Quantification without Cardinality

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Recent treatments of quantification in natural language have been, for the most part, denotational in character. This is unsurprising, perhaps, since it has been fashionable for some time in semantics to regard virtually every expression in a language as denoting. While this lends itself readily to the investigation of certain semantic questions, it closes the door to others—for example: how does one account for semantic variation *within* a basic category C. The only answer available in a denotational framework is that the expressions of C denote objects which are distinguished by properties characteristic of their kind. For example, a quantifier-expression *Q* denotes a quantifier Q--i.e., a mapping which assigns truth-values to possible extensions of predicates. The expression 'most', then, denotes the quantifier M which assigns *truth* to a pair of sets (A,B) just in case $|A \cap B| > |A - B|$.[1]

It is a familiar fact that elementary quantification theory fails to meet the needs of natural language semantics. We describe a method of expanding a first-order language L to a language L^* in which a variety of non-elementary quantifiers can be defined. In brief, L^* is obtained from L by introducing substitutional quantifiers and a class of non-denotational terms (QS-terms) which may appear as subscripts in quantifier-expressions. QS-terms are built from $\bar{0}$ and a stock of substitutional variables with numeral substituends. The chief distinction of L^* is that its semantics incorporates a set of explicit computation procedures defined on QS-terms which induce variation in quantifier sense.

L is a first-order language with variables v_0, v_1,... and a (finite) stock of descriptive symbols. The following is a description of an expansion L^* of L. (Boldface variables abbreviate sequences of indefinite (finite) length.)

(A) Vocabulary.

In addition to that of L:

1. $\bar{0}$, ', $\geqslant$.
2. f_i, for $i \geqslant 0$.
3. x_i, for $i \geqslant 0$.
4. a_i, for $i \geqslant 0$ (substitutional variables).
5. Π (universal substitutional-quantifier symbol).

(B) Quantifier-subscript terms (QS-terms).

1. $\bar{0}$ and a_i, for $i \geqslant 0$.
2. If t is a QS-term, so is t'.
3. If $t_1,\ldots,t_r$ are QS-terms, so is $f_i(t_1,\ldots,t_r)$.

Among the QS-terms, the numerals $\bar{0}$, $\bar{0}'$, $\bar{0}''$,... are distinguished as the substituends of the variables a_i. ($\bar{0}^{(n)}$ will be abbreviated by the appropriate arabic numeral.)

A QS-term is closed if it contains no occurrences of the variables a_i.

(C) Quantifier expressions.

In addition to those of L:

$$\exists_{\geq t} x_i \text{ and } \Pi a_i.$$

(D) Formulae.

1. L-formulae are L^*-formulae.
2. Any expression obtained by substituting a variable x_i for every free occurrence of an L-variable in an L-formula is an L^*-formula.
3. If Q is a quantifier expression and A is an L^*-formula, then QA is an L^*-formula.

(To 1-3 add any rules for the formation of L-formulae—e.g., those for sentential connectives.)

An elementary formula (e-formula) is an L^*-formula which contains no occurrences of a substitutional variable.

A pure substitutional formula (ps-formula) is an L^*-formula in which no substitutional quantifier occurs within the scope of an objectual quantifier.

L(**f**) is the sublanguage of L^* obtained by restricting (A)2 to the symbols **f**. (Below, **f** will be an initial segment of the list (A)2.)

(E) Some abbreviations:

Σa_i for $\neg \Pi a_i \neg$.
$\exists_{=t} xA$ for $\exists_{\geq t} xA \ \& \ \neg \exists_{\geq t'} xA$.
$\exists_{>t} xA$ for $\exists_{\geq t'} xA$.

$\exists_{<t}xA$ for $\neg \exists_{\geqslant t}xA$.
$\exists_{\leqslant t}xA$ for $\neg \exists_{\geqslant t'}xA$.
$\forall_{\geqslant t}xA$ for $\neg \exists_{\geqslant t}x \neg A$.

(F) Examples.

1. $\Pi a[\ \exists_{\geqslant a}xAx \dashrightarrow \exists_{\geqslant f(p,q,a)}x(Ax \ \&\ Bx)]$.

2. $\exists v\, \Pi a[\ \exists_{\geqslant a}xAxv \dashrightarrow \exists_{\geqslant f(a)}x(Axv \ \&\ Bxv)]$.

(G) A (partial) <u>f-interpretation</u> of L^* is a list $\mathcal{E}$ of equations in the f-symbols of a finite initial segment of the list (A)2. We assume here that each equation in $\mathcal{E}$ is of one of the following types.

1. $f_i(\mathbf{u}) \equiv \bar{n}$, $\bar{n}$ a fixed numeral.
2. $f_i(\mathbf{u}) \equiv u_j$, u_j among $\mathbf{u}$.
3. $f_i(\mathbf{u}) \equiv g(h_1(\mathbf{u}),\ldots,h_r(\mathbf{u}))$, where
   g and h_j are among $f_0,\ldots, f_{i-1}$, and '.
4. $f_i(0,\mathbf{u}) \equiv g(\mathbf{u})$,
   $f_i(u',\mathbf{u}) \equiv h(f_i(u,\mathbf{u}),u,\mathbf{u})$,
   where g and h are among $f_0,\ldots, f_{i-1}$, and '.

If $\mathcal{E}$ contains an equation for f_i it must contain one for each f_j, $j < i$. (The variables u_i are auxiliary notations used in specifying f-interpretations.) An f-interpretation in $f_0,\ldots, f_i$, together with an interpretation of L, affords a complete interpretation of $L(f_0,\ldots,f_i)$.

(H) A relation $\vdash$ of reduction of QS-terms to numerals is defined (relative to $\mathcal{E}$) as follows.

1. $\bar{n} \vdash \bar{n}$, $\bar{n}$ a numeral.
2. If $t \vdash \bar{n}$, then $t' \vdash \bar{n}'$.
3. $f_i(\mathbf{t}) \vdash \bar{n}$, if the equation for f_i is of type

(G)1.

4. $f_i(t) \vdash \bar{n}$, if the equation for f_i is of type (G)2 and $t_j \vdash \bar{n}$.

5. $f_i(t) \vdash \bar{n}$, if the equation for f_i is of type (G)3 and $t_j \vdash \bar{l}_j$, $h_k(\bar{\mathbf{l}}) \vdash \bar{m}_k$, and $g(\bar{\mathbf{m}}) \vdash \bar{n}$.

6. If the equations for f_i are of type (G)4, $t \vdash \bar{l}$ and $t_j \vdash \bar{l}_j$, then
   a. if $\bar{l}$ is $\bar{0}$ and $g(\bar{\mathbf{m}}) \vdash \bar{n}$, then $f_i(t,\mathbf{t}) \vdash \bar{n}$, and
   b. if $\bar{l}$ is $\bar{k}'$, $f_i(\bar{k},\bar{\mathbf{m}}) \vdash \bar{m}$, and $h(\bar{m},\bar{k},\bar{\mathbf{m}}) \vdash \bar{n}$, then $f(t,\mathbf{t}) \vdash \bar{n}$.

(I) Truth for e-sentences and ps-sentences can be defined in terms of an f-interpretation and a primitive predicate T_0 for truth in L. Let $\mathcal{E}$ be an f-interpretation in $f_0, \ldots, f_i$ and S be a e-sentence of $L(f_0, \ldots, f_i)$.

1. If S contains no QS-terms other than numerals obtain S^* as follows: starting with the innermost, replace each subformula of S of the form $\exists_{\geqslant \bar{n}} xA$ by $\exists v_1, \ldots, \exists v_n[(\&_{i\neq j} v_i \neq v_j)$ & $(\&_{1\leqslant i\leqslant n} A(v_i)]$, where $v_1, \ldots, v_n$ are the first n L-variables which do not occur in A and $A(v_i)$ is the result of substituting v_i for x in A. Then,

$$T(S) \text{ iff } T_0(S^*).$$

2. If $t_0, \ldots, t_k$ are all the QS-terms occurring in S and $t_i \vdash \bar{n}_i$, let $S^\#$ be the result of substituting $\bar{n}_i$ for every occurrence of t_i in S, $0 \leqslant i \leqslant k$. Then,

$$T(S) \text{ iff } T(S^\#).$$

3. If S is of the form $\Pi aS'$, then

T(S) iff for every numeral $\bar{n}$, T(S'($\bar{n}$)),

where S'($\bar{n}$) is the result of substituting $\bar{n}$ for a in S'.

4. (Clauses for connectives, as usual.)

(J) Satisfaction for L(f).

Let $\mathcal{E}$ be an f-interpretation in f, and M be a model for L. (For simplicity, assume that the descriptive vocabulary of L consists of a sole unary predicate symbol R. Subscripts on '$\models$' are dropped.) Below, A and B contain no free substitutional variables unless otherwise noted.

1. $s \models Rv_i$ iff $s_i \in R^M$.
2. $s \models \neg A$ iff $s \not\models A$.
3. $s \models A \mathbin{\&} B$ iff $s \models A$ and $s \models B$.
4. $s \models \exists v_i A$ iff for some $s' \approx_i s$, $s' \models A$.
5. $s \models \Pi a_i A$ iff for all numerals $\bar{n}$, $s \models A(\bar{n})$,
   where A contains at most a_i free.
6. $s \models \exists_{\geqslant \bar{0}} xA$.
7. $s \models \exists_{\geqslant \bar{1}} xA$ iff $s \models \exists v_i A(v_i)$, where v_i is the first L-variable which does not occur in A.
8. $s \models \exists_{\geqslant \bar{n}} xA$ iff
   $s \models \exists v_i \ \exists_{\geqslant \overline{n-1}} x(v_i \neq x \mathbin{\&} A(v_i) \mathbin{\&} A)$,
   where v_i is the first L-variable which does not occur in A and $\bar{n}$ is other than $\bar{0}$ or $\bar{1}$.
9. $s \models \exists_{\geqslant t} xA$ iff for some $\bar{n}$, $t \vdash \bar{n}$ and $s \models \exists_{\geqslant \bar{n}} xA$.

Remarks and explanation

f-interpretations are primitive recursive "computation sequences". A function $g: \omega^k \longrightarrow \omega$ is primitive recursive just in case there is an f-interpretation $\mathcal{E}$, in $f_0, \ldots, f_s$ say, such that for any m and n,

$$g(\mathbf{m}) = n \ \text{ iff } \ f_s(\overline{\mathbf{m}}) \vdash_{\mathcal{E}} \bar{n}.$$

($\mathcal{E}$ introduces the function g.) Consider the function g defined by

$$g(p,q,n) = \begin{cases} [p\cdot n/q] + 1 & \text{if, for some } k, \ p\cdot n = q\cdot k + r \text{ where } r < q \text{ and } r \neq 0, \\ [p\cdot n/q] & \text{if, for some } k, \ p\cdot n = q\cdot k. \end{cases}$$

([x] is the greatest integer $\leqslant x$.) g is primitive recursive. Let $\mathcal{E}$ introduce g. Relative to $\mathcal{E}$, example (F)1 expresses that more than p/q of the A's are B's.

There is no special reason to restrict functions which may be introduced by f-interpretations to those which are primitive recursive. We might have used a stronger collection of algorithms. However, this would not enhance the expressive power of L^*. (In fact, it is possible to significantly restrict the the class of functions which can be introduced by f-interpretations without affecting the expressive power of L^*.)

We distinguished e- and ps-formulas in order to illustrate (in (I)) the way in which the semantics of L^* depends of that of the base language L. e-formulas amount to little more than abbreviations of first-order formulas. ps-formulas are quite a bit more than abbreviations of first-order formulas. (The sentence S in the proof of proposition (2) below is a ps-sentence.) There are English statements, however, which cannot be expressed by ps-sentences. Example (F)2 has no ps-equivalent. Something like (F)2 is needed to capture the sense of, "There is someone whom most people employed by him dislike."

Quantifiers definable in L^*

A unary (unrestricted) quantifier on a set D is a map Q:P(D)--->2 which respects permutations of D--i.e., for any bijection m:D--->D,

$Q(m''A) = Q(A)$.

($P(D)$ is the set of subsets of D. $m''A = \{m(x) \mid x \in A\}$ is the image of A under m.)

Let L contain a unary predicate R, and $M(L,D)$ be the set of L-structures with domain D. A unary quantifier Q on D is definable in $L(\mathbf{f})$ relative to an f-interpretation $\mathcal{E}$ if there is a sentence S of $L(\mathbf{f})$ such that for each $M \in M(L,D)$,

$$(M, \mathcal{E}) \models S \text{ iff } Q(R^M) = 0.$$

(S is a definition of Q. We use '0' for truth, '1' for falsehood.)

Mostowski supplied the following simple (but useful) characterization of quantifiers on D. If κ is a cardinal, let $p(\kappa)$ be the set of pairs of cardinals (α, β) with $\alpha + \beta = \kappa$. Let $\kappa = |D|$. For each $T: p(\kappa) \longrightarrow 2$, define Q_T by

$$Q_T(A) = T(|A|, |D - A|).$$

Proposition 1. The map $T \longmapsto Q_T$ is a bijection between $2^{p(\kappa)}$ and the set of unary quantifiers on D.

If $Q = Q_T$, Q is said to be determined by T. We shall call T (and Q_T) finitary if $T(\alpha, \beta) = T(\kappa, \kappa)$ whenever $\alpha, \beta \geq \omega$. If T is finitary, it can be identified with the triple (A,B,i), where $A = \{ n \in \omega \mid T(n, \kappa) = 0\}$, $B = \{ n \in \omega \mid T(\kappa, n) = 0\}$, and $i = T(\kappa, \kappa)$. A (finitary) quantifier Q on D is arithmetical if it is determined by a triple (A,B,i) where A and B are both arithmetical. Q is Σ_n if A and B are both Σ_n. (For a discussion of the arithmetical hierarchy see Rogers.)

Proposition 2. For each n, there is a fixed f-interpretation $\mathcal{E}$ such that every Σ_n quantifier on D is definable in $L(\mathbf{f})$ relative to $\mathcal{E}$.

Proof. By the enumeration theorem for Σ_n predicates, there is a primitive recursive (n+2)-ary relation C such that for every Σ_n set A there is an e such that

$$k \in A \text{ iff } \exists x_1 \forall x_2 \ldots Qx_n C(e,k,x_1,\ldots,x_n)$$

(where Q is $\exists$ if n is odd, and $\forall$ if n is even). e is called an <u>index</u> for A.

Let $\mathcal{E}$ be an f-interpretation in $f_0,\ldots,f_s$ such that

$$f_s(\bar{e},\bar{k},\bar{\mathbf{m}}) \vdash_{\mathcal{E}} \bar{0} \text{ if } C(e,k,\mathbf{m}),$$

$$f_s(\bar{e},\bar{k},\bar{\mathbf{m}}) \vdash_{\mathcal{E}} \bar{1} \text{ if } \text{not-}C(e,k,\mathbf{m}).$$

Now let Q be determined by (A,B,i) where A and B are Σ_n sets, and e_0 and e_1 be indices for A and B, respectively. Let R_0 be R and R_1 be $\neg R$, and let S_j, j = 0,1, be the sentence

$$\Sigma a[\exists_{=a} xR_j \;\&\; \Sigma a_1 \Pi a_2 \ldots \Delta a_n \exists_{\geq f(\bar{e}_j,a,a_1,\ldots,a_n)} xF]$$

(where Δ is Σ if n is odd, Π if n is even, and F is the propositional constant for falsehood.) Let S_2 be the sentence

$$\Pi a \exists_{\geq a} xR \;\&\; \Pi a \exists_{\geq a} x \neg R.$$

Finally, let

$$S = \begin{cases} S_0 \vee S_1 \vee S_2 & \text{if } i = 0, \\ S_0 \vee S_1 & \text{if } i = 1. \end{cases}$$

Then, for all $M \in M(L,D)$, $(M,\mathcal{E}) \models S$ iff $Q(R^M) = 0$. []

<u>Remarks</u>. The arithmetical quantifiers on D are precisely the quantifiers on D definable over the collection of

hereditarily finite sets on D, HF(D) (using membership, identity, a predicate for D, and a unary predicate R). (See Barwise, ch.II.2.) Also, any quantifier definable in L(f) relative to an f-interpretation $\mathfrak{E}$ is definable over HF(D). It follows from these facts, together with proposition (2), that the quantifiers on D which can be defined in L^* relative to f-interpretations are precisely the arithmetical ones.

A quantifier Q:P(D)--->2 is monotone increasing (decreasing) if Q(A) = 0 and $A \subset B$ ($B \subset A$) imply that Q(B) = 0. For quantifiers definable in L^*, monotonicity can be expressed as a syntactic property. An L^* definable Q is monotone increasing (decreasing) if it has a definition S(R) in which R has only positive (negative) occurrences. This is easy to see for unary unrestricted quantifiers. The only L^* definable monotone quantifiers of this type are those which are defined by one of the following sentences: $\exists_{\geqslant \bar{n}} xR'$, $\forall_{\geqslant \bar{n}} xR'$ ($n \in \omega$), and $\Pi a\, \exists_{\geqslant a} xR'$, where R' is either R or $\neg R$.

The preceding discussion can be adapted, more or less straightforwardly, to restricted unary, and binary quantifiers on D. (For analogues of proposition 1, see the appendix of Higginbotham and May.)

FOOTNOTE

We have chosen to confine our attention to describing the language L^* and some of its properties. Our introductory remarks are meant only to be suggestive. We plan to explore L^* more broadly in a future paper.

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