	0	2	13	92	643	4502	G0N121k10A
10	256	1	8	1	0	2	13	92	643	4502	G0N256k10A
10	256	1	8	1	0	2	13	92	643	4502	G0N256k10B
10	256	1	9	1	0	3	13	93	643	4503	G0N256k10C
10	256	1	8	2	0	2	14	92	644	4502	G0N256k10D
10	288	1	9	1	0	3	13	93	643	4503	G0N288k10A
10	288	1	8	1	0	2	13	92	643	4502	G0N288k10B
10	288	1	8	1	0	2	13	92	643	4502	G0N288k10C
11	8	1	9	1	0	3	19	135	943	6603	G1N8k11A
11	11	1	9	1	0	3	19	135	943	6603	G1N11k11B
11	15	1	9	1	0	3	19	135	943	6603	G1N15k11C
11	15	1	10	1	0	4	19	136	943	6604	G1N15k11D
11	16	1	9	1	0	3	19	135	943	6603	G1N16k11A
11	20	2	9	1	0	3	19	135	943	6603	G1N20k11E
11	23	3	9	1	0	3	19	135	943	6603	G1N23k11B
11	23	3,2	9	1	0	3	19	135	943	6603	G1N23k11B
11	32	1	9	1	0	3	19	135	943	6603	G1N32k11D
11	36	1	9	1	0	3	19	135	943	6603	G1N36k11A
11	36	1	9	1	0	3	19	135	943	6603	G1N36k11B
11	39	2	9	1	0	3	19	135	943	6603	G1N39k11I
11	39	2	9	1	0	3	19	135	943	6603	G1N39k11J

Table B.16: λ-invariants at $p = 7$.

k	N	$d, \mathfrak{p}$	λ^+	λ^-	0	1	2	3	4	5	label
12	20	2	10	1	0	4	25	178	1243	8704	G1N20k12C
12	24	2	10	1	0	4	25	178	1243	8704	G1N24k12F
12	32	1	10	1	0	4	25	178	1243	8704	G0N32k12A
12	33	2	10	1	0	4	25	178	1243	8704	G1N33k12H
12	36	2	10	1	0	4	25	178	1243	8704	G1N36k12G
12	64	1	10	1	0	4	25	178	1243	8704	G0N64k12A
12	121	1	10	1	0	4	25	178	1243	8704	G0N121k12A
12	256	1	11	2	0	5	26	179	1244	8705	G0N256k12A
12	256	1	10	1	0	4	25	178	1243	8704	G0N256k12B
12	256	1	10	1	0	4	25	178	1243	8704	G0N256k12C
12	256	1	10	1	0	4	25	178	1243	8704	G0N256k12D
13	11	1	11	1	0	5	31	221	1543	10805	G1N11k13B
13	15	1	11	1	0	5	31	221	1543	10805	G1N15k13C
13	15	1	11	1	0	5	31	221	1543	10805	G1N15k13D
13	23	1	11	1	0	5	31	221	1543	10805	G1N23k13B
13	23	2	11	1	0	5	31	221	1543	10805	G1N23k13C
13	32	1	11	1	0	5	31	221	1543	10805	G1N32k13E
13	36	1	11	1	0	5	31	221	1543	10805	G1N36k13A
13	39	1	11	1	0	5	31	221	1543	10805	G1N39k13G
13	39	1	11	2	0	5	32	221	1544	10805	G1N39k13H
13	39	2	11	1	0	5	31	221	1543	10805	G1N39k13J
13	39	2,2	11	1	0	5	31	221	1543	10805	G1N39k13J
14	20	2	12	1	0	6	37	264	1843	12906	G1N20k14C
14	24	2	12	1	0	6	37	264	1843	12906	G1N24k14F
14	32	1	13	1	0	1	37	265	1843	12907	G0N32k14A
14	33	2	13	1	0	1	37	265	1843	12907	G1N33k14I
14	36	2	13	1	0	1	37	265	1843	12907	G1N36k14H
14	64	1	12	2	0	6	38	264	1844	12906	G0N64k14A
14	121	1	13	1	0	1	37	265	1843	12907	G0N121k14A

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