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# *Blowup with Small BV Data in Hyperbolic Conservation Laws*

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## **Abstract**

We construct weak solutions of  $3 \times 3$  conservation laws which blow up in finite time. The system is strictly hyperbolic at every state in the solution, and the data can be chosen to have arbitrarily small total variation. This is thus an example where Glimm's existence theorem fails to apply, and it implies the necessity of uniform hyperbolicity in Glimm's theorem. Because our system is very simple, we can carry out explicit calculations and understand the global geometry of wave curves.

## **1. Introduction**

The first author has recently constructed examples of hyperbolic conservation laws with solutions which blow up in finite time [8]. This blowup is effected by choosing a system whose nonlinear wave interactions are locally unstable (so reflected waves are stronger than incident waves), and in which the wavespeed increases with wave strength. As these wavespeeds increase, the waves interact more frequently and strengthen further. This coupled increase of strength and speed leads to infinitely many interactions in finite time and blowup of the solution. In that example the solution blows up on an entire interval and there is no possibility of continuing the solution beyond blowup time. We control the interactions by building a  $3 \times 3$  system consisting of different  $2 \times 2$  fluxes coupled by a linearly degenerate coordinate, carefully chosen so that interactions are unstable.

A simpler approach to the same construction is given in [7]: in a scalar conservation law (say Burgers' equation) we begin with an infinite compression wave which focuses at a single point  $(x_*, t_*)$ ; this is regarded as an instance of blowup which comes from infinite initial data. By symmetry we generate a backward infinite compression which focuses at  $(-x_*, t_*)$ . We now couple these infinite compressions through a third degenerate field, by adding stationary waves at  $x = \pm\ell$ , where  $\ell < x_*$ . If the fluxes are carefully chosen, the data can be taken to be constant

outside the interval  $-\ell < x < \ell$ , so that the initial data for the  $3 \times 3$  system is finite, while the solution blows up at  $(x, t_*)$  for any  $-\ell < x < \ell$ .

It turns out that the flux exterior to the strip  $-\ell < x < \ell$  can be taken to be linear. The full system is given by

$$\begin{aligned} u_t + f(u, s, w)_x &= 0, \\ s_t &= 0, \\ w_t - f(w, s, u)_x &= 0, \end{aligned} \tag{1}$$

where  $f$  is given by

$$f(p, s, q) = s p^2/2 + (1 - s) a p + (1 - s) b q,$$

and  $a$  and  $b$  are constants satisfying  $a < b < 0$ . The solution can be described by

$$\begin{aligned} u(x, t) &= \frac{x_* - x}{t_* - t}, \\ w(x, t) &= \frac{x_* + x}{t_* - t}, \end{aligned}$$

and  $s = 1$  inside the strip, and by  $s = 0$  with outgoing linear waves outside the strip, see Fig. 1.

By choosing the constants  $x_*$ ,  $t_*$  and  $\ell$ , we can control the time of existence and the size of the initial data, and indeed the data can be taken to have small total variation in both  $u$  and  $w$ , without reference to the flux, while the jumps in  $s$  are fixed. A natural question arises: at what point does the transition from stability of solutions with small data, given by GLIMM's theorem [1], to instability of solutions, as constructed above, take place? Moreover, there is an obvious candidate for a bifurcation parameter, namely the degenerate coordinate  $s$ , which can be regarded as an interpolation parameter between  $2 \times 2$  Burgers and linear fluxes.

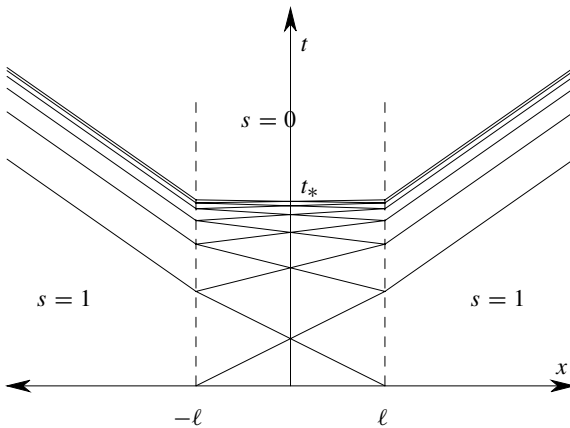


Fig. 1. Wave pattern in the characteristic plane.

In beginning to answer this question, we must understand the global structure of wave curves and interactions. We find that the system is not everywhere hyperbolic, and has an elliptic region with two components separated by the plane  $s = 1$ . Our data outside the strip lies in the interior of the hyperbolic region, while the data in the strip lies near the boundary of the elliptic region. A heuristic perturbation argument suggests that the interesting behavior is concentrated near  $s = 1$ , where the elliptic region collapses.

In order to focus our attention on this interesting behavior, we study the simplified system

$$\begin{aligned} u_t + \left(\frac{1}{2}u^2 - s w\right)_x &= 0, \\ s_t &= 0, \\ w_t - \left(\frac{1}{2}w^2 - s u\right)_x &= 0, \end{aligned} \tag{2}$$

which is quadratic homogeneous. The hyperbolic region is  $|u + w| \geq 2|s|$ , and because of the homogeneity, state space can be scaled and the behavior is focused at the origin. Moreover, this homogeneity carries over to the wave curves, which are self-similar. This simplified global geometry means that we can study solutions in some detail and get exact formulae for wave curves and interactions. Moreover, self-similarity implies that we can generate solutions analogous to those of Fig. 1, but we can scale the data to be small. We thus obtain the following theorem.

**Theorem 1.** *For any  $\varepsilon > 0$ , there are initial data  $\mathbf{v}_0(x)$  satisfying*

$$T.V.(\mathbf{v}_0) < \varepsilon,$$

*for which the weak solution  $\mathbf{v}(x, t)$  of (2) blows up in finite time. The solution is piecewise constant, with two stationary jump discontinuities and all other waves being shocks. The data consists of three constant states, and the solution blows up in amplitude on an entire interval. In particular, there is some time  $t_*$  depending explicitly on the data beyond which the solution cannot be defined.*

We conclude that although our  $3 \times 3$  system (2) is quite simple, it is one for which GLIMM's celebrated existence theorem fails [1]. This is surprising, but a careful reading of Glimm's theorem reveals no contradiction. Indeed, one of the conditions of that theorem is that the initial data be entirely contained in a neighborhood  $\mathcal{V}$  of some data point, and that the system be strictly hyperbolic throughout the neighborhood  $\mathcal{V}$ . Here, the data lies in an arbitrary neighborhood of the origin, and the system is hyperbolic at every state of the solution. However, there is no neighborhood of the origin on which the system is uniformly hyperbolic. We have thus demonstrated the necessity of uniform hyperbolicity in Glimm's theorem.

An immediate question that arises is to what extent the blowup depends on the discontinuous nature of the solution, or whether it can be extended to classical solutions of (2). By slightly modifying our construction, we show that blowup can occur with all shocks replaced by compressions. Thus the formation of shocks is a separate nonlinear phenomenon. However, we note that the two jump discontinuities are essential in our construction, the crucial point being that these jump discontinuities connect separate open components of the hyperbolic region.

**Theorem 2.** *There are initial data for system (2), for which the solution becomes infinite in finite time before any shocks form. The data can be chosen to have arbitrarily small total variation, and the solution again blows up on an entire interval.*

Technically, the proof of Glimm's theorem depends on asymptotic interaction estimates, which are obtained by means of Taylor expansions along wave curves. These Taylor expansions are valid only in neighborhoods, so they are unavailable around all points of our solution. Stated differently, in [5, 6] YOUNG showed that there is a critical threshold for the total variation of the initial data, provided the oscillation of the data is small enough. This threshold is determined by the geometry of wave curves, being given by the Lie bracket of vector fields. Here the eigenvectors are not differentiable at the origin, so in particular the threshold vanishes for any neighborhood of the origin.

Because the system is  $3 \times 3$ , we always have one real eigenvalue (in our case 0, corresponding to the stationary waves), whereas the other two eigenvalues may be real or complex. This zero eigenvalue is degenerate, and states can be connected by such a wave if they lie on the same stationary Hugoniot curve. These Hugoniot curves connect the hyperbolic and elliptic regions, and have a single discontinuity. The curve goes out to infinity on one side of this discontinuity, and reenters a separate hyperbolic region from infinity on the other side. This connection of disjoint components of the hyperbolic region by stationary waves is the source of instability in our example.

## 2. The system

### 2.1. Geometry of the hyperbolic region

In order to study the global properties of wave curves in system (1), we calculate the wavespeeds and eigensystem. Written in quasilinear form, our system becomes

$$\mathbf{v}_t + A(\mathbf{v}) \mathbf{v}_x = 0, \quad (3)$$

where  $\mathbf{v} = (u, s, w)^\dagger$ , and the flux matrix is

$$A(\mathbf{v}) = \begin{pmatrix} s u + (1-s)a & \frac{1}{2}u^2 - a u - b w & (1-s)b \\ 0 & 0 & 0 \\ -(1-s)b & -\frac{1}{2}w^2 + a w + b u & -s w - (1-s)a \end{pmatrix}.$$

The wavespeeds are the eigenvalues of  $A$ , which are  $\lambda = 0$  and

$$\lambda = s \frac{u-w}{2} \pm \sqrt{\left(s \frac{u+w}{2} + (1-s)a\right)^2 - ((1-s)b)^2}.$$

The calculation of eigenvalues naturally yields two surfaces in state space, namely the boundary of the elliptic region, given by

$$\left(s \frac{u+w}{2} + (1-s)a\right)^2 - ((1-s)b)^2 = 0 \quad (4)$$

and the nonstrictly hyperbolic surface, on which

$$\left(s \frac{u+w}{2} + (1-s)a\right)^2 - ((1-s)b)^2 = \left(s \frac{u-w}{2}\right)^2, \quad (5)$$

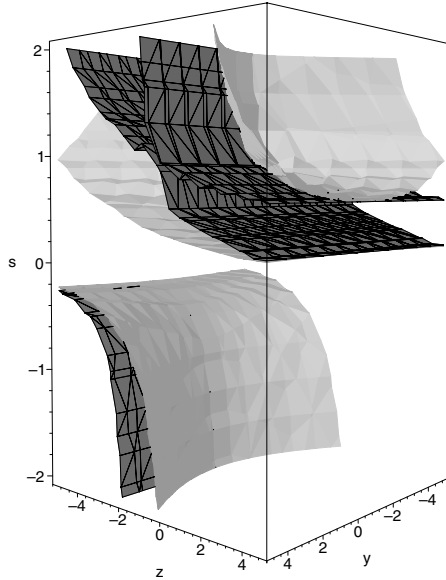
so that two of the wavespeeds coincide. These surfaces are easiest to visualize in rotated variables,

$$y = \frac{u-w}{2} \quad \text{and} \quad z = \frac{u+w}{2}, \quad (6)$$

where (4) is the cylinder over the slice of (5) with  $y = 0$ . These two surfaces are shown in Fig. 2; the dark surface is the boundary of the elliptic region.

Observe that the plane  $s = 0$  is contained entirely inside the hyperbolic region (indeed, the flux is linear for  $s = 0$ ), whereas the elliptic region degenerates at  $s = 1$ . Since these are the only  $s$ -values taken on in the solution, it is evident that the interesting behavior is focused around the plane  $s = 1$  at which the elliptic region collapses, rather than near the plane  $s = 0$  which is contained inside the (open) hyperbolic region.

Instead of working with (1) and (3), for which the hyperbolic region and wave curves are quite complicated, we investigate a simpler flux. Perturbing away from  $s = 0$ , which corresponds to the linear flux, we observe that we can divide the flux by  $s \neq 0$  to get simpler fluxes. By continuity, we expect that the behavior will not change much provided  $s$  stays away from  $s = 1$ . Also, the parameters  $a$  and  $b$  determine the angle between the surfaces in Fig. 2 without changing the topology of those surfaces, so that we can make convenient choices without affecting the structure of solutions.



**Fig. 2.** Elliptic region and nonstrictly hyperbolic surface.

## 2.2. Flux and coordinates

Following these observations, we simplify our system and look for similar structures and behavior. We thus perturb a pair of Burgers' equations by a linear flux, and add an extra equation for the perturbation variable. Since we wish to take advantage of symmetry, in (1) we take the scalar flux function to be

$$f(u, s, w) = \frac{1}{2} u^2 - s w, \quad (7)$$

so our simplified system becomes (2), namely

$$\begin{aligned} u_t + \left(\frac{1}{2} u^2 - s w\right)_x &= 0, \\ s_t &= 0, \\ w_t - \left(\frac{1}{2} w^2 - s u\right)_x &= 0. \end{aligned}$$

Note that this flux is quadratic and homogeneous. This homogeneity is also present in the eigensystem, and will allow us to scale the states  $(u, s, w)$  so that the data can be made arbitrarily small.

The eigensystem is calculated as follows: the flux matrix is

$$A(\mathbf{v}) = Df(\mathbf{v}) = \begin{pmatrix} u - w & -s \\ 0 & 0 & 0 \\ s & u & -w \end{pmatrix},$$

and the nonzero eigenvalues are easily calculated to be

$$\lambda = \frac{u - w}{2} \pm \sqrt{\left(\frac{u + w}{2}\right)^2 - s^2}.$$

It is again convenient to rotate the coordinates using (6), so that

$$u = y + z \quad \text{and} \quad w = z - y. \quad (8)$$

We can then rewrite the system (2) as

$$\begin{aligned} y_t + \left(\frac{1}{2} y^2 + \frac{1}{2} z^2 - s z\right)_x &= 0, \\ s_t &= 0, \\ z_t - (y s + y z)_x &= 0. \end{aligned} \quad (9)$$

Since the transformations (6) and (8) are linear, weak solutions are respected by the change of coordinates and the flux remains homogeneous. The transformed flux matrix is

$$B = \begin{pmatrix} y & -z & z - s \\ 0 & 0 & 0 \\ z + s & y & y \end{pmatrix}, \quad (10)$$

and the nonzero eigenvalues satisfy the equation

$$(y - \lambda)^2 - z^2 + s^2 = 0. \quad (11)$$

Equation (11) immediately yields the elliptic and hyperbolic regions: namely, the system is hyperbolic if  $z^2 > s^2$ , and the eigenvalues are 0 and

$$\lambda = y \pm \sqrt{z^2 - s^2}. \quad (12)$$

Note that there is a loss of strict hyperbolicity when

$$z^2 = s^2 + y^2,$$

which is a double cone with vertex at the origin. On this cone, the stationary wave-speed coincides with the forward wavespeed for  $y < 0$ , and with the backward wavespeed for  $y > 0$ . Note that the cone intersects the boundary of the elliptic region in the plane  $y = 0$ .

### 3. Wave curves

We now describe the waves of our system. We describe the global wave curves without reference to the strength of the waves, which may be arbitrary. Again homogeneity of the flux allows us to scale the curves, which makes the calculation much simpler.

#### 3.1. Simple waves

The forward and backward simple waves are integrals of the (right) eigenvectors; these eigenvectors can be written as

$$r_{\pm} = \begin{pmatrix} \pm(z - s) \\ 0 \\ \sqrt{z^2 - s^2} \end{pmatrix},$$

and the simple wave curves are their integrals. Using  $z$  itself as the parameter, and noting that the middle equation gives  $s$  constant, the simple wave curve reduces to

$$\frac{dy}{dz} = \pm \frac{z - s}{\sqrt{z^2 - s^2}}. \quad (13)$$

Integrating, we get

$$\begin{aligned} y_r - y_l &= \int_{z_b}^{z_a} \frac{z - s}{\sqrt{z^2 - s^2}} dz \\ &= \frac{\bar{z} - s}{\sqrt{\bar{z}^2 - s^2}} (z_a - z_b), \end{aligned} \quad (14)$$

where the subscripts denote right, left, ahead and behind states, respectively, and  $\bar{z}$  defined by

$$\frac{\bar{z} - s}{\sqrt{\bar{z}^2 - s^2}} = \bar{h} = \frac{1}{z_a - z_b} \int_{z_b}^{z_a} \frac{z - s}{\sqrt{z^2 - s^2}} dz \quad (15)$$

is a weighted average of  $z_a$  and  $z_b$ .

Alternatively, we can integrate directly to get

$$y_r - y_l = h_s(z_a) - h_s(z_b),$$

where

$$h_s(z) = \sqrt{z^2 - s^2} - s \operatorname{sgn}(z) \log(|z| + \sqrt{z^2 - s^2}).$$

After adjusting  $h_s$  by the constant  $s \operatorname{sgn}(z) \log |s|$  ( $z$  does not change sign), we have the scaling relation

$$h_s(z) = |s| h_1\left(\frac{z}{s}\right)$$

for  $s \neq 0$ . From (15), we get an explicit expression for  $\bar{z}$ , namely

$$\bar{h} = \frac{h_s(z_a) - h_s(z_b)}{z_a - z_b} \quad \text{and} \quad \bar{z} = \frac{1 + \bar{h}^2}{1 - \bar{h}^2} s. \quad (16)$$

The simple wave can be either a rarefaction or a compression, depending how the wavespeed changes across the wave. We thus check the type of the simple wave by differentiating (12) along (13), to get

$$\frac{d\lambda}{dz} = \pm \left( \frac{z - s}{\sqrt{z^2 - s^2}} + \frac{z}{\sqrt{z^2 - s^2}} \right).$$

Since our simple wave curve must be wholly contained in the hyperbolic region  $|z| > |s|$ , the two terms in parentheses have the same sign, which is that of  $z$ . For a rarefaction, the wavespeed increases from left to right across the wave. Thus for a forward wave to be a rarefaction,  $z$  increases from left to right if  $z > 0$ , while  $z$  decreases if  $z < 0$ ; the opposite is true for backward waves. In either case, we see that the wave is a rarefaction provided  $|z|$  increases from behind to ahead of the wave,  $|z_b| < |z_a|$ . Similarly, the wave is a compression if  $|z|$  decreases from behind to ahead, so  $|z_b| > |z_a|$ .

### 3.2. Shocks

The forward and backward wave families are genuinely nonlinear, so they will admit shock waves. We calculate the shocks by solving the Rankine-Hugoniot jump conditions [3, 4]. These are

$$\sigma \begin{bmatrix} [y] \\ [s] \\ [z] \end{bmatrix} = \begin{bmatrix} \left( \frac{1}{2} y^2 + \frac{1}{2} z^2 - s z \right) \\ 0 \\ y s + y z \end{bmatrix}, \quad (17)$$

where  $[\cdot]$  denotes the jump in a quantity, and  $\sigma$  the shock speed. Since the flux is quadratic homogeneous, we can rewrite (17) as an eigenvalue problem, namely

$$\sigma \begin{pmatrix} [y] \\ [s] \\ [z] \end{pmatrix} = \bar{B} \begin{pmatrix} [y] \\ [s] \\ [z] \end{pmatrix},$$

with the matrix

$$\bar{B} = \begin{pmatrix} \bar{y} & -\bar{z} & \bar{z} - \bar{s} \\ 0 & 0 & 0 \\ \bar{z} + \bar{s} & \bar{y} & \bar{y} \end{pmatrix},$$

and  $\bar{\cdot}$  is the arithmetic average of the left and right states,

$$\bar{z} = \frac{z_l + z_r}{2} \quad \text{and} \quad \bar{y} = \frac{y_l + y_r}{2}. \quad (18)$$

Comparing with (10), we see that the shock speed is

$$\sigma = \bar{y} \pm \sqrt{\bar{z}^2 - \bar{s}^2},$$

and that the jump is some multiple of  $\bar{r}$ ,

$$\begin{pmatrix} [y] \\ [s] \\ [z] \end{pmatrix} = \mu \begin{pmatrix} \pm(\bar{z} - \bar{s}) \\ 0 \\ \sqrt{\bar{z}^2 - \bar{s}^2} \end{pmatrix},$$

or equivalently  $\bar{s} = s_l = s_r = s$  and

$$[y] = \pm \frac{\bar{z} - s}{\sqrt{\bar{z}^2 - s^2}} [z]. \quad (19)$$

Fixing one of the states (say the left state), the curve is fully described with either  $z_r$ ,  $\bar{z}$  or  $[z] = z_r - z_l$  as the parameter.

The entropy condition implies that the shock is admissible only if the characteristic speeds on either side of the shock impinge on it,

$$\lambda_l > \sigma > \lambda_r.$$

After using (12) and (19), this reduces to the condition

$$\pm \frac{\bar{z} - s}{\sqrt{\bar{z}^2 - s^2}} [z] \pm \left( \sqrt{z_r^2 - s^2} - \sqrt{z_l^2 - s^2} \right) < 0,$$

and by the mean value theorem, this is

$$\pm \left( \frac{\bar{z} - s}{\sqrt{\bar{z}^2 - s^2}} + \frac{\tilde{z}}{\sqrt{\tilde{z}^2 - s^2}} \right) [z] < 0, \quad (20)$$

for some  $\tilde{z}$  between  $z_l$  and  $z_r$ . Since  $z$  cannot change sign, each of  $z_l$ ,  $z_r$ ,  $\bar{z}$  and  $\tilde{z}$  have the same sign, and both terms in parentheses have that sign. Thus (20) is equivalent to the condition  $\pm[|z|] < 0$ , which for forward and backward waves translates to

$$|z_a| - |z_b| < 0, \quad \text{that is} \quad |z_b| > |z_a|.$$

As in (14), we can describe both forward and backward shocks by

$$y_r - y_l = \frac{\bar{z} - s}{\sqrt{\bar{z}^2 - s^2}} (z_a - z_b), \quad (21)$$

with  $|z_b| > |z_a|$ , and  $\bar{z} = (z_a + z_b)/2$ . Note that the entropy condition holds for shocks of arbitrary size, and is consistent with our condition for rarefactions and compressions.

### 3.3. Stationary waves

We now describe the stationary waves, which are those with zero speed, corresponding to our “entropy” field  $s$ . Since the field is linearly degenerate, the simple wave curves coincide with the Hugoniot locus, which is the solution of (17) with  $\sigma = 0$ , giving two algebraic equations,

$$\begin{aligned} \left[ \frac{1}{2} y^2 + \frac{1}{2} z^2 - s z \right] &= 0, \\ [y s + y z] &= 0. \end{aligned}$$

To solve these, it is convenient to rewrite them as

$$\begin{aligned} \frac{1}{2} y^2 + \frac{1}{2} z^2 - s z &\equiv \alpha \equiv \frac{1}{2} y_0^2 + \frac{1}{2} z_0^2 - s_0 z_0, \\ y s + y z &\equiv \beta \equiv y_0 s_0 + y_0 z_0, \end{aligned} \quad (22)$$

where the subscript  $\cdot_0$  refers to a fixed left or right state, which defines the constant fluxes  $\alpha$  and  $\beta$ , so that we are solving for the adjacent (unsubscripted) state. Eliminating  $s$ , we get

$$\frac{1}{2} y^3 + \frac{3}{2} y z^2 = \alpha y + \beta z,$$

which is a quadratic in  $z$ . Solving, we get

$$3 y z = \beta \pm \sqrt{\beta^2 + 6 \alpha y^2 - 3 y^4}, \quad (23)$$

and from (22),

$$3 y s = 2 \beta \mp \sqrt{\beta^2 + 6 \alpha y^2 - 3 y^4}. \quad (24)$$

Since we are interested only in the hyperbolic region, we require that

$$0 < 9 y^2 (z^2 - s^2) = -3 \beta^2 \pm 6 \beta \sqrt{\beta^2 + 6 \alpha y^2 - 3 y^4}, \quad (25)$$

so we can choose  $\text{sgn}(\beta)$  in (23), which yields

$$\begin{aligned} 3 y z &= \beta + \text{sgn}(\beta) \sqrt{\beta^2 + 6 \alpha y^2 - 3 y^4}, \\ 3 y s &= 2 \beta - \text{sgn}(\beta) \sqrt{\beta^2 + 6 \alpha y^2 - 3 y^4}. \end{aligned} \quad (26)$$

Here we regard  $y$  as the parameter across the stationary wave, taking values in the range

$$-\sqrt{\alpha + \sqrt{\alpha^2 + \beta^2/3}} < y < \sqrt{\alpha + \sqrt{\alpha^2 + \beta^2/3}}.$$

By (25), the curve is in the hyperbolic region if

$$\beta^2/4 + 2 \alpha y^2 - y^4 > 0,$$

which becomes

$$-\sqrt{\alpha + \sqrt{\alpha^2 + \beta^2/4}} < y < \sqrt{\alpha + \sqrt{\alpha^2 + \beta^2/4}}.$$

Note that curve (26) is not continuous at  $y = 0$ , and as  $y$  crosses 0, the Hugoniot locus jumps from one component of the hyperbolic region into the other. Also,  $\alpha$  and  $\beta$  are homogeneous quadratic in  $y_0$ ,  $s_0$  and  $z_0$ , and so also are all terms in (26). This means that the curves are invariant under a uniform scaling of state space, as were the simple wave and shock curves. Fig. 3 shows two representative curves in the  $s$ - $z$  plane, parametrized by  $y$ . Recall that the elliptic region is  $|z| < |s|$ , and note that the curves end on the line  $z = s/2$ .

In our construction, we will be considering a fixed ‘‘entropy’’ jump, so  $s_l$  and  $s_r$  will be fixed. Fixing one state, we look for the value of  $y$  which gives a prescribed value of  $s$ . In general this means solving the quartic (24) for  $y$ , but we can simplify this by prescribing  $s = 0$ , corresponding to a pair of Burgers’ equations inside the strip  $-\ell < x < \ell$ . Setting  $s = 0$  in (24) yields

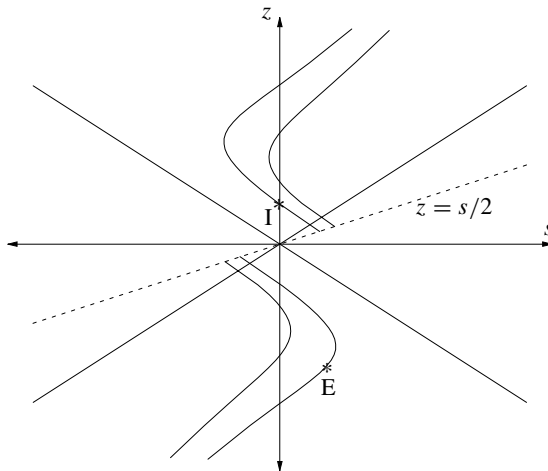
$$y^4 - 2\alpha y^2 + \beta^2 = 0,$$

which has solutions

$$y = \pm \sqrt{\alpha \pm \sqrt{\alpha^2 - \beta^2}}, \quad (27)$$

and plugging  $s = 0$  into (22), we get  $z = \beta/y$ . In order for such a jump to be possible, we require that the flux in (22) satisfies

$$\alpha > |\beta| > 0. \quad (28)$$



**Fig. 3.** Stationary Hugoniot curves.

We choose the signs in (27) by choosing

$$z = \beta/y > |y| > 0$$

inside the strip  $-\ell < x < \ell$ , which by (8) corresponds to both  $u$  and  $w$  being positive there. This becomes  $y^2 < |\beta| < \alpha$ , which holds only for the negative sign inside the root. We must also have  $\text{sgn}(y) = \text{sgn}(\beta)$ , so that in (27) we can take

$$y = \text{sgn}(\beta) \sqrt{\alpha - \sqrt{\alpha^2 - \beta^2}} \quad \text{and} \quad z = \beta/y. \quad (29)$$

This solution is labeled ‘I’ in Fig. 3, where ‘E’ labels the state across the jump (with  $s > 0$ ).

We get an equivalent expression for the solution by substituting  $s = 0$  directly into (22) and solving: this gives

$$\frac{\alpha + \beta}{2} = \left(\frac{y + z}{2}\right)^2 \quad \text{and} \quad \frac{\alpha - \beta}{2} = \left(\frac{z - y}{2}\right)^2,$$

where we require that  $\alpha > |\beta|$ , and sign considerations as above yield the solution

$$z = \sqrt{\frac{\alpha + \beta}{2}} + \sqrt{\frac{\alpha - \beta}{2}} \quad \text{and} \quad y = \sqrt{\frac{\alpha + \beta}{2}} - \sqrt{\frac{\alpha - \beta}{2}}. \quad (30)$$

This solution is obvious in the variables  $(u, w)$  if we plug  $s = 0$  into the flux (7).

#### 4. The construction

We now use the wave curve information to construct a solution similar to that of Fig. 1 which blows up. If we carefully enforce reflection symmetry, we need only analyze one type of interaction, namely that which occurs on the line  $x = \ell$ . We construct two solutions, the first consisting of shocks, and the second being smooth away from the jump discontinuities at  $x = \pm\ell$ .

The reflection symmetry is used as follows: suppose that a (weak) solution of our original system (1) is defined on the right half-plane  $x > 0$ , and suppose also that

$$u(0, t) = w(0, t) \quad \text{for almost every } t > 0.$$

This solution can then be extended to a weak solution on all of  $t > 0$  by setting

$$u(x, t) = w(-x, t), \quad w(x, t) = u(-x, t)$$

and  $s(x, t) = s(-x, t)$  for  $x < 0$ . In the rotated coordinates (6), this reflection is

$$y(x, t) = -y(-x, t), \quad z(x, t) = z(-x, t) \quad (31)$$

and  $s(x, t) = s(-x, t)$  for  $x < 0$ , where now the necessary condition is simply

$$y(0, t) = 0 \quad \text{for } t > 0. \quad (32)$$

We thus concentrate on the wave interactions at the right edge  $x = \ell$  and enforce this symmetry condition (32).

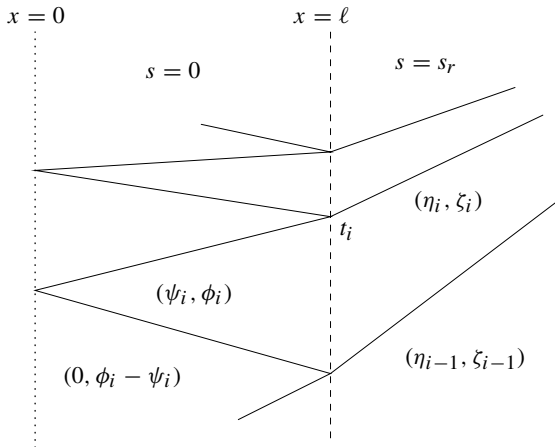
Note in Fig. 1 that on the right of the strip,  $x > \ell$ , any unbounded wave must be an outgoing forward wave, i.e., any infinite backward wave could be traced back to  $t = 0$ , yielding some unbounded component in the initial data.

Our strategy for constructing the solution which blows up is as follows: first, we look for a sequence of outgoing waves parametrized by  $\zeta = z_b$  in the region  $x > \ell$ , and along which  $|\zeta| \rightarrow \infty$ . Such waves are described by (14) or (21), with  $s = s_r$  constant. Each such state  $(y(\zeta), s_r, \zeta)$  occurs at the boundary of the strip at some (as yet unknown) point  $(\ell+, t)$ . These states in turn determine the fluxes  $\alpha(\zeta)$  and  $\beta(\zeta)$  given in (22). We now solve the jump discontinuity at  $x = \ell$ , enforcing  $s = 0$  inside the strip: this is (29). Next, fill in the interior solution by tracing the forward and backward (decoupled Burgers') characteristics from  $x = \ell-$  to  $x = 0$ . Note that these interior states are also parametrized by  $\zeta$ . Our final step is to choose times  $t(\zeta)$  at which these states are taken on at the jump discontinuity, which is accomplished by enforcing our symmetry condition (32). This defines a functional relation for  $t(\zeta)$ , and we get finite time blowup if  $t(\zeta) \rightarrow t_*$  as  $|\zeta| \rightarrow \infty$ . We then choose the unstable initial data by taking the trace of the solution on  $t = 0$ .

#### 4.1. Blowup with Shocks

Referring to Fig. 4, we construct a solution with shocks as follows: starting with a constant initial state  $(y_r, s_r, z_r)$  in the region  $x > \ell$  which will be chosen later, set  $\zeta_0 = z_r$  and  $\eta_0 = y_r$ . Now suppose we have a sequence  $\zeta_i$ , having the same sign as  $\zeta_0$  and satisfying

$$|\zeta_{i+1}| > |\zeta_i| \quad \text{for each } i \geq 1,$$



**Fig. 4.** Construction with shocks.

which parameterizes the outgoing shocks. Equation (21) determines the state  $\eta_i = y(\zeta_i)$  behind each of these shocks, namely

$$\eta_{i+1} = \eta_i + \frac{\bar{\zeta}_i - s_r}{\sqrt{\bar{\zeta}_i^2 - s_r^2}} (\zeta_{i+1} - \zeta_i). \quad (33)$$

These states in (22) determine the fluxes which are constant across the jump discontinuity,

$$\begin{aligned} \alpha_i &= \frac{1}{2} \eta_i^2 + \frac{1}{2} \zeta_i^2 - s_r \zeta_i, \\ \beta_i &= \eta_i s_r + \eta_i \zeta_i, \end{aligned} \quad (34)$$

and which must satisfy (28). Equations (30) give the state  $(\psi_i, 0, \phi_i)$  across the jump on the inside of the strip  $x < \ell$ , namely

$$\begin{aligned} \psi_i &= y = \left( \sqrt{\alpha_i + \beta_i} - \sqrt{\alpha_i - \beta_i} \right) / \sqrt{2}, \\ \phi_i &= z = \left( \sqrt{\alpha_i + \beta_i} + \sqrt{\alpha_i - \beta_i} \right) / \sqrt{2}. \end{aligned} \quad (35)$$

In Fig. 3, the exterior state  $\zeta_i$  is labeled ‘E’, and the interior state  $\phi_i$  is labeled ‘I’; by scaling, this picture holds for any  $i$ .

We now resolve the solution inside the strip in such a way that we can impose the reflection symmetry (31). We have  $s = 0$  in the strip, so we are dealing with a pair of uncoupled Burgers’ equations in the variables  $u$  and  $w$ . We take a piecewise constant solution with a forward shock meeting the jump discontinuity and a backward reflected shock, as in Fig. 4. The state ahead of the incoming forward shock is  $(\psi_i, 0, \phi_i)$ , and that behind the reflected shock is  $(\psi_{i+1}, 0, \phi_{i+1})$ . For the symmetry condition, we want the state between these two waves to have  $y = 0$ . Since  $w = z - y$  is constant across the incident shock and  $u = y + z$  is constant across the reflected shock, the intermediate state has  $y = 0$  and

$$z = \phi_i - \psi_i = \psi_{i+1} + \phi_{i+1}, \quad (36)$$

which implicitly defines  $\zeta_{i+1}$  in terms of  $\zeta_i$ . Using (35), this reduces to the equation

$$\alpha_{i+1} + \beta_{i+1} = \alpha_i - \beta_i, \quad (37)$$

which together with (33) and (34) determines  $\zeta_{i+1}$  (and  $\eta_{i+1}$ ) from  $\zeta_i$  and  $\eta_i$ .

Finally, in order to be able to use reflection symmetry, we must ensure that the incoming shock meets the previous reflected shock exactly at  $x = 0$ ; this determines the times at which the shocks meet the line  $x = \ell$ , and hence the exact points  $(\ell, t_i)$  from which the outgoing shocks outside the strip emanate. Since the reflected shock has speed  $\bar{w} = \bar{z} - \bar{y}$ , the reflected wave leaving  $(\ell, t_i)$  meets  $x = 0$  at time

$$\bar{t} = t_i + \frac{\ell}{\bar{z} - \bar{y}} = t_i + \frac{2\ell}{\phi_{i+1} + \phi_i - \psi_{i+1}}, \quad (38)$$

and similarly the forward characteristic from  $(0, \bar{t})$  with speed  $\bar{u} = \bar{y} + \bar{z}$  meets  $x = \ell$  at time

$$t_{i+1} = \bar{t} + \frac{\ell}{\bar{y} + \bar{z}} = \bar{t} + \frac{2\ell}{\psi_{i+2} + \phi_{i+2} + \phi_{i+1}}, \quad (39)$$

Equations (38) and (39) determine an increasing sequence of times  $t_i$  of interaction explicitly once the  $\zeta_i$  are known. Our finite time blowup occurs if both  $|\zeta_i| \rightarrow \infty$  (or  $\phi_i \rightarrow \infty$ ) and  $t_i \rightarrow t_* < \infty$ , and we have blowup for small data provided we can choose the initial values arbitrarily small.

**Theorem 3.** *Let  $y_r > 0$ ,  $s_r > 0$  and  $z_r < -s_r$  be otherwise arbitrary, set*

$$z_m = \sqrt{(y_r - z_r)^2 - 2s_r(y_r + z_r)} > 0, \quad (40)$$

and define initial data for (9) by

$$(y_0, s_0, z_0) = \begin{cases} (-y_r, s_r, z_r) & \text{for } x < -\ell, \\ (0, 0, z_m) & \text{for } -\ell < x < \ell, \text{ and} \\ (y_r, s_r, z_r) & \text{for } x > \ell. \end{cases}$$

The corresponding weak solution is piecewise constant and blows up in amplitude on the interval  $(-\ell, \ell)$  in finite time  $t_*$ , where

$$t_* \leq \frac{4\ell}{|z_r|} \frac{1+\gamma}{\gamma} \quad \text{and} \quad \gamma = \frac{z_r + s_r}{s_r z_r} \min \left\{ -\frac{4}{5}(z_r + s_r), y_r \right\}.$$

Since the initial data can be chosen to have arbitrarily small total variation, Glimm's existence theorem fails to apply.

**Proof.** We inductively define  $\zeta_{i+1}$  in terms of  $\zeta_i$  as described above, using (33), (34) and (37). We must show that  $|\zeta_i|$  is an increasing unbounded sequence, and that our consistency condition

$$\alpha_i > |\beta_i| \quad (41)$$

holds for each  $i$ . According to our construction,  $z = \phi_i > 0$  inside the strip  $-\ell < x < \ell$ , and  $z$  is an even function of  $x$ . Since the total integral  $\int z dx$  is conserved and  $\phi_i$  grows, we must have  $\zeta_{i+1} < \zeta_i < 0$  for each  $i$ . Thus taking  $\zeta_i < -s$  in (33), we see that  $\eta_{i+1} - \eta_i > 0$ , and to ensure growth we take also  $\eta_0 > 0$ . With the conditions

$$\eta_i > 0 \quad \text{and} \quad \zeta_i < -s < 0 \quad (42)$$

in (34), we see that  $\beta_i < 0$  and

$$\alpha_i + \beta_i = \frac{1}{2} (\zeta_i + \eta_i)^2 + s_r (\eta_i - \zeta_i) > 0,$$

so that (41) is always satisfied.

We show in Lemma 1 below that (37) has a unique solution  $(\eta_{i+1}, \zeta_{i+1})$  satisfying

$$\zeta_{i+1} < (1 + \gamma) \zeta_i < 0 \quad \text{and} \quad \eta_{i+1} - \eta_i > \zeta_i - \zeta_{i+1},$$

where  $\gamma > 0$  is given in (73) below, and therefore

$$\zeta_i < (1 + \gamma)^i \zeta_0 \rightarrow -\infty \quad \text{and} \quad \eta_i > -(1 + \gamma)^i \zeta_0 \rightarrow \infty \quad (43)$$

as  $i \rightarrow \infty$ . It follows from (34), (35) that  $\alpha_i, \beta_i$  and  $\phi_i$  are also unbounded, so that the solution blows up on the entire strip  $-\ell < x < \ell$ .

Next, we set  $t_0 = 0$  and define  $t_i$  inductively by (38) and (39). We must show that  $t_i$  is a bounded increasing sequence. Since  $\phi_i > 0$ , we have

$$\begin{aligned} t_{i+1} &= t_i + \frac{2\ell}{\phi_{i+1} + \phi_i - \psi_{i+1}} + \frac{2\ell}{\psi_{i+2} + \phi_{i+2} + \phi_{i+1}} \\ &< t_i + \frac{2\ell}{\phi_{i+1} - \psi_{i+1}} + \frac{2\ell}{\psi_{i+2} + \phi_{i+2}} \\ &= t_i + \frac{2\sqrt{2}\ell}{\sqrt{\alpha_{i+1} - \beta_{i+1}}}, \end{aligned}$$

where we have used (35) and (36). To complete the proof, we thus need to show that the series

$$\sum_j \frac{2\sqrt{2}\ell}{\sqrt{\alpha_j - \beta_j}} \quad (44)$$

converges. From (34) and (42), we have

$$\alpha_i - \beta_i > \alpha_i > \frac{1}{2} \zeta_i^2$$

so that

$$\frac{1}{\sqrt{\alpha_j - \beta_j}} < \frac{\sqrt{2}}{|\zeta_j|} < \frac{\sqrt{2}}{|\zeta_0|} \frac{1}{(1 + \gamma)^j},$$

using (43). We now sum the series (44) to get

$$t_i < \sum_j \frac{4\ell}{|\zeta_0| (1 + \gamma)^j} = \frac{4\ell}{|\zeta_0|} \frac{1 + \gamma}{\gamma},$$

so that all interactions take place, and blowup occurs, in finite time.

Finally, we explicitly describe the initial data. In the region  $x > \ell$ , we choose data satisfying  $s_r = s > 0$ ,

$$y_r = \eta_0 > 0 \quad \text{and} \quad z_r = \zeta_0 < -s.$$

These determine the fluxes  $\alpha_0$  and  $\beta_0$  in (34), and (35) gives the states  $\phi_0$  and  $\psi_0$  across the stationary jump. Now we use (36) to determine the initial data inside the strip, namely  $s_m = 0$ ,  $y_m = 0$  and  $z_m = \phi_0 - \psi_0$ . Explicitly, we have

$$z_m = \sqrt{2(\alpha_0 - \beta_0)} = \sqrt{(\eta_0 - \zeta_0)^2 - 2s(\eta_0 + \zeta_0)},$$

which is (40). This completely specifies the initial data, recalling that  $s$  and  $z$  are even functions and  $y$  is an odd function.  $\square$

#### 4.2. Blowup without shocks

To construct a solution without shocks, we need a continuous analog of the above calculation. Since the outgoing wave is a smooth compression, we parameterize it by the smooth variable  $\zeta$ . We define the state at points across the wave by (14),

$$\eta(\zeta) = y(\zeta) = \eta_0 + h(\zeta) - h(\zeta_0), \quad (45)$$

and define the corresponding fluxes by (22),

$$\begin{aligned} \alpha(\zeta) &= \frac{1}{2} \eta(\zeta)^2 + \frac{1}{2} \zeta^2 - s_r \zeta, \\ \beta(\zeta) &= \eta(\zeta) s_r + \eta(\zeta) \zeta. \end{aligned}$$

The state across the jump discontinuity is then given by (30),

$$\begin{aligned} \psi(\zeta) &= y = \left( \sqrt{\alpha(\zeta) + \beta(\zeta)} - \sqrt{\alpha(\zeta) - \beta(\zeta)} \right) / \sqrt{2}, \\ \phi(\zeta) &= z = \left( \sqrt{\alpha(\zeta) + \beta(\zeta)} + \sqrt{\alpha(\zeta) - \beta(\zeta)} \right) / \sqrt{2}, \end{aligned} \quad (46)$$

where we again require the consistency condition (28).

These functions describe the states across the outgoing wave and across the jump discontinuity. We must now resolve the solution inside the strip  $0 \leq x < \ell$  and identify the point at which the state parametrized by  $\zeta$  meets the jump discontinuity. This is done by defining a function  $t(\zeta)$ , representing the start of the outgoing characteristic corresponding to  $\zeta$ . We find a defining relation for the function  $t(\zeta)$  by again enforcing the condition  $y(0, t) = 0$ . The reflected (backward) characteristic from  $(\ell-, t)$  has (constant) speed  $w = z - y$ , so it meets  $x = 0$  at time

$$\bar{t} = t + \frac{\ell}{z - y}, \quad (47)$$

and  $w$  is constant along this characteristic. We take  $y = \psi(\zeta)$  and  $z = \phi(\zeta)$  as the state at  $(\ell-, t)$ , and follow the characteristic to  $(0, \bar{t})$ , at which point we want  $y = 0$ , so we set

$$z(0, \bar{t}) = \phi(\zeta) - \psi(\zeta). \quad (48)$$

Similarly, the forward characteristic from  $(0, \bar{t})$  has constant speed  $u = y + z$ , so it meets the jump discontinuity at  $(\ell, t_+)$ , where

$$t_+ = \bar{t} + \frac{\ell}{z + y} = \bar{t} + \frac{\ell}{\phi(\zeta) - \psi(\zeta)}, \quad (49)$$

and we have used (48) and  $y(0, \bar{t}) = 0$ . Since  $u = y + z$  is constant along this characteristic, we must also have

$$y(\ell, t_+) + z(\ell, t_+) = \phi(\zeta) - \psi(\zeta), \quad (50)$$

where  $t_+$  is given by (49) and (47)

$$t_+ = t + \frac{2\ell}{\phi(\zeta) - \psi(\zeta)}. \quad (51)$$

In order to have a consistently defined continuous solution, we must also have

$$y(\ell, t_+) = \psi(\zeta_+) \quad \text{and} \quad z(\ell, t_+) = \phi(\zeta_+) \quad (52)$$

for some parameter value  $\zeta_+$ . We now have a defining relation for the function  $t(\zeta)$ , the time at which the outgoing characteristic corresponding to  $\zeta$  leaves  $x = \ell$ , given by (50), (52) and (51): for any  $\zeta$ , defining  $\zeta_+$  implicitly by the equation

$$\psi(\zeta_+) + \phi(\zeta_+) = \phi(\zeta) - \psi(\zeta), \quad (53)$$

we must have

$$t(\zeta_+) = t(\zeta) + \frac{2\ell}{\phi(\zeta) - \psi(\zeta)}. \quad (54)$$

After use of (46), (53) simplifies to

$$\alpha(\zeta_+) + \beta(\zeta_+) = \alpha(\zeta) - \beta(\zeta), \quad (55)$$

which implicitly determines  $\zeta_+$ .

Now, we are ready to prove the previously stated Theorem 2.

**Theorem 2.** *There are initial data for system (2), for which the solution becomes infinite in finite time before any shocks form. The data can be chosen to have arbitrarily small total variation, and the solution again blows up on an entire interval.*

**Proof.** The proof follows exactly that of Theorem 3, the main difference being the use of a continuous monotone function  $t(\zeta)$  in place of the sequence  $t_i$ . We shall show that (54) has a continuous monotone solution  $t(\zeta)$ . We must also check that the characteristics are consistent after reflection: this means that the absolute speeds of the forward and backward characteristics meeting at  $(0, \bar{t})$  must coincide. Since these speeds are  $y + z$  and  $z - y$ , respectively, this is implied by (53).

We make the change of variables  $x = -1/\zeta > 0$ , and define

$$f(x) = \frac{-1}{\zeta_+} < x,$$

where  $\zeta_+$  is the solution of (55) given in Lemma 1 below. Also, define

$$h(x) = \frac{2\ell}{\phi(\zeta) - \psi(\zeta)} \quad \text{and} \quad \tau(x) = t(\zeta).$$

Then  $f$  is continuous and monotone,  $h$  is monotonic with  $h(x) \rightarrow 0$  as  $x \rightarrow 0$ , and the functional equation (54) can be written as

$$\tau(f(x)) = \tau(x) + h(x).$$

According to [2], Theorem 2.3.6, this has the monotone solution

$$\tau(x) = c - \sum_{j=0}^{\infty} (h(f^j(x)) - h(f^j(x_0))), \quad (56)$$

where  $x_0$  is arbitrary and  $c = \tau(x_0)$  uniquely determines the solution up to a constant. This constant corresponds to a choice of initial time  $t_0$ .

The solution (56) determines  $t(\zeta)$ , and we now show that this is bounded. To this end, we choose  $\zeta_0$ , define  $\zeta_i$  inductively by (55), so that in particular

$$\zeta_i \rightarrow \infty,$$

and set  $t_i = t(\zeta_i)$ . Then by (54),

$$t_{i+1} = t_i + \frac{2\ell}{\phi(\zeta_i) - \psi(\zeta_i)},$$

and so each  $t_i$  is bounded if the series

$$\sum_j \frac{2\ell}{\phi(\zeta_j) - \psi(\zeta_j)} = \sum_j \frac{\sqrt{2}\ell}{\sqrt{\alpha(\zeta_j) - \beta(\zeta_j)}}$$

converges, where we have used (46). Convergence of this sequence follows exactly as in the proof of Theorem 3. It now follows by continuity that the function  $t(\zeta)$  is bounded, and the proof is complete.  $\square$

We obtain initial data leading to the shockless solution which blows up by taking the trace of the solution at time  $t_0 = 0$  inside the strip, and setting the exterior data to be the corresponding constant value of  $\zeta$ . This has the effect of chopping off the outgoing simple wave. Note that we can trace the continuous solution back in time, and by doing so for long enough our initial data will again have arbitrarily small total variation.

### 4.3. Interaction Estimate

Here we solve (37) and (55), and find the necessary estimates on the solution. Given  $y$  and  $z$ , we look for  $z_+$  satisfying

$$\alpha(z_+) + \beta(z_+) = \alpha(z) - \beta(z),$$

and where  $y_+$  is given by (33) or (45), respectively. From (22), this becomes

$$\frac{1}{2} (y_+ + z_+)^2 + s (y_+ - z_+) = \frac{1}{2} (y - z)^2 - s (y + z). \quad (57)$$

**Lemma 1.** *Suppose  $z < z_0 < -s < 0$  and  $y > y_0 > 0$ . The equations (57) and (33) or (45) have a unique solution  $(y_+, z_+)$  satisfying*

$$y_+ > y \quad \text{and} \quad z_+ < z.$$

*Moreover, there is a constant  $\gamma > 0$  depending only on  $y_0$ ,  $z_0$  and  $s$ , so that the solution satisfies*

$$z_+ < (1 + \gamma) z \quad \text{and} \quad y_+ - y > z - z_+.$$

**Proof.** We write

$$y_+ = y + dy \quad \text{and} \quad z_+ = z + dz, \quad (58)$$

and solve for  $dy$  and  $dz$ . Using (58) and (22), equation (57) becomes

$$\begin{aligned} y \, dy + \frac{1}{2} \, dy^2 + z \, dz + \frac{1}{2} \, dz^2 - s \, dz \\ + 2 \, y \, (s + z) + (s + z) \, dy + y \, dz + dy \, dz = 0, \end{aligned}$$

which simplifies to

$$\frac{1}{2} \, (dz + dy)^2 + (y + z) \, (dz + dy) - s \, (dz - dy) + 2 \, y \, (s + z) = 0. \quad (59)$$

Equations (33) and (45) can both be written as

$$dy = \kappa \, dz, \quad \text{where} \quad \kappa = \frac{\bar{z} - s}{\sqrt{\bar{z}^2 - s^2}} \quad (60)$$

and  $\bar{z}$  is given by (18) for shocks, and by (16) for compressions. Note that since  $\bar{z} < -s < 0$ , we have

$$\kappa = -\sqrt{\frac{\bar{z} - s}{\bar{z} + s}} = -\sqrt{1 - \frac{2s}{\bar{z} + s}} < -1. \quad (61)$$

Using (60), equation (59) becomes

$$\frac{1}{2} \, ((1 + \kappa) \, dz)^2 + (y + z - s \, \frac{1-\kappa}{1+\kappa}) \, (1 + \kappa) \, dz + 2 \, y \, (s + z) = 0, \quad (62)$$

which we now regard as a quadratic in  $dz$ , with  $\kappa$  a parameter to be chosen. Solving, we get

$$(1 + \kappa) \, dz(\kappa) + y + z - s \, \frac{1 - \kappa}{1 + \kappa} = \pm \sqrt{\left(y + z - s \, \frac{1 - \kappa}{1 + \kappa}\right)^2 - 4 \, y \, (s + z)} \quad (63)$$

for any value of  $\kappa < -1$ . Because  $y \, (s + z) < 0$ , (62) always has a (unique) solution  $dz < 0$ , given by taking the positive square root. Thus (58) determines a solution  $z_+(\kappa)$ , and we must find the correct value of  $\kappa$  so that (60) is satisfied.

It is convenient to make another change of variables, by setting

$$\mu = -\frac{2s}{1 + \kappa}, \quad (64)$$

which is in the range  $(0, \infty)$ . After some manipulation, (63) can be written as

$$\begin{aligned} dz(\mu) &= -\frac{\mu}{2s} \left( \sqrt{(a + \mu)^2 + b^2} - (a + \mu) \right) \\ &= -\frac{b^2}{2s} \, g(\mu), \end{aligned} \quad (65)$$

where  $a$  and  $b$  are regarded as constants,

$$a = y + z + s \quad \text{and} \quad b^2 = -4 \, y \, (s + z), \quad (66)$$

and the function  $g$  is defined by

$$g(\mu) = \frac{\mu}{\sqrt{(a + \mu)^2 + b^2} + a + \mu}. \quad (67)$$

With  $z$  fixed,  $dz(\mu)$  in (58) determines  $z_+$ , and (18) or (16) gives  $\bar{z}$  as a function of  $\mu$ , namely

$$\bar{z}_s(\mu) = \frac{z_+(\mu) + z}{2} = z + \frac{1}{2} dz(\mu) \quad (68)$$

in the case of a shock, or

$$\bar{h} = \frac{h_s(z + dz(\mu)) - h_s(z)}{dz(\mu)} \quad \text{and} \quad \bar{z}_c(\mu) = \frac{1 + \bar{h}^2}{1 - \bar{h}^2} s \quad (69)$$

for a compression. According to (15),  $\bar{z}_c(\mu)$  is a weighted average of  $z$  and  $z_+$ . Since (61) must also hold, we solve that equation to get

$$\bar{z}_* = s \frac{1 + \kappa^2}{1 - \kappa^2} = -s \left( 1 + \frac{2}{\kappa^2 - 1} \right),$$

which after use of (64) translates to

$$\bar{z}_*(\mu) = -\frac{\mu^2 + 2\mu s + 2s^2}{2(\mu + s)} = -\frac{1}{2}(\mu + s) - \frac{s^2}{2(\mu + s)}. \quad (70)$$

We now equate (70) with either (68) or (69), as appropriate, to get the correct value of  $\mu$ , which in turn determines the solution  $(y_+, z_+)$ .

First, we examine the function  $dz(\mu)$ : calculus reveals that it is bounded and negative, with

$$dz(0) = 0 \quad \text{and} \quad dz(\infty) = \lim_{\mu \rightarrow \infty} dz(\mu) = -\frac{b^2}{4s}.$$

If  $a > 0$ ,  $dz$  is decreasing and concave up for  $\mu > 0$ , while if  $a < 0$ ,  $dz$  has a single minimum at

$$\mu_m = -\frac{a^2 + b^2}{2a} \quad \text{with} \quad dz(\mu_m) = \frac{a\mu_m}{2s} = -\frac{a^2 + b^2}{4s},$$

and the graph is increasing for  $\mu > \mu_m$ . Similar conclusions clearly follow for  $\bar{z}_s$ , and also for  $\bar{z}_c$  since  $\bar{z}_c(\bar{h})$  is monotone and  $h_s$  is convex down.

On the other hand,  $\bar{z}_*(\mu)$  satisfies  $\bar{z}_*(0) = -s$ , decreases without bound and is concave down for  $\mu > 0$ . Thus the intermediate value theorem implies that equations (70) and (68) or (69) have a common solution  $\mu_* > 0$ , and convexity arguments show that it is unique.

We now estimate the size of the solution. From (61) and (64), we derive

$$\begin{aligned} \mu(\bar{z}) &= -(\bar{z} + s) \left( 1 + \sqrt{\frac{\bar{z} - s}{\bar{z} + s}} \right) \\ &= -2(\bar{z} + s) + \frac{2s}{1 + \sqrt{\frac{\bar{z} - s}{\bar{z} + s}}}, \end{aligned}$$

which gives

$$\mu(\bar{z}) > -2(\bar{z} + s) > -2(z + s), \tag{71}$$

and therefore by (66),

$$a + \mu > y - (z + s) > 0.$$

On the other hand, in (67) we have  $g(0) = 0$  and  $g(\infty) = 1/2$ , and it is easy to see that

$$g(\mu) > \min \{ g(\mu_0), 1/2 \} \quad \text{for any } \mu > \mu_0 > 0. \tag{72}$$

Using (71), we set  $\mu_0 = -2(z + s)$ , and using (66) in (67), we get

$$\begin{aligned} g(-2(z + s)) &= \frac{-2(z + s)}{\sqrt{y^2 - 4y(z + s) + (z + s)^2} + y - (z + s)} \\ &= \frac{2r}{\sqrt{1 + 4r + r^2} + 1 + r} = \tilde{g}(r), \end{aligned}$$

where  $r = -(z + s)/y > 0$ . It follows from (65) and (72) that

$$\frac{dz(\mu)}{z} > \min \left\{ \frac{2y(z + s)}{sz} \tilde{g}(r), \frac{y}{s} \frac{z + s}{z} \right\},$$

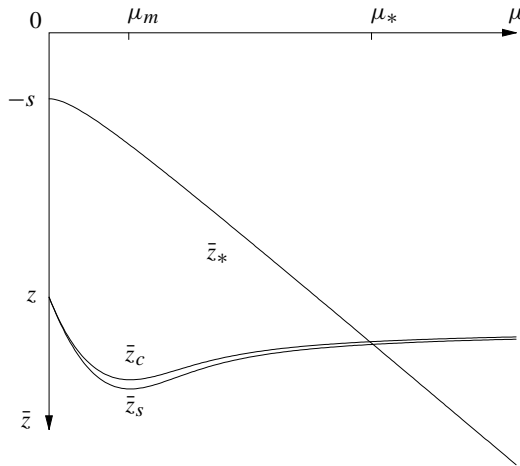


Fig. 5. Typical functions  $\bar{z}_s$ ,  $\bar{z}_c$  and  $\bar{z}_*$ .

and we wish to bound this away from zero. Observe that  $\tilde{g}(5/4) = 1/2$ , so we need only consider  $r < 5/4$ . Using  $r = -(z + s)/y$ , we have

$$\begin{aligned} \frac{2y(z+s)}{sz} \tilde{g}(r) &= -\frac{(z+s)^2}{sz} \frac{4}{\sqrt{1+4r+r^2+1+r}} \\ &> -\frac{(z+s)^2}{sz} \frac{4}{5} \\ &> -\frac{4}{5} \frac{(z_0+s)^2}{sz_0} > 0, \end{aligned}$$

while also

$$\frac{y}{s} \frac{z+s}{z} > \frac{y_0}{s} \frac{z_0+s}{z_0} > 0.$$

Thus setting

$$\gamma = \frac{z_0+s}{sz_0} \min \left\{ -\frac{4}{5} (z_0+s), y_0 \right\}, \quad (73)$$

we have

$$z_+ = z + dz < (1 + \gamma)z < 0,$$

and since  $dy = \kappa dy$  with  $\kappa < -1$ , the proof is complete.  $\square$

#### 4.4. Concluding remarks

It is clear that blowup is occurring independent of shock formation, although the solutions constructed here are discontinuous. Indeed, in our construction the presence of jump discontinuities is essential in that they connect states in different hyperbolic regions by a curve which “passes through  $\infty$ ”.

Our construction is very specific as we have enforced symmetry for ease of calculation. However, it is evident that we can easily obtain instability in many other situations, say beginning with a single shock or compression inside a strip, or by adding several other stationary jumps which connect the two separate hyperbolic regions. The key to generating the blowup is the instability of interactions of nonlinear waves with such jumps, together with genuine nonlinearity, which means that stronger waves have higher speeds and interact more often.

We conjecture that if the initial data is constrained to lie in one single connected component of the hyperbolic region, then the analog of Glimm’s theorem will apply and global solutions will exist and be stable in BV. Our reasoning is as follows: given such data in BV, we use the homogeneity and corresponding scale invariance in state space to scale the data into a small neighborhood in which the system is uniformly hyperbolic. Now we can apply Glimm’s theorem, which guarantees global existence. Finally, we rescale the solution in state space to retrieve a global solution of the original data.

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