A NUMERICAL FLUTTER PREDICTOR FOR 3D AIRFOILS USING THE ONERA DYNAMIC STALL MODEL

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A Thesis Presented

by

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ABSTRACT

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To be able to harness more power from the wind, wind turbine blades are getting longer. As they get longer, they get more flexible. This creates issues that have until recently not been of concern. Long flexible wind turbine blades can lose their stability to flow induced instabilities such as coupled-mode flutter. This type of flutter occurs when increasing wind speed causes a coupling of a bending and a torsional mode, which create limit cycle oscillations that can lead to blade failure. To be able to make the design of larger blades possible, it is important to be able to predict the critical flutter and post critical flutter behaviors of wind turbine blades.

Most numerical research concerning coupled-mode wind turbine is focused on predicting the critical flutter point, and less focused on the post critical behavior. This is because of the mathematical complexities associated with the coupled, nonlinear wind turbine blade systems. Here, a numerical model is presented that predicts the critical flutter velocity and post critical flutter behavior for 3D airfoils with third order structural nonlinearities. The numerical model can account for the attached flow and separated flow region by using the ONERA dynamic stall model. By retaining higher-order structural nonlinearities, lateral and torsional displacements can be predicted, which makes it
possible to use this model in the future to control wind turbine blade flutter. Furthermore, by using a dynamic stall model to simulate the flow, the solver is able to predict accurate limit cycle oscillations when the effective angle of attack is larger than the stall angle.

The coupled, nonlinear equations of motion are two coupled nonlinear PDEs and are determined using Hamilton’s principle. In order to solve the equations of motion, they are discretized using the Galerkin technique into a set of ODEs. The motion of the airfoil is used as an input to ONERA. The airfoil is sectioned with the lateral position and angle of attack known as well as the velocity and acceleration of the section at an instance of time. This information is used by ONERA to calculate lift and moment coefficients for each section which are then used to calculate the total lift and moment forces of the airfoil. Then, a Fortran code solves the system by using Houbolt’s finite difference method.

A theoretical NACA 0012 airfoil has been designed to define the parameters used by the equations of motion. Third bending and first torsional coupling occurs after the critical flutter point and dynamic lift and moment coefficients were observed. Dynamic stall was also observed at wind velocities farther away from the bifurcation point. Bifurcation diagrams, time histories, and phase planes have been created that represent the flutter behavior.
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CHAPTER 1
INTRODUCTION

1.1 Introduction

Energy production through wind turbines is a popular method of harnessing energy. Global wind power capacity has been increasing by 5-10% every six months since 2012 reaching a worldwide capacity of almost 500 GW in 2016 (WWEA Half-Year Report 2016, 2016) (Renewables 2017 Global Status Report, 2017). This figure makes wind power alone responsible for nearly 12% of total energy production worldwide, with over 90 countries participating in commercial wind power activity and 29 countries producing at least 1 GW (Renewables 2017 Global Status Report, 2017).

Wind turbines are found in areas with regular, high speed winds; either onshore or offshore. Offshore wind power is more efficient, since winds blow stronger and 40% more often (Hodges & Dowell, 1974). Despite this, the largest wind turbine farms are found inland, and currently produce an order of magnitude more power than offshore wind (Watts, 2012) (First foundation installed at London Array, 2011). This is because constructing and maintaining wind turbines on land is easier and cheaper than offshore. To maximize the amount of power harnessed by offshore wind turbines and to offset installation and maintenance costs, the largest, highest capacity wind turbines are used (Aarhus, 2017).

An offshore wind turbine is able to produce more power than an onshore wind turbine because a wind turbine’s power is proportional to the cube of wind velocity. Furthermore; the longer a wind turbine’s blade, the more power it can harness from the
wind. This is because the amount of power generated by a turbine is proportional to the swept area of the blade. Since the wind velocity for an area cannot be changed, wind turbines are increasing their swept area by using longer blades.

Some wind turbine configurations use flexible blades to reduce cost and loading on the tower and blades (Ning & Petch, 2016). Downwind turbines are able to use flexible blades because the blades are configured aft of the tower and wind, instead of the more traditional forward of the tower and wind configuration.

As the blades get longer and more flexible, they can become more susceptible to aerodynamic instabilities like coupled-mode flutter. Until recently, the most commonly studied aerodynamic instability is that caused by vortex shedding and flow separation and is referred to as stall flutter (Hansen, 2007) (Dowell, et al., 2004). This type of instability is related to the geometry of the turbine blade and is what traditionally has been the major concern (Hansen, 2007). Coupled-mode flutter is caused when a parameter, such as wind speed, is changed and causes the coupling of a bending and a torsional mode of the structure. This creates limit cycle oscillations (LCOs), which can cause failure. Furthermore, this type of instability can occur before the stall flutter point, in longer, more flexible blades, making it an increasing priority for design of longer blades.

Determining the flutter onset point has been a main focus for many numerical predictors (Nayfehet al., 2011) (Breitbach, 1978) (Pourazam, 2016) (Hansen, 2004) (Lee & Wong, 1999) (Alighanbari et al., 1994) (Tang & Dowell, 2001) (Bertrand et al., 2017) (Drazumeric, Gjerek, Kosel, & Marzocca, 2014) (Drazumeric et al., 2014) (Ladge & Modarres-Sadeghi, 2015). Behavior such as the amplitude of limit cycle oscillations beyond the flutter point are not often studied. This is because of the complexities
associated with the numerical modeling of the nonlinear behavior of bending and twisting airfoils. Many models make assumptions to simplify the behavior after the flutter onset point. Frequently, flutter is studied based on airfoils that are modeled as 2D and linear. Some 2D models address the nonlinear behavior of a 3D airfoil by adding a nonlinear restoring force to represent the torsional and flapwise stiffness. 2D models can also be used to predict the flutter onset point. However, a wind turbine blade is physically 3D and nonlinearities associated with bending and twisting must be considered when predicting its LCO magnitude and frequency. Common 3D structural models are usually nonlinear to the second order. This ordering scheme is appropriate for airfoils undergoing moderate displacements (Hodges & Dowell, 1974). For larger displacements, a higher order nonlinear system is needed (Freno, 2010).

Other simplifications have been made regarding the aeroelastic response of the system. The simplified steady flow assumption assumes that the angle of attack of an airfoil relative to the incoming wind is the same as the pitch angle. This assumption is not true when the airfoil is oscillating rapidly. For rapidly oscillating airfoils, the aeroelastic response becomes nonlinear. Furthermore, for higher angles of attack, the flow itself can change for attached (unstalled) to separated (stalled).

No nonlinear numerical model exists for predicting the post-critical response of airfoils at high angles of attack. This means that the current methods used to predict flutter and LCOs are based on models that do not accurately represent the physical system.
1.2 Background

Coupled-mode flutter occurs when a torsional mode and a bending mode couple and cause oscillations after a critical air speed. This oscillation causes the blade or airfoil to bend in the flapwise direction and twist in the torsional direction with equal oscillating frequencies. Previous numerical studies have made assumptions to simplify their models such as (i) modeling a 2D airfoil, (ii) assuming a simplified steady or unsteady flow, or (iii) assuming linear behaviors, while other studies take into account the structural nonlinearities of a 3D airfoil or unattached flow caused by high angles of (Nayfeh et al., 2011) (Breitbach, 1978) (Pourazam, 2016) (Hansen, 2004) (Lee & Wong, 1999) (Alighanbari et al., 1994) (Tang & Dowell, 2001) (Bertrand et al., 2017) (Drazumeric et al., 2014) (Ladge & Modarres-Sadeghi, 2015).

1.3 Models for the Structure

1.3.1 2D Airfoil

The 2D airfoil model consists of two parts, the equations that represent the structure and the forcing function, which represents the flow forces. The equations of motion are derived by modeling a 2D airfoil whose pitching and plunge directions are modeled with springs fixed at the elastic axis. The nature of these springs then determines if the structure is linear or nonlinear (Dowell, et al., 2004) (Lee & Wong, 1999). This method does not take into account the shape of the 3D airfoil, which means it ignores the strain of the bending and twisting blade.
The equations of motion for the system in Figure 1 in the flapwise and torsional
directions respectively are

\[
\begin{align*}
    m\ddot{v} + m e \ddot{\theta} + C_v \dot{v} + D_v v &= F_L, \\
    (J + me^2) \ddot{\theta} + m e \ddot{v} + C_{\phi} \dot{\theta} + D_{\phi} \theta &= M_T,
\end{align*}
\]

where \( m \) is the mass, \( e \) is the distance from the elastic axis to the center of gravity, \( C_v \) and
\( C_{\phi} \) are the overall structural damping in the flapwise and torsional directions respectively,
\( D_v \) and \( D_{\phi} \) are the spring stiffnesses in the flapwise and torsional directions respectively,
\( F_L \) is the lift force, and \( M_T \) is the moment force.

These coupled equations represent the motion of a 2D airfoil and are well known.
Classical flutter studies use this model with or without structural damping and an unsteady,
attached flow model such as that proposed by Theodorsen (Theodorsen, 1935).

1.3.2 3D Airfoil

Hodges and Dowell (1974) have derived the nonlinear equations of motion for long,
straight, slender, homogenous, isentropic beams undergoing moderate displacements with
applications to helicopter rotor blades. Their equations of motion are derived through two
methods: Hamilton’s extended principle and Newtonian methods. For simplicity and for
easier comparison to other methods, only Hamilton’s principle is discussed here, although
both methods yield the same results. See Appendix A and B for further explanation of
Hamilton’s principle and an example of the derivation of the equations of motion for a
beam.

Hamilton’s principle is expressed as

\[ \int_{t_i}^{t_f} \left[ \delta (T - U) - \delta W \right] dt \]  \tag{1.3}

where \( U \) is the potential energy in the form of strain energy, \( T \) is the kinetic energy, and
\( \delta W \) is the virtual work done by the forces at the boundaries and the external forces (Hodges

The structural nonlinear terms are brought in through the strain energy term. The
bending and twisting of the blade brings in terms that are up to fourth order partially
differentiable terms in space and up to second order partially differentiable terms in time.
It is because of the complexities associated with these nonlinearities that 2D structural
models with no bending are more popular. The kinetic energy brings in second order
partially differentiable terms in time and linear spatial terms.

The equations that Hodges and Dowell have derived contain structural
nonlinearities up to second order. In order to apply their equations of motion to flexible
airfoils with large deflections, up to third order nonlinearities must be retained (Freno,
2010). This criterion has been applied to airfoils (Freno, 2010), (Nayfehet al., 2011) and
Euler-Bernoulli beams (Crespo da Silva, 1988), (Arafat, 1999), (Delgado-Velazquez,
2007) with all concerning motion in the flapwise, edgewise, and torsional directions. For
simplicity, motion in the edgewise direction is assumed to be small when compared to the torsional and flapwise motions and is ignored.

The equations of motion can be derived for an airfoil and a blade using the derivation from Crespo da Silva (1988), Delgado-Velazquez (2007), and Arafat (1999). In order to validate the solution method, first an airfoil with a uniform, non-twisting cross section is assumed. This allows for the assumption that the elongation is zero. For a wind turbine blade, the elongation would be related to the untwisting of the blade as it flutters due to centripetal forces. The following is an overview of the equations of motion from Crespo da Silva (1988), Delgado-Velazquez (2007), and Arafat (1999).

The equations of motion for an airfoil can be approximated as an Euler-Bernoulli beam. In order to derive the equations of motion, a coordinate system that is orthogonal to a cross-section must be defined. Figure 2 represents this coordinate system where \( \hat{\xi}, \hat{\eta}, \) and \( \hat{\zeta} \) represent unit vectors in the \( x, y, \) and \( z \) directions of the deformed plane, respectively.

The equations of motion defined by Crespo da Silva (1988) in the flapwise, edgewise, and torsional respectively are

\[
G_u' = \left[ A_{\tilde{q}} \frac{\partial \tilde{\theta}}{\partial u} + A_{\tilde{q}} \frac{\partial \tilde{\theta}}{\partial \lambda} \right]' = m \ddot{\alpha} - Q_u \tag{1.4}
\]

and

\[
A_{\tilde{\theta}} = Q_{\tilde{\theta}} \tag{1.5}
\]

where

\[
\alpha = v, w
\]

and
\[ \lambda = \frac{E Ae_0}{(1+e_0)}. \]

\( \lambda \) is the Lagrange multiplier and can be determined by solving the inextendable constraint and the \( Q \) terms are the nonconservative forces in their respective directions. Furthermore, the \( A \) terms are equal to

\[ A_{\alpha} = \frac{\partial^2 l}{\partial \alpha \partial t} + \frac{\partial^2 l}{\partial \alpha \partial x} - \frac{\partial l}{\partial \alpha}. \]

where

\[ \alpha = \theta_z, \theta_y, \theta_x \]

and the Lagrangian \( (l) \) is

\[ l = T - U + \frac{E Ae_0^2}{2}. \]

\( T \) and \( U \) are the kinetic and potential energy of the beam. Equations (1.4) and (1.5) can be expanded using the Taylor series to third order and integrated over the length of the beam.
Figure 2. Undeformed and deformed beam showing reference coordinate system perpendicular to the cross section and relation to deformed and undeformed case (Crespo da Silva, 1988).

Expanding on this, the Lagrangian is equal to (Arafat, 1999) (Delgado-Velazquez, 2007):

\[
l = \frac{1}{2} \int_{0}^{L} \left[ m \left( \frac{\partial}{\partial t} \int_{0}^{1} \frac{1}{2} (w'^2) \, ds \right) \right]^2 + m\dot{\omega}^2 + J_{\xi} \psi'^2 + J_{\eta} \left( \omega'^2 - \theta^2 \omega'^2 + w'^2 \omega'^2 \right) \\
+ J_{\zeta} \theta'^2 \dot{w}'^2 - D_{\xi} \theta'^2 - D_{\eta} \left( w''^2 - \theta^2 w''^2 + w'^2 w''^2 \right) - D_{\zeta} \theta'^2 w''^2 \right] \, ds
\]

where \( m \) is the mass per length, \( J \) is the mass moment of inertia, and \( D \) is the stiffness. Relations for determining these parameters can be found in Section 2.1.1. The subscripts \( \xi, \eta, \) and \( \zeta \) determine about what axis the inertia and stiffness must be taken. About the \( \zeta \) axis represents flapwise motion, about the \( \eta \) axis represents edgewise motion and about the \( \xi \) axis represents torsional motion. Substituted into Hamilton’s principle, the equations of motion become in the flapwise direction.
\[ m\ddot{w} + c_w \dot{w} + D_g w^{(4)} - J_g \ddot{w}'' = Q_w - \left( D_\eta - D_\zeta \right) \left( \theta^2 w'' \right)'' - D_\zeta \left( w' (w'w'')' \right)'
\]

\[-(J_\eta - J_\zeta) \left( \frac{\partial}{\partial t} (\dot{w}'\theta^2) \right)' + J_\zeta \left( w' (\dot{w}'^2) \right)' - \frac{m}{2} \left( w' \int_0^L \frac{\partial^2}{\partial t^2} \int_0^t (w'^2) ds \right) ds \right)'
\]

and in the torsional direction

\[ J_\zeta \ddot{\theta} + c_\theta \dot{\theta} - D_\zeta \theta'' = Q_\theta + \left( D_\eta - D_\zeta \right) (-\theta w'^2) + \left( J_\eta - J_\zeta \right) (\theta \dot{w}'^2).\]

The damping terms \( c_w \) and \( c_\theta \) and external force terms \( Q_w \) and \( Q_\theta \) are brought in through the virtual work term \( \delta W \) used in Hamilton’s principle (Delgado-Velazquez, 2007).

\[ \delta W = \int_0^L \left[ (Q_w - c_w) \delta w + (Q_\theta - c_\theta) \delta \theta \right] ds \]

This method is valid for deriving the fully third order nonlinear equations of motion for a beam with its elastic axis (shear center) and center of gravity occupying the same point. This is not the case for an airfoil where the two points are offset. The derivation for this situation will be discussed in Section 2.1.1.

The 2D airfoil is not an accurate representation of the motion of a physical, 3D airfoil because it does not take into account the strain energy of the bending-twisting blade found in the potential energy term. By taking into account the strain energy, the system quickly becomes nonlinear. Structural nonlinearities such as the bending-twisting strain energy must be accounted for in order to accurately represent an airfoil (Breitbach, 1978).
1.3.3 Flow Models

1.3.3.1 Steady, Attached Flow

For slowly oscillating airfoils at low angles of attack, a simplified flow model can be used. This model assumes that the effective angle of attack and the pitch angle are the same and that the angle of attack is low enough to keep the airfoil out of the stall region. The equations for this type of flow are (Pourazam, 2016):

\[ L = C_L \rho c U^2 \theta, \]  
\[ M = M_{1/4} + b L. \]  

Equation (1.6) represents the lift force and Equation (1.7) represents the moment. In these equations, \( C_L \) represents the lift coefficient of a flat plate for low angles of attack and is equal to \( 2\pi \), \( \rho \) is the air density, \( c \) is the chord length, \( U \) is the flow velocity, and \( \theta \) is the angle of attack. \( M_{1/4} \) represents the pitching moment at the quarter chord and \( b \) is the distance from the elastic axis to the center of lift. This model is not realistic for an airfoil undergoing flutter and is only valid for small angles of attack.

1.3.3.2 Unsteady, Attached Flow

The effective angle of attack of an airfoil is not equal to the pitching angle for rapidly oscillating airfoils, such as those undergoing flutter. The velocity of the leading edge of the blade adds a vertical component to the oncoming air flow which changes the effective angle of attacked. Because the motion of the airfoil is not constant, the vertical component is not constant either, making the effective angle of attack unsteady. This then changes the lift in an unsteady manner and makes the flow unsteady.
The flow region before the stall angle of attack is the attached, linear flow region. This means that the Kutta condition is valid, and the flow does not separate at the trailing edge. This type of flow when used with a 2D airfoil is referred to as classical flutter. For unsteady, attached flow, the Theodorsen thin airfoil theory has been developed and used (Pourazam, 2016), (Theodorsen, 1935), (Bisplinghoff et al., 1955). Equations (1.8) and (1.9) represent the Theodorsen lift and moment respectively.

\[ L = C_{La} \rho_\infty C(k)b \left[ U\dot{w} - (1 - 2a)\frac{b}{2} U\dot{\theta} - U^2 \theta \right] \]
\[ -C_{La} \rho_\infty \frac{b^2}{2} \left[ -U\dot{\theta} + \dot{w} + ab\dot{\theta} \right] \]
\[ (1.8) \]

\[ M = -C_{La} \rho_\infty C(k)b^2 U \left[ \left( \frac{1}{2} + a \right) \dot{w} - (1 + 2a) \left( \frac{1}{2} - a \right) b\dot{\theta} - \left( \frac{1}{2} + a \right) Ua \right] \]
\[ -C_{La} \rho_\infty \frac{b^3}{2} \left[ \left( \frac{1}{2} - a \right) U\dot{\theta} + \alpha \dot{w} + b \left( \frac{1}{8} + a^2 \right) \dot{\theta} \right] \]
\[ (1.9) \]

In these equations, \( L \) is the lift, \( M \) is the moment, \( C_{La} \) is the slope of the lift coefficient verses the angle of attack in the unstalled region, \( \rho_\infty \) is the air density, \( C(k) \) is the Theodorsen function as a function of the reduced frequency \( k \), \( b \) is the half chord length, \( U \) is the flow velocity, and \( a \) is the distance between the elastic axis and the mid chord. This theory is valid for angles of attack before the stall angle. After the onset of oscillations, a turbine blade could enter the stall region, meaning that a flow theory should be used that accounts for this region.
1.3.3.3 Unsteady, Separated Flow

For higher angles of attack, an airfoil can enter the stalled region, and since the airfoil is moving, dynamic stall can occur. It is known that wind turbine blades can experience dynamic stall (Holierhoek et al., 2012). Dynamic stall occurs when time-dependent motion, such as pitching, causes the relative angle of attack to exceed the normal steady stall angle. The effects of dynamic stall are considerable. Dynamic stall causes a delay in the stall onset (Holierhoek et al., 2012) (Beedy et al., 2003). This means that stall will occur at a higher angle of attack than is predicted with the static stall case. A strong vortex is created at the leading edge of the airfoil that travels down the chord length of the airfoil. This creates a temporary strong lift force that is rapidly lost as the vortex is shed at the trailing edge, which causes an oscillating pressure loading that is not considered in that static stall model.

Experiments have concluded that it is necessary to use a dynamic stall model when designing wind turbine blades (Holierhoek et al., 2012). Dynamic stall has a significant effect on the aeroelastic damping and pressure load. Several models exist for predicting the dynamic stall behavior, which will be discussed in Section 1.3.4.

1.3.4 Airfoil Flutter

Two-dimensional airfoil flutter is a well studied problem. Multiple tests involving experimental and numerical simulations have been carried out for airfoils with different structures and aerodynamic forces. Research into flutter has mostly been concerned with determining the air speed at which an airfoil will lose its stability. Multiple methods have
been developed to determine this point while methods to determine flutter characteristics beyond that are limited and focus mostly on flapwise oscillations.

Hansen (2004) developed a design tool for analyzing the stability of wind turbine blades using an eigenvalue approach. He used a linear structural model and nonlinear aerodynamic model. They found the natural frequencies, damping, and modal shapes of the system which matched closely to experimental results. By looking at the damping of various wind speeds, the point at which damping became negative implied that the blade had lost its stability.

Holierhoek et al. (2012) compared three nonlinear dynamic stall flow models. These models include the ONERA model, Beddoes-Leishman model, and the Snel model. They concluded that the ONERA model is often the method of choice because its differential equations that define the system are relatively easy to couple with a structural model. Furthermore, its prediction of aerodynamic forces is acceptable. The authors also highlighted some drawbacks of this method. The ONERA model assumes that the difference between the dynamic and static lift coefficients is small, which is not always true. This method also requires a large amount of empirical data that must be found through wind tunnel experiments. Although the Beddoes-Leishman model also requires empirical data, its main advantage over the ONERA model is that it requires fewer inputs. This is because it uses four subsystems that feed into each other to determine aerodynamic forces. These subsystems examine different flow regimes that are first attached and then separated. It then determines the dynamic stall onset point and then the forces caused by dynamic stall. Unlike the other two methods, the Snel model requires no inputs from wind tunnel data because it uses no airfoil-specific parameters. They
compared these models with a physical experiment and found that the accuracy of the method depended on the airfoil. They also highlighted the importance of differentiating between dynamic and static stall. The lift coefficient differs dramatically between the two cases. Figure 3 shows how the lift coefficient changes for different reduced frequencies (k) for dynamic and static stall. The cyclic lift coefficient caused by the changing angle of attack is not accounted for in static stall, which means that a dynamic stall model must be used to properly capture the post critical behavior.

Beedy et al. (2003) used ONERA to calculate stall flutter in helicopter blades. They compared CFD simulations calculating lift coefficients for a range of angles of attack to a structural model coupled with ONERA. They could then fit the ONERA data with a fluid solver which lead to acceptable approximations for their CFD or experimental data. They found that their methods were efficient and robust enough to be able to provide adequate results for their preliminary analysis of stall flutter. Furthermore,
the authors were able to use ONERA to correctly calculate the flutter onset point and their frequencies.

Lee et al. (1999) used a 2D airfoil to study the effects of using springs with cubic restoring forces and different stiffness constants on predicting flutter. This restoring force is related to the restoring force from the structure of a 3D airfoil. Their numerical results concluded that the presence of a restoring force affects the flutter onset point. Their stiffer springs were related to thin airfoils with higher torsional displacements, while their softer springs were related to airfoil buckling. Their results concluded that an airfoil with a stiffer spring would undergo limit cycle oscillations after the flutter point, while softer springs would diverge after the critical flutter point. Price et al. (1994) also used a cubic and bilinear structural model to study the aeroelastic response of a 2D airfoil numerically and semi-analytically with a linear flow model. Their numerical simulation was solved using the Houbolt finite difference scheme, a method that will be described later. One of their conclusions was that for larger limit cycle amplitudes after the flutter point, their numerical analysis was no longer able to settle on a stable solution.

Tang and Dowell (2001) compared a theoretical model to a physical model of a 3D, high aspect ratio, NACA 0012 airfoil to predict flapwise limit cycle oscillations. They used the Hodges and Dowell’s second order nonlinear equations of motion for their structural model and the ONERA dynamic stall model for their aerodynamic model. Although their experimental results and theoretical results correlated within a margin of error, there were some disagreements as the airspeed increased and the flutter oscillations increased (Figure 3).
Figure 4. Experimental and numerical results for high aspect ratio NACA 0012 wing. Amplitude of limit cycle oscillations in flapwise direction vs flow velocity shown. As flow velocity increases, theory and experiment diverge (Tang & Dowell, 2001).

Bertrand et al. (2017) studied a NACA 0012 with an end mass in a wind tunnel. They studied the effects of angle of attack and aspect ratio on the flutter onset point. They concluded that as the aspect ratio was increased, the flutter onset point decreased. Figure 4 illustrates their findings with multiple starting angles of attack.
Drazumeric et al. (2014) used an eigenvalue approach to predict the flutter onset point for a classical flutter case. Their goal was to increase the required wind speed for flutter to occur by changing the flexibility of the chordwise stiffness. Their numerical predictions were compared to experimental results. They concluded that the critical flutter point could be increased by increasing the plunge stiffness and decreasing the torsional stiffness.

Ladge and Modarres-Sadeghi (2015) experimentally tested small-scale wind turbine blades to study subcritical and supercritical flutter instabilities. In the subcritical case, there exists a wide range of wind velocities where two stable solutions exist, the trivial solution and a dynamic solution. They concluded that linear models could not reliably predict the flutter onset point in subcritical cases and that nonlinear models were needed to predict the flutter point of wind turbine blades. Furthermore, they concluded that the twist of a blade can have a significant effect on the subcritical flutter onset point and should also be studied.
Nayfeh et al. (2011) used an eigenvalue approach to predict the flutter onset point of a 3D airfoil with two degrees of freedom. They used Hamilton’s principle to derive the equations of motion and retained up to the third order structural nonlinearities. They concluded that when using the Galerkin discretization method to create a set of partial differential equations, one mode was acceptable to predict the flutter onset point but more were needed to predict the behavior after this point.

1.4 Motivation

One of the major barriers to increasing the power produced by wind turbines comes from flow-induced instabilities such as coupled-mode flutter. In order to reduce cost and weight, it is important that wind turbine blades are light, which can reduce the rigidity of turbine blades. Furthermore, longer turbine blades are used in offshore applications so that they generate more power and offset their higher installation and maintenance costs. Longer, flexible wind turbine blades are more susceptible to flow-induced instabilities such as flutter. This instability is a self-excited instability which causes oscillations in the torsional and flapwise directions, which can result in failure of the blade. In this thesis, a nonlinear model is presented to understand the behavior of an airfoil at the onset of flutter and post-critical wind speeds.
CHAPTER 2

METHOD

2.1 Method

The goal of this research is to couple the nonlinear equations of motion of a 3D airfoil with an unsteady, attached/separated flow model. This means that the bending-twisting motion of an airfoil caused by aeroelastic instabilities can be modeled and better understood. This model can then be used in the design process of future long and flexible wind turbine blades.

2.1.1 Structure Equations of Motion

This section defines the equations of motion for a 3D airfoil with third order nonlinearities using the extended form of Hamilton’s principle. The equations of motion are defined using methods from Hodges & Dowell (1974), Nayfeh et al. (2011), Freno (2010), and Crespo da Silva (1988) with a particular focus on Nayfeh et al. (2011).

Two coordinate systems are used to define the deformed and undeformed airfoil. The Cartesian XYZ coordinate system defines the undeformed airfoil, while the orthogonal ξηζ defines the deformed coordinate system. The unit vectors describing the relationship of the two systems are defined as,

$$
\begin{bmatrix}
\xi \\
\eta \\
\zeta \\
\end{bmatrix}
= [T]
\begin{bmatrix}
\hat{X} \\
\hat{Y} \\
\hat{Z} \\
\end{bmatrix},
$$

(2.1)

where $T$ describes the Euler angles in terms of $\phi$, $\psi$, and $\theta$. 

\[20\]
\[
[T] = \begin{bmatrix}
c\theta c\psi & c\theta s\psi c\phi + s\theta s\phi & c\theta s\psi s\phi - s\theta c\phi \\
-s\psi & c\phi c\psi & s\phi c\psi \\
s\theta c\psi & s\theta s\psi c\phi - c\theta s\phi & s\theta s\psi s\phi + c\theta s\phi
\end{bmatrix}
\]

(2.2)

where \( c \) and \( s \) represent \( \cos \) and \( \sin \) respectively. Furthermore, the bending curvatures for the \( \xi\eta\zeta \) coordinate system and the relationship between displacements and Euler angles are defined as (Nayfeh et al., 2011):

\[
\rho_z = \phi' c\theta c\psi - \psi' s\theta ,
\]

(2.3.1)

\[
\rho_\zeta = -\phi' s\psi + \theta' ,
\]

(2.3.2)

\[
\rho_\eta = \phi' s\theta c\psi + \psi' c\theta ,
\]

(2.3.3)

\[
c\phi = \frac{1 + v'}{\sqrt{(1 + v')^2 + w'^2}} ,
\]

(2.4.1)

\[
s\phi = \frac{w'}{\sqrt{(1 + v')^2 + w'^2}} ,
\]

(2.4.2)

\[
c\psi = \frac{\sqrt{(1 + v')^2 + w'^2}}{\sqrt{(1 + v')^2 + u'^2 + w'^2}} ,
\]

(2.4.3)

\[
s\psi = \frac{-u'}{\sqrt{(1 + v')^2 + u'^2 + w'^2}} .
\]

(2.4.4)

Terms with a prime denote a spatial derivative in the airfoil length direction. The axial strain along this direction is equal to zero, which means the elongation is zero and the airfoil is inextendable. This allows for the following relation, derived from the axial strain equation,

\[
v = -\frac{1}{2} \int_0^L w'^2 ds ,
\]

(2.5)
where s is a position on the length of the airfoil. With these relations defined, the
equations of motion can be determined. Recalling Hamilton’s principle, defined in (1.3),
first the kinetic and potential energies of the system must be determined.

2.1.1.1 Kinetic Energy

The kinetic energy of the airfoil is defined as

$$KE = \frac{1}{2} \int_0^L \rho V \cdot V \, dA \, dy$$  \hspace{1cm} (2.6)

where \(V\) is the velocity of a point along the length of the airfoil, \(\rho\) is the density of the
airfoil, \(L\) is the airfoil length, and \(A\) is the cross-sectional area. Velocity, \(V\), is found by
taking the time derivative, denoted with an over dot, of the deformed position vector, \(R\).
This vector is written in terms of unit vectors defined previously as,

$$R = u\hat{X} + (s + v)\hat{Y} + w\hat{Z} + \zeta\hat{\zeta} + \eta\hat{\eta}.$$  \hspace{1cm} (2.7)

By substituting the appropriate \(T\) relationship from (2.2) into (2.7) and taking a
time derivative, vector \(V\) can be defined as

$$V = (\zeta s\theta + \eta c\theta)\dot{\theta}\hat{X} + \left[\dot{v} + (\zeta c\theta + \eta s\theta)\dot{s}\phi + (\zeta s\theta - \eta c\theta)\dot{c}\phi\right]\hat{Y}$$

$$+ \left[\dot{w} - (\zeta c\theta + \eta s\theta)\dot{c}\phi + (\zeta s\theta - \eta c\theta)\dot{s}\phi\right]\hat{Z}.$$  \hspace{1cm} (2.8)

From here, (2.4.1) and (2.4.2) can be substituted into (2.8) and expanded
assuming small values of \(w\) and \(\theta\) and retaining up to third order nonlinearities.

2.1.1.2 Potential Energy

The potential energy of the system comes from the strain energy of the bending
airfoil. This energy is defined as
\[ PE = \frac{1}{2} \int_0^L \left( D_\xi \rho_\xi^2 + D_\eta \rho_\eta^2 + D_\zeta \rho_\zeta^2 \right) dy, \quad (2.9) \]

where \( D \) represents the bending and torsional stiffness about the \( \xi, \eta, \) and \( \zeta \) axes.

The potential can be found by substituting (2.3.1) to (2.3.3) and (2.4.1) to (2.4.2) and expanding again assuming small values of \( w \) and \( \theta \), and retaining up to the third order nonlinearities.

**2.1.1.3 Final Equations of Motion**

The final equations of motion for a bending-twisting airfoil have been derived using the aforementioned methods in (Nayfeh et al., 2011). The time bounds mentioned in Hamilton’s principle (1.3) create variations in the \( w \) and \( \theta \) directions that must be equal to zero i.e. \( \theta(y, t_1) = \delta \theta(y, t_2) = \delta w(y, t_1) = \delta w(y, t_2) = 0 \). Furthermore, the boundary conditions must be equal to those of a cantilever Euler-Bernoulli beam. These conditions are:

\[
\begin{align*}
\quad w(0, t) &= 0, \\
w'(0, t) &= 0, \\
\quad w''(L, t) &= 0, \\
\quad w'''(L, t) &= 0, \\
\quad \theta(0, t) &= 0, \\
\quad \theta'(L, t) &= 0.
\end{align*}
\]

The equations can then be non-dimensionalized using the following parameters:

\[
\begin{align*}
\quad w^* &= \frac{w}{L}, \\
\quad \theta^* &= \theta, \\
\quad \zeta^* &= \frac{\zeta}{L}, \\
\quad e^*_\zeta &= \frac{e_\zeta}{L}, \\
\quad J^*_\zeta &= \frac{J_\zeta}{mL^2}, \\
\quad J^*_\zeta &= \frac{J_\zeta}{mL^2}, \\
\quad \tau &= L^2 \sqrt{\frac{m}{D_\zeta}}, \\
\quad \beta_1 &= \frac{D_\eta}{D_\zeta}, \\
\quad \beta_2 &= \frac{D_\zeta}{D_\zeta}, \\
\quad F_L^* &= \frac{F_L}{D_\zeta L^3}, \\
\quad M_T^* &= \frac{M_T}{D_\zeta L^2}.
\end{align*}
\]
For these equations, the following from Hodges & Dowell (1974) and Crespo da Silva (1988) are used to define the physical properties:

\[ m = \int_A \rho d\eta d\zeta, \quad J_{\eta} = \int_A \zeta^2 \rho d\eta d\zeta, \quad J_{\zeta} = \int_A \eta^2 \rho d\eta d\zeta, \quad J_{\zeta} = J_{\eta} + J_{\zeta}, \]

\[ D_{\eta} = \int_A E\zeta^2 \rho d\eta d\zeta, \quad D_{\zeta} = \int_A E\eta^2 \rho d\eta d\zeta, \quad D_{\zeta} = \int_A E(\zeta^2 + \eta^2) \rho d\eta d\zeta. \]

The nondimensional equations of motion for a cantilevered airfoil with third order structural nonlinearities are

\[ \ddot{w} - e_{\zeta} \dot{\theta} + w^{(4)} - J_{\zeta} \dddot{w} - (1 - \beta_1) \left( \theta^2 w^2 \right)'' \\
+ \left( w'(w'^2)' \right)' + \frac{1}{2} \left[ \dot{w}' \int_0^w \frac{\partial^2}{\partial t^2} \left( \int_0^\zeta w^2 d\zeta \right) d\zeta \right]' = -F_L \quad (2.10) \]

\[ J_{\zeta} \ddot{\theta} - e_{\zeta} \dot{w} - \beta_2 \dot{\theta}'' - (1 - \beta_1) \theta w'^2 = M_r \quad (2.11) \]

in the flapwise and torsional directions. For simplicity, the star denotation has been dropped.

These equations of motion are for a symmetric, inextendable, homogenous airfoil with a uniform cross section where \( w \) and \( \theta \) are functions of both time and space. This means that the elastic axis and center of gravity are located on the axis-symmetric centerline of the airfoil’s cross section. It is possible to derive equations of motion that account for nonsymmetrical airfoils or airfoils without a uniform cross section, but in order to simplify the problem and to first prove the concept, this scenario is not yet considered. Furthermore, edgewise displacement is ignored and only motion in the flapwise and torsional directions are considered.
Equations (2.10) and (2.11) are the equations of motion in normal form. This means that they only represent the dynamics of the system close to the critical flutter point. In the future, new equations of motion will be derived to represent the dynamics of the system past the critical flutter velocity.

### 2.1.1.4 Discretizing using Galerkin Method

In order to solve Equations (2.10) and (2.11), the time and spatial dependent terms $w(t,s)$ and $\theta(t,s)$ are separated into terms that are functions of only time or space. The Galerkin discretization method can be used to separate the terms by assuming that sum of the product of a time dependent term and a spatial dependent term is equal to the time and spatial dependent terms (Dowell, et al., 2004), (Hodges & Dowell, 1974), (Freno, 2010), (Crespo da Silva, 1988), (Wadham-Gagnon, et al., 2007), (Modarres-Sadehi, et al., 2007). The spatial dependent term is an assumed function that satisfies the boundary conditions of the system. In the case of a cantilevered airfoil, these functions are equal to the Euler-Bernoulli beam vibration mode shape functions (Freno, 2010) (Bisplinghoff, et al., 1955). The sum of the product then refers to the number of modes used to calculate the solution. The method is applied so that

$$w(s,t) = \sum_{i=1}^{N} q_i(t)W_i(s)$$

and

$$\theta(s,t) = \sum_{i=1}^{N} p_i(t)\Theta_i(s)$$
where \( N \) is the total number of mode shapes, \( i \) is the mode number and \( W_i(s) \) and \( \Theta_i(s) \) are dimensionless eigenfunctions or mode shape functions that represent the natural mode shapes of the airfoil (Lee & Wong, 1999).

The mode shape functions are derived by assuming that the shape functions in the bending and torsional directions of an airfoil are equal to the shape functions of a Euler-Bernoulli beam in both directions. This assumption can be made because the boundary conditions of the Euler-Bernoulli beam and the airfoil are the same (Freno, 2010) (Bisplinghoff et al., 1955), (Crespo da Silva, 1988). The mode shape functions for flapwise and torsional directions respectively are (Bisplinghoff et al., 1955):

\[
W_i(s) = \cosh \left( \frac{\beta_i s}{L} \right) - \cos \left( \frac{\beta_i s}{L} \right) - \sigma_i \left[ \sinh \left( \frac{\beta_i s}{L} \right) - \sin \left( \frac{\beta_i s}{L} \right) \right],
\]

\[
\Theta_i(s) = \sqrt{2} \sin \left( \frac{\gamma_i s}{L} \right),
\]

where \( \beta_i \), \( \sigma_i \) and \( \gamma_i \) are constants, depending on the mode number \( i \). The equations and values for the first 6 modes are given in Table 1. The first 4 mode shapes in the bending and torsional directions are also shown in Figure 5. The spatial derivatives are then easily obtained in both directions and can be used in the discretized equations of motion.
Table 1. List of parameters to be used in the mode shape function formulation

<table>
<thead>
<tr>
<th>$i$</th>
<th>$\beta_i = \cos^{-1}(-\frac{1}{\cosh(\beta_i)})$</th>
<th>$\sigma_i = \frac{\cosh(\beta_i) + \cos(\beta_i)}{\sinh(\beta_i) + \sin(\beta_i)}$</th>
<th>$\gamma_i = \frac{(2i-1)\pi}{2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.597$\pi$</td>
<td>0.7340956</td>
<td>1.5707963</td>
</tr>
<tr>
<td>2</td>
<td>1.49$\pi$</td>
<td>1.0184641</td>
<td>4.7123890</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{5}{2}\pi$</td>
<td>0.9992245</td>
<td>7.8539816</td>
</tr>
<tr>
<td>4</td>
<td>$\frac{7}{2}\pi$</td>
<td>1.0000336</td>
<td>10.9955743</td>
</tr>
<tr>
<td>5</td>
<td>$\frac{9}{2}\pi$</td>
<td>0.9999986</td>
<td>14.13716694</td>
</tr>
<tr>
<td>6</td>
<td>$\frac{11}{2}\pi$</td>
<td>1.0000001</td>
<td>17.27875960</td>
</tr>
</tbody>
</table>
Figure 6. The first four mode shapes. 
Top; Bending direction. Bottom; Torsional direction.

Discretizing the equations of motion changes the system of PDEs into a system of ODEs that can be solved numerically. The spatially dependent terms become constant for all time and are represented as coefficients in front of the time dependent terms. The
expanded, non-dimensionalized, discretized equations of motion in the flapwise and torsional directions respectively are

\[
\left( MB_q \right) \ddot{q}_j + \left( MB_p \right) \ddot{p}_j + \left( KB \right) q_j + \left( Bppq \right) p_j p_k q_l + \left( Bqqq \right) q_j q_k q_l + \left( Bqdqdq \right) q_j q_k \dot{q}_l + \left( Bqqddq \right) q_j q_k q_l = -F_L
\]

and

\[
\left( MT_q \right) \ddot{q}_j + \left( MT_p \right) \ddot{p}_j + \left( KT \right) p_j + \left( Tppq \right) p_j q_k q_l = M_T
\]

where the constant coefficients are defined as

\[
MB_q = \int_0^1 W_0 W ds - J_z \int_0^1 W_0 W_0'' ds
MB_p = -e_z \int_0^1 W_0 \Theta ds
KB = \int_0^1 W_0 W^{(4)} ds
Bppq = -\left( 1 - \beta_1 \right) \int_0^1 W_0 \left( 2 \Theta_j \Theta_j'' W_0'' + \Theta_j \Theta_j W^{(4)}_j \right) ds
Bqqq = \int_0^1 W_0 \left( 4 W_0 W_0 W_0'' W_0'' + W_0 W_0'' W_0'' + W_0 W_0'' W_0'' \right) ds
Bqdqdq = \int_0^1 W_0 W_0 W_0 W_0' dsds - \int_0^1 W_0 W_0 W_0' \int_0^1 W_0 W_0' dsds
Bqqddq = Bqdqdq
\]

and

\[
MT_q = -J_z \int_0^1 \Theta_1 \Theta_1 ds
MT_p = -e_z \int_0^1 \Theta_1 W ds
KT = \beta_2 \int_0^1 \Theta_1 \Theta_1'' ds
Tppq = \left( 1 - \beta_1 \right) \int_0^1 \Theta_1 \Theta_1 W_0 W_0'' ds
\]

When the forcing function has been defined, the system of ODEs can then be solved using a finite difference method.
2.1.2 Forcing Function

The ONERA Dynamic stall model is a flow model that can be used to simulate unsteady flow in the separated flow region. Dynamic stall is associated with stall caused by a moving airfoil, i.e. bending and twisting. Static stall is associated with stall caused by a non-moving airfoil at a fixed angle of attack. The angle at which dynamic stall occurs is different from the angle at which static stall occurs. See Section 1.3.3.3 for an explanation.

ONERA uses the characteristics of differential equations to simulate the aerodynamic response (McAlister et al., 1984). This makes it easier to couple with the partial differential equations of the structural model. The model uses two differential equations to model the aerodynamic effects on an airfoil section element. The first accounts for the attached, unstalled flow region while the second accounts for the separated, stalled flow region (Amandolses, 2017), (Beedy et al., 2003), (Tang & Dowell, 1996). The attached results that ONERA produces should be equal to the Theodorsen Theory results. The stalled region is modeled using the Beddoes model to first calculate the static stall.

The ONERA dynamic stall model discussed here is based on methods described in Amandolses (2017) and Beedy et al. (2003). The Beddoes static stall model assumes that the trailing edge flow separation, a dimensionless parameter $f$, that causes nonlinearities in the flow can be modeled as a flat plate. The Beddoes model then improves on this by predicting the evolution of the flow separation point as it moves away from the trailing edge towards the leading edge. The flow separation point can then be used to calculate the normal lift coefficient, $C_N$, and the center of pressure, $x_{cp}$, which is used to find the moment coefficient. The center of pressure for a NACA 0012 airfoil can be predicted as

$$x_{cp} = 0.25 - k_0 - k_1 f - k_2 f^4$$

(2.13)
where

\[ f = \begin{cases} 
1 - 0.3 \exp \left( \frac{(\alpha - \alpha_0) - \alpha_1}{S_1} \right) & \text{if } \alpha \leq \alpha_1 \\
0.04 + 0.66 \exp \left( \frac{\alpha_1 - (\alpha - \alpha_0)}{S_2} \right) & \text{if } \alpha > \alpha_1 
\end{cases} \]

and where \( \alpha_1 \) is the static stall angle, \( \alpha_0 \) is the zero lift angle of attack, \( \alpha \) is the angle of attack, \( S_1 \) and \( S_2 \) are static stall characteristics that must be determined experimentally (Holierhoek et al., 2012), (Larsen et al., 2007). The constants \( k_0, k_1, \) and \( k_2 \) are derived by curve fitting center of pressure data (Larsen et al., 2007).

The normal lift coefficient is defined as

\[ C_{n} = \left( \frac{dC_N}{d\alpha} \right)_0 \left( \frac{1 + \sqrt{f}}{2} \right)^2 (\alpha - \alpha_0) \quad (2.13) \]

After the static parameters have been defined, they can be used to calculate the dynamic stall in the ONERA model.

Due to the semi-empirical nature of this method, some information about the airfoil must be based on wind tunnel experiments. For the attached part, if airfoil data cannot be obtained, data for a flat plate can be used. For the separated part data for the mean airfoil can be used, at a cost of some accuracy (Holierhoek et al., 2012). The formulas for the lift and moment coefficients are:

\[ C_L = \gamma \hat{\phi}_0 + \gamma \hat{\phi}_1 + C_{L1} + C_{L2}, \quad (2.14) \]

\[ C_M = C_{M1} + C_{M2}. \quad (2.15) \]

The first three terms in Equation (2.14) account for the attached flow and separated static flow region of the system, while the last term accounts for the separated dynamic
stall region. Similarly, the first term in Equation (2.15) accounts for the attached flow region and separated static region of the system, while the last term accounts for the separated dynamic stall region. The first two terms are relatively easy to find. \( \theta_0 \) and \( \theta_1 \) are respectively the effective angle of attack at the center of rotation and the effective angle of attack for the pitch rate contribution and are used in the lift and moment formulations below. Their time derivatives are also used in Equation (2.14) and in the lift and moment formulations below. \( \gamma_s \) and \( \gamma_k \) are parameters used for the pitch-plunge formulation, which must be determined experimentally. \( C_{L1}, C_{L2}, \) and \( C_{M2} \) are found by solving the lift and moment formulations.

**Lift Formulation**

\[
\dot{C}_{L1} + \lambda C_{L1} = \lambda \left( \frac{dC_L}{d\alpha} \right)_A \theta_0 + \lambda \sigma_s \theta_1 + \left[ \beta \left( \frac{dC_L}{d\alpha} \right)_A + \sigma_l |\Delta C_L| \right] \dot{\theta}_0 + \beta \sigma_s \dot{\theta}_1
\]

\[
\ddot{C}_{L2} + c_a C_{L2} + c_r C_{L2} = -c_r \Delta C_L (\theta_0) - c_{\theta} \dot{\theta}_0
\]

**Moment Formulation**

\[
C_{M1} = \left( \frac{dC_L}{d\alpha} \right)_A \theta_0 + \sigma_p \theta_1 + (\sigma_b + \sigma_l |\Delta C_L|) \dot{\theta}_0 + \sigma_s \dot{\theta}_1
\]

\[
\ddot{C}_{M2} + c_a \dot{C}_{M2} + c_r C_{M2} = -c_r \Delta C_M - c_{\theta} \dot{\theta}_0
\]

These formulations contain terms that must be derived experimentally. The following two tables contain data for a NACA 0012 airfoil (Holierhoek et al., 2012), (Amandolse, 2017).
Table 2. Airfoil-specific parameters to be used in ONERA dynamic stall model. Parameters are for a NACA 0012 airfoil.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Lift</th>
<th>Moment</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda$</td>
<td>$\approx 0.17$</td>
<td></td>
</tr>
<tr>
<td>$\sigma_s$</td>
<td>$\approx 2\pi$</td>
<td>$\approx 3\pi/16$</td>
</tr>
<tr>
<td>$\sigma_0$</td>
<td>$\approx 1.185\pi$</td>
<td>$-\frac{5\pi}{16}$</td>
</tr>
<tr>
<td>$\sigma_1$</td>
<td>$-0.15 \leq \sigma_1 \leq 0$</td>
<td>$0 \leq \sigma_1 \leq 0.15$</td>
</tr>
<tr>
<td>$\gamma_s$</td>
<td>$\approx \pi$</td>
<td></td>
</tr>
<tr>
<td>$\gamma_k$</td>
<td>$\approx \pi/2$</td>
<td></td>
</tr>
<tr>
<td>$\beta$</td>
<td>$\approx 0.53$</td>
<td></td>
</tr>
<tr>
<td>$\sigma_b$</td>
<td>$\approx -\pi/4$</td>
<td></td>
</tr>
<tr>
<td>$\sigma_p$</td>
<td>$\approx -\pi/4$</td>
<td></td>
</tr>
</tbody>
</table>

The formulations can be rearranged into a system of equations and solved using a solver such as Runge-Kutta 4 for $C_{L_1}$, $C_{L_2}$, and $C_{M_2}$.

The lift and moment forces are then defined respectively as (Beedy et al., 2003),

$$F_L = \frac{1}{2} \rho_a U^2 c C_L$$

and

$$M_T = \frac{1}{2} \rho_a U^2 c^2 C_M.$$  

Here, $\rho_a$ is the air density, $U$ is the wind speed, and $c$ is the chord length.

### 2.1.3 Solution Methods

After the equations of motion have been established, they are discretized into a set of coupled, nonlinear ODEs using the Galerkin technique. From there, their time derivatives are approximated using Houbolt’s method, and then solved numerically using Newton’s method.
2.1.3.1 Houbolt’s Method

Houbolt’s method is an iterative solver that can be applied to first and second order time-dependent nonlinear systems. According to Semler et al. (1996), Wu & Witmer (1973), and Bisplinghoff et al. (1955), Houbolt’s method is an efficient finite difference method (FDM) for time integration of nonlinear structural dynamic systems and has been used to numerically predict the behavior of nonlinear systems (Modarres-Sadahi et al., 2007). Similar to other schemes, it introduces some numerical damping and frequency errors; but is still able to yield an acceptable, “geometrically” similar solution when compared to other FDM solvers. This allows for the assumption that the motion of the system is accurately represented (Semler et al., 1996).

The method approximates the time derivative terms as:

\[
\ddot{x}_{n,j+1} = \left[ 2x_{n,j+1} - 5x_{n,j} + 4x_{n,j-1} - x_{n,j-2} \right] / (\Delta t)^2,
\]

\[
\dot{x}_{n,j+1} = \left[ 11x_{n,j+1} - 18x_{n,j} + 9x_{n,j-1} - 2x_{n,j-2} \right] / (6\Delta t),
\]

where \(x_{n,j}\) is equal to \(x_n(i\Delta t)\), where \(\Delta t\) is the time step. The subscript \(n\) refers to the number of mode shapes for a degree of freedom. By substituting the \(x\) ’s in Equations (2.16) and (2.17) with the \(q\) ’s and \(p\) ’s from the Galerkin discretized equations of motion, the time derivative values for these values can then be estimated. Attention must be paid to the size of the time step; if it is below a critical value, the solution may become unstable; if it is too large, the solution will not be accurate (Semler et al., 1996).
2.1.3.2 Newton’s Method

If the time step is small, it can be assumed that the solutions for $x_{j,n}$ and $x_{j,n+1}$ are close together. This allows for the use of the rapidly converging Newton’s method. For this method, the Jacobian must be determined which can be done numerically. After it has been determined, Newton’s method (2.18) is used:

$$x_{i+1} = x_i + F'(x_i)^{-1} F(x_i)$$  \hspace{1cm} (2.18)

After the system has been approximated by Houbolt’s method and the Jacobian of the system has been determined, a guess for the next iteration of the $x$, $\dot{x}$, and $\ddot{x}$ can be determined. To determine convergence, the change between the next iteration and the current iteration must be small. If the change falls below a defined tolerance, the solution for the next iteration can be assumed to have converged. If not, the process is repeated using the guessed next iteration as the starting point.

2.1.4 Numerical Code Outline

2.1.4.1 Structural Code

A Fortran code has been written that uses Houbolt’s and Newton’s Method to solve the discretized nonlinear structural model. First, the bending and torsional mode shapes discussed in Section 2.1.1 are used to solve the state-dependent terms of the discretized equations. Then, initial conditions provided by the user define the initial flapwise and torsional displacements. Houbolt’s method is then used to approximate the system using first these initial conditions and then the solution that is derived through its finite difference method. Newton’s method is used as a root finder for the system of
equations. This process is repeated for the desired time history. See Figure 7 for a block diagram describing the structural solution method with no forcing function where the solution in the figure refers to the position solution of the model.

2.1.4.2 Flow Code

The ONERA dynamic flow model uses empirical data to find the lift and moment coefficients for a stalled and unstalled fluttering airfoil. Figure 8 outlines the steps used to solve a system of first order equations with the form \( \dot{X} = AX + B \). This system can be solved using an ODE solver such as Runge-Kutta 4. Rearranging lift and moment formulations in Section 2.1.2 creates the following system (Amandolses, 2017), (Beedy et al., 2003),

\[
\begin{bmatrix}
\dot{C}_{L1} \\
\dot{C}_{L2} \\
\dot{C}_{L2} \\
\dot{C}_{M2} \\
\dot{C}_{M2}
\end{bmatrix} = 
\begin{bmatrix}
-\lambda & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & -c_r & -c_{a,L} & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & -c_r & -c_{a,M}
\end{bmatrix}
\begin{bmatrix}
C_{L1} \\
C_{L2} \\
C_{L2} \\
C_{M2} \\
C_{M2}
\end{bmatrix} + 
\begin{bmatrix}
B_1 (\theta_0, \dot{\theta}_0, \theta_1, \dot{\theta}_1) \\
0 \\
-c_r \Delta C_L (\theta_0) - c_{e,L} \dot{\theta}_0 \\
0 \\
-c_r \Delta C_L (\theta_0) - c_{e,M} \dot{\theta}_0
\end{bmatrix}
\]

where

\[
B_1 (\theta_0, \dot{\theta}_0, \theta_1, \dot{\theta}_1) = \lambda \left( \frac{dC_L}{d\alpha} \right)_{\text{attach}} \theta_0 + \lambda \sigma_{s,L} \theta_1 + \beta \left( \frac{dC_L}{d\alpha} \right)_{\text{attach}} + \sigma_{t,L} [\Delta C_L (\theta_0)] \dot{\theta}_0 + \beta \sigma_{s,L} \dot{\theta}_1
\]

and where the lowercase \( c \) terms are nonlinear parameters unique to the airfoil and are different for lift and moment.
2.1.4.3 Coupling ONERA and Structure

The equations of motion are solved using the Houbolt finite difference method and the mode shape function. This method requires the solution at a time step to be in the form

\[
X = \begin{bmatrix}
q_{1,i+1} & q_{1,i} & q_{1,i-1} & q_{1,i-2} \\
\vdots & \vdots & \vdots & \vdots \\
q_{N,j+1} & q_{N,j} & q_{N,j-1} & q_{N,j-2} \\
p_{1,i+1} & p_{1,i} & p_{1,i-1} & p_{1,i-2} \\
\vdots & \vdots & \vdots & \vdots \\
p_{N,j+1} & p_{N,j} & p_{N,j-1} & p_{N,j-2} \quad _{i=0} \quad tT \\
\end{bmatrix} , \quad \dot{X} = \begin{bmatrix}
\dot{q}_{N,j+1} \\
\vdots \\
\dot{q}_{N,j} \\
\dot{p}_{1,i+1} \\
\vdots \\
\dot{p}_{N,j+1} \quad _{i=0} \quad tT \\
\end{bmatrix} , \text{ and } \ddot{X} = \begin{bmatrix}
\ddot{q}_{N,j+1} \\
\vdots \\
\ddot{q}_{N,j} \\
\ddot{p}_{1,i+1} \\
\vdots \\
\ddot{p}_{N,j+1} \quad _{i=0} \quad tT \\
\end{bmatrix}
\]

where \( N \) is the number of modes in directions \( q \) and \( p \) which are flapwise and torsional directions respectively. The initial conditions first populate this matrix. The time derivatives of this matrix can be approximated using Equations 2.16 and 2.17. The first column of the resulting first and second time derivative along with the first column of the position matrix are used by ONERA to calculate the lift and moment coefficients of the airfoil. These matrices are used to determine \( \theta_0, \dot{\theta}_0, \theta_1, \dot{\theta}_1 \) found in the lift and moment formulations in Section 2.1.4.2. At time step \( t = T \) they are equal to

\[
\theta_0 \bigg|_{t=T} = \alpha_n(s) + \frac{\dot{h}_n(s)}{b} \bigg|_{t=T}
\]

\[
\theta_1 \bigg|_{t=T} = \left( \frac{1}{2} - a \right) \dot{\alpha}_n(s) \bigg|_{t=T}
\]

with their time derivatives equal to

\[
\dot{\theta}_0 \bigg|_{t=T} = \ddot{\alpha}_n(s) + \frac{\ddot{h}_n(s)}{b} \bigg|_{t=T}
\]
and

\[ \dot{\theta}_l \bigg|_{t=T} = \left( \frac{1}{2} - a \right) \ddot{\alpha}_n(s) \bigg|_{t=T} \cdot \]

Here, \( b \) is equal to the half chord and \( a \) is the dimensionless distance between the elastic axis and the center of lift. \( h \) and \( \alpha \) are replaced by their respective time dependent term from the left most column of matrix \( X, \dot{X}, \ddot{X} \). The subscript \( n \) refers to the mode number and \( s \) refers to the position along the blade length. Position, velocity, and acceleration matrices for flapwise and torsional motion can then be created for each mode shape at each interval along the blade length. At time step \( t = T \), the following is derived using position, velocity, and acceleration information.

\[
\begin{align*}
    h_n(s) \bigg|_{t=T} &= (q_{n,j+1})(W_n(s)) \\
    \dot{h}_n(s) \bigg|_{t=T} &= (\dot{q}_{n,j+1})(W_n(s)) \\
    \ddot{h}_n(s) \bigg|_{t=T} &= (\ddot{q}_{n,j+1})(W_n(s)) \\
    \alpha_n(s) \bigg|_{t=T} &= (p_{n,j+1})(\Theta_n(s)) \\
    \dot{\alpha}_n(s) \bigg|_{t=T} &= (\dot{p}_{n,j+1})(\Theta_n(s)) \\
    \ddot{\alpha}_n(s) \bigg|_{t=T} &= (\ddot{p}_{n,j+1})(\Theta_n(s))
\end{align*}
\]

With these relations defined, they can be used by ONERA to predict lift and moment coefficient for each \( s \) position along the airfoil. For each mode shape, the airfoil is divided into sections with each section having its own lift and moment coefficient. By integrating over the length of the airfoil, the total lift and moment coefficients of the airfoil are obtained for each mode shape. These coefficients can then be used in the equations of motion to predict the next \( X \) vector for the next time step. See Figure 9 for a block diagram of this process.
Figure 7. Numerical code outline to solve for structural system
Figure 8. Block diagram outline to solve for dynamic flow
Figure 9. Coupling structure and ONERA block diagram
CHAPTER 3
PARAMETERS AND RESULTS

A Fortran code has been written following the block diagrams of Figures 7-9. A hollow NACA 0012 airfoil with parameters given in Table 3 is considered. The material properties of Verowhite plastic are listed in Table 4.

### Table 3. Parameters for NACA 0012 airfoil

<table>
<thead>
<tr>
<th>Area (mm$^2$)</th>
<th>Length (m)</th>
<th>Chord (m)</th>
<th>$e_\xi$ (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>26.21</td>
<td>0.3</td>
<td>0.03</td>
<td>1.26</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$D_\eta$ (Nm$^2$)</th>
<th>$D_\xi$ (Nm$^2$)</th>
<th>$D_\zeta$ (Nm$^2$)</th>
<th>$j_\xi$ (kgm)</th>
<th>$j_\zeta$ (kgm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.17</td>
<td>0.077</td>
<td>0.22</td>
<td>3.627e-8</td>
<td>2.466e-6</td>
</tr>
</tbody>
</table>

### Table 4. Material properties for Verowhite plastic

<table>
<thead>
<tr>
<th>Density (kg/m$^3$)</th>
<th>Young’s Modulus (MPa)</th>
<th>Shear Modulus (MPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1175</td>
<td>2500</td>
<td>1000</td>
</tr>
</tbody>
</table>

Results are presented in the form of bifurcation diagrams (Figure 10) and sample time histories (Figures 11-13). The critical point occurs at the nondimensional wind speed, $U^* = U \tau / L$, of 0.95. In Figure 10, dimensionless wind speed vs. nondimensional flapwise displacement and angle of twist at the tip are plotted. Limit cycles are produced after this critical point. Figures 11-14 represent sample time histories, phase planes, and FFTs at low wind speeds, at the critical point, just after the critical point, and beyond the critical point. Subplots (a) and (b) represent the nondimensional tip displacement in the flapwise direction. (a) is the time history and (b) is the phase plane of the tip at steady state. Subplots (c) and (d) are similar except they represent the torsional displacement of...
the tip. Subplots (e) and (f) represent the FFTs of the tip from which what modes are present can be determined.

![Figure 10. Bifurcation for Nondimensional Flapwise (a) and Torsional (b) Directions vs non-dimensional wind speed](image)

The number of mode shapes used in the Galerkin technique depends on the system. Nayfeh et al. (2011) used three flapwise and three torsional modes for their nonlinear airfoil. They found that the correlation with this number of modes between their numerical and analytical solution was excellent (Nayfeh et al., 2011). Wadham-Gagnon et al. 2007 used two to four modes for their analysis of a cantilevered pipe conveying fluid. They found that this number of modes was sufficient to obtain reliable results. Delgado-Velazques (2007) and Arafat (1999) both studied vibrations of cantilever beams. Delgado-Velazques used four mode shapes in each bending direction and Arafat used three in each bending direction. They concluded that their numbers were sufficient to obtain accurate solutions. Freno (2010) used five mode shapes in the edgewise, flapwise, and torsional directions of his analysis of beams with various cross
sections, material properties, and geometries. He also concluded that this amount of modes yielded accurate results. For the system discussed in this thesis, six modes were used in both torsional and flapwise directions with the literature implying that this amount should be enough to capture the nonlinear structural effects. A proper convergence test must be done in the future to show that six modes are enough for this system.

Linear structural damping is assumed to be zero with damping coming from the flow. For low wind speeds, flow damping is very low, resulting in time histories that damp very slowly to zero. Figure 11 represents the behavior at $U^* = 0.57$. For this sample case, it can be clearly seen that the torsional direction (Figure 11c) damps to zero for the length of time observed. The damping is much lower in the flapwise direction (Figure 11a), resulting in a behavior that doesn’t damp out to zero. Instead, the solution appears to be linear in this direction until after the bifurcation point.

![Figure 11. Dimensionless tip behavior at $U^* = 0.57$](image)

(a) Flapwise displacement, (b) Flapwise phase plane, (c) Torsion angle, (d) Flapwise FFT, (e) Flapwise FFT, (f) Torsion FFT
Figure 12 represents the behavior at the bifurcation point $U^* = 0.95$. Subplots (a) and (b) represent the nondimensional tip and torsional displacements while (b) and (d) represent their directions respective phase plane. At this wind speed, the flow damping is still low in the flapwise direction, resulting in a time series that does not damp out to zero. Furthermore, the torsional time series takes longer to damp out to zero. Immediately after this point, the solution no longer damps to zero and limit cycles are produced in both directions. Phase planes (b) and (d) represent the behavior of the system towards the end of the time series to avoid any transient behavior at the beginning of the series. The lack of damping in the flapwise direction can be seen in subplot (b), which is almost perfectly cyclical. Furthermore, it can be seen that the extremes on the x-axis of this subplot are almost the same as the initial conditions seen in subplot (a). Subplot (d) shows how the torsional direction has damped to a very small number when compared to the initial displacement in the torsional direction seen in subplot (b).

**Figure 12. Dimensionless tip behavior at $U^* = 0.95$**
(a) Flapwise displacement, (b) Flapwise phase plane, (c) Torsion angle, (d) Flapwise FFT, (e) Flapwise FFT, (f) Torsion FFT
Figure 13. Dimensionless tip behavior at $U^* = 1.14$
(a) Flapwise displacement, (b) Flapwise phase plane, (c) Torsion angle, (d) Flapwise FFT, (e) Flapwise FFT, (f) Torsion FFT

After the bifurcation point, the overall damping becomes negative and limit cycles can be observed in the tip behavior (Figure 13a, c). The coupling of modes can be seen in subplots (e) and (f). The third bending and first torsional mode coupling is observed at a frequency of 14 Hz. The third bending and first torsional mode coupling has been observed in previous work for a similar system (Pourazam, 2016). Subplots (a) and (b) show how the amplitude of oscillation after the transient is larger than at the beginning of the time series. Subplots (b) and (d) are again representing the behavior of the system towards the end of the time series. At velocities after the bifurcation point, the amplitude of the limit cycles increases with increasing wind speed.

At the highest wind speeds shown in the bifurcation diagram, the angle of attack did not increase above the static stall angle which implies that this airfoil doesn’t stall dynamically. Higher velocities were tested, and results did show dynamic stall; however,
because the structural equations are in the normal form, the results were not representative of the system and are not included. The time histories did show lift coefficients that were increasing past the static stall point. This is because dynamic stall delays the stall onset point and the angle of attack is able to exceed the static stall angle and produce larger coefficients until the airfoil stalls dynamically. A sharp drop was then observed indicating that the flow had separated and low angles of attack and lift and moment coefficients were calculated after this point. After the dynamic stall angle, the flow does not re-attach and the motion does not recover. This behavior was observed by Holierhoek et al. (2012) and can be seen in Figure 3. This implies that the system is able to account for dynamic stall when presented with the correct structural equations of motion.
CHAPTER 4

CONCLUSIONS

Wind turbine blades are susceptible to flow-induced instabilities such as coupled-mode flutter. In this thesis, a third order nonlinear structural equation of motion has been used to represent the blade. These equations of motion were discretized using a Galerkin technique and the modal shape functions of a cantilevered beam. This changes the system from a PDE to an ODE which can be solved using a finite different method. Houbolt’s finite difference method was used to solve the system because it is one of the most accurate and efficient time solvers for elastic nonlinear IVPs. The structural equations were coupled with ONERA by sectioning the airfoil and using the angular position, velocity, and acceleration and lateral position and velocity of each section to determine the effective angle of attack for that section. This was then used to calculate lift and moment coefficients for each airfoil section, which could then be used to calculate the lift and moment forces of the airfoil.

With the equations of motion defined and assuming all damping comes from the flow, time histories and phase planes for various wind speeds were calculated, which were then used to produce bifurcation diagrams. Bifurcation was predicted at $U^* = 0.95$ and limit cycles with increasing amplitudes were observed after. The third bending and the first torsional mode were observed to be coupled, indicating a coupled-mode flutter. Dynamic lift and moment coefficients were observed as well as dynamic stall. However, dynamic stall was observed at wind speeds well past the bifurcation point. Because the structural equations are in the normal form, they do not represent the dynamic behavior of the system far from the point of instability and those results are not included. Overall,
a nonlinear model for predicting flow–induced instabilities of a flexible airfoil placed in wind and sample results are produced.
CHAPTER 5

FUTURE WORK

The current state of the solver couples a structural equation with a forcing function. The solver must be further refined by doing the following:

- Create fully nonlinear model. Include all nonlinear structural terms to account for flutter far beyond the critical point.
- Include proper linear damping terms. Find the correct damping of each mode and apply them.
- Numerical results must be compared to experiments.
APPENDIX A

HAMILTON’S PRINCIPLE EXPLAINED

Hamilton’s principle is described in (Meirovitch, 1967) and summarized here. The principle considers the entire motion of the system between two time instants and is invariant with respect to the coordinate system used. The Hamilton’s principle solution system is equal to the solution determined using Lagrangian methods. Consider a system of \( N \) particles subjected to kinematical conditions. The work of the system is expressed as

\[
\sum_{i=1}^{N} \left( m_i \ddot{r}_i - \vec{F}_i \right) \cdot \delta \vec{r}_i = 0. \tag{A.1}
\]

Also note that the virtual work of the system is equal to

\[
\delta W = \sum_{i=1}^{N} \vec{F}_i \cdot \delta \vec{r}_i \tag{A.2}
\]

Assume that \( \frac{d}{dt} \) and \( \delta \) are interchangeable. This allows part of (A.1) to be rewritten as

\[
\ddot{r} \cdot \delta \dot{r} = \frac{d}{dt} \left( \dot{r} \cdot \delta \dot{r} \right) - \delta \left( \frac{1}{2} \dot{r} \cdot \dot{r} \right). \tag{A.3}
\]

Plugging (A.3) and (A.2) into (A.1) allows the system to be solved in terms of \( \delta T \) and \( \delta W \) where \( T \) is the kinetic energy of the system:

\[
\delta T + \delta W = \sum_{i=1}^{N} m_i \frac{d}{dt} \left( \dot{r}_i \cdot \delta \vec{r}_i \right) \tag{A.4}
\]

where \( \vec{r}_i \) is a position vector. If a small variation is allowed in \( \delta \vec{r}_i \) with no change in time, i.e. \( \delta t = 0 \), then the true path of the particle has been varied slightly. If we assume that the varied path and the true path intersect at two arbitrary points in time, then
$\delta \bar{r}_i$ is equal to zero at those points in time. With this information, (A.4) can be integrated with respect to those two points in time. The result of this integration is that the right hand side of equation (A.4) is equal to zero,

$$\int_{t_1}^{t_2} (\delta T + \delta W) dt = \sum_{i=1}^{N} \int_{t_i}^{t} m_i \frac{d}{dt} (\dot{\bar{r}} \cdot \delta \bar{r}) dt = \sum_{i=1}^{N} m_i (\dot{\bar{r}} \cdot \delta \bar{r}) \bigg|_{t_1}^{t_2} = 0$$

(A.5)

If $\delta W$ is comprised of conservative and nonconservative forces, (A.5) becomes

$$\int_{t_1}^{t_2} (\delta T - \delta U + \delta W) dt = 0$$

(A.6)

which is the extended form of Hamilton’s principle and is used to derive all the equations of motion for a system. In (A.6), $T$ is the kinetic energy, $U$ is the potential energy, and $W$ is the virtual work done by the nonconservative external forces.
APPENDIX B

DERIVATION OF THE EQUATIONS OF MOTION FOR A BEAM USING HAMILTON’S PRINCIPLE

The following is an example of the derivation of the equation of motion for a vibrating cantilever beam using Hamilton’s principle, which has been taken from (Meirovitch, 1967).

Figure 11 represents a nonuniform bar in transverse vibration that is allowed to bend and twist. The kinetic energy and potential energy of this beam are expressed respectively as

\[ T(t) = \frac{1}{2} \int_0^L \left( \frac{\partial y(x,t)}{\partial t} \right)^2 m(x)dx + \frac{1}{2} \int_0^L \left( \frac{\partial \psi(x,t)}{\partial t} \right)^2 k(x)^2 m(x)dx, \]  

\[ (B.1) \]

and

\[ U(t) = \frac{1}{2} \int_0^L EI(x) \left( \frac{\partial \psi(x,t)}{\partial x} \right)^2 dx + \frac{1}{2} \int_0^L k'GA(x) \beta(x,t)^2 dx. \]  

\[ (B.2) \]

Their variations follow respectively as

\[ \delta T = \int_0^L \left( \frac{\partial \psi}{\partial t} \right) \delta \left( \frac{\partial y}{\partial t} \right) m(x)dx + \int_0^L \left( \frac{\partial \psi}{\partial t} \right) \delta k(x)^2 m(x)dx, \]  

\[ (B.3) \]

and

\[ \delta U = \int_0^L EI(x) \frac{\partial \psi}{\partial x} \delta \left( \frac{\partial \psi}{\partial x} \right) dx + \int_0^L k'GA(x) \beta \delta \beta dx. \]  

\[ (B.4) \]

The variation due to nonconservative virtual work is defined as

\[ \delta W = \int_0^L p \delta y dx. \]  

\[ (B.5) \]
In these equations, $E$ is the modulus of elasticity, $I(x)$ is the moment of inertia about the neutral axis as a function of position, $k'$ is a numerical factor depending on the cross section, $k$ is the radius of gyration, $G$ is the shear modulus, $A(x)$ is the cross section as a function of position, $L$ is the length of the beam, and $p$ is the distributed load, $\beta$ and $\psi$ are equal to the angle of distortion due to shear and angle of rotation due to bending respectively. They are related to each by $\partial y(x,t)/\partial t = \psi(x,t) + \beta(x,t)$.

Figure 14. Nonuniform beam in transverse vibration (Meirovitch, 1967).

Next, equations (B.1) to (B.5) are plugged into the extended form of Hamilton’s principle, (A.6). This leads to

$$
\int_0^L \left[ \int_0^L m \frac{\partial y}{\partial t} \delta \left( \frac{\partial y}{\partial t} \right) \, dx \right] + \int_0^L k^2 m \frac{\partial \psi}{\partial t} \delta \left( \frac{\partial \psi}{\partial t} \right) \, dx
$$

$$
= -\int_0^L EI \frac{\partial \psi}{\partial x} \delta \left( \frac{\partial \psi}{\partial x} \right) \, dx - \int_0^L k'GA \left( \frac{\partial y}{\partial x} - \psi \right) \delta \left( \frac{\partial y}{\partial x} - \psi \right) \, dx + \int_0^L p\delta y \, dx \, dt = 0.
$$

(B.6)
Integrating the time derivative terms $\frac{\partial}{\partial t}$ by parts over the domain $t_1$ to $t_2$ and recognizing that $\delta y = \delta \psi = 0$ at $t_1$ and $t_2$ allows for the following expressions.

\[
\int_{t_1}^{t_2} m \frac{\partial y}{\partial t} \frac{\partial}{\partial t} \left( \frac{\partial y}{\partial t} \right) dt = -\int_{t_1}^{t_2} m \frac{\partial^2 y}{\partial t^2} \delta y dt, \tag{B.7}
\]

\[
\int_{t_1}^{t_2} k^2 m \frac{\partial \psi}{\partial t} \frac{\partial}{\partial t} \left( \frac{\partial \psi}{\partial t} \right) dt = -\int_{t_1}^{t_2} k^2 m \frac{\partial^2 \psi}{\partial t^2} \delta \psi dt. \tag{B.8}
\]

Furthermore, integrating the spatial terms $\frac{\partial}{\partial x}$ by parts yields the following expressions.

\[
\int_0^L EI \frac{\partial \psi}{\partial x} \frac{\partial}{\partial x} \left( \frac{\partial \psi}{\partial x} \right) = \left( EI \frac{\partial \psi}{\partial x} \right) \delta \frac{\partial \psi}{\partial x} \bigg|_0^L - \int_0^L \frac{\partial}{\partial x} \left( EI \frac{\partial \psi}{\partial x} \right) \delta \psi dx \tag{B.9}
\]

\[
\int_0^L k'GA \left( \frac{\partial y}{\partial x} - \psi \right) \frac{\partial}{\partial x} \left( \frac{\partial y}{\partial x} - \psi \right) = \left( k'GA \left( \frac{\partial y}{\partial x} - \psi \right) \right) \delta y \bigg|_0^L - \int_0^L \frac{\partial}{\partial x} \left( k'GA \left( \frac{\partial y}{\partial x} - \psi \right) \right) \delta y dx - \int_0^L \left( k'GA \left( \frac{\partial y}{\partial x} - \psi \right) \right) \delta y dx \tag{B.10}
\]

Because $\delta \psi$ and $\delta y$ are zero at $x = 0$ and $x = L$, and are arbitrary between these bounds, the equations of motion and the boundary conditions can be extracted from equations (B.7) to (B.10). The bending and torsional equations of motion are respectively

\[
\frac{\partial}{\partial x} \left[ k'GA \left( \frac{\partial y}{\partial x} - \psi \right) \right] - m \frac{\partial^2 y}{\partial t^2} + p = 0 \tag{B.11}
\]

and

\[
\frac{\partial}{\partial x} \left( EI \frac{\partial \psi}{\partial x} \right) + k'GA \left( \frac{\partial y}{\partial x} - \psi \right) - k^2 m \frac{\partial^2 \psi}{\partial x^2} = 0. \tag{B.12}
\]

The boundary conditions are respectively

\[
\left( EI \frac{\partial \psi}{\partial x} \right) \delta \psi \bigg|_0^L = 0 \tag{B.13}
\]
and

\[
\left( k'GA \left( \frac{\partial y}{\partial x} - \psi \right) \right) \delta y \bigg|_0^L = 0. \tag{B.14}
\]

Equations (B.13) and (B.14) can be better defined by applying the conditions of a cantilever beam. This means that at \( x = 0 \) and \( \psi = y = 0 \) for all \( t \) and at \( x = L \),

\[
EI \partial \psi / \partial x = k'GA(\partial y / \partial x - \psi) = 0. \]

The equations of motion and the boundary conditions comprise the boundary value problem.
BIBLIOGRAPHY


