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# SCHRÖDINGER MAPS AND THEIR ASSOCIATED FRAME SYSTEMS

ANDREA NAHMOD<sup>1</sup>, JALAL SHATAH<sup>2</sup>, LUIS VEGA<sup>3</sup>, AND CHONGCHUN ZENG<sup>4</sup>

**ABSTRACT.** In this paper we establish the equivalence of solutions between Schrödinger maps into  $\mathbb{S}^2$  or  $\mathbb{H}^2$  and their associated gauge invariant Schrödinger equations. We also establish the existence of global weak solutions into  $\mathbb{H}^2$  in two space dimensions. We extend these ideas for maps into compact hermitian symmetric manifolds with trivial first cohomology.

## 1. INTRODUCTION

Schrödinger maps are maps from space-time into a Kähler manifold with metric  $h$  and complex structure  $J$  satisfying:  $u : \mathbb{R}^d \times \mathbb{R} \rightarrow (M, h, J)$

$$(SM) \quad \partial_t u = J \sum_{\ell} D_{\ell} \partial^{\ell} u,$$

where  $D$  denotes the covariant derivative on  $u^{-1}TM$ . These maps are a generalization of the Heisenberg model describing the magnetization spin  $m \in \mathbb{S}^2 \subset \mathbb{R}^3$  in a ferromagnetic material

$$\partial_t m = m \times \Delta m.$$

For  $m \in \mathbb{S}^2$  the operator  $J = m \times$  acting on  $T_m \mathbb{S}^2$  is equivalent to complex multiplication by  $i$  on  $\mathbb{C}$ . Thus

$$\partial_t m = m \times \Delta m = m \times (\Delta m + |\nabla m|^2 m) = J \sum_i D_i \partial^i m$$

where as before  $D_i = \partial_i + \langle \partial_i m, \cdot \rangle m$  denotes the covariant derivative on  $m^{-1}T\mathbb{S}^2$ ; and the Heisenberg model can be written as

$$\partial_t m = J \sum_i D_i \partial^i m.$$

In one-space dimension the Heisenberg model can be transformed into the focusing NLS

$$(NLS) \quad i\partial_t q - \partial_x^2 q - \frac{1}{2}|q|^2 q = 0$$

via the Hasimoto transformation. This transformation was later generalized by N. H. Chang, J. Shatah, and K. Uhlenbeck [CSU00] to study the regularity of Schrödinger maps. The idea in [CSU00] was to disregard the customary coordinates representation of the (SM) system and to introduce instead a gauge invariant nonlinear Schrödinger equations (GNLS) derived by using a pull-back frame on  $u^{-1}TM$ . The GNLS is given schematically by

$$\mathcal{D}_t q = i \sum_k \mathcal{D}_k^2 q + iFq.$$

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Using the Coulomb gauge, this system can be written as

$$i\partial_t q = \Delta q + \Delta^{-1}[\partial(O(|q|^2))\partial q + O(|q|^3)].$$

One of the consequences of such a representation was to reveal the semilinear nature of the Schrödinger maps systems which led to the first regularity proof in 1 and 2-dimensions for finite energy equivariant data [CSU00]. Here we would like to note that the 1-dimensional Cauchy problem for (SM) is subcritical with respect to the energy space  $\partial u \in L^2$  and as such should be solvable for data  $\partial u \in L^2$ . However the only proof of global well-posedness in this case was given for data  $\partial u \in H^1$  and uses the GNLS system [CSU00]. The desired goal would be to solve the Cauchy problem and to show equivalence when the derivative of the data behaves like  $\delta(x)$ ; i.e. data scaling as  $\partial u \in \dot{H}^{-1/2}$ .

Another consequence of introducing the GNLS was to show that for constant curvature  $M$  the GNLS system doesn't depend explicitly on  $u$ , and therefore can be solved without any reference to the SM system.

Using this last observation a natural question to ask in the constant curvature case is: When do solutions of the GNLS represent solutions of SM? For smooth solutions this question was answered in one dimension by Terng and Uhlenbeck [TU06] and in two dimensions, for a special case, by N.H. Chang and O. Pashev [CP05].

In this paper we are interested in studying the correspondence between solutions  $u$  of the Schrödinger map system and solutions  $q$  of its associated gauge invariant nonlinear Schrödinger equations for low regularity data. In particular we show the equivalence of the two systems for solutions where the problems are expected to be well posed, i.e.,  $\partial u \in H^{\frac{d}{2}-1}$  plus Strichartz estimates for  $d = 2$ . One should remark that the interesting cases for the equivalence of the SM system and GNLS system correspond to  $d = 1, 2$  or  $3$  since in  $d \geq 4$ ,  $\partial u \in H^{\frac{d}{2}-1}$  and equation (SM) holds a.e.; thus there is little difference between smooth and  $\partial u \in H^{\frac{d}{2}-1}$  solutions.

The outline of the paper is as follows: In section 2 we present the frame system. In section 3 we study the equivalence problem when the target is the sphere. For  $d = 1$  we show the equivalence under the condition  $q \in L^2_{t,x}$  and  $|q|^2 \in L^2(H^{-1})$ . For higher dimension we show the equivalence of GNLS and SM for data in critical spaces, i.e., invariant under the scaling  $u(t, x) \rightarrow u(\lambda^2 t, \lambda x)$ . In section 4 we study the problem when the target is  $\mathbb{H}^2$ , the hyperbolic space. Here we show equivalence of smooth solutions and for two space dimensions we show global existence of finite energy solutions. Finally in section 5 we describe the extension of these results for maps into compact hermitian symmetric manifolds with trivial first cohomology.

Throughout this paper we sum over repeated indices unless we explicitly state the contrary, and we follow the convention that Greek subscripts vary from 0 to  $d$  while roman subscripts vary from 1 to  $d$  or  $n$  depending on the context.

## 2. FRAME SYSTEM

The use of frames on the pullback bundle was introduced in [CSU00], and was later used successfully to study the Cauchy problem for wave maps [SS02, NSU03b]. In [NSU03a] similar ideas as in [CSU00] were also used, starting with the pull-back of the conformal frame of  $\mathbb{S}^2$  -which amounts to the stereographic projection- followed by the Coulomb gauge transformation.

**Frames on the pullback bundle.** Let  $\phi : \mathbb{R}^d \rightarrow (M, h, J)$  be a map into a  $2n$ -dimensional Kähler manifold and let  $D$  denote the covariant derivative on  $\phi^{-1}TM$ . Since  $M$  is Kähler then  $D_\ell J(\phi(x)) = 0$  for  $\ell = 1, \dots, d$ . With a slight abuse of language we will refer to sections on  $\phi^{-1}TM$  as vectors. Let  $\{e_a\}_{a=1}^{2n}$  denote an orthonormal frame on  $\phi^{-1}TM$  such that  $e_{a+n} = J e_a$  for  $a = 1, \dots, n$ . Such a frame always exists since  $\mathbb{R}^d$  is contractible and  $M$  is Kähler.

**Proposition 2.1.** Fix the origin  $0 \in \mathbb{R}^d$  and introduce polar coordinates  $(r, \omega)$  on  $\mathbb{R}^d$ . Given a smooth  $\phi : \mathbb{R}^d \rightarrow (M, h, J)$ , let  $\{e_1^*, \dots, e_{2n}^*\}$  denote an orthonormal set of vectors on  $\phi^{-1}TM$  at  $x = 0$  such that  $e_{a+n}^* = Je_a^*$  and let  $\{e_1, \dots, e_{2n}\}$  be the solution to the ODEs

$$D_r e_a = 0, \quad e_a(0, \omega) = e_a^*.$$

Then  $\{e_1, \dots, e_{2n}\}$  is an orthonormal frame for  $\phi^{-1}TM$  with  $e_{a+n} = Je_a$  for  $a = 1, \dots, n$ .

*Proof.* Solve the linear ODEs and use the fact that  $D_r J e_a = J D_r e_a$  since  $M$  is Kähler.  $\square$

Write the frame as  $\{e_1, \dots, e_{2n}\} = \{e_1, \dots, e_n, J e_1, \dots, J e_n\} \stackrel{\text{def}}{=} \{e, J e\}$ . For any vector  $v \in \phi^{-1}TM$  with coordinates  $v = \sum_{\ell=1}^{2n} v_\ell e_\ell$ , we introduce complex coordinates  $w = (w_1, \dots, w_n) \in \mathbb{C}^n$ , where  $w_\ell = v_\ell + i v_{\ell+n}$ , on  $\phi^{-1}TM$  and write  $v = w \cdot e$  where

$$v = \sum_{\ell=1}^{2n} v_\ell e_\ell = \sum_{\ell=1}^n (v_\ell + v_{\ell+n} J) e_\ell = \sum_{\ell=1}^n (v_\ell + i v_{\ell+n}) e_\ell = \sum_{\ell=1}^n w_\ell e_\ell = w \cdot e.$$

In these complex coordinates,  $J \rightarrow i$  on  $\phi^{-1}TM$ .

The covariant derivative on  $M$  introduces a connection  $\{A_\ell\}$  on  $\phi^{-1}TM$  given by  $D_\ell e_a = A_{a\ell}^b e_b$  for  $a = 1, \dots, n$ ,  $\ell = 1, \dots, d$ . We simply write

$$(2.1) \quad D_\ell e = A_\ell \cdot e.$$

where the  $n \times n$  matrices  $A_\ell = (A_{a\ell}^b) \in \mathfrak{su}(n)$ . For any vector  $v = w \cdot e \in \phi^{-1}TM$ , with coordinates  $w \in \mathbb{C}^n$  we have

$$D_\ell v = D_\ell (w \cdot e) = (\partial_\ell w + A_\ell w) \cdot e \stackrel{\text{def}}{=} (\mathcal{D}_\ell w) \cdot e$$

where  $\mathcal{D}$  denotes the covariant derivative on  $\phi^{-1}TM$  expressed in terms of the frame  $\{e, J e\}$ .

If one chooses another frame  $\{\hat{e}, J \hat{e}\}$  related to  $\{e, J e\}$  by a transformation  $g \in \mathbb{S}\mathbb{U}(n)$ , i.e.,  $\hat{e} = g \cdot e$  then

$$\begin{aligned} D_\ell \hat{e} &= \hat{A}_\ell \cdot \hat{e} \\ \hat{A}_\ell &= g^{-1} A_\ell g + g^{-1} \partial_\ell g. \end{aligned}$$

Thus fixing a frame is equivalent to fixing the connection  $A$ ; i.e. fixing a gauge. The matrices  $\{A_\ell\}_{\ell=0}^d$ , which are given by (2.1), have to verify the curvature equation. That is, if we let  $\partial_k \phi = q_k \cdot e$  and denote by

$$[D_\ell, D_k] e_a = R(\partial_k \phi, \partial_\ell \phi) e_a = R(q_k \cdot e, q_\ell \cdot e) e_a \stackrel{\text{def}}{=} F(q_\ell, q_k) \cdot e_a = F_{\ell k} \cdot e_a$$

then we have

$$(2.2) \quad [\mathcal{D}_\ell, \mathcal{D}_k] = \partial_\ell A_k - \partial_k A_\ell + [A_\ell, A_k] = F_{\ell k}.$$

Here it is worth mentioning that the frame constructed in proposition 2.1 corresponds to choosing a connection such that  $x^k A_k(x) = 0$ . This gauge is referred to as the exponential (or Crömstrom) gauge [Uh83]. For this gauge the connection  $A$  can be easily recovered from  $F$  by the formula

$$(2.3) \quad A_k(x) = \int_0^1 x^\ell F_{\ell k}(sx) s ds.$$

Throughout this paper we are interested in a special frame which corresponds to the Coulomb gauge, i.e., a frame for which  $\sum_1^d \partial_\ell \hat{A}_\ell = 0$ . Local smooth Coulomb frames can always be constructed as was demonstrated by K. Uhlenbeck in [Uh83]. This is done by solving the elliptic equation for  $g$

$$0 = \partial_\ell \hat{A}_\ell = \partial_\ell (g^{-1} A_\ell g + g^{-1} \partial_\ell g),$$

locally on balls in  $\mathbb{R}^d$ . For  $d > 1$  gluing these local solutions does not necessarily yield a global Coulomb frame. Of course if  $n = 1$  then  $g = \exp(i\theta)$  and the above equation is linear and can be solved globally. For

the general problem Dell Antonio and Zwanziger [DZ91] showed that the existence of a global  $\dot{H}^1$  Coulomb frame.

**Proposition 2.2.** *Given a smooth map  $\phi : \mathbb{R}^d \rightarrow (M, h, J)$  there exists a frame  $\{\hat{e}, J\hat{e}\}$  such that*

$$\begin{aligned} D_\ell \hat{e}_a &= \hat{A}_\ell \cdot e \\ \sum_{\ell=1}^d \partial_\ell \hat{A}_\ell &= 0. \end{aligned}$$

*Sketch of the proof.* Fix a frame  $\{e, Je\}$  of  $\phi^{-1}TM$  and let  $A_\ell$  be given by  $D_\ell e = A_\ell \cdot e$ . For any  $g \in \mathbb{S}\mathbb{U}(n)$  let  $\hat{A}_\ell = g^{-1}A_\ell g + g^{-1}\partial_\ell g$  and consider the variational problem

$$\inf_g \int |\hat{A}|^2 dx = \inf_g \int \sum_{\ell=1}^d |g^{-1}A_\ell g + g^{-1}\partial_\ell g|^2 dx.$$

It is easy to verify that the infimum is achieved and that  $\sum_{\ell=1}^d \partial_\ell \hat{A}_\ell = 0$  [DZ91]. Thus the frame  $\{\hat{e}, J\hat{e}\}$  is a Coulomb frame with  $\hat{e} = g \cdot e$ .

*Remark.* If  $\phi \in W^{1,d}(\mathbb{R}^d, M)$  and  $M$  is compact then by a result of Schoen and Uhlenbeck [SU83a, SU83b]  $\phi$  can be approximated by smooth functions. Therefore by proposition 2.1 and equation (2.3), the exponential frame  $\{e, Je\}$  on  $\phi^{-1}TM$  belongs to  $e \in W_{\text{loc}}^{1, \frac{d}{2}}$ . The local Coulomb gauge in [Uh83] which satisfies

$$\begin{aligned} \partial_\ell A_k - \partial_k A_\ell + [A_\ell, A_k] &= F_{\ell k} \in L^{\frac{d}{2}} \\ \partial_k A_k &= 0 \end{aligned}$$

belongs to  $L_x^d$  whence  $e \in W_{\text{loc}}^{1,d}$  for  $d > 2$ . For  $d = 2$  we need to require  $\phi \in W^{1,p}$  for some  $p > 2$ .

**GNLS.** The relation of Schrödinger maps to Gauge invariant Schrödinger equations is given through the frame coordinates

$$(2.4) \quad \begin{cases} \partial_\alpha u = q_\alpha \cdot e, \\ D_\alpha e = A_\alpha \cdot e. \end{cases}$$

for  $\alpha = 0, 1, \dots, d$ . Given such  $\{q_\alpha, A_\alpha\}$ , let  $F_{\alpha\beta} = F(q_\beta, q_\alpha) = R(q_\alpha \cdot e, q_\beta \cdot e)$  where  $R$  denotes the Riemann curvature tensor of  $M$ . We have,

**Proposition 2.3.** *Given a smooth Schrödinger map  $u : \mathbb{R}^d \times \mathbb{R} \rightarrow M$  and a frame  $\{e, Je\}$  on  $u^{-1}TM$ ; the coordinates  $(q_\alpha, A_\alpha)$  for  $\alpha = 0, 1, \dots, d$ , given by (2.4) satisfy*

$$(GNLS) \quad \begin{cases} q_0 = i\mathcal{D}_\ell q_\ell \\ \mathcal{D}_t q_\ell = i\mathcal{D}_k^2 q_\ell + iF_{\ell k} q_k, \\ \mathcal{D}_\ell q_k = \mathcal{D}_k q_\ell, \\ \partial_\alpha A_\beta - \partial_\beta A_\alpha + [A_\alpha, A_\beta] = F_{\alpha\beta} \end{cases}$$

for  $k, \ell = 1, \dots, d$  and  $\alpha, \beta = 0, 1, \dots, d$ ; and where we summed on repeated indices.

*Proof.* Write  $q = (q_0, q_1, \dots, q_d) \in \mathbb{C}^{n \times (d+1)}$ . The  $\mathbb{C}^n$  valued functions  $q_\alpha$ ,  $\alpha = 0, 1, \dots, d$  have to satisfy

$$(2.5) \quad q_0 = i\mathcal{D}_\ell q_\ell \quad \text{since} \quad \partial_t u = J\mathcal{D}_\ell \partial^\ell u$$

$$(2.6) \quad \mathcal{D}_\alpha q_\beta = \mathcal{D}_\beta q_\alpha \quad \text{since} \quad D_\alpha \partial_\beta u = D_\beta \partial_\alpha u$$

The equations for the matrices  $\{A_\alpha\}_{\alpha=0}^d$  can be derived from the curvature equation

$$[D_\alpha, D_\beta]e_a = R(\partial_\beta u, \partial_\alpha u)e_a = R(q_\beta \cdot e, q_\alpha \cdot e)e_a = F(q_\alpha, q_\beta) \cdot e_a = F_{\alpha\beta} \cdot e_a.$$

Note that  $F_{\alpha\beta}$  is bilinear in  $(q_\alpha, q_\beta)$  and is calculated from the Riemannian curvature and the frame on  $u^{-1}TM$ . Moreover in terms of the given frame we have

$$[\mathcal{D}_\alpha, \mathcal{D}_\beta] = \partial_\alpha A_\beta - \partial_\beta A_\alpha + [A_\alpha, A_\beta] = F_{\alpha\beta}.$$

Equations (2.5) and (2.6) can be simplified by substituting (2.5) in equation (2.6) for  $\alpha = 0$  to obtain

$$\mathcal{D}_t q_\ell = \mathcal{D}_\ell q_0 = i\mathcal{D}_\ell \mathcal{D}_k q_k.$$

By commuting  $[\mathcal{D}_\ell, \mathcal{D}_k]$  and using the fact that  $\mathcal{D}_\ell q_k = \mathcal{D}_k q_\ell$  we obtain the (GNLS) system.  $\square$

*Remarks.* 1. Given  $u$  a solution of (SM) and a choice of frames  $\{e, Je\}$  we can compute  $A_\alpha$  from  $D_\alpha e_a = A_{a\alpha}^b e_b$ . By choosing another frame  $\{\hat{e}, J\hat{e}\}$ , where  $\hat{e}_a = g_a^b e_b$  and  $g \in SU(n)$ , the connection  $\partial_\alpha \hat{e}_a = \hat{A}_{a\alpha}^b \hat{e}_b$  can be determined from  $A_\alpha$  by  $\partial_\alpha g_a^b + A_{c\alpha}^b g_a^c = g_c^b \hat{A}_{a\alpha}^c$ , or in matrix notation

$$\hat{A}_\alpha = g^{-1} A_\alpha g + g^{-1} \partial_\alpha g \quad \alpha = 0, 1, \dots, d.$$

Thus the equations for  $A_\alpha$  in (GNLS) are underdetermined unless we fix a choice of the orthonormal basis  $\{e, Je\}$ . Throughout this paper we fix the frame by choosing the Coulomb gauge which is given by  $\sum_1^d \partial_\ell A_\ell = 0$ .

2. For  $M$  a Riemann surface, the gauge group is  $\mathbb{U}(1)$ ,  $q_\ell \in \mathbb{C}$ ,  $A_\alpha = ia_\alpha$  and  $F(q_\alpha, q_\beta) = F_{\alpha\beta} = if_{\alpha\beta}$  where  $a_\alpha, f_{\alpha\beta} \in \mathbb{R}$ . In this case the (GNLS) system simplifies to

$$(2.7) \quad \begin{aligned} \mathcal{D}_t q_\ell &= i\mathcal{D}_k^2 q_\ell - \kappa(u) \langle q_\ell, iq_k \rangle q_k \\ \mathcal{D}_\ell q_k &= \mathcal{D}_k q_\ell \\ \partial_\ell a_j - \partial_j a_\ell &= f_{\ell j} = \kappa(u) \langle q_\ell, iq_j \rangle \\ \partial_\ell a_0 - \partial_t a_\ell &= f_{\ell 0} = -\kappa(u) \langle q_\ell, \mathcal{D}_j q_j \rangle \end{aligned}$$

where  $\kappa$  is the Gauss curvature of  $M$ , and where for two complex numbers  $z$  and  $w$  we used the notation  $\langle z, w \rangle = \text{Re}(z\bar{w})$ . In this case it is always possible to put the above system in the Coulomb gauge globally by the gauge transformation  $\hat{q}_\ell = (\exp i\theta)q_\ell$  and  $\hat{a}_\alpha = a_\alpha + \partial_\alpha \theta$  where  $\Delta\theta = -\partial_\ell a_\ell$ . In this Coulomb gauge equations (2.7) transform into

$$(2.8) \quad \begin{aligned} \mathcal{D}_t q_\ell &= i\mathcal{D}_k^2 q_\ell - \kappa(u) \langle q_\ell, iq_k \rangle q_k \\ \mathcal{D}_\ell q_k &= \mathcal{D}_k q_\ell \\ \Delta a_j &= \partial_k f_{kj} = \partial_k (\kappa(u) \langle q_k, iq_j \rangle) \\ \Delta a_0 &= \partial_k f_{k0} = -\partial_k \left( \kappa(u) (\partial_j \langle q_k, q_j \rangle - \frac{1}{2} \partial_k |q_j|^2) \right) \\ \partial_k a_k &= 0. \end{aligned}$$

3. In general the system (GNLS) depends on  $u$  which appears in  $F_{\alpha\beta}$ . For constant curvature  $M$ , the Schrödinger map  $u$  does not appear explicitly in (GNLS). Thus we can consider the system (GNLS) on its own as an evolution problem. In this case the equations  $\mathcal{D}_\ell q_k = \mathcal{D}_k q_\ell$  should be viewed as a compatibility condition which will be satisfied under the evolutions of  $q_\ell$  provided they are satisfied initially. Thus one of the questions we are interested in here is : Given  $(q_\ell, A_\alpha)$  solutions of the (GNLS) in the Coulomb gauge, is there a Schrödinger map  $u$  and a frame  $\{e, Je\}$  such that  $\partial_\ell u = q_\ell \cdot e$  and  $D_\alpha e = A_\alpha \cdot e$ ?

3. MAPS INTO  $\mathbb{S}^2$ 

**One space dimension.** A Schrödinger map  $u : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{S}^2 \subset \mathbb{R}^3$  is a solution to

$$(3.1) \quad \begin{cases} \partial_t u = JD_x \partial_x u = u \times u_{xx} = \partial_x(u \times u_x), & u \in \mathbb{S}^2 \subset \mathbb{R}^3 \\ u(0) = u_0. \end{cases}$$

In this case the associated GNLS system in the Coulomb gauge  $A_1 = 0$  is the nonlinear Schrödinger equation

$$i\partial_t q - \partial_x^2 q - \frac{1}{2}|q|^2 q = 0,$$

and the transformation between  $u$  and  $q$  is given by

$$(3.2) \quad \begin{aligned} \partial_t u &= p \cdot e = p_1 e + p_2 u \times e & u(0, x) &= u_0(x) \\ \partial_x u &= q \cdot e = q_1 e + q_2 u \times e \\ D_t e &= \partial_t e + p_1 u = -\frac{1}{2}|q|^2 u \times e & e(0, x) &= e_0(x) \\ D_x e &= \partial_x e + q_1 u = 0 \end{aligned}$$

where  $p = iq_x$ .

For smooth solutions one can easily show the equivalence between solutions to the (SM) and solutions to the (NLS).

**Proposition 3.1.** *1. Given a smooth solution  $u$  of (3.1) there exist a frame  $\{e, u \times e\}$  for  $u^{-1}T\mathbb{S}^2$  and a solution to the NLS  $\partial_t q = i(\partial_x^2 q + \frac{1}{2}|q|^2 q)$  such that*

$$\begin{aligned} \partial_x u &= q \cdot e = q_1 e + q_2 u \times e \\ D_x e &= \partial_x e + q_1 u = 0. \end{aligned}$$

*2. Conversely given a smooth solution  $q$  to (NLS) with data  $q_0$ , a point  $m \in \mathbb{S}^2$  and  $v_0 \in T_m \mathbb{S}^2$  with  $|v_0| = 1$ , there exists a unique solution  $u$  to (3.1) and a frame  $\{e, Je\}$  for  $u^{-1}T\mathbb{S}^2$  such that (3.2) is satisfied with  $u(0, 0) = m$  and  $e(0, 0) = v_0$ .*

*Proof.* 1. Let  $u$  be a solution of (3.1),  $\{e, u \times e\}$  be any frame on  $u^{-1}T\mathbb{S}^2$ , and let  $\partial_\alpha u = q_\alpha \cdot e$  and  $D_\alpha e = (ia_\alpha) \cdot e$ . Apply the gauge transformation  $\partial_x \theta = -a_1$  to put the system in the Coulomb gauge  $\hat{a}_1 = 0$ . Since in this case the scalar curvature  $\kappa = 1$  we conclude from (2.8) that  $\hat{a}_0 = -\frac{1}{2}|q|^2$  and that  $\hat{q}$  satisfies

$$(\partial_t + i\hat{a}_0)\hat{q} = i\partial_x^2 \hat{q}$$

which is the (NLS).

2. We start by constructing  $u(0, x) \stackrel{\text{def}}{=} u_0(x)$  and  $e(0, x) \stackrel{\text{def}}{=} e_0(x)$  by solving the ODEs

$$\begin{aligned} \partial_x u_0 &= q_{01} e_0 + q_{02} u_0 \times e_0 \\ \partial_x e_0 + q_{01} u_0 &= 0 \\ u_0(0) &= m \quad e_0(0) = v_0. \end{aligned}$$

where  $q_{01}(x) + iq_{02}(x) = q_0(x) = q(0, x)$ . It is easy to check that  $e_0(x) \perp u_0(x)$  and that  $|u_0(x)| = |e_0(x)| = 1$ .

To construct  $u$  and  $e$  we evolve the data in time using (3.2)

$$\begin{aligned} \partial_t u &= p_1 e + p_2 u \times e \\ \partial_t e + p_1 u &= -\frac{1}{2}|q|^2 u \times e \\ u(0, x) &= u_0(x) \quad e(0, x) = e_0(x). \end{aligned}$$

where  $p = iq_x$ , to find  $u(t, x) \in \mathbb{S}^2$  and  $e \in T_u\mathbb{S}^2$ ,  $|e(t, x)| = 1$ . To verify that  $u$  solves (SM) and that  $D_x e = 0$  we set  $\partial_x u = \tilde{q} \cdot e$  and  $D_x e = au \times e$ . Then  $\tilde{q}(0, x) = q(0, x)$  and  $a(0, x) = 0$  by construction. By commuting derivatives, we have

$$\begin{aligned} D_t \partial_x u &= D_x \partial_t u \Rightarrow \tilde{q}_t = i\partial_x^2 q - q_x a + i\frac{1}{2}|q|^2 \tilde{q} \\ D_t D_x e - D_x D_t e &= R(u_t, u_x)e \Rightarrow \partial_t a + \partial_x \frac{1}{2}|q|^2 = \tilde{q}_1 q_{1x} + \tilde{q}_2 q_{2x}. \end{aligned}$$

Therefore  $\tilde{q} - q$  and  $a$  satisfy the ODEs

$$\begin{aligned} \partial_t(\tilde{q} - q) &= i\frac{|q|^2}{2}(\tilde{q} - q) + ipa \\ \partial_t a &= (\tilde{q}_1 - q_1)q_{1x} + (\tilde{q}_2 - q_2)q_{2x}, \\ (\tilde{q} - q)(0, x) &= 0 \quad a(0, x) = 0, \end{aligned}$$

which imply  $\tilde{q} \equiv q$  and  $a \equiv 0$ . Since  $\partial_t u = (iq_x) \cdot e$  and  $D_x e = 0$  we conclude that  $u$  solves (3.1). The uniqueness of  $u$  follows from the uniqueness of the solutions to the ODEs and NLS.  $\square$

For rough initial data we can show equivalence of solutions under weak integrability conditions.

**Theorem 3.1.** *Let  $q \in L^2_{|t|<T} L^2_{x,\text{loc}}$  satisfying  $|q|^2 \in L^2_{|t|<T}(H^{-1})$  be the limit of smooth solutions, i.e.,  $\exists q_k$  smooth solutions of (NLS) such that*

$$q_k \rightarrow q \in L^2_{tx,\text{loc}} \quad \text{and} \quad |q_k|^2 \rightarrow |q|^2 \in L^2_{t,\text{loc}} H^{-1}.$$

*Then there exists a weak solution  $u \in L^2_{|t|<T}(H^1_{\text{loc}}) \cap C_{|t|<T}(L^2_{\text{loc}})$  to (3.1) and a frame  $\{e, u \times e\}$  of  $u^{-1}T\mathbb{S}^2$  such that  $e \in L^2_{|t|<T}(H^1_{\text{loc}}) \cap C_{|t|<T}(L^2_{\text{loc}})$ . Moreover the solution is unique up to isometries on the sphere.*

**Remarks.** 1) In one dimension, Vargas-Vega [VV01] showed local well posedness for the cubic NLS in a space containing  $L^2$  and scaling like  $\dot{H}^{-1/3}(\mathbb{R})$ . Their solutions belong to  $L^3_{|t|<T} L^6_x$  (or  $L^4_{|t|<T} L^4_x$ ). The critical scaling for the 1d NLS is that of  $\dot{H}^{-1/2}(\mathbb{R})$ . Below  $L^2$  however, the Galilean transformations are not preserved and the problem is ill-posed in the Sobolev class [KPV01].

2) In [GRV03] it is proved that a vortex filament can develop a singularity in the shape of a corner in finite time. This shows the existence of a Dirac delta singularity for the corresponding cubic NLS solution. For NLS data in  $L^2$  such a behavior cannot occur due to mass conservation.

*Proof.* By proposition 3.1 we can construct from  $\{q_k\}$  smooth solutions  $u_k$  of (3.1) and frames  $\{e_k, u_k \times e_k\}$  of  $u_k^{-1}T\mathbb{S}^2$  such that

$$(3.3) \quad \begin{aligned} \partial_t u_k &= p_k \cdot e_k = p_{k1} e_k + p_{k2} u_k \times e_k & u_k(0, x) &= u_0(x) \\ \partial_x u_k &= q_k \cdot e_k = q_{k1} e_k + q_{k2} u_k \times e_k \\ D_t e_k &= \partial_t e_k + p_{k1} u_k = -\frac{1}{2}|q_k|^2 u_k \times e_k & e_k(0, x) &= e_0(x) \\ D_x e_k &= \partial_x e_k + q_{k1} u_k = 0 \end{aligned}$$

where  $p_k = iq_{kx}$ . By the hypothesis of the theorem we can pass to the limit in (3.3) and thus  $u$  and  $e$  satisfy equation (3.2) in the sense of distributions. From the equations for  $\partial_x u$  and  $\partial_x e$  we conclude that  $u$  and  $e$  are in  $L^2_{|t|<T}(H^1_{\text{loc}})$ . From the equations for  $\partial_t u$  and  $\partial_t e$  we have  $\forall \varphi \in C_0^\infty(\mathbb{R})$   $\varphi u$  and  $\varphi e$  are in  $C_{|t|<T}(H^1)$ . From computing

$$\partial_t \int |u(t, x) - u(t_0, x)|^2 \varphi(x) dx = 2 \int \langle p \cdot e, u(t, x) - u(t_0, x) \rangle \varphi(x) dx,$$

where  $\langle \cdot, \cdot \rangle$  is the inner product in  $\mathbb{R}^3$ , and a similar expression for  $e$  we conclude that  $u$  and  $e$  are in  $C_{|t|<T}(L^2_{\text{loc}})$ . Note that in this case (3.2) implies that for every  $t \in (-T, T)$ ,  $u(t, \cdot)$  and  $e(t, \cdot)$  are in  $\dot{H}^1$ .

To show uniqueness, let  $(u, e)$  and  $(\tilde{u}, \tilde{e})$  be two such solutions that satisfy (3.2). Then by using the isometries on  $\mathbb{S}^2$  we can assume that  $u(0, 0) = \tilde{u}(0, 0)$  and  $e(0, 0) = \tilde{e}(0, 0)$ . Equation (3.2) implies that  $u(0, x) = \tilde{u}(0, x)$  and  $e(0, x) = \tilde{e}(0, x)$  for all  $x$ . Set  $\delta u = u - \tilde{u}$ ,  $\delta e = e - \tilde{e}$ , and  $\delta f = u \times e - \tilde{u} \times \tilde{e}$  then

$$(3.4) \quad \begin{aligned} \partial_t \delta u &= p_1 \delta e + p_2 \delta f \\ \partial_t \delta e &= -p_1 \delta u - \frac{1}{2} |q|^2 \delta f \\ \partial_t \delta f &= -p_2 \delta u + \frac{1}{2} |q|^2 \delta e \\ \delta u(0) &= 0, \quad \delta e(0, x) = \delta f(0, x) = 0 \end{aligned}$$

which can be written in matrix notation as

$$\partial_t V = BV \quad V(0) = 0.$$

Since  $B$  is skew symmetric and is locally in  $L^2 H^{-1}$  and  $V$  is locally in  $L^2 H^1 \cap L^\infty$  then for any  $C_0^\infty \ni \varphi$

$$\frac{d}{dt} \int |V(t, x)|^2 \varphi(x) dx = 2 \int \langle BV, V \rangle \varphi dx = 0$$

and therefore  $V \equiv 0$ . □

**Higher dimensional maps into  $\mathbb{S}^2$ .** A Schrödinger map  $u : \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{S}^2 \hookrightarrow \mathbb{R}^3$  is a solution to

$$(3.5) \quad \partial_t u = u \times \Delta u = \partial_{x_j} (u \times \partial_{x_j} u).$$

In this case, since  $n = 1$ , we have  $q_\alpha \in \mathbb{C}$ ,  $A_\alpha = ia_\alpha$  and  $F(q_\alpha, q_\beta) = F_{\alpha\beta} = if_{\alpha\beta}$  where  $a_\alpha, f_{\alpha\beta} \in \mathbb{R}$ . Given a Schrödinger map  $u$  into  $\mathbb{S}^2$  and a frame  $\{e, J_e\}$  we recall (GNLS) for  $(q_k, a_k)$

$$\begin{aligned} \mathcal{D}_t q_\ell &= i\mathcal{D}_k^2 q_\ell - i\langle q_\ell, iq_k \rangle q_k \\ \mathcal{D}_k q_\ell &= \mathcal{D}_\ell q_k \\ \partial_\ell a_j - \partial_j a_\ell &= f_{\ell j} = \langle q_\ell, iq_j \rangle \\ \partial_\ell a_0 - \partial_t a_\ell &= f_{\ell 0} = -\langle q_\ell, \mathcal{D}_j q_j \rangle \end{aligned}$$

and the transformation between  $u$  and  $q$

$$(3.6) \quad \begin{aligned} \partial_t u &= q_0 \cdot e \\ \partial_\ell u &= q_\ell \cdot e \\ \mathcal{D}_t e &= a_0 u \times e \\ \mathcal{D}_\ell e &= a_\ell u \times e \end{aligned}$$

where  $q_0 = i\mathcal{D}_k q_k$ . In the Coulomb frame this system simplifies to

$$(3.7) \quad \begin{aligned} \mathcal{D}_t q_\ell &= i\mathcal{D}_k^2 q_\ell - i\langle q_\ell, iq_k \rangle q_k \\ \Delta a_j &= \partial_k f_{kj} = \partial_k \langle q_k, iq_j \rangle \\ \Delta a_0 &= \partial_k f_{k0} = -\partial_\ell \partial_j \langle q_\ell, q_j \rangle + \frac{1}{2} \Delta |q_j|^2 \end{aligned}$$

along with the Coulomb frame equation and compatibility conditions

$$(3.8) \quad \begin{aligned} \partial_k a_k &= 0 \\ \mathcal{D}_k q_\ell &= \mathcal{D}_\ell q_k \\ \partial_\ell a_j - \partial_j a_\ell &= f_{\ell j} = \langle q_\ell, i q_j \rangle \\ \partial_\ell a_0 - \partial_t a_\ell &= f_{\ell 0} = -\langle q_\ell, \mathcal{D}_j q_j \rangle. \end{aligned}$$

It is easy to verify that (3.8) are satisfied by smooth solutions of (3.7) for all  $t$  if they are satisfied at  $t = 0$  and  $a$  decays at infinity.

**Proposition 3.2.** *Given a smooth solution  $(q, a)$  of (3.7) satisfying (3.8), a point  $m \in \mathbb{S}^2$  and a vector  $v_0 \in T_m \mathbb{S}^2$  with  $|v_0| = 1$ , then there exists a unique solution  $u$  of the (3.5) system and a frame  $\{e, J_e\}$  for  $u^{-1} T \mathbb{S}^2$  such that (2.4), i.e.,*

$$\begin{aligned} \partial_\alpha u &= q_\alpha \cdot e \\ D_\alpha e &= a_\alpha u \times e \end{aligned}$$

holds with  $u(0, 0) = m$  and  $e(0, 0) = v_0$ .

*Proof.* Given  $(q, a)$  solution to (3.7) we first construct the initial data for  $u$  and for the frame  $\{e, u \times e\}$ . This will be done inductively on every coordinate  $x_1, x_2, \dots, x_d$ . We start by solving

$$\begin{aligned} \partial_1 w_1(x_1) &= q_1(x_1, 0, \dots, 0) \cdot e_1(x_1) \\ D_1 e_1(x_1) &= \partial_1 e_1(x_1) + \langle e_1(x_1), \partial_1 w_1(x_1) \rangle w_1(x_1) = a_1(x_1, 0, \dots, 0) w_1 \times e_1(x_1) \\ w_1(0) &= m, \quad e_1(0) = v_0. \end{aligned}$$

It is easy to verify that  $\{e_1, w_1 \times e_1\}$  are a frame along the curve  $w_1^{-1} T \mathbb{S}^2$ . Repeat this process to construct  $w_2(x_1, x_2)$  and  $\{e_2, w_2 \times e_2\}$  from

$$\begin{aligned} \partial_2 w_2 &= q_2(x_1, x_2, 0, \dots, 0) \cdot e_2 \\ D_2 e_2 &= \partial_2 e_2 + \langle e_2, \partial_1 w_2 \rangle w_2 = a_2(x_1, x_2, 0, \dots, 0) w_2 \times e_2 \\ w_2(x_1, 0) &= w_1(x_1), \quad e_2(x_1, 0) = e_1(x_1). \end{aligned}$$

This construction terminates by constructing  $u_0 \stackrel{\text{def}}{=} w_d(x_1, \dots, x_d)$  and  $\{e_0, u_0 \times e_0\} \stackrel{\text{def}}{=} \{e_d, w_d \times e_d\}$ .

To verify that  $u_0$  and  $\{e_0, J_{e_0}\}$  satisfy (2.4) at  $t = 0$ , we note that by construction the equations hold on  $(x_1, 0, \dots, 0) \in \mathbb{R}^2$ . Here  $\mathbb{R}^2$  denotes the  $x_1 x_2$ -plane. To show that the same holds for on  $(x_1, x_2, 0, \dots, 0) \in \mathbb{R}^2$  we compute

$$(3.9) \quad \begin{aligned} D_2 \partial_1 u_0 &= D_1 \partial_2 u_0 \\ D_2 D_1 e_0 &= D_1 D_2 e_0 + R(\partial_1 u_0, \partial_2 u_0) e_0. \end{aligned}$$

By our construction we have

$$(3.10) \quad \begin{aligned} \partial_2 u_0 &= q_2 \cdot e_0 & D_2 e_0 &= a_2 u_0 \times e_0 & (x_1, x_2) &\in \mathbb{R}^2 \\ \partial_1 u_0 &= \hat{q}_1 \cdot e_0 & D_1 e_0 &= \hat{a}_1 u_0 \times e_0 & (x_1, x_2) &\in \mathbb{R}^2 \\ \hat{q}_1(x_1, 0) &= q_1(x_1, \dots, 0) & \hat{a}_1(x_1, 0) &= a_1(x_1, \dots, 0) \end{aligned}$$

Substituting (3.10) in (3.9) we obtain the following ODEs

$$\begin{aligned} \partial_2(\hat{q}_1 - q_1) + ia_2(\hat{q}_1 - q_1) &= i(\hat{a}_1 - a_1)q_2 \\ \partial_2(\hat{a}_1 - a_1) &= \hat{f}_{12} - f_{12} = \langle \hat{q}_1 - q_1, q_2 \rangle \end{aligned}$$

where  $\hat{f}_{12} = (\hat{q}_1, iq_2)$ . Since 0 is a solution of this ODE by uniqueness we have  $\hat{q}_1 = q_1$  and  $\hat{q}_2 = q_2$  in the  $x_1x_2$ -plane. Repeating this process for  $x_3, \dots, x_d$ , we obtain the desired result.

To construct  $u(t, x)$  and  $\{e, u \times e\}$  we solve the ODEs

$$\begin{aligned}\partial_t u &= q_0 \cdot e \\ D_t e &= \partial_t e + q_{01} u = a_0 u \times e \\ u(0, x) &= u_0(x) \quad e(0, x) = e_0(x)\end{aligned}$$

To verify that  $u$  solves (SM) and (2.4) holds we set  $\partial_\ell u = \tilde{q}_\ell \cdot e$  and  $D_\ell e = \tilde{a}_\ell u \times e$  and define  $\tilde{\mathcal{D}}_\ell = \partial_\ell + i\tilde{a}_\ell$ . Then  $\tilde{q}(0, x) = q(0, x)$  and  $\tilde{a}(0, x) = a(0, x)$  by construction. By commuting derivatives, we have

$$\begin{aligned}D_t \partial_\ell u &= D_\ell \partial_t u \Rightarrow \mathcal{D}_t \tilde{q}_\ell = i\tilde{\mathcal{D}}_\ell q_0 = i\tilde{\mathcal{D}}_\ell \mathcal{D}_k q_k \\ D_t D_\ell e - D_\ell D_t e &= R(u_\ell, u_t) e \Rightarrow \partial_t \tilde{a}_\ell - \partial_\ell a_0 = \tilde{f}_{0\ell},\end{aligned}$$

where  $\tilde{f}_{0\ell} = -iF(q_0, \tilde{q}_\ell)$ . Therefore  $\tilde{q}_\ell - q_\ell$  and  $\tilde{a}_\ell - a_\ell$  satisfy the ODEs

$$\begin{aligned}\mathcal{D}_t(\tilde{q}_\ell - q_\ell) &= -(\tilde{a}_\ell - a_\ell) \mathcal{D}_k q_k \\ \partial_t(\tilde{a}_\ell - a_\ell) &= \tilde{f}_{0\ell} - f_{0\ell} = \langle q_0, i(\tilde{q}_\ell - q_\ell) \rangle, \\ (\tilde{q}_\ell - q_\ell)(0, x) &= 0 \quad (\tilde{a}_\ell - a_\ell)a(0, x) = 0.\end{aligned}$$

which imply  $\tilde{q}_\ell \equiv q_\ell$  and  $\tilde{a}_\ell \equiv a_\ell$ , and thus we conclude that  $u$  solves (3.1). The uniqueness of  $u$  follows from the uniqueness of the solutions to the ODEs and NLS.  $\square$

**Theorem 3.2.** *Given a solution  $q_k$  to (3.7) such that*

$$\begin{aligned}q_k &\in C([0, T], L^2(\mathbb{R}^d)) \cap L^6([0, T], L^3(\mathbb{R}^d)) \quad \text{for } d = 2 \\ q_k &\in C([0, T], L^2(\mathbb{R}^d) \cap L^d(\mathbb{R}^d)) \quad \text{for } d \geq 3\end{aligned}$$

and assume that  $q_k$  is the  $C(L^2)$  limit of smooth solutions, i.e.,  $\exists \{q_{k(j)}\}$  where  $q_{k(j)} \rightarrow q_k$  in  $C([0, T], L^2_x)$ . Then there exists a solution  $u \in L^\infty(\dot{H}^1) \cap C(L^2_{\text{loc}})$  of the Schrödinger maps equation (3.5) and a frame  $\{e, Je\}$ , where  $e \in L^\infty(\dot{H}^1) \cap C(L^2_{\text{loc}})$  for  $d > 2$  and  $e \in L^\infty(\dot{H}^1_{\text{loc}}) \cap C(L^2_{\text{loc}})$  for  $d = 2$ , that satisfies the coordinates equation (2.4)

$$\begin{cases} \partial_\alpha u = q_\alpha \cdot e \\ D_\alpha e = \partial_\alpha e + q_{\alpha 1} u = a_\alpha u \times e, \end{cases}$$

Moreover if there are two such  $\{u, e\}$  and  $\{\tilde{u}, \tilde{e}\}$  that have the same coordinates given by (2.4) with initial data  $u(0, x_0) = \tilde{u}(0, x_0)$ ,  $e(0, x_0) = \tilde{e}(0, x_0)$  at one Lebesgue point  $(0, x_0)$ ,  $x_0 \in \mathbb{R}^d$  of  $u - \tilde{u}$  and  $e - \tilde{e}$ , then  $u \equiv \tilde{u}$  and  $e \equiv \tilde{e}$ .

**Remarks.** 1) The assumption that  $q_k \in C(L^2) \cap L^6(L^3)$  in two dimensions guarantees finite energy plus a Strichartz norm. This is necessary to make sense of all the terms in (2.4), such as  $a_0$  and is not needed for the existence of weak solutions. Other Strichartz choices are also possible.

2) The assumption that  $q_k \in C(L^d)$  for  $d \geq 3$  is much weaker than the space  $C(H^{\frac{d}{2}-1})$  which is the optimal space for existence of solutions to (3.7).

3) The assumption  $\tilde{u} = u$  and  $\tilde{e} = e$  at a Lebesgue point in the uniqueness statement can also be replaced by any decay to 0 of  $\tilde{u}(0, \cdot) - u(0, \cdot)$  and  $\tilde{e}(0, \cdot) - e(0, \cdot)$  as  $|x| \rightarrow \infty$ .

*Proof.* From the expression for  $a_0$  and  $a_j$  in (3.7)

$$\begin{aligned}\Delta a_j &= \partial_k f_{kj} = \partial_k \langle q_k, iq_j \rangle \\ \Delta a_0 &= \partial_k f_{k0} = -\partial_\ell \partial_j \langle q_\ell, iq_j \rangle + \frac{1}{2} \Delta |q_j|^2\end{aligned}$$

we have  $a_j \in C(L^d)$ ,  $a_0 \in C(L^{d/2})$  for  $d > 2$  and  $a_j \in L^3(L^6)$ ,  $a_0 \in L^3(L^{3/2})$  for  $d = 2$ . By proposition 3.2 we can construct from  $\{q_{k(j)}\}$  smooth solutions  $u_{(j)}$  of (3.5) and frames  $\{e_{(j)}, u_{(j)} \times e_{(j)}\}$  of  $u_{(j)}^{-1}T\mathbb{S}^2$  such that

$$(3.11) \quad \begin{aligned} \partial_\alpha u_{(j)} &= q_{\alpha(j)} \cdot e_{(j)} \\ \partial_\alpha e_{(j)} &= -q_{\alpha 1(j)} u_{(j)} + a_{\alpha(j)} u_{(j)} \times e_{(j)} \end{aligned}$$

By the regularity hypothesis on  $q_k$  given in the theorem we can pass to the limit in (3.11) and thus  $u$  and  $e$  are in  $L^\infty(\dot{H}^1) \cap C(L_{\text{loc}}^2)$  (if  $d = 2$ ,  $e \in L^\infty(\dot{H}_{\text{loc}}^1)$ ), they satisfy equation (2.4) in the sense of distribution, and  $u$  solves (3.5).

To show uniqueness assume  $u$  and  $\tilde{u}$  are two solutions that satisfy (2.4) and agree at a point say  $(0, 0)$ . We first show that the data for  $u$  and  $\tilde{u}$  are the same. Let  $f = u \times e$  and  $\tilde{f} = \tilde{u} \times \tilde{e}$ , then from (2.4) we have at  $t = 0$

$$\begin{aligned} \partial_k u &= q_k \cdot e = q_{k1} e + q_{k2} f \\ \partial_k e &= -q_{k1} u + a_k f \\ \partial_k f &= -q_{k2} u - a_k f \end{aligned}$$

and the same for  $\{\tilde{u}, \tilde{e}, \tilde{f}\}$ . Since  $q_k(0, \cdot) \in L^d(\mathbb{R}^d)$ , for  $d = 2$ , it is straight forward to prove that  $a_k(0, \cdot)$  is in the dual space of  $L^r \cap L^{\frac{2r}{r+2}}$  for any  $r > 2$  and for  $d > 2$ ,  $a_k(0, \cdot) \in L^d$ . Thus  $a_k(0, \cdot)$  is in  $L_{\text{loc}}^1$ . Therefore we may take differences in the above linear equations to obtain

$$|u - \tilde{u}|^2 + |e - \tilde{e}|^2 + |f - \tilde{f}|^2 = \text{constant}.$$

Since  $u(0, 0) = \tilde{u}(0, 0)$  and  $e(0, 0) = \tilde{e}(0, 0)$  then  $u \equiv \tilde{u}$  and  $e \equiv \tilde{e}$  at  $t = 0$ .

To show that  $u \equiv \tilde{u}$  for all  $t$  we use the time derivative part of (2.4)

$$\begin{aligned} \partial_t u &= p \cdot e = q_{01} e + q_{02} f \\ \partial_t e &= -q_{01} u + a_0 f \\ \partial_t f &= -q_{02} u - a_0 e \end{aligned}$$

and the same for  $\{\tilde{u}, \tilde{e}, \tilde{f}\}$ . Again since  $p$  and  $a_0$  are in  $L_{\text{loc}}^1(H^{-1})$  we have

$$\partial_t (|u - \tilde{u}|^2 + |e - \tilde{e}|^2 + |f - \tilde{f}|^2) = 0$$

and since at  $t = 0$ ,  $u = \tilde{u}$  and  $e = \tilde{e}$ , then  $u \equiv \tilde{u}$  and  $e \equiv \tilde{e}$ ,  $\forall (t, x) \in \mathbb{R} \times \mathbb{R}^d$ .  $\square$

#### 4. SCHRÖDINGER MAPS INTO $\mathbb{H}^2$

The Cauchy problem for Schrödinger maps into the hyperbolic plane  $u : \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{H}^2$  has two difficulties that are not present when the target is  $\mathbb{S}^2$ . The first difficulty is due to the fact that  $\mathbb{H}^2$  cannot be embedded isometrically and equivariantly in  $\mathbb{R}^k$ . The second is due to the non compactness of  $\mathbb{H}^2$ , which makes controlling  $u$  an issue.

The first difficulty can be avoided by embedding  $\mathbb{H}^2$  in the Lorentz space  $(\mathbb{R}^3, \eta)$  where  $\eta = \text{dia}(-1, 1, 1)$  and the embedding is given by

$$\mathbb{H}^2 = \{u \in (\mathbb{R}^3, \eta); -u_0^2 + u_1^2 + u_2^2 = -1, u_0 > 0\}.$$

The embedding is isometric and equivariant as becomes apparent after introducing the coordinates  $u_0 = \cosh \chi$ ,  $u_1 = \sinh \chi \cos \theta$  and  $u_2 = \sinh \chi \sin \theta$ . The tangent space and the normal space for this embedding are given by

$$\begin{aligned} T_u \mathbb{H}^2 &= \{v \in \mathbb{R}^3; \langle \eta u, v \rangle = 0\} \\ N_u \mathbb{H}^2 &= \{\gamma u; \gamma \in \mathbb{R}\}. \end{aligned}$$

The unit normal at  $u \in \mathbb{H}^2$  is the vector  $u$  since  $\langle \eta u, u \rangle = -1$ . For a vector  $v \in T_u \mathbb{H}^2$  we introduce the notation

$$\|v\|^2 = |\langle v, \eta v \rangle| = -v_0^2 + v_1^2 + v_2^2,$$

and for a map  $u : \mathbb{R}^d \rightarrow \mathbb{H}^2$  with  $w \in u^{-1}T\mathbb{H}^2$

$$\|w\|_{L^2}^2 = \int \|w(x)\|^2 dx.$$

Given a map  $\phi : \mathbb{R}^d \rightarrow \mathbb{H}^2 \subset \mathbb{R}^3$ , the covariant derivative on  $\phi^{-1}T\mathbb{H}^2$  is given by

$$D_k V = \partial_k V - \langle V, \eta \partial_k \phi \rangle \phi.$$

The complex structure on  $T\mathbb{H}^2$  can be represented by

$$Jv = \eta(u \times v)$$

where  $\times$  is the usual cross product on  $\mathbb{R}^3$ . This is a consequence of  $\langle u, \eta Jv \rangle = \langle v, \eta Jv \rangle = 0$  and  $J^2 = -I$ .

Using the embedding  $\mathbb{H}^2 \subset (\mathbb{R}^3, \eta)$  Schrödinger maps  $u : \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{H}^2$  can be written in divergence form as

$$(4.1) \quad \frac{\partial u}{\partial t} = \eta(u \times (\Delta u - \langle \nabla u, \eta \nabla u \rangle)) = \eta(u \times \Delta u) = \eta \partial_k (u \times \partial_k u)$$

or equivalently

$$(4.2) \quad \eta\left(u \times \frac{\partial u}{\partial t}\right) = -\Delta u + (\nabla u, \eta \nabla u)u$$

In hyperbolic coordinates this system reduces to

$$\begin{aligned} (\sinh \chi) \theta_t &= \Delta \chi - \sinh \chi \cosh \chi |\nabla \theta|^2 \\ (\sinh \chi) \chi_t &= -\operatorname{div}(\sinh^2 \chi \nabla \theta). \end{aligned}$$

Given a smooth solution to (4.2) we can easily construct a frame  $\{e\}$  in the Coulomb gauge and from section 2 the coordinates  $\partial_\ell u = q_\ell e$  satisfy

$$(4.3) \quad \begin{aligned} \mathcal{D}_t q_\ell &= i \mathcal{D}_k^2 q_\ell + i \langle q_\ell, i q_k \rangle q_k \\ \mathcal{D}_\ell q_k &= \mathcal{D}_k q_\ell \\ \partial_\ell a_j - \partial_j a_\ell &= f_{\ell j} = -\langle q_\ell, i q_j \rangle \\ \partial_\ell a_0 - \partial_t a_\ell &= f_{\ell 0} = -\langle q_\ell, i q_0 \rangle \\ \partial_k a_k &= 0 \end{aligned}$$

where  $q_0 = i \mathcal{D}_j q_j$ . Conversely given a solution to (4.3) one can repeat the construction given for the sphere in proposition 3.1 to obtain

**Proposition 4.1.** *Given a smooth solution to (4.3), a point  $m \in \mathbb{H}^2$  and a vector  $v_0 \in T_m \mathbb{H}^2$  with  $\|v_0\| = 1$ , then there exists a unique smooth solution to the Schrödinger maps equation*

$$\begin{aligned} \partial_t u &= \eta \partial_\ell (u \times \partial_\ell u) \\ u &\in \mathbb{H}^2 \subset (\mathbb{R}^3, \eta) \end{aligned}$$

and a frame  $\{e, Je\}$  for  $u^{-1}T\mathbb{H}^2$  such that  $u(0, 0) = m$ ,  $e(0, 0) = v_0$ , and (2.4) holds.

**Weak finite energy solutions from  $\mathbb{R}^{2+1}$  into  $\mathbb{H}^2$ .** The difficulty of the non compactness of  $\mathbb{H}^2$  appears in constructing weak solutions and it can be overcome by requiring the map  $u$  to converge to a point as  $x \rightarrow \infty$ . In particular, fix a point  $o \in \mathbb{H}^2$  and embed  $\mathbb{H}^2$  into Lorentz space with  $o \rightarrow (1, 0, 0)$ . We will consider maps  $u : \mathbb{R}^2 \rightarrow \mathbb{H}^2 \subset (\mathbb{R}^3, \eta)$  such that  $u \rightarrow (1, 0, 0)$  as  $x \rightarrow \infty$  and

$$\int (u_0 - 1)dx = \int (\cosh \chi - 1)dx < \infty.$$

This is a reasonable assumption since, like the energy

$$\|\nabla u\|_{L^2}^2 = \int |\nabla \chi|^2 + \sinh^2 \chi |\nabla \theta|^2 dx,$$

$\int (u_0 - 1)dx$  is also a conserved quantity of the Schrödinger maps.

Consider the Cauchy problem

$$(4.4) \quad \begin{aligned} \partial_t u &= \eta(u \times \Delta u) = \eta \partial_k (u \times \partial_k u) \\ \|\nabla u(0)\| &\in L^2(\mathbb{R}^2), \quad u_0(0) - 1 \in L^1(\mathbb{R}^2). \end{aligned}$$

Since the equation is in divergence form then it is easy to conclude that the weak limit of finite energy smooth solutions is a weak solution.

**Proposition 4.2.** *Let  $\{u_k\}$  be a sequence of smooth solutions to the Schrödinger maps equations (4.4) such that*

$$\|\nabla u_k\|_{L^2}^2 \leq C \quad \text{and} \quad \int (u_{0k} - 1)dx \leq C$$

*then  $\exists$  a subsequence that converges  $w^*$  to a weak solution of (4.4)  $u \in L^\infty(W_{\text{loc}}^{1,p})$  for any  $p < 2$ .*

*Proof.* From conservation of energy and the divergence form of the equation we have

$$\int \|\nabla u_k(t)\|^2 dx \leq C, \quad \int (u_{0k}(t) - 1)dx \leq C.$$

In hyperbolic coordinates we have

$$\int |\nabla \chi_k(t)|^2 dx \leq C \quad \int (\cosh \chi_k(t) - 1)dx \leq C.$$

Thus  $|\chi_k(t)|_{H^1(\mathbb{R}^2)} \leq C$  and from Moser-Trudinger inequality we have  $\forall$  compact sets  $\Omega \subset \mathbb{R}^2$

$$\int_{\Omega} e^{a\chi_k^2(t)} dx \leq C(\Omega, |\chi_k|_{H^1}) \leq C(\Omega),$$

for some  $a > 0$ . These bounds on  $\chi_k(t)$  imply the following Euclidean bounds on  $u_k(t)$

$$|u_k(t)|_{L^p(\Omega)} \leq C \quad \forall 1 \leq p < \infty$$

which in turn gives the Euclidean bounds

$$\begin{aligned} |u_k|_{L^\infty(W^{1,p}(\Omega))} &\leq C \quad \forall 1 \leq p < 2 \\ |\partial_t u_k|_{L^\infty(W^{-1,p}(\Omega))} &\leq C \quad \forall 1 \leq p < 2. \end{aligned}$$

Thus by going to a subsequence and a diagonalization argument we have  $\forall$  compact  $\Omega \subset \mathbb{R}^2$

$$\begin{aligned} u_k &\rightharpoonup u \in L^\infty(W^{1,p}(\Omega)) \quad 1 \leq p < 2 \\ u_k &\rightarrow u \in C(L^p(\Omega)) \quad 1 \leq p < \infty. \end{aligned}$$

and this implies

$$u_k \wedge \nabla u_k \rightharpoonup u \wedge \nabla u \in L^\infty(L^p(\Omega)) \quad 1 < p < 2.$$

From the above and Fatou's lemma we conclude that  $u$  is a weak solution of the Schrödinger maps equation with

$$\int (u_0(t) - 1) dx \leq C.$$

In order to show

$$\int \|\nabla u(t)\|^2 dx \leq C,$$

we take an isometric embedding  $\Phi : \mathbb{H}^2 \rightarrow \mathbb{R}^n$  satisfying  $\Phi(o) = 0$  and consider  $\tilde{u}_k = \Phi \circ u_k$ . Since  $\chi$  is the geodesic distance to  $o$  on  $\mathbb{H}^2$  and the intrinsic metric  $\|\cdot\|$  on  $T\mathbb{H}^2$  coincides with the metric induced by  $\Phi$ , we have  $|\tilde{u}_k(t)|_{H^1}^2 \leq C$ . Due to the pointwise convergence of  $u_k$  to  $u$ , we have  $\tilde{u}_k \rightarrow \tilde{u} \triangleq \Phi \circ u$  in  $H^1$  and

$$\|\nabla u\|_{L^2}^2 = |\nabla \tilde{u}|_{L^2}^2 \leq C.$$

□

To construct a sequence  $\{u_k\}$  such that  $\partial_t u_k - \eta \partial_\ell (u_k \times \partial_\ell u_k) \rightarrow 0$  in the sense of distribution and such that  $\|\nabla u_k(t)\|_{L^2} < C$  and  $|u_{0k}(t) - 1|_{L^1} < C$ , we introduce the parabolic perturbation

$$\varepsilon \partial_t u - \eta(u \times \partial_t u) = \Delta u - \langle \nabla u, \eta \nabla u \rangle u$$

and show by using the frame coordinates  $qs$  that the above equation has global smooth solutions with the desired bounds.

**Proposition 4.3.** *Given  $\varepsilon > 0$  and a function  $u_*(y) \in \mathbb{H}^2$  such that*

$$u_* - (1, 0, 0) \in L^1(\mathbb{R}^2), \quad \nabla u_* \in L^2(\mathbb{R}^2, T\mathbb{H}^2), \quad D\partial u_* \in L^2(\mathbb{R}^2, T\mathbb{H}^2),$$

*there exists a unique global classical solution to*

$$(4.5) \quad \begin{aligned} \varepsilon \partial_t u - \eta(u \times \partial_t u) &= \Delta u - \langle \nabla u, \eta \nabla u \rangle u \\ u(0, y) &= u_*(y) \in \mathbb{H}^2, \end{aligned}$$

*such that  $u \in \mathbb{H}^2$  and*

$$\int \|\nabla u(t)\|^2 dx \leq C, \quad \int (u_0(t) - 1) dx \leq C, \quad \varepsilon \int \int \|\partial_t u(t)\|^2 dx dt \leq C, \quad \text{and } \varepsilon \int \int \|D\partial u(t)\|^2 dx dt \leq C.$$

*Proof.* To show that solutions to equation (4.5) stay in  $\mathbb{H}^2$  we take the inner product of the equation with  $\eta u$  to obtain

$$\begin{aligned} \frac{1}{2} \varepsilon \partial_t \langle u, \eta u \rangle &= \frac{1}{2} \Delta \langle u, \eta u \rangle + \langle \nabla u, \eta \nabla u \rangle (1 + \langle u, \eta u \rangle) \\ \langle u, \eta u \rangle|_{t=0} &= -1. \end{aligned}$$

which implies that  $u(t) \in \mathbb{H}^2$ . To construct solutions let  $q$  be the Coulomb frame coordinates of  $\partial u$ , then

$$(4.6) \quad \begin{aligned} (\varepsilon - i) \mathcal{D}_t q_\ell &= \mathcal{D}_k^2 q_\ell - i \langle q_\ell, iq_k \rangle q_k \\ \Delta a_j &= -\partial_k \langle q_k, iq_j \rangle \\ \Delta a_0 &= \partial_\ell \langle q_\ell, q_0 \rangle. \end{aligned}$$

where  $(\varepsilon - i)q_0 = \mathcal{D}_j q_j$ . By standard fixed point argument system (4.6) has local smooth solutions for initial data in  $H^s$  for  $s$  sufficiently large. Moreover the system has a conserved energy which can be obtained by dividing the above equation by  $(\varepsilon - i)$ , multiplying by  $\bar{q}_\ell$  and taking the real part

$$\frac{d}{dt} \int \frac{1}{2} |q_\ell|^2 = \frac{-\varepsilon}{1 + \varepsilon^2} \int |\mathcal{D}_k q_\ell|^2 + |\langle q_k, iq_\ell \rangle|^2.$$

This implies global bounds

$$E_0 = \frac{1}{2} \int |q_\ell(t)|^2 dx + \frac{\varepsilon}{1 + \varepsilon^2} \int_0^t \int |\mathcal{D}_k q_\ell(t)|^2 + |\langle q_k(t), i q_\ell(t) \rangle|^2 dx dt.$$

We will obtain the  $H^1(\mathbb{R}^2)$  estimate on  $q$  by looking at  $\mathcal{D}q$ . In fact,

$$|\partial_k q_\ell|^2 \leq |\mathcal{D}_k q_\ell|^2 + |a_k q_\ell|^2.$$

Using the equation for  $a_k$  and Sobolev inequalities we conclude

$$|\partial q|_{L^2}^2 \leq |\mathcal{D}q|_{L^2}^2 + (|q|_{L^4}^2 |q|_{L^2})^2 \leq C(E_0) |\mathcal{D}q|_{L^2}^2$$

where the Sobolev inequality was used in the last step with  $\partial q$  replaced by  $\mathcal{D}q$  which is true due to the observation

$$|\partial_k |q|^2| = 2|\langle q, \mathcal{D}_k q \rangle| \leq 2|q| |\mathcal{D}_k q| \in L^1.$$

To obtain  $H^1$  bounds on  $q$  multiply equation (4.6) by  $\overline{\mathcal{D}_t q_\ell}$  and take the real part to obtain

$$(4.7) \quad \frac{1}{2} \frac{d}{dt} \int |\mathcal{D}_k q_\ell|^2 + \int \langle q_k, q_0 \rangle \langle \mathcal{D}_k q_\ell, i q_\ell \rangle + \langle q_\ell, i q_k \rangle \langle \mathcal{D}_t q_\ell, i q_k \rangle + \varepsilon |\mathcal{D}_t q_\ell|^2 = 0.$$

Writing  $\mathcal{D}$  for the spatial covariant derivative, the second term can be bounded by

$$\begin{aligned} \int |\langle q_k, q_0 \rangle \langle \mathcal{D}_k q_\ell, i q_\ell \rangle| dx &\leq |\mathcal{D}_k q_\ell|_{L^2} |q_0|_{L^6} |q|_{L^6}^2 \leq |\mathcal{D}_k q_\ell|_{L^2} |q_0|_{L^2}^{\frac{1}{3}} |\mathcal{D}q_0|_{L^2}^{\frac{2}{3}} |q|_{L^2}^{\frac{2}{3}} |\mathcal{D}q|_{L^2}^{\frac{4}{3}} \\ &\leq C(E_0) |\mathcal{D}q|_{L^2}^{\frac{7}{3}} |q_0|_{L^2}^{\frac{1}{3}} |\mathcal{D}q_0|_{L^2}^{\frac{2}{3}} \\ &\leq C(E_0) |\mathcal{D}q|_{L^2}^{\frac{8}{3}} |\mathcal{D}q_0|_{L^2}^{\frac{2}{3}} \quad \text{since } (\varepsilon - i)q_0 = \mathcal{D}_j q_j \\ &\leq \frac{1}{\sqrt{\varepsilon}} C(E_0) |\mathcal{D}q|_{L^2}^4 + \frac{\varepsilon}{4} |\mathcal{D}q_0|_{L^2}^2, \end{aligned}$$

and the third term by

$$\int |\langle q_\ell, i q_k \rangle \langle \mathcal{D}_t q_\ell, i q_k \rangle| dx \leq C |q|_{L^6}^3 |\mathcal{D}_t q_\ell|_{L^2} \leq \frac{C(E_0)}{\varepsilon} |\mathcal{D}q|_{L^2}^4 + \frac{\varepsilon}{4} |\mathcal{D}q_0|_{L^2}^2$$

Using the identities  $\mathcal{D}_k q_0 = \mathcal{D}_t q_k$ , the above inequality, and equation (4.7), we have

$$\frac{1}{2} \frac{d}{dt} \int |\mathcal{D}_k q_\ell|^2 + \frac{\varepsilon}{2} \int |\mathcal{D}_t q_\ell|^2 \leq \frac{C(E_0)}{\varepsilon} |\mathcal{D}q|_{L^2}^2 \int |\mathcal{D}_k q_\ell|^2.$$

Since by the energy identity  $\varepsilon \int_0^\infty |\mathcal{D}q(t)|_{L^2}^2 dt \leq E_0$  we obtain global bounds on  $\mathcal{D}q$

$$\int |\mathcal{D}_k q_\ell(t)|^2 + \varepsilon \int_0^t \int |\mathcal{D}_t q_\ell|^2 \leq C e^{\frac{C(E_0)}{\varepsilon^2}},$$

which implies the desired bound on  $\partial q$ .

Using this smooth solution  $q$  we can construct a global smooth solution  $u$  by means of proposition 4.1. To show that  $u$  belongs to the stated spaces we only need to show that

$$\int (u_0 - 1) dx \leq C.$$

In fact, equation (4.5) is equivalent to

$$\partial_t u = \frac{1}{1 + \varepsilon^2} (\eta \partial_k (u \times \partial_k u) + \varepsilon (\Delta u - \langle \nabla u, \eta \nabla u \rangle)).$$

Integrating the first component we obtain

$$\frac{d}{dt} \int (u_0 - 1) = -\frac{\varepsilon}{1 + \varepsilon^2} \int \|\nabla u\|^2 \cosh \chi dx \leq 0.$$

□

Weak solutions to Schrödinger maps into  $\mathbb{H}^2$  can be constructed as weak limits of the above solutions as  $\varepsilon \rightarrow 0$ .

**Theorem 4.1.** *Given  $u_* \in \dot{H}^1(\mathbb{R}^2, \mathbb{H}^2)$  such that  $\int (u_{*0} - 1)dx < \infty$  there exists a global weak solution to the Schrödinger maps system*

$$\frac{\partial u}{\partial t} = \eta \partial_\ell (u \times \partial_\ell u) \quad u(0) = u_*$$

with  $\|\nabla u\| \in L^\infty(L^2(\mathbb{R}^2))$ ,  $u \in C(L^2_{\text{loc}}(\mathbb{R}^2))$  and  $u_0 - 1 \in C(L^1(\mathbb{R}^2))$ .

*Proof.* Approximate the initial data by smooth functions  $u_{*k}$  so that  $\|\nabla u_{*k}\|_{L^2(\mathbb{R}^2)}^2$  and  $|u_{*k0} - 1|_{L^1(\mathbb{R}^2)}$  are uniformly bounded and the geodesic distance between  $u_{*k}(x)$  and  $u_*(x)$  on  $\mathbb{H}^2$  converges to 0 in  $L^2(\mathbb{R}^2)$ . Even though  $\mathbb{H}^2$  is not compact, this can still be done since  $\mathbb{H}^2$  is diffeomorphic to  $\mathbb{R}^2$ . In fact, using hyperbolic coordinates  $u_0 = \cosh \chi$ ,  $u_1 = \sinh \chi \cos \theta$  and  $u_2 = \sinh \chi \sin \theta$ , one can first approximate  $u_*$  by a map whose image is in a compact set and then modify it into a smooth map by standard methods. In the hyperbolic coordinates, the boundedness of  $\|\nabla u_{*k}\|_{L^2(\mathbb{R}^2)}^2$  and  $|u_{*k0} - 1|_{L^1(\mathbb{R}^2)}$  takes the form

$$\int (|\nabla \chi_{*k}|^2 + \sinh^2 \chi_{*k} |\nabla \theta_{*k}|^2) dx < C \quad \text{and} \quad \int (\cosh \chi_{*k} - 1) dx < C.$$

From Proposition 4.3 we have a global smooth solution to

$$(4.8) \quad \begin{aligned} \partial_t u_k + \frac{1}{k} \eta (u_k \times u_{k_t}) &= \eta \partial_\ell (u_k \times \partial_\ell u_k) \\ u_k(0, x) &= u_{*k}(0, x) \end{aligned}$$

such that

$$\begin{aligned} \int \|\nabla u_k(t)\|^2 dx &= \int |\nabla \chi_k(t)|^2 + \sinh^2 \chi_k(t) |\nabla \theta_k(t)|^2 dx \leq E_0 \\ \int (u_{0k}(t) - 1) dx &= \int (\cosh \chi_k - 1) dx \leq C. \end{aligned}$$

Thus  $\chi_k$  is bounded in  $L^\infty(H^1)$  and by Moser-Trudinger inequality  $\forall$  compact  $\Omega \subset \mathbb{R}^2$

$$\int_{\Omega} \exp\left(\alpha \frac{\chi_k^2(t)}{E_0}\right) dx \leq C(\Omega)$$

for some positive  $\alpha$ . This implies as in Proposition 4.2 that for a subsequence

$$\begin{aligned} \chi_k &\rightharpoonup \chi \quad \text{weak}^* \text{ in } L^\infty(H^1) \\ u_k &\rightharpoonup u \quad \text{weak}^* \text{ in } L^\infty(W_{\text{loc}}^{1,p}), \quad p \in [1, 2) \end{aligned}$$

where  $\|\nabla u(t)\|_{L^2(\mathbb{R}^2)} \leq C$  and  $\int (u_0(t) - 1)dx < C$ . Moreover for every cut off function  $\varphi \in C_0^\infty(\mathbb{R}^2 \times \mathbb{R})$

$$\partial_t(\varphi u_k) - \partial_t \varphi u_k + \frac{1}{k} \eta(\varphi u_k \times u_{k_t}) = \eta \partial_\ell(\varphi u_k \times \partial_\ell u_k) - \eta \partial_\ell \varphi u_k \times \partial_\ell u_k$$

which implies that  $\varphi u_k$  is bounded in  $H_{\text{loc}}^1(\mathbb{R}, W^{-1,p}(\mathbb{R}^2))$ ,  $1 \leq p < 2$ . Consequently we have a subsequence where

$$u_k \rightarrow u \quad \text{in } C(L^p) \quad \text{locally and a.e.}$$

These bounds allow us to pass to the limit in equation (4.8) to obtain

$$\begin{aligned} \partial_t u &= \eta \partial_\ell (u \times \partial_\ell u) \\ u(0, x) &= u_*(x) \end{aligned}$$

in the sense of distributions.  $\square$

## 5. EPILOGUE

The results stated in this paper can be generalized to compact Hermitian symmetric Kähler manifolds  $(M, g, J)$ . The equivalence of the Schrödinger maps system and the frame system can be done in an identical manner provided there exist global smooth Coulomb frames when the dimension of  $M$  is greater than 2. To show global existence of weak solutions in any space dimension we need to write the Schrödinger map system in divergence form. Therefore we have to restrict ourselves to the case when  $M$  has vanishing first cohomology group. In such a setting one uses the Killing vector fields to define weak solutions to the Schrödinger map system (SM)

$$\partial_t u = JD_k u \partial_k u.$$

in the following manner:

A vector field  $X \in TM$  is called Killing if  $\mathcal{L}_X g = 0$  and  $\mathcal{L}_X J = 0$ . Consequently if one considers the one form  $\omega$  defined by  $\omega(V) = g(JX, V)$  then  $\omega$  is a closed one form since  $M$  is Kähler. Moreover since the first cohomology vanishes  $\omega$  is exact. Whence there is a function  $f_X$  such that  $\omega = df_X$ , and for a solution  $u$  to the (SM) system we have

$$\begin{aligned} \partial_t f_X(u) &= \omega(u_t) = g(JX(u), u_t) = -\langle X(u), D_k \partial_k u \rangle \\ &= -\partial_k \langle X(u), \partial_k u \rangle + \langle D_k X(u), \partial_k u \rangle = -\partial_k \langle X(u), \partial_k u \rangle \end{aligned}$$

since  $X$  is Killing. If the  $2n$ -dimensional manifold  $M$  is compact and has  $m$  Killing vector fields  $\{X_a\}_{a=1}^m$  such that  $TM = \text{span}\{X_1, \dots, X_m\}$ , then the (SM) system is equivalent to

$$\partial_t f_{X_a}(u) = -\partial_k \langle X_a(u), \partial_k u \rangle, \quad a = 1, \dots, m.$$

**Remarks.** Though  $\mathbb{H}^2$  is not compact, actually the definition (4.1) of weak solutions of Schrödinger maps targeted on  $\mathbb{H}^2$  can also be viewed in this formulation with two Killing vector fields  $X_1 = J\nabla(\sinh \chi \cos \theta)$  and  $X_2 = J\nabla(\sinh \chi \sin \theta)$ .

Weak solutions in higher dimensions can also be constructed using the idea in [Sh88, Sh97, Fr96]. In this case we 1) embed  $M$  isometrically and equivariantly in  $\mathbb{R}^L$  [MS80], and 2) define  $d(u)$  the distance function from  $M$  to  $u$  and let  $\sigma > 0$  be so that  $d(u)$  is smooth in the tubular neighborhood  $O = \{u \in \mathbb{R}^L \mid d(u) < \sigma\}$  of  $M$ . Extend  $d$  globally as a smooth function

$$F(u) = \varphi(d)d + (1 - \varphi(d))\sigma, \quad d = d(u)$$

where  $\varphi \in C_0^\infty(-\sigma, \sigma)$  and  $0 \leq \varphi \leq 1$  and  $\varphi|_{[-\frac{\sigma}{2}, \frac{\sigma}{2}]} = 1$ . 3) Extend  $J$  smoothly to act on  $T\mathbb{R}^L$ . This can be achieved by first extending  $J(p)$  for  $p \in M$  to act on  $T\mathbb{R}^L = T_p M \oplus T_p M^\perp$  by first projecting on  $TM$  and then applying  $J$ . This operator can be extended to  $O$  as a constant in the directions normal to  $M$ , i.e.,  $\forall u \in O$  decompose  $u = p + n$  where  $p \in M$  and  $n \perp T_p M$  and define  $\tilde{J}(u) = J(p)$  acting on  $T_u O$ . Finally define  $\hat{J}(u) = \varphi(d(u))\tilde{J}(u)$  for  $u \in \mathbb{R}^L$ . It is clear that  $\hat{J}$  is skew-symmetric. 4) Solve the equation

$$\begin{aligned} \varepsilon \partial_t^2 u - \hat{J}(u) \partial_t u - \Delta u + \frac{1}{\delta} F(u) F'(u) &= 0 \\ u(0, x) &= u_*(x) \in M, \quad \partial_t u(0, x) = 0. \end{aligned}$$

which has conserved energy

$$\int \varepsilon |\partial_t u|^2 + |\nabla u|^2 + \frac{1}{\delta} |F(u)|^2 dx = \int |\nabla u_*|^2$$

By the energy method, the above equation has global solutions in  $H^1$ . For any Killing vector field  $X \in TM$ , from the equivariance of the embedding,  $X$  can be extended to a vector field  $X : \mathbb{R}^L \rightarrow T\mathbb{R}^L$  which generates an isometry on  $\mathbb{R}^L$  and satisfies  $X \perp F(u)F'(u)$ . Therefore, we have

$$\varepsilon \partial_t \langle X(u), \partial_t u \rangle - \langle X(u), \hat{J}(u) \partial_t u \rangle - \partial_k \langle X(u), \partial_k u \rangle = 0$$

By letting  $\delta \rightarrow 0$  we have from the energy identity  $u \rightarrow M$  in the  $L^2$  sense and whence the limit satisfies

$$\varepsilon \partial_t \langle X(u), \partial_t u \rangle + \partial_t f_X(u) - \partial_k \langle X(u), \partial_k u \rangle = 0.$$

Finally as  $\varepsilon \rightarrow 0$  we obtain the Schrödinger map system in conservation form.

**Theorem.** *Given  $u_* : \mathbb{R}^d \rightarrow M$  such that  $\nabla u_* \in L^2$ , the Schrödinger map system*

$$\begin{aligned} \partial_t u &= JD_k \partial_k u \\ u(0, x) &= u_*(x), \end{aligned}$$

*has a global weak solution such that  $u \in C(\mathbb{R}, L^2) \cap L^\infty(\dot{H}^1)$ .*

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