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Covariate Benchmarking for Sensitivity Analysis when the Confounder is Correlated with Observed Covariates

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Abstract

Covariate benchmarking is an important part of sensitivity analysis about omitted variable bias and can be used to bound the strength of the unobserved confounder using information and judgments about observed covariates. It is common to carry out formal covariate benchmarking under the assumption that the unobserved confounder is orthogonal to the observed covariates. This assumption is restrictive and will be difficult to defended in most empirical analyses. In this paper I show that relaxing the orthogonality assumption leads to a breakdown of a recently proposed innovative formal covariate benchmarking methodology.

JEL Codes: C01.

Keywords: confounding; omitted variable bias; sensitivity analysis.

1 Introduction

Omitted variable bias is a serious problem in observational studies (De Luca et al., 2018; Oster, 2019; De Luca et al., 2019; Basu, 2020; Cinelli and Hazlett, 2020). Developing methodologies to conduct sensitivity analysis related to omitted variable bias is an important area of research in statistics and econometrics. Among recent proposal for sensitivity analysis, a most innovative and promising approach has been presented by Cinelli and Hazlett (2020). The key novelty in the proposal of Cinelli and Hazlett (2020) was to rewrite the traditional omitted variable bias expression using partial R^2.[1] This opened up a fruitful way to conduct sensitivity analysis about omitted variable bias using a three-step procedure.

In the first step, the researcher computes 'robustness values' of two parameters, which are understood as the magnitude of the minimum strength of association (measured with the partial R^2) that an unobserved confounder would need to have, both with the treatment and with the outcome, to change the research

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[1] Partial R^2 measures the strength of association between two random variables conditional on a set of other random variables. For instance, the partial R^2 of Z and Y conditional on the set of random variables X is defined as follows (Cinelli and Hazlett, 2020, equation 17, page 51): $R^2_{Y\sim Z|X} = (R^2_{Y\sim Z+X} - R^2_{Y\sim X})/(1 - R^2_{Y\sim X})$, with the implicit assumption that $R^2_{Y\sim X} < 1$. Therefore, the partial R^2 is the increment in the total R^2 when Z is added as a covariate to the regression of Y on X, normalized by the distance of the total R^2 of the regression of Y on X from 1. This immediately shows that the partial R^2 must lie between 0 and 1.

conclusions. The robustness values are of limited use because the confounder is not observed. Hence, it is difficult for a researcher to judge whether the robustness values are relevant for her specific study.

This is where formal covariate benchmarking can prove useful, and that is precisely the second step of the sensitivity analysis proposed by Cinelli and Hazlett (2020)—the most difficult and crucial part. In this step, the researcher uses information about an *observed* covariate (or set of covariates) to compute *upper bounds* for the strength of association, measured once again using the partial R^2, between the *unobserved* confounder and treatment (and outcome).[2]

In the final and third step, the researcher compares the magnitudes of the robustness values computed in the first step with the upper bounds computed in the second step. If the upper bounds are lower than the robustness values, then the researcher is able to conclude that her conclusions are robust to omitted variable bias.

In this elegant methodology, the crucial step of covariate benchmarking relies on a rather restrictive assumption: *the unobserved confounder is assumed to be orthogonal to the observed covariates.* This assumption is difficult to defend in most observational studies, where all covariates (observed and unobserved) are likely to be correlated with each other. Hence, it is necessary to investigate the implications of relaxing this orthogonality assumption.

The first contribution of this paper is to show that once we relax this orthogonality assumption, the benchmarking analysis in Cinelli and Hazlett (2020, section 4.4, and appendix B) breaks down. This is because upper bounds for the relevant measures of association between the *unobserved* confounder and treatment (and outcome) cannot be generated without additional information. Hence, the second step of the sensitivity analysis is undermined. In deriving my results, I develop a result about the decomposition of the R^2 in a linear regression (theorem 1). To the best of my knowledge, this is a novel result—and constitutes the second contribution of this paper.

Before concluding this introductory section, I need to highlight one technical issue. The second step of the sensitivity analysis, briefly discussed above, needs to generate upper bounds for two parameters: $R^2_{D\sim Z|X}$ (partial R^2 of the confounder, Z, with the treatment, D, conditional on the observed covariates, X) and $R^2_{Y\sim Z|D,X}$ (partial R^2 of the confounder, Z, with the outcome, Y, conditional on the observed covariates, X, and the treatment, D). For my argument, it suffices to show that upper bounds cannot be generated for *at least one* of these two unknown parameters, $R^2_{D\sim Z|X}$ and $R^2_{Y\sim Z|D,X}$. For the sake of brevity, I will focus on $R^2_{D\sim Z|X}$, but the argument could equally well be applied to $R^2_{Y\sim Z|D,X}$.

[2] Cinelli and Hazlett (2020, section 4.4) argue persuasively that informal covariate benchmarking can often be misleading because it does not solve the correct identification problem.

The rest of the paper is organized as follows: in section 2, I present the basic set up and the key expression for bias; in section 3, I discuss the implications of relaxing the orthogonality assumption for covariate benchmarking with the total R^2; in section 4, I discuss the implications of relaxing the orthogonality assumption for covariate benchmarking with the partial R^2; I conclude in section 5. Proofs are collected in the appendix. Throughout this paper, I will follow the notation used in Cinelli and Hazlett (2020) to facilitate easy comparison.

2 The Setup

2.1 Expression of relative bias

Consider the linear regression of an outcome on a treatment, controlling for a set of covariates given by X and Z,

$$Y = \hat{\tau} D + X\hat{\beta} + \hat{\gamma} Z + \hat{\varepsilon}_{\text{full}} \tag{1}$$

where Y is the $n \times 1$ vector of the outcome (dependent variable), X is the $n \times k$ matrix of observed covariates, including a constant, Z is the $n \times 1$ (unobserved) confounder vector, and all hat-quantities denote estimated (sample, and not population) quantities. Since Z is unobserved, the researcher cannot estimate (1) but is forced to estimate the following restricted regression

$$Y = \hat{\tau}_{\text{res}} D + X\hat{\beta}_{\text{res}} + \hat{\varepsilon}_{\text{res}} \tag{2}$$

Letting $\widehat{\text{bias}} = \hat{\tau}_{\text{res}} - \hat{\tau}$ denote the bias of the treatment effect arising from the restricted model, Cinelli and Hazlett (2020, page 48) show, by combining the Frisch-Waugh-Lovell theorem and definitions of partial R^2, that

$$\left|\widehat{\text{bias}}\right| = \text{se}\left(\hat{\tau}\right) \sqrt{\frac{\text{df} \times R^2_{Y\sim D,X} \times R^2_{D\sim Z|X}}{1 - R^2_{D\sim Z|X}}} \tag{3}$$

where 'se' denotes standard error, 'df' denotes the degrees of freedom of the restricted regression in (2), $R^2_{Y\sim D,X}$ denotes the total R^2 from a regression of Y on D and X, $R^2_{D\sim Z|X}$ refers to the partial R^2 from a regression of D on Z conditioning on X and we assume that $0 \leq R^2_{D\sim Z|X} < 1$ (to make sure we do not attempt to divide by zero). The expression for bias can be further manipulated to derive the expression for 'relative bias' (Cinelli and Hazlett, 2020, page 49)

$$\left|\frac{\widehat{\text{bias}}}{\hat{\tau}_{\text{res}}}\right| = \frac{\left|R_{Y\sim Z|D,X} \times f_{D\sim Z|X}\right|}{\left|f_{Y\sim D|X}\right|} \tag{4}$$

where $f^2_{Y\sim D|X} = R^2_{Y\sim D|X}/(1 - R^2_{Y\sim D|X})$ is Cohen's f-statistic, and $f_{D\sim Z|X}$ is defined accordingly.[3]

2.2 Importance of relative bias for sensitivity analysis

What is the importance of relative bias for sensitivity analysis? Relative bias is crucial because it helps a researcher address the question whether taking account of omitted variable bias can overturn the conclusions of an observational study. For, if relative bias is equal to or larger than 1, then the magnitude of the bias can large enough to nullify any nonzero treatment effect that might have been estimated by a researcher, i.e. she would not be able to rule out the possibility that taking account of omitted variable bias would make the estimated treatment effect zero. Sensitivity analysis will, therefore, boil down to seeing if relative bias is larger or smaller than unity. Hence, to investigate how sensitive an estimate of a treatment effect is to omitted variable bias, a researcher should analyze the conditions under which relative bias might equal or exceed unity. Cinelli and Hazlett (2020) propose a three-step sensitivity analysis for this purpose that I explain next.

2.3 Sensitivity analysis

2.3.1 Step 1: Compute robustness values

The first step of the sensitivity analysis is to compute two 'robustness values'. The first robustness value is RV_q, which asks us to answer the following question: If the partial R^2 of the confounder with the treatment, $R^2_{D\sim Z|X}$ and of the confounder with the outcome $R^2_{Y\sim Z|D,X}$ were equal in magnitude, how strong would this partial R^2 need to be to reduce the estimated treatment effect by $100 \times q\%$ and thereby open up the possibility that the treatment effect is zero when $q = 1$? Since $q = 1$ is the most commonly used and relevant, I will focus on this case. Note that RV_1 is denoted simply as RV.

The second robustness value is $RV_{q,\alpha}$, which asks us to answer the following question: If the partial R^2 of the confounder with the treatment, $R^2_{D\sim Z|X}$ and of the confounder with the outcome $R^2_{Y\sim Z|D,X}$ were equal in magnitude, how strong would this partial R^2 need to be to make the adjusted t-test not reject the

[3] In this paper, I will only deal with the case of a single confounder. The case of multiple confounders does not need a separate treatment because the bias with a single confounder is an upper bound for the bias with multiple confounders (Cinelli and Hazlett, 2020, section 4.5).

null hypothesis that the true treatment effect is $(1-q)|\hat{\tau}|$ at the α level? Thus, $RV_{q,\alpha}$ allows a researcher to see whether the estimated treatment effect is zero, after taking account of the uncertainty associated with estimation. Hence, it is superior to RV_q. My focus will be on the case with $q = 1$, where $RV_{q,\alpha}$ will refer to the null hypothesis that the true treatment effect is zero. For notational simplicity, $RV_{1,\alpha}$ is denoted simply as RV_α.

2.3.2 Step 2: Compute bounds using covariate benchmarking

The second step of the sensitivity analysis is the difficult and crucial step of formal covariate benchmarking.

> Arguably, the most difficult part of a sensitivity analysis is taking the description of a confounder that would be problematic from the formal results [e.g., the robustness values], and reasoning about whether a confounder with such strength plausibly exists in one's study, given its design and the investigator's contextual knowledge. (Cinelli and Hazlett, 2020, page 13).

In this step, the researcher needs to investigate the question whether she can reasonably rule out the possibility that $R^2_{D\sim Z|X}$ and $R^2_{Y\sim Z|D,X}$ are higher than the robustness values. Since these two partial R^2 values cannot be computed—because Z is unobserved—she must use information about *observed* covariates to find *upper bounds* for them.

At this point, Cinelli and Hazlett (2020) introduce two parameters, k_D and k_Y, to assist in the process. The first parameter, k_D, captures the relative strength of the confounder in explaining variation in the treatment as compared to a chosen, observed covariate (or set of covariates); the second parameter, k_Y, captures the corresponding relative strength of the unobserved confounder for explaining variation in the outcome. Both parameters can be defined with and without conditioning on observed covariates and the treatment.

These parameters capture the judgment of the researcher based on her knowledge of the context of the research. Once the values of k_D and k_Y have been chosen, Cinelli and Hazlett (2020) show that we can generate *upper bounds* for $R^2_{D\sim Z|X}$ and $R^2_{Y\sim Z|D,X}$ as functions of known quantities and k_D (or k_Y).

2.3.3 Step 3: Compare robustness values with bounds

In the third and final step, the researcher needs to compare the magnitudes of $R^2_{D\sim Z|X}$ and $R^2_{Y\sim Z|D,X}$, or their upper bounds, computed in the second step with the magnitudes of the robustness values computed in the first step. This comparison can then allow the researcher to assess the robustness of the results to omitted variable bias. In particular:

1. If

$$\max\left\{R^2_{D\sim Z|X}, R^2_{Y\sim Z|D,X}\right\} < RV$$

then the researcher can conclude that the estimate of the treatment effect is robust to omitted variable bias;

2. If

$$\max\left\{R^2_{D\sim Z|X}, R^2_{Y\sim Z|D,X}\right\} < RV_\alpha$$

then the researcher can conclude that the bias-adjusted t-test of the null hypothesis that the treatment effect is zero can be rejected at the $\alpha\%$ level of significance;

3. If

$$R^2_{D\sim Z|X} < R^2_{D\sim X}$$

then the researcher can conclude that the "worst case confounder" (a confounder that explains all the residual variance in the outcome, i.e. $R^2_{Y\sim Z|D,X} = 1$) would not eliminate the the estimated treatment effect.

2.4 The role of the orthogonality assumption

The crucial part of the whole sensitivity analysis is the formal benchmarking that generates *upper bounds* for both $R^2_{D\sim Z|X}$ and $R^2_{Y\sim Z|D,X}$. In Cinelli and Hazlett (2020, section 4.4), these bounds have been generated *under the assumption that the confounder is uncorrelated with the observed covariates*, i.e. $Z \perp X$. This seems to be a restrictive assumption. In most, if not all, observational studies, the unobserved confounder is likely to be correlated with included regressors. Hence, we need to ask: what are the implications of relaxing the orthogonality assumption? In the rest of the paper, I will show that the orthogonality assumption is not innocuous. Once we relax that assumption, the covariate benchmarking methodology breaks down.

3 Total R^2-based covariate benchmarking

I will first investigate the implications of relaxing the assumption about the orthogonality of the confounder, X, and the included covariates, X, for the total R^2 approach (Cinelli and Hazlett, 2020, appendix B.1).

3.1 A result about the decomposition of R^2

Consider the following three regressions estimated by ordinary least squares (OLS),

$$Y \text{ on } X, Z \tag{5}$$

$$Y \text{ on } X \tag{6}$$

$$Y \text{ on } Z \tag{7}$$

Let $R^2_{Y\sim X+Z}$, $R^2_{Y\sim X}$, and $R^2_{Y\sim Z}$, denote the total R-squared (coefficient of determination) for the regressions in (5), (6), and (7), respectively; and let $W = (X : Z)$ denote the $n \times (k+1)$ augmented matrix.

Using the definition of the R-squared (Greene, 2012, page 41), we have

$$R^2_{Y\sim X+Z} = \frac{(P_W Y)' M^0 (P_W Y)}{Y' M^0 Y} = \frac{Y' P_W M^0 P_W Y}{Y' M^0 Y} \tag{8}$$

$$R^2_{Y\sim X} = \frac{(P_X Y)' M^0 (P_X Y)}{Y' M^0 Y} = \frac{Y' P_X M^0 P_X Y}{Y' M^0 Y} \tag{9}$$

$$R^2_{Y\sim Z} = \frac{(P_Z Y)' M^0 (P_Z Y)}{Y' M^0 Y} = \frac{Y' P_Z M^0 P_Z Y}{Y' M^0 Y} \tag{10}$$

where P_W, P_X, P_Z denote $n \times n$ projection matrices onto the column spaces of W, X, Z, respectively, so that, for instance,

$$P_W = W (W'W)^{-1} W',$$

and M^0 is the $n \times n$ matrix that generates deviations from means when pre-multiplied to a n vector (Greene, 2012, page 978–79), i.e.,

$$M^0 = \left[I - \frac{1}{n} ii' \right],$$

where I is the identity matrix of dimension n and i denotes a column vector of 1s. Note that projection matrices are symmetric and idempotent. The first property allowed me to write the second equality in (8), (9), and (10), respectively. Note that M^0 is also symmetric and idempotent (Greene, 2012, page 978–79).

Theorem 1. *(Decomposition of R^2). Let $Z^{\perp X} = Z - P_X Z = (I - P_X) Z$ denote the OLS residual obtained from a regression of Z on X. Consider the regression of*

$$Y \textit{ on } Z^{\perp X}, \tag{11}$$

and denote by

$$\eta_{X,Y,Z} = R^2_{Y\sim Z^{\perp X}} - R^2_{Y\sim Z}, \tag{12}$$

the relative magnitudes of the powers (in the sense of R^2) of $Z^{\perp X}$ and Z in explaining variations in Y. Then,

$$R^2_{Y\sim X+Z} - R^2_{Y\sim X} - R^2_{Y\sim Z} = \eta_{X,Y,Z}. \tag{13}$$

The main implication of theorem 1 is to show that the sign of $R^2_{Y\sim X+Z} - R^2_{Y\sim X} - R^2_{Y\sim Z}$ is indeterminate. This is because $\eta_{X,Y,Z}$, which is the difference of the R^2 from two separate regressions, the first a regression of Y on $Z^{\perp X}$, and the second a regression of Y on Z, cannot be signed. Without more information about the relationship between X and Z, it is not possible to assert which of these two R^2 are greater in magnitude. One such special, and rather restrictive, case has been used to derive the first equalities in equations (51) and (52) in Cinelli and Hazlett (2020), and is given in the next result.

Corollary 1. *If $Z \perp X$, then $R^2_{Y\sim X+Z} - R^2_{Y\sim X} - R^2_{Y\sim Z} = 0$.*

With the results of theorem 1 and corollary 1 in place, I am now ready to revisit the benchmarking exercise using total R^2 that was presented in Cinelli and Hazlett (2020, appendix B.1).

3.2 Breakdown of bounding exercise

Our primary task is to generate an upper bound for $R^2_{D\sim Z|X}$. Following Cinelli and Hazlett (2020, appendix B.1), let us define

$$k_D := \frac{R^2_{D\sim Z}}{R^2_{D\sim X_j}} \tag{14}$$

to capture the relative importance of the unobserved confounder in explaining variation in treatment assignment, compared to the chosen, observed covariate, X_j, where relative importance is judged in terms of the total R^2, and we assume that $R^2_{D\sim X_j} > 0$.

Then, using the definition of partial R^2 (Cinelli and Hazlett, 2020, equation 17, page 51), the result in Theorem 1, and the definition of k_D, we get,

$$\begin{aligned} R^2_{D\sim Z|X} &= \frac{R^2_{D\sim Z+X} - R^2_{D\sim X}}{1 - R^2_{D\sim X}} \\ &= \frac{k_D R^2_{D\sim X_j}}{1 - R^2_{D\sim X}} + \frac{\eta_{X,Y,Z}}{1 - R^2_{D\sim X}} \end{aligned} \tag{15}$$

According to the result in Theorem 1, the sign of $\eta_{X,Y,Z}$ is indeterminate. This implies that the exact magnitude of $R^2_{D\sim Z|X}$ in (15) is indeterminate too. Hence, equation (22) in Cinelli and Hazlett (2020, appendix B.1) does not hold—there is an extra term, the second term in (15), whose sign is indeterminate. Thus, the bounding exercise in terms of total R^2 discussed in appendix B.1 in Cinelli and Hazlett (2020) fails, unless more information is used. We need to consider three cases.

Case 1. Suppose $Z \perp X$. In this case, $\eta_{X,Y,Z} = 0$, as shown in corollary 1, and we can pin down the magnitude of $R^2_{D\sim Z|X}$ and $R^2_{Y\sim Z|X}$ exactly. This is the setting in which the proposal of Cinelli and Hazlett (2020, appendix B.1) is located. Since the orthogonality assumption might be difficult to defend in actual observational studies, this limits the relevance of this setting.

Case 2. Suppose the unobserved confounder, Z, has strictly *more* power (in the sense of R^2) in explaining the variations in the outcome variable, Y, than the part of Z that is orthogonal to the set of included covariates, X. In this case, using (12) we can see that $\eta_{X,Y,Z} < 0$, and so using (15) we have an upper bound for $R^2_{D\sim Z|X}$:

$$R^2_{D\sim Z|X} < \frac{k_D R^2_{D\sim X_j}}{1 - R^2_{D\sim X}} \tag{16}$$

The bounding exercise proposed in Cinelli and Hazlett (2020, appendix B.1) works, though the bound is not sharp. Moreover, it is difficult to see how researchers can argue plausibly about the relative explanatory powers of Z and $Z^{\perp X}$ in explaining variation in Y.

Case 3. Suppose the unobserved confounder, Z, has strictly *less* power (in the sense of R^2) in explaining variations in the outcome variable, Y, than the part of Z that is orthogonal to the set of included covariates, X. In this case, using (12) we can see that $\eta_{X,Y,Z} > 0$, and so we have a lower, and not an upper, bound for $R^2_{D\sim Z|X}$:

$$R^2_{D\sim Z|X} > \frac{k_D R^2_{D\sim X_j}}{1 - R^2_{D\sim X}} \tag{17}$$

Hence, in this case, the bounding exercise proposed in Cinelli and Hazlett (2020, appendix B.1) does not work.

Let me summarize my findings about the efficacy of the bounding exercise using comparisons of total R^2: if a researcher can plausibly argue either that $Z \perp X$ or that $Z^{\perp X}$ has more power—in the sense of R^2—than Z in explaining the variation in Y, then the bounding exercise using the total R^2 proposed in Cinelli and Hazlett (2020, appendix B.1) can be used; if not, then the bounding exercise is not feasible.

4 Partial R^2-based covariate benchmarking

I will now investigate the implications of relaxing the assumption about the orthogonality of the confounder, X, and the included covariates, X, for the partial R^2 approach to covariate benchmarking proposed in Cinelli and Hazlett (2020, appendix B.2). I will limit my analysis to the case where the treatment variable, D, is not used for conditioning. The analysis could equally well apply to the case where D is included in the conditioning set.

4.1 Single covariate used for benchmarking

We would like, as in the total R^2 case, to generate an upper bound for $R^2_{D\sim Z|X}$. Suppose there are j covariates, $\{X_1, X_2, \ldots, X_j\}$, and the researcher wishes to use the j-th observed covariate, X_j, for benchmarking. Let X_{-j} refer to the set of observed covariates that is not used for benchmarking, and define

$$k_D := \frac{R^2_{D\sim Z|X_{-j}}}{R^2_{D\sim X_j|X_{-j}}} \tag{18}$$

to capture the relative importance of the unobserved confounder in explaining variation in treatment assignment, compared to the observed covariate, X_j, conditional on X_{-j}, where, as before, relative importance is judged using the total R^2, and we assume that $R^2_{D\sim X_j|X_{-j}} > 0$.

Theorem 2. *Suppose* $0 \leq R^2_{Z\sim X_j|X_{-j}} < 1$ *and* $0 \leq R^2_{D\sim X_j|X_{-j}} < 1$. *Then, we have the following lower bound for* $R^2_{D\sim Z|X}$*:*

$$R^2_{D\sim Z|X} \geq \alpha k_D f^2_{D\sim X_j|X_{-j}}, \tag{19}$$

where

$$\alpha = \frac{\left(1 - |R_{Z\sim X_j|X_{-j}}|\right)^2}{1 - R^2_{Z\sim X_j|X_{-j}}}$$

and we have $0 \leq \alpha \leq 1$.

The implication of theorem 2 is particularly damaging for sensitivity analysis because here we have a lower bound for $R^2_{D\sim Z|X}$. For the sensitivity analysis we instead need an upper bound. Theorem 2 shows that once we give up the assumption about the orthogonality of Z and X, we are no longer able to generate an upper bound for $R^2_{D\sim Z|X}$. We end up with a lower bound. Hence, the sensitivity analysis proposed in Cinelli and Hazlett (2020, section 4.4, and appendix B.2) using comparisons of partial R^2 will not work unless we can use additional information to generate an upper bound for $R^2_{D\sim Z|X}$.

4.2 Multiple covariates used for benchmarking

If a researcher wishes to use multiple observed covariates for benchmarking, then she will have to face, in addition to the problem of not being able to generate an upper bound, an infeasible bounding exercise. To see this, one only needs to note that the expression on the RHS of equation (67) in Cinelli and Hazlett (2020, appendix B.2) cannot arise without the assumption that Z is orthogonal to X. Each step of a j-step recursive process that leads to equation (67) will have additional terms, first and foremost on the numerator, that conveniently become zero when $Z \perp X$, as I now show.

Let $X_{(1,2,\ldots,j)}$ denote the set of covariates; and let $X_{-(1,2,\ldots,j)}$ denote the complement of that set. The first step of the recursion starts by using the recursive definition for partial correlations (Cinelli and Hazlett, 2020, equation 16, page 50) to get

$$R_{D\sim Z|X} = \frac{R_{D\sim Z|X_{-(1)}} - R_{D\sim X_{(1)}|X_{-(1)}} R_{Z\sim X_{(1)}|X_{-(1)}}}{\sqrt{1 - R^2_{D\sim X_{(1)}|X_{-(1)}}}\sqrt{1 - R^2_{Z\sim X_{(1)}|X_{-(1)}}}} \tag{20}$$

If $Z \perp X$, then $R^2_{Z\sim X_{(1)}|X_{-(1)}} = 0$. Hence, the second term on the numerator becomes zero and we get

$$R_{D\sim Z|X} = \frac{R_{D\sim Z|X_{-(1)}}}{\sqrt{1 - R^2_{D\sim X_{(1)}|X_{-(1)}}}} \tag{21}$$

We can now move to the next step of the recursive process by applying the recursive formula for partial R^2 to the numerator of (21):

$$R_{D\sim Z|X_{-(1)}} = \frac{R_{D\sim Z|X_{-(1,2)}} - R_{D\sim X_{(2)}|X_{-(1,2)}} R_{Z\sim X_{(2)}|X_{-(1,2)}}}{\sqrt{1 - R^2_{D\sim X_{(2)}|X_{-(1,2)}}}\sqrt{1 - R^2_{Z\sim X_{(2)}|X_{-(1,2)}}}} \tag{22}$$

If $Z \perp X$, then $R^2_{Z\sim X_{(2)}|X_{-(1,2)}} = 0$. Hence, the second term on the numerator is zero, once again, and we get

$$R_{D\sim Z|X_{-(1)}} = \frac{R_{D\sim Z|X_{-(1,2)}}}{\sqrt{1 - R^2_{D\sim X_{(2)}|X_{-(1,2)}}}} \tag{23}$$

Plugging (23) into (21), we get

$$R_{D\sim Z|X} = \frac{R_{D\sim Z|X_{-(1,2)}}}{\sqrt{1 - R^2_{D\sim X_{(1)}|X_{-(1)}}}\sqrt{1 - R^2_{D\sim X_{(2)}|X_{-(1,2)}}}} \tag{24}$$

Repeating this process j times, we get

$$R_{D\sim Z|X} = \frac{R_{D\sim Z|X_{-(1,2,\ldots,j)}}}{\sqrt{1-R^2_{D\sim X_{(1)}|X_{-(1)}}}\sqrt{1-R^2_{D\sim X_{(2)}|X_{-(1,2)}}}\cdots\sqrt{1-R^2_{D\sim X_{(j)}|X_{-(1,2,\ldots,j)}}}} \tag{25}$$

which is equation (67) in Cinelli and Hazlett (2020, appendix). Defining

$$k_D := \frac{R^2_{D\sim Z|X_{-(1,2,\ldots,j)}}}{R^2_{D\sim X_{(1,2\ldots,j)}|X_{-(1,2,\ldots,j)}}} \tag{26}$$

as the parameter to capture the relative strength of the unobserved confounder, Cinelli and Hazlett (2020) then proceeds with the bounding exercise.

The whole argument hinges crucially on the orthogonality of Z and X. If $Z \not\perp X$, then the very first step of the argument fails because we cannot get to (21) from (20). Instead, plugging (22) into (20), we get

$$\begin{aligned} R_{D\sim Z|X} = & \frac{R_{D\sim Z|X_{-(1,2)}}}{\sqrt{1-R^2_{D\sim X_{(2)}|X_{-(1)}}}\sqrt{1-R^2_{Z\sim X_{(2)}|X_{-(1,2)}}}\sqrt{1-R^2_{D\sim X_{(1)}|X_{-(1)}}}\sqrt{1-R^2_{Z\sim X_{(1)}|X_{-(1)}}}} \\ & - \frac{R_{D\sim X_{(2)}|X_{-(1,2)}}R_{Z\sim X_{(2)}|X_{-(1,2)}}}{\sqrt{1-R^2_{D\sim X_{(2)}|X_{-(1)}}}\sqrt{1-R^2_{Z\sim X_{(2)}|X_{-(1,2)}}}\sqrt{1-R^2_{D\sim X_{(1)}|X_{-(1)}}}\sqrt{1-R^2_{Z\sim X_{(1)}|X_{-(1)}}}} \\ & - \frac{R_{D\sim X_{(1)}|X_{-(1)}}R_{Z\sim X_{(1)}|X_{-(1)}}}{\sqrt{1-R^2_{D\sim X_{(1)}|X_{-(1)}}}\sqrt{1-R^2_{Z\sim X_{(1)}|X_{-(1)}}}} \end{aligned}$$

If one wished to use covariate benchmarking to replace the terms involving Z in the above expression, one would run into the problem of a proliferation of parameters. Moreover, carrying out the analysis to the next step of the recursion quickly becomes unwieldy. In essence, once we relax the assumption that Z is orthogonal to X, the benchmarking exercise using multiple covariates becomes infeasible.

5 Conclusion

In an innovative and important contribution to the literature on omitted variable bias, Cinelli and Hazlett (2020) have proposed a methodology for conducting sensitivity analysis using partial R^2 measures. In their proposed methodology, the key step of covariate benchmarking to generate upper bounds for measures of association between the unobserved confounder and the treatment (and outcome) relies on the assumption that the unobserved confounder is orthogonal to the set of observed covariates. This is a restrictive assumption and will be difficult to defend in most observational studies. In this paper I have demonstrated that once we relax the orthogonality assumption, covariate benchmarking for sensitivity analysis to omitted variable bias as proposed in Cinelli and Hazlett (2020) breaks down.

What is the implication of the theoretical findings of this paper? This paper is not trying to undermine the overall methodology proposed in Cinelli and Hazlett (2020). Rather it tries to show a weakness in a key step of the overall argument so that researchers can address it and strengthen the overall methodology. In particular, it needs to be highlighted that the first step of the sensitivity analysis proposed in Cinelli and Hazlett (2020), whereby omitted variable bias is related to measures of association between the unobserved confounder and the treatment (and outcome) using partial R^2, is extremely innovative and useful. It is a superior approach not only in comparison with informal covariate benchmarking but also in comparison to the proposal in Oster (2019) that relies on a parameter, δ, that is difficult to interpret. The partial R^2 formulation takes the long-standing discussion on omitted variable bias forward. But the second step of covariate benchmarking is weak, as demonstrated by the theoretical results reported in this paper. Therefore, it would seem that finding alternative methods of formal covariate benchmarking would be one of the fruitful avenues for future research in sensitivity analysis of omitted variable bias.

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Appendix A Proofs

A.1 Proof of Theorem 1

I will need a result on the decomposition of projection matrices that is given in Rao et al. (2008, page 323).

Lemma 1. *Let $Z^{\perp X} = Z - P_X Z = (I - P_X) Z$ denote the OLS residual obtained from a regression of Z on X. Then,*

$$P_W = P_X + P_{Z^{\perp X}}. \tag{27}$$

Proof. Using results on the inverse of partitioned matrices, it can be shown (Rao et al., 2008, page 323) that

$$P_W = P_X + \frac{(I - P_X) ZZ' (I - P_X)}{Z' (I - P_X) Z}. \tag{28}$$

Using the definition of $Z^{\perp X}$, we see that

$$\frac{(I - P_X) ZZ' (I - P_X)}{Z' (I - P_X) z} = P_{Z^{\perp X}}, \tag{29}$$

where I use the fact that $(I - P_X)$ is also a projection matrix (onto the orthogonal complement of the column space of X) and hence symmetric and idempotent. □

The proof of theorem 1 now follows.

Proof. Using lemma 1, we have

$$P_W M^0 P_W = P_W M^0 M^0 P_W = \left(M^0 P_X + M^0 P_{Z^{\perp X}}\right)' \left(M^0 P_X + M^0 P_{Z^{\perp X}}\right).$$

This becomes

$$P_W M^0 P_W = P_X M^0 P_X + P_{Z^{\perp X}} M^0 P_{Z^{\perp X}} \tag{30}$$

because the cross product terms on the extreme right hand side is

$$P_X M^0 M^0 P_{Z^{\perp X}} = P_X M^0 P_{Z^{\perp X}} = P_X P_{Z^{\perp X}} = 0.$$

Note that the penultimate equality is true because

$$\begin{aligned} M^0 P_{Z^{\perp X}} &= M^0 Z^{\perp X} \left[(Z^{\perp X})' Z^{\perp X} \right]^{-1} (Z^{\perp X})' \\ &= Z^{\perp X} \left[(Z^{\perp X})' Z^{\perp X} \right]^{-1} (Z^{\perp X})' \\ &= P_{Z^{\perp X}} \end{aligned}$$

where, because $Z^{\perp X}$ is a regression residual vector, we have $M^0 Z^{\perp X} = Z^{\perp X}$ (Greene, 2012, page 40). The final equality is true because

$$P_X P_{Z^{\perp X}} = X \left(X'X \right)^{-1} X' Z^{\perp X} \left[(Z^{\perp X})' Z^{\perp X} \right]^{-1} (Z^{\perp X})' = 0,$$

where I have used $X' Z^{\perp X} = 0$ (i.e. residuals are orthogonal to the regressors).

I pre-multiply (30) by Y', then post-multiply the result by Y, and finally divide through by $Y' M^0 Y$ to get

$$\frac{Y' P_W M^0 P_W Y}{Y' M^0 Y} = \frac{Y' P_X M^0 P_X Y}{Y' M^0 Y} + \frac{Y' P_{Z^{\perp X}} M^0 P_{Z^{\perp X}} Y}{Y' M^0 Y}.$$

Using (8) and (9), we get

$$R^2_{Y \sim X+Z} - R^2_{Y \sim X} = \frac{Y' P_{Z^{\perp X}} M^0 P_{Z^{\perp X}} Y}{Y' M^0 Y}.$$

Subtracting $R^2_{Y \sim Z}$ from both sides and using (10), we get

$$R^2_{Y \sim X+Z} - R^2_{Y \sim X} - R^2_{Y \sim Z} = \frac{Y' P_{Z^{\perp X}} M^0 P_{Z^{\perp X}} Y - Y' P_Z M^0 P_Z Y}{Y' M^0 Y} \tag{31}$$

Now, using the definition of $\eta_{X,Y,Z}$ in (12), we see that the RHS of (31) is $\eta_{X,Y,Z}$. Hence, we get

$$R^2_{Y \sim X+Z} - R^2_{Y \sim X} - R^2_{Y \sim Z} = \eta_{X,Y,Z}.$$

$\square$

A.2 Proof of Corollary 1

Proof. If $Z \perp X$, then $Z^{\perp X} = Z$. Hence $P_{Z^{\perp X}} = P_Z$. Hence, the RHS of (31) becomes

$$\frac{Y'P_{Z^{\perp X}}M^0P_{Z^{\perp X}}Y - Y'P_ZM^0P_ZY}{Y'M^0Y} = \frac{Y'P_ZM^0P_ZY - Y'P_ZM^0P_ZY}{Y'M^0Y} = 0;$$

so, we have

$$R^2_{Y\sim X+Z} - R^2_{Y\sim X} - R^2_{Y\sim Z} = 0.$$

□

A.3 Proof of Theorem 2

Proof. We know $\left|R_{D\sim Z|X_{-j}}\right| = \sqrt{k_D}\left|R_{D\sim X_j|X_{-j}}\right|$. Now using the recursive definition of partial correlations (Cinelli and Hazlett, 2020, equation 16, page 50), we have

$$\begin{aligned}
\left|R_{D\sim Z|X}\right| &= \frac{\left|R_{D\sim Z|X_{-j}} - R_{D\sim X_j|X_{-j}}R_{Z\sim X_j|X_{-j}}\right|}{\sqrt{1-R^2_{D\sim X_j|X_{-j}}}\sqrt{1-R^2_{Z\sim X_j|X_{-j}}}} \\
&\geq \frac{\left|R_{D\sim Z|X_{-j}}\right| - \left|R_{D\sim X_j|X_{-j}}R_{Z\sim X_j|X_{-j}}\right|}{\sqrt{1-R^2_{D\sim X_j|X_{-j}}}\sqrt{1-R^2_{Z\sim X_j|X_{-j}}}} \\
&= \frac{\sqrt{k_D}\left|R_{D\sim X_j|X_{-j}}\right| - \left|R_{D\sim X_j|X_{-j}}R_{Z\sim X_j|X_{-j}}\right|}{\sqrt{1-R^2_{D\sim X_j|X_{-j}}}\sqrt{1-R^2_{Z\sim X_j|X_{-j}}}} \\
&= \frac{\sqrt{k_D}\left|R_{D\sim X_j|X_{-j}}\right|\left(1-\left|R_{Z\sim X_j|X_{-j}}\right|\right)}{\sqrt{1-R^2_{D\sim X_j|X_{-j}}}\sqrt{1-R^2_{Z\sim X_j|X_{-j}}}}
\end{aligned}$$

where the second step uses the well-known result for real numbers: $|a-b| \geq |a| - |b|$. Hence, taking the square of both sides of the above inequality, using the definition of k_D in (18), and noting that

$$f^2_{D\sim X_j|X_{-j}} = \frac{R^2_{D\sim X_j|X_{-j}}}{1-R^2_{D\sim X_j|X_{-j}}}$$

we get

$$R^2_{D\sim Z|X} \geq \alpha k_D f^2_{D\sim X_j|X_{-j}} \tag{32}$$

where

$$\alpha = \frac{\left(1-\left|R_{Z\sim X_j|X_{-j}}\right|\right)^2}{1-R^2_{Z\sim X_j|X_{-j}}}.$$

If $0 \le R_{Z\sim X_j|X_{-j}} < 1$, then

$$\alpha = \frac{\left(1 - \left|R_{Z\sim X_j|X_{-j}}\right|\right)^2}{1 - R^2_{Z\sim X_j|X_{-j}}} = \frac{1 - R_{Z\sim X_j|X_{-j}}}{1 + R_{Z\sim X_j|X_{-j}}} \le 1.$$

If $-1 < R_{Z\sim X_j|X_{-j}} < 0$, then

$$\alpha = \frac{\left(1 - \left|R_{Z\sim X_j|X_{-j}}\right|\right)^2}{1 - R^2_{Z\sim X_j|X_{-j}}} = \frac{1 + R_{Z\sim X_j|X_{-j}}}{1 - R_{Z\sim X_j|X_{-j}}} \le 1.$$

$\square$